Non-concave Profit, Multiple Equilibria and Catastrophes in Monopolistic Competition

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Abstract

Our model is a Dixit-Stiglitz type, but for general, unspecified preferences. We also relax conditions traditionally responsible for profit concavity or quasi-concavity. Therefore, equilibria can appear asymmetric, set-valued. We find "weakest" conditions when equilibria exist. In comparative statics, we guarantee emergence of asymmetric equilibria (in finite number of moments) during population growth—if and only if the demand generates non-monotone marginal revenue. At such points firms ambiguously split into small producers and big producers. In spite of related catastrophic jumps in consumption, price, and mass of firms—the direction of changes is determined unambiguously.

JEL codes: L11, L13, F12, F15.

Keywords: monopolistic competition, non-concave profit, multiple equilibria, catastrophes, discontinuous comparative statics.

1 Introduction

Our **motivation** for studying non-concave producers' profit function includes: (a) extending general market theory to new situations; (b) finding new market effects.

So far, in modelling any markets, global profit concavity remains a widespread technical assumption, though it lacks clear empirical support or intuitive motivation. For instance, when the demand curve results from combining the curves of two distinct consumer groups having a "chock-price", our profit appears non-concave. We would like to model such natural situations, that means dropping the traditional concavity assumption. Can we really drop it and thus expand the applicability of monopolistic competition concept? Would our usual conclusions about equilibria existence, uniqueness and comparative statics remain true? This paper answers to all the questions more or less *positively* but changes the equilibrium concept.

To outline the theoretical context, we should recall that monopolistic competition model has become by now the main work-horse of international trade, economic geography, growth theory, and other fields, gradually replacing

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the concepts of perfect competition and oligopoly. Its essence is the assumption that a firm or brand-owner behaves as a price-maker but free entry drives all profits to zero. In the presence of increasing returns to scale and consumers' love for variety (incomplete substitution), such market does not degenerate into perfect competition or natural monopoly. This idea became really productive after its formalization by Dixit and Stiglitz (1977). Later on, too peculiar effects of specific CES functional form of utility and related criticism drived the theorists to other tractable specifications: quadratic (Ottaviano et al., 2002) and exponential (Behrens and Murata, 2007). At some stage, this tendency for generalizaion revitalized the initial Dixit, Stiglitz and Krugman's (1979) approach of studying monopolistic competition in quite general *unspecified* form. Several researchers were thinking about such generalization independently, that resulted in subsequent working papers and articles: Bertoletti et al. (2008), Zhelobodko et al. (2010), Dhingra and Morrow (2011), Mrazova and Neary (2012)—achieving various findings in comparative statics.[1] To continue this line of generalizations, the present paper makes one more step in expanding the range of markets suitable for monopolistic competition modelling. Namely, we extend recent Zhelobodko et al. (2012, called further ZKPT) in the direction of non-concave profit. This case was never studied under monopolistc competition, up to our knowledge. Can there arise multiple or/and asymmetric equilibria? How can they change when market increases?

These questions looking technical, still worth clarification: first, to make all theoretical predictions robust against any changes in utilities; second, to uncover new effects possible. To argue that non-concave profit function is not something too peculiar and unrealistic, we repeat that it *always* arise when the demand curve is piece-wise smooth, including typical and realistic combinations of (linear or non-linear) demand curves having a chock-price. Such aggregate demand curve *has a kink*, that makes the marginal revenue non-monotone, which imples non-concave profit. One should not think that such non-monotonicity can be cured by small demand variations smoothening such kinks. No, non-monotonicity remains for all smooth approximations of a kinked demand, and the family of non-monotone marginal revenues is *generic*, as well as monotone family.

Thus, we cannot see any reasons to exclude non-monotone marginal revenue in reality and therefore to exclude it from theory. One more reason for considering non-concave profit arise from the supply side. Indeed, in settings with endogenous R&D (Vives, 2008) profits are often non-convex, because small R&D brings small effect, then higher one. (Yet, in this publication we confine ourselves to simple linear cost, hoping that the same equilibrum concept will be used later in richer settings with R&D.) These considerations make us believe that non-concave profit functions were avoided in theory only for technical hardships—hardships to be overcome now.

Our **setting** repeats Zhelobodko et al. (2012), called further ZKPT, but without profit concavity and without non-linear costs. Namely, we study a closed economy with one diversified sector, one homogenous production factor—labor, homogeneous monopolistically-competitive firms. The representative consumer's elementary utility is unspecified, satisfuing weak natural restrictions, and gross utility is the sum of elementary utilities.

Among **results** characterizing such equilibria, Theorem 1 finds the weakest conditions when monopolistically-competitive equilibria *do exist*. Existence is guaranteed mainly by a natural boundary condition on elementary utility $u(x)$: function $R(x) \equiv xu'(x)$ called "elementary revenue" must become zero at the origin (that exclude utilities like $u(x) = \ln(x)$), and sufficiently decrease at infinity (that exclude utilities like $u(x) = \ln(x + a)$). Thus we define utilities *suitable for monopolistic competition* modelling. This is a very broad class. Theorists may feel happy with such a modest assumption, instead of several doubtful and unnecessary restrictions imposed in typical papers. However, the equilibria studied can be set-valued and asymmetric. Asymmetry means coexistence of two or more kinds of equally-profitable behavior of firms: those with big outputs and ones with small outputs. The masses of both types remain ambiguous, up to their weighed sum, satisfying the labor market clearing. Such set-valued asymmetric equilibrium is the main conceptual achievement of this paper. To analyse such equilibria, we use "ordinal" technique of comparative statics from Milgrom and Shannon (1994), the technique being applied to

[1]The first draft of this paper is Alexey Gorn's diploma (2009) at Novosibirsk State University completed under S.Kokovin's supervision.

monopolistic competition in the manner like in Mrazova and Neary (2011).

Further, for achieving more definite equilibrium structure, we impose a "regularity" restriction on our elementary utility $u(x)$: there must be a finite number of "kinks" in the elementary inverse demand $u'(x)$ and they must be non-degenerate. A "kink" is an interval where the "elementary marginal revenue" $R'(x) \equiv u'(x) + xu''(x)$ is not (strictly) decreasing. "Non-degenerate" means, that under the regularity assumption there cannot be more than *two* kinds of equally-profitable coexisting behavior of firms: big outputs and small outputs.

Using the regularity assumption, Theorem 2 states the structure of a set-valued equilibrium. We show that the equilibrium value of the "intensity of competition" (marginal utility of income) is always *unique*. However, there can be multiple equilibrium outputs/prices, nemely, a couple of possible outputs and a couple of prices, then all possible masses of firms constitute an interval. Thereby we show that equilibria multiplicity and asymmetry always arise *together*.

Further, we turn to comparative statics w.r.t. the market size. We consider "complete path" of evolution, when the economy population continuously grows from zero to infinity (or, equivalently, the evolution can result from a monotone decrease in costs, because only relative market size matters). Theorem 3 shows that *if and only if utility $u(x)$ generates non-monotone elementary marginal revenue R', the multiple-asymmetric equilibria do arise* at some moment during the market growth. Somewhat non-trivial here is the idea, that monotonic decrease in R' "everywhere" is equivalent to "profit concavity in all possible market situations" and simultaneously equivalent to profit quasi-concavity (by contrast, in a *given* situation, quasi-concavity is a weaker assumption). Rather surprizing for us also became the necessity side of Theorem 3: that equilibria multiplicity *must* arise under non-monotone marginal revenue (non-concave profit) during market evolution, though such multiplicities are rather degenerate situatons—points on the path of growth (as numerous as the number of demand kinks). The explanation lies in continuity of such comparative statics.

Intuitively, that such market evolution works through the entry of new competitors and resulting gradually growing "intensity of competition." Geometricaly, this affects a firm as if the marginal-cost horizontal line were gradually rising from zero to infinity against the immobile marginal revenue curve R', and the intersections determine the locally-optimal outputs. When R' is non-monotone, it must be that at some moment rising marginal cost hits this interval of non-monotonicity, and then sooner or later equalizes two local maxima of profit. Here the firms *must* split into asymmetric groups. And indeed, we give such numerical examples: multiplicity/asymmetry situations really do arise under some reasonable utilties.

Importantly, comparing situations before and after this split of firms into groups, we see that outputs and prices make a finitely-big *jump* in response to infinitely-small increase in the market size. In our numerical example, the mass of firms (varieties) suddenly jumped up as much as 10 times and the price doubled! We call such abrupt changes *catastrophic*, regardless, are they good or bad for consumers (actually, the jumps are good under expanding market, bad under shrinking market). Moreover, at the moment of ambiguity, i.e., during the catastroph, all admissible equilibria are *non-equivalent* for consumers' welfare.

Studying in more detail the growing market, Theorem 4 establishes the direction of changes in prices/outputs. Relying on the "ordinal" technique developed by Milgrom and coauthors, we extend the comparative statics conclusions from ZKPT onto set-valued equilibria (the extension amounts to studying the jumps because in single-valued points ZKPT applies directly). Namely, we find that under growing market the mass of firms anyway goes *up*, with or without jumps, and the individual consumption of each variety always goes *down*. As to prices, they jump *up* during a catastroph, that looks paradoxical under increasing competition of firms. Explaining similar paradox in smooth comparative statics, ZKPT exploits the Arrow-Pratt measure $-xu''(x)/u'(x)$ of concavity, called "relative love for variety" (RLV) or elasticity of the inverse demand. According to ZKPT, decreasingly-elastic demands, called super-convex (Mrazova and Neary, 2011), necessarily make the prices go up together with the population and mass of firms. The same idea applied in this paper means, that the demand curve at the zone of any kink appears too much convex. That's why the upward jump of the equilibrium mass of firms should bring an *increase* in prices instead of decrease, that seemed natural. As in ZKPT, under market growth, firm's output and price both follow the distinction governed by RLV. Namely, most realistic sub-convex demands—show price-decreasing competition with increasing outputs, whereas super-convex demands display the opposite pattern, CES utility being the neutral borderline.[2]

As theorists, we are satisfied that all general market regularities, found in ZKPT only under global profit concavity and unique symmetric equilibrium, are now extended to set-valed equilibria and more general situation. Can a *practical* economists also learn something from our findings? It depends upon realism of our new effect found: a possibility and even substantial probability of "market jumps". They can be supposed unrealistic only if one provide an empirical evidence that non-monotone marginal revenue (very convex interval of the demand curve) is excluded in reality. The essence of our warning is that *small causes can bring big market consequences*, i.e., abrupt changes in the path of market development. One can object that formally jumps do depend on our assumption of homogenous firms but firms' heterogeneity should only smoothen this effect.

