Being Central on the Cheap: Stability in Heterogeneous Multiagent Centrality Games

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ABSTRACT

We study strategic network formation games in which agents attempt to form (costly) links in order to maximize their network centrality. Our model derives from Jackson and Wolinsky's *symmetric connection model*, but allows for heterogeneity in agent utilities by replacing *decay centrality* (implicit in the J-W model) by a variety of classical centrality measures, as well as game-theoretic measures of centrality. We are primarily interested in characterizing the asymptotically pairwise stable networks, i.e. those networks that are pairwise stable for all sufficiently small, positive edge costs. We uncover a rich typology of stability:

- we give an axiomatic approach to network centrality that allows us to predict the stable network for a rich set of combination of centrality utility functions, yielding stable networks with features reminiscent of structural properties such as "core periphery" and "rich club" networks.
- We show that a simple variation on the model renders it universal, i.e. every network may be a stable network.
- We also show that often we can infer a significant amount about agent utilities from the structure of stable networks.

KEYWORDS

Strategic Network Formation; Asymptotic Pairwise Stability; Network Centrality Measures

1 INTRODUCTION

Centrality in social networks is a topic that has seen an overwhelming amount of recent work at the intersection of Social Network Analysis [55], Physics of Complex Systems [45], Economics [36], Theoretical Computer Science and Artificial Intelligence [22]. Many of the existing models of network formation are stochastic. In reality, networks form and evolve as a consequence of *agent incentives*. Among these incentives *centrality maximization* is certainly a pervasive one. To give just one too familiar example: the increasing competitive nature of the scientific enterprise, coupled perhaps with an ever more common[1] reliance on quantitative measures[2]

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[1]and, in our opinion, unfortunate
[2]of "centrality" in the citation network (!), such as the H-index

and rankings of individuals and publishing venues as proxies for research quality, has often resulted in a significant explosion in the number of submissions to venues perceived as "top ones" (e.g., A.I. conferences such as NeurIPS, IJCAI or AAAI). The decisive factor seems to be, of course, authors' perception that publication in top venues is a way to increase their papers' impact, which ultimately increases their own "centrality", that they want maximized.

Less clear are the *strategic consequences* of agents' propensity for competing for central positions. By this we mean understanding the manner in which agents' preferences for central network positions influence the actual unraveling of network structure. Until recently, models of strategic network formation (e.g. [4, 37]) were unable to reproduce the rich typology of emerging networks uncovered by social network analysis (witnessed e.g. by concepts such as *core-periphery structure* [10], *the rich-club effect* [58], or *small-world networks* [56]). Very interesting recent work [7] showed that strategic network formation models can reproduce the basic characteristics of social networks, low diameter, a power-law degree distribution and high clustering in its equilibrium networks. The model of Biló et al. assumes that the cost of an agent u is a sum of (a). The sum of costs of all adjacent edges uv, assumed to be an arbitrary convex function of the distance between u and v, should the edge uv not be present, and (b). The sum of distances to all other agents. Interesting as this result is, it assumes that agents' utilities have a quite specific nature, **fairly similar for all agents:** what all agents attempt to minimize is a (generalized form) of sum of distances to all other networks. Real agents may have objectives that are not distance based (e.g. intermediate communication between various parts of the network). Furthermore, **different agents may have different, unrelated, centrality objectives.**

The goal of this paper is to contribute to understanding network formation from a game-theoretic perspective that assumes that agents are willing to modify network structure (at a small cost per extra link) in order to improve their centrality. We start from the realization that the most well-known model of strategic network formation, the *symmetric connection model* [36, 37] can be seen as maximizing agents' *decay centrality* [17, 53], subject to constant edge cost. Our model accomodates heterogeneity in centrality objectives: we show via an axiomatic approach to network centrality [9], that salient features of centrality measures can influence the structure of emerging networks in predictible ways.

A key design choice for models of network formation is their handling of *tie strength*: It is well-known that weak ties play an important role in social dynamics, being extremely useful for information dissemination [29]. Our model restricts agents tie manipulation attempts to *weak ties only*, whose cost of establishing/maintainance can realistically be assumed to be a tiny positive constant. An

explanation for such a restriction is not merely modeling convenience, but the existing tension between breaking strong links and centrality maximization: in real life strong links could be either exogeneously imposed (e.g. agents interacting as being part of the same organization, e.g. being coworkers) or have such a high intrinsic cost of breaking (e.g. family ties) that this renders the objective of centrality maximization unrealistic.

A second key design choice concerns *the limits of centrality maximization*: So far we have assumed that agents want to increase their centrality indefinitely, subject to maintaining reasonable costs for their direct contacts. In real life increasing centrality may be subject to *diminishing returns*: the marginal benefit from increased centrality diminishes (or even plateaus) beyond a certain point. A simple way of incorporating this observation into the model is assuming (see Definition 5 below) that each agent is endowed with a *target centrality threshold* θ_i (possibly infinite) beyond which centrality increases bring no utility to the agent.

The main questions that motivate us are:

Q1. Given a set of agents with a heterogeneous set of centrality objectives, can one predict the stable networks ?

Q2. Can we obtain richer classes of stable networks than in the classic symmetric connections model ? Are observed features of real-world graphs (e.g. rich-club, core-periphery) compatible with explanations based on centrality maximization ?

Q3. Can we infer (at least something about) agents' centrality objectives from the (set of) stable networks ?

Given the extreme potential heterogeneity of agent objectives, it may seem that no substantive positive answer could be given to the three questions above. **A first contribution of our paper is to show that this is not the case**, by providing well-behaved examples where the answers to the three previous questions are affirmative. Remarkably, stable networks in our models display (stylized versions of) some features such as core-periphery and rich club effect. A second contribution of our paper is the identification of a **fundamental limit of prediction in centrality maximization models**, in the form of a plausible variation in the model specification that renders it *universal*, in the sense that **every network** can arise as an equilibrium. The property we highlight is natural, assuming that agents' utility is subject to a threshold beyond which no extra gain in centrality increases their utility.

