Convexity of Hypergraph Matching Game

Soh Kumabe
The University of Tokyo
Tokyo, Japan
soh_kumabe@mist.i.u-tokyo.ac.jp

Takanori Maehara
RIKEN Center for Advanced Intelligence Projec
Tokyo, Japan
takanori.maehara@riken.jp

ABSTRACT

The hypergraph matching game is a cooperative game defined on a hypergraph such that the vertices are the players, and the characteristic function is the value of a maximum hypergraph matching on a hypergraph induced by a coalition. This game models the nature of group formation and will have applications in, e.g., organ exchange and joint purchasing. The hypergraph matching game is intractable in general because evaluating its characteristic function is already NP-hard. Thus, we study a more tractable condition, called the convexity. First, we prove that the problem of checking whether a given hypergraph matching game is convex or not is solvable in polynomial time. Second, we prove that the Shapley value of a given convex hypergraph matching game is exactly computable in polynomial time. Third, we show that the problem of finding a minimum compensation to make a given hypergraph matching game convex is NP-hard, even if the input is a graph, and is 2-approximable in polynomial time if the input is an antichain. Finally, we consider the fractional hypergraph matching game and prove that if the fractional hypergraph matching game is convex, then its characteristic function coincides with the characteristic function of the corresponding (integral) hypergraph matching game.

KEYWORDS

hypergraph matching game; synergy coalition group; convexity; Shapley value

ACM Reference Format:
Soh Kumabe and Takanori Maehara. 2020. Convexity of Hypergraph Matching Game. In *Proc. of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2020), Auckland, New Zealand, May 9–13, 2020,* IFAAMAS, 9 pages.

1 INTRODUCTION

1.1 Background and Motivation

A *cooperative game* (V, v) is given by a finite set V and a function $v\colon 2^V \to \mathbb{R}$ with $v(\emptyset) = 0$, where V is the set of *n players* and v is the *characteristic function* such that $v(X)$ denotes the value if players $X \subseteq V$ form a coalition. The main problem of a cooperative game is to find a "fair way" to distribute the collective value to the players. Here, the way is referred to as a *solution concept*, which includes the stable set [32], core [13], Shapley value [27], kernel [9], and nucleolus [25]. See [7] for the details of cooperative game theory.

In this study, we consider the *hypergraph matching game* (V, v), which is also known as the *packing game* [10] or *synergy coalition group representation* [8]; see Section 2 for the terminology of hypergraphs. Let $G = (V, E)$ be a hypergraph, and $c\colon E \to \mathbb{R}_+$ be the weights of the hyperedges. A *matching* M is a collection of hyperedges that are pairwise disjoint. Then, the characteristic function $v : 2^V \to \mathbb{R}$ of the hypergraph matching game is given by

$$v(X) = \max_{M\text{:matching of } G[X]} \sum_{e \in M} c(e), \tag{1.1}$$

where $G[X]$ is the subgraph induced by X. Note that v is superadditive, i.e., for all disjoint $X, Y \subseteq V$, the following inequality holds.

$$v(X) + v(Y) \leq v(X \cup Y). \tag{1.2}$$

Intuitively, this game models the nature of group formation. Suppose V is the set of players, and $E \subseteq 2^V$ is the set of all the possible groups of V. When players $X \subseteq V$ form a coalition, they find subgroups (i.e., hypergraph matching) in the coalition to maximize their total profit $v(X)$. Then, the behavior of the players is captured by the hypergraph matching game.

The hypergraph matching game is a generalization of the assignment game [29] and the matching game [10] whose inputs are bipartite and general graphs, respectively. These games have been studied extensively for a long time [1, 2, 6, 15, 16]. Moreover, they have many practical applications such as trading markets [29], kidney exchange [5], and spectrum sharing [22].

The hypergraph case is much difficult than the graph case because evaluating $v(X)$ is already NP-hard [12]. Consequently, most computational problems about this game are NP-hard. Deng et al. [10] gave a necessary and sufficient condition of the core non-emptiness (see Remark 4.4). Conitzer and Sandholm [8] proposed (1.1) as a compact representation of a characteristic function and named *synergy coalition group representation* (with transferrable utility). They proved that checking the core non-emptiness is NP-hard [8, Theorem 2] but in P if the optimal hypergraph matching of G is provided [8, Theorem 4].

Because the general hypergraph matching game is intractable, we here consider a (possibly) computationally tractable subclass of the hypergraph matching game. A cooperative game is *convex* if its characteristic function $v\colon 2^V \to \mathbb{R}$ is *supermodular* [29], i.e., the following inequality holds for all $X, Y \subseteq V$ [11]:

$$v(X) + v(Y) \leq v(X \cap Y) + v(X \cup Y). \tag{1.3}$$

Note that the supermodularity (1.3) is a stronger requirement than the superadditivity (1.2). Convex hypergraph matching games have natural applications in drug manufacturing; see Section 7 in [8].

A convex game has several desirable properties. For example, the core is always non-empty [28], coincides with the bargaining set [20], is the unique von Neumann–Morgenstern stable set [28, 32], and contains the Shapley value [28]. A core element is obtained in oracle polynomial time (i.e., polynomial number of characteristic

Proc. of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2020), B. An, N. Yorke-Smith, A. El Fallah Seghrouchni, G. Sukthankar (eds.), May 9–13, 2020, Auckland, New Zealand.

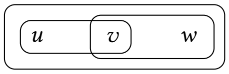

Figure 1: A convex game. The hyperedge weights are $c(e) = 2^{|e|-1}$. Then $\nu(\{u,v\}) = 1$, $\nu(\{v,w\}) = 1$, $\nu(\{v\}) = 0$, and $\nu(\{u,v,w\}) = 4$. Therefore, $\nu(\{u,v\}) + \nu(\{v,w\}) \leq \nu(\{v\}) + \nu(\{u,v,w\})$.

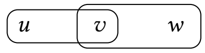

Figure 2: A non-convex game. All the hyperedge weights are 1. Then $\nu(\{u,v\}) = 1$, $\nu(\{v,w\}) = 1$, $\nu(v) = 0$, and $\nu(\{u,v,w,\}) = 1$ (attained by $\{u,v\}$ or $\{v,w\}$). Therefore, $\nu(\{u,v\}) + \nu(\{v,w\}) \not\leq \nu(\{v\}) + \nu(\{u,v,w\})$.

function evaluations), and the core membership problem is solvable in oracle polynomial time [29]. The nucleolus coincides with the kernel [20], and is computable in oracle polynomial time [18], etc. In particular, for the hypergraph matching game, Conitzer and Sandholm showed that a solution in the core is obtained in $O(|V|^2|E|)$ time. Therefore, researchers have been interested in conditions of the convexity of several games such as minimum base games [21], communication games [31], and maximum spanning tree games [17, 23].

A hypergraph matching game is sometimes convex (Figure 1) and sometimes non-convex (Figure 2). Thus, we are interested in a condition under which a hypergraph matching game is convex. We are also interested in whether computational problems on convex hypergraph matching game are tractable or not. There is a polynomial-time algorithm to compute a solution in the core [8], but the tractability of other problems are still open.

1.2 Our Problems and Contributions

In this paper, we consider three computational problems and one structural property related to the convexity of the hypergraph matching game. The first problem is the following.