The next section introduces the model, developing all the notions of "set-valued equilibria". Section 3 displays existence of multipe and asymmetric equilibria, Section 4 presents comparative statics of set-valued equilibria.

2 Model and examples

We study one-sector closed economy with monopolistic competition a'la Dixit-Stiglitz but with general (unspecified) utility function. In doing so, we follow ZKPT but impose less restrictions.

Our economy involves one production factor—labor. One differentiated good is split into continuum $[0, n]$ of firms/varieties with same index $i \in [0, n]$ because each variety is produced by one single-product firm (n is endogenous).

Demand. Labor is chosen as the numéraire. There are L identical consumers who supply each one unit of labor inelastically, so that 1 is both a worker's income and expenditure. Every consumer chooses an infinite-dimensional consumption vector $\mathbf{x} = x_{i\leq n}$ (a measurable consumption function) to maximize her utility subject to the budget constraint:[3]

$$\max_{\mathbf{x}(.)} \mathcal{U} \equiv \int_0^n u(x_i)\mathrm{d}i \qquad \text{s.t.} \qquad \int_0^n p_i x_i \mathrm{d}i = 1. \tag{1}$$

Here we use infinite-dimensional price vector $\mathbf{p} : [0, n] \to \mathbb{R}_+$, where $p_i \equiv p(i)$ is the price of i-th variety; $x_i \equiv x(i)$ denotes i-th consumption. One of our goals is to formulate the weakest restriction on utilities suitable for possibility of monopolistic competition in the market, as follows.

Assumption 1 (*non-Inada conditions*). Elementary utility function $u(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is thrice continuously differentiable and strictly concave on $(0, \infty)$, increasing on some non-empty interval $[0, x_{max})$, where $x_{max} \leq \infty$ denotes (finite or infinite) argmaximum of function $R_u(x) \equiv xu'(x)$, called "elementary revenue". At the origin, utility $u(\cdot)$ is normalized as $u(0) = 0$, and generates increasing strictly concave bounded revenue $R_u(0)$, in the sense

$$\lim_{x\to+0} R_u(x) = 0, \;\; \overline{MR} \equiv \lim_{x\to+0} R'_u(x) > 0, \;\; \lim_{x\to+0} R''_u(x) < 0. \tag{2}$$

At the elementary revenue's argmaximum point x_{max}, we require concavity: $\lim_{x\to x_{max}} R''_u(x) < 0$ and condition:

$$\text{either } \underline{MR} \equiv \lim_{x\to x_{max}} R'_u(x) = 0 \quad \text{or} \quad \lim_{c\to+u'_{min}} \max_x [R_u(x) - cx] = \infty. \tag{3}$$

[2] As to welfare, normally in our examples welfare goes up: increasing variety outweigh increasing prices. However, we know some paradoxical counter-examples.

[3] Profits are not included into income, because they vanish under free entry.

We call all such utilities suitable for monopolistic competition modelling *(MC-suitable)* and maintain this assumption throughout.[4]

Strict concavity of u implies that gross utility $\mathcal{U}$ displays some "love for variety", because, as in risk-taking theory, strictly concave elementary utility entails strictly convex preferences. Thereby under uniform prices across varieties, the consumer strictly prefers buying a mixture of varieties rather than any single variety. Like in risk-taking, to express this preference for mixture, we can exploit Arrow-Pratt measure r_u of concavity for any function u:

$$r_u(x_i) \equiv -\frac{x_i u''(x_i)}{u'(x_i)} > 0. \tag{4}$$

Applied to utility, r_u in ZKPT is called *relative love for variety* (hereafter, RLV). Using the standard definition of the elasticity of substitution σ, it is easy to show that at a symmetric consumption pattern *the RLV is the inverse of the elasticity of substitution across varieties* $\sigma(x)$*, i.e.,* $r_u(z) = 1/\sigma(z)$. Importantly for price-making, at any x_i the demand elasticity is also equal to $\sigma(x_i)$. (In particular, CES utility $u(x_i) = x_i^\rho$ implies constant RLV $r_u = 1 - \rho \equiv \frac{1}{\sigma}$; $0 < \rho < 1$).

Further, using FOC of consumers' optimization and denoting the Lagrange multiplier as λ, we obtain the inverse demand function p^*:

$$p^*(x_i, \lambda) = u'(x_i)/\lambda. \tag{5}$$

Thus, marginal utility of income λ makes each demand shrinking, and therefore measures the intensity of competition.

Supply. We assume identical firms: to produce output $q_i = Lx_i$ each firm i must spend $cq_i + f$ units of labor, where c is the marginal cost and f is the fixed cost of business. Taking as given the inverse demand function $p^*(x_i, \lambda)$ for i-th variety and given λ, i-th firm maximize its per-consumer operational profit π w.r.t. quantity:[5]

$$\pi(x_i, \lambda, c) \equiv (\frac{u'(x_i)}{\lambda} - c)x_i \to \max_{x_i \in \mathbb{R}_+} . \tag{6}$$

We see that current marginal utility of money λ becomes the single market statistic expressing *intensity of competition*, like price index under CES modelling.

Now we can introduce notations for the maximal value π_u^* of the per-consumer profit function, for the set X_u^* of profit-maximizers, and formulate the "firm's survival" or free-entry condition:

$$\pi_u^*(\lambda, c) \equiv \max_{x_i \in \mathbb{R}_+} \pi(x_i, \lambda, c), \qquad X_u^*(\lambda, c) \equiv \arg \max_{x_i \in \mathbb{R}_+} \pi(x_i, \lambda, c). \tag{7}$$

$$f \leq L\pi_u^*(\lambda, c). \tag{8}$$

[4]Assumption 1 rules out some neoclassical utilities like $\log(x) \pm bx$ or $-1/x$. The assumption is *not* technical: the functions excluded really do not suit any monopolistic competition model, because related profit may remain positive and increasing at $x \to 0$, or increases at infinite x. But, unlike restrictive Inada conditions, our formulation allows for all utilities useful in monopolistic competition models: quadratic utilties $u = ax - bx^2$ that have a satiation point $\check{x}_0 = 0.5a/b$ and chock-price $u'(0) < \infty$, CES utilties $u(x) = x^\rho : \rho < 1$ that have infinite derivative at 0, AHARA utilties $u(x) = (a+x)^\rho - a^\rho \pm bx$ that may have positive limiting derivative $u'_{min} \equiv \lim_{x\to\infty} u'(x) = b > 0$, and many others.

[5]

Standardly, profit maximization w.r.t. *price* gives an equivalent result. Also, equivalent is maximization of *gross* profit $L\pi(q_i/L, \lambda, c) - f$ w.r.t. output q_i.

Note that both mappings π^*_u and X^*_u do not have index i because the optimization program is the *same* across firms, so both mappings are just characteristics of function u. Still, the program above *allows for asymmetry among firms' decisions* x_i when X^*_u is not a singleton. Such possibe assymetry is the essence of our paper. To explain it, we recall usual FOC for maximizing profit $\pi(x, \lambda, c)$ formulated as equality between marginal cost and marginal revenue m_R:

$$m_R(x_i, \lambda) \equiv \frac{x_i u''(x_i) + u'(x_i)}{\lambda} = c. \tag{9}$$

As to SOC, the second derivative of profit (6) being the first derivative of marginal revenue (9), it standardly follows that *profit is strictly concave* at x if and only if the elementary marginal revenue $m_R(x, 1)$ *is strictly decreasing* at x. In particular, when u''' exists, such restriction on utility in terms of concavity of u' looks like (see ZKPT):

$$r_{u'}(x) \equiv -\frac{xu'''(x)}{u''(x)} < 2. \tag{10}$$

In ZKPT this requirement is imposed globally ($\forall x > 0$), being a condition for unique symmetric equilibrium, but our goal here is the opposite and we do not require it (it naturally holds locally at local optima).

Now we are ready to construct a complicated notion of set-valued equilibria. But to motivate it, we first present an example illustrating everything throughout. In some sense, it immediately presents all main ideas of our paper.

(Guiding) example 1. To illustrate our reasoning, we use throughout the utility

$$u(x) = x + \sqrt{x} + 1.4 \arctan(2x + 0.05) - 1.4 \arctan(0.05). \tag{11}$$

Algebraically, this function u looks rather exotic, but nevertheless it satisfies Assumption 1, i.e., it is increasing at 0, strictly concave, smooth, etc. The only specific feature differing from textbook examples like CES or quadratic functions, is that related marginal revenue becomes non-monotone and thereby related profit is non-concave. Specifically, in Fig.1, we take parameters $L/f \approx 10.04$, $\lambda c \approx 0.00038$ suitable for two global argmaxima of profit (see Section 4 for numerical details). One can see that the producer's operational profit $x_i(p^*(x_i, \lambda) - c)$ is not locally strictly concave only at those points where the normalized marginal revenue $MR(x) \equiv [x_i p^*(x_i, 1)]' = u'(x_i) + x_i u''(x_i)$ is increasing: approximately from 0.9 to 4.8.