The following is an outline of our main results: we first show (Theorem 3) that for mixtures of agents satisfying one of four monotonicity axioms stable networks may contain an unique complex component consisting of a "core" (clique) and a "periphery" of nodes only connected to core nodes. On the other hand (Corollary 2), for degree homophilic measures stable networks **display a rich club effect**: they consist of a sequence of cliques of rapidly decreasing sizes, together with isolated nodes. For betweenness centrality games we show (Theorem 7) that connected stable networks have small diameter and are characterized by an interesting property related to *neighborhood domination* in graphs. We provide some futher analytical results and conjectures (based on simulations) for some less well-behaved centralities. Finally, we show that adding thresholds to the model makes it universal (Theorem 4). Some of these cases are complemented by results (Corollary 1 and Theorem 9) on learning agent types from the (set of) stable networks.

2 PRELIMINARIES

We assume familiarity with basics of graph theory, coalitional game theory (see Chalkiadakis et al. [14]), strategic network formation (e.g. Jackson [36]) and centrality measures in social networks [19, 40]. In particular, given network g and vertex i of g, we will denote by $deg(i)$ the degree of i in g, by $N(i)$ the set of neighbors of i in g, and by $\widehat{N(i)}$ the set $\widehat{N(i)} = \{i\} \cup N(i)$. We also denote by $g + ij$ the network obtained by adding missing edge ij, by $d(i, j)$ the distance between i and j, and by $Conn(i)$ the connected component of i in g. We will write $g_1 + g_2$ for the disjoint union of two networks g_1, g_2. An edge is called *a bridge edge* if its removal disconnects the graph. The neighborhood domination relation is a classical concept in graph theory (e.g. Definition 1.16 in Brandstädt et al. [12]), first formulated (under the name *vicinal preorder*) in Foldes and Hammer [25] and formalized as follows:

DEFINITION 1. *Given vertices x, y in graph g, we say that y dominates x (and write $x \leq y$) iff $N(x) \subseteq N(y) \cup \{y\}$.*

Given a set of vertices V, denote by $\mathcal{G}_V$ the set of graphs on vertex set V. A *centrality measure* is a function $C : V \times \mathcal{G}_V \to \mathcal{R}$. We will force notation and write $C_v[g]$ instead of $C(v, g)$. We also review the following special cases:

DEFINITION 2. *Given node i in network g, define:*
- *The degree centrality of i is $C_{deg}[i] = \frac{1}{n-1} deg(i)$.*
- *The closeness centrality of i is $C_{close}[i] = \frac{1}{\sum\limits_{j \in Conn(i)} d(i,j)}$.*
- *The eccentricity centrality of i is defined[3] as $C_{ecc}[i] = min((n-1)/d(i,j) : j \in Conn(g))$.*
- *The random walk closeness centrality of i is defined [57] as $C_{close}[i] = \frac{1}{\sum\limits_{j \in Conn(i)} ht[j,i]}$, where $ht[j,i]$ is the expected time for a random walk started at j to first hit i.*
- *The decay centrality of i is defined as $C_{dec}[i] = \sum\limits_{j \in g} \beta^{d(i,j)}$, where β is a fixed parameter, $0 < \beta < 1$.*
- *The harmonic centrality of i is $C_{harm}[i] = \sum\limits_{j \in g} \frac{1}{d[i,j]}$.*
- *The betweenness centrality of i is $C_{between}[i] = \sum\limits_{y \neq i \neq z} \frac{\sigma_{y,z}(i)}{\sigma_{yz}}$, that is the sum of percentages of shortest paths between arbitrary vertices y, z that pass through i.*
- *The random walk (a.k.a. current flow) betweenness centrality of i is defined [46] as $C_{RWB}[i] = \sum\limits_{j \neq i \neq k} r_{j,k}$, where $r_{j,k}$ is the probability that a random walk starting at node j with absorbing node k passes through node i.*
- *The eigenvector centrality of i is defined as $C_{eig}[i] = w[i]$, where w is the eigenvector corresponding to the largest eigenvalue of the adj. matrix of g.*
- *The Katz centrality is defined as $C_{Katz}[i] = \sum\limits_{k=1}^{\infty} \sum\limits_{j=1}^{n} \alpha^k (A^k)_{ji}$. where α is a parameter, $0 < \alpha < 1$.*
- *Pagerank. See e.g. [8] for formal definitions and some properties.*

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[3] in many papers eccentricity is defined as $max(d(i,j) : j \in Conn(g))$. Defining eccentricity centrality like we do has been done before, and has the advantage that bigger values correspond to "more central nodes" .

We also need centralities defined using coalitional games (see also [35]):

Definition 3. *The* Michalak et al. centrality *of i [42] is defined as the Shapley value of node i in the coalitional game (N, v), where $v(S) = |S \cup N(S)|$. It has the formula $C_{GT}[i] = \sum\limits_{j \in \widehat{N(i)}} \frac{1}{deg(j)+1}$.*

We will also use a variant (to our knowledge first considered here) based on the Banzhaf, rather than the Shapley value:

Definition 4. *The* Banzhaf-Michalak centrality *of i is defined as the Banzhaf value of node i in the coalitional game (N, v), where $v(S) = |S \cup N(S)|$. An easy computation shows that it has the formula $C_{BGT}[i] = \sum\limits_{j \in \widehat{N(i)}} \frac{1}{2^{deg(j)}}$.*

Definition 5. *Given centrality measure $C = (C_i)$ and threshold $\theta = (\theta_i) \in (\mathbb{R} \cup \{+\infty\})^n$, the θ-truncation of C is the centrality C_θ defined by*

$$C_{i,\theta}[g] = \begin{cases} C_i[g], & if\ C_i[g] < \theta_i \\ \theta, & otherwise. \end{cases}$$