Problem 1. *Given a hypergraph $G = (V, E)$ and hyperedge weights $c\colon E \to \mathbb{R}$, determine whether the hypergraph matching game is convex or not.*

Our first contribution is a polynomial-time algorithm to solve Problem 1 (Theorem 4.1). We first establish a necessary and sufficient condition of the convexity of a hypergraph matching game (Theorem 3.3). Then, we construct an algorithm that checks the condition efficiently. Note that, during the proof of this theorem, we also show that the characteristic function of a convex hypergraph matching game is evaluated in polynomial time (Lemma 4.3).

After obtaining the characterization and algorithm for the convexity of a hypergraph matching game, we consider solution concepts to a convex hypergraph matching game. The convexity and the polynomial-time evaluation of ν imply that the stable set, core, kernel, and nucleolus are tractable [18, 20, 28]. In particular, there is an efficient algorithm to get a solution in the core [8]. Thus, we focus on another standard solution concept, the Shapley value.

The *Shapley value* [27] $\phi\colon V \to \mathbb{R}$ is defined by

$$\phi(u) = \sum_{X \subseteq V \setminus \{u\}} \frac{|X|!(|V| - |X| - 1)!}{|V|!} (\nu(X \cup \{u\}) - \nu(X)). \tag{1.4}$$

The Shapley value is one of the most standard solution concepts. Computing the Shapley value of a matching game is #P-complete, but it admits a fully-randomized polynomial-time approximation scheme (FPRAS) [3]. There exists a convex game of which the Shapley value is not computable in polynomial time unless P = NP, and any convex game admits a FPRAS [19]. Therefore, an exact algorithm for computing the Shapley value of a convex hypergraph matching game is still an open problem. This is our second problem.

Problem 2. *Given a hypergraph $G = (V, E)$ and hyperedge weight $c\colon E \to \mathbb{R}$ that form a convex hypergraph matching game. Compute the Shapley value.*

Our second contribution is a polynomial-time algorithm for solving this problem (Theorem 5.1). The algorithm is derived as follows. We observe that the marginal gain, $\nu(X \cup \{u\}) - \nu(X)$, takes at most $O(|E|)$ different values associated with the hyperedges (Lemma 5.4). Thus, the subsets are grouped by their marginal gain, and their contributions are computed independently. Here, the cardinality of each group is computed by the double-counting argument (Lemma 5.5) with the Möbius inversion formula [30] (Equation (5.2)).

Thus far, we see that a convex hypergraph matching game has several desirable properties. However, because the convexity is a strong condition, most instances of the hypergraph matching game may be non-convex (cf: Corollary 3.4). Thus, we consider the following problem, called the *minimum compensation problem*.

Let us consider the following scenario: We hope the players form the grand coalition V. Because the convexity is a sufficient condition that the players do not deviate from the grand coalition [28], we slightly modify the game to make it convex. Here, we provide a *compensation* $h(u) \in \mathbb{R}_+$ to each user u if he/she affiliates with a coalition X but does not affiliate with a maximum matching. In this case, the value of the coalition X is given by

$$\nu_h(X) = \max_{M:\text{matching of } G[X]} \left(\sum_{e \in M} c(e) + \sum_{u \in X \setminus V_M} h(u) \right), \tag{1.5}$$

where $V_M = \bigcup_{e \in M} e$ is the set of vertices covered by M. If the compensation h is sufficiently high, the grand coalition will be formed but no players affiliate with a matching because they are satisfied by getting the compensation. Thus, we are interested in a smallest compensation h such that the game (V, ν_h) is convex. The third problem is given as follows.

Problem 3. *Given a hypergraph $G = (V, E)$ and hyperedge weight $c\colon E \to \mathbb{R}$, find a vertex weight $h\colon V \to \mathbb{R}_+$ such that the game (V, ν_h) is convex and $\sum_{u \in V} h(u)$ is minimized.*

Our third contribution is the following: We first prove that Problem 3 is NP-hard, even if the input is a graph (Theorem 6.1) by reducing the 3-SAT problem. Thereafter, we construct a polynomial-time 2-approximation algorithm if the input is an antichain (Theorem 6.4). The algorithm is based on the LP-relaxation of the problem.

Note that Problem 3 is a variation of the *vertex stabilizing problem* [1, 15], which removes a smallest vertex subset to make the game have a non-empty core. This problem is polynomial-time solvable if the vertices have the same removal cost, and in general case, it is NP-hard but 2-approximable. Our Problem 3 requires removal (compensation) of a vertex subset to make the game convex.

Because the convexity is stronger than the core non-emptiness, our problem requires modifying the game with more than the vertex stabilizing problem.

Finally, we consider the *fractional hypergraph matching game*, whose characteristic function $\nu'(X)$ is given by the optimal value of the following linear programming problem:

$$\begin{array}{lll} \text{maximize} & \sum_{e \in E} c(e)x(e) & \\ \text{subject to} & \sum_{e \in E: e \ni u} x(e) \leq 1, & (u \in V), \\ & x(e) = 0, & (e \notin E[X]), \\ & 0 \leq x(e) \leq 1, & (e \in E). \end{array} \tag{1.6}$$

Note that this game is a special case of the *linear production game* [24]. The characteristic function $\nu'(X)$ is evaluated in polynomial time via linear programming [14]. Thus, it will be much tractable than the integral hypergraph matching game.

If the integral hypergraph matching game is convex, the core of the game is non-empty. Hence, the LP-relaxation (1.6) is integral (Deng et al. [10, Theorem 1]). Therefore, we have $\nu = \nu'$, and hence, the fractional hypergraph matching game is also convex.

Our fourth contribution shows the converse direction: if the fractional hypergraph matching game is convex, then we have $\nu' = \nu$ (Theorem 7.1), i.e., the integral hypergraph matching game is also convex (Corollary 7.7). This implies that relaxing the integrality does not change the situation in convex hypergraph matching game.

2 PRELIMINARIES

We introduce the basic terminology of hypergraphs. See [4] for a theory of hypergraphs.

A *hypergraph* $G = (V, E)$ is a pair of finite set V and a collection of its subsets $E \subseteq 2^V$. Each $u \in V$ is called a *vertex*, and each $e \in E$ is called a *hyperedge*. We say that e and e' are *adjacent* if $e \cap e' \neq \emptyset$.

For a vertex subset $X \subseteq V$, we denote by $E[X] = \{e \in E : e \subseteq X\}$. We also denote by $G[X] = (X, E[X])$ the hypergraph *induced by* X. For a hyperedge subset $F \subseteq E$, we denote by $V_F = \bigcup_{e \in F} e$ the vertices incident to F.

A set M of hyperedges is a *matching* if the hyperedges in M are pairwise non-adjacent. A hypergraph $G = (V, E)$ is a matching if E is a matching.

A k-hypergraph is a hypergraph whose hyperedges have the same cardinality k. A hypergraph $G = (V, E)$ is an *antichain* if there are no $e, e' \in E$ with $e \subseteq e'$. Any matching is an antichain, and any k-hypergraph is an antichain.

3 CHARACTERIZATION OF CONVEXITY

We give a characterization of the convexity of a hypergraph matching game. This characterization forms the basis for all the results that appear in the subsequent sections.

We assume that the weights of the hyperedges of cardinality one are zero. This condition is satisfied by subtracting a modular function $X \mapsto \sum_{\{u\} \in E; u \in X} c(\{u\})$ from ν; this operation preserves the supermodularity of ν.

