One-Sided Matching Markets with Endowments: Equilibria and Algorithms

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ABSTRACT

The Arrow-Debreu extension of the classic Hylland-Zeckhauser scheme [27] for a one-sided matching market – called ADHZ in this paper – has natural applications but has instances which do not admit equilibria. By introducing approximation, we define the *ϵ-approximate ADHZ model*, and we give the following results.

(1) Existence of equilibrium under linear utility functions. We prove that the equilibrium satisfies Pareto optimality, approximate envy-freeness, and approximate weak core stability.
(2) A combinatorial polynomial time algorithm for an ϵ- approximate ADHZ equilibrium for the case of dichotomous, and more generally bi-valued, utilities.
(3) An instance of ADHZ, with dichotomous utilities and a strongly connected demand graph, which does not admit an equilibrium.
(4) A rational convex program for HZ under dichotomous utilities; a combinatorial polynomial time algorithm for this case was given in [35].

The ϵ-approximate ADHZ model fills a void in the space of general mechanisms for one-sided matching markets; see details in the paper.

KEYWORDS

Arrow-Debreu model; Hylland-Zeckhauser scheme; one-sided matching markets; rational convex program; dichotomous utilities

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1 INTRODUCTION

In this paper, we define an Arrow-Debreu extension of the classic Hylland-Zeckhauser (HZ) mechanism [27] for one-sided matching markets. This fills a void in the space of general[1] mechanisms for one-sided matching markets. Such mechanisms are classified according to two criteria: whether they use cardinal or ordinal utility functions, and whether they are in the Fisher or Arrow-Debreu[2] setting. The other three possibilities are covered as follows: (cardinal, Fisher) by the Hylland-Zeckhauser scheme [27]; (ordinal, Fisher) by Probabilistic Serial [7] and Random Priority [31]; and (ordinal, Arrow-Debreu) by Top Trading Cycles [33]. Details about these mechanisms are given in Section 1.1.

The two ways of expressing utilities of goods – ordinal and cardinal – have their own pros and cons and neither dominates the other. On the one hand, the former is easier to elicit from agents and on the other, the latter is far more expressive, enabling an agent to not only report if she prefers good A to good B but also by how much. [1] exploits this greater expressivity of cardinal utilities to give mechanisms for school choice which are superior to ordinal-utility-based mechanisms.

The following example illustrates the advantage of cardinal vs ordinal utilities. The instance has three types of goods, T_1, T_2, T_3, and these goods are present in the proportion of $(1\%,\ 97\%,\ 2\%)$. Based on their utility functions, the agents are partitioned into two sets A_1 and A_2, where A_1 constitute 1% of the agents and A_2, 99%. The utility functions of agents in A_1 and A_2 for the three types of goods are $(1,\ \epsilon,\ 0)$ and $(1,\ 1-\epsilon,\ 0)$, respectively, for a small number $\epsilon > 0$. The main point is that whereas agents in A_2 marginally prefer T_1 to T_2, those in A_1 overwhelmingly prefer T_1 to T_2.

Clearly, the ordinal utilities of all agents in $A_1 \cup A_2$ are the same. Therefore, a mechanism based on such utilities will not be able to make a distinction between the two types of agents. On the other hand, the HZ mechanism, which uses cardinal utilities, will fix the price of goods in T_3 to be zero and those in T_1 and T_2 appropriately so that by-and-large the bundles of A_1 and A_2 consist of goods from T_1 and T_2, respectively.

The Arrow-Debreu setting of one-sided matching markets has several natural applications beyond the Fisher setting, e.g., allocating students to rooms in a dorm for the next academic year, assuming their current room is their initial endowment. Similarly, school choice, when a student's initial endowment is a seat in a school which they already have. The issue of obtaining such an extension of the HZ mechanism, called *ADHZ* in this paper, was studied by Hylland and Zeckhauser. However, this culminated in an example which inherently does not admit an equilibrium [27].

One recourse to this was given by Echenique, Miralles and Zhang [17] via their notion of an *α-slack Walrasian equilibrium*: This is a hybrid between the Fisher and Arrow-Debreu settings. Agents have initial endowments of goods and for a fixed $\alpha \in (0, 1]$, the budget of each agent, for given prices of goods, is $\alpha + (1 - \alpha) \cdot m$, where m is the value for her initial endowment; the agent spends this budget to obtain an optimal bundle of goods. Via a non-trivial proof, using the Kakutani Fixed Point Theorem, they proved that an α-slack equilibrium always exists.

[1] As opposed to mechanisms for specific one-sided matching markets.
[2] This is also called the Walrasian or exchange setting.

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In this paper, we show that we can remain with a pure Arrow-Debreu setting provided we relax the notion of equilibrium to an *approximate equilibrium*, a notion that has become common-place in the study of equilibria within computer science. We call this the *ϵ-approximate ADHZ model.* For this model, we give the following results.

We prove the existence of an equilibrium for arbitrary cardinal utility functions, using the fact from [17] that an α-slack equilibrium always exists for $\alpha > 0$.

We prove that the equilibrium in our ϵ-approximate ADHZ model is Pareto optimal, approximately envy free, and approximately weak core stable. In contrast, the allocation found by an HZ equilibrium is Pareto optimal and envy-free [27].

For an Arrow-Debreu market under linear utilities, Gale [20] defined a *demand graph:* a directed graph on agents with an edge (i, j) if agent i likes a good that agent j has in her initial endowment. He proved that a sufficiency condition for the existence of equilibrium is that this graph be strongly connected. The following question arises naturally: Is this a sufficiency condition for equilibrium existence in ADHZ as well? We provide a negative answer to this question. We give an instance of ADHZ whose demand graph is not only strongly connected but also has dichotomous utilities, and yet it does not admit an equilibrium.

For the case of dichotomous utilities, we give a combinatorial polynomial-time algorithm for computing an equilibrium for our ϵ-approximate ADHZ model. This result also extends to the case of bi-valued utilities, i.e., each agent's utility for individual goods comes from a set of cardinality two, though the sets may be different for different agents. These utilities are well-studied (see, e.g., [5, 8, 16, 21, 35]), mainly due to their significance in practical applications. For example, it might be simpler for agents to answer whether their desire for a good is "high" or "low" with numerical values. We note that the polynomial-time algorithm of [14, 15] for Arrow-Debreu markets under linear utilities, as well as the recent strongly polynomial-time algorithm for the same problem [22] are quite complicated, in particular because they resort to the use of balanced flows, which uses the l_2 norm. In contrast, we managed to avoid the use of l_2 norm and hence we obtain a simple algorithm.