Note that for $\theta = \infty$ (or just a large integer) we get the original utility. So truncations really extend our previous framework. An agent threshold θ_i is called individually feasible *if $\theta_i = 0$ or $\theta_i > 0$ and there exists some network h on the set of vertices $\{j \in N : \theta_j > 0\}$ such that $C_i[h] = \theta_i$. A network w is called* feasible *iff $C_i[w] \le \theta_i$ for all agents i.*

3 MODEL AND AXIOMATIC SETTING

Our framework, which extends the Jackson-Wolinsky symmetric connection model, is specified as follows:

Definition 6. *The* symmetric connection model with generalized centralities *(C_i), thresholds θ_i and edge cost c is defined as follows: the utility of player i on network g is*

$$u_i(g) = C_{i,\theta_i}(g) - c \cdot deg(i). \tag{1}$$

where C_{i,θ_i} is the θ_i-truncation of centrality C_i. We will occasionally avoid mentioning the family of centralities (C_i) and thresholds θ_i when they are clear from the context.

An *edge flip* of a pair of nodes i, j of a network g is the addition of ij to g, if $ij \notin g$, or its removal from g, if $ij \in g$. The *outcome of an edge flip* is the resulting network h.

Definition 7. *An edge flip is a* weakly improving move *for player i if $u_i(h) \ge u_i(g)$, and a* strongly improving move *if $u_i(h) > u_i(g)$. An edge flip is* an improving move *iff:*

- *it is an edge addition that is strongly improving for at least one endpoint and at least weakly improving for both, or*
- *is an edge deletion, strongly improving for some endpoint.*

The main model of the emerging network structure employed in the area of strategic network formation, defined in Jackson and Wolinsky [37] is:

Definition 8. *Network g is called* pairwise stable *if no edge flip is an improving move.*

In this paper we use a version of pairwise stability that is appropriate to our setting that assumes weak ties only, in which the edge cost is a tiny (but positive) value $\epsilon > 0$. Therefore, the following variant of pairwise stability will be our main notion of interest:

Definition 9. *Consider the symmetric connection model with generalized centralities. Network g is called* **asymptotically pairwise stable** *(APSN) if there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, g is pairwise stable in the model instantiation with edge cost ϵ.*

Since APSN is a version of pairwise stability, results on APSN relate to existing literature. For instance the original Jackson-Wolinsky result can be interpreted as stating that for decay centrality games the unique family of APSN consists of complete graphs K_n.

3.1 Axioms for network centralities

The first axiom is a simple one and has been discussed before in the literature [9]. It formalizes the intuition that adding edges always improves the centrality of adjacent nodes:

Axiom 1. *A centrality measure C is* increasing *if whenever $ij \notin g$, $C[i, g + ij] > C[i, g]$.*

If C is a centrality measure satisfying some axiom then $C' = 1/(1 + C)$ (or even $C' = 1/C$, when C is strictly positive) satisfies a correspondingly modified "dual" axiom. For instance, here's the dual of Axiom 1:

Axiom 1'. *A centrality measure C is* decreasing *if whenever $ij \notin g$, $C[i, g + ij] < C[i, g]$.*

Our next axiom represents a different type of monotonicity: the benefit of extra links only incurs for agents already in the same connected component. It is a more precise version of an axiom due to Boldi et al. [8]:

Axiom 2. *A centrality measure C is* locally increasing *if it satisfies the following conditions: If $ij \notin g$ and i, j are in the same connected component then $C[i, g + ij] > C[i, g]$. If $ij \notin g$ and i, j are not in the same connected component in g then $C[i, g + ij] \le C[i, g]$.*

In the dual scenario agent only benefit when forming bridges between previously disconnected components:

Axiom 2'. *A centrality measure C is* peripherally decreasing *if it satisfies the following conditions: If $ij \notin g$ and i, j are in the same connected component then $C[i, g + ij] \le C[i, g]$. If $ij \notin g$ and i, j are not in the same connected component in g then $C[i, g + ij] > C[i, g]$.*

The setting of Axiom 2 is not vacuous, as we have:

Theorem 1. *Closeness centrality and random walk closeness centrality satisfy Axiom 2.*

The next axiom has a different flavor, and encode a scenario when agents benefit by connecting only when they were "of the same/different types". In our particular setting homophily is assessed with respect to agents' degree.

Axiom 3. *A centrality measure C is* degree homophilic *if the following is true: there exists a strictly increasing function f such that for any network g and edge $ij \notin g$, adding edge ij to g is an improving move for i iff $deg(i) \le f(deg(j))$.*

Centrality	Axm.	Reference
Degree	1,4,5	trivial
Harmonic	1	Boldi and Vigna [9]
Katz	1	
Decay	1	
Pagerank	1	Chien et al. [16]
		Boldi et al. [8]
Closeness	2	Theorem 1
r.w. Closeness	2	Theorem 1
Michalak et al.	3	Theorem 2

Figure 1: Axioms for centrality measures.

An example of measure satisfying Axiom 3 is:

THEOREM 2. *Michalak et al. centrality satisfies Axiom 3 with* $f(n) = (n+1)(n+2) - 3$.