Here, as usual, the marginal revenue MR intersects the normalized marginal cost $MC = \lambda c$ at points where the profit function could reach local maxima or minima. The middle intersection, approximately $x \approx 2.5$, is the local minimum. For us important are only the leftmost and the rightmost intersections – local argmaxima $\hat{x}(\lambda c) \approx 0.65 < \check{x}(\lambda c) \approx 12.9$. More generally, under K intersections, local maxima must be among the *odd* roots of FOC equation (9), which can be reformulated as:

$$(u''(x) + xu'(x) - \lambda c) = 0.$$

Under some value λc, like in this picture, two or more local maxima become global, bringing the same profit. Then each producer becomes *indifferent* which optimal quantity to produce: $\hat{x}$ or $\check{x}$.

In the case of such ambiguity, we denote by $\hat{n}$ the unknown mass of firms who chose the left optimum $\hat{x}$ (small output), and by $\check{n}$ the mass of firms choosing the bigger output $\check{x}$. Using these notations, we express the total costs of both firm types as $(c\hat{x}L + f)$, $(c\check{x}L + f)$, and formulate the labor balance (labor market clearing condition):

$$(c\hat{x}L + f)\hat{n} + (c\check{x}L + f)\check{n} = L. \tag{12}$$

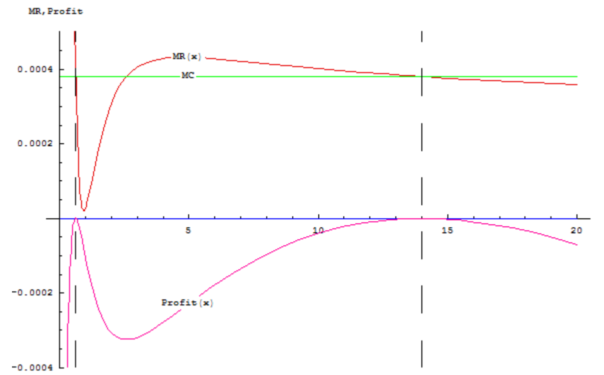

Figure 1: Non-monotone marginal revenue and two peaks of profit under utility $u(x) = x + \sqrt{x} + 1.4\arctan(2x + 0.05) - 1.4\arctan(0.05)$.

To define a certain kind of market equilibrium, this balance equation should be combined with free entry, consumers' and producers' FOC. Luckily, the equilibrium equations are not simultaneous. One can first find the equilibrium value of marginal utility of income λ from inequality (7) turning into the free-entry equation:

$$\pi_u^*(\lambda, c) = f/L. \tag{13}$$

Then, having λ, we find equilibrium consumptions $(\hat{x}, \check{x})$ from (9), then both prices are determined from (5). Finally, we seek for the masses of groups of firms $\check{n}$ and $\hat{n}$ from the labor balance (12). Here some difficulty arise: finding two variables from one equation is impossible. Thereby some indeterminacy always remains in the firms' masses $(\hat{n}, \check{n})$. In other words, there exists the whole interval of possible couples $(\hat{n}, \check{n})$ that satisfy all equilibrium conditions, ambiguity cannot be excluded. Thereby, instead of a single equilibrium arising under strictly concave profit, in any situation with two argmaxima $\hat{x} < \check{x}$, we get a continuous *set of equilibria*: any couple $\hat{n}, \check{n}$ satisfying the labor balance (and therefore the consumers' budget constraint) satisfies the idea of asymmetric equilibrium. Thus, we see that identical firms may behave *differently*. Besides, *asymmetry and multiplicity of equilibria always come together*.

Now we can generalize this example from two argmaxima to finitely-many profit argmaxima $X = (x_1, ..., x_K)$ $K \geq 1$, and formulate related notion of a set-valued equilibrium.[6]

Asymmetric equilibria and set-valued equilibria. Consider $K \geq 1$ types of firms' behavior, and a bundle $z = (\lambda, X, P, N)$ consisting of $\lambda > 0$ – the level of competition, $X = (x_1, ..., x_K) \in \mathbb{R}_+^K$ – the vector of consumptions bringing maximal profit, $P = (p_1, ..., p_K) \in \mathbb{R}_+^K$ – the vector of prices, and $N = (n_1, ..., n_K) \in \mathbb{R}_+^K$ – the masses of firms' types. This z is called a ***(free-entry) equilibrium*** when λ satisfies the free-entry equation (13), X satisfy optimization necessary conditions (FOC: – (9)) and sufficient conditions (SOC: $m'_R(x, 1) < 0$), prices fit the demand rule $p_k = p^*(x_k, \hat{\lambda})$ and masses of firms fit the labor balance:

$$\sum_k^K (cx_k L + f) n_k = L. \tag{14}$$

[6] An extension of this definition is possible: instead of finite K, and finite-dimensional argmaxima vector $X = (x_1, x_2, x_3, ...)$, one can use the same definition for infinite-dimensional X arising when profit function includes linear intervals. Then the summation in condition (14) should be understood as an integral. Probably, we can include this infinite-dimensional case into our further theorems without changing anything in formulations and proofs but the extension is not too important.

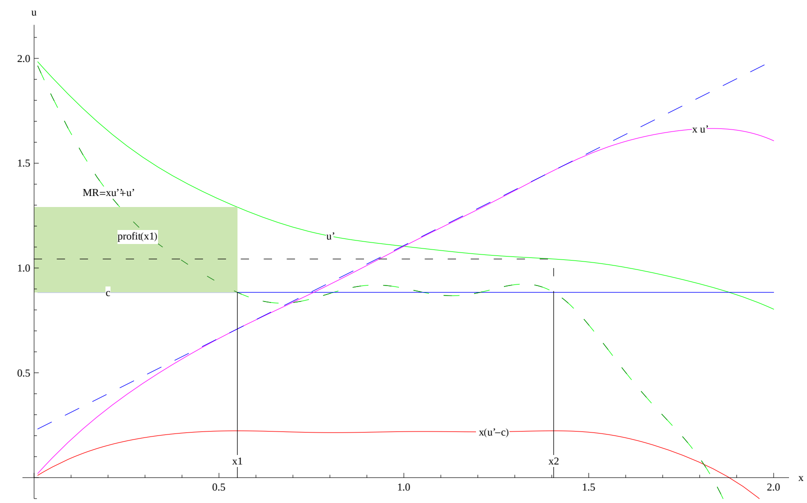

Figure 2: Three-peaks non-concave profit under utility $u(x) = 2x - x^2 + 0.5x^3 - 0.1x^4 + \frac{1}{3600}\sin[4\pi x]$.

When $K \geq 2,\ 0 < x_i < x_j\ \exists i, j$, this equilibrium is called ***asymmetric***; then simplex Z of all bundles satisfying all the named conditions under given exogeneous parameters (u, c, L, f) is called a *set-valued equilibrium.* In the special case when $K = 1$ this $Z = \{z\}$ becomes a singleton and such equilibrium is called *unique and symmetric.*

After stating the existence theorem, in our comparative statics we would like to reduce generality of possible equilibrium structures to $K \leq 2$. Then the following regularity restriction on utility u will be used to rule out any degenerate outcomes like three or more global maxima, lying on the same line.

Assumption 2 (*κ-regular u*). (i) Under $\lambda = 1, \forall c$, the number of profit argmaxima do not exceed 2:

$$\#|\arg\max_{x \in \mathbb{R}_+} \pi(x, 1, c)| \leq 2\ \ \forall c$$

(i.e., any line $f(x) = ax + b$ cannot be tangent to the elementary revenue $\pi(x, 1, 0) \equiv xu'(x)$ more than at two points simultaneously, dominating the elementary revenue in the sense $f(x) \geq \pi(x, 1, 0)\ \forall x$). (ii) There can be only a finite number $\kappa \geq 0$ of magnitudes $c_{k \leq \kappa} : \#|\arg\max_{x \in \mathbb{R}_+} \pi(x, 1, c_k)| = 2$ bringing multiplicity of argmaxima (i.e., there can be only a finite number $\kappa \geq 0$ of the dominating lines with double tangency to $\pi(x, 1, 0)$). Such utility is called *κ-regular.*

Geometricaly, $\kappa \geq 0$ is the number of sags (non-concave intervals) in the curve of elementary revenue; whenever $\kappa = 0$, the revenue appears strictly concave. More precisely, we can take the convex hull of the undergraph of the elementary revenue $xu'(x)$, and define function R_{conv} as the upper envelope of this convex hull, expressed as

$$R_{conv}(x) \equiv \max_r conv\{r \geq 0 |\, r \leq xu'(x)\}. \tag{15}$$

Then, each "sug" is the flat (linear) interval of R_{conv}, and κ is the number of such intervals. These notions and the regularity assumption help us in the next section to economize notations by ensuring specific dimentionality of all asymmetric equilibria $(\hat{\lambda}, X, P, N) = (\hat{\lambda}, (\hat{x}, \check{x}), (\hat{p}, \check{p}), (\check{n}, \hat{n})) \in R^7_+$ for all values (Lc/f).

Example 2. Figure 2 further explaines Assumption 2. It shows that the number of increasing intervals in the marginal revenue is not necessarily the number κ of "sags", i.e., possible asymmetry situations.

