Kernel Contraction and Base Dependence: Redundancy in the Base Resulting in Different Types of Dependence

Mehrdad Oveisi **James P. Delgrande** **Fred Popowich**
School of Computing Science
Simon Fraser University
Burnaby, BC, Canada
{oveisi,jim,popowich}@cs.sfu.ca

Francis Jeffry Pelletier
Department of Philosophy
University of Alberta
Edmonton, AB, Canada
francisp@ualberta.ca

Abstract

The *AGM* paradigm of *belief change* studies the dynamics of belief states in light of new information. Finding, or even approximating, dependent or relevant beliefs to a change is valuable because, for example, it can narrow the set of beliefs considered during belief change operations. Gärdenfors' preservation criterion (GPC) suggests that formulas independent of a belief change should remain intact. GPC allows to build dependence relations that are theoretically linked with belief change. Such dependence relations can in turn be used as a theoretical benchmark against which to evaluate other approximate dependence or relevance relations. There are already some studies, based on GPC, on the parallelism between belief change and dependence. One study offers a dependence relation parallel to *AGM contraction* for belief sets. Another study links *base dependence* relation to a more general belief base contraction, *saturated kernel contraction*. Here we offer yet a more general parallelism between *kernel contraction* and base dependence. At this level of generalization, different types of base dependence emerge. We prove that this differentiation of base dependence types is a result of possible redundancy in the base. This provides a theoretical means to distinguish between redundant and informative parts of a belief base.

1 Introduction

Research into *belief change* provides formal means for incorporating new and changing information. Alchourrón, Gärdenfors and Makinson [1985] provide the *AGM* paradigm of belief change that idealizes a belief state as a *belief set* K: a set of logical formulas that is closed under implication. One example of belief change operation on K is *contraction* which retracts α and other formulas from K as necessary to ensure α is not implied by the remaining formulas.

During a belief change operation, beliefs independent of a change should remain intact. Gärdenfors [1990] states this intuition in the following *preservation criterion*:

"If a belief state is revised by a sentence A, then all sentences in K that are independent of the validity of A should be retained in the revised state of belief" [Gärdenfors, 1990]. (GPC)

Then, based on GPC, Fariñas del Cerro and Herzig [1996] (FH) axiomatize a *dependence* relation, and formalize the connection between *dependence* and *AGM contraction*.

A more practical and important variant of the original AGM approach uses *belief bases* instead of belief sets. Belief bases need not be deductively closed, and are usually *finite*. One very general class of base contraction is *kernel contraction* [Hansson, 1994], which is a superclass of *saturated kernel contraction*, itself a superclass of AGM contraction.

In our previous work [Oveisi *et al.*, 2014], using belief bases instead of belief sets, we introduced *base dependence* as a (reversible) generalization of FH's dependence. Based on the definitions presented in §6.1, here we will refer to this new dependence relation as *strong* base dependence. Thus, based on GPC, we had indeed established the correspondence between *strong base dependence* and *saturated kernel contraction* in that work.

Indeed, as seen in both studies based on GPC mentioned above, GPC allows to build dependence relations that are theoretically linked with belief change. Such dependence relations can in turn be used, for example, as *a theoretical benchmark* against which to evaluate other approximate dependence or relevance relations.

Therefore, in this work, we aim to capture a yet more general base dependence relation to correspond to (full) kernel contraction, once again based on GPC. After providing the necessary background in §2, we present an in-depth motivational example in §3 to show why connecting *base dependence* and *kernel contraction* via GPC is desirable.

At this level of generalization, as discussed in §4, different types of base dependence emerge, namely strong base dependence and weak base dependence. In §5, weak base dependence is shown to be a result of redundancy in the base. Indeed, this is *a second result from our study that can be used as a theoretical benchmark in other studies.* The fact that weak base dependence captures redundancy may be exploited for various purposes. For example, one may use weak base dependence to distinguish between redundant and informative formulas in a belief base. In §6, we offer a generalization of the formalism provided in our previous work. We will finally discuss related works in §7, and conclude in §8.

