Multiwinner Candidacy Games

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ABSTRACT

In strategic candidacy games, both voters and candidates have preferences over the set of possible election outcomes, and candidates may strategically withdraw from the election in order to manipulate the result in their favor. In this work, we extend the candidacy game model to the setting of multiwinner elections, where the goal is to select a fixed-size subset of candidates (a committee), rather than a single winner. We examine the existence and properties of Nash equilibria in the resulting class of games, under various voting rules and voter preference structures.

KEYWORDS

Social choice theory; Non-cooperative games: theory & analysis

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1 INTRODUCTION

The problem of selecting a subset of alternatives based on the preferences of a number of agents (voters) is ubiquitous: democratic societies choose representatives to govern on their behalf, companies select products and services to promote to their customers [23, 24, 32], local governments decide which potential projects should get funded [3, 4], and working groups shortlist tasks to perform or applicants to join their team.

In many of these scenarios, candidates are free to join or leave the election, and may themselves have preferences over its possible outcomes. They may therefore make strategic decisions concerning their participation. The resulting interaction can then be viewed as a non-cooperative game among the candidates: the players decide whether to run in the election or abstain, and the outcome is decided by voting. The voters are typically assumed to report their preferences sincerely (see, however, the work of [8], where both candidates and voters may be strategic). In the context of single-winner elections, such games are known as *strategic candidacy games*; they were introduced by Dutta et al. [11, 12], and have subsequently been studied by a number of authors [13, 17, 21, 22, 28, 30, 31]. However, to the best of our knowledge, strategic candidacy games have not been investigated in the context of multiwinner elections.

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Against this background, in this paper we initiate the study of *multiwinner candidacy games* (MCGs), and study the properties of their equilibria under a number of common multiwinner rules.

Contributions We study two types of elections, with different formats of voter ballots: approval-based elections and ordinal elections. In both cases, we assume that all candidates prefer to be selected.

Under approval preferences a voter's approval of a candidate is independent of the set of candidates participating in the election. One may then expect that in approval-based elections no candidate has an incentive to withdraw. We show that this is indeed the case for a large class of rules known as Thiele rules [32, 35], as well as for their sequential variants: the action profile where every candidate participates in the election is an equilibrium. In fact, sequential Thiele rules are truthful in an even stronger sense: not only is full participation an equilibrium, but in every equilibrium the winning committee is the same as under full participation; we say that candidacy games with this property are *genuine*. However, games that are based on simultaneous versions of Thiele rules may fail to be genuine. Moreover, there exist approval-based rules for which full participation may fail to be an equilibrium: e.g., this is the case under the Minimax Approval voting rule [7].

In ordinal elections, pure strategy Nash equilibria may fail to exist even for simple single-winner voting rules [22]. Interestingly, while one expects non-existence results to extend easily from $k = 1$ to $k \geq 2$ (where k is the target committee size), this is not always the case: for the family of k-*CC* voting rules [10] we obtain an existence result that holds for $k \geq 2$, but not for $k = 1$. However, while this positive result is encouraging, it only applies to a limited class of rules. To obtain positive results for other rules, we follow the approach of [22, 27] for the single-winner setting, and focus on restricted preference domains. For profiles that admit k sequential Condorcet winners (where k is the target committee size), we extend the result of Lang et al. [22] for single-winner Condorcet-consistent rules to a multiwinner version of Maximin. However, the analogous result for Plurality [27] does not extend to the multiwinner setting: we provide a counterexample for the Bloc rule. We then consider elections with single-peaked voter preferences. We establish several structural results on the collective ranking of candidates induced by a single-peaked profile, and then use them to prove the existence of equilibria for the family of top-k-counting rules [18]. However, we show that the respective games are not genuine.

2 MODEL

An election is given by a set $C = \{c_1, c_2, \ldots, c_m\}$ of *potential candidates* and a set $V = \{v_1, v_2, \ldots, v_n\}$ of *voters*; we assume $C \cap V = \emptyset$.

Let $\mathcal{L}(C)$ be the set of all linear orders over C, and let $\mathcal{P}(C)$ be the power set of C. We consider two types of elections: in *ordinal* elections, the *preference* of voter $v \in V$ is a ranking $P_v \in \mathcal{L}(C)$, ordering the candidates from most to least preferred, and in *approval-based* elections each voter $v \in V$ indicates a subset $P_v \in \mathcal{P}(C)$ of candidates that she approves. In the latter case, abusing notation, we write aP_vb whenever $a \in P_v$ and $b \notin P_v$. The list $P^V = (P_v)_{v\in V}$ is called the *voter preference profile.*

We consider voting procedures that operate as follows. There is a desired committee size $k \le m$, and an order $\rhd \in \mathcal{L}(C)$. First, a subset of candidates $A \subseteq C$ announce that they will participate in the election; we refer to the candidates in A as the *actual candidates*, and denote their number by $m_A = |A|$. We focus on the case where $m_A \ge k$.[1] Each voter $v \in V$ reports her preferences over the candidates in A; these preferences are obtained by restricting P_v to A. We assume that all voters are sincere. A *multiwinner voting rule* takes the set A and the list of voter preferences over A as input, and outputs a size-k committee $W \subseteq A$ that is declared to be the *election winner.*

2.1 Multiwinner Voting Rules

We will now define several procedures that, given an election with a set V of n voters, a set A of m_A actual candidates, and a target committee size $k \le m_A$, output a non-empty collection of k-element subsets of A (committees) that are tied for winning. Any such procedure $\mathcal{R}$ can be turned into a multiwinner voting rule by means of a tie-breaking scheme, $\rhd_{\mathcal{R}}$, which is a partial order on size-k subsets of A. The only assumption we make with regard to $\rhd_{\mathcal{R}}$ is that for any two committees that differ from each other by exactly one element, the selection is made according to a predetermined order $\rhd$ over the candidate set: for $S \subseteq A$ with $|S| = k-1$, and $a, b \in A$, we have $S \cup \{a\} \rhd_{\mathcal{R}} S \cup \{b\}$ iff $a \rhd b$.

Many such rules assign each committee S a score $\mathrm{sc}^{\mathcal{R}}(S)$, and select those with the highest (or the lowest) score to be tied for winning. Below we list several rules in this category.

Approval-Based Rules We first consider approval-based elections. In what follows, we write $\mathbb{I}_q = 1$ if condition q is true and $\mathbb{I}_q = 0$ otherwise.