In Fig.2 we use exotic but regular and concave utility function $u(x) = 2x - x^2 + 0.5x^3 - 0.1x^4 + \frac{1}{3600}\sin[4\pi x]$. So, the (normalized) inverse demand u' decreases. Here the cost value $c = 0.883965$ crosses marginal revenue (MR) as much as *five* times, but only *two* of these crossings relate to global argmaxima $\{\hat{x}, \check{x}\} = \{0.54966, 1.40512\}$ of operational profit $xu'(x) - cx$. This function is the red curve, whereas cost $c(x) = 0.883965$ is the solid blue line, and the sloped dashed line shows the tangent line $ax + b$ "dominating" the elementary revenue $\pi(x, 1, 0) = xu'(x)$. This revenue has *two* intervals where it is not concave (i.e., where MR increases), that generates *three local* maxima of profit ($\hat{k} = 3$). However, there is only *one essential* sag in the revenue graph ($\kappa = 1$) and it generates only *two global* profit maxima $\hat{x}, \check{x}$ ($K = 2$). Indeed, the middle peak in the revenue curve $xu'(x)$ is too small, *inessential*, because the dashed line tangent to $xu'(x)$ does not touch the middle peak. In other words, those profit peaks that cannot become global profit maxima are inessential.

To understand regularity of asymmetric equilibria phenomenon, note that this twin-peak example is *non-degenerate.* Ineed, any small changes in utility u do not change the essence of the picture: there *always* exists some c yielding twin maxima $\#|X^*(c)| = 2$. Similarly, our guiding example is non-degenerate. More generally, all functions generating non-monotone MR is a broad class, where each function generates some assymmetric equilibria under some c with *necessity.* In contrast, any example with 3 equivalent peaks $\#|X^*| = 3$ is degenerate.

Our further plan is to explore existence of equilibria, reveal their structure, and study their responses to the market expansion, e.g., population growth, or countries' integration, or technological shocks.

3 Equilibrium existence and structure

We start with the properties of the profit argmaxima $X^*_u(\lambda, c)$ and optimal per-consumer profit $\pi^*_u(\lambda, c)$, defined in (7). Rather obviously, for determining these argmaxima under given λ, c, the following formulations are equivalent:

$$X^*_u(\lambda, c) \equiv \arg\max_{x\in\mathbb{R}_+} \pi(x, \lambda, c) = \arg\max_{x\in\mathbb{R}_+} \lambda\pi(x, \lambda, c) \ : \tag{16}$$

$$\lambda\pi(x, \lambda, c) \equiv \pi(x, 1, \lambda c) \equiv [u'(x) - \lambda c]x. \tag{17}$$

This reformulation allows us to use further the single-argument mapping $X^*_u(\lambda c) \equiv X^*_u(\lambda, c)$ interchangeably with two-argument one (with a little abuse in notation). Such trick simplifies our reasoning about reaction of X^*_u to changing arguments λ or c, or both. Economically, this equivalency means that increasing k-times the intensity of competition λ affects the firm's optimal output Lx exactly in the same way as increasing k-times the marginal cost c.

Geometricaly (see Fig.2), finding $\arg\max_{x\in\mathbb{R}_+}[u'(x) - \lambda c]x$ means the following. Slope λc given, the firm should vary a for finding the highest line $a + \lambda cx$ tangent to the elementary revenue curve $R_u(x) \equiv xu'(x)$ (painted purple in Fig.2, whereas $a + \lambda cx$ is painted dashing-blue). Then the tangency points are the argmaxima $X^*_u(\lambda c)$. Obviously, they *exist if and only if*

$$\lambda c > \underline{MR} \equiv \lim_{x\to x_{max}} R'_u(x). \tag{18}$$

At smaller $\lambda c < \underline{MR}$, the objective function $[u'(x) - \lambda c]x$ is undounded when $x \to \infty$; even under $\lambda c = \underline{MR} > 0$ this function goes to infinity by Assumption 1.

Now consider the argmaxima comparative statics. When we increase the slope λc, we always induce *decrease* in our (set-valued) argmaxima, irrespectively, is the undergraph of $[u'(x) - \lambda c]x$ convex or not.

To express this idea rigorously, let us define three kinds of "decreasing" mapping (set-valued function) $X : \mathbb{R} \to 2^{\mathbb{R}}$. We call $X(\lambda)$ *monotone nonincreasing* if a bigger argument $\bar{\lambda} > \tilde{\lambda}$ implies $x \wedge \tilde{x} \in X(\tilde{\lambda})$ and $x \vee \tilde{x} \in X(\bar{\lambda})$ for every $x \in X(\bar{\lambda})$ and $\tilde{x} \in X(\tilde{\lambda})$ (where $\wedge$ denotes minimum and $\vee$ is maximum, thereby the extreme members do not increase).[7] We call a mapping $X(\lambda)$ (strictly) *decreasing*, when its extreme members decrease in the sense

$$\bar{\lambda} > \tilde{\lambda} \Rightarrow \min_{x \in X(\bar{\lambda})} < \min_{\tilde{x} \in X(\tilde{\lambda})} \quad \text{and} \quad \max_{x \in X(\bar{\lambda})} < \max_{\tilde{x} \in X(\tilde{\lambda})} . \tag{19}$$

We call X *strongly decreasing*, when all its selections decrease in the sense

$$\bar{\lambda} > \tilde{\lambda} \Rightarrow \bar{x} < \tilde{x}, \, \forall \bar{x} \in X(\bar{\lambda}) \, \forall \tilde{x} \in X(\tilde{\lambda}). \tag{20}$$

The latter (strongest) version of negative monotonicity implies mapping X single-valued everywhere, excluding isolated points (downward jumps).

To reveal, step by step, all these types of monotonicity (in (λc)) of our argmaximum X^*_u, we shall argue as if the cost were $c = 1$ (we just economize notation, the same logic works for any c or any changes in λc). We denote by $\bar{X}(\lambda)$ all roots of the FOC equation:

$$\bar{X}(\lambda) \equiv \{x \geq 0 | \; u'(x) + xu''(x) - \lambda = 0\}, \tag{21}$$

this set including global argmaxima $X^*_u(\lambda) \subset \bar{X}(\lambda)$, and maybe some other extrema. Now we can apply the following lemma, which is a version of a theorem from Milgrom and Roberts (1994, Theorem 1).[8] It predicts monotone comparative statics of both extreme roots $\hat{x} \leq \check{x}$ of any equation $g(x, \lambda) = 0$ with a parameter λ.

Lemma 1. (Monotone roots, Milgrom and Roberts): *Assume a partially ordered set Λ, some bounds $\overline{x} > \underline{x}$ of the domain and a parameterized function $g(.,.) = g(x, \lambda) : [\underline{x}, \overline{x}] \times \Lambda \to \mathbb{R}$ which is continuous and weakly changes the sign, in the sense $[g(\underline{x}, \lambda) \geq 0 \; \& \; g(\overline{x}, \lambda) \leq 0 \; \forall \lambda \in \Lambda]$. Then for all $\lambda \in \Lambda$:*

(i) there exist some non-negative roots of equation $g(x, \lambda) = 0$, including the lowest solution $\hat{x} \equiv \sup\{x | g(x, \lambda) \geq 0$ and the highest solution $\check{x} \equiv \inf\{x | g(x, \lambda) \leq 0\}$, these can coincide;[9]

(ii) if our function $g(x, \lambda)$ is non-increasing w.r.t. λ everywhere, then both extreme roots $\hat{x}(\lambda)$, $\check{x}(\lambda)$ are non-increasing w.r.t. λ, i.e., mapping $\bar{X}(\lambda)$ is a nonincreasing one;

(iii) if, moreover, $g(x, \lambda)$ is decreasing in λ and strictly changes the sign $[g(\underline{x}, \lambda) > 0 \; \& \; g(\overline{x}, \lambda) < 0 \; \forall \lambda \in \Lambda]$, then both extreme roots $\hat{x}, \check{x}$ are decreasing, i.e., mapping $\bar{X}(\lambda)$ is a decreasing one.

The intuition behind this lemma is simple: when we shift down any continuous curve whose left wing is above zero and the right one is below—the roots should decline. More subtile fact is that when some isolated root $\check{x}$ disappears or emerges, the jump goes in the same direction as all continuous changes, i.e., downward.

We apply this lemma to the (continuous) auxiliary function g gained from FOC of $\pi(x, 1, \lambda)$:

$$g(x, \lambda) \equiv [u'(x) + xu''(x) - \lambda],$$

using domain $\Lambda = [0, \infty)$. We conclude that mapping $\bar{X}$ is "nonincreasing". We would like to enforce this property; to find "decreasing" $\bar{X}$ at those λ and domains $[\underline{x}, \overline{x}]$, where we can apply claim (iii). Locally, this task is easy: at a given λ, we can apply (iii) to any vicinity $(\underline{x}, \overline{x}) \ni \acute{x} > 0$ of any positive local argmaximum $\acute{x} | g(\acute{x}, \lambda) = 0$—whenever strict SOC holds. The latter means that $u'(x) + xu''(x)$ decreases at $\acute{x}$, i.e., it is an isolated argmaximum. Thereby, *any positive local argmaximum $\acute{x} = \acute{x}(\lambda)$ satisfying strict SOC—locally decreases w.r.t. λ.*

Searching for globally decreasing $\bar{X}$, on a positive ray we would like to identify a subinterval $(\lambda_{min}, \lambda_{max}) \subset [0, \infty)$ where claim (iii) is applicable. This amounts to finding an area where all roots of equation (21) are positive and finite, under Assumption 1.

[7] This terminology follows Milgrom and Shannon (1994), who use it for lattices: "strong set order" $\leq_s$ says that $X \leq_s Y$ (Y is higher than X), if for every $x \in X$ and $y \in Y$, $x \wedge y \in X$ and $x \vee y \in Y$. Such order requires our simple mapping $(\lambda, X) \subset R^2$ to be a lane without any increases in its extreme members. Similar is the notion of *nondecreasing* mapping, actually used in Milgrom and Shannon.