A corollary of the last result is that the equilibrium of the dichotomous utilities case of the ϵ-approximate ADHZ model involves only rational numbers. In contrast we give an instance of ADHZ whose unique equilibrium has irrational prices and allocations. This instance is obtained by appropriately modifying an instance for the HZ model, given in [35], whose (unique) equilibrium has irrational prices and allocations. This led us to ask if there is a *rational convex program (RCP)* that captures the equilibrium in this setting.

An RCP, defined in [34], is a nonlinear convex program all of whose parameters are rational numbers and which always admits a rational solution in which the denominators are polynomially bounded. The quintessential such program is the Eisenberg-Gale convex program [19] for a linear Fisher market. The significance of finding such a program for a problem is that it directly implies existence of a polynomial time algorithm for the underlying problem, since using the ellipsoid algorithm and Diophantine approximation [23, 28], an RCP can be solved exactly in polynomial time. As a result, it gives practitioners a direct way to compute a solution using general-purpose convex programming solvers. Although we were not able to answer this question, we did find an RCP for HZ equilibrium under dichotomous utilities. A combinatorial polynomial time algorithm for this case was given in [35].

1.1 Related Results

Matching markets have found many applications in various multi-agent settings, e.g., see the recent works [3, 4, 13].

We start by stating the properties of mechanisms for one-sided matching markets listed in the Introduction. Random Priority [31] is strategyproof though not efficient or envy-free; Probabilistic Serial [7] is efficient and envy-free but not strategyproof; and Top Trading Cycles [33] is efficient, strategyproof and core-stable.

Recently, [35] undertook a comprehensive study of the computational complexity of the HZ scheme. They gave a combinatorial polynomial time algorithm for dichotomous utilities and an example which has only irrational equilibria; as a consequence, this problem is not in PPAD. They showed that the problem of computing an exact HZ equilibrium is in the class FIXP and the problem of computing an approximate equilibrium is in PPAD. Very recently, [10] showed that computing an approximate HZ equilibrium is PPAD-hard. In order to deal with the computational intractability of HZ, a Nash-bargaining-based mechanism was proposed in [26].

The study of the dichotomous case of matching markets was initiated by Bogomolnaia and Moulin [8]. They studied a two-sided matching market and they called it an "important special case of the bilateral matching problem." Using the Gallai-Edmonds decomposition of a bipartite graph, they gave a mechanism that is Pareto optimal and group strategyproof. They also gave a number of applications of their setting, some of which are natural applications of one-sided markets as well, e.g., housemates distributing rooms, having different features, in a house. As in the HZ scheme, their mechanism also outputs a doubly-stochastic matrix whose entries represent probability shares of allocations. However, they give another interesting interpretation of this matrix. They say, "Time sharing is the simplest way to deal fairly with indivisibilities of matching markets: think of a set of workers sharing their time among a set of employers." Roth, Sönmez and Ünver [32] extended these results to general graph matching under dichotomous utilities; this setting is applicable to the kidney exchange marketplace.

An interesting recent paper [2] defines the notion of a random partial improvement mechanism for a one-sided matching market. This mechanism truthfully elicits the cardinal preferences of the agents and outputs a distribution over matchings that approximates every agent's utility in the Nash bargaining solution.

Several researchers have proposed Hylland-Zeckhauser-type mechanisms for a number of applications, for instance [9, 25, 29, 30]. The basic scheme has also been generalized in several different directions, including two-sided matching markets, adding quantitative constraints, and to the setting in which agents have initial endowments of goods instead of money, see [17, 18].

2 THE HYLLAND-ZECKHAUSER MECHANISM

The Hylland-Zeckhauser (HZ) mechanism can be viewed as a marriage between a fractional perfect matching and a linear Fisher market, which consists of a set A of agents and a set G of goods. Each agent i comes to the market with a budget b_i and has utilities

$u_{ij} \geq 0$ for each good j. In the case of linear utilities, agent i's utility from allocation $(x_{ij})_{j \in G}$ is $\sum_j u_{ij} x_{ij}$. By fixing the units for each good, we may assume without loss of generality that there is a unit of each good in the market.

Definition 1. A *Fisher equilibrium* is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and *prices* $(p_j)_{j \in G}$ with the following properties.

(1) Each agent i spends at most their budget, i.e., $\sum_{j \in G} p_j x_{ij} \leq b_i$.
(2) Each agent i gets an *optimal bundle*, i.e., utility maximizing bundle at prices p. Formally:

$$\sum_{j \in G} u_{ij} x_{ij} = \max \left\{ \sum_{j \in G} u_{ij} y_j \;\middle|\; y \in \mathbb{R}^G_{\geq 0}, \sum_{j \in G} p_j y_j \leq b_i \right\}.$$

(3) The market clears, i.e., each good with positive price is fully allocated to the agents.

The set of equilibria of a linear Fisher market corresponds to the set of optimal solutions of the Eisenberg-Gale convex program [19], which is a *rational convex program* (RCP) and in fact it motivated the definition of this concept [34].

Fisher equilibria satisfy various nice properties, including equal-type envy-freeness and Pareto optimality.

Definition 2 (Envy-freeness and Pareto optimality). An allocation is *envy-free* if for any two agents $i, i' \in A$, agent i weakly prefers their allocation than those that i' gets, i.e., $\sum_{j \in G} u_{ij} x_{ij} \geq \sum_{j \in G} u_{ij} x_{i'j}$. It is *equal-type envy-free* if the above holds for any two agents with identical budgets.

An allocation x *weakly dominates* another allocation x' if no agent prefers x' to x. It *strongly* dominates x' if it weakly dominates it and some agent prefers x to x'. An allocation x is *Pareto efficient* or *Pareto optimal* if there is no other allocation x' which strongly dominates it.

Definition 3. A *one-sided matching market* consists of a set A of *agents* and a set G of *goods*. Each agent has preferences over goods, expressed either using cardinal or ordinal utility functions. An *allocation* is a perfect matching of agents to goods. The goal of the market is to find an allocation so that the underlying mechanism has some desirable game-theoretic properties.