We say that a matching M *attains* $\nu(Z)$ *for* $Z \subseteq V$ if $e \subseteq Z$ for all $e \in M$ and $\nu(Z) = \sum_{e \in M} c(e)$. A hyperedge $e \in E$ is *synergetic* [8] if $\nu(e)$ is *uniquely* attained by $\{e\}$. Note that the value of ν is preserved if we remove non-synergetic edges [8, Lemma 6]. The following is easily verified.

Lemma 3.1. *For a synergetic hyperedge $e \in E$, we have $\nu(e) > \nu(X_1) + \cdots + \nu(X_k)$ for all subpartitions $\{X_1, \ldots, X_k\}$ of e.*

From now, we give a necessary and sufficient condition of the convexity of a hypergraph matching game. We say that two hyperedges e, e' are *intersecting* if $e \setminus e' \neq \emptyset$, $e' \setminus e \neq \emptyset$, and $e \cap e' \neq \emptyset$. We repeatedly use the following lemma.

Lemma 3.2 ([8, Lemma 3]). *If the characteristic function ν is supermodular, for any two intersecting synergetic hyperedges $e, e' \in E$, their union $e \cup e'$ is also a synergetic hyperedge in E.*

Our main theorem is stated as follows.

Theorem 3.3. *The characteristic function ν is supermodular if and only if for any intersecting synergetic hyperedges $e, e' \in E$ their union $e \cup e'$ is a synergetic hyperedge in E and satisfies $c(e) + c(e') \leq \nu(e \cap e') + c(e \cup e')$.*

Proof. The "only-if" part follows from Lemma 3.2. Now we will prove the "if" part. Suppose that there exist two subsets $X, Y \in V$ such that $\nu(X) + \nu(Y) > \nu(X \cap Y) + \nu(X \cup Y)$. We choose X, Y with the minimum value of $|X \cup Y|$. If there are many such candidates, we choose a pair with the minimum value of $|X \cap Y|$. We can assume that X and Y are intersecting. Otherwise, if $X \cap Y = \emptyset$, then $\nu(X) + \nu(Y) \leq \nu(X \cup Y)$ from the definition of ν, which shows the supermodularity of ν. Moreover, if $Y \subseteq X$, then $\nu(X) + \nu(Y) = \nu(X \cap Y) + \nu(X \cup Y)$, which shows the supermodularity of ν.

Let $M = \{e_1, \ldots, e_{|M|}\} \subseteq E$ and $N = \{f_1, \ldots, f_{|N|}\} \subseteq E$ be matchings that attain $\nu(X)$ and $\nu(Y)$, respectively. By the minimality of X and Y, we have $X = V_M$ and $Y = V_N$; otherwise, we have

$$\nu(V_M) + \nu(V_N) = \nu(X) + \nu(Y) \tag{3.1}$$
$$> \nu(X \cap Y) + \nu(X \cup Y) \tag{3.2}$$
$$\geq \nu(V_M \cap V_N) + \nu(V_M \cup V_N), \tag{3.3}$$

where the first line is the definition of M and N, the second line is the definition of X and Y, and the third line is from the monotonicity of ν with $V_M \subseteq X$ and $V_N \subseteq Y$. This means that the pair V_M and V_N is a candidate of X and Y with a smaller union and intersection, but this contradicts the choice of X and Y.

If each of M and N consists of one edge, i.e., $M = \{e_1\}$ and $N = \{f_1\}$, we have

$$c(e_1) + c(f_1) = \nu(e_1) + \nu(f_1) \tag{3.4}$$
$$> \nu(e_1 \cap f_1) + \nu(e_1 \cup f_1) \tag{3.5}$$
$$= \nu(e_1 \cap f_1) + c(e_1 \cup f_1), \tag{3.6}$$

where the last line uses the assumption that $e_1 \cup f_1$ is a synergetic hyperedge. However, this contradicts the assumption.

Now we consider the case that M contains two or more hyperedges. If $e_1 \cap Y = X \cap Y$, we have

$$\nu(X) + \nu(Y) = \nu(e_1) + \nu(X \setminus e_1) + \nu(Y) \tag{3.7}$$
$$\leq \nu(e_1 \cup Y) + \nu(X \setminus e_1) + \nu(e_1 \cap Y) \tag{3.8}$$
$$\leq \nu(X \cup Y) + \nu(X \cap Y), \tag{3.9}$$

where the second line is the supermodularity on e_1 and Y, and the last line is the supermodularity on $e_1 \cup Y$ and $X \setminus e_1$, and $(X \setminus e_1) \cap$

$(e_1 \cup Y) = \emptyset$ by the assumption. This shows the supermodularity of v. Otherwise, i.e., if $e_1 \cap Y \subsetneq X \cap Y$, we have

$$v(X) + v(Y) = v(e_1) + v(X \setminus e_1) + v(Y) \tag{3.10}$$

$$\leq v(e_1) + v((X \setminus e_1) \cap Y) + v((X \setminus e_1) \cup Y) \tag{3.11}$$

where the last line is the supermodularity. Because $|e_1 \cup ((X \setminus e_1) \cup Y)| = |X \cup Y|$ and $|e_1 \cap ((X \setminus e_1) \cup Y)| = |e_1 \cap Y| < |X \cap Y|$, by the minimality of X and Y, we have

$$v(e_1) + v((X \setminus e_1) \cup Y) \leq v(X \cup Y) + v(e_1 \cap Y). \tag{3.12}$$

Therefore,

$$v(X) + v(Y) \leq v(X \cup Y) + v(e_1 \cap Y) + v((X \setminus e_1) \cap Y) \tag{3.13}$$

$$\leq v(X \cup Y) + v(X \cap Y), \tag{3.14}$$

where the last line is from superadditivity of v (1.2). This shows the supermodularity of v. □

Corollary 3.4. *If $G = (V, E)$ is an antichain, the hypergraph matching game is convex if and only if G is a matching.*

Proof. We first mention that any hyperedge e of an antichain G is synergetic because $E[e] = \{e\}$.

If the game is convex, and if G has a pair of adjacent hyperedges $e, e' \in E$, by Lemma 3.2 with the above observation, G must contain hyperedge $e \cup e'$, which includes e and e'. However, this contradicts the assumption that G is an antichain.

Conversely, if G is a matching, we obtain a closed formula of the characteristic function:

$$v(X) = \sum_{e \in E[X]} c(e). \tag{3.15}$$

We verify the supermodularity using this formula as follows:

$$v(X) + v(Y) = \sum_{e \in E[X]} c(e) + \sum_{e \in E[Y]} c(e) \tag{3.16}$$

$$= \sum_{e \in E[X] \cup E[Y]} c(e) + \sum_{e \in E[X] \cap E[Y]} c(e) \tag{3.17}$$

$$\leq \sum_{e \in E[X \cup Y]} c(e) + \sum_{e \in E[X \cap Y]} c(e) \tag{3.18}$$

$$= v(X \cup Y) + v(X \cap Y), \tag{3.19}$$

where $E[X] \cup E[Y] \subseteq E[X \cup Y]$ and $E[X] \cap E[Y] = E[X \cap Y]$ were used in the third line. □

4 ALGORITHM TO CHECK CONVEXITY

We consider the problem of checking the convexity of a given hypergraph matching game (Problem 1). The result in this section is the following.

