Popularity and Strict Popularity in Altruistic Hedonic Games and Minimum-Based Altruistic Hedonic Games

Extended Abstract

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ABSTRACT

We consider average-based and min-based altruistic hedonic games and study the problem of verifying popular and strictly popular coalition structures. While strict popularity verification has been shown to be coNP-complete in min-based altruistic hedonic games, this problem has been open for equal-treatment and altruistic-treatment average-based altruistic hedonic games. We solve these two open cases of strict popularity verification and then provide the first complexity results for popularity verification in (both average- and min-based) altruistic hedonic games, where we cover all three degrees of altruism.

KEYWORDS

Cooperative Game Theory; Coalition Formation; Hedonic Game; Popularity; Altruism; Computational Complexity

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1 INTRODUCTION

Much work has been done in recent years to study *hedonic games*, coalition formation games where players express their preferences over those coalitions that contain them. Drèze and Greenberg [9] were the first to propose hedonic games and Bogomolnaia and Jackson [4] and Banerjee et al. [3] formally defined and investigated them. For more background and the rich literature on hedonic games, we refer to the book chapters by Aziz and Savani [2] and Elkind and Rothe [10] and the survey by Woeginger [20].

We focus on *altruistic* hedonic games (AHGs) that, based on the friend-and-enemy encoding of the players' preferences due to Dimitrov et al. [8], were introduced by Nguyen et al. [16]. Schlueter and Goldsmith [18] generalized them to *"super AHGs,"* using ideas of the *"social distance games"* due to Brânzei and Larson [6]. Bullinger and Kober [7] introduced the related notion of *loyalty in hedonic games.* Nguyen et al. [16] defined three degrees of altruism depending on the order in which players take their own or their friends' preferences into account. They chose to model players' utilities by taking the *average* of these friends' valuations in the same coalition. Wiechers and Rothe [19] studied the same three degrees of altruism for *minimum-based* utilities, and Kerkmann and Rothe [15] applied the original model to *coalition formation games* in general. For an overview of various notions of altruism in both cooperative and noncooperative game theory, we refer to the survey by Rothe [17].

We study both average-based and min-based AHGs. For these two classes of games (and for hedonic games in general), many stability notions have been studied, including stability based on single-player deviations (such as Nash stability) or on deviations by groups of players (such as core stability) (see, e.g., [2, 10, 20]).

By contrast, for *popularity* and *strict popularity* we look at entire coalition structures (i.e., partitions of the players into coalitions) and ask—similarly to the notion of (weak) Condorcet winner in voting—whether a (strict) majority of players prefer a given coalition structure to every other coalition structure. Previous literature on popularity in hedonic games is, e.g., due to Aziz et al. [1], Brandt and Bullinger [5], and Kerkmann et al. [13]. We study the complexity of the problem of verifying (strictly) popular coalition structures in AHGs. While strict popularity verification is known to be coNP-complete in all three degrees of min-based AHGs [19] and for so-called selfish-first average-based AHGs [16], its complexity was open for the other two degrees of average-based altruism.

We solve these two missing cases via technically rather involved constructions in Section 3. In addition, in Section 4 we provide the first complexity results for popularity verification in average-based and min-based AHGs, covering for both all three degrees of altruism. We show that the problem in all cases is coNP-complete.

2 PRELIMINARIES

Let $N = \{1, \dots, n\}$ be a set of *n players.* A subset of players is a *coalition.* For any player $i \in N$, $\mathcal{N}^i = \{C \subseteq N \mid i \in C\}$ denotes the set of coalitions containing i. A *coalition structure* is a partition $\Gamma = \{C_1, \dots, C_k\}$ of the players into coalitions (i.e., $\bigcup_{i=1}^{k} C_i = N$ and $C_i \cap C_j = \emptyset$ for $i \neq j$), where the coalition containing player i is denoted by $\Gamma(i)$. The set of all coalition structures for a set of agents N is given by $\mathcal{C}_N$. A *hedonic game* is a pair $(N, \succeq)$, where N is a set of agents, $\succeq = (\succeq_1, \dots, \succeq_n)$ is a profile of preferences, and the preference $\succeq_i$ of any agent $i \in N$ is a complete weak order over $\mathcal{N}^i$. For coalitions $A, B \in \mathcal{N}^i$, player i *weakly prefers* A *to* B if $A \succeq_i B$, and analogously for *(strict) preference* $\succ_i$ and *indifference* $\sim_i$.