2 Background

2.1 Formal Preliminaries

We assume $\mathcal{L}$ to be a propositional language defined on a finite set of propositional variables or atoms $\mathcal{V}$ with Boolean operators negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and implication $\rightarrow$. For meta variables over sentences in $\mathcal{L}$, we use Greek letters α, β, δ, etc. We introduce the sentential constants $\top$ and $\bot$ for convenience, representing truth and falsity respectively. $B \vdash \alpha$ represents a logical consequence α of a set of formulas B. Cn, defined as $\mathrm{Cn}(B) = \{\alpha \mid B \vdash \alpha\}$, is a consequence operator, a total function taking sets of formulas to sets of formulas.

2.2 Belief Contraction

To model rational belief change, AGM uses *rationality postulates* to describe what constitute operators for belief change, and it also specifies how to *construct* such operators. Two example belief contraction postulates, which an AGM contraction operator $\div$ on K should satisfy, are as follows:

$K \div \alpha$ is a belief set (closure)

$K \div \alpha \subseteq K$ (inclusion)

If $\nvdash \alpha$ then $\alpha \notin K \div \alpha$ (success)

Some of our beliefs are more *epistemically entrenched* than others, making them harder to give up. Based on this intuition, Gärdenfors [1988] introduced epistemic entrenchment, and defined the properties of an order relation $\leq$ between sentences. Gärdenfors and Makinson [1988] show that an AGM contraction operator $\div$ can be constructed using a $\leq$ relation, and that, conversely, an epistemic entrenchment relation $\leq$ can be constructed using an AGM contraction operator $\div$.

Turning now to belief base contraction, the following are some other important axioms for belief base contraction:

$B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ (relative closure)

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$
then $B \div \alpha = B \div \beta$ (uniformity)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then (core-retainment)
$\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$

Different combinations of the contraction axioms specify different contraction operations. In particular, *kernel contraction* operators satisfy success, inclusion, core-retainment and uniformity. *Saturated kernel contraction* operators satisfy success, inclusion, core-retainment, uniformity and relative closure [Hansson, 1994].

2.3 AGM Contraction and Dependence

Fariñas del Cerro and Herzig [1996] formalize the notion of dependence and its connection with belief change. They investigate a binary relation $\rightsquigarrow$ on formulas, where $\alpha \rightsquigarrow \beta$ reads as "β depends on α" (or equivalently "α is relevant to β"). Independence, then, is denoted by $\not\rightsquigarrow$, which is the complement of $\rightsquigarrow$, so $\alpha \not\rightsquigarrow \beta$ reads as "β is independent of α" (or "α is irrelevant to β"). They then provide a set of postulates that any dependence relation $\rightsquigarrow$ needs to satisfy.

Similar to epistemic entrenchment, to provide the connection between dependence and contraction, FH introduce the two following conditions, Cond$\rightsquigarrow$ and Cond$\div$. As mentioned in §1, they use GPC as a guiding principle to study the relation between dependence and belief change. For example, if $\beta \in K$ to begin with, but $\beta \notin K \div \alpha$, then we can say that β depends on α, or $\alpha \rightsquigarrow \beta$.

$\alpha \rightsquigarrow \beta$ iff $\beta \in K$ and $\beta \notin K \div \alpha$ (Cond$\rightsquigarrow$)

$\beta \in K \div \alpha$ iff either $\vdash \beta$ or $\beta \rightsquigarrow \beta$ and $\alpha \not\rightsquigarrow \beta$ (Cond$\div$)

They then show that Cond$\rightsquigarrow$ allows constructing $\rightsquigarrow$ using an AGM contraction $\div$, and that Cond$\div$ allows constructing AGM contraction $\div$ using dependence $\rightsquigarrow$. Finally they present an axiomatic characterization theorem via Cond$\rightsquigarrow$.