Theorem 4.1. *There exists a polynomial-time algorithm that determines whether a given hypergraph matching game is convex or not.*

Our algorithm, as shown in Algorithm 1, enumerates all the synergetic hyperedges (stored in E^*) and verifies the condition in Theorem 3.3 simultaneously in ascending order of the cardinality of the hyperedges. Here, $I(e)$ stores the pairs of intersecting synergetic hyperedges e', e'' such that $e' \cup e'' = e$. The correctness of the algorithm is clear from the construction. The only remaining issue is how to implement the lines 4 and 6.

Algorithm 1 Checking the convexity of the game

Require: Hypergraph $G = (V, E)$
Ensure: Decide whether the game (V, v) is convex
1: $E^* \leftarrow \emptyset$
2: $I(e) \leftarrow \emptyset$ for all $e \in E$
3: **for** $e \in E$ in ascending order of the cardinality **do**
4:   **if** e is synergetic **then**
5:     **for** $(e', e'') \in I(e)$ **do**
6:       **if** the condition of Theorem 3.3 does not hold **then**
7:         **return** `false`
8:       **end if**
9:     **end for**
10:     **for** $e' \in E^*$ **do**
11:       **if** e and e' are intersecting **then**
12:         **if** $e \cup e' \in E$ **then**
13:           $I(e \cup e') \leftarrow I(e \cup e') \cup \{(e, e')\}$
14:         **else**
15:           **return** `false`
16:         **end if**
17:       **end if**
18:     **end for**
19:     $E^* \leftarrow E^* \cup \{e\}$
20:   **else if** $I(e) \neq \emptyset$ **then**
21:     **return** `false`
22:   **end if**
23: **end for**
24: **return** `true`

We say that a matching M in $G[e]$ is *proper* if $M \neq \{e\}$. Line 4 requires checking whether a hyperedge e is synergetic. This is performed by computing a maximum proper matching in $G[e]$. Line 6 is performed by computing a maximum matching in $G[e \cap e']$. Computing a maximum (proper) hypergraph matching is NP-hard in general [12]; however, in both lines, because the algorithm examines the hyperedges in their ascending order of the cardinalities, we can suppose that all the synergetic hyperedges in the candidate subset are enumerated, and v is supermodular in its proper subsets. We prove that, under these conditions, a maximum matching is computable in polynomial time. We first prove the following structural property.

Lemma 4.2. *Let $Z \subseteq V$. Suppose that the supermodular inequality* (1.3) *holds for all $X, Y \subseteq Z$ with $X \cup Y \subsetneq Z$. Let M be a maximum proper matching in Z. Then, any synergetic hyperedge in Z is a subset of a member of M or adjacent to all the members of M.*

Proof. Let $M = \{e_1, \ldots, e_{|M|}\}$ be a maximum proper matching in Z. Then, all the hyperedges in M are synergetic. Let $e \in E[Z]$ be a synergetic hyperedge. If $e \subseteq e_i$ for some $e_i \in M$ then there is nothing to prove. Otherwise, let $M' = \{e_i \in M : e_i \cap e \neq \emptyset\}$. If $M' = M$ then the lemma is proved. Otherwise, $F = e \cup \bigcup_{e_i \in M'} e_i$ is a proper subset of Z. Thus, by Lemma 3.2, it forms a synergetic hyperedge. By the construction, F is disjoint with all $e_j \in M \setminus M'$; therefore, $(M \setminus M') \cup \{F\}$ forms a matching in Z having a higher objective value than M. This is a contradiction. □

Then, we construct a polynomial-time algorithm to compute a maximum matching as follows.

Lemma 4.3. *Under the same assumption as for Lemma 4.2, there exists a polynomial-time algorithm that computes a proper maximum matching in Z.*

Proof. We guess a synergetic hyperedge $e_1 \in M$ in a maximum proper matching. Then, any synergetic hyperedge that is not a subset of a member of M is adjacent to e_1. Therefore, by removing them, the set of the remaining maximal hyperedges forms a matching. □

By using the algorithm in Lemma 4.3 at lines 4 and 6 in Algorithm 1, we obtain a polynomial-time algorithm to solve the problem. This proves Theorem 4.1.

Remark 4.4. Conitzer and Sandholm [8] proved Lemma 4.3 by proposing a combinatorial algorithm based on Lemma 3.2. Our Algorithm 1 is similar to their algorithm because ours is based on Theorem 3.3, and its "only-if" part relies on the same lemma. Note that ours is more involved because it also contains a procedure for the "if" part.

Lemma 4.3 can also be proved by combining classical results as follows. We first guess a hyperedge that is in a maximum matching, and we consider the remaining problem. By the assumption, the remaining problem forms a convex game. Here, we recall that the core of a convex game is non-empty [28], and the core of a hypergraph matching game is non-empty if and only if the LP-relaxation of the corresponding hypergraph matching problem has an integral optimal solution [10, Theorem 1]. Therefore, $\nu(X)$ is obtained by the LP-relaxation, which is solvable in polynomial time [14].

5 SHAPLEY VALUE

We consider the problem of computing the Shapley value for a given convex hypergraph matching game (Problem 2). The result in this section is the following theorem.

Theorem 5.1. *There is a polynomial-time algorithm that computes the Shapley value of a convex hypergraph matching game.*

As a preprocessing, we remove all the non-synergetic hyperedges from the given hypergraph. This is performed in polynomial time by running Algorithm 1. Consequently, for the rest of this section, we can assume that all the hyperedges are synergetic.

We first give a presentation of the marginal gain $\nu(X \cup \{u\}) - \nu(X)$ for a vertex u and a subset $X \subseteq V \setminus \{u\}$. The next lemma is easily verified.

Lemma 5.2. *If there is no hyperedge e such that $u \in e \subseteq X \cup \{u\}$ then $\nu(X \cup \{u\}) - \nu(X) = 0$.*

Proof. The assumption implies $E[X \cup \{u\}] = E[X]$. Hence, $\nu(X \cup \{u\}) = \nu(X)$. □

Now we concentrate on the counter case, i.e., there exists $e \in E$ such that $u \in e \subseteq X \cup \{u\}$. Let $e_{X,u}$ be a maximal hyperedge among them. Then, the next lemma holds.

Lemma 5.3. *$e_{X,u}$ is uniquely determined.*

Proof. If there exist two such hyperedges e and e', by Lemma 3.2, $e \cup e'$ also satisfies the condition. This contradicts the maximality of e and e'. □

An explicit representation of the marginal gain is given by using $e_{X,u}$ as follows.

Lemma 5.4. $\nu(X \cup \{u\}) - \nu(X) = \nu(e_{X,u}) - \nu(e_{X,u} \setminus \{u\})$.