The following axiom introduces a weak version of a fairness axiom that has been considered in previous literature ([43, 47]):

AXIOM 4. *Centrality measure C is called* weakly fair *if the following are true: whenever g is a network, $S \subseteq V(g)$, $i, j \in S$ such that $ij \notin E(g)$ and $C_i[g + ij] - C_i[g] = max\{C_k[g+kl] - C_k[g] : k, l \in S, kl \notin E(g)\}$, we have $C_j[g+ij] \geq C_j[g+rj]$ for all $r \in S, rj \notin E(g)$. In other words edges that are (global) maximizers of the increase in network centrality are (local) maximizers of the increase in network centrality for both endpoints.*

AXIOM 5. *Increasing centrality measure C is called* ordered *if the following are true: whenever g, h are networks on the same set of vertices such that $E(g) \subseteq E(h)$ and i, j, k are vertices of g (and h) such that $ij, ik \notin E(h)$, we have: if $C_i[g + ij] \geq C_i[g + ik]$ then $C_i[h + ij] \geq C_i[h + ik]$ In other words the relative effect of edges is closed under the addition of (unrelated) edges.*

EXAMPLE 1. *Degree centrality satisfies axioms 1,4 and 5. So do related variations such as $C_i[g] = deg_g(i)^2$, weighted versions of the degree, or the following (more interesting) centrality measure: $C_v^*[g] = 0$ if v is isolated in g, $C_v^*[g] = \frac{deg_g(v)}{|E(g)|}$ otherwise.*

For space reasons, we refer for some of the missing proofs to the longer version [34]. On the other hand, many of our results (even theoretical ones) arose from implementing our model in Python using the *networkx* package [31] and performing computational experiments.

4 APSN FOR MONOTONE CENTRALITIES

In this section we consider mixtures of agents satisfying axioms $1, 1'$, and $2, 2'$. Our first result deals with the case when all agents thresholds are $\theta_i = \infty$. In this case we can characterize the APSN as those graphs whose connected components are (perhaps single node) cliques plus, maybe, one *complex component*, which is not a complete graph. This complex component displays an extreme form of *core-periphery structure*: it consists of a *core*, a clique of nodes of type 1 and 2, and a *periphery* consisting of nodes of type $2'$ attached to core nodes only:

THEOREM 3. *The APSN in the centrality model with agents satisfying one of Axioms $1,1'$, $2,2'$ are precisely those networks g satisfying the following rules:*

- *All agents of type $1'$ are isolated.*
- *There is a single connected component that contains all agents of type 1. Agents of type 1 and 2 in this component form a clique ("the core"). All agents of type $2'$ belong to this component, and are pendant vertices attached to vertices of type 1 of the clique ("the periphery").*
- *All other connected components are complete graphs (including isolated nodes) containing agents of type 2 only.*

PROOF. Consider an APSN. All agents of type 1 must be connected in a clique, since adding an edge is an improving move for all of them. For agents of type 2 belonging to this component it is beneficial to connect to all nodes of type 1 (and among themselves), hence they are also part of the clique core. Agents of type $2'$ want to stay connected to the component, but only minimally: once they are connected to a node in the component, adding any extra edge is not improving for them. Their contacts must be of type 1: if they were of type 2 they'd benefit from severing the connection.

All other components consist of agents of type 2 only, for which it is beneficial to fully connect. □
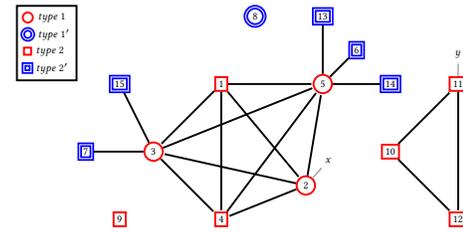
Figure 2: APSN for mixtures of monotone centralities.

As a corollary of the previous result, we can infer quite a lot about agent types from the structure of stable networks:

COROLLARY 1. *Consider, in the setting of Theorem 3, an asymptotically pairwise stable network g. Then:*

- a). *Isolated nodes are either of type $1'$ or 2.*
- b). *An agent y in clique components of size ≥ 2 may be of type 1 or 2. It is guaranteed to be of type 2 when there exists a complex component in g which doesn't contain y, or when some agent x in a different component is known to be of type 1 (Fig. 2).*
- c). *In the complex component pendant vertices are of type $2'$ (Fig. 2) and their neighbors are of type 1. All other nodes are of types 1 or 2.*
- d). *One cannot distinguish between centrality measures of the same type, nor between nodes of different types in the same listing (a)-(c).*

As the next result shows, even for monotone measures, adding finite thresholds has a profound effect on the structure of APSN, changing them from complete graphs to all graphs:

THEOREM 4. *The following are true:*

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Start with the empty graph g = ∅_n.
while (not done):
    let e = ij ∉ E(g) maximize quantity C_i[g + ij] − C_i[g].
    among those satisfying C_i[g] < θ_i and C_j[g] < θ_j.
    g = g ∪ {e}
return g.
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Figure 3: Greedy algorithm for finding APSN.

a). For **every network** g and centralities $(C_i)_{i \in V(g)}$ satisfying axiom 1 there exist thresholds (θ_i) s.t. g is an APSN for the truncated centrality game with thresholds θ_i.

b). If agents' original centrality measures satisfy axiom 1, then for all families (θ_i) of thresholds, the APSN for the truncated centrality games with thresholds θ_i, **if they exist**, can be characterized as the graphs with "Pareto optimal centralities", i.e. graphs h satisfying:
- for every $ij \notin E(h)$, $C_i[h] \geq \theta_i$ or $C_j[h] \geq \theta_j$, and
- for every edge $ij \in E(h)$, removing ij from h would yield a network l with $C_i[l] < \theta_i$ and $C_j[l] < \theta_j$.

c). For all $\theta_i \geq 0$, **APSN exist** in all truncated centrality games with centralities (C_i) satisfying Axioms 1, 4 and 5.