[8] Their original Theorem 1 differs in using function $g(x, t)$ *non-decreasing* in t, continuous "but for upward jumps", and domain $[\underline{x}, \overline{x}] = [0, 1]$. This makes a minor difference.

[9] Naturally, when the roots are finite, $\hat{x}(\lambda) = \min\{x | g(x, \lambda) = 0\}$, $\check{x}(\lambda) = \max\{x | g(x, \lambda) = 0\}$).

Lowest λ yielding $x \in (0,\infty)$. Consider the case when our elementary revenue $R_u(x) = xu'(x)$ has a finite global argmaximum x_{max} (that implies satiable demand). Then, obviously, all positive λ enable *finite* solutions to (21), i.e., we can take the lower bound of the needed interval as $\lambda_{min} = \underline{MR} = u'(x_{max}) + x_{max}u''(x_{max}) = 0$ (using notations from (3)). Similarly, under unsatiable demand ($x_{max} = \infty$) but zero limiting value $\lim_{x\to\infty}(u'(x)+xu''(x)) = 0$, the infimum of all λ bringing finite roots is $\lambda_{min} = \underline{MR} = 0$. The third possible case is when at infinity R'_u remains positive: our parameter $\underline{MR} > 0$ (an example is utility $u = \sqrt{x} + x$, in such cases this $\underline{MR}$ becomes the parameter delimiting the situations λ where mapping $X(\lambda)$ is finite). However, the outcome is the same and we conclude that anyway we must take (zero or positive) lower bound $\lambda_{min} = \underline{MR}$ when we search for an interval $(\lambda_{min}, \lambda_{max})$ bringing positive finite roots of function g.

Highest λ yielding $x \in (0,\infty)$. Recall notation $\overline{MR}$ from (2) and consider the case of finite derivative at the origin ($\overline{MR} < \infty$), that implies chock-price. Then all sufficiently high parameters $\lambda \geq \overline{MR}$ should bring zero solutions $\hat{x}(\lambda) = \check{x}(\lambda) = 0$ to (21), for lower parameters the solutions are positive. In the case of infinite derivative $\overline{MR} = \infty$ all λ must bring positive x. We conclude that anyway we must take finite or infinite $\lambda_{max} = \overline{MR}$ as an upper boundary, that determines the open interval

$$\hat{\Lambda} \equiv (\lambda_{min}, \lambda_{max}) \equiv (\underline{MR}, \overline{MR}),$$

which brings positive finite roots of g. It also strictly decreases in x at both boundaries $(\underline{x}, \overline{x})$, because of strict concavity of $xu'(x)$ at 0 and at x_{max} (Assumption 1). This ensures that outside interval $\hat{\Lambda}$ the roots of g cannot be positive and finite, that we use in Theorem 1.

Now we can apply claim (iii) to this interval $\hat{\Lambda}$, because our function $g(x,\lambda) \equiv [u'(x)+xu''(x)-\lambda]$ takes positive value $\overline{MR} - \lambda > 0$ at the lower boundary $\underline{x} = 0$ and negative value $\underline{MR} - \lambda < 0$ at $\overline{x} = x_{max}$ (for all $\lambda \in \hat{\Lambda}$). Moreover, g remains strictly decreasing in λ. Thus, our function $g(x,\lambda)$ satisfies the boundary conditions and monotonicity conditions needed for Lemma 1-(iii). This implies *strict decrease of the extreme roots* $\hat{x}(\lambda) \leq \check{x}(\lambda)$ on $\hat{\Lambda}$.

It must be added that both extreme roots $\hat{x}(\lambda) \leq \check{x}(\lambda)$ of (21) are the local maxima (not minima) of function $\pi(x,1,\lambda) \equiv xu'(x) - \lambda x$, because of SOC. Indeed, by definition of $\hat{x}, \check{x}$, function $g(x,\lambda) > 0$ must be (strictly) decreasing in some left vicinity of the left point $\hat{x}$, and in some right vicinity of $\check{x}$. Using continuous differentiability of $xu'(x)$ (Assumption 1) we expand this decrease to complete (left and right) vicinities of each point $\hat{x}, \check{x}$. This decrease of $g(x,\lambda) \equiv \pi'(x,1,\lambda)$ means SOC. We can summarize our arguments as follows.

Proposition 1 (Monotone local argmaxima). *Each local argmaximum of the normalized profit $\pi(x,1,\lambda)$ is nonincreasing w.r.t. parameter $\lambda \geq 0$. Moreover, the local argmaximum decreases when being positive and finite, which is guaranteed only on interval $\hat{\Lambda} \equiv (\underline{MR}, \overline{MR})$. In the case of (sufficiently small) positive parameters $\lambda \in (0, \underline{MR}]$ all argmaxima are infinite, under (sufficiently big) finite $\lambda \in [\overline{MR}, \infty)$ all argmaxima are zero.*

Now, to establish similar monotonic behavior of *global* argmaxima set X^*_u we use "single crossing" notion and Theorems 4, 4' from Milgrom and Shannon (1994) simplified here for our case of real parameter t and unidimensional real domain $S(t)$ of maximizers.

Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$. If $g(x', t") \geq g(x", t")$ implies inequality $g(x', t') > g(x", t')\, \forall (x' > x", t' > t")$, then g satisfies the *strict single crossing* property w.r.t. couple $(x; t)$ of arguments. Similarly, *single crossing* property means

$$[g(x', t") \geq g(x", t") \Rightarrow g(x', t') \geq g(x", t')\, \forall (x' > x", t' > t") \text{ and}$$
$$g(x', t") > g(x", t") \Rightarrow g(x', t') > g(x", t')\, \forall (x' > x", t' > t")]$$

(essentially, in these two versions of single-crossing notion, parameter t strictly or weakly amplifies monotonicity of g w.r.t. x, alike supermodularity).

Lemma 2 (Monotone argmaxima, Milgrom and Shannon). *Consider a domain $S(t) : \mathbb{R} \to 2^{\mathbb{R}}$ which is non-shrinking w.r.t. t (nondecreasing by inclusion) and a function $g : \mathbb{R}^2 \to \mathbb{R}$. If g satisfies the single crossing property* *in $(x;t)$, then* $\arg\max_{x\in S(t)} g(x,t)$ *is monotone nondecreasing in* t. *If* g *satisfies the strict single crossing property in* $(x;t)$, *then every selection* $x^*(t)$ *from* $\arg\max_{x\in S(t)} g(x,t)$ *is monotone nondecreasing in* t.

Unfortunately, this lemma does not predict strict increase that we need. But, it is important for us that the latter claim about all selection implies that all points of multi-valued $g(.,t)$ are "isolated", in the sense that there is no *open* interval of multi-valuedness (single-valuedness holds "everywhere except isolated poins"). The third result that we need—is similar to "envelope" Theorems 1, 2 from Milgrom and Segal (2002) but formulated here for more simple conditions as (trivial) Lemma 3.

Lemma 3 (Monotone maxima, Milgrom, Segal) *Consider a compact choice set* X, *a continuous function* $g(x,t) : X \times [0,1] \to \mathbb{R}$ *and its maximal value* $\pi^*(t) = \sup_{x\in X} g(x,t)$. *If* $\pi(x,t)$ *continuously decreases in* t *for all* $x \in X$, $t \in (0,1)$, *then its maximal value* $\pi^*(t)$ *continuously decreases in* $t \in (0,1)$.

Now, using our notations $\underline{MR}, \overline{MR}$, x_{max} and new notions

$$\lambda_{cmin} \equiv \underline{MR}/c,\ \ \lambda_{cmax} \equiv \overline{MR}/c,\ \ R_{max} \equiv x_{max}u'(x_{max}),$$

(that can become infinite) we formulate and prove the result we were long driving to.

Proposition 2. (Monotone argmaxima and maxima): *Consider some given* $c > 0$ *and parameter* λ *increasing in the open interval* $\Lambda(c) = (\lambda_{cmin},\, \lambda_{cmax})$. *Then on* $\Lambda(c)$ *the argmaxima set* $X_u^*(\lambda c) \equiv \arg\max_x[xu'(x) - \lambda cx]$ *is non-empty and strongly decreases from* x_{max} *to 0 (i.e., all its selections decrease); simultaneously the objective function* $\pi_u^*(1,\lambda c) \equiv x[u'(x) - \lambda c]$ *continuously decreases from* R_{max} *to 0. Outside* $\Lambda(c)$, *under smaller parameter* $\lambda \le \lambda_{cmin}$ *all argmaxima and maxima remain infinite, under bigger* $\lambda \ge \lambda_{cmax}$ *all argmaxima and maxima remain zero.*

Proof. Using $S \equiv \mathbb{R}_+$, we can apply Lemma 2 to our auxiliary function $g(x,t) \equiv \pi(x,1,-t)$ with argument $\lambda c = -t > 0$ because evident is strict single crossing property: it means increase of $\pi'_x(x,1,-t) = u'(x) + xu''(x) + t$ w.r.t. t. Thereby, whenever $X_u^*(\lambda c)$ exists, *every selection* $x^*(\lambda c)$ from $X_u^*(\lambda c)$ is monotone *nonincreasing* when $x > 0$. This yields almost-everywhere single-valued $X_u^*(\lambda c)$, i.e., absence of any open intervals for λ maintaining multi-valued $X_u^*(\lambda c)$. In other words, $X_u^*(\lambda c)$ is single-valued, except for some "isolated" downward jumps. In essence, this fact follows from smoothness of $xu'(x)$ (Assumption 1). Smoothness makes function $\pi'_x(x,1,-t)$ single-valued and strict single crossing property applicable (geometricaly, the reason for strictly decreasing argmaximum is that a smooth set—undergraph of $xu'(x)$—cannot have multiple tangent slopes λc at a given point x).

