Research Methods of Multiparameter Spectral Problems and the Nonlinear Algebraic Equations

Written by

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Horizon Research Publishing, USA

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Introduction

Spectral theory of operators is the essential direction of the functional analysis. Development of physical sciences set all new and new tasks for mathematicians. In particular, solutions of the boundary problems for the physical processes, a research of partial differential equations and equations of mathematical physics demanded new approach. The method of a separation of variables in many cases was the only accepted as it reduced finding of the solution of the complex equation with many variables to finding of the solutions of system of ordinary differential equations which research was the simpler. For example, problems in quantum mechanics [90], [85], [83], [116]; diffraction theory[72], [115], the theories of elastic envelopes(of shells), calculation of nuclear reactor (theories of resilient envelopes, calculation of nuclear reactors) [85]; stochastic diffusion processes [83], Brownian motion [83], boundary value problems for equations of elliptic-parabolic type, the Cauchy problem of ultra parabolic equations [68], and etc. lead to multivariable problems.

The spectral theory of multiparameter systems was studied insufficiently, despite of importance and prescription of researches. Until recently the available results in this area apply only to multivariable systems of a selfadjoint operators depending linearly on spectral parameters

F.V. Atkinson [12] created multiparameter spectral theory of selfadjoint systems in finite dimensional Euclidean spaces, studied the fragmentary results for multiparameter symmetric differential systems. Further, by taking the limit Atkinson generalized the results, obtained for the multiparameter systems with selfadjoint operators in finite dimensional space, on the case of the multiparameter system with selfadjoint completely continuous operators in Hilbert spaces.

Further, Browne, Sleeman, Roch and other mathematicians built the spectral theory of selfadjoint multiparameter system in infinite-dimensional Hilbert spaces [32], [92], [106].

But, unfortunately, the research technique in these works significantly uses a selfadjointness of the all operators entering the system. This technique does not allow resolving the most prime problems which are traditionally studied in the spectral theory of operators.

In this monograph the author offers new approach to the research of multiparameter problems, gives also new way of a separation of variables in partial differential equations that allows expanding significantly a class of the studying tasks.

For multiparameter system of operators are introduced the concepts of the associated vectors, the multiple completeness of system of eigen and associated vectors, essential in the spectral theory of operators, and obtained the main results, traditional in spectral theory of operators in Hilbert space.

It is studied the application of the results of multiparameter spectral theory to the study of nonlinear algebraic system with complex dependence on several unknowns. The nonlinear algebraic systems with many variables are the special case of multiparameter systems of operators.

Linear algebra is central to modern mathematics and its applications. Linear algebra had its origin in the study of vectors in Cartesian 2-space and 3-space. The modern linear algebra has been extended to consider spaces of arbitrary or infinite dimension and linear mappings between such spaces.

Since linear algebra is a successful theory, its methods have been developed and generalized in other parts of mathematics. The study of linear algebra first emerged from the study of determinants, which were used to solve systems of linear equations. Determinants were used by Leibniz in 1693, and subsequently, Gabriel Cramer devised Cramer's rule for solving linear systems in 1750.

Solving for one unknown in one equation was already studied. It is more interesting the situation when more than one unknown variable present in more than one equation. An elementary application of linear algebra is to find the solution of a system of linear equations in several unknowns .When in a given problem more than one of algebraic equation are true in one and the same time, then it is said there is a system of equations which are true simultaneous . The word "system" indicates that the equations are to be considered collectively, rather than individually.

The principal objectives of linear algebra are the resolution of the systems of linear equations. Such an investigation is initially motivated by a system of linear equations with several unknown variables. These equations are naturally represented using the formalism of matrices and vectors. The majority of problems of linear algebra suppose an equivalent statement in each of these theories.

Naturally, the study of linear algebra includes the topics of vector algebra, matrix algebra, and the theory of vector spaces. It includes a range of theorems and applications in different branches of linear algebra, such as linear systems, matrices, operators, etc. Functional analysis studies the infinite-dimensional version of the theory of vector spaces. Techniques of linear algebra are also used in analytic geometry, in engineering, in physics, in natural sciences, computer science, in computer animation, and in the social sciences (particularly in economics).

In module theory one replaces the field of scalars by a ring. The concepts of linear span, basis, and dimension (called rank in module theory) still make sense.

In 1882, Hüseyin Tevfik Pasha has written the book titled "Linear Algebra"[70], [71] in which has included bases of linear algebra.

The first modern and more precise definition of a vector space was introduced by Peano in 1888- 1900, a theory of linear transformations of finite-dimensional vector spaces had emerged. Linear algebra first took its modern form in the first half of the twentieth century, when many ideas and methods of previous centuries were generalized as abstract algebra. The use of matrices in quantum mechanics, special relativity, and statistics helped spread the subject of linear algebra beyond pure mathematics. The development of computers led to increased research in efficient algorithms for Gaussian elimination and matrix decompositions, and linear algebra became an essential tool for modeling and simulations.