The HZ mechanism uses cardinal utility functions, in which each good is rendered divisible by viewing it as one unit of *probability shares*. An HZ equilibrium is defined as follows.

Definition 4. A *Hylland-Zeckhauser (HZ) equilibrium* is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and *prices* $(p_j)_{j \in G}$ with the following properties.

(1) x is a fractional perfect matching, i.e., $\sum_{j \in G} x_{ij} = 1$ for all i and $\sum_{i \in A} x_{ij} = 1$ for all j.
(2) Each agent i spends at most their budget, i.e., $\sum_{j \in G} p_j x_{ij} \leq b_i$ (usually $b_i = 1$).
(3) Each agent i gets an *optimal bundle*, which is defined to be a cheapest utility maximizing bundle, i.e., $\sum_{j \in G} u_{ij} x_{ij} = \max \left\{ \sum_j u_{ij} y_j \;\middle|\; \sum_j y_j = 1;\ \sum_j p_j y_j \leq b_i \right\}$ and $\sum_{j \in G} p_j x_{ij} = \min \left\{ \sum_j p_j y_j \;\middle|\; \sum_j y_j = 1; \sum_j u_{ij} y_j \geq \sum_j u_{ij} x_{ij} \right\}$.

Like Fisher equilibria, HZ equilibria are Pareto optimal and envy-free (assuming unit budgets).[3] The allocation x found by the HZ mechanism is a fractional perfect matching or a doubly-stochastic matrix. In order to get an integral perfect matching from x, a lottery can be carried out using the Theorem of Birkhoff [6] and von Neumann [36]. It states that any doubly-stochastic matrix can be written as a convex combination of integral perfect matchings; moreover, this decomposition can be found efficiently. Picking a perfect matching according to the discrete probability distribution determined by this convex combination yields the resulting allocation in the HZ mechanism.

3 THE ϵ-APPROXIMATE ADHZ MODEL

In this paper we are interested in an exchange version of the HZ mechanism. Before defining it, we introduce the Arrow-Debreu (exchange) market under linear utility functions, which consists of a set A of *agents* and a set G of *goods*. Each agent i comes to the market with an *endowment* $e_{ij} \geq 0$ of each good j and also has a utility $u_{ij} \geq 0$. Each good j must be fully owned by the agents, i.e., $\sum_{i \in A} e_{ij} = 1$ for all $j \in G$.

Definition 5. An *Arrow-Debreu (AD) equilibrium* for a given AD market is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and *prices* $(p_j)_{j \in G}$ with the following properties.

(1) Each agent spends at most the budget earned from the endowment, i.e., $\sum_j p_j x_{ij} \leq b_i := \sum_j p_j e_{ij}$.
(2) Each agent i gets an *optimal bundle*, i.e., $\sum_{j \in G} u_{ij} x_{ij} = \max \left\{ \sum_{j \in G} u_{ij} y_j \;\middle|\; y \in \mathbb{R}^G_{\geq 0}, \sum_{j \in G} p_j y_j \leq b_i \right\}$.
(3) The market clears, i.e., each good with positive price is fully allocated to the agents.

The AD model generalizes Fisher model in the sense that any Fisher market can be easily transformed into an AD market by giving each agent a fixed proportion of every good. Clearly, AD equilibria satisfy the condition of individual rationality, defined below, since every agent could always buy back their endowment.

Definition 6. An allocation in an AD market is *individually rational* if for every agent i we have $\sum_j u_{ij} x_{ij} \geq \sum_j u_{ij} e_{ij}$, i.e., no agent loses utility by participating in the market.

However, individual rationality fundamentally clashes with envy-freeness. Consider a market consisting of two agents each owning a distinct good. Assume that both agents prefer the good of agent 2 over the good of agent 1, then in any allocation either agent 1 envies agent 2 or agent 2's individual rationality is violated. For this reason we primarily consider a version of equal-type envy-freeness in exchange markets, which demands envy-freeness only for agents with the same initial endowment.

AD equilibria do not always exist. However, there is a simple necessary and sufficient condition for their existence based on *strong connectivity of demand graph*, due to Gale [20]. An RCP for this problem was given by Devanur, Garg and Végh [11].

[3] Pareto optimality for HZ equilibria requires that each agent receives a *cheapest* utility maximizing bundle. If this condition is dropped, we get counter-examples to Pareto optimality: Consider an instance with two agents a_1 and a_2, and two goods g_1 and g_2 with $u_{11} = u_{21} = u_{22} = 1; u_{12} = 0$. The prices $(2, 0)$ together with the allocation $x_{11} = x_{12} = x_{21} = x_{22} = 0.5$ are optimal bundles, though not cheapest. The utilities in this equilibrium are 0.5 for agent a_1 and 1 for agent a_2. However, there is another HZ equilibrium with prices $(1, p)$, for any $p \in [0, 1]$ with utility 1 for both agents.

We now turn to the extension of the HZ mechanism to exchange markets. In the *ADHZ market*, we have a set A of *agents* and a set G of *goods* with $|A| = |G| = n$. Each agent i comes with an *endowment* $e_{ij} \geq 0$ of each good j and utilities $u_{ij} \geq 0$. The endowment vector e is a fractional perfect matching.

Definition 7. An *ADHZ equilibrium* for a given ADHZ market is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and *prices* $(p_j)_{j \in G}$ with the following properties.

(1) x is a fractional perfect matching, i.e., $\sum_{j \in G} x_{ij} = 1$ for all i and $\sum_{i \in A} x_{ij} = 1$ for all j.
(2) Each agent spends at most the budget earned from the endowment, i.e., $\sum_j p_j x_{ij} \leq b_i := \sum_j p_j e_{ij}$.
(3) Each agent i gets an *optimal bundle*, which is defined to be a cheapest utility maximizing bundle, i.e., $\sum_{j \in G} u_{ij} x_{ij} = \max\left\{\sum_j u_{ij} y_j \mid \sum_j y_j = 1;\ \sum_j p_j y_j \leq b_i\right\}$ and $\sum_{j \in G} p_j x_{ij} = \min\left\{\sum_{j \in G} p_j y_j \mid \sum_j y_j = 1;\ \sum_{j \in G} u_{ij} y_j \geq \sum_{j \in G} u_{ij} x_{ij}\right\}$.