Proof.   a. Let $\theta_i = C_i[g]$. We claim that g is an APSN for the truncated centrality game with centralities C_i and thresholds θ_i. Indeed, consider an edge ij of g. Nodes i, j don't want to drop edge ij, since their current centrality values are θ_i, θ_j while, by Axiom 1, their centralities would decrease below these values if they dropped ij. Let now i, j be vertices such that $ij \notin g$. Since their current centrality values are θ_i, θ_j (at the threshold), adding edge ij would not increase their truncated centralities, while incurring the extra (positive) cost of edge ij. So adding edge ij is not an improving move.

b. First, by essentially repeating the proof at point a., it is easy to see that graphs with Pareto optimal centralities are APSN. The opposite direction is equally easy: consider an APSN h and two vertices i, j. If $ij \notin E(h)$ then adding edge ij must not be an improving move for at least one of i, j. Since C_i, C_j are increasing, the only possibility is that $C_i[h] \geq \theta_i$ (so that adding edge ij does not increase the truncated centrality of i and, in fact, decrease its utility, because of the extra cost of edge ij) or, similarly, $C_j[h] \geq \theta_j$. Consider now the case when $ij \in E(h)$. Because centralities are increasing and removing edge ij is not an improving move, removing ij must strictly decrease truncated centralities for both nodes i, j. This is only possible if the centralities of i, j in the resulting network l satisfy $C_i[l] < \theta_i$ and $C_j[l] < \theta_j$.

c. First of all, a comment about the result at point b.: it does **not** establish the existence of APSN, since it is not clear that the conditions in the characterization are actually feasible. This is what we show next, under the hypothesis that all centralities satisfy axioms 1,4,5.

We will prove the existence of APSN as follows. Consider the algorithm in Figure 3. We claim that its outcome g is an APSN. Indeed, none of the missing edges could be added to g: if $ab \notin E(g)$ then $C_a[g] \geq \theta_a$ or $C_b[g] \geq \theta_b$ at the moment when edge ab was considered for inclusion. Since centralities only increase during the algorithm, the condition is valid at the end of the algorithm as well.

On the other hand, consider an edge $ab \in E(g)$ with $C_a[g] \geq \theta_a$. In order to apply point (b), we aim to prove that for every edge $ad \in E(g)$, $C_a[g - ad] < \theta_a$. Let af be the last edge adjacent to a added to g by the algorithm in Figure 3. By the algorithm, adding af is the first moment the centrality of node a increases beyond value $\geq \theta_a$. So $C_a[g - af] < \theta_a$. We will prove that in fact $C_a[g - ad] \leq C_a[g - af] < \theta_a$. If $d = f$ then our claim is true. Otherwise, edge ad must have been added to g before edge af. Consider the moment when adding edge ad. Let g_0 be the network before the addition. By Axiom 4, adding edge ad maximized the centrality increases of both nodes a, d. Since af was a candidate for edge addition, $C_a[g_0 + af] \leq C_a[g_0 + ad]$. By the fact that C_a satisfies Axiom 5, $C_a[g_1 + af] \leq C_a[g_1 + ad]$, where $g_1 \supseteq g_0$ is the graph $g - \{af, ad\}$. But $g_1 + af = g - ad$ and $g_1 + ad = g - af$.   □

5 DEGREE HOMOPHILY YIELDS RICH-CLUB APSN

Next we study centrality games for degree-homophilic centrality measures. The following result shows that APSN in this case have a "rich club" hierarchical structure:

Theorem 5. *Let h be an APSN for the centrality game with upward degree homophilic centralities with function $f(\cdot)$ satisfying $f(0) = -1$ and $f(x) \geq x$ for every $x \geq 1$.*

Let m be the maximum degree of a node in h. Let $n_1^* = \min\{k : f(k) \geq m\}$ and, for $i \geq 2$, $n_i^* = \min\{r : f(r) \geq n_{i-1}^*\}$. Clearly $n_1^* \geq n_2^* \geq \ldots$ (and one can assume w.l.o.g., by removing multiple copies of the same value, that $n_1^* > \ldots > n_r^* = 1$ for some $r \geq 1$)

a). If $deg(i), deg(j) \geq n_1^*$ then $ij \in E(h)$.

b). If $deg(i), deg(j) \in [n_k^*, n_{k-1}^*]$ for some $k \geq 2$ then $ij \in E(h)$ ("alike nodes connect to each other")

c). If $k \geq 2$, $deg(i) \leq n_k^*$, $deg(j) > n_{k-1}^*$ then $ij \notin E(h)$.

Proof. We use the definition of degree homophily:

a). Since $deg(i) \geq n_1^*$ and f is monotonic, $f(deg(i)) \geq f(n_1^*) \geq m \geq deg(j)$, and similarly $f(deg(j)) \geq deg(i)$. If i, j were not connected, then adding ij would be an improving move for both of them.

b). Similar to (a): as $deg(i) \geq n_k^*$ and f is monotonic, $f(deg(i)) \geq f(n_k^*) = n_{k-1}^* \geq deg(j)$, so $f(deg(i)) \geq deg(j)$, and similarly $f(deg(i)) \geq deg(j)$. If i, j were not connected, then adding ij would be an improving move.

c). We have $deg(i) \leq n_k^*$ so $f(deg(i) - 1) \leq f(n_k^* - 1) < n_{k-1}^* \leq deg(j) - 1$. Hence $deg(j) - 1 > f(deg(i) - 1)$, so removing edge ij is an improving move for j, since in the graph $h = g - ij$ adding edge ij is not an improving move for j.

□

Corollary 2. *Let $a_1 > a_2 > \ldots > a_p > a_{p+1} = 1$ be a sequence of integers such that $a_i - 1 > f(a_{i+1} - 1)$ for all $i = 1, \ldots, p$. Then all graphs of type $K_{a_1} + K_{a_2} + \ldots + K_{a_p} + rK_0$ ("stratified clique graphs") are APSN for upward degree homophilic centrality games*

with function f and conversely, all APSN are unions of cliques with this structure.