This monograph is devoted to researches of the problems of nonlinear algebraic system of equations. The new approach of the decision of nonlinear algebraic systems of the equations is offered when unknown variables are included into this system as polynomials. In case of nonlinear algebraic system when the number of unknowns is equal to the numbers of the equations author gives the method of constructing of the analogue of Cramer's determinants. It is known that the linear algebraic system of equations has a solution if and only if the basic determinant of Cramer of this system is distinct from zero. In addition, we also confirm that in the case of nonlinear algebraic system if the constructed analogue of determinant of Cramer is not equal to zero then it is a necessary and sufficient condition for the existence of solutions of the nonlinear algebraic system of equations. Here we suppose that the number of equations is equal to the number of unknowns.

The new way for research of systems of the algebraic equations allows finding of the solutions of nonlinear algebraic system under some conditions, naturally, when the number of unknowns coincides or less of the number of the equations of algebraic system. It is studied also the properties of the solutions of nonlinear algebraic systems of equations, namely, under certain conditions it is proved the reality of them.

Besides of, we consider multiparameter system of operators in Euclidian space, and for them the different theorems about completeness of eigen and associated vectors are proved.

Monograph may be of interest to senior students of mathematical faculties of high schools and universities, graduate students, and professionals in the field of spectral theory of operators.

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Chapter 1. Basic Concepts and Definitions Needed for Understanding of the Content of the Monograph

1.1. Hilbert Space

The linear set of elements $x, y, z, \ldots$ when for each couple of elements x, y is defined the complex number (x, y) satisfying the following conditions

$$\begin{aligned} &1)(y,x)=\overline{(x,y)} \\ &2)(\lambda x_1+\mu x_2 y)=\lambda(x_1,y)+\mu(x_2,y) \\ &3)(x,x)\geq 0;(x,x)=0\rightarrow x=0 \end{aligned},$$

is called by pre-Hilbert space, and the number (x, y) is called by inner product of the space.

The Cauchy-Bunyakovskii inequality takes place

$$|(x,y)|^2 \leq (x,x)(y,y)$$

If the linear set with the given inner product is full (contains the limit points), then it call Hilbert space. The Hilbert space H is separable if in it there is an everywhere dense countable subset. In this case the space H contains the countable sequence of elements $\{x_n\}_1^\infty$ such, that any element uniquely decomposes in series in the elements of $\{x_n\}_1^\infty$. If the maximal number of linearly independent vectors of space is finite, then this Hilbert space is finite dimensional. The maximal number of linearly independent elements of space is accepted as its dimension.

Completeness of space - an essential factor in order that any vector space with the given inner product could be included in the complete Hilbert space.

If in space H inner product is $\|x\| = \sqrt{(x, x)},\ (x \in H)$, then H becomes the normed space.

1.1.1. Orthogonal Sum of Subspaces

Let be m_μ closed mutual orthogonal subspaces in a Hilbert space. The minimum subspace containing all m_μ is called the orthogonal sum of subspaces, and designated through $\sum_\nu \oplus m_\nu$. The orthogonal sum $m = m_1 \oplus m_2 + \ldots \oplus m_n$ of a finite number of subspaces m_μ is set of all vectors $x = x_1 + \ldots + x_n$, $x_k \in m_k, k = 1,2,\ldots,n$, respectively; the orthogonal sum of countable number of subspaces is set of vectors $x = x_1 + \ldots + x_n + \ldots$, where $x_k \in m_\mu$, and the

the product of functions $f_1(x_1)f_2(x_2)(f_1 \in L_2(\overline{G}), f_2 \in L_2(\overline{\overline{G}}))$; the closure of all linear combinations of the products of these functions $f_1(x_1)f_2(x_2)$ coincides with the entire space $L_2(\overline{G}\times\overline{\overline{G}}) = L_2(\overline{G})\otimes L_2(\overline{\overline{G}})$.

If the operator $A_1(A_2)$ is a bounded operator, then the operator $A_1\otimes A_2$ from space $H_1\otimes H_2$ to space $\overline{G}\otimes\overline{\overline{G}}$ is also a bounded operator. The operator $A_1\otimes A_2$ is defined as $A_1\otimes A_2\sum f_{1,j}\otimes f_{2,j} = \sum A f_{1,j}\otimes A_2 f_{2,j}$ on the elements $\sum f_{1,j}\otimes f_{2,j}$ $(f_{1,j}\in H_1, f_{2,j}\in H_2)$. So the definition of operator does not depend of form the element $\sum f_{1,j}\otimes f_{2,j}$. If operators A_1 and A_2 are bounded in the space H_1 and H_2, respectively, then the operator $A_1\otimes A_2$ in the space $H_1\otimes H_2$ is also bounded, in addition $\|A_1\otimes A_2\| \le \|A_1\|\|A_2\|$. Main properties of the tensor product:

$$(A_1 - B_1)\otimes C_2 = A_1\otimes C_2 - B_1\otimes C_2$$

$$A_1\otimes(B_2 - C_2) = A_1\otimes B_2 - A_1\otimes C_2$$

$$\lambda(A_1\otimes B_2) = (\lambda A_1)\otimes B_2 = A_1\otimes(\lambda B_2)$$

$$(A_1\otimes A_2)(B_1\otimes B_2) = A_1B_1\otimes A_2B_2$$

$$(A_1\otimes A_2)^* = (A_1)^*\otimes(A_2)^*$$

In the future, we will also use the tensor product of operators $A_1\otimes A_2$ when the operators A_1 and A_2 act in space H_1 and H_2, respectively, and their domain of definitions $D(A_1)$ and $D(A_2)$ are unrestricted [18].

Chapter 2. Abstract Separation of Variables

2.1. Aspects of the Use of the Method of Separation of Variables

In the works of Morse Feshbach [88], Roche [93], Sleeman [109], Artsgolts [9], [10] have paid a lot of attention, and have showed in detail the essence of the origin and development of multiparameter system as a result of applying of the method of separation of variables in the equations of mathematical physics, and in partial differential equations.

The method of separation of variables applied to various boundary problems. The initial conditions is considered as a special case of the border.

[88] In particular, it is identified a number of coordinate systems, which allow separating of variables in the equations. Then there is the need to study of obtained after the separation of variables of several ordinary differential equations.

To understand and not be limited to general observations, in [88] carried out a study for the equation

$$(\Delta^2 + k^2)\phi = \phi \tag{2.1}$$

ϕ -scalar field, Δ^2 - Laplace operator.

If $k = 0$ in (2.1), then (2.1) is the Laplace equation for the potential static fields.

If $k > 0$ in (2.1), then (2.1) is a sinusoidal wave equation for the function of time (Helmholtz), or the diffusion equation for exponential function of time.

If k^2 is a function of the coordinates, then (2.1) is the Schrödinger equation for a particle with constant energy E; k^2 is proportional to the difference between E and the potential energy of the particle.

Since the equation has an infinite number of solutions, the question is about finding private solutions, or solutions that correspond to particular problem which we want to solve. These different specific problems differ from each other in the nature of the imposed conditions: changing in the form of the boundary or the behavior of the field at the boundaries.

The boundary conditions, imposed on the field, consist in setting the field value in each of the boundary surfaces, or the normal component gradient setting of this surface. The Dirichlet problem is given by the values on the surface ϕ; boundary problem of Neumann is given by the set of values $\partial\phi / \partial n$; in the Cauchy problem are given both as field values, and the values of the normal component of its gradient. For finding of a clear definition of solution on the border it is necessary to build coordinate system adapted to this boundary. We choose such coordinate system ξ_1, ξ_2, ξ_3, to the boundary surface has become one of the coordinate

Chapter 3. Selfadjoint Multiparameter Systems

3.1. Research Methods in Spectral Theory of Selfadjoint Multiparameter Systems

The first works on spectral theory of multiparameter system of operators were the investigations of Faierman concerning to the system of differentional equations. Faierman [60], [61] considered the system

$$\frac{d^2 y_r(x_r)}{dx_r^2} + q_r(x_r)y_r(x_r) + \sum_{s=1}^{k} \lambda_s a_{r,s} y_r(x_r) = 0, r = 1,2,...,k , \quad (3.1)$$

when $q_r(x_r), a_{r,s}(x_r), r, s = 1,2,...,k$ are continuous, real valued and differentiable on the interval $[a_r, b_r]$ of real axis.

System (3.1) with the common boundary conditions

$$\begin{gathered} y_r(a_r)Cos\alpha_r + y_r^{/} Sin\alpha_r = 0; \alpha_r \in [0,\pi) \\ y_r(b_r)Cos\beta_r + y_r^{/} Sin\beta_r = 0; \beta_r \in (0,\pi] \\ r = 1,2,...,k \end{gathered} \quad (3.2)$$

is the $k-$ parameter Sturm-Luiville system.

If condition $a(x) = \det\{a_{r,s}(x_r)\} > 0$ for $\forall x = \{x_1, x_2, ..., x_k\}$ in the systems (3.1), and (3.2) are satisfied, then their eigenvalues belong to R^k and do not have a finite limit point.

Let $\{p_1, p_2, ..., p_k\}$ be the set of nonnegative integers, then there is exactly one eigenvalue $\lambda \in R^k$ of the problem (3.1) and (3.2), for which the function $y_r(x_r)$ has exactly p_r zeroes in the interval of $[a_r, b_r]$.