Theorem 8. *ADHZ equilibria are Pareto optimal, individually rational, and equal-type envy-free.*

Proof. Pareto optimality follows from the fact that any ADHZ equilibrium is an HZ equilibrium with certain budgets b. Since any HZ equilibrium is Pareto optimal, we get the same for ADHZ.

Note that the budget of any agent is always enough to buy back their initial endowment. Since they get an optimal bundle, they must get something which they value at least as high as their initial endowment. Thus individual rationality is guaranteed.

If two agents, say 1 and 2, have the same endowment, then their budget will be the same and so agent 1 will never value the 2's bundle higher than their own. Thus ADHZ equilibria are equal-type envy-free. □

In addition, ADHZ equilibria also satisfy the following notion of core-stability.

Definition 9. An allocation x in an ADHZ market is *weakly core-stable* if for any subsets $A' \subseteq A$ and $G' \subseteq G$, there does not exist an allocation $x' \in \mathbb{R}_{\geq 0}^{A' \times G'}$ such that

- x' allocates at most one unit of goods to every agent in A',
- every good $j \in G'$ is allocated at most to the extent of the endowments of the agents in A', i.e. $\sum_{i \in A'} x'_{ij} \leq \sum_{i \in A'} e_{ij}$, and
- every agent in A' receives strictly better utility in x' than in x.

Theorem 10. *ADHZ equilibria are weakly core-stable.*

Proof. Let (x, p) be some ADHZ equilibrium. For the sake of a contradiction, assume that there are $A' \subseteq A$, $G' \subseteq G$ and $x' \in \mathbb{R}_{\geq 0}^{A' \times G'}$ as excluded by the definition of weak core-stability. Now consider the total money spent "along allocation x'", i.e., the quantity $\sum_{i \in A'} \sum_{j \in G'} p_j x'_{ij}$.

On the one hand we know that only the endowment of the agents in A' is allocated by x'. Thus

$$\sum_{i \in A'} \sum_{j \in G'} p_j x'_{ij} \leq \sum_{i \in A'} \sum_{j \in G'} p_j e_{ij}.$$

On the other hand, every agent i receives strictly better utility from x' than from x. But since agents buy optimal bundles in (x, p), this implies that the bundles in x' must be worth more than their budget, i.e.,

$$\sum_{j \in G'} p_j x'_{ij} > \sum_{j \in G} p_j e_{ij} \geq \sum_{j \in G'} p_j e_{ij}.$$

Summing this inequality over all $i \in A'$ yields a contradiction to the previous inequality. □

Like in the case of HZ, equilibrium prices in ADHZ are invariant under the operation of *scaling* the difference of prices from 1, as shown in the following lemma.

Lemma 11. *Suppose p be an equilibrium price vector. For any $r > 0$, let p' be such that $p'_j - 1 = r(p_j - 1)$ for all $j \in G$. Then p' is also an equilibrium price vector.*

Proof. Let x be an equilibrium allocation at prices p. For any agent i, we have $\sum_{j \in G} x_{ij} p_j \leq \sum_{j \in G} e_{ij} p_j$. We show that the pair (x, p') is also an equilibrium.

Since (x, p) is an equilibrium, for each $i \in A$, we have

$$\sum_{j \in G} u_{ij} x_{ij} = \max\left\{\sum_{j \in G} u_{ij} y_j \mid y \in \mathbb{R}_{\geq 0}^{G},\ \sum_{j \in G} y_j = 1,\ \sum_{j \in G} y_j p_j \leq \sum_{j \in G} e_{ij} p_j\right\}.$$

Replacing p_j by $(p'_j - 1)/r + 1$ for all $j \in G$, we get:

$$\sum_{j \in G} u_{ij} x_{ij} = \max\left\{\sum_{j \in G} u_{ij} y_j \mid y \in \mathbb{R}_{\geq 0}^{G},\ \sum_{j \in G} y_j = 1,\right.$$
$$\left.\sum_{j \in G} y_j \left(\frac{p'_j - 1}{r} + 1\right) \leq \sum_{j \in G} e_{ij} \left(\frac{p'_j - 1}{r} + 1\right)\right\}.$$

Simplifying the above using $\sum_{j \in G} e_{ij} = 1$ and $\sum_{j \in G} y_j = 1$ for all $i \in A$, we get:

$$\sum_{j \in G} u_{ij} x_{ij} = \max\left\{\sum_{j \in G} u_{ij} y_j \mid y \in \mathbb{R}_{\geq 0}^{G},\ \sum_{j \in G} y_j = 1,\ \sum_{j \in G} y_j p'_j \leq \sum_{j \in G} e_{ij} p'_j\right\}.$$

The above implies that x gives each agent an optimal bundle at prices p'. This, together with the fact that x is a fractional perfect matching, shows that (x, p') is also an equilibrium. □

Unlike HZ, which always admits an equilibrium, ADHZ has instances which do not admit an equilibrium, as observed by Hylland and Zeckhauser [27]. Below we give a counterexample in which the demand graph is strongly connected and utilities are dichotomous.

Proposition 12. *The ADHZ market with dichotomous utilities in Figure 1 does not admit an equilibrium.*

Proof. Assume there is an equilibrium (x, p) in this market. Further, using Lemma 11, we can assume that the minimum price is zero at p. This implies that no agent will buy a zero utility good at a positive price.

Each agent buys a total of one unit of goods and s is the only agent having positive utility for goods a and b. Therefore, at least one of these goods is not fully sold to s and must be sold to an agent deriving zero utility from it. Therefore this good must have zero price. Without loss of generality, assume $p_a = 0$. Since a has no budget and c and d are desired only by a, $p_c = p_d = 0$, otherwise c and d cannot be sold. For the same reason, $p_e = 0$. Now observe

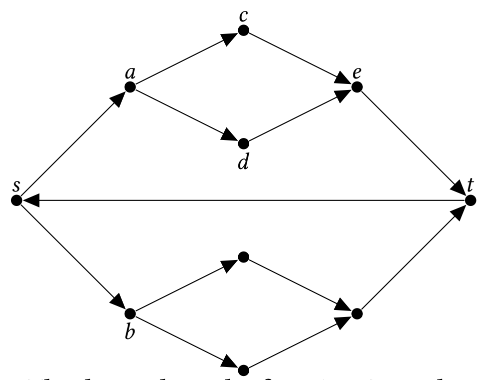

Figure 1: The demand graph of an ADHZ market with dichotomous utilities and no equilibrium. Each node represents an agent as well as the good possessed by this agent in her initial endowment. An arrow from i to j represents $u_{ij} = 1$; the rest of the edges have utility 0.

that both agents c and d have a utility 1 edge to a good of price zero, namely e. Therefore, the optimal bundle of both c and d is e. But then e would have to be matched twice which is a contradiction. □

Even if ADHZ equilibria *do* exist, computing them is at least as hard as computing HZ equilibria. This follows from the following reduction.