Proportional Approval Voting (PAV). Voter v's satisfaction increases with the number of her approved candidates included in the committee, with the marginal value for each additional member decreasing harmonically:

$$\mathrm{sc}^{PAV}(S) = \sum_{v\in V} \sum_{j=1}^{|S\cap P_v|} \frac{1}{j}.$$

Chamberlin–Courant (CC). Voter v is satisfied with a committee as long as it contains at least one of her approved candidates:

$$\mathrm{sc}^{CC}(S) = \sum_{v\in V} \mathbb{I}_{|S\cap P_v|\neq\emptyset}.$$

Both PAV and CC fall into the general class of Thiele rules [35]. These rules can be defined in terms of *ordered weighted averaging* (OWA) operators [32], and hence are also referred to as OWA rules. An OWA operator is defined by a (length-k) sequence of real numbers $\mathbf{w} = (w_1, \dots, w_k)$, termed the *weight sequence.*

Thiele rules (T). The score that voter v assigns to a committee S under the Thiele rule with weight sequence $\mathbf{w}$ is given by the sum of the first $|S \cap P_v|$ weights in $\mathbf{w}$:

$$\mathrm{sc}^{\mathbf{w}}(S) = \sum_{v\in V} \sum_{j=1}^{|S\cap P_v|} w_j.$$

Thus, CC is the Thiele rule defined by the weight sequence $\mathbf{w}_{CC} = (1, 0, 0, \dots, 0)$, and PAV is the Thiele rule with the weight sequence $\mathbf{w}_{PAV} = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k})$.

Sequential Thiele rules (seq-T). We also consider the sequential variants of Thiele rules, which output all committees that can be obtained by starting with the empty committee ($S = \emptyset$), and then, in k consecutive steps, adding to S a candidate c so as to maximize the value $\mathrm{sc}^{\mathbf{w}}(S \cup \{c\}) - \mathrm{sc}^{\mathbf{w}}(S)$.

Minimax Approval Voting (MAV). Finally, we provide an example of an approval-based rule that is not a (sequential) Thiele rule. The Minimax Approval Voting (MAV) rule is based on egalitarian principles. It defines the score of a committee S as $\mathrm{sc}^{MAV}(S) = \max_{v\in V}(|P_v \setminus S| + |S \setminus P_v|)$, and outputs all committees with the minimum score.

Ordinal Rules We now consider the case where voters' preferences are ordinal. In particular, we redefine PAV and CC for this case, and consider a few variants of the latter.

Let $\mathrm{pos}_v(c)$ denote the position of candidate c in the preference list $P_v \in \mathcal{L}(C)$ of voter v, restricted to the set of actual candidates $A \subseteq C$, and let $\mathrm{top}_k(v) = \{c \in A \mid \mathrm{pos}_v(c) \le k\}$.

***k*-Maximin (*k*-MM).** To compute the Maximin score of candidate c, for every other candidate a we compute the number of voters that rank c over a; the score of c is then the smallest among these values. We extend this rule to the multiwinner setting by considering pairwise contests between members of a given committee S and candidates outside of S:

$$\mathrm{sc}^{k\text{-}MM}(S) = \min_{c\in S} \min_{a\notin S} |\{v \in V \mid cP_va\}|.$$

Bloc. Under this rule, each voter gives one point to each candidate in the top k positions of her preference list, where k is the target committee size:

$$\mathrm{sc}^{Bloc}(S) = \sum_{v\in V} \sum_{c\in S} \mathbb{I}_{\mathrm{pos}_v(c)\le k}.$$

Ordinal PAV (O-PAV). As above, voter v would like the committee to include as many of her top k candidates as possible; however, her marginal value for each additional member decreases:

$$\mathrm{sc}^{O\text{-}PAV}(S) = \sum_{v\in V} \sum_{j=1}^{|S\cap \mathrm{top}_k(v)|} \frac{1}{j}.$$

Ordinal CC (O-CC). Under the O-CC rule, the score that a committee S receives from a voter v depends on her most preferred member of S, called her *representative.* There are several variants of this rule, which can be defined by a *positional scoring function*, $f_{\mathrm{pos}} : \{1, \dots, m_A\} \to \mathbb{R}$,

[1] We assume that if $m_A < k$, the output consists of A together with top $m_A - k$ candidates in $C \setminus A$ with respect to $\rhd$.

where f_{pos} is non-increasing and $f_{\text{pos}}(1) > f_{\text{pos}}(m_A)$:

$$\text{sc}^{O\text{-}CC}(S) = \sum_{v \in V} \max_{c \in S} f_{\text{pos}}\left(\text{pos}_v(c)\right).$$

In particular, the most common are the following two rules:

Borda-CC. The contribution of voter v to the score of S is the Borda score of her representative in S:

$$\text{sc}^{Borda\text{-}CC}(S) = \sum_{v \in V} \max_{c \in S} \left(m_A - \text{pos}_v(c)\right).$$

k-**CC.** Voter v contributes one point to the score of S if her representative is ranked in top k positions of P_v:

$$\text{sc}^{k\text{-}CC}(S) = \sum_{v \in V} \max_{c \in S} \mathbb{I}_{\text{pos}_v(c) \leq k} = \sum_{v \in V} \mathbb{I}_{S \cap \text{top}_k(v) \neq \emptyset}.$$

ϵ-**Bloc-CC.** Finally, we define a hybrid of the k-CC and Bloc rules, by letting voter v contribute one point to the score of S if her representative is ranked in top k positions, but also add another $\epsilon > 0$ points for every additional member of S with this property (we assume $\epsilon < \frac{1}{kn}$):

$$\text{sc}^{\epsilon\text{-}Bloc\text{-}CC}(S) = \sum_{v \in V} (\mathbb{I}_{S \cap \text{top}_k(v) \neq \emptyset} + \epsilon \cdot \max\{|S \cap \text{top}_k(v)| - 1, 0\}).$$

For some of the voting rules listed above, namely for Bloc, O-PAV, k-CC and ϵ-Bloc-CC, the contribution of a voter v to the score of a committee S only depends on the size of the set $\text{top}_k(v) \cap S$. Such rules form the following class.

Top-k-Counting rules (k-TC). These voting rules are defined by a non-decreasing *top-k-counting scoring function*, $f_{\text{top}} : \{1, \ldots, m_A\} \to \mathbb{R}$, as follows:

$$\text{sc}^{k\text{-}TC}(S) = \sum_{v \in V} f_{\text{top}}\left(|\text{top}_k(v) \cap S|\right).$$

We now turn to complete the formal definition of our model for candidacy games in multiwinner elections.