Proof. From the maximality of $e_{X,u}$ with Lemma 3.2, any hyperedge $e' \in E[X]$ are a subset of $e_{X,u}$ or disjoint with $e_{X,u}$. Thus, $\nu(X \cup \{u\}) = \nu(e_{X,u}) + \nu(X \setminus e_{X,u})$ and $\nu(X) = \nu(e_{X,u} \setminus \{u\}) + \nu(X \setminus e_{X,u})$. Therefore, $\nu(X \cup \{u\}) - \nu(X) = \nu(e_{X,u}) - \nu(e_{X,u} \setminus \{u\})$. □

For a vertex u, hyperedge e, and integer k, we define $E(u, e, k) = \{X \subseteq V \setminus \{u\} : |X| = k, e_{X,u} = e\}$ and $L(u, e, k) = |E(u, e, k)|$. Lemma 5.4 implies that all the subsets in $E(u, e, k)$ have the same contribution to the Shapley value. Therefore, by grouping the subsets by $E(u, e, k)$, we obtain the following representation of the Shapley value.

$$\phi(u) = \sum_{k=0}^{|V|-1} \frac{k!(|V| - k - 1)!}{|V|!} \times \left(\sum_{\substack{u \in V, e \in E \\ u \in e}} L(u, e, k)(\nu(e) - \nu(e \setminus \{u\})) \right). \quad (5.1)$$

In this representation, the number of summands is polynomial. Moreover, $\nu(e)$ is computable in polynomial time (Lemma 4.3). Hence, if we can calculate $L(u, e, k)$ in polynomial time, the Shapley value is obtained in polynomial time. The next lemma shows a combinatorial identity of $L(u, e, k)$.

Lemma 5.5. *For each $u \in V$ and $e \in E$,*

$$\sum_{e \subseteq e'} L(u, e', k) = \binom{|V| - |e|}{k - |e|}. \quad (5.2)$$

Proof. We prove this by the double-counting argument. The left-hand side is the number of subsets X such that $u \notin X$, $e \subseteq X \cup \{u\}$, and $|X| = k$. This quantity is equal to the number of ways to choose $k - |e|$ vertices from $V \setminus e$, which is the right-hand side. □

Now we apply the *Möbius inversion formula* [30] to Equation (5.2) to obtain $L(u, e, k)$ as follows: By rearranging the summation, the identity becomes

$$L(u, e, k) = \binom{|V| - |e|}{k - |e|} - \sum_{e \subsetneq e'} L(u, e', k). \quad (5.3)$$

Here, the right-hand side depends on $L(u, e', k)$ with larger e'. Hence, by examining the hyperedges in their decreasing order of cardinalities, we can compute $L(u, e, k)$ for all $e \in E$ in polynomial-time. This proves Theorem 5.1.

6 MINIMUM COMPENSATION

We consider the minimum compensation problem (Problem 3). We say that a vertex u is *compensated* by h if $h(u) > 0$. Also, we say that an edge e is *disabled* by h if $c(e) - \sum_{u \in e} h(u) \leq 0$.

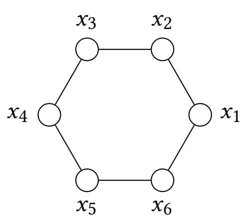

Figure 3: A variable gadget.

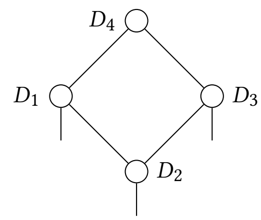

Figure 4: A clause gadget.

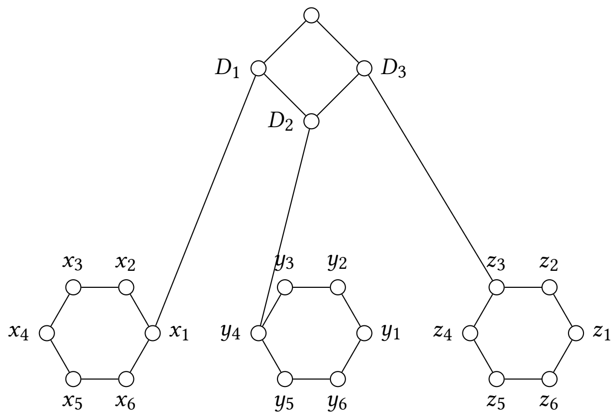

Figure 5: An instance of $D = x \vee y \vee \bar{z}$.

6.1 Hardness

We first prove that the problem is NP-hard, even if the input is a bipartite graph. In this case, by Corollary 3.4, the problem is equivalent to finding h that minimizes $\sum_{u \in V} h(u)$ under the condition that the set of non-disabled edges forms a matching.

Theorem 6.1. *The minimum compensation problem is NP-hard, even if the input is a bipartite graph whose edge weights are* 1.

Before proceeding to the proof of this theorem, we show the integrality of the problem on bipartite graphs.

Lemma 6.2. *If G is a bipartite graph, there exists an integral optimal solution h. Moreover, if all the edge weights are one, we can choose $h(u) \in \{0, 1\}$.*

Proof. Let M be the set of edges that are not disabled by h, which forms a matching by Corollary 3.4. Then, h is a fractional minimum vertex cover of $G \setminus M$ because of the definition of "disable." Because $G \setminus M$ is bipartite, it has an integral optimal solution [26]. The last claim follows because there is no advantage to put $h(u) \geq 2$. □

Now we prove Theorem 6.1 by a reduction from 3-SAT, which is known to be NP-complete [12]. Let $\mathcal{I}$ be an instance of the 3-SAT problem. For each variable x in $\mathcal{I}$, we put a *variable gadget*, which is a cycle of length six whose vertices are named $x_1, \ldots, x_6$ in consecutive order. For each clause D, we put a *clause gadget*, which is a cycle of length four whose vertices are named $D_1, \ldots, D_4$ in consecutive order. Let z_1, z_2, and z_3 be the literals in D. If z_1 is a positive literal x, we connect D_1 to x_1; otherwise, we connect D_1 to x_3. Similarly, if z_3 is a positive literal x, we connect D_3 to x_1; otherwise, we connect D_3 to x_3. If z_2 is a positive literal x, we connect D_2 to x_4; otherwise, we connect D_2 to x_6. Note that the above "non-symmetric" connection of z_2 is for bipartiteness. See Figure 5 for an example.

By the construction, the obtained instance $\mathcal{I}'$ is a bipartite graph. It satisfies the following property.

Lemma 6.3. *There is a feasible solution to $\mathcal{I}$ if and only if the optimal value of $\mathcal{I}'$ is $2n + 2m$.*

Proof. Any instance $\mathcal{I}'$ has the objective value of at least $2n+2m$ because each variable gadget must have at least two compensated vertices, and each clause gadget also must have at least two compensated vertices.

We first prove the "if" part. Let h be a solution to $\mathcal{I}'$ whose objective value is exactly $2n + 2m$. By Lemma 6.2, we assume that $h(u) \in \{0, 1\}$ for all $u \in V$. In this case, by the above discussion, each variable gadget has exactly two compensated vertices. These must be located at the opposite vertices. If x_1 and x_4 are compensated, we set $x = \texttt{true}$; otherwise $x = \texttt{false}$. The assignment of these values gives a feasible solution to $\mathcal{I}$. For each clause D, by the above discussion, the clause gadget has exactly two compensated vertices, i.e., there exists at least one connecting edge that is not disabled by the vertices in the clause gadget. Therefore, the corresponding variable gadget must have a compensated vertex at the terminal. This implies that the literal appears `true` in the clause, i.e., D is satisfied.

The "only-if" part is proved similarly by following the above construction. Thus, we obtain the lemma. □

This proves Theorem 6.1.

6.2 Approximation Algorithm

Because the problem is NP-hard (Theorem 6.1), we consider an approximation algorithm. Here, we show that if the input is an antichain, there exists a polynomial-time 2-approximation algorithm.