When introducing *altruistic hedonic games,* Nguyen et al. [16] used the *friends-and-enemies encoding* by Dimitrov et al. [8]: Each player i partitions the set of other players into a set of friends F_i and a set of enemies E_i, and assigns the friend-oriented value $v_i(A) = n|A \cap F_i| - |A \cap E_i|$ to any coalition $A \in \mathcal{N}^i$. The friendship relations, which are assumed to be mutual, can then be represented

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by a *network of friends*, an undirected graph where two players are connected by an edge if and only if they are friends of each other.

Nguyen et al. [16] introduce altruism into a player i's preference by incorporating the valuations of those friends of i's that are in the same coalition into i's utility, considering the average of these friends' valuations. Wiechers and Rothe [19] vary this model by considering the minimum of those friends' valuations instead. For any coalition $A \in \mathcal{N}^i$, we use the following notations: $\mathrm{avg}_i^F(A) = \sum_{a \in A \cap F_i} \frac{v_a(A)}{|A \cap F_i|}$; $\mathrm{avg}_i^{F+}(A) = \sum_{a \in (A \cap F_i) \cup \{i\}} \frac{v_a(A)}{|(A \cap F_i) \cup \{i\}|}$; $\min_i^F(A) = \min_{a \in A \cap F_i} v_a(A)$; $\min_i^{F+}(A) = \min_{a \in (A \cap F_i) \cup \{i\}} v_a(A)$, where the minimum of the empty set is defined as zero. We also define these values for coalition structures $\Gamma \in \mathcal{C}_N$ by the value of the coalition that agent i belongs to, e.g., $\mathrm{avg}_i^F(\Gamma) = \mathrm{avg}_i^F(\Gamma(i))$.

Nguyen et al. [16] introduced their three *degrees of altruism* by defining, for a constant $M \geq n^5$ and any $A, B \in \mathcal{N}^i$, player i's

- *selfish-first (SF) preference* by $A \succeq_i^{SF} B \Leftrightarrow u_i^{SF}(A) \geq u_i^{SF}(B)$, with the SF utility $u_i^{SF}(A) = M \cdot v_i(A) + \mathrm{avg}_i^F(A)$;
- *equal-treatment (EQ) preference* by $A \succeq_i^{EQ} B \Leftrightarrow u_i^{EQ}(A) \geq u_i^{EQ}(B)$, with the EQ utility $u_i^{EQ}(A) = \mathrm{avg}_i^{F+}(A)$; and
- *altruistic-treatment (AL) preference* by $A \succeq_i^{AL} B \Leftrightarrow u_i^{AL}(A) \geq u_i^{AL}(B)$, with the AL utility $u_i^{AL}(A) = v_i(A) + M \cdot \mathrm{avg}_i^F(A)$.

The min-based altruistic preferences, denoted by $\succeq^{minSF}$, $\succeq^{minEQ}$, and $\succeq^{minAL}$, are defined analogously, using the minimum instead of the average. A pair $(N, \succeq)$, where $\succeq$ is a profile of preferences defined by one of the average-based degrees of altruism, is called an *altruistic hedonic game* (AHG) with *average-based altruistic preferences* $\succeq$. A game $(N, \succeq^{\min})$ with *min-based altruistic preferences* $\succeq^{\min}$ is said to be a *min-based altruistic hedonic game* (MBAHG).