Assuming $a_{r,s}(x) \in C[0,1]$, $a_{r,s}(x)$ and $q(x)$ are differentiable functions, Faerman [60] also proved that the eigenfunctions of the problem (3.1) and (3.2) form a complete orthogonal system with respect to the weight function $\det\{a_{r,s}(x_s)\} \in L^2(I_k)$, where $\forall x = \{x_1, x_2, ..., x_k\} \in I_k$ the Cartesian product of intervals.

Later Browne [31] establishes this result without conditions of differentiable of functions.

A brief proof of the fundamental theorem of Browne [32] gives research methods for study of selfadjoint multiparameter systems.

For the purpose of presenting of the method of investigation of multiparameter selfadjoint spectral theory we present some results in this direction.

means that the condition on the determinant for system (3.18) is satisfied. Really,

$$\left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 & A_2 \\ A_2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 & B_2 \\ B_2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] =$$

$$= [(A_1x_1, x_1) + (A_2x_2, x_2)].[(B_1y_1, y_1) + (B_2y_2, y_2)] -$$

$$- [(A_2x_1, x_2) + (A_2x_2, x_1)].[(B_2y_1, y_2) + (B_2y_2, y_1)] =$$

$$= \sum_{i=1}^{2}(A_ix_i, x_i)\sum_{i=1}^{2}(B_iy_i, y_i) - 4\operatorname{Re}(B_2y_1, y_2)\operatorname{Re}(A_2x_1, x_2) \geq \mu\sum_{i=1}^{2}\|x_i\|^2\sum_{i=1}^{2}\|y_i\|^2.$$

Then for any vector $\widetilde{f} \in \widetilde{H}$ equality of Parseval-Atkinson takes place $\widetilde{f} = \sum_{i=1}^{k}\begin{pmatrix} f_{1i} \\ f_{2i} \end{pmatrix} \otimes \begin{pmatrix} g_{1i} \\ g_{2i} \end{pmatrix} = \int_{\lambda} Ed\lambda \sum_{i=1}^{k}\begin{pmatrix} f_{1i} \\ f_{2i} \end{pmatrix} \otimes \begin{pmatrix} g_{1i} \\ g_{2i} \end{pmatrix}$}

For elements of space H by virtue of representation

$$\widetilde{f} = \sum_{i=1}^{m}(f_{1i} \otimes g_{1i}, f_{2i} \otimes g_{1i}, f_{1i} \otimes g_{2i}, f_{2i} \otimes g_{2i})$$

we have $\sum_{i=1}^{m} f_{1i} \otimes g_{1i} = \int_{\lambda} E(d\lambda)\sum_{i=1}^{m} f_{1i} \otimes g_{1i}$. Thus, the Theorem 3.7 is proved.

Chapter 4. Research Methods of Nonselfadjoint Multiparameter Systems

4.1. Statements and Definitions

The spectral theory of operators is one of essential directions of a functional analysis. Many partial differential equations and the equations of mathematical physics, connected with physical processes, demanded the new approach for a solution of these problems. Research of the equation of mathematical physics, partial differential equations, in particular, a solution of a Cauchy problem for the operationally-differential equation lead to research of multiparameter system

$$A_i(\lambda) = A_{0,i} + \lambda_1 A_{1,i} + \lambda_2 A_{2,i} + \ldots + \lambda_n A_{n,i}$$

$$i = 1,2,\ldots n, \tag{4.1}$$

In (4.1) operators $A_{i,k}$ act in separable Hilbert space H_k and bounded.

Despite of an urgency and prescription of researches, the spectral theory of multiparameter systems was studied insufficiently. Available outcomes in this area until recently are obtained only for multiparameter selfadjoint systems of the operators linearly depending on spectral parameters.

The founder of researches of spectral problems of the multiparameter selfadjoint systems was F.V. Atkinson [12]. Studied the outcomes which are available for multiparameter symmetrical differential systems, Atkinson had constructed the spectral theory of multiparameter systems in finite-dimensional spaces. Further, by means of passage to the limit Atkinson could generalize the received outcomes on a case of multiparameter systems with the selfadjoint compact operators in infinite-dimensional Hilbert spaces.

In the further, the method that had introduced by Atkinson for study of multiparameter systems in finite-dimensional spaces, it has appeared possible for building and in infinite-dimensional spaces. Last has allowed to construct the spectral theory of multiparameter selfadjoint systems in Hilbert spaces [32], etc.

But, unfortunately, the technique of research in these works essentially used a selfadjointness of operators entering into system that is why, this technique of investigation of selfadjoint multiparameter systems did not allow to solve the simplest problems of the spectral theory of operators. We give a series of known positions from the spectral theory of multiparameter systems.