Theorem 6.4. *There is a polynomial-time 2-approximation algorithm for the minimum compensation problem if the input is an antichain.*

Proof. When the input is an antichain, by Corollary 3.4, the problem is represented as the following integer linear programming problem (ILP):

$$\begin{aligned}
\text{minimize} \quad & \sum_{u \in V} h(u) & & \\
\text{subject to} \quad & \sum_{e \ni v} y(e) \leq 1, & & (v \in V), \\
& \sum_{v \in e} h(u) \geq (1 - y(e))c(e), & & (e \in E), \\
& y(e) \in \{0, 1\}, & & (e \in E), \\
& h(u) \geq 0, & & (u \in V).
\end{aligned} \tag{6.1}$$

Here, variable $y(e)$ represents whether e is disabled or not. We relax the integral constraint "$y(e) \in \{0, 1\}$" to the nonnegative constraint "$y(e) \geq 0$" to obtain a LP-relaxation. Our algorithm computes an optimal solution h to the LP-relaxation, and outputs $2h$ as a solution to the problem.

We show that $2h$ is a feasible solution to the problem. For any pair of adjacent hyperedges $e, e' \in E$, by the first constraint of the LP, we have $y(e) + y(e') \leq 1$. By symmetry, we can assume $y(e) \leq y(e')$. Then, $y(e) \leq 1/2$. Therefore, by the second constraint of the above LP, we have $\sum_{u \in e} 2h(u) \geq c(e)$, i.e., $2h$ disables e. This means that the set of non-disabled edges by $2h$ forms a matching. Hence, $2h$ is a feasible solution to the problem.

The approximation factor is clearly 2. The running time is polynomial in the size of the input because the above LP has polynomial size. Therefore, we obtain the theorem. □

We have less hope to break the 2-approximation if we use the LP-relaxation of (6.1).

Corollary 6.5. *The integrality gap of the LP relaxation of* (6.1) *is* 2.

Proof. Theorem 6.4 shows that the integrality gap is at most 2. We prove that there exists an instance whose integrality gap is exactly 2. Let us consider an unweighted cycle of length four as an input. Then, the optimal value of the LP relaxation of (6.1) is 1, where the corresponding optimal solution is $y(e) = 1/2$ for all $e \in E$ and $h(u) = 1/4$ for all $u \in V$. On the other hand, by Theorem 6.2, the optimal value of the ILP (6.1) is 2 because at least two vertices must be compensated. □

7 FRACTIONAL HYPERGRAPH MATCHING GAME

We consider the fractional hypergraph matching game. We prove the following theorem, which shows that the convexity of the fractional hypergraph matching game implies the "integrality" of the game.

Theorem 7.1. *If the characteristic function ν' of the fractional hypergraph matching game, defined by* (1.6), *is supermodular, we have $\nu(X) = \nu'(X)$ for all $X \subseteq V$, where ν is the characteristic function of the hypergraph matching game.*

We say that a vector $x \in \mathbb{R}^E$ *attains* $\nu'(X)$ if it is the optimal solution to the fractional hypergraph matching problem (1.6). We denote by $c(x) = \sum_{e \in E} c(e)x(e)$.

We prove Theorem 7.1 by the induction on $|X|$. The statement clearly holds on $|X| = 1$. Let $s = |X| \geq 2$ and suppose that the statement holds for all subset $Y \subseteq V$ of size $|Y| \leq s - 1$. Let $x \in \mathbb{R}^E$ be the vector that maximizes the auxiliary function $f(x) = \sum_{e \in E} 3^{|e|} x(e)$ among all vectors that attain $\nu'(X)$. Let M be the set of hyperedges e with $x(e) > 0$. Note that each $e \in M$ is synergetic (in the sense of the integral hypergraph matching game) because, otherwise, we can distribute $x(e)$ to the hyperedges contained in e to obtain a fractional matching of higher value. The next lemma exploits the structure of x.

Lemma 7.2. *If $\nu(Y) = \nu'(Y)$ for all $Y \subsetneq X$, then for all intersecting two hyperedges $e_1, e_2 \in M$, we have $e_1 \cup e_2 = X$.*

Proof. Let $e_1, e_2 \in M$ be two intersecting hyperedges such that $e_1 \cup e_2 \subsetneq X$. From the induction hypothesis on Theorem 7.1, $\nu(Y) = \nu'(Y)$ holds for all $Y \subseteq e_1 \cup e_2$. As mentioned in the above, e_1 and e_2 are synergetic. Therefore, by Lemma 3.2, $e_1 \cup e_2$ is synergetic. Let $\epsilon = \min(x(e_1), x(e_2))$ and let y be a vector that attains $\nu'(e_1 \cap e_2)$. Let x' be a vector defined by

$$x'(e_1) = x(e_1) - \epsilon, \tag{7.1}$$

$$x'(e_2) = x(e_2) - \epsilon, \tag{7.2}$$

$$x'(e_1 \cup e_2) = x(e_1 \cup e_2) + \epsilon, \tag{7.3}$$

$$x'(e) = x(e) + \epsilon y(e) \quad (e \in E \setminus \{e_1, e_2, e_1 \cup e_2\}). \tag{7.4}$$

Then x' is also a fractional hypergraph matching of X. We have

$$\begin{aligned} c(x') &= c(x) + \epsilon\,(c(e_1 \cup e_2) + c(y) - c(e_1) - c(e_2)) && (7.5)\\ &= c(x) + \epsilon\left(\nu'(e_1 \cup e_2) + \nu'(e_1 \cap e_2) - \nu'(e_1) - \nu'(e_2)\right) && (7.6)\\ &\geq c(x). && (7.7) \end{aligned}$$

Here, the first equality uses $y(e_1) = y(e_2) = y(e_1 \cup e_2) = 0$ because y is zero outside of $e_1 \cap e_2$. The second equality uses $c(e_1) = \nu(e_1)$, $c(e_2) = \nu(e_2)$, and $c(e_1 \cup e_2) = \nu(e_1 \cup e_2)$ because these are synergetic, and $\nu(e_1) = \nu'(e_1)$, $\nu(e_2) = \nu'(e_2)$, and $\nu(e_1 \cup e_2) = \nu'(e_1 \cup e_2)$ because of the induction hypothesis. The last inequality is the supermodularity of ν'. Moreover, we have

$$\begin{aligned} f(x') - f(x) &= \epsilon(3^{|e_1 \cup e_2|} - 3^{|e_1|} - 3^{|e_2|} + \sum_{e \in E} 3^{|e|} y_e) && (7.8)\\ &\geq \epsilon(3^{|e_1 \cup e_2|} - 3^{|e_1|} - 3^{|e_2|}) > 0. && (7.9) \end{aligned}$$

Here, the last inequality is from $|e_1|, |e_2| \leq |e_1 \cup e_2| - 1$ because e_1 and e_2 are intersecting. This contradicts to the definition of x. □

We further investigate the structure of x. Let $M' \subseteq M$ be the set of hyperedges that does not intersect with other hyperedges in M. By definition, any hyperedges in M' has no intersection, i.e., M' forms a laminar. Let

$$l(v) = \sum_{\substack{e \in M \setminus M' \\ e \ni v}} x(e). \tag{7.10}$$

Lemma 7.3. *Suppose that $\nu(Y) = \nu'(Y)$ holds for all $Y \subsetneq X$. Then, $l(v) = 1$ holds for some vertex $v \in X$, or $\nu(X) = \nu'(X)$ holds.*