2.2 Multiwinner Candidacy Games

In a candidacy game, candidates themselves also have preferences over the set C, and each candidate can choose whether to run in the election. Formally, each candidate $c \in C$ has two available actions: 1 (run) and 0 (abstain), and is endowed with either an ordinal or an approval-based preference P_c over C. We adopt the assumption of *self-supporting preferences*, which is common in the candidacy games literature: in ordinal elections, where $P_c \in \mathcal{L}(C)$, each candidate c strictly prefers herself over the others, and in approval-based elections where $P_c \in \mathcal{P}(C)$, each candidate c approves herself. The list $P^C = (P_c)_{c \in C}$ is the *candidate preference profile.*

The tuple $\langle C, V, P^V, P^C, k, \mathcal{R}, \rhd \rangle$ defines a strategic game, $\Gamma^{\mathcal{R}}$, termed the *multiwinner candidacy game (MCG).* In this game, the set of players is C and each player's set of actions is $\{0, 1\}$. We denote the action (or, strategy) of a player $c \in C$ by s_c and call the vector $\mathbf{s} = (s_c)_{c \in C}$ the *strategy profile.* A strategy profile $\mathbf{s}$ defines the set of actual candidates $A(\mathbf{s}) = \{c \in C \mid s_c = 1\}$ and hence an outcome $W(\mathbf{s}) \subseteq C$; we will sometimes identify a strategy profile $\mathbf{s}$ with the set $A(\mathbf{s})$. For $|A(\mathbf{s})| \geq k$, the outcome is the winning committee under rule $\mathcal{R}$, with ties broken according to $\rhd$ as defined in Section 2.1, computed based on the votes of voters in V over candidates in $A(\mathbf{s})$; we denote the score obtained by a committee $S \subseteq A(\mathbf{s})$ in this election by $\text{sc}^{\mathcal{R}}(S, \mathbf{s})$. If $|A(\mathbf{s})| < k$, the outcome $W(\mathbf{s})$ is the set $A(\mathbf{s}) \cup X$ where X consists of the top $k - |A(\mathbf{s})|$ candidates in the restriction of $\rhd \in \mathcal{L}(C)$ to $C \setminus A(\mathbf{s})$.

The players' preferences over the states of the game are determined by their most preferred candidates in respective elected committees. For every committee $S \subseteq C$ and player $c \in C$, let $top_c(S)$ denote the highest ranked member of S, according to the full preference P_c of c over the set C of potential candidates. Then, in game $\Gamma^{\mathcal{R}}$, player c prefers state $\mathbf{s}$ to state $\mathbf{t}$ (denoted by $\mathbf{s} >_c \mathbf{t}$) if and only if $top_c(W(\mathbf{s}))\, P_c\, top_c(W(\mathbf{t}))$. Otherwise, player c is indifferent between $\mathbf{s}$ and $\mathbf{t}$.[2]

We will be interested in Nash equilibria of MCGs, i.e., the states of the game that no player can profitably deviate from. Formally, given a game $\Gamma^{\mathcal{R}}$, we say that a strategy profile $\mathbf{s}$ is a *pure strategy Nash equilibrium* (PSNE) of $\Gamma^{\mathcal{R}}$ if for every candidate $c \in C$ it is not the case that $\mathbf{t} >_c \mathbf{s}$, where $\mathbf{t}$ is the strategy profile given by $t_c = 1 - s_c$ and $t_a = s_a$ for all $a \in C \setminus \{c\}$.

3 APPROVAL-BASED ELECTIONS

To start, we consider the arguably simpler setting of MCGs where voters have approval-based preferences; the candidates' preferences may be either ordinal or also approval-based.

An important difference between approval-based and ordinal elections is that in the former a voter's approval of a candidate is independent of the set of actual candidates running in the election, so whenever a candidate is present in the ballot, it will be approved by the same group of supporters, whereas under ordinal preferences the candidates' positions in restricted rankings depend on the set of candidates running in the election.

Intuitively, this suggests that for the setting of approval ballots in games with self-supporting preferences the *truthful* strategy profile, where all candidates are present, should be an equilibrium. We will now see that this intuition is indeed correct for (sequential) Thiele rules, but not for MAV. In fact, we will show that sequential Thiele rules have an even stronger truthfulness property: in every equilibrium the outcome is the same as under full participation. We first need to formalize this idea.

Definition 3.1. We say that a given MCG $\Gamma^{\mathcal{R}}$ is *genuine* if:

(1) its set of PSNE contains the truthful profile $\mathbf{s}^* = C$, and
(2) every equilibrium state $\mathbf{s}$ of $\Gamma^{\mathcal{R}}$ produces the same outcome as the truthful state: $W(\mathbf{s}) = W(\mathbf{s}^*)$.

Our first result is that multiwinner candidacy games under sequential Thiele rules are genuine.

Theorem 3.2. *$\Gamma^{seq\text{-}T}$ is genuine.*

Proof. First, we observe that the truthful profile $\mathbf{s}^* = C$ is an equilibrium. Indeed, since candidates have self-supporting preferences, no member of the truthful winning committee $W^* = W(\mathbf{s}^*)$ wants to leave the election, and the withdrawal of any candidate in $C \setminus W^*$ has no effect on how other candidates are evaluated

[2] We choose this Chamberlin–Courant-like definition of player utilities for its simplicity and consistency with the self-supporting candidate preference assumption. In fact, all our results hold for any utility definition under which for all $S \subseteq C$ with $|S| = k - 1$, and all $a, b \in C$, if player c prefers committee $S \cup \{a\}$ to committee $S \cup \{b\}$ then it must be that $a P_c b$.

throughout the selection process, and hence does not affect the outcome.

We will now argue that any other equilibrium state, if it exists, produces the same outcome, W^*, as the truthful state. It suffices to show that any such state contains the set W^*.

Recall that under sequential Thiele rules, the committee is formed in k consecutive steps, where a candidate with the largest marginal contribution is added to the committee at each step. Let $c_1, c_2, \ldots, c_k$ be the order in which the members of W^* join the committee.

Assume to the contrary that $\mathbf{s}$ is an equilibrium profile not containing W^*, and let c_i be the first candidate in the above sequence that is not present in $\mathbf{s}$. That is, candidates $c_1, c_2, \ldots, c_{i-1}$ are running in the election, and are included in the winning committee $W(\mathbf{s})$ in the first $i-1$ steps of its formation. Then, if candidate c_i were to join the election, it would be added to the winning committee at the following step i, which is beneficial for c_i due to self-supporting preferences; a contradiction. □

In contrast, Γ^{PAV} and Γ^{CC} may fail to be genuine: even though in any such game full participation is an equilibrium, there exist equilibria that result in different outcomes.

Theorem 3.3. *Γ^{CC} and Γ^{PAV} admit the truthful equilibrium, but may also have PSNE with different outcomes.*

Proof. Our first claim is proved similarly to the first claim of Theorem 3.2: in both games Γ^{CC} and Γ^{PAV}, the truthful profile $\mathbf{s}^* = C$ is a PSNE, as the elected candidates would stay due to their self-supporting preferences, and the non-elected candidates have no incentive to leave, as this would not change the outcome.

To prove the second claim, we construct an MCG with $n = 26$ voters, $m = 10$ candidates, and committee size $k = 5$. The candidate set is $C = \{c_1, c_2, \ldots, c_{10}\}$, and the tie-breaking order on C is given by $\rhd = c_1c_2c_3c_4c_{10}c_9c_8c_7c_6c_5$ (this will only be relevant for the PAV rule as there are no ties under CC in this election).

Table 1 summarizes the voters' preferences by listing, for each candidate c_i, $i = 1, \ldots, 10$, its supporters—the voters who approve this candidate. The candidates' preferences are irrelevant.