2.4 Saturated Kernel Contraction and Strong Base Dependence

In our previous work, *Belief Change and Base Dependence*, we further generalized FH's work, introducing *base dependence* $\bar{\rightsquigarrow}$ as a relation between formulas w.r.t. a *belief base*, instead of a belief set. $\alpha \bar{\rightsquigarrow} \beta$ is read as "β *base-depends* on α," which is the same as $\alpha \rightsquigarrow \beta$ except that it also implies that $\beta \in B$:

If $\alpha \bar{\rightsquigarrow} \beta$ then $\beta \in B$. (1)

Indeed, in FH's work, dependence can only happen between (contingent) sentences from K, or $\mathrm{Cn}(B)$ if $K = \mathrm{Cn}(B)$: If $\alpha \rightsquigarrow \beta$ then $\alpha \in \mathrm{Cn}(B)$ and $\beta \in \mathrm{Cn}(B)$. One way to generalize the dependence relation $\rightsquigarrow$ is to make α or β be from B instead of $\mathrm{Cn}(B)$. We had adopted the later: If $\alpha \bar{\rightsquigarrow} \beta$ then $\alpha \in \mathrm{Cn}(B)$ and $\beta \in B$, which implies (1).

The following are the basic postulates of base dependence:

$\beta \in B$ iff either $\vdash_B \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some α. (Def-B)

If $\alpha \bar{\rightsquigarrow} \beta$ then $\beta \bar{\rightsquigarrow} \beta$. (Cond-ID^B)

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$
then $\alpha \bar{\rightsquigarrow} \delta$ iff $\beta \bar{\rightsquigarrow} \delta$. (conjugation)

If $\alpha \bar{\rightsquigarrow} \beta$ then (contribution)
$\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$.

If $\alpha \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then (modularity)
either $\vdash \alpha$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\beta \in B'$.

If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then (redundancy)
either $\alpha \bar{\not\rightsquigarrow} \beta$ or $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$.

Please see [Oveisi *et al.*, 2014] for motivation and interpretation of these postulates. In some of the postulates and conditions, *base entailment* $\vdash_B$ is used as a simplifying notation to help represent *tautologies present in the base*: $\vdash_B \beta$ means: $\beta \in B$ and $\vdash \beta$.

We next provided the following conditions, again similar to the epistemic entrenchment approach, and similar to FH's conditions Cond$\rightsquigarrow$ and Cond$\div$:

$\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$. (Cond$\bar{\rightsquigarrow}$)

$\beta \in B \div \alpha$ iff either $\vdash_B \beta$ or $\beta \bar{\rightsquigarrow} \beta$ and $\alpha \bar{\not\rightsquigarrow} \beta$. (Cond$\bar{\div}$)

Cond$\bar{\rightsquigarrow}$ is fairly intuitive. For Cond$\bar{\div}$, $\beta \in B \div \alpha$ means either that β is a tautology in B, $\vdash_B \beta$, or that β is a contingent truth in B, $\beta \bar{\rightsquigarrow} \beta$, but contraction by α does not lead to retraction of β from B, $\alpha \not\bar{\rightsquigarrow} \beta$.

Finally using these conditions, we presented characterization theorems to link a base dependence satisfying the six axioms above, and saturated kernel contraction.

Furthermore, we also proved that a base dependence satisfying all the six axioms above is a *reversible generalization* of FH's dependence. That is, in the special case when the underlying belief base is deductively closed (i.e., it is a belief set), base dependence reduces to dependence.

Note that in the current work, we have a more general definition for base dependence (see Definition 11), compared to the definition of base dependence used above. Therefore, based on these new definitions, we can say that our previous study discussed the correspondence between *saturated kernel contraction* and ***strong*** *base dependence* (see Definition 12).

3 Motivation

Both *AGM Contraction and Dependence*, §2.3, and *Saturated Kernel Contraction and Strong Base Dependence*, §2.4, connect the notions of dependence and belief change as two sides of the same coin. This provides a theoretically sound definition of dependence in the context of belief change—a *theoretical benchmark* that, for example, other approximating approaches can be compared against. For instance, one may study how (in)compatible a given approximate dependence relation is with saturated kernel contraction.

As a motivating example, consider the *syntactical relevance* relation R provided by Riani and Wassermann [2004]: $R(\alpha, \beta)$ if and only if the formulas α and β share an atom. Simply put, they consider formulas that share atoms as related (a.k.a. variable sharing). Then, for instance, to speed up belief change operations, one may want to find out to what extent R is compatible with belief change. In the following example, we consider only one base dependence axiom redundancy (see §2.4), and we denote $R(\alpha, \beta)$ with $\alpha \bar{\rightsquigarrow}_R \beta$.