Definition 4.1. Operator $A_i^+ (i = 1,2,\ldots,n)$ is called by induced operator in the space H by operator A_i, acting in the space H_i, defined on decomposable elements of H with help of the formulae $A_i^+ z = A_i^+ x_1 \otimes x_2 \otimes \ldots \otimes x_n = x_1 \otimes x_2 \otimes \ldots \otimes A_i x_i \otimes x_{i+1} \otimes \ldots \otimes x_n$, when $z = x_1 \otimes x_2 \otimes \ldots \otimes x_n$. On all other elements of space

Chapter 5. Multiparameter System of Operators, Nonlinearly Depending on Parameters

5.1. Multiple Basis of Eigen and Associated Vectors of a Two-parameter System in Finite-dimensional Spaces

Concepts of basis, multiple basis, the associated vectors of operator, completeness and multiple completeness of system of eigen and associated vectors of operator are essential in the spectral theory of multiparameter system. In the Chapter 5 all necessary notions are introduced and the new approach to the study of multiparameter problems is given.

Let

$$A(\lambda,\mu)x = (A_0 + \lambda A_1 + \ldots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \ldots + \mu^{n_1} A_{m_1+n_1})x = 0$$

$$B(\lambda,\mu)y = (B_0 + \lambda B_1 + \ldots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \ldots + \mu^{n_2} B_{m_2+n_2})y = 0 \quad (5.1)$$

be multiparameter system in two parameters. In (5.1) linear operators $A_i (i = 0,1,\ldots,m_1 + n_1)$ act in finite-dimensional space H_1; and linear operators $B_i (i = 0,1,\ldots,m_2 + n_2)$ act in finite-dimensional space H_2.

$H_1 \otimes H_2$ is tensor product of finite-dimensional spaces H_1 and H_2. Dimension of space H is the product of dimensions of spaces H_1 and H_2. If $f_1 \otimes f_2 \in H_1 \otimes H_2$ and $g_1 \otimes g_2 \in H_1 \otimes H_2$, then inner product of these elements in space $H_1 \otimes H_2$ is defined by means of the formulae $[f_1 \otimes f_2, g_1 \otimes g_2]_{H_1 \otimes H_2} = (f_1, g_1)_{H_1} \cdot (g_1, g_2)_{H_2}$

This definition is spread to other elements of tensor product spaces on linearity.

By means of the approach, stated in this monograph, it manages to establish completeness, multiple completeness of system of eigen and associated vectors of (5.1), a possibility of multiple expansion at system of eigen and associated vectors of multiparameter system. Let's present a number of known positions from the spectral theory of multiparameter systems.

In the special case when the number of parameters in (5.1) is equal to 2 the definition of associated vector of system (5.1) looks like

Definition 5.1. A tensor z_{m_1,m_2} is called (m_1, m_2)- the associated vector to an eigenvector $z_{0,0} = x \otimes y$ of the (5.1) if following conditions are satisfied

$$\sum_{0 \le r_i \le k_i} \frac{1}{r_1! r_2!} \frac{\partial^{r_1+r_2} A^+(\lambda,\mu)}{\partial \lambda^{r_1} \partial \mu^{r_2}} z_{k_1-r_1,k_2-r_2} = 0$$

Chapter 6. Analoque of a Resolvent of a Multiparameter System

An abstract concept of the resolvent of a multiparameter system of operators in Hilbert space was introduced for the first time. It is known that the concept of the resolvent of an operator is closely connected with the concept of the existence of an inverse operator, which is problematic in the case of a multiparameter system. In Chapter 6, the decomposition of the introduced resolvent in the vicinity of an isolated eigenvalue of a multiparameter system is given.

The problem of existence of a resolvent of a linear operator A is tightly connected with existence of an inverse operator $A-\lambda E$ $(\lambda \in C^n)$ which exists in only case when λ is not eigenvalue of an operator. Thus the inverse operator can be as bounded, or unbounded. If the inverse operator exists and bounded, λ is a point of a resolvent set $\lambda \in \rho(A)$. In a multiparameter case situation is a little differently. Even if $\lambda^0 \in C^n$ is not an eigenvalue of system, the inverse operator of the system may not exists.

Let be

$$A_k(\lambda)x_k = \sum_{0\le i_s\le k_s} \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n} A_{i_1 i_2\ldots i_n}^{(k)} x_k ,$$

$$k,s = 1,2,\ldots,n , \qquad (6.1)$$

$A_{i_1 i_2\ldots i_n}^{(k)}$ in (6.1) are the linear bounded operators acting in Hilbert space H_k, an index i_s specifies on a degree of parameter λ_s, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in C^n$, k - an index of space H_k; $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ - tensor product of spaces $H_1, H_2, \ldots, H_n$.

Nevertheless, for the any set of n elements $y_1, \ldots, y_n$, where $y_i \in R_{A_i(\lambda_0)}$, it is not always possible to restore a pre-image of this system $y_1, \ldots, y_n$ since λ^0 can be an eigenvalue for one from equations (6.1).