Candidates	Approving voters
c_1	v_1 v_2 v_3 v_4 v_5
c_2	v_6 v_7 v_8 v_9 v_{10}
c_3	v_{11} v_{12} v_{13} v_{14} v_{15}
c_4	v_{16} v_{17} v_{18} v_{19} v_{20}
c_5	v_{21} v_{22} v_{23} v_{24} v_{25} v_{26}
c_6	v_1 v_6 v_{11} v_{16} v_{21}
c_7	v_7 v_{12} v_{17} v_{22}
c_8	v_3 v_8 v_{13} v_{18}
c_9	v_9 v_{14} v_{19} v_{24}
c_{10}	v_4 v_5 v_{10} v_{15} v_{20} v_{25} v_{26}

Table 1: Proof of Theorem 3.3—voters' approval preferences

Recall that under the CC voting rule, a voter is satisfied with a committee W (and contributes one point to its score) as long as W contains at least one of her approved candidates (otherwise, she gives the committee zero points).

Under the truthful PSNE profile $\mathbf{s}^* = C$, the winning committee is $W^* = \{c_1, c_2, c_3, c_4, c_5\}$ with the score of 26; indeed, this committee satisfies all voters. However, there are other equilibria in this game, which, as we will now show, produce outcomes different from W^*.

Consider state $\mathbf{s}$ with $A(\mathbf{s}) = \{c_6, c_7, c_8, c_9, c_{10}\} \not\supseteq W^*$ where the winning committee $W(\mathbf{s})$ consists of all actual candidates and gains the score of 24 (as voters v_2 and v_{23} do not approve of any of its members). We claim that $\mathbf{s}$ is a PSNE.

Indeed, none of the actual candidates wish to leave the election as they are all included in the winning committee. Now, none of the candidates c_2, c_3, c_4 can change the outcome by joining the election, as all their supporters already have some approved candidate in $W(\mathbf{s})$. Finally, if c_1 or c_5 decided to join, then any committee where any of them would replace one of the current winning committee members, would gain one additional point from either v_2 or v_{23}, but lose at least three points from the supporters of the excluded member who will now be left unsatisfied. Hence, $W(\mathbf{s})$ remains the winning committee, thus implying the stability of $\mathbf{s}$.

Under PAV, we have the same truthful outcome and the same non-truthful equilibrium, albeit the analysis is somewhat more tedious due to tie-breaking. We omit the details due to space constraints. □

In contrast, under MAV the truthful profile may fail to be an equilibrium. In fact, this is the case even for $k = 1$, i.e., for the case where MAV is used to select a single winner.

Example 3.4. Let $C = \{a, b, c\}$, $V = \{v_1, v_2, v_3\}$, and suppose that voters have the following preferences: $P_{v_1} = \{c\}$, $P_{v_2} = \{b\}$, $P_{v_3} = \{a, c\}$. Let $k = 1$. Suppose also that candidate b prefers c to a, and ties are broken lexicographically.

If all candidates are present, we have $\mathrm{sc}^{MAV}(\{a\}) = \mathrm{sc}^{MAV}(\{c\}) = 2$, $\mathrm{sc}^{MAV}(\{b\}) = 3$, so a wins due to lexicographic tie-breaking. However, if b withdraws from the election, we have $\mathrm{sc}^{MAV}(\{a\}) = 2$, $\mathrm{sc}^{MAV}(\{c\}) = 1$, so c becomes the winner; as b prefers c to a, this means that the truthful profile is not an equilibrium.

We note that this candidacy game does not have an equilibrium where only a single candidate participates in the election: if a runs alone, then c wants to join, if b runs alone, then a wants to join, and if c runs alone, then b wants to join. However, it does have an equilibrium with two actual candidates, namely, $\{a, c\}$.

It remains an open question whether Γ^{MAV} always has a PSNE; in fact, this problem appears to be challenging even for $k = 1$, i.e., for the single-winner scenario.

4 ORDINAL ELECTIONS

We now move to the setting where both voters and candidates have ordinal preferences.

In this case, the very existence of pure strategy Nash equilibria is a challenging question. For single-winner elections it is known that, in general, under most voting rules candidacy games may fail to have PSNE [22]; however, the presence of a Condorcet winner guarantees the existence of PSNE for Condorcet-consistent rules [22] and for Plurality [27]. Here we explore whether similar results can be obtained for multiwinner elections.

While one would expect non-existence results of Lang et al. [22] to extend easily to the multiwinner setting, we show that this is not necessarily the case: while for $k = 1$ the k-CC rule is simply the Plurality rule and hence the associated MCG may fail to have a PSNE, it turns out that PSNE always exist in MCGs under k-CC rules with $k \geq 2$. In fact, this result extends to all O-CC rules with $k \geq 2$ and $f_{\text{pos}}(1) = f_{\text{pos}}(2)$ (Theorem 4.1); our proof is constructive. While we conjecture that this result no longer holds if $f_{\text{pos}}(1) \neq f_{\text{pos}}(2)$, we have not been able to prove this.

Next, we extend the definition of a Condorcet winner to a sequence of k Condorcet winners (where k is the target committee size), and examine k-Maximin and Bloc—multiwinner generalizations of a well-known Condorcet-consistent rule and Plurality, respectively. For the former, we prove that the set consisting of k sequential Condorcet winners forms a PSNE (Theorem 4.5); however, for the latter we construct a game with no PSNE (Theorem 4.6).

Based on this negative result, we place further structural constraints on voters' preferences, and prove that for single-peaked preferences the existence of a PSNE is guaranteed not just for Bloc, but for most k-TC rules (Theorem 4.13). Our proof is constructive and builds on a sequence of structural results on the order of candidates that is induced by a single-peaked profile (Lemmas 4.8–4.12). These auxiliary results do not depend on the choice of the voting rule and are interesting for their own merit.

Finally, for all of the cases with PSNE, we show that the respective games are not genuine (Examples 4.3 and 4.15), even though k-Maximin does allow for the truthful equilibrium (Theorem 4.7).

4.1 Chamberlin–Courant Rules

In this section, we prove that multiwinner candidacy games based on O-CC rules with $f_{\text{pos}}(1) = f_{\text{pos}}(2)$ always have a PSNE for $k \geq 2$. Let *k-TOP* denote the set of top k candidates in the tie-breaking order $\rhd$.

Theorem 4.1. *In $\Gamma^{O\text{-}CC}$ defined by a scoring function f_{pos} with $f_{\text{pos}}(1) = f_{\text{pos}}(2)$ and $k \geq 2$, the set k-TOP is a PSNE.*

Proof. Clearly, none of the members of *k-TOP* has an incentive to withdraw from the election, so assume to the contrary that a candidate $c \in C \setminus k\text{-}TOP$ wants to join.