Proof. We need the following simple

Lemma 1. *Let g be a network and $ij \in E(g)$. Then removing edge ij from g is an improving move iff $deg(j) - 1 > f(deg(i) - 1)$ or $deg(i) - 1 > f(deg(j) - 1)$.*

Proof. Removing edge ij is an improving move **iff** for at least one of the two nodes i, j its betweenness centrality stays the same when removing the edge. In this case adding edge ij to $h = g - ij$ is **not** an improving move and vice-versa: if adding edge ij to h is not an improving then one of i, j has the same betweenness centrality in g as in h, hence removing edge ij is an improving move in g.

By definition, adding edge ij to h is not improving iff $deg_h(i) > f(deg_h(j))$ or $deg_h(j) > f(deg_h(i))$. □

Consider now a sequence $a_1 > a_2 > \ldots > a_p > a_{p+1} = 1$ be a sequence of integers such that $a_i - 1 > f(a_{i+1} - 1)$ for all $i = 1, \ldots, p$. We first need to prove that all graphs of type $K_{a_1} + K_{a_2} + \ldots + K_{a_p} + lK_0$, $l \geq 0$, are APSN.

This is easy, by applying points a),b),c) of the theorem: let, indeed, y, z be nodes in the same clique K_{a_r}, $1 \leq r \leq p$. We need to show that removing edge yz is not an improving move. Since they are in the same clique, the degrees of y, z are both equal to $a_r - 1$. Since $a_r - 1 \leq f(a_r - 1)$ (because $a_r \geq 2$ and $f(x) \geq x$ for $x \geq 1$), the desired conclusion follows by Lemma 1.

Let now y, z be nodes in different cliques, $y \in K_{a_r}$, $z \in K_{a_s}$, $a_r > a_s$. We have $deg(y) = a_r - 1 > f(a_s - 1) = f(deg(z))$. By the definition, adding edge yz is **not** an improving move.

Since $0 > f(0) = -1$ connecting any isolated node to any other node is **not** an improving move. So g is an APSN.

Conversely, let g be an APSN. By applying points a) and b) of the Theorem, we get that g has edges between every two vertices whose degrees are in the same interval $[n_i^*, n_{i-1}^*]$, where by convention $n_0^* = m$.

To infer the fact that g has the structure claimed in the corollary we need to prove that no other edges are present. Point c) of the theorem excludes edges between node whose degrees are **not** in the same interval.

The only potential trouble is that there might be a node x of degree n_i^* who is connected with nodes whose degrees are in both intervals $[n_{i+1}^*, n_i^*]$ and $[n_i^*, n_{i-1}^*]$, thus "joining two cliques". We will show that something like this doesn't happen by induction on i.

Case $i = 1$: Let z be a node of maximum degree m. Let A be the set of nodes with degree in the range $[n_1^*, n_0^*]$. Then all the nodes in A are connected to each other. z is not connected to any node outside A. If there were some other node w in A that is connected to a node outside A then w would have degree higher than m, a contradiction. Hence nodes in A form a connected component that is a clique.

The induction step: Assume we have obtained $l - 1$ connected components that are cliques of size $a_1 > a_2 > \ldots > a_{l-1}$ satisfying the condition $a_i - 1 > f(a_{i+1} - 1)$ for $i = 1, \ldots, l - 2$. Applying the reasoning in the induction case $i = 1$ to the remaining graph we obtain a connected component of size a_l that is a clique. Furthermore

$a_{l-1} - 1 > f(a_l - 1)$, since nodes in the l'th clique component are not connected to those in the $l - 1$'st component.

It is possible that the tail of the resulting sequence $a_1, \ldots, a_s$ is composed of components of size 1, that is isolated nodes. The required condition is satisfied, since $1 - 1 > f(1 - 1) = f(0) = -1$. □

In the previous theorem the condition $f(x) \geq x$ is necessary: as the next result shows, without it the structure of APSN is much simpler:

Theorem 6. *Consider a centrality game with upward degree homophilic centralities with common function $f(x) = x - 1$. Then no edge addition can be an improving move.*

Assume that, additionally, for every agents i, j such that $ij \notin g$ and $deg(i) \leq deg(j)$ we have $C_i[g + ij] \leq C_i[g]$. Then the unique APSN for the centrality game is the empty network $\emptyset_n$.

Proof. Adding a missing edge ij can never be an improving move: to be so, one would need, simultaneously that $deg(i) \leq deg(j) - 1$ and $deg(j) \leq deg(i) - 1$, which is impossible.

For similar reasons, removing an existing edge ij is always improving for one of the endpoints. Indeed, assume that h is a network containing edge ij and, w.l.o.g. $deg(i) \leq deg(j)$. Let $g = h - ij$. Then $u_i(g) - u_i(h) = C_i[g] - C_i[h] + c > 0$. So removing edge ij is an improving move for i. □

Observation 1. *The Banzhaf-Michalak centrality satisfies the conditions of Theorem 6. Indeed, assume $ij \notin g$ and $deg(i) \leq deg(j)$. Then $C_i[g + ij] - C_i[g] = \frac{1}{2^{deg(i)+1}} + \frac{1}{2^{deg(j)+1}} - \frac{1}{2^{deg(i)}} = \frac{1}{2^{deg(j)+1}} - \frac{1}{2^{deg(i)+1}} \leq 0$.*

6 DOMINATION AND APSN IN BETWENNESS CENTRALITY GAMES

In this section we completely characterize APSN for betweenness centrality games. First, simple computations provide examples of APSN with components that are not complete graphs: networks $C_4 + nK_1$, $n \geq 0$. What about the general structure of APSN ? We will show that the domination relation plays a decisive role in their characterization. To accomplish this, we first prove:

Lemma 2. *The following statements are true:*
  - *Adding any bridge edge ij weakly increases i's betweenness centrality, strictly unless i was isolated. Consequently adding a bridge edge is improving for i, unless i was isolated. Conversely, a disconnecting edge removal is improving for i iff i was a pendant node.*
  - *Adding any non-bridge edge ij weakly increases i's betweenness centrality.*

We now prove the following result, which gives an unexpected (and fairly elegant) algorithmic characterization of APSN for betweenness games using the domination relation:

Theorem 7. *Graphs g that are APSN for betweenness centrality games consist of isolated vertices plus at most one connected component C with at least two vertices which satisfies the following condition: $deg(l) \geq 2$ for every $l \in C$, $diam(C) = 2$ and for every $i \neq j \in C$, $ij \in E(g)$ **if and only if** sets $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$ are incomparable, i.e. if **none of** i, j **dominates the other.***

Proof. First, it is easy to see that the networks that satisfy the condition of Theorem 7 are APSN: indeed, by Lemma 2 isolated vertices have no incentive to connect to anyone else, as their utility would decrease. Consider, on the other hand two vertices i, j in a large component C.