We now define popularity, which is based on the pairwise comparison of coalition structures. For a hedonic game $(N, \succeq)$ and two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, let $\#_{\Gamma > \Delta} = |\{i \in N \mid \Gamma \succ_i \Delta\}|$ be the number of players that prefer Γ to Δ. A coalition structure $\Gamma \in \mathcal{C}_N$ is *popular* (respectively, *strictly popular*) if, for every other coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$, it holds that $\#_{\Gamma > \Delta} \geq \#_{\Delta > \Gamma}$ (respectively, $\#_{\Gamma > \Delta} > \#_{\Delta > \Gamma}$). Define the verification problem P-Veri: Given a hedonic game $(N, \succeq)$ and a coalition structure Γ, is Γ popular in $(N, \succeq)$? Relatedly, define the existence problem P-Exi: Given a hedonic game $(N, \succeq)$, does there exist a popular coalition structure in $(N, \succeq)$? The strict variants of the problems, SP-Veri and SP-Exi, are defined analogously. It is easy to see that all these verification problems are in coNP. To show their coNP-hardness, we reduce from the complement of a restricted variant of the exact cover by 3-sets problem that is denoted by RX3C and defined as follows: Given a set $B = \{1, \dots, 3k\}$ (for some integer $k \geq 2$) and a collection $\mathscr{S} = \{S_1, \dots, S_{3k}\}$ of 3-element subsets of B, where each element of B occurs in exactly three sets in $\mathscr{S}$, does there exist an exact cover of B in $\mathscr{S}$, i.e., a subset $\mathscr{S}' \subseteq \mathscr{S}$ of size k such that every element of B occurs in exactly one set in $\mathscr{S}'$? RX3C is still NP-complete [11, 12]. In all (omitted) proofs of the coNP-hardness of (strict) popularity verification, given an RX3C instance $(B, \mathscr{S})$, we construct an instance of our problem, i.e., a hedonic game $(N, \succeq)$ represented by its network of friends and a coalition structure Γ. We then show that Γ is not (strictly) popular under the considered model if and only if there exists an exact cover of B in $\mathscr{S}$.

3 STRICT POPULARITY IN AHGS

While Wiechers and Rothe [19] showed that SP-Veri is coNP-complete for all three degrees of altruism in MBAHGs, Nguyen et al. [16] showed the same result only for SF AHGs. We solve the two missing cases (i.e., for EQ and AL).

Theorem 1. *SP-Veri is* coNP*-complete for EQ and AL AHGs.*

In the proof of Theorem 1, we use a tie between two most popular coalition structures to show that one of them is not strictly popular. We can use the same construction while not giving any coalition structure as a part of the instance to show the hardness of SP-Exi.

Corollary 2. *SP-Exi is* coNP*-hard for EQ and AL AHGs.*

4 POPULARITY IN AHGS AND MBAHGS

Now, we provide the first complexity results for P-Veri in AHGs and MBAHGs, and we cover for both all three degrees of altruism. As mentioned earlier, Nguyen et al. [16, Thm. 12] showed that SP-Veri is coNP-complete for SF AHGs and Wiechers and Rothe [19, Thm. 4] showed the same result for SF MBAHGs. We modify their proofs to establish the same results for P-Veri.

Theorem 3. *P-Veri is* coNP*-complete for SF AHGs and SF MBAHGs.*

Since the altruistic tie-breaker is never used in our construction of Theorem 3 (there never occur indifferences that are broken altruistically), the proof also holds for friend-oriented preferences and P-Veri is coNP-complete for friend-oriented hedonic games.

With an adaptation of our construction in the proof of Theorem 1, we further get results for P-Veri in EQ and AL AHGs.

Theorem 4. *P-Veri is* coNP*-complete for EQ and AL AHGs.*

The last result for P-Veri is again inspired by a proof by Wiechers and Rothe [19, Thm. 4].

Theorem 5. *P-Veri is* coNP*-complete for EQ and AL MBAHGs.*

Finally, we turn to P-Exi. Note that we cannot simply modify the proofs of the preceding theorems in order to show the hardness of P-Exi (similarly to how we used Theorem 1 to obtain Corollary 2) because a tie between two most popular coalition structures would not suffice to show the nonexistence of a popular coalition structure. However, for both AHGs and MBAHGs and in all three degrees of altruism, there exist examples where no popular coalition structures exist, and we suspect that P-Exi is hard for all considered models.

5 CONCLUSIONS

We have solved the two remaining open problems regarding the complexity of strict popularity verification in AHGs, namely for equal treatment and altruistic treatment (Theorem 1). The proofs of these results required new ideas and are technically demanding.

In addition, we have provided the first complexity results for popularity verification in AHGs and MBAHGs, covering for both all three degrees of altruism (Theorems 3, 4, and 5). Hence, the complexity of popularity verification and strict popularity verification is now settled in AHGs and MBAHGs; the picture is complete.

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