Let's show that if λ^0 is not eigenvalue (6.1), then a tensor $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ uniquely is restored. For this purpose we define some analogue of a resolvent for our system (6.1). By means of system (6.1) we build an operator $A^+(\lambda)$ acting from space $H = H_1 \otimes \ldots \otimes H_n$ into the space $H^n = \bigotimes_1^n H_1 \otimes \ldots \otimes H_n$, made of elements

$$\{ y_1 \otimes x_2 \otimes \ldots \otimes x_n;\ x_1 \otimes y_2 \otimes x_3 \otimes \ldots \otimes x_n; \ldots;\ x_1 \otimes x_2 \otimes \ldots \otimes y_n \} \qquad (6.2)$$

and their of linear combinations.

From (6.2) we have

$$A^+(\lambda)\sum_{i=1}^{s} x_1^i \otimes \ldots \otimes x_n^i = \left(A_1^+(\lambda)\sum_{i=1}^{s} x_1^i \otimes \ldots \otimes x_n^i, \ldots, A_n^+(\lambda)\sum_{i=1}^{s} x_1^i \otimes \ldots \otimes x_n^i \right) =$$

$$= \left(\sum_{i=1}^{s} y_1^i \otimes x_2^i \otimes \ldots \otimes x_n^i, \ldots, \sum_{i=1}^{s} x_1^i \otimes x_2^i \otimes \ldots \otimes y_n^i \right) \qquad (6.3)$$

Chapter 7. Nonlinear Algebraic Systems

7.1. Research Methods of Nonlinear Algebraic Systems

7.2. Use of a Notion of Resultant in the Solving of the Simple Linear Algebraic Equation

7.3. Common Roots of Several Polynomials in One or More Variables

7.4. Nonlinear Algebraic Systems with Three Variables

7.5. The Nonlinear Algebraic System of Equations with Many Variables

7.1. Research Methods of Nonlinear Algebraic Systems

Chapter 7 of this monograph is devoted to researches of the problems associated with nonlinear algebraic equations. In the case when algebraic system is linear, and the number of variables coincides with the number of equations we have a fairly well-studied the system of algebraic equations.

Linear algebra is central to modern mathematics and its applications. Linear algebra is a successful theory; its methods have been developed and generalized in other parts of mathematics. In module theory, one replaces the field of scalars by a ring. The concepts of linear independence, span, basis, and dimension (which are called the rank in module theory) still make sense.

An elementary application of linear algebra is to find the solution of a system of linear algebraic equations in several unknowns.

The study of linear algebra first emerged from the study of determinants, which were used to solve systems of linear equations. Determinants were used by Leibniz in 1693, and subsequently, Gabriel Cramer devised Cramer's Rule for solving linear systems in 1750.

If the problem with the number of equations is greater than one is performed at one and the same time, we speak of the existence of solutions of this system.

In this monograph it is offered the new approach of the decision of nonlinear algebraic systems of the equations with the variables which enter the system nonlinearly. Solving for one unknown in one equation was already studied. It is more interesting the situation when more than one unknown variable present in more than one equation.

It is naturally, that in case of linear algebraic system when the number of unknowns is equal to numbers of the equations we may construct the Cramer's determinants.

Linear algebra had its beginnings in the study of vectors in Cartesian 2-space and 3-space. Modern linear algebra has been extended to consider spaces of arbitrary or infinite dimension, and the linear mappings between such spaces. The principal objectives of linear algebra are the resolution of the systems of linear equations. Such an investigation is initially motivated by a system of linear equations in several unknown variables. These equations are naturally represented using the formalism of matrices and vectors.

Naturally, the study of linear algebra includes the topics of vector algebra, matrix algebra, and the theory of vector spaces. It includes a range of theorems and applications in different branches of linear algebra, such as linear systems, matrices, operators, etc. Functional analysis studies the infinite-dimensional version of the theory of vector spaces. Techniques from linear algebra are also used in analytic

Chapter 8. Appications

8.1. Bases in Banach Spaces

8.2. σ_p Class in Banach Space

8.3. Analoque of Selfadjoint Operator in Banach Space

8.1. Bases in Banach Spaces

The problem of the bases in Banach spaces has been and remains one of the fundamental questions of functional analysis. In the case when the Hilbert space, issues related to the bases, widely sanctified in [16], [83], etc.

Definitions and provisions, known from the literature and necessary for the understanding and presentation of the results of this paper are following.

Definion 8.1. Sequence of vectors in Banach space is called linearly independent if the equality $\sum_{i=0}^{\infty} c_i \psi_i = 0$ holds if and only if $c_i = 0$ for all $i = 1,2,...$

Definition 8.2. ψ_j is called a Riesz basis if there exists a bounded and boundedly inverse of operator A for which to some orthonormal basis φ_i in the Hilbert space equalities $A\varphi_j = \psi_j,\ A^{-1}\psi_j = \varphi_j,\ j = 1,2,...$ are true.