Example 1. Assume that $B = \{p, q, p \vee q\}$, and that $\bar{\rightsquigarrow}_R$ is a relation constructed based on variable sharing such that, for example, $p \bar{\rightsquigarrow}_R p$, $p \not\bar{\rightsquigarrow}_R q$ and $p \bar{\rightsquigarrow}_R (p \vee q)$ hold. We show that $\bar{\rightsquigarrow}_R$ violates redundancy. Let $B' = \{q\}$. Clearly $B' \subseteq B$ and $p \vee q \in \mathrm{Cn}(B')$, and, as stated above, $p \bar{\rightsquigarrow}_R (p \vee q)$. Thus, by redundancy, $p \bar{\rightsquigarrow}_R \delta$ for some $\delta \in B'$. Since $B' = \{q\}$, δ can only be q, so $p \bar{\rightsquigarrow}_R q$. This contradicts $p \not\bar{\rightsquigarrow}_R q$. □

Now, let us relax R (as Riani and Wassermann do), and even though p and q do not share any atoms, we allow $R(p, q)$ because after all $R(p, p \vee q)$ and $R(p \vee q, q)$. Then in Example 1, we have $p \bar{\rightsquigarrow}_R q$, and it will not violate redundancy any more. That is, we can study to what extent a relation is compatible with belief change *without referring to belief change axioms* at all. It is far more convenient to conduct a study about dependence relations using only dependence axioms.

The above example relies on the correspondence between base dependence and saturated kernel contraction. As the next step, we would like to be able to make a similar comparison using (full) kernel contraction. This is further motivated by the fact that kernel contraction is a very general class of contractions [Hansson, 1999].

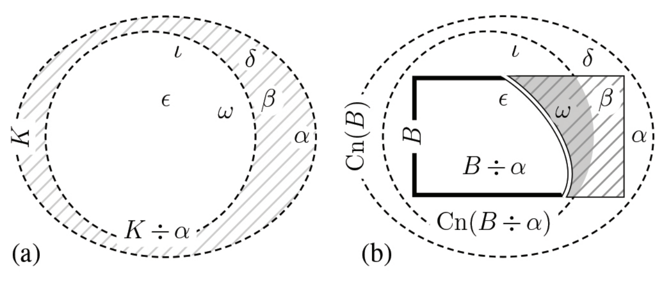


Figure 1: Building (b) *base dependence* via Cond$\bar{\rightsquigarrow}$ is more complex than (a) *dependence* via Cond$\rightsquigarrow$. Example formulas α, β, ω, δ, ι, ϵ fall into different subareas. Let $K = \mathrm{Cn}(B)$.

4 Different Types of Base Dependence Construction

To achieve our high-level goal of generalizing previous works based on GPC to establish a new parallelism between kernel contraction and base dependence, we start by specifying what stays the same. We use the same meaning for base dependence (or base relevance) as before. That is, $\bar{\rightsquigarrow}$ is similar to $\rightsquigarrow$, and as in §2.3: $\alpha \bar{\rightsquigarrow} \beta$ means that "β depends on α" or "α is relevant to β" or "doubting in α leads to doubting in β." Additionally, by (1) in §2.4, $\alpha \bar{\rightsquigarrow} \beta$ implies that $\beta \in B$.

Again similar to previous works, we will need to specify the axioms of our sought-after base dependence, and to specify how to construct it using kernel contraction. As mentioned above and further discussed in §6.1 (see Definition 11), it turns out that, being a "generalization," this new base dependence relation will need to satisfy a subset of the six base dependence axioms in §2.4. Thus, there will not be any "new" axioms for the new, generalized base dependence.

Next we consider the construction method for this base dependence relation. We start by noting that our previous formalism was constrained with the goal that base dependence should be a *reversible* generalization of dependence. That is, where a base dependence relation corresponds to a belief set instead of a belief base, the base dependence relation should reduce to FH's dependence. Although this is a nice property, here we aim to further generalize it to the next level. Thus, our first try is to fully explore the base dependence relations constructed via Cond$\bar{\rightsquigarrow}$ *without* constraining the results in any way. In particular, we will show in this section that there is more than one way to construct base dependence using base contraction.