Let $S \neq k\text{-}TOP$ be the winning committee in the game with $A = k\text{-}TOP \cup \{c\}$. Such a committee must include the candidate c and exclude some candidate $c' \in k\text{-}TOP$ so that $|k\text{-}TOP \cap S| = k - 1$. We claim that $\text{sc}^{O\text{-}CC}(k\text{-}TOP) = \text{sc}^{O\text{-}CC}(S)$. Indeed, since there are exactly $k + 1$ actual candidates, for each voter $v \in V$, her representative in any size-k committee is either in the first or in the second position of her (restricted) preference list, and we have $f_{\text{pos}}(1) = f_{\text{pos}}(2)$. As $c' \rhd c$, the tie-breaking rule favors *k-TOP* over S, so *k-TOP* is the winning committee. Hence, c has no incentive to join, a contradiction. □

Corollary 4.2. *k-TOP is a PSNE in $\Gamma^{k\text{-}CC}$ with $k \geq 2$.*

These results are in contrast with the negative result for Plurality [22], which can be seen as a special case of k-CC for $k = 1$.

We can show, however, that $\Gamma^{k\text{-}CC}$ is not genuine.

Example 4.3. We construct a game with $m = 4$ candidates $C = \{a, b, c, d\}$, tie-breaking rule $a \rhd b \rhd c \rhd d$, and committee size $k = 2$. We have a single voter v who ranks the candidates as $d \succ c \succ b \succ a$.

Observe that in $\Gamma^{k\text{-}CC}$, in the truthful profile $\mathbf{s}^* = C$ all committees except $\{a, b\}$ are tied for winning, so we have $W(\mathbf{s}^*) = \{a, c\}$ by the tie-breaking rule. However, by Theorem 4.1, there is a PSNE state $\mathbf{s} = k\text{-}TOP$, whose outcome $W(\mathbf{s}) = \{a, b\} \neq W(\mathbf{s}^*)$ is untruthful, and so $\Gamma^{k\text{-}CC}$ is not genuine.

4.2 Sequential Condorcet Winners

In this section, our goal is to understand whether the positive results for single-winner elections with a Condorcet winner extend to the multiwinner setting.

Given a preference profile P^V over a candidate set C, we say that a candidate a *beats* another candidate b in a pairwise election if a strict majority of voters prefer a to b; if exactly half of the voters prefer a to b, then a and b are *tied.* A candidate is a *(weak) Condorcet winner* if it does not lose in any of its pairwise elections; it is a *strong* Condorcet winner if it beats all other candidates. Each election has at most one Condorcet winner, but many have none.

Several approaches to generalize the concept of a Condorcet winner to multiwinner elections have been proposed [16, 19, 20, 29]; we refer the reader to [2] for a detailed analysis of their merits and applications. In this work, we take a sequential approach, as captured by the following definition.

Definition 4.4. Given a voter preference profile P^V over a candidate set C, we say that candidates $c_1, c_2, \dots, c_\ell$, where $\ell \leq m$, form a *sequence of Condorcet winners* of length ℓ if candidate c_1 is a Condorcet winner in P^V, and for every $i = 1, 2, \dots, \ell - 1$ it holds that candidate c_{i+1} is a Condorcet winner in the restriction of P^V to $C \setminus \{c_1, \dots, c_i\}$.

Let $\Gamma^{\mathcal{R}}$ be a multiwinner candidacy game defined by a tuple $\langle C, V, P^V, P^C, k, \mathcal{R}, \rhd \rangle$, where k is the target committee size, and assume that P^V induces a sequence of (at least) k Condorcet winners $c_1, \dots, c_k$ as above. Let $k\text{-}SCW = \{c_1, \dots, c_k\}$ denote the set of these candidates.

The following theorem suggests that Definition 4.4 is useful for extending the positive result for single-winner Condorcet-consistent rules [22] to the multiwinner setting.

Theorem 4.5. *In $\Gamma^{k\text{-}MM}$ with k sequential Condorcet winners, the set k-SCW is a PSNE.*

Proof. Observe that the outcome of $\mathbf{s} = k\text{-}SCW$ is *k-SCW* itself. As candidates have self-supporting preferences, none of the candidates in *k-SCW* wants to withdraw from the election. On the other hand, no candidate in $C \setminus k\text{-}SCW$ would join the election, as this will not change the outcome. Indeed, note that by definition, each member c of *k-SCW* beats every candidate a in $C \setminus k\text{-}SCW$ in a pairwise election; that is, there are more than $\frac{n}{2}$ voters who prefer c to a. This implies $\text{sc}^{k\text{-}MM}(k\text{-}SCW) > \frac{n}{2}$. On the other hand, consider the committee $S = k\text{-}SCW \setminus \{c\} \cup \{a\}$ obtained by replacing c with a candidate $a \in C \setminus k\text{-}SCW$. Since c beats a, we have $\text{sc}^{k\text{-}MM}(S) < \frac{n}{2} < \text{sc}^{k\text{-}MM}(k\text{-}SCW)$. □

However, a related result for Plurality [27] does not seem to extend easily to multiwinner elections under Definition 4.4. Recall that under Plurality voting, each voter indicates her most preferred candidate and contributes one point to her score; the Bloc rule can thus be interpreted as a generalization of Plurality where a voter indicates her ideal committee of k candidates. As we show in Theorem 4.6 below, the presence of k sequential Condorcet winners appears insufficient to guarantee equilibrium existence in MCGs under this rule.

Theorem 4.6. *Γ^{Bloc} with k sequential Condorcet winners may have no PSNE.*

Proof. We construct an example of an MCG with $n = 13$ voters, $m = 5$ candidates, and committee size $k = 2$.[3] The set of candidates is $C = \{a, b, c, d, e\}$, and the voters' preferences are given in Table 2 below. There are four voter blocks containing 4, 3, 4 and 2 voters, respectively; for each block, the most preferred candidate is listed at the top, and the least preferred one at the bottom.

4 voters	3 voters	4 voters	2 voters
a	a	b	e
d	e	c	b
b	c	d	a
c	b	e	d
e	d	a	c

Table 2: Proof of Theorem 4.6—voters' preferences

For the candidates, we only need to specify some of their preferences, which can then be arbitrarily completed to full rankings. These are given in Table 3. For readability, we use $>$ to indicate the preference relation of each candidate.

Candidates	Preferences
b	$c > d$
c	$d > b,\ d > e > a$
d	$c > b,\ a > e$
e	$b > c,\ b > d$

Table 3: Proof of Theorem 4.6—candidates' preferences

The tie-breaking rule is irrelevant as there is a unique winning committee in each state.

In this game, candidate a is a Condorcet winner, candidate b wins in pairwise elections against c, d and e, candidate c beats d and e, and d beats e. That is, a, b, c, d, e is a sequence of Condorcet winners. However, there is no PSNE under Bloc. To see this, note first that if the number of participating candidates is 0 or 1, then any non-participating candidate would like to join the election, as she will be included in the winning committee. In Table 4 we list the larger states of the game, as given by their sets of actual candidates A, the respective outcomes W, and a candidate who has an incentive to deviate (denoted by $\rightarrow$) by either joining (+) or leaving (−) the election at a given state. For the sake of readability, we write sets as strings, i.e., we write xyz for $\{x, y, z\}$. This table shows that no state is a PSNE.