If $ij \in E(g)$ then, by the condition of the theorem, there exist vertices $k \in N(i) \setminus \{j\}$ and $l \in N(j) \setminus \{i\}$. Since $d(k, j) \leq 2$ it follows that $k - i - j$ is a shortest path between i and j that would disappear if we dropped edge ij, decreasing the betweenness centrality of i and ultimately its utility. Similarly, if we dropped edge ij the utility of j would also decrease, hence it is **not** an improving move.

On the other hand if $ij \notin E(g)$ then $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$ are comparable. Assume w.l.o.g. that $N(i) \setminus \{j\} \subseteq N(j) \setminus \{i\}$. Then every shortest path between two vertices $s, t \neq i$ that goes through i stays a shortest path when we add edge ij: This is clear when $s, t \neq j$, so assume w.l.o.g. $t = j$. Then $d(s, j) = 1$. Adding edge ij creates no new shortest paths, hence it is not an improving move for i.

Let us now prove the converse direction, that APSN satisfy the conditions in the theorem. A first statement to prove is that any APSN has at most one component with at least two vertices. Indeed, if there were more than two such connected components then, by Lemma 2, joining them by an edge ij would be an improving move for both i, j. Second, we claim that this nontrivial component has diameter 2: indeed, it cannot have diameter 1, as complete graphs are not APSN. Assume there was a (shortest) path of length 3 $p - q - r - s$ between two vertices p, s. Then p, s would increase their utility by connecting since, for instance, now there is a shortest path from q to s going through p. Third, this component has no pendant vertices: if a node had degree 1, it would have, by Lemma 2, an incentive to disconnect.

Consider a connected APSN g and a pair $ij \notin g$, and assume w.l.o.g. that adding edge ij decreases the utility of i for small $\epsilon > 0$, so that the move is not improving. Hence adding edge ij does not increase the betweenness of i. Paths contributing positively to the betweenness of i **before adding** ij are between nodes $k_1, k_2 \in N(i)$ that are not connected, so that a shortest path between k_1, k_2 goes through i. Then adding edge ij does not change the fraction corresponding to k_1, k_2 in the betweenness of i. Consider now the shortest paths between $k_1 \in N(i) \setminus \{j\}$ and j. Since the betweenness of i does not increase as a result of adding edge ij, k_1 must be connected to j. Hence j dominates i.

Consider now an edge $ij \in g$. Since the removal of edge ij is not improving for either i or j, it means that there exists a shortest path between some vertices $s_1 \neq i \neq t_1$ that employs edge ij. As the diameter of g is two, one of s_1, t_1 (say t_1) must be j, hence $s = s_1$ is a neighbor of i that is not a neighbor of j. Similarly, there must be a vertex $t = t_2$ that is a neighbor of j that is not a neighbor of i. Hence none of i, j dominates the other. □

Observation 2. *Complete bipartite graphs $K_{a,b}$, $a, b \geq 2$ satisfy the conditions of the theorem, hence they are APSN. One could believe that these are **all** connected APSN with at least 2 vertices, but this is **not** true: a counterexample, found using computer simulations, is the graph g in Figure 4. g is not bipartite as it has, e.g. triangle 459.*
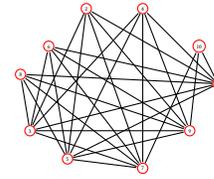
Figure 4: Non-bipartite APSN for betweenness games.

7 OTHER RESULTS/CONJECTURES

In this section we study centrality games for some measures that appear to satisfy none of Axioms 1,2,3: eccentricity centrality, random walk betweenness and eigenvector centrality. First, we show that eccentricity centrality is very close to obeying Axiom 2:

Lemma 3. *Let g be a network, i a node in g and j another node such that $ij \notin g$. The following are true:*

- *If $j \notin Conn(i)$ then $EC(i, g + ij) \leq EC(i, g)$.*
- *If $j \in Conn(i)$ then $EC(i, g + ij) \geq EC(i, g)$. The inequality is strict iff j is on all shortest paths to all vertices k farthest in g from i.*

In spite of this result, the structure of APSN for eccentricity centrality games is quite different from the one for centrality games with measures satisfying Axiom 2:

Theorem 8. *All vertices in connected components of size at least three of an APSN have degree at least two. On the other hand all connected, eccentricity-one graphs with min. degree 2 and at least two nodes with degree at most $n - 2$ are APSN. There exist (Fig. 5) APSN with eccentricity two.*
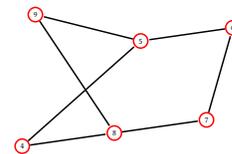
Figure 5: APSN for eccentricity centrality games.