$\alpha \rightsquigarrow \alpha$	$\alpha \rightsquigarrow \beta$	$\alpha \not\rightsquigarrow \omega$	$\alpha \rightsquigarrow \delta$	$\alpha \not\rightsquigarrow \iota$	$\alpha \not\rightsquigarrow \epsilon$
$\alpha \not\bar{\rightsquigarrow} \alpha$	$\alpha \bar{\rightsquigarrow} \beta$	$\alpha \bar{\rightsquigarrow} \omega$	$\alpha \not\bar{\rightsquigarrow} \delta$	$\alpha \not\bar{\rightsquigarrow} \iota$	$\alpha \not\bar{\rightsquigarrow} \epsilon$
$\alpha \not\hat{\rightsquigarrow} \alpha$	$\alpha \hat{\rightsquigarrow} \beta$	$\alpha \not\hat{\rightsquigarrow} \omega$	$\alpha \not\hat{\rightsquigarrow} \delta$	$\alpha \not\hat{\rightsquigarrow} \iota$	$\alpha \not\hat{\rightsquigarrow} \epsilon$
$\alpha \not\check{\rightsquigarrow} \alpha$	$\alpha \not\check{\rightsquigarrow} \beta$	$\alpha \check{\rightsquigarrow} \omega$	$\alpha \not\check{\rightsquigarrow} \delta$	$\alpha \not\check{\rightsquigarrow} \iota$	$\alpha \not\check{\rightsquigarrow} \epsilon$

Table 1: Examples of different types of (base) dependence, obtained via Cond$\rightsquigarrow$, Cond$\bar{\rightsquigarrow}$, Cond$\hat{\rightsquigarrow}$ and Cond$\check{\rightsquigarrow}$, for the illustrated formulas in Figure 1.

Throughout this section, Figure 1 will provide a running example to help illustrate different existing or new concepts discussed or introduced. To start with the simpler case, consider FH's use of Cond$\rightsquigarrow$ (§2.3) to construct a dependence relation using a given AGM contraction. An example application of Cond$\rightsquigarrow$ is depicted in Figure 1a. It shows a belief set K along with some set formulas: α, β, ω, δ, ι and ϵ in K. It further shows how contracting K by α also results in retraction of some other formulas: β and δ, and not the rest of the example formulas. Thus, we conclude by Cond$\rightsquigarrow$ that: $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \delta$, and that $\alpha \not\rightsquigarrow \omega$, $\alpha \not\rightsquigarrow \iota$ and $\alpha \not\rightsquigarrow \epsilon$. These results are summarized on the first row of Table 1.

4.1 Base Dependence Constructions

Construction 1: Base Dependence

We have already seen the condition in Cond$\bar{\rightsquigarrow}$ in §2.4:

$$\alpha \bar{\rightsquigarrow} \beta \ \text{ iff } \ \beta \in B \text{ and } \beta \notin B \div \alpha. \qquad (\mathsf{Cond}\bar{\rightsquigarrow})$$

Figure 1b shows a belief base B and its logical closure $\mathrm{Cn}(B)$. Here, some formulas from base B have been retracted, namely, β and ω, so that the remaining set, $B \div \alpha$, does not imply α anymore. By Cond$\bar{\rightsquigarrow}$, then, we conclude that β and ω base-depend on α: $\alpha \bar{\rightsquigarrow} \beta$ and $\alpha \bar{\rightsquigarrow} \omega$. Cond$\bar{\rightsquigarrow}$ maintains its intuitive appeal as a reasonable formalization of GPC. Nevertheless, there remains some subtleties that we will explore next.

Construction 2: Strong Base Dependence

Before discussing the next base dependence construct, let us once again consider FH's Cond$\rightsquigarrow$, but this time transformed to an equivalent base-generated representation:

$$\alpha \rightsquigarrow \beta \ \text{ iff } \ \beta \in \mathrm{Cn}(B) \text{ and } \beta \notin \mathrm{Cn}(B \div \alpha). \qquad (\mathsf{Cond}\rightsquigarrow)$$

Now, comparing Cond$\rightsquigarrow$ above and Cond$\bar{\rightsquigarrow}$ makes it clear that indeed there is another possible formalization of GPC for belief bases as follows:

$$\alpha \hat{\rightsquigarrow} \beta \ \text{ iff } \ \beta \in B \text{ and } \beta \notin \mathrm{Cn}(B \div \alpha). \qquad (\mathsf{Cond}\hat{\rightsquigarrow})$$

This provides a stronger condition for base dependence than Cond$\bar{\rightsquigarrow}$ because $B \div \alpha \subseteq \mathrm{Cn}(B \div \alpha)$ by the standard inclusion property of the Cn operator.