Proposition 13. *Consider an HZ market with unit budgets. Define an ADHZ market by giving every agent as endowment an equal amount of every good. Then every HZ equilibrium in which the prices sum up to n is an ADHZ equilibrium and every ADHZ equilibrium yields an HZ equilibrium by rescaling all prices by* $n/\sum_{j\in G} p_j$.

[35] gave an instance of HZ with four agents and four goods which has one equilibrium in which all agents fully spend their budgets, and allocations and prices are irrational. Since this example satisfies the conditions of Proposition 13, we get that the modification of the example of [35], as stated in the Proposition, is an instance for ADHZ having only irrational equilibria.

3.1 Existence and Properties of ϵ-Approximate ADHZ Equilibria

Since ADHZ equilibria do not always exist, we study the following approximate equilibrium notion instead.

Definition 14. An *ϵ-approximate ADHZ equilibrium* is an HZ equilibrium (x, p) for a budget vector b with

$$(1-\epsilon)\sum_{j\in G} p_j e_{ij} \le b_i \le \epsilon + \sum_{j\in G} p_j e_{ij} \quad \text{for all } i \in A \ .$$

We also require that if two agents have the same endowment, then their budget should also be the same.

The additive error term in the upper bound is necessary since otherwise the counterexample from Proposition 12 still works. On the other hand, the multiplicative lower bound is useful to get approximate individual rationality. However, one can always find approximate equilibria in which the sum of prices is bounded by n using Lemma 11, so we also get

$$\sum_{j\in G} p_j e_{ij} - \epsilon' \le b_i \le \sum_{j\in G} p_j e_{ij} + \epsilon' \quad \text{for } \epsilon' := n\epsilon.$$

This implies that we can equivalently define the above notion with additive error terms on both upper and lower bounds.

In our notion of approximate equilibrium, we do not relax the fractional perfect matching constraints or the optimum bundle condition. We only allow the budgets of agents to be slightly different from the money they would normally obtain in an ADHZ market. Hence the step of randomly rounding the equilibrium allocation to an integral perfect matching is the same as in the HZ scheme.

Theorem 15. *Any ϵ-approximate ADHZ equilibrium is Pareto optimal, ϵ-approximately individually rational, equal-type envy-free.*

Proof. Pareto optimality follows just as for the non-approximate ADHZ setting from the fact that an ϵ-approximate ADHZ equilibrium is first and foremost an HZ equilibrium. For approximate individual rationality note that every agent gets a budget of at least $(1-\epsilon)$ times the cost of their endowment. Hence their utility can decrease by at most a factor of $(1-\epsilon)$. Equal-type envy-freeness follows immediately from the condition that agents with the same endowment have the same budget. □

One can also define a suitably ϵ-approximate notion of weak core-stability, where instead of demanding that every agent strictly improves in the seceding coalition, we instead require that every agent improves by a factor of more than $\frac{1}{1-\epsilon}$.

Theorem 16. *Any ϵ-approximate ADHZ equilibrium is ϵ- approximately weak-core stable.*

Proof. Let (x, p) be an ϵ-approximate ADHZ equilibrium for some budget vector b. Then in order for some other allocation x' to improve agent i's utility by a factor of more than $\frac{1}{1-\epsilon}$, i must spend more than $\frac{b_i}{1-\epsilon}$. But note that $\frac{b_i}{1-\epsilon} \ge \sum_{j\in G} p_j e_{ij}$. From here the proof is identical to that of Theorem 10. □

While approximate equilibrium notions are more amenable to computation, they generally do not lend themselves well to existence proofs. However, our notion of ϵ-approximate ADHZ equilibrium is a slight relaxation of the notion of an α-slack equilibrium introduced in [17].

Definition 17. An *α-slack ADHZ equilibrium* for $\alpha \in (0, 1]$ is an HZ equilibrium (x, p) for a budget vector b in which $b_i = \alpha + (1-\alpha)\sum_{j\in G} p_j e_{ij}$ for all $i \in A$.

Theorem 18 (Theorem 2 in [17]). *In any ADHZ market, α-slack equilibria always exist if $\alpha > 0$.*

Note that any α-slack equilibrium is automatically also an α-approximate equilibrium. Thus we get:

Theorem 19. *In any ADHZ market, ϵ-approximate equilibria always exist if $\epsilon > 0$.*

4 ALGORITHM FOR ϵ-APPROXIMATE ADHZ UNDER DICHOTOMOUS UTILITIES

Before we can tackle the ADHZ setting, let us first give an algorithm that can compute HZ equilibria with non-uniform budgets. This is an extension of the algorithm presented in [35]. In the following, fix some HZ market consisting of n agents and goods with $u_{ij} \in \{0,1\}$ for all $i \in A$ and $j \in G$. If $u_{ij} = 1$, we will say that i *likes* j (and *dislikes* otherwise). We assume that every agent likes at least one good. [4]

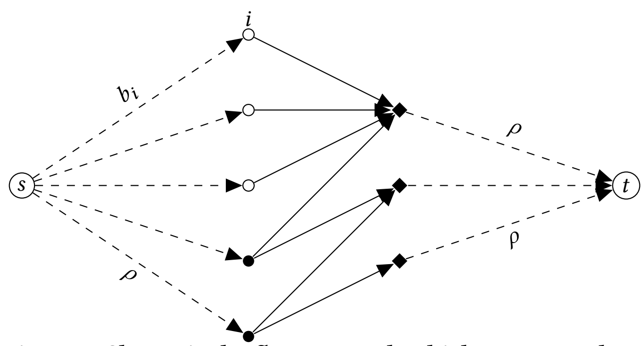

Figure 2: Shown is the flow network which corresponds to finding an equilibrium allocation in price class ρ. Filled circles represent agents in $A(\rho)$ with $b_i < \rho$, empty circles are agents in $A(\rho)$ with $b_i \geq \rho$, and diamond vertices are goods in $G(\rho)$. The contiguous edges represent all utility 1 edges and have infinity capacity (utility 0 edges are not part of the network). Dashed edges to empty circle vertices i have capacity b_i whereas the other dashed edges have capacity ρ.