We weren't able to obtain a full characterization of APSN in this case, or analytical results for random walk betweenness and eigenvector centrality. However, computer simulations suggest that the following statements are true. The first one is interesting due to apparent difference with the case of betweenness:

Conjecture 1. *For random walk betweenness centrality the only APSN are the empty graph $\emptyset_n$ and the complete K_n.*

As for eigenvector centrality, although it seems not to have any monotonicity properties, experimental evidence is consistent with the following conjecture, that seems to situate this measure together with the monotonic ones:

Conjecture 2. *The complete graphs K_n are the only asymptotically pairwise stable networks for eigenvector centrality.*

8 LEARNING AGENT THRESHOLDS

In spite of the previous result, we can still talk about learning agent utility functions. However, the problem that we will deal with is **not** that of learning agent centralities (which we will, in fact, assume known), but **agent thresholds.** In other words, we want to answer the following variant of Q3: *Can we learn (something about) agents' thresholds from the structure of stable networks ?*

The learning model we will assume is a type of *oracle learning* [1]. Specifically[4], oracle queries are pairs (g, i) consisting of a network g and an agent index i. Given query (g, i) the oracle will either reply with an APSN h such that $C_i[h] > C_i[g]$, or with "NONE", in the case such an APSN h does not exist.

It is important to realize that thresholds may fail to be fully identifiable simply due to *the coarse resolution of centralities*: for instance, any values between two consecutive integers (e.g. 2.3 and 2.7) are completely equivalent as thresholds for degree centrality, since degrees in graphs are integral, and jumps in centrality (as a result of an edge flip) have a magnitude at least one. The best we can hope for in such a scenario is to identify the interval [2,3] as an interval that contains the threshold. The interval corresponds to a single edge flip in a network that decreased the centrality of the given node below the threshold value.

A potential issue with the identification of thresholds is the fact that (consistent with the model in our Corollary 1) we only get APSN as oracle answers. If for all APSN g, $C_i[g] < \theta_i$ then **all** estimates provided by the oracle on the value of the threshold are too low. If this doesn't happen, we can prove:

THEOREM 9. *Given an agent i, assume that there exists an APSN h with $\theta_i \leq C_i[h]$. Then there exists an algorithm that uses oracle queries and outputs an APSN g and edge ij s.t. $C_i[g-ij] \leq \theta_i \leq C_i[g]$. For linear centralities the algorithm runs in polynomial time.*

9 RELATED LITERATURE

The area of network games is quite large, and a comprehensive survey is impossible. We list here two such overviews: the first one, most relevant to our interest is [38]. Another one with an algorithmic bent is [51]. The model that we are concerned with is a variant of the symmetric connection model [37] (see also [36]). Some notable subsequent work includes [21, 28, 39]. Many alternative models have been investigated. A more preeminent one is [4].

Our work owes much to the axiomatic approach to network centralities. For significant work in this area see [5, 8, 9, 47–49, 54].

More related work exists in the theoretical computer science literature: for example, Hopcroft and Sheldon [32] discuss an oriented model in which nodes have control over outgoing edges. There is no cost for changing their links, and their purpose is to increase their Pagerank. They show that the Nash equilibria in this game have a fairly sophisticated structure (see also Chen et al. [15]). Undirected versions of this game have been studied [3]. Recently Kouroupas et al. [41] have studied a model in which the utility of a node is a product of two factors: content quality multiplied by the traffic level. In this model pure Nash equilibria always exist. On the other

hand Avin et al. [2] prove that preferential attachment models can be seen as Nash equilibria of some network games.

Other related work comes from the sociology literature [13, 33, 44]. For instance, in the Buskens and Van De Rijt model every node strives to fill "structural holes" (including lack of connnectedness) between nodes. This is somewhat analogous to maximizing betweenness, but the precise model (and the results) are different.

Our model allows heterogeneity in agents' utilities, corresponding to distinct measures of centrality. Heterogeneous network formation models have been studied before, e.g. Galeotti et al. [26].

Finally, several papers (e.g. [6, 18]) have treated the problem of improving the centrality of a node by adding or removing links. Our work is different in several respects: first of all, in our setting *all* agents aim to improve their respective centralities. Second, in our model maintaining a link has a (small) cost.

10 CONCLUSIONS, POSSIBLE EXTENSIONS

We have shown that our models can accommodate a wide range of agent centrality objectives. Still, we do not see our results as adequate enough yet for the analysis of real-life networks. They have, instead, more of a proof-of-concept nature, and could conceivably be made more realistic in many ways. Some variations (we believe) worth investigating are listed below:

Probabilistic edge addition/removal: In real life an edge may only form with some probability even though both agents would benefit from it. Studying such a variation could produce networks with core structures that are dense but not quite complete.

Strong and weak links, forced links, affiliation models: In the model we have discussed all the links are weak links. A natural extension allows for both strong and weak links. This would entail using two types of costs: fixed, constant costs for the strong links, small ("ϵ") costs for weak links. A second, orthogonal, distinction that could be useful is that of forced versus free links. We assumed implicitly that link formation is completely under the control of the agent. Often this is not so: there are social ties in real life that could be regarded as fixed, since severing them entails a significant cost. Forced links may be a consequence of *affiliation*: people meet as the result of joining the same clubs. A possibly relevant model is the *social effort model* of [11]. Another one is the *social clubs* model of [24]. For centrality in affiliation networks see [23].

Manipulating link strength: agents could manipulate *link strength*, rather than completely severing them.

Tagged networks: agents have a *tag* and care about the tags of their neighbors, like in Schelling's segregation model.

Spatial agents: Agents interaction may result from placement in space. A standard reference for spatial connection models is [28].

Multilayer networks: Sometimes (e.g. [20]) link formation may encompass multiple, correlated, link types. E.g. two coworkers may end up being friends as well. It would be interesting to formulate multilayer extensions of the Jackson-Wolinsky model.

Overlapping communities: For centrality in such models see [27, 30, 50, 52].

Dynamic models: Finally, our concepts of network stability are steady-state concepts. It would be interesting to study the emerging networks in *dynamic* models of network formation with a similar philosophy.

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[4]perhaps a more natural model would be one that centrality of node i in APSN g, $C_i[g]$) can be bigger/smaller than some arbitrary target value. The point is that our restricted query model is good enough.

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