Definition 8.3. [16]. If ψ_j is ω linearly independent system of the Hilbert space H , and φ_i is an orthonormal basis in H , in addition $\sum_{j=1}^{\infty} \left\| \psi_j - \varphi_j \right\|^2 < \infty$, then ψ_j is also a Riesz basis in H In the literature these bases are called bases Bari.

Definition 8.4. Let be a matrix

$$A = \{a_{ik}\}_{i,k=1}^{\infty} = \begin{pmatrix} a_{11}\, a_{12} \ldots a_{1n} \ldots \\ a_{21}\, a_{22} \ldots a_{2n} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix} \tag{8.1}$$

and $\vec{\xi} = (\xi_1, \xi_2, ...) \in l_p$, $\eta_i = \sum_{j=1}^{\infty} a_{ij}\xi_j$, $\vec{\eta} = (\eta_1, \eta_2, ...,) \in l_r$.

The necessary and sufficient conditions for the matrix A to be a linear operation from l_p to l_r are the following: some operations U_{nn} , defined by the matrix

REFERENCES

1. AllakhverdievJ.E. On approaching the speed of compact operators by finite-dimensional operators, Uch. Zap., Azerbaijan. University 2, 1957, 27-35 (in Russian).

2. Allakhverdiev J.E., Dzhabarzadeh R.M. Abstract separation of variables. DAN USSR, 1988, v.300, 2. (in Russian).

3. Allakhverdiyev J.E., Dzhabarzadeh R.M. Multiparameter system operators nonlinearly depending on the spectral parameters. DAN USSR, 1988, t.299, 6.

4. Allakhverdiev J.E., Dzhabarzadeh R.M. On the spectrum of multiparameter system and spectral theorem. DAN USSR, 1988, i.e, 300, №1 (in Russian)

5. Allakhverdiev J.E., Dzhabarzadeh R.M. Completeness of the system of eigen elements of multiparameter system of operators. Dokl.AN Azerbaijan SSR, 1982, v.38, 10, ppp.3-7(in Russian).

6. Allahkverdiev J.E., Dzhabarzadeh R.M. Multiple completeness of eigenvectors of some nonself two-parameter system operators. Dokl.AN Azerbaidzhan SSR, 1984, t.40, 5, pp.7-9(in Russian).

7. Akhiezer N.I., Glazman I.M. Theory of linear operators in Hilbert space.M.-Nauka, 1966 (in Russian)

8. Askerov N.G., Krein S.G. and Laptev G.I. On class of selfadjoint problem, DAN, 155, 3(1964), pp.499-502 (in Russian)

9. Arscott F.M. Two-parameter eigenvalue problems in differential equations. Proc. London Math. Soc., 1964, 14, pp. 459-470.

10. Arscott F.M. Transform theorems for two-parameter eigenvalue problems in Hilbert space. Proc. Conf. on the theory of ord. and part. diff. equation. Lect. Notes in Math.,415, 1974, 302-307, Berlin, Springer-Verloig.

11. Atkinson F.V. Multivariable spectral theory the linked eigenvalue problem for matrices. Technical Summary Report № 431, V.S.Army Math. Res. Center, Madison, Wesconsin,1964

12. Atkinson F.V. Multiparameter spectral theory. Bull.Amer.Math.Soc.1968, 74, 1-27

13. Atkinson F.V. Multiparameter eigenvalue problems: Matrices and compact operators. Academic Press: New York, London, 1972

14. Atkinson F.V. Deficiency – index theory on the multiparameter Sturm – Ziouville case. Diff. Equat. Symposia Unif. Upsal. Stockholm, 1977, 1-10

15. Atkinson F.V. Boundary problems leading to orthogonal polynomials in several Variables. Bull.Amer.Math.Soc. 1963, 69, 345-351.

16. Barii N.K. Bases in Hilbert space, DAN, 54 (1946), 383-386(in Russian).

17. Balinsky A.I. A generalization of the concept of the resultant Bezoutiant. DAN Ukr. SSR ser.fiz-mat. and Technical Sciences(in Russian).3

18. Berezanskiy Y. M. Expansion in eigenfunctions selfadjoint operators in Hilbert space. Scientific notes of the Moscow State University, 1951, vol. 148, v.4, pp. 69-107. Naukova Dumka (in Russian).

19. Binding P., Browne P.J. Avariational approach to multiparameter eigenvalue problems for matrices. S.I., A.M.Journ. Math.Anal. 1977, 8, 763-777.