□

$A = W$	$\rightarrow$	A	W	$\rightarrow$	A	W	$\rightarrow$
ab	$e+$	abc	ab	$d+$	$abcd$	ac	$e+$
ac	$b+$	abd	ab	$c+$	$abce$	ab	$c-$
ad	$b+$	abe	be	$d+$	$abde$	ad	$e-$
ae	$b+$	acd	ad	$b+$	$acde$	ad	$e+$
bc	$a+$	ace	ae	$b+$	$bcde$	bc	$a+$
bd	$a+$	ade	ae	$c+$	$abcde$	ab	$c-$
be	$c+$	bcd	bc	$a+$			
cd	$b+$	bce	bc	$a+$			
ce	$b+$	bde	bd	$a+$			
de	$b+$	cde	cd	$a+$			

Table 4: Proof of Theorem 4.6—no PSNE

Note that the game Γ^{Bloc} is therefore not genuine, by definition. For $\Gamma^{k\text{-}MM}$, we prove the following result.

Theorem 4.7. *$\Gamma^{k\text{-}MM}$ with k sequential Condorcet winners admits the truthful equilibrium, but may also have PSNE with non-truthful outcomes.*

Proof. We start by showing that for every set A that contains $k\text{-}SCW$ the committee $k\text{-}SCW$ wins. Let $A \supseteq k\text{-}SCW$, and let $S \subseteq A$ be a committee that contains candidate $c \in S$ such that $c \notin k\text{-}SCW$. Note that for every candidate $c' \in k\text{-}SCW$, the number of voters that rank c' over c is greater than $\frac{n}{2}$ (and vice versa, the number of voters that rank c over c' is smaller than $\frac{n}{2}$), and so $\mathrm{sc}^{k\text{-}MM}(k\text{-}SCW) > \frac{n}{2} > \mathrm{sc}^{k\text{-}MM}(S)$. It then follows that the truthful state $\mathbf{s}^* = C$ is a PSNE of $\Gamma^{k\text{-}MM}$, and the truthful outcome is $k\text{-}SCW$.

However, as we show below, there may also be equilibrium states with untruthful outcomes in this game. To this end, we construct an example with $m = 6$ candidates $C = \{a, b, c, d, e, f\}$, tie-breaking rule $a \rhd b \rhd c \rhd d \rhd e \rhd f$, and committee size $k = 3$. There are $n = 3$ voters $V = \{v_1, v_2, v_3\}$ whose preferences are given in Table 5 where, for each voter, the most preferred candidate is listed at the top, and the least preferred one at the bottom.

voter v_1	voter v_2	voter v_3
a	f	f
b	e	e
c	d	d
d	b	c
f	c	a
e	a	b

Table 5: Proof of Theorem 4.7—voters' preferences

We claim that there is an equilibrium state $\mathbf{s} = \{a, b, c\}$ whose outcome is different from the truthful outcome $k\text{-}SCW = \{d, e, f\}$. Indeed, none of the candidates in $\mathbf{s}$ has an incentive to withdraw. Now, if any of the candidates d, e or f would join the election, we get a set A with 4 actual candidates. As any size-3 committee under A has a score of exactly 1, the winning committee is $\{a, b, c\}$ by the tie-breaking rule, so the state $\mathbf{s}$ is stable. □

4.3 Single-peaked Preferences

Based on the negative result of Theorem 4.6, in this section we further restrict the voters' preferences and focus on single-peaked profiles. The concept of single-peaked preferences, which was first proposed by Black [6] and Arrow [1], captures settings where voters' preferences are essentially single-dimensional, and found many applications in (computational) social choice [15]. It turns out to be relevant for our analysis as well.

Given a preference profile P^V over C, let top_v denote the most preferred candidate of voter v. Given a linear order $\rhd^{SP}$ of C, we

[3] We thank the anonymous IJCAI-2019 reviewer who has simplified our original example.

say that P_v is *single-peaked with respect to* $\rhd^{SP}$ if for all $a, b \in C$ such that $top_v \rhd^{SP} a \rhd^{SP} b$ or $b \rhd^{SP} a \rhd^{SP} top_v$ we have that $aP_v b$. In other words, the order P_v is decreasing as we move in either direction from top_v; we refer to $\rhd^{SP}$ as an *axis* for C. We say that a profile P^V is *single-peaked* if there exists an axis $\rhd^{SP}$ such that for each $v \in V$ the vote P_v is single-peaked with respect to $\rhd^{SP}$.

Importantly, the (weak) majority relation is transitive for single-peaked profiles, and hence all the candidates form a sequence of Condorcet winners. Specifically, the Median Voter Theorem [25, 26] suggests that a single-peaked profile induces an order on the candidate set, which is strict for an odd number of voters (so all sequential Condorcet winners are strong) and weak for an even number of voters (so there may be weak Condorcet winners). However, it does not specify the structure of this order. To this end, in Lemmas 4.8–4.12 we prove a few structural results in this context (see the work of Smeulders [34], which contains some results in the same spirit), which enable us to prove equilibrium existence for k-TC in Theorem 4.13.

In the remainder of this section, we use the following notation. Let P^V be a single-peaked profile with axis $\rhd^{SP}$ over C and let $C_0 \supsetneq C_1 \supsetneq C_2 \ldots \supsetneq C_\ell$ be a concentric sequence of candidate sets, where $C_0 = C$ and each C_i, $i = 0, 1, \ldots, \ell$, corresponds to a unique set of Condorcet winners, SCW_i, in the restriction P_i^V of P^V to C_i (similarly, we denote the restriction of $\rhd^{SP}$ to C_i by $\rhd_i^{SP}$). That is, for every $a \in SCW_{i-1}$ and $b \in SCW_i$, $i = 1, \ldots, \ell$ it holds that a beats b in their pairwise election, and for each $i = 0, 1, \ldots, \ell$ all candidates $a, b \in SCW_i$ are tied. The sets SCW_i form a partition of C where singletons contain strong Condorcet winners and non-singletons are composed of weak Condorcet winners in respective restrictions; the overall sequence of Condorcet winners is given by $(\pi(SCW_i))_{i=0}^{\ell}$ where $\pi(SCW_i)$ denotes a permutation over SCW_i. Finally, let $\succ$ indicate the preference relation of an arbitrary voter.