Proof. Let $\epsilon = \min(1 - \max_{v \in X} l(v), \min_{e \in M'} x(e))$. Suppose that $l(v) \neq 1$ for all $v \in X$. Then $\epsilon > 0$. We define a vector x' by

$$x'(e) = \begin{cases} \epsilon, & e \text{ is a maximal hyperedge in } M', \\ 0, & \text{otherwise.} \end{cases} \tag{7.11}$$

From the definition of ϵ and the fact that M' forms a laminar, we see that both $x_1 = \frac{1}{\epsilon} x'$ and $x_2 = \frac{1}{1-\epsilon}(x - x')$ are fractional hypergraph matchings of X. Hence, for any $\lambda \in [0, 1]$, their convex combination $\lambda x_1 + (1 - \lambda)x_2$ is also feasible. Let $c'(\lambda) := c(\lambda x_1 + (1 - \lambda)x_2)$. Then, this function attains the maximum at $\lambda = \epsilon$ because $x = \epsilon x_1 + (1 - \epsilon)x_2$ and x attains $\nu'(X)$. Therefore, because of the linearlity of c', we have $c(x_1) = c(x)$, i.e., x_1 also attains $\nu'(X)$. Because x_1 is an integral vector, we have $\nu(X) = \nu'(X)$. □

Next, we prove a stronger version of the above lemma.

Lemma 7.4. *Suppose that $\nu(Y) = \nu'(Y)$ holds for all $Y \subsetneq X$. Then, $l(v) = 1$ holds for all $v \in X$, or $\nu(X) = \nu'(X)$ holds.*

Proof. From Lemma 7.3, if $\nu(X) \neq \nu'(X)$, there is $u \in X$ with $l(u) = 1$. If there is $v \in X$ with $l(v) < 1$, then there is a hyperedge

$e \in M \setminus M'$ such that $v \notin e$. From the definition of e, for all $w \in X \setminus e$, $\sum_{w \in e' \in (M \setminus M')} x(e') \leq 1 - x(e)$.

Let $\epsilon = \min_{e' \in M} x(e')$. We define a vector x' by

$$x'(e') = \begin{cases} \epsilon, & e' = e, \text{ or } e' \text{ is a maximal hyperedge in } M' \\ & \text{that is disjoint with } e, \\ 0, & \text{otherwise.} \end{cases} \tag{7.12}$$

Then, the rest of the proof is similar to Lemma 7.3. Both $x_1 = \frac{1}{\epsilon} x'$ and $x_2 = \frac{1}{1-\epsilon}(x - x')$ are feasible solutions to the fractional hypergraph matching problem for X. Hence, for any $\lambda \in [0, 1]$, their convex combination $\lambda x_1 + (1 - \lambda) x_2$ is also feasible. The function $c'(\lambda) := c(\lambda x_1 + (1 - \lambda) x_2)$ is linear and attains maximum at $\lambda = \epsilon$. Therefore, it also attains maximum at $\lambda = 0$, i.e., the integral vector x_1 attains $v'(X)$. This means that $v(X) = v'(X)$. □

Lemma 7.4 shows that if $v(X) \neq v'(X)$ then $M' = \emptyset$. Now we have the following structure.

Lemma 7.5. *If $v(Y) = v'(Y)$ for all $Y \subsetneq X$ and $v(X) \neq v'(X)$, then there is an integer $k \geq 2$ and a partition of X into k disjoint subsets $X_1, \ldots, X_k$ such that $M = \{X \setminus X_i : i = 1, \ldots, k\}$. Furthermore, $x(e) = \frac{1}{k-1}$ for all $e \in M$.*

Proof. From Lemma 7.2, $\bar{M} = \{X \setminus e : e \in M\}$ forms a laminar. If there are two hyperedges $e, e' \in M$ such that $e \subsetneq e'$, then for any $v \in e' \setminus e$, $l(v) < 1$ holds, which contradicts to Lemma 7.4. If there is a vertex v that is contained in all the hyperedges in M, then we have $M = \{X\}$. Otherwise, i.e., if there is $e \in M$ with $e \neq X$, then for any $u \in X \setminus e$, $l(u) \leq \sum_{e' \in M \setminus e} x(e') < \sum_{e' \in M} x(e') = l(v)$, which contradicts to Lemma 7.4. This implies that x is integral, i.e., $v(X) = v'(X)$, which is a contradiction. Thus, $\bar{M}$ is a partition of X.

For each $i = 1, \ldots, k$, by evaluating $l(v)$ at $v \in X_i$, we have

$$1 = l(v) = \sum_{t \in \{1, \ldots, k\} \setminus \{i\}} x(X_t). \tag{7.13}$$

This linear equation has a unique solution $x(e) = \frac{1}{k-1}$ for all $e \in M$. Thus the lemma is proved. □

Finally, we show the induction step of the proof of Theorem 7.1.

Lemma 7.6. *If $v(Y) = v'(Y)$ for all $Y \subsetneq X$, then, $v(X) = v'(X)$.*

Proof. Suppose the contrary. From Lemma 7.5, there is a partition of X into $k \geq 2$ disjoint subsets $X_1, \ldots, X_k$, such that $M = \{X \setminus X_i : i = 1, \ldots, k\}$. We treat the index of X_i in $\bmod k$, i.e., $X_{k+t} = X_t$ for all t. Let $X_{i,j} = X \setminus \{X_i, \ldots, X_{i+j-1}\}$.

We prove the following claim by the induction: For all $j = 1, \ldots, k-1$, the following equality holds:

$$\sum_{i=1}^{k} v'(X_{i,j}) = (k - j) v'(X). \tag{7.14}$$

Lemma 7.5 shows the case of $j = 1$. In the general case, we have

$$v'(X_{1,t}) + v'(X_{2,t}) \leq v'(X_{2,t-1}) + v'(X_{1,t+1}), \tag{7.15}$$

$$v'(X_{2,t}) + v'(X_{3,t}) \leq v'(X_{3,t-1}) + v'(X_{2,t+1}), \tag{7.16}$$

$$\vdots$$

$$v'(X_{k,t}) + v'(X_{1,t}) \leq v'(X_{1,t-1}) + v'(X_{k,t+1}). \tag{7.17}$$

from the supermodularity of v'. Adding these inequalities yields

$$2 \sum_{i=1}^{k} v'(X_{i,t}) \leq \sum_{i=1}^{k} v'(X_{i,t-1}) + \sum_{i=1}^{k} v'(X_{i,t+1}). \tag{7.18}$$

By the induction hypothesis of the claim, this can be written as

$$(k - t - 1) v'(X) \leq \sum_{i=1}^{k} v'(X_{i,t+1}). \tag{7.19}$$

Let $x_{i,t+1}$ be the vector that attains $v'(X_{i,t+1})$. Then, the vector $x = \frac{1}{k-t-1} \sum_{i=1}^{k} x_{i,t+1}$ is a feasible fractional hypergraph matching of X. Thus,

$$(k - t - 1) v'(X) \geq \sum_{i=1}^{k} v'(X_{i,t+1}). \tag{7.20}$$

Therefore, the equality holds and the claim is proved.