Going back to Figure 1b, we saw that by Cond$\bar{\rightsquigarrow}$: $\alpha \bar{\rightsquigarrow} \beta$ and $\alpha \bar{\rightsquigarrow} \omega$. However, according to Cond$\hat{\rightsquigarrow}$, β has strong base dependence on α, $\alpha \hat{\rightsquigarrow} \beta$, but ω does not, $\alpha \hat{\not\rightsquigarrow} \omega$. When contracting B by α, β is retracted whether we consider the contraction remaining set, $B \div \alpha$, or its closure, $\mathrm{Cn}(B \div \alpha)$. This is not the case for ω, which we will study next.

Construction 3: Weak Base Dependence

To further investigate the difference between Cond$\bar{\rightsquigarrow}$ and Cond$\hat{\rightsquigarrow}$, observe that ω is originally in B, and it is then retracted as a result of contracting B by α, $\omega \notin B \div \alpha$, but later it is reintroduced as a logical implication of the contracted set, $\omega \in \mathrm{Cn}(B \div \alpha)$. We refer to this non-persistent base dependence of ω on α as *weak base dependence*, denoted by $\alpha \check{\rightsquigarrow} \omega$. On the one hand, $\alpha \check{\rightsquigarrow} \omega$ refers to a kind of base dependence in the sense that ω is removed from the base as a result of contracting by α. On the other hand, it does not fully capture the concept of dependence because ω is still implicitly present in the consequences of the contracted set. Thus even though it is a kind of base dependence, it is a *weak* dependence. Basically a base dependence which is not a strong base dependence is a weak base dependence, which can be specified as follows:

$$\alpha \check{\rightsquigarrow} \beta \ \text{ iff } \ \beta \in B \text{ and } \beta \notin B \div \alpha \text{ and } \beta \in \mathrm{Cn}(B \div \alpha). \qquad (\mathsf{Cond}\check{\rightsquigarrow})$$

4.2 Connections Among (Base) Dependence Constructions

The above-mentioned base dependence constructions as well as FH's dependence construction Cond$\rightsquigarrow$ are all connected. For example, a base dependence between two formulas, $\alpha \bar{\rightsquigarrow} \beta$, either is a strong base dependence, $\alpha \hat{\rightsquigarrow} \beta$, or is a weak base dependence, $\alpha \check{\rightsquigarrow} \beta$.

Theorem 2. *Given relations $\bar{\rightsquigarrow}$, $\hat{\rightsquigarrow}$, $\check{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\bar{\rightsquigarrow}$, Cond$\hat{\rightsquigarrow}$ *and* Cond$\check{\rightsquigarrow}$ *hold, the following also holds:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$ or $\alpha \check{\rightsquigarrow} \beta$

The proof is rather straightforward by comparing the right hand sides of the three conditions. Indeed, base dependence and strong base dependence become equivalent when there is no weak base dependence. This is guaranteed when relative closure is satisfied (as also shown by Theorem 10).

Theorem 3. *Given relations $\bar{\rightsquigarrow}$, $\hat{\rightsquigarrow}$, $\check{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\bar{\rightsquigarrow}$, Cond$\hat{\rightsquigarrow}$ *and* Cond$\check{\rightsquigarrow}$ *hold and* relative closure *is satisfied, the base dependence relation $\bar{\rightsquigarrow}$ and the strong base dependence relation $\hat{\rightsquigarrow}$ are equivalent:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$.

If $\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ then by relative closure, $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$, $\beta \in B \div \alpha$ also holds. Thus weak base dependence cannot happen. Then, by Theorem 2, we have $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$.

Finally, in this section, we establish the connection between dependence and base dependence. In the presence of relative closure or equivalently in the absence of weak base dependence, base dependence is equivalent to dependence for the formulas in the base.

Theorem 4. *Given relations $\bar{\rightsquigarrow}$, $\rightsquigarrow$ and $\div$ for base B such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* relative closure *is satisfied, the following holds too:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\alpha \rightsquigarrow \beta$.