To transform the monotonicity found into *strongly decreasing* $X_u^*(\lambda c)$ on interval $\Lambda(c)$ (at finite positive X_u^*), we apply Proposition 1 used for any local maximum. Since global maxima should be among the local ones, in the intervals of single-valued $X_u^* = \hat{x} = \check{x}$ they must strictly decrease. The remaining isolated points of multi-valued X_u^* are the points of *downward* jumps, as we have found. Thus, we conclude that mapping $X_u^*(\lambda)$ *strongly* decreases on interval $\Lambda(c)$, remaining infinite for smaller λ and remaining zero for higher λ.

Now we turn to the value function and apply Lemma 3 to ensure monotonicity of maximal $\pi_u^*(1,\lambda c)$.[10] Indeed, the objective function $\pi_u(x,1,\lambda c)$ continuously decreases w.r.t. λ everywhere under positive x. Thereby its optimal value π_u^* also continuously decreases when positive, i.e., on our interval $(\lambda_{cmin},\, \lambda_{cmax})$. The optimal value $\pi_u^* \to 0$ when $\lambda \to \lambda_{cmax}$ because of monotonicity and zero lower bound of profit found in Proposition 1. So, continuity at the upper boundary of our interval $\Lambda(c)$ is maintained. Similar logic proves continuity at the lower boundary when $\underline{MR} = 0$. Only in special case when $\underline{MR} > 0$, the continuity of π_u^* becomes more delicate, questionable at point $\lambda_{cmin} = \underline{MR}/c$. Continuity could be violated for a utility like $u(x) = \ln(x+a) - \ln(a) + bx$ $(a,b > 0)$, because here the maximal profit value remains *bounded* from above for all $\lambda \ge \underline{MR}/c = b/c$ but abruptly jumps to infinity under any lower parameter $\lambda < \underline{MR}/c$. However, "revenue unboundedness" requirement (3) in Assumption 1 excludes such exotic utilities. Using Lemma 3 with any compactified domain $X \equiv [0,\bar{x}]$ under *any* $\bar{x}$, we guarantee a *continuous*

[10]For revealing monotonicity of $\pi_u^*(\lambda,c)$ we cannot use more standard envelope theorems since π_u^* appears non-differentiable at the points where X^* make jumps (see Section 4).

response of $\pi_u^*(1,\lambda c)$ to decreasing $\lambda \to 0$. Therefore, if there were a jump of π_u^* under open domain $X \equiv [0,\infty)$ it would occur also on some compactified domain, that contradicts our findings. Thus, the maximal value of $\pi_u^*(1,\lambda c)$ decreases *continuously* from $\overline{MR}$ to 0 under increasing $\lambda c \in [0,\infty)$. This completes the proof. **Q.E.D**

Corollary 1. *The value of maximal per-consumer profit* $\pi_u^*(\lambda,c) \equiv x^*_{(\lambda)}[u'(x^*_{(\lambda)})/\lambda - c]$ *continuously decreases, changing from value* $\overline{MR}/\lambda_{cmin}$ *to 0 on interval* $\Lambda(c)$*; it remains zero on* $[\lambda_{cmax},\infty)$ *(whenever this interval exists); it remains infinite on* $[0,\lambda_{cmin})$ *(whenever this interval exists); and anyway*

$$\lim_{\lambda\to 0}\pi_u^*(\lambda,c)=\infty,\ \lim_{\lambda\to\infty}\pi_u^*(\lambda,c)=0. \tag{22}$$

This corollary is obvious, we just transform function $\pi_u^*(1,\lambda c)$ into $\pi_u^*(\lambda,c)$.

Based on these facts, equilibrium existence can be stated, together with uniqueness of the equilibrium λ.

Theorem 1. *Under Assumption 1 and any cost/population parameters (*$c>0,\ L>0,\ f>0$*) there exists a unique equilibrium value* $\hat{\lambda}$ *that generates some nonempty set* $(\hat{\lambda},X,P,N)$ *of (possibly-asymmetric) equilibria. Each equilibrium in this set is positive in the sense* $\hat{\lambda}>0,\ \pi_u^*(\hat{\lambda},c)>0,\ X\not\ni 0$.

Proof. We can apply Lemma 1 to decreasing maximal per-consumer profit function $g(\lambda)\equiv\pi_u^*(\lambda,c)-f/L$ on interval $(\lambda_{cmin},\lambda_{cmax})$, based on g continuity and its bounds of change (Corollary 1). So, some positive root $\hat{\lambda}\in(\lambda_{cmin},\lambda_{cmax})$ of free-entry equation must exist. In essence, it equalizes a positive investment f with optimal operational profit in the sense $L\pi_u^*(\hat{\lambda},c)=f>0$, that's why $\hat{\lambda}$ is positive. However, to apply Lemma 1, we must ensure that all equilibrium notions are valid. Indeed, on $(\lambda_{cmin},\lambda_{cmax})$ the inverse demand $p^*(x,\lambda,c)$ is well-defined, as well as profit argmaxima and maxima.

To reveal positivity of all equilibrium variables, from *positive* $X_u^*(\hat{\lambda},c)\gg 0$, one can calculate *positive* equilibrium prices $p_k=p^*(x_k,\hat{\lambda},c)\,\forall x_k\in X$, using the inverse demand function. The remaining element of the equilibrium definition is such a vector $N=(n_1,...,n_K)$ that satisfy the labor balance (14). All coefficients of this linear equation being positive, we confirm existence of *a hyperplane* of admissible vectors $(n_1,...,n_K)$, which is truncated to an admissible polygone by the positivity requirement $(n_1,...,n_K)\geq 0$. Thus, all equilibrium components exist.

Uniqueness of equilibrium $\hat{\lambda}$ follows from *strict* monotonicity of π_u^* at points with positive finite profit: $0<\pi_u^*(\lambda,c)<\infty$. **Q.E.D**.

Based on these facts and "regular utility" assumption, it is easy to establish now more definite equilibrium structure: unique $\hat{\lambda}$, *two* groups of firms and an *interval* $(\hat{n},\check{n})$ of firms' masses.

Theorem 2. *Under Assumption 1 and Assumption 2 (*κ*-regular u), the set-valued equilibrium contains one or two types of firms behavior. It consists of a unique marginal utility of income* $\hat{\lambda}$*, a unique couple* $((\hat{x},\hat{p}),(\check{x},\check{p}))$ *of quantity-price bundles:* $\hat{x}\leq\check{x},\ \hat{p}\geq\check{p}$*, and an interval of firms' masses* $(\hat{n},\check{n}):\ \hat{n}\geq\check{n}$*, namely, all the masses that satisfy the labor balance (12) with the couple* $((\hat{x},\hat{p}),(\check{x},\check{p}))$*. This interval of masses degenerates into a point under coincidence of points* $(\hat{x},\hat{p})=(\check{x},\check{p})$*.*[11]

Proof. The uniqueness of $\hat{\lambda}$ is already stated. We also had discussed already (in connection with the equilibrium definition) a polygone of admissible vectors $(n_1,...,n_K)$ of firms' masses. What remains is to ensure the dimensionality $K\leq 2$: not more than two global maxima of profit, i.e., not more than two types of firms' behavior. This fact was already explained when discussing Example 2: it amounts to Assumption 2 on regular utility. Thus, the polygone of masses is an interval. **Q.E.D**.

[11] Thus, n becomes a point only when equilibrium becomes symmetric but under asymmetry $\hat{x}<\check{x}$ the masses $(\hat{n},\check{n})$ and the total mass of firms $n=\hat{n}+\check{n}$ remain ambiguous.

4 Comparative statics under growing market: monotonicity and catastrophes

This section shows an example of multiplicity and asymmetry, finds how numerous can be the asymmetry moments under growing population (or decreasing fixed cost) and clarifies the direction of changes in consumption, price and variety.

Number of jumps. Based on κ-regular utility function, the following theorem establishes the number of jumps (catastrophes) in consumption and price during the "complete" path of comparative statics, i.e., when the relative market size L/f grows from zero to infinity. Surprisingly, under non-globally concave profit such jumps are guaranteed, being accompanied by equilibria asymmetry. After proving this, we explain the underlying behavior of the local argmaxima, and turn to equilibria monotonicity during such market evolution.

Theorem 3. *Let Assumption 1 and Assumption 2 hold, with $\kappa \geq 0$. Then under any constant marginal cost c, there are exactly κ critical values of relative market size $L/f \in (0, \infty)$ that bring equilibria multiplicity and asymmetry, being also points of discontinuity of equilibria w.r.t L/f.*

Corollary 2. *(i) For equilibrium uniqueness and symmetry under all $c, f, L > 0$, sufficient is strict concavity of the elementary revenue function $xu'(x)$ (i.e., $\kappa = 0$). (ii) This strict concavity condition is also necessary, in the sense that under non-concave $xu'(x)$, for any cost $c > 0$ there exists some relative market size $\hat{L}/f > 0$ that generates equilibria assymetry, multiplicity and discontinuity.*

Proof. We have shown already that the equilibrium value $\hat{\lambda} = \hat{\lambda}(L/f)$ of marginal utility of income is unique and exists, i.e., the equilibrium mapping $\hat{\lambda}(L/f) : R_+ \mapsto R_+$ is well-defined and single-valued. Now we would like to ensure that $\hat{\lambda}(L/f)$ continuously changes from $\lambda_{cmin} \geq 0$ to λ_{cmax} over the domain $L/f \in [0, \infty)$. Obviously, well-defined and continuous is the (equilibrium per-consumer) profit mapping $\bar{\pi}_{u,c}(L/f) \equiv \pi^*_u(\hat{\lambda}(L/f), c) \equiv f/L$, inversely dependent on the market size. Our Proposition 2 claims that function $\pi^*_u(\lambda, c)$ continuously decreases in λ. Then, we can apply Lemma 1 to equation $g(\lambda, L/f) \equiv \pi^*_u(\lambda, c) - f/L = 0$ (with L/f as a parameter and λ as argument). Therefore, the unique solution $\hat{\lambda}(L/f)$ to this zero-profit equation must *increase* w.r.t. L/f from value $\lambda_{cmin} \geq 0$ to $\lambda_{cmax} \leq \infty$, without reaching these borders (using Theorem 1). Additionally, $\hat{\lambda}(L/f)$ increases continuously, because the maximal value function $\pi^*_u(\lambda, c)$ continuously decreases.