Lemma 20. *Let $(p_j)_{j \in G}$ be non-negative prices. For any $\rho \geq 0$, let $G(\rho)$ be the goods which are sold at price ρ and let $A(\rho)$ be those agents for which the cheapest price of any liked good is ρ. Assume that*

- *there is a matching in the utility 1 edges on $A(0) \cup G(0)$ which covers all agents in $A(0)$ and*
- *if $\rho > 0$ is equal to the price of some good, then the flow network shown in Figure 2 has a maximum flow of size $\rho|G(\rho)|$.*

Then we can find a fractional perfect matching x which makes (x, p) an HZ equilibrium in polynomial time.

Proof. Allocate every agent in $A(0)$ to some good in $G(0)$ according to the matching which exists by assumption. Let $\rho > 0$, be the price of some good. Then we compute the maximum flow $f^{(\rho)}$ in the flow network from Figure 2 and allocate $x_{ij} = f^{(\rho)}_{i,j}/\rho$ for all $i \in A(\rho)$ and $j \in G(\rho)$. Lastly, extend x to a fractional perfect matching by matching the remaining capacity of the agents to the remaining capacity of goods in $G(0)$.

Clearly, no agent exceeds their budget. To see that this yields an HZ equilibrium, note that every agent only spends money on cheapest liked goods and if they do not get allocated entirely to liked goods, then they additionally spend all of their budget. This ensures that every agent gets an optimum bundle. □

[4] Any HZ equilibrium (x, p) for the utilities u_{ij} is also an equilibrium for $\tilde{u}_{ij}$ where $\tilde{u}_{ij} = a_i$ if $u_{ij} = 0$ and b_i if $u_{ij} = 1$ for for all agents i, goods j, and arbitrary $0 \leq a_i < b_i$ for every agent. This is because $\sum_{j \in G} \tilde{u}_{ij} x_{ij} = a_i + (b_i - a_i) \sum_{j \in G} u_{ij} x_{ij}$ since x is a fractional perfect matching. Hence utility function $\tilde{u}$ is an affine transformation of utility function u; the former is called a *bi-valued utility function*.

Theorem 21. *For any rational budget vector b, we can compute an HZ equilibrium in polynomial time.*

Proof. We start in the same way is the algorithm in [35]: by computing a minimum vertex cover in the graph of utility 1 edges, we partition $A = A_1 \cup A_2$ and $G_1 \cup G_2$ such that

- every agent in A_2 can be matched to a distinct liked good in G_2,
- every agent in A_1 only has liked goods in G_1, and
- for every $S \subseteq G_2$ we have $|N^-(S)| \geq |S|$ where $N^-(S)|$ are the agents that have a liked good in S.

Set $p_j = 0$ for all $j \in G_2$ and $p_j = \min_{i \in A_1} b_i$ for all $j \in G_1$. Now we run a DPSV-like [12] algorithm on $A_1 \cup G_1$ to raise prices until certain sets of goods become tight.

For each $i \in A$, let β_i be its *effective* budget at current prices p, that is the minimum of its actual budget b_i and the price of its cheapest liked good. The algorithm will now raise all prices p at the same rate until there is a set $S \subseteq G_1$ which goes *tight* in the sense that $\sum_{i \in \Gamma(S)} \beta_i = \sum_{j \in S} p_j$ where Γ is the collection of agents which have a cheapest liked good in S. At this point, we freeze the prices of the goods in S. If all prices have been frozen we are done. Otherwise, we continue raising all unfrozen prices of goods in G_1.

It is easy to see that if the prices keep rising, eventually each agents' effective budget will be their real budget and so a set must become tight at some point. We will not go into detail here but it is possible to find the next set which will go tight in polynomial time similar as in DPSV. Finally, since we never unfreeze prices, there will be at most n iterations of the algorithm and hence it runs in polynomial time overall.

We observe that as in the proof of the DPSV algorithm, for any $S \subseteq G_1$, we have that $\sum_{i \in \Gamma(S)} \beta_i \geq \sum_{j \in A} p_j$ and $\sum_{i \in A_1} \beta_i = \sum_{j \in G_1} p_j$. It is then easy to show that this implies that for any price ρ above 0, the corresponding flow network from Figure 2 supports a flow of value $\rho|G(\rho)|$ by the max-flow min-cut theorem. Thus we can apply Lemma 20 to get an equilibrium allocation. □

Lemma 22. *Let b and b' be two budget vectors with $0 \leq b \leq b'$. Assume we are given an HZ equilibrium (x, p) for the budgets b. Then we can compute in polynomial time a new HZ equilibrium (x', p') with $p \leq p'$ for the budgets b'.*

Proof. We will simply run the same algorithm as in the proof of Theorem 21, except that this time we start with the prices p. More precisely, we increase the lowest non-zero price until a set goes tight or it becomes equal to the next higher price, then repeat this process until we once again get $\sum_{i \in \Gamma(S)} \beta_i \geq \sum_{j \in A} p_j$ and $\sum_{i \in \Gamma(G_1)} \beta_i = \sum_{j \in G_1} p_j$ where G_1 is now defined as the set of goods with positive prices in (x, p). As in the proof of Theorem 21, this will freeze all prices in polynomial time at which point we can use a max-flow min-cut argument to construct the new equilibrium allocation x' in polynomial time. □

Let us now return to the approximate ADHZ setting. Instead of budgets, fix now some fractional perfect matching of endowments $(e_{ij})_{i \in A, j \in G}$.

Theorem 23. *An ϵ-approximate ADHZ equilibrium for rational $\epsilon \in (0,1)$, can be computed in time polynomial in $\frac{1}{\epsilon}$ and n, i.e. by a fully polynomial time approximation scheme.*

Proof. We will iteratively apply Lemma 22. Start by setting $b_i^{(1)} := \frac{\epsilon}{2}$ for all $i \in A$ and computing an HZ equilibrium $(x^{(1)}, p^{(1)})$ according to Theorem 21. Beginning with $k := 1$, we run the following algorithm.