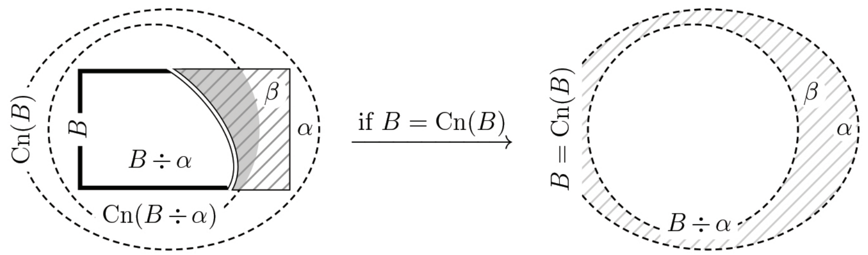
This result paves the way to show next that when B is logically closed, $B = \mathrm{Cn}(B)$, base dependence and dependence become equivalent. This is depicted in Figure 2 on the following page.

Theorem 5. *Given relations $\bar{\rightsquigarrow}$, $\rightsquigarrow$ and $\div$ for base B such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* closure *is satisfied, in the special case where B is logically closed, $\bar{\rightsquigarrow}$ reduces to $\rightsquigarrow$:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \rightsquigarrow \beta$.

Note that this is consistent with our previous results, stating that base dependence is a reversible generalization of dependence.

5 Redundancy Resulting in Different Types of Base Dependence

Based on the results from §4, we are now interested to know when there is no weak base dependence because then, for example, we can say when base dependence and strong base



(a) Base dependence for base B, which is generally not closed

(b) When B is closed though, base dependence reduces to dependence

Figure 2: Dependence is a *special case* of base dependence.

dependence are equivalent. Formally, we define absence of weak base dependence as follows:

Definition 6. Given relations $\check{\rightsquigarrow}$ and $\div$ for base B such that Cond$\check{\rightsquigarrow}$ holds, we say that: there is *no weak base dependence* if and only if $\alpha \not\check{\rightsquigarrow} \beta$ for all formulas α and β.

In this section we show that there is a powerful correspondence between two seemingly different concepts: weak base dependence and redundancy of belief bases.

We offer the following definition to clarify what redundancy in a base means.

Definition 7. β is *redundant* in B with respect to B' if and only if $B' \subseteq B$ and $\beta \in B$ and $\beta \notin B'$ and $\beta \in \mathrm{Cn}(B')$.

The following theorem shows that weak base dependence exists exactly when some redundant contracted statements are still implied by the remaining statements.

Theorem 8. *Given relations $\check{\rightsquigarrow}$ and $\div$ for base B, where* inclusion *holds,* Cond$\check{\rightsquigarrow}$ *is equivalent to the following:*
$\alpha \check{\rightsquigarrow} \beta$ iff *β is redundant in B with respect to $B \div \alpha$.*

One immediate and interesting implication of this theorem is that weak base dependence cannot occur in a belief base that contains no redundancy.

Corollary 9. *Given relations $\check{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\check{\rightsquigarrow}$ *and* inclusion *hold, the following also holds: if B has no redundan*cy, *then it contains no weak base dependence.*

To summarize the results so far, let us consider Definition 6. Note that for any $\beta \notin B$, it trivially holds by Cond$\check{\rightsquigarrow}$ that $\alpha \not\check{\rightsquigarrow} \beta$ for all α. More interesting instances of absence of weak base dependence can occur when $\beta \in B$. It is a property of the belief base B and/or the contraction operator $\div$ used that determines whether any weak base dependence can exist. By Corollary 9, if base B does not contain any redundancy, then there will be no weak base dependence involving any of its formulas. Also by Theorem 8, neither will there be any weak base dependence via a contraction operation $\div$ using Cond$\check{\rightsquigarrow}$ that can *properly* handle any redundancy that may exist in the base B.

The following theorem formally identifies relative closure as the condition under which there does not exist weak base dependence between any given pair of sentences.