Lemma 4.8. *The set SCW_i forms a contiguous segment of $\rhd_i^{SP}$.*

Proof. If the number of voters n is odd, SCW_i is a singleton, and our claim is trivially true, so assume that n is even. If the statement of the lemma is not true, there exist two candidates $a, b \in SCW_i$ and a candidate $d \notin SCW_i$ such that $a \rhd_i^{SP} \cdots \rhd_i^{SP} d \rhd_i^{SP} b$. Since a and b are Condorcet winners in C_i, there is a set of votes V_1, $|V_1| = \frac{n}{2}$, such that $a \succ b$ in each vote in V_1 and a set of votes V_2, $|V_2| = \frac{n}{2}$, such that $b \succ a$ in each vote in V_2. In every vote in V_1 candidate d is ranked above b and all candidates to the right of b with respect to $\rhd_i^{SP}$; similarly, in every vote in V_2 candidate d is ranked above a and all candidates to the left of a with respect to $\rhd_i^{SP}$. It remains to consider candidates that appear between a and d on $\rhd_i^{SP}$. Let c be some such candidate. Since $b \in SCW_i$, at least half of the voters prefer b to c, and all such voters prefer d to c. Thus, d is preferred to every other candidate in C_i by at least half of the voters. □

Lemma 4.9. *For every $j = 0, 1, \ldots, \ell$, the elements of $\cup_{i=0}^{j} SCW_i$ form a contiguous segment of $\rhd^{SP}$.*

Proof. Assume for brevity that n is odd. Suppose for the sake of contradiction that there are $a, b \in \cup_{i=0}^{j} SCW_i$ and $d \notin \cup_{i=0}^{j} SCW_i$ such that $a \rhd^{SP} \ldots \rhd^{SP} d \rhd^{SP} \ldots \rhd^{SP} b$. We can assume without loss of generality that $a \succ b$ in more than $n/2$ votes; but then also $d \succ b$ in more than $n/2$ votes; a contradiction. □

Lemmas 4.8 and 4.9 imply that the sets SCW_i form concentric intervals around SCW_0 on the original axis $\rhd^{SP}$, where the elements within each SCW_i can be freely permuted.

Lemma 4.10. *Let $a \in SCW_i$ for some $i = 0, 1, \ldots, \ell$, and let candidate $b \in C_i$ tie in a pairwise election with a. Then, either $b \in SCW_i$ or there exists another $c \in SCW_i$ such that c lies between a and b on $\rhd_i^{SP}$; moreover, a is tied with any candidate on the interval between a and b on $\rhd_i^{SP}$.*

Proof. Let $b \notin SCW_i$ be the closest on the axis candidate to a in a tie with it. W.l.o.g., let a be to the left from b. In the case where there are no other candidates between them, since $b \succ a$ in half of the votes then for any d on the left from a we have $b \succ d$ in at least half of the votes. Let e be an arbitrary candidate to the right from b. Then, in all the votes where $a \succ e$ we also have $b \succ e$, and there are at least $\frac{n}{2}$ of those. It thus follows that $b \in SCW_i$; a contradiction.

Consider now the case where there are candidates between a and b. By construction, they all belong to SCW_i. Similarly to the previous case, for any candidate d to the right from b or to the left from a we have $b \succ d$ in at least $\frac{n}{2}$ votes. Thus, the only candidates that are preferred to b in more than half of the votes are those members of SCW_i that lie between a and b, as required. □

Lemma 4.11. *Let $SCW_i = \{c_1, c_2, \ldots, c_{|SCW_i|}\}$, $i = 0, \ldots, \ell$, such that $c_1 \rhd^{SP} c_2 \rhd^{SP} \ldots \rhd^{SP} c_{|SCW_i|}$. Then, in half of the votes in P_i^V we have $c_1 \succ c_2 \succ \ldots \succ c_{|SCW_i|}$, and in the other half we have $c_{|SCW_i|} \succ c_{|SCW_i|-1} \succ \ldots \succ c_1$.*

Proof. By Lemma 4.10, the candidates $c_1, c_2, \ldots, c_{|SCW_i|}$ are tied with each other. Note that a candidate which is listed in the bottom of a vote in a single-peaked profile, can be either the first or the last one on the axis $\rhd^{SP}$: i.e., only c_1 or $c_{|SCW_i|}$ can be listed in the bottom of any vote. Now, there can be at most $\frac{n}{2}$ votes, where c_1 (resp., $c_{|SCW_i|}$) loses to the other candidates $c_2, \ldots, c_{|SCW_i|-1}$, and so there are exactly $\frac{n}{2}$ such votes where c_1 (resp., $c_{|SCW_i|}$) is listed last. Hence, in the other half of the votes c_1 (resp., $c_{|SCW_i|}$) must appear in the top. The proof is complete by induction. □

Lemma 4.12. *If $|SCW_i| \geq 3$ then $i = 0$.*

Proof. Assume to the contrary that for some $i \geq 1$ there exist three candidates c_1, c_2, c_3 such that $c_1 \rhd^{SP} c_2 \rhd^{SP} c_3$ and $SCW_i \supseteq \{c_1, c_2, c_3\}$. Consider a candidate $b \in SCW_{i-1}$ such that $b \succ c_1$ in more than $\frac{n}{2}$ votes (such a candidate exists by Lemma 4.10). Note that b must lie between c_1 and c_2 on the axis, as otherwise we would have $c_1 \succ b$ in at least $\frac{n}{2}$ votes.

Now consider the $\frac{n}{2}$ votes where $c_3 \succ c_2$. In all these votes we have $c_3 \succ b$ and $c_2 \succ b$. Hence, b is tied with c_2 and c_3 in P_{i-1}^V, and so there is a candidate $a \in SCW_{i-1}$ that lies between b and c_2, c_3 on the axis, and $a \succ c_2$, $a \succ c_3$ in more than $\frac{n}{2}$ votes. However, in all the votes where $a \succ c_3$ we also have $c_2 \succ c_3$, and we can only have at most $\frac{n}{2}$ such votes; a contradiction. □

We are now ready to state the main result of this section.

Theorem 4.13. *Let $\Gamma^{k\text{-}TC}$ be defined by a scoring function f_{top} with $f_{\text{top}}(k) \neq f_{\text{top}}(k-1)$, and let P^V be single-peaked. Then, $\Gamma^{k\text{-}TC}$ always has a PSNE if n is odd, and whenever $\rhd^{SP} = \rhd$ if n is even.*

Proof. By Lemmas 4.8–4.12, there are three possible cases:

(1) $|SCW_0| \geq k$;
(2) $|SCW_0| < k$ and $\exists j \geq 1$ such that $|SCW_i| \leq 2$ for all $i = 1, \dots, j$ and $|\cup_{i=0}^{j} SCW_i| = k$;
(3) $|SCW_0| < k$ and $\exists j \geq 1$ such that $|SCW_i| \leq 2$ for all $i = 1, \dots, j$, $|\cup_{i=0}^{j-1} SCW_i| = k-1$, and $|\cup_{i=0}^{j} SCW_i| = k+1$.