Now, we have

$$\sum_{i=1}^{k} v'(X_i) = \sum_{i=1}^{k} v'(X_{i,k-1}) = v'(X). \tag{7.21}$$

By the assumption of the lemma, $v'(X_i)$ is attained by an integral vector. Summing up these vectors for all i yields an integral vector that attains $v'(X)$. Thus the lemma is proved. □

Therefore, Theorem 7.1 is proved. As mentioned in Section 1.1, the convexity of the integral hypergraph matching game implies the convexity of the corresponding fractional hypergraph matching game. The above theorem shows the converse direction. Therefore, we obtain the following corollary.

Corollary 7.7. *A fractional hypergraph matching game is convex if and only if the corresponding (integral) hypergraph matching is convex.*

8 CONCLUSION

We studied the hypergraph matching game. Because the game is intractable due to the NP-hardness of the hypergraph matching problem, we focused on a tractable subclass of the game, a "convex" hypergraph matching game.

We solved three problems related to the convexity of the hypergraph matching game: (1) Determining the convexity of a hypergraph matching game, (2) Computing the Shapley value of a convex hypergraph matching game, and (3) Computing the minimum compensation that makes a hypergraph matching game convex. We showed that (1) and (2) are solvable in polynomial time (Theorem 4.1 and Theorem 5.1, respectively), and (3) is NP-hard, even if the input is a bipartite graph (Theorem 6.1), and it is 2-approximable if the input is an antichain (Theorem 6.4). We also considered the fractional hypergraph matching game and proved that if the fractional game is convex, then its characteristic function coincides with the characteristic function of the integral one (Theorem 7.1).

We conclude this paper by giving a few open problems. Firstly, the hypergraph matching game is a subclass of the integral version of the linear production game [24]. Thus, the most promising future work will be extending these results to the integral linear production game. Secondly, in the minimum compensation problem, we derived an algorithm for antichains. Finding an algorithm for the general case will be an interesting and useful problem.

REFERENCES

[1] Sara Ahmadian, Hamideh Hosseinzadeh, and Laura Sanità. 2018. Stabilizing network bargaining games by blocking players. *Mathematical Programming* 172, 1-2 (2018), 249–275.
[2] Haris Aziz and Bart De Keijzer. 2011. Complexity of coalition structure generation. In *The 10th International Conference on Autonomous Agents and Multiagent Systems (AAMAS'11)*. 191–198.
[3] Haris Aziz and Bart de Keijzer. 2014. Shapley meets Shapley. In *Proceedings of the 31st International Symposium on Theoretical Aspects of Computer Science (STACS'14)*. 99–111.
[4] Claude Berge. 1984. *Hypergraphs: combinatorics of finite sets*. Vol. 45. Elsevier.
[5] Péter Biró, Walter Kern, Dömötör Pálvölgyi, and Daniel Paulusma. 2019. Generalized Matching Games for International Kidney Exchange. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 413–421.
[6] Péter Biró, Walter Kern, and Daniël Paulusma. 2012. Computing solutions for matching games. *International journal of game theory* 41, 1 (2012), 75–90.
[7] Georgios Chalkiadakis, Edith Elkind, and Michael Wooldridge. 2011. Computational aspects of cooperative game theory. *Synthesis Lectures on Artificial Intelligence and Machine Learning* 5, 6 (2011), 1–168.
[8] Vincent Conitzer and Tuomas Sandholm. 2006. Complexity of constructing solutions in the core based on synergies among coalitions. *Artificial Intelligence* 170, 6-7 (2006), 607–619.
[9] Morton Davis and Michael Maschler. 1965. The kernel of a cooperative game. *Naval Research Logistics Quarterly* 12, 3 (1965), 223–259.
[10] Xiaotie Deng, Toshihide Ibaraki, and Hiroshi Nagamochi. 1999. Algorithmic aspects of the core of combinatorial optimization games. *Mathematics of Operations Research* 24, 3 (1999), 751–766.
[11] Satoru Fujishige. 2005. *Submodular functions and optimization*. Vol. 58. Elsevier.
[12] Michael R Garey and David S Johnson. 2002. *Computers and intractability*. Vol. 29. wh freeman New York.
[13] Donald B Gillies. 1959. Solutions to general non-zero-sum games. *Contributions to the Theory of Games* 4 (1959), 47–85.
[14] Martin Grötschel, László Lovász, and Alexander Schrijver. 1981. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* 1, 2 (1981), 169–197.
[15] Takehiro Ito, Naonori Kakimura, Naoyuki Kamiyama, Yusuke Kobayashi, and Yoshio Okamoto. 2017. Efficient stabilization of cooperative matching games. *Theoretical Computer Science* 677 (2017), 69–82.
[16] Walter Kern and Daniël Paulusma. 2003. Matching games: the least core and the nucleolus. *Mathematics of operations research* 28, 2 (2003), 294–308.
[17] Zhuan Khye Koh and Laura Sanità. 2019. An Efficient Characterization of Submodular Spanning Tree Games. In *International Conference on Integer Programming and Combinatorial Optimization*. Springer, 275–287.
[18] Jeroen Kuipers. 1996. *A polynomial time algorithm for computing the nucleolus of convex games*. Technical Report. Maastricht University.
[19] David Liben-Nowell, Alexa Sharp, Tom Wexler, and Kevin Woods. 2012. Computing Shapley value in supermodular coalitional games. In *International Computing and Combinatorics Conference (CCC'12)*. 568–579.
[20] Michael Maschler, Bezalel Peleg, and Lloyd S Shapley. 1971. The kernel and bargaining set for convex games. *International Journal of Game Theory* 1, 1 (1971), 73–93.
[21] Hiroshi Nagamochi, Dao-Zhi Zeng, Naohiśa Kabutoya, and Toshihide Ibaraki. 1997. Complexity of the minimum base game on matroids. *Mathematics of Operations Research* 22, 1 (1997), 146–164.
[22] Nima Namvar and Fatemeh Afghah. 2015. Spectrum sharing in cooperative cognitive radio networks: A matching game framework. In *2015 49th Annual Conference on Information Sciences and Systems (CISS)*. IEEE, 1–5.
[23] Yoshio Okamoto. 2003. Submodularity of some classes of the combinatorial optimization games. *Mathematical Methods of Operations Research* 58, 1 (2003), 131–139.
[24] Guillermo Owen. 1975. On the core of linear production games. *Mathematical programming* 9, 1 (1975), 358–370.
[25] David Schmeidler. 1969. The nucleolus of a characteristic function game. *SIAM J. Appl. Math.* 17, 6 (1969), 1163–1170.
[26] Alexander Schrijver. 2003. *Combinatorial optimization: polyhedra and efficiency*. Vol. 24. Springer Science & Business Media.
[27] Lloyd S Shapley. 1953. A value for n-person games. *Contributions to the Theory of Games* 2, 28 (1953), 307–317.
[28] Lloyd S Shapley. 1971. Cores of convex games. *International Journal of Game Theory* 1, 1 (1971), 11–26.
[29] Lloyd S Shapley and Martin Shubik. 1971. The assignment game I: The core. *International Journal of game theory* 1, 1 (1971), 111–130.
[30] Richard P. Stanley. 2011. *Enumerative Combinatorics: Volume 1* (2nd ed.). Cambridge University Press.
[31] Anne van den Nouweland and Peter Borm. 1991. On the convexity of communication games. *International Journal of Game Theory* 19, 4 (1991), 421–430.
[32] John Von Neumann and Oskar Morgenstern. 1947. *Theory of games and economic behavior*. Princeton University Press.