(1) Let $b_i^{(k+1)} := \frac{\epsilon}{2} + (1 - \frac{\epsilon}{2}) \sum_{j \in G} p_j^{(k)} e_{ij}$ for all $i \in A$.
(2) Compute a new HZ equilibrium $(x^{(k+1)}, p^{(k+1)})$ for budgets $b^{(k+1)}$ according to Lemma 22 using the old equilibrium $(x^{(k)}, p^{(k)})$ as the starting point. Note that since $p^{(k)} \geq p^{(k-1)}$ we always have $b^{(k+1)} \geq b^{(k)}$ and so this is well-defined.
(3) Set $k := k + 1$ and go back to step 1.

Note that

$$\sum_{i \in A} b_i^{(k+1)} = \frac{\epsilon}{2} n + \left(1 - \frac{\epsilon}{2}\right) \sum_{j \in G} p_j^{(k)} \quad \leq \frac{\epsilon}{2} n + \left(1 - \frac{\epsilon}{2}\right) \sum_{i \in A} b_i^{(k)}$$

and thus

$$\sum_{j \in G} p_j^{(k)} \leq \sum_{i \in A} b_i^{(k)} \leq n$$

as otherwise we would get $\sum_{i \in A} b_i^{(k+1)} < \sum_{i \in A} b_i^{(k)}$.

Let K be the first iteration such that $p^{(K)} \leq \frac{1-\epsilon/2}{1-\epsilon} p^{(K-1)}$. Note that

$$K \leq n \log_{\frac{1-\epsilon/2}{1-\epsilon}} \left(\frac{n}{\epsilon}\right) = O\left(\frac{n}{\epsilon} \log\left(\frac{n}{\epsilon}\right)\right)$$

since all non-zero prices are initialized to at least ϵ but are bounded by n. Then $(x^{(K)}, p^{(K)})$ is an ϵ-approximate ADHZ equilibrium with budget vector $b^{(K)}$ because for all $i \in A$ we have

$$b_i^{(K)} = \frac{\epsilon}{2} + \left(1 - \frac{\epsilon}{2}\right) \sum_{j \in G} p_j^{(K-1)} e_{ij} \in \left[(1-\epsilon) \sum_{j \in G} p_j^{(K)} e_{ij}, \epsilon + \sum_{j \in G} p_j^{(K)} e_{ij}\right].$$

Lastly, we note that since the number of iterations is bounded by $O(\frac{n}{\epsilon} \log(\frac{n}{\epsilon}))$ and each iteration runs in polynomial time, the total runtime is polynomial in $\frac{1}{\epsilon}$ and n as claimed. □

5 AN RCP FOR THE HZ SCHEME UNDER DICHOTOMOUS UTILITIES

We will assume without loss of generality that each agent $i \in A$ likes some good $j \in G$, i.e. $u_{ij} = 1$. We will show that program (1) given below is the required RCP.

$$\begin{array}{lcc} & \max & \sum_{i \in A} \log \sum_{j \in G} u_{ij} x_{ij} \\ \text{subject to} & \forall j \in G: & \sum_{i \in A} x_{ij} \leq 1 \\ & \forall i \in A: & \sum_{j \in G} x_{ij} \leq 1 \\ & \forall i \in A, j \in G: & x_{ij} \geq 0 \end{array} . \qquad (1)$$

Let p_j's and α_i's denote the non-negative dual variables for the first and second constraints, respectively.

Theorem 24. *Any HZ equilibrium is an optimal solution to (1), and every optimal solution of (1) can be trivially extended to an HZ equilibrium. Furthermore, the latter can be expressed via rational numbers whose denominators have polynomial, in n, number of bits, thereby showing that (1) is a rational convex program.*

Proof. Let $u_i := \sum_{j \in G} u_{ij} x_{ij}$. Clearly, in any HZ equilibrium, since each agent i is allocated an optimal bundle of goods, she will be allocated a non-zero amount of a unit-utility good and hence will satisfy $u_i > 0$. Furthermore, in an optimal solution x of (1), every agent must have positive utility, because otherwise the objective function value will be $-\infty$. Therefore, $\forall i \in A: \; u_i > 0$.

The KKT conditions of this program are:

(1) $\forall i \in A : \alpha_i \geq 0$.
(2) $\forall j \in G : p_j \geq 0$.
(3) $\forall i \in A$: If $\alpha_i > 0$ then $\sum_j x_{ij} = 1$.
(4) $\forall j \in G$: If $p_j > 0$ then $\sum_i x_{ij} = 1$.
(5) $\forall i \in A, j \in G : u_{ij} \leq u_i(p_j + \alpha_i)$.
(6) $\forall i \in A, j \in G : \quad x_{ij} > 0 \Rightarrow u_{ij} = u_i(p_j + \alpha_i)$.

To prove the forward direction of the first statement, let (x, p) be an HZ equilibrium. Since x is a fractional perfect matching on agents and goods, it satisfies the constraints of (1) and is hence a feasible solution for it. We are left with proving optimality.

The KKT conditions 2, 3 and 4 are clearly satisfied by (x, p). Next, consider agent i. If there is a good j such that $p_j \leq 1$ and $u_{ij} = 1$, then i will be allocated one unit of the cheapest such goods. Assume the price of the latter is p. Define $\alpha_i = 1 - p$. Clearly $u_i = 1$. Now, it is easy to check that Conditions 1, 5 and 6 are also holding.

Next assume that every good j such that $u_{ij} = 1$ has $p_j > 1$ and let p be the cheapest such price. Clearly, i's optimal bundle will contain $1/p$ amount of these goods, giving her total utility $1/p$. Since the equilibrium always has a zero-priced good, that good, say j, must have $u_{ij} = 0$. Now, i must be buying such zero-utility zero-priced goods to get to one unit of goods. We will define $\alpha_i = 0$. Again, it is easy to check that Conditions 1, 5 and 6 are holding. Hence we get that (x, p) is an optimal solution to (1).