Therefore, λ takes *all* values from $(\lambda_{cmin}, \lambda_{cmax})$. Further, to get the number of jumps, we can use equivalency (16) between the real profit maximization and the maximization of the auxiliary function (17). We obtain the number of possible multiplicity instances directly from Assumption 2 on κ-regularity (see also geometry reasoning below). Corollary is evident. **Q.E.D**.

Let us explain now non-differentiability of $\bar{\pi}^*_{u,c}(L/f)$ and the geometry of our comparative statics. For any value of λ that brings two local maxima, we can define two functions, which are locally-maximal values of (multiplied by λ) profit (16), i.e., auxiliary functions[12]

$$\hat{\pi}^*(\lambda c) \equiv \lambda \pi(\hat{x}(\lambda c), \lambda, c), \qquad \check{\pi}^*(\lambda c) \equiv \lambda \pi(\check{x}(\lambda c), \lambda, c).$$

The first function uses the left *local* argmaximum $\hat{x}$ (the leftmost solution to FOC described in Lemma 1) while $\check{\pi}^*$ exploits the right local argmaximum $\check{x}$. Somewhat loosely, we use the same notations $\hat{x}, \check{x}$ without specifying are they argmaxima or not.

To describe the behavior of $\hat{\pi}^*(.), \check{\pi}^*(.)$, we apply the usual envelope theorem, and conclude (standardly) that the absolute value of the derivative of profit w.r.t. cost—is equal to the demand value x at the point studied. I.e.,

[12] As we have seen, there is one-to-one correspondence between the argmaxima of three functions $x \in \arg\max_{x \in \mathbb{R}_+} \pi(x, 1, \lambda) = \lambda \pi(x, \lambda, 1) \Leftrightarrow x \in \arg\max_{x \in \mathbb{R}_+} \pi(x, \lambda, 1)$.

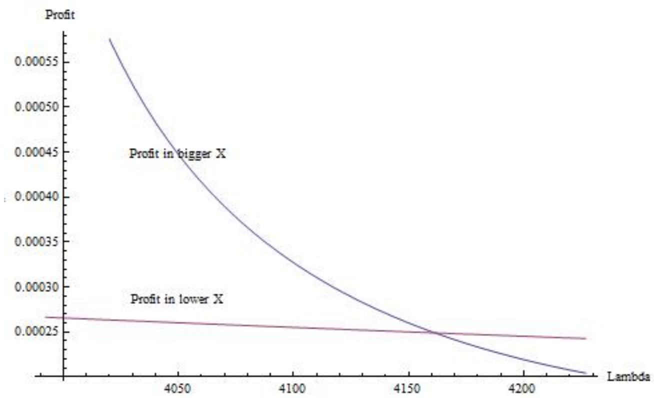

Figure 3: Two locally-optimal λ-adjusted profit functions $\hat{\pi}^*(\lambda c), \breve{\pi}^*(\lambda c)$.

whenever there are two local argmaxima $\hat{x} < \breve{x}$, it must be that

$$\frac{d\hat{\pi}^*(\lambda c)}{d\lambda} = -\hat{x}(\lambda c) > \frac{d\breve{\pi}^*(\lambda c)}{d\lambda} = -\breve{x}(\lambda c). \tag{23}$$

In other words, the left locally-optimal value $\hat{\pi}^*(\lambda c)$ everywhere decreases *slower* than the right value $\breve{\pi}^*(\lambda c)$.

Guiding example continued. Such decrease is illustrated by Fig. 3, where the global maximum of $\lambda\pi_u^*(\lambda, c)$ is the *upper envelope* of these two locally-optimal loci $\hat{\pi}^*(\lambda c), \breve{\pi}^*(\lambda c)$.

Because of unequal slopes (23), these two loci can cross only *once* at some point $\hat{\lambda}$ (this proves the number κ of jumps). Besides, for all small $\lambda < \hat{\lambda}$ the bigger root $x = \breve{x}$ of FOC remains the true global argmaximum, whereas for all big $\lambda > \hat{\lambda}$ the smaller root $\hat{x}$ becomes the global argmaximum. Then the jump poes downward. Thus, under regular utility with $\kappa = 1$ there *is exactly one point* $\hat{\lambda}$ generating multiple profit argmaxima during the evolution of λ from 0 to ∞. It is also evident why our profit function is not differentiable at this point.

Equilibria monotonicity. Generally, both valid roots of equation (9), i.e., both local argmaxima $\hat{x}(\lambda), \breve{x}(\lambda)$ must *decrease* w.r.t. λ (Proposition 1). Really, one can observe such negative monotonicity of both local argmaxima $\hat{x}(\lambda), \breve{x}(\lambda)$ in our example in Fig. 4. Here, we illustrate multiplicity of equilibria with costs $f = 0.0025$, $c = 0.0002555$. To derive comparative statics, we vary population L from 0 to ∞.[13]

Resulting comparative statics of consumption x of each variety w.r.t. L is presented in Figures 4, 5 together with related evolution of price and mass of firms:

Specifically, we have found the unique switching point $\hat{L}/f \approx 10.04$ and related two boundary equilibria among the interval of them: $(\hat{\lambda}, \breve{x}, \hat{x}, \breve{p}, \hat{p}, \hat{n}, 0)$ and $(\hat{\lambda}, \breve{x}, \hat{x}, \breve{p}, \hat{p}, 0, \breve{n})$, differing only in vector $(\hat{n}, \breve{n})$ and gross utility U:

[13]Calculation technique is as follows: we first compute related marginal revenue function $m_R(x_j)$. Inverting it on its intervals of monotonicity, we find two locally-optimal quantities $\hat{x}(\lambda c)$, $\breve{x}(\lambda c)$. Substituting these $\hat{x}(\lambda c)$, $\breve{x}(\lambda c)$ into the operational profit function (13) we obtain two branches of the adjusted profit function, $\hat{\pi}^*(\lambda c), \breve{\pi}^*(\lambda c)$ and their upper envelope $\pi^*(\lambda, c) = \max\{\hat{\pi}^*(\lambda c)/\lambda, \breve{\pi}^*(\lambda c)/\lambda\}$. It appears monotone decreasing and continuous, so from equation (13) we calculate the unique root, equilibrium value $\hat{\lambda}(L/f)$ which is an increasing continuous function of L/f, and derive related consumptions and prices. On each interval of L we select the valid (small or big) consumption value relying on smaller or bigger $\hat{\pi}^*(\lambda c) \lesseqgtr \breve{\pi}^*(\lambda c)$.

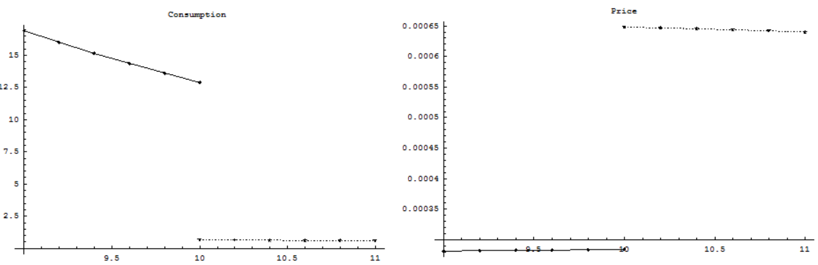

Figure 4: Dependence of consumption and price upon population L.

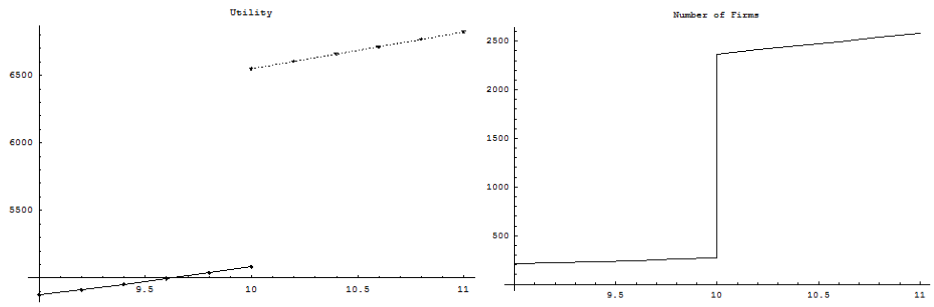

Figure 5: Changes in welfare and mass of firms.

(1) $\hat{x} = 0.652644395317$, $\check{x} = 12.8961077968$, $\hat{p} = 0.000284323031025$, $\hat{n}_1 = 272.727734586$, $U_1 = 5081.514933136$;

(2) $\hat{x} = 0.652644395317$,$\check{x} = 12.8961077968$, $\check{p} = 0.0006479943306508$, $\check{n}_2 = 2364.570031198$, $U_2 = 6549.082752038$.