For every $X \subseteq C$ let let $k\text{-}TOP(X)$ denote the set of top k candidates in the restriction of the tie-breaking order $\rhd$ to X. We will now construct the set $k\text{-}SCW$ of k sequential Condorcet winners for each of the above cases in the following way:

(1) $k\text{-}SCW = k\text{-}TOP(SCW_0)$;
(2) $k\text{-}SCW = \cup_{i=0}^{j} SCW_i$;
(3) $k\text{-}SCW = k\text{-}TOP\left(\cup_{i=0}^{j} SCW_i\right)$.

We show that $k\text{-}SCW$ is a PSNE of $\Gamma^{k\text{-}TC}$. Clearly, no member of $k\text{-}SCW$ wants to withdraw, so it remains to show that none of the candidates in $C \setminus k\text{-}SCW$ wants to join the election.

First, consider the cases (1) and (2). Let $k\text{-}SCW = \{c_1, c_2, \dots, c_k\}$ and $b \in C \setminus k\text{-}SCW$. We will now argue that if the set of actual candidates is $A = k\text{-}SCW \cup \{b\}$ then the winning committee is $k\text{-}SCW$, so b has no incentive to join.

Indeed, if in case (1) we have $|SCW_0| > k$ and $b \in SCW_0$, then since for any $c \in k\text{-}SCW$, $c \rhd b$, and $\rhd = \rhd^{SP}$, then by Lemmas 4.8–4.12 the members of A are ordered on the axis $\rhd^{SP}$ as follows: $c_1, \dots, c_k, b$. That is, b is in the bottom position in $\frac{n}{2}$ votes over A. This implies that for any committee $S = (k\text{-}SCW \setminus \{c\}) \cup \{b\}$ where $c \in k\text{-}SCW$, its score is exactly $\text{sc}^{k\text{-}TC}(S) = \frac{n}{2} f_{\text{top}}(k) + \frac{n}{2} f_{\text{top}}(k-1) = \text{sc}^{k\text{-}TC}(k\text{-}SCW)$, but $k\text{-}SCW$ wins by the tie-breaking. Otherwise, if b is not in a tie with any of $c_1, \dots, c_k$, then $\text{sc}^{k\text{-}TC}(S) \leq \frac{n}{2} f_{\text{top}}(k) + \frac{n}{2} f_{\text{top}}(k-1) = \text{sc}^{k\text{-}TC}(k\text{-}SCW)$, so $k\text{-}SCW$ remains the winner.

We now consider case (3). In this case, $|SCW_j| = 2$, so let $\cup_{i=0}^{j-1} SCW_i = \{c_1, c_2, \dots, c_{k-1}\}$, $SCW_j = \{a, b\}$. By Lemmas 4.8–4.12, the possible orders of candidates on the axis $\rhd^{SP}$ are as follows: (a) $c_1, \dots, c_{k-1}, a, b$; (b) $a, c_1, \dots, c_{k-1}, b$; and (c) $a, b, c_1, \dots, c_{k-1}$. Since $\rhd = \rhd^{SP}$, for (a) and (b) we have $k\text{-}SCW = \{a, c_1, \dots, c_{k-1}\}$, and the proof follows the same lines as for cases (1) and (2) above. For (c), $k\text{-}SCW = \{a, b, c_1, \dots, c_{k-2}\}$. Now, for any candidate $c \in C \setminus \{a, b, c_1, \dots, c_{k-2}, c_{k-1}\}$, there is no incentive to participate, for the same reasons as above. It remains to show that c_{k-1} does not want to join the the election either. To this end, we observe that for any committee $S = (k\text{-}SCW \setminus \{c\}) \cup \{c_{k-1}\}$ where $c \in k\text{-}SCW$, its score is $\text{sc}^{k\text{-}TC}(S) \leq \frac{n}{2} f_{\text{top}}(k) + \frac{n}{2} f_{\text{top}}(k-1) = \text{sc}^{k\text{-}TC}(k\text{-}SCW)$, with the equality for $c \in \{a, b\}$. To complete the proof, we observe that $a, b \rhd c_{k-1}$. □

Corollary 4.14. *Theorem 4.13 implies equilibrium existence for Γ^{Bloc}, $\Gamma^{O\text{-}PAV}$, and $\Gamma^{\epsilon\text{-}Bloc\text{-}CC}$ with single-peaked preferences.*

However, we show that $\Gamma^{k\text{-}TC}$ is not necessarily genuine, even if voters have single-peaked preferences: for Γ^{Bloc}, $\Gamma^{O\text{-}PAV}$, and $\Gamma^{\epsilon\text{-}Bloc\text{-}CC}$ the truthful state may be unstable.

Example 4.15. We construct an example with $n = 5$ voters, $m = 5$ candidates, and a committee size $k = 2$. The candidate set is $C = \{a, b, c, d, e\}$, with tie-breaking order $a \rhd b \rhd c \rhd d \rhd e \rhd f$. The single-peaked voter preference profile over axis $a \rhd^{SP} b \rhd^{SP} c \rhd^{SP} d \rhd^{SP} e$ is specified in Table 5. There are three voter blocks containing 2, 2, 1 voters, respectively; for each block, the most preferred candidate is listed at the top, and the least preferred one at the bottom.

2 voters	2 voters	1 voter
a	a	e
c	d	b
d	b	d
b	c	a
e	e	c

Table 6: Example 4.15—voters' preferences

Under the truthful profile $\mathbf{s}^* = C$, the winning committee is $W(\mathbf{s}^*) = \{a, c\}$. However, if candidate d withdraws from the election, the winner changes to $\{a, b\}$, which is a beneficial move for d is she prefers b over c. Hence, $\mathbf{s}^* = C$ is unstable.

5 CONCLUSIONS

We have initiated the study of strategic candidacy games in multi-winner elections, both for the approval-based setting and for the setting with ranked ballots.

For approval-based elections, we developed a good understanding of equilibria of games that correspond to Thiele rules as well as their sequential variants, and showed that for both classes of games full participation is an equilibrium strategy profile. However, we have shown that this is not necessarily the case for other approval-based voting rules, such as MAV. In fact, a natural open problem suggested by our work is whether strategic candidacy games under the MAV rule always possess an equilibrium. It would also be interesting to investigate more sophisticated voting rules for this setting, such as the Phragmén rule and its sequential variants [9].

For the ordinal setting, we inherit equilibrium non-existence results from the single-winner case. However, in contrast with the single-winner case, we obtain positive results for CC rules. Moreover, for a number of voting rules, we identify structural properties of voters' preferences that guarantee the existence of an equilibrium. On the negative side, for all rules we consider there is an equilibrium where the winning committee is different from the one chosen under full participation.

An obvious open direction is to investigate the complexity of finding equilibria in multiwinner candidacy games. Of course, we expect this problem to be hard when even computing a winning committee with respect to the underlying voting rule is computationally demanding, and this is the case for many voting rules that we consider. However, we can study the parameterized complexity of this problem or focus on restricted preference domains, where the respective rules are known to be tractable [5, 14, 33].

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