Next, we prove the reverse direction of the first statement. Let (x, p) be an optimal solution to (1). Assume that agent i is allocated good j, i.e. $x_{ij} > 0$. We consider the following two cases:

(*a*) $u_{ij} = 0$. Using Condition 6 and $u_i > 0$, we get that $p_j = \alpha_i = 0$.
(*b*) $u_{ij} = 1$. Using Conditions 5 and 6 and $u_i > 0$, we get that the price of good j is the cheapest among all goods for which i's utility is 1.

For each agent i, multiply the equality in Condition 6 by x_{ij} and sum over all j to get:

$$\sum_j x_{ij} u_{ij} \; = \; u_i \sum_j x_{ij}(p_j + \alpha_i)$$

After canceling u_i from both sides we obtain

$$\sum_j x_{ij}(p_j + \alpha_i) = 1 = \sum_j x_{ij} p_j + \alpha_i \sum_j x_{ij}.$$

Now, if $\alpha_i > 0$, then $\sum_j x_{ij} = 1$ and if $\alpha_i = 0$, then $\alpha_i \sum_j x_{ij} = 0 = \alpha_i$. Therefore, in both cases $\alpha_i \sum_j x_{ij} = \alpha_i$. Hence,

$$\sum_j x_{ij} p_j = 1 - \alpha_i. \qquad (2)$$

We will view the dual variables p of the optimal solution (x, p) as prices of goods. The above statement then implies that agent i's bundle costs $1 - \alpha_i$.

Let S denote the set of agents who get less than one unit of goods, i.e. $S := \{i \in A \mid \sum_j x_{ij} < 1\}$, and let T denote the set of partially allocated goods, i.e. $T := \{j \in G \mid \sum_i x_{ij} < 1\}$. By Condition 4, $p_j = 0$ for each $j \in T$. Observe that if for $i \in S$ and $j \in T$, $u_{ij} = 1$, then by allocating a positive amount of good j to i, the objective function value of program (1) strictly increases, giving a contradiction. Therefore, $u_{ij} = 0$.

Since the number of agents equals the number of goods, the total deficiency of agents in solution x equals the total amount of unallocated goods. Therefore, we can arbitrarily allocate unallocated goods in T to deficient agents in S so as to obtain a fractional perfect matching, say x'. Clearly, (x', p) is still an optimal solution to (1) and is also an HZ equilibrium.

For the second statement, we will start with this solution (x', p). Let $G' \subseteq G$ denote the set of goods with prices bigger than 1, i.e. $G' = \{j \in G \mid p_j > 1\}$ and let $A' \subseteq A$ be the set of agents who have allocations from G'. By Cases (a) and (b), for each $i \in A'$, there is a $j \in G'$ such that $u_{ij} = 1$; moreover this is the cheapest good for which i has utility 1. We first show that each agent $i \in A'$ satisfies $\alpha_i = 0$. If $\sum_{j \in G} x_{ij} < 1$, this follows from KKT Condition 3. Otherwise, there exists $j \in G$ such that $x_{ij} > 0$ and $u_{ij} = 0$. The last statement follow from the fact that $\sum_j x_{ij} p_j \leq 1$, which follows from (2). Again, by Case (a), $\alpha_i = 0$. Now, by (2), the money spent by each agent in A' is exactly 1 dollar on goods in G'.

Consider the connected components of bipartite graph (A', G', E), where the set $E = \{(i, j) \in (A', G') \mid x_{ij} > 0\}$. Cases (a) and (b) imply that all goods in a connected component C must have the same price, say p_C. Clearly, the sum of prices of all goods in C equals the total money of agents in C; the latter is simply the number of agents in C. This implies that p_C is rational. Clearly, there is a rational allocation of $1/p_C$ amount of goods to every agent in C.

Let $i \in A$ such that the cheapest good for which i has utility 1 has price 1. If $\alpha_i = 0$, by (2), i buys 1 dollar, and hence 1 unit, of such goods. If $\alpha > 0$, by KKT Condition 3, $\sum_{j \in G} x_{ij} = 1$ and therefore again i has bought 1 unit of such goods. Now, without loss of generality, we will assign to i an entire unit of one such good.

Finally, let $G'' \subseteq G$ denote the set of goods with prices in the interval $(0, 1)$, i.e. $G'' = \{j \in G \mid 0 < p_j < 1\}$ and let $A'' \subseteq A$ be the set of agents who have allocations from G''. Let $i \in A''$. Since $\sum_j x_{ij} p_j < 1$, by (2) $\alpha > 0$. Therefore each agent in A'' buys one unit of goods from G''. Hence the allocation of goods from G'' to A'' forms a fractional perfect matching on (G'', A''). Therefore, we can pick any perfect matching consistent with this fractional perfect matching and allocate goods from G'' integrally to A''.

Hence in all cases, the allocation consists of rational numbers, completing the proof. □

Remark 25. *The proof of Theorem 24 shows that for the dichotomous case, the dual of (1) yields equilibrium prices. In contrast, for arbitrary utilities, there is no known mathematical construct, no matter how inefficient its computation, that yields equilibrium prices. In a sense, this should not be surprising, since there is a polynomial time algorithm for computing an equilibrium for the dichotomous case [35].*

Since the objective function in (1) is strictly concave, the utility derived by each agent i must be the same in all solutions of (1). Hence, we get the following corollary which can be seen as a variant of the well-known *Rural Hospital Theorem*; see [24] for the latter.

Corollary 26. *Each agent gets the same utility under all HZ equilibria with dichotomous utilities.*

6 DISCUSSION

In this paper, we defined an ϵ-approximate ADHZ model for one-sided matching markets with endowments. We showed that ϵ-approximate ADHZ equilibrium always exists for every $\epsilon > 0$. We strengthened the non-existence of ADHZ equilibrium for the case when the demand graph is not strongly connected and agents have dichotomous utilities. We derived a novel combinatorial polynomial-time algorithm for computing an ϵ-ADHZ equilibrium under dichotomous utilities. Finally, we presented a rational convex program (RCP) for the HZ model under dichotomous utilities, which also implies that the problem is polynomial-time solvable.

Since finding an HZ equilibrium is PPAD-complete [10, 35], it will be interesting to obtain a similar result for the ϵ-approximate ADHZ model. In Section 1.1 we stated a number of results that build on the HZ scheme and others that are generalizations of the HZ scheme. It will be interesting to explore similar extensions of the ϵ-approximate ADHZ model as well.

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