20. Binding P., Browne P.J. Spectral properties of two – parameter eigenvalue problems. Proc. Roy.Soc.Edinburgh. 1981, 89A, № 1-2, pp. 157-173.

21. Binding P. Multiparameter definiteness conditions. Proc. Roy. Soc.Edinburgh, 89 A, 1981, 319-332.

22. Binding P., Browne P.J., Tyrin L. Existence conditions for two-parameter eigenvalue problems. Proc. RoyaL. Soc.Edinburgh 91 A, 1981, 15-30.

23. Binding P. On generalized and quadratic eigenvalue problems. Applicable analysis. V. 12, 1981, pp.27-45.

24. Binding P. Left definite multiparameter eigenvalue problems. Trans.Amer.Math. Soc.v.272, №2, 1982, 475-486.

25. Binding P., Kalstrom A., Sleeman B. D. An abstract multiparameter spectral theory. Proc. Roy. Soc. Edinburgh, 92 A, № 3-4, 1982, 193-194.

26. Binding P. Multiparameter definiteness conditions II, Proc. Royal Soc. Edinburgh, 93 A, № 1-2, 1982, 47-61.

27. Binding P., Browne P.J., Tyrin L. Existence conditions for eigenvalue problems generated by compact multiparameter operator. Proc.Royal Soc. Edinburgh, 96A, №3-4, 1984, 261-274.

28. Binding P., Browne P.J., Tyrin L. Spectral properties of compact multiparameter operators. Proc. Royal. Soc. Edinburgh, 98A, №3-4, 1984, 291-303.

29. Bragat P. Eigenfunction expansions associated with a pair of second order differential equations, Proc, Nat. Inst. Sei. India(A), 35, (1963), 161-174.

30. Browne P.J. A multiparameter eigenvalue problem J. Math. Anal. Appl. 38, 1972, 553-568.

31. Browne P.J. A singular multiparameter eigenvalue problem in second order ord. diff. equat. J. diff. Equat. 12, 1972, 81-94.

32. Browne P.J. Multiparameter spectral theory. Indiana Univ. Math. J, №24, №3, 1974.

33. Browne P.J. Commuting k-tuples of selfadjoint operators and matrix measures. Canad. Math. Bull, V.18, 3, 1975.

34. Browne P.J., Sleeman B.D. Non-linear multiparameter eigenvalue problems for ordinary differential equations. J. Math. anal. appl. 77, 1980, 425-432.

35. Browne P.J. Multiparameter problems: the last decade. Lec. Notes. Math. 964, 1982, 95-109.

36. Camp C. C. An expansions involving p inseparable parameter associated with a partial differential equations. Amer.J. Math., 1928, 50, 259-268.

37. Carnichael R.D. Boundary value and expansions problems. Algebraic basis of the theory. Amer.J. Math. 43, 1921, 69-101, ibid 232-279, ibid 44, 1922, 129-152.3I. 38.Chandrasechar I.Hydrodynamic hydromagnetic. Clarendo Press. Oxford, 1961, 3I

39. Carrol R. Problems in linked operators. Math. Ann, 151, (1963), 272-282

40. Dzhabarzadeh R.M. On existence of common eigenvalues of some operator bundles, polynomial depending on parameter. International Topoloji Conference, Baku, 3-9oct., part 2, 1987, p.93

41. Dzhabarzadeh R.M. Spectral theory of two parameter system in finite dimensional space, Transactions of AS Azerbaijan, V.XVIII, №3-4, 1998, p. 12-18.

42. Dzhabarzadeh R.M. Spectral theory of multiparameter system of operators in Hilbert space, Transactions of Sciences of Azerbaijan, V.XIX, №1-2, 1999, pp.33-40, Baku “Elm” Publishing House.

43. Dzhabarzadeh R.M. Spectral theory of multiparameter system, polinomially depending on parameters. Proc. Inst. Math. a Mech Azerb. Acad. Sci, 2000, Vol. 13(21), pp. 39-45.

Overview of the Book

In this monograph, the author proposes a new approach to the study of multiparameter problems and also gives a new method for separating variables in partial differential equations, which makes it possible to significantly expand the class of problems under study. For a multiparameter nonselfadjoint system of operators, the author introduces the concepts of associated vectors, the multiple completeness of a system of eigen and associated vectors, and all other necessary concepts, and under some conditions proves the main spectral results in Hilbert space. In this monograph, the problems of nonlinear algebraic systems of equations are also considered. The new approach of the decision of nonlinear algebraic systems of the equations is offered when unknown variables are included into this system as polynomials. In a new way for research, the algebraic system under some conditions, naturally, when the number of unknowns coincides or less of the number of the equations of algebraic system, the properties of the solutions of nonlinear algebraic systems of equations are also studied. Namely, under certain conditions, the reality of them is proved. Besides, the author considers a multiparameter system of operators in Euclidean space and some problems of functional analyses.

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