There is an interval of asymmetric equilibria between these two extreme equilibria under this $(\hat{L}/f)$ and related $\hat{\lambda}$. On this interval low and high consumptions $\hat{x}$, $\check{x}$ remain the same, whereas the set for masses of firms has the form:

$$N = \{(\check{n}, \hat{n}) \geq 0 | \, \check{n} = 2364.570031198 - 8.670075431758 \, \hat{n}\}.$$

Here some firms choose producing high quantity $\check{x}$ whereas other firms choose low quantity $\hat{x}$.

In this example one can observe that growing market and related entry of new firms push per-variety consumption down *monotonically* with only one jump (dicontinuity) at some point $\hat{L}/f \approx 10.04$ of this evolution. Under small market size $L/f < 10.04$, the global profit maximum is attained at higher consumption $\check{x}$ and the mass of firms is relatively low. But when L/f exceeds 10.04, all firms switch to producing smaller quantity $\hat{x}$ because their mass jumps up. By Theorem 3, such dicontinuous behavior of equilibria is not a degenerate example, but *a general rule in all situations with non-concave profit.*

Turning from this example of market size impact to general case, we formulate now a theorem about set-valued monotone comparative statics, similar to single-valued comparative statics in ZKPT. We use again our monotonicity notions for mappings: $X(L)$ *strongly* decreases when all its selections decrease.

Theorem 4. *Under Assumption 1, an increase in the relative market size L/f induces several changes in the set-valued equilibria: (i) the marginal utility of money $\hat{\lambda}(L/f)$ increases with elasticity $r_u(x)$; (ii) the consumption mapping $\bar{X}_u(L/f)$ for each variety is a closed one and strongly decreases; (iii) the set-valued total mass of firms*

$(n_1+n_2+...): \forall n \in N$ *strongly increases; (iv) the price* $p(L/f)$ *strongly decreases on any interval of single-valued* $\bar{X}_u(L/f)$ *where* $r'_u(x) > 0$*, and conversely, price strongly increases on any interval where* $r'_u(x) < 0$*; (v) at any point* (L/f) *with multi-valued* $X^*_u(L/f)$*—the price always jumps up*; *(vi) whenever a firm's output* Lx *is single-valued, it changes oppositly to price, whereas at the jumpt it jumpes down.*[14]

Proof. Claim (i) about increasing λ has been verified when proving Theorem 3. Moreover, we can derive the elasticity $\mathop{E}\limits_{L}\lambda(L)$ of equilibrium λ. We totally differentiate w.r.t. L the free entry condition

$$\pi = q\frac{u'(q/L)}{\lambda(L)} - cq - f = 0.$$

We can ignore q'_L due to the envelope theorem. We get

$$\pi'_L = -q^2\frac{u''(q/L)}{\lambda(L)L^2} - \frac{qu'(q/L)}{\lambda^2(L)}\lambda'(L) = 0 \Rightarrow L\frac{\lambda'(L)}{\lambda(L)} = -q\frac{u''(q/L)}{u'(q/L)L},$$

and obtain the needed elasticity

$$\mathop{E}\limits_{L}\lambda(L) = L\frac{\lambda'(L)}{\lambda(L)} = -\frac{q}{L}\cdot\frac{u''(q/L)}{u'(q/L)} = r_u(q/L).$$

Now, using $\bar{X}_u(L/f) = X^*_u(\hat{\lambda}(L/f))$, claim (ii) actually becomes equivalent to strongly decreasing mapping of profit argmaxima $X^*_u(\lambda)$ which was stated in Proposition 2. Additionally, mapping $X^*_u(\lambda)$ under our assumptions is closed (upper-semi-continuous on every compact set), being the argmaximum of a continuous function on a compact set, or, more correctly, a set $[0,\infty)$ compactifiable under any given λ. Since $\hat{\lambda}(L/f)$ is continuous, so, $\bar{X}_u(L/f) = X^*_u(\hat{\lambda}(L/f))$ is closed too.

Claim (iii) about the firms' masses follows from "isolated" points of multi-valued argmaximum $X^*_u(\lambda)$, the same property being transferred to $\bar{X}_u(L/f)$. Respectively, at the open intervals of single-valuedness, unique $\bar{x}(L/f)$ decreases in such a way, that (using labor balance (12)) related mass $n = N$ increases, by Proposition 2 from ZKPT.

What remains is to find how vector N of admissible masses $(n_1, n_2, ...)$ of different firms behave at any point of jump, denoted here $\bar{L}_f \equiv \bar{L}/\bar{f}$. It is sufficient to note that to the left and to the right from such point $\bar{L}_f$, unique consumption $\bar{x}(L/f)$ satisfies the labor balance in the form $c\bar{x} + f/L = 1/n$, and $\bar{x}$ makes a *downward* jump. At the same time, parameter f/L in the left and right vicinities of this point $\bar{L}_f$ remains essentially the same. Thereby, the jump in unique equilibrium n in these vicinities is *upward* in the sense $\underline{n} \equiv \lim_{L/f\to\bar{L}_f(-)} < \bar{n} \equiv \lim_{L/f\to\bar{L}_f(+)}$. Using closedness of the equilibrium mapping, both these limits belong to the vector $\bar{N} = (n_1, n_2, ...)$ of admissible masses of firms at the limiting point $\bar{L}_f$ itself. Therefore, amongst all $\bar{N} = (n_1, n_2, ...)$ satisfying the labor balance (12), the supremum $\bar{n} = 1/(c\hat{x} + 1/\bar{L}_f)$ and the infimum $\underline{n} = 1/(c\check{x} + 1/\bar{L}_f)$ also must belong to the right and the left limiting values of N, respectively.

(iv) Similar reasoning with limits can be applied to prices, therefore the set of possible prices $P(\bar{L}_f)$ at the jumping point—contains the limits of prices taken from the left and from the right. The conclusion about prices is simple on any intervals where X^*_u is a singleton, because the direction of price changes just follows from Proposition 2 from ZKPT: prices go down when $r'_u > 0$ and up in the opposite case.[15]

[14] Additionally, the equilibrium markup $M = (p-c)/p = r_u(x)$ always behaves like price.

[15] It seemingly contradicts the picture in Fig.4: here price increases *before* the jump but decreases *after* it. However, actually function $r_u(x)$ can be checked to decrease at all arguments $x(L/f): L < \hat{L}$ to the left from the jump, and to increase on the right interval $L > \hat{L}$. Thereby on the left interval including the jump point $\hat{L}/f \approx 10.04$ the claim holds true with $r'_u < 0$, and on the right interval (excluding the jump point where $r'_u(x) = 0$) it holds true either, with $r'_u > 0$.

(v) By contrast, at the multi-valued situation, any price jump always occurs *upward*, because two (equi-profitable) consumptions are compared as $\hat{x} < \check{x}$, whereas the inverse demand function decreases, so, $\hat{p} = p^*(\hat{x}) > \check{p} = p^*(\check{x})$. Thus, under growing market, this jump goes from the point $\check{p}$ to the higher point $\hat{p}$.

(vi) As to the behavior of output, on intervals of single-valuedness it is revealed in ZKPT: $q(L)$ increases under $r'_u > 0$ (increasingly-elastic demand) and decreases under opposite condition. Further, to establish the direction of jump at a multi-valued point, we recall that $q = Lx$ and L remains constant at a point, whereas x jumps down. So, q jumpts down.

This completes the proof. Q.E.D.

Let us explain again the strange direction of the price changes (following ZKPT and Intro). Under $r'_u < 0$ we observe some counter-intuitive *anti-competitive effect*: growing market attracts more firms but still all prices go up. The explanation lies in the demand convexity: (only) whenever convexity is too strong, the firms compensate their decreasing output with growing prices. Now we extended the *same* mechanism of anti-competitive effect to set-valued equilibria also. Indeed, convexity is even stronger at the points of jumps. In addition, at such points the price increase becomes *abrupt*.

Such discontinuity of equilibria evolution also seems a paradoxical outcome. It means *catastrophic jumps* in consumptions and prices in response to smooth shifts in exogenous parameters. Though the jump itself occurs typically only at one point among the continuum of parameters L determining the market evolution, but we have found that under non-monotone marginal revenue—hitting such point sooner or later during complete evolution is *guaranteed*! Even on any finite interval $[\underline{L}, \overline{L}]$ of changing parameters *the probability of catastrophes is not negligible.*

Confirming or exculding catastrophes appears now as an empirical question. Economically, the possibility of non-monotone marginal revenue looks quite plausible. Mathematicaly, any demand curve that reminds a piece-wise linear function (flat, then having a kink and again flat) must generate a non-monotone marginal revenue. Then, any gross demand summed up from linear demands of two distinct consumer groups—*must generate a non-monotone marginal revenue.* These considerations increase our faith in the possibility of catastrofhic effects reactions in a monopolistically-competitive market. They explain, that a jump may happen, in particular, when a large group of consumers coherently comes out of the market in response to changes (not in our homogenous model).

5 Conclusion

First of all, this paper satisfied our curiousity about robustness of the market theory: what happens when we get rid of technically convenient assumption of profit "concave everywhere", or, equivalently, of monotone marginal revenue? It turns out that equilibria existence remains, but equilibria become asymmetric and get set-valued structure. Importantly, the comparative statics of equilibrium-set in response to growing market (for instance, population growth or countries integration) remains *similar* to what we know about single-valed equilibria.

Second, new effects found are *catastrophes,* i.e., jumps of outputs and prices in response to small shifts in population or costs. Surprisingly, such jumps *must* happen whenever marginal revenue is non-monotone, i.e., the demand has kinks. This case looks natural under distinct groups of demands. Thus, abrupt market reactions to small parameters shifts look now not quite unrealistic.

If we expand the approach with non-concave profit to heterogenous firms, the same effect of "catastrophes" should manifestate itself in the "gap" between exporting and non-exporting firms, or those engaged in R&D and others. Such clusterization looks interesting.

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