Theorem 10. *Given relations $\check{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\check{\rightsquigarrow}$ *holds, there is no weak base dependence if and only if* relative closure *holds for $\div$.*

Thus, avoidance of weak base dependence can be achieved solely based on the properties of the corresponding contraction operator. A base dependence constructed using a contraction operator that satisfies relative closure is guaranteed to avoid weak base dependence altogether.

6 Kernel Contraction and Base Dependence

We are now at a position to further generalize the formalism we offered in [Oveisi *et al.*, 2014]. That is, we want to move from a parallelism between *saturated kernel contraction* and *strong base dependence* to another more general parallelism between *kernel contraction* and *base dependence*, based on GPC.

6.1 Definitions and Assumptions

We know that a subset of saturated kernel contraction axioms specifies the more general kernel contraction [Hansson, 1994]. We may expect something similar for the two respective base dependence relations as well: starting from the six base dependence axioms in §2.4, and finding the appropriate subset of these axioms. First observation is that the saturated kernel contraction has exactly one axiom more than kernel contraction: relative closure. Another observation is that in many important theorems relative closure turns out to have a very substantial role.

For example, in §4.2, we saw that, in the presence of relative closure, Cond$\hat{\rightsquigarrow}$ and Cond$\bar{\rightsquigarrow}$ become equivalent by Theorem 3. This in turn means that for the results in our previous work, Cond$\bar{\rightsquigarrow}$ can be replaced with Cond$\hat{\rightsquigarrow}$. That is because relative closure was present in all of our theorems that used Cond$\bar{\rightsquigarrow}$. Indeed, that is why it sounds reasonable to rename the base dependence relations in that study "strong base dependence" because they are always constructed using the stronger condition Cond$\hat{\rightsquigarrow}$, either explicitly or implicitly (by combining Cond$\bar{\rightsquigarrow}$ and relative closure).

One last observation is that the combined results from Theorems 8 and 10 from §5 show that any contraction operator that satisfies relative closure can handle any redundancy in the base such that weak base dependency cannot happen. Figure 1b may help to see this point at a more intuitive level. Basically, the darker shaded area, which is the result of redundancy and gives rise to weak base dependence, cannot exist in the presence of relative closure.

All of these observations suggest that at least one of the strong base dependence axioms that is crucial for handling redundancy is redundancy! (Please refer to [Oveisi *et al.*, 2014] for a more detailed discussion and example on this axiom.)

Indeed, the theorems presented in this section confirm this point. That is, redundancy is the only axiom different between base dependence (built by Cond$\tilde{\rightsquigarrow}$) and strong base dependence (built by Cond$\hat{\rightsquigarrow}$). Therefore, we offer the following definitions to simplify the expressions of the upcoming theorems.

Definition 11. A relation $\tilde{\rightsquigarrow}$ is a *base dependence* if and only if it satisfies the axioms Def-B, Cond-ID^B, conjugation, contribution and modularity.

Definition 12. A relation $\hat{\rightsquigarrow}$ is a *strong base dependence* if and only if it is a base dependence that satisfies redundancy.

Therefore, the base dependence studied in our previous work was indeed strong base dependence by Definition 12.

In the remaining of this section, we present the theorems that show that there is a mutual correspondence between kernel contraction and base dependence (Definition 11).

Remark 13. Please note that in any place that we follow mostly what has been done in previous studies [Fariñas del Cerro and Herzig, 1996; Oveisi *et al.*, 2014], we avoid repeating the explanations provided in those studies, and we may just state so.

6.2 From Base Dependence to Contraction

To construct a contraction operator $\div$, assume all the following are present: a base dependence relation $\tilde{\rightsquigarrow}$ (Definition 11), $\vdash_B$ (§2.4), and Cond$\bar{\div}$. $\vdash_B$ is effectively used to determine the tautologies present in the base $T \subseteq B$ where $T = \{\beta \mid \vdash_B \beta\}$.

In common with the previous studies (Remark 13), we can obtain B using $\tilde{\rightsquigarrow}$ and $\vdash_B$:

$$B = B_{\tilde{\rightsquigarrow}} = \{\beta \mid \vdash_B \beta \text{ or } \alpha \tilde{\rightsquigarrow} \beta \text{ for some } \alpha\}\,.$$

Theorem 14 states that the contraction operator $\div$ obtained from $\tilde{\rightsquigarrow}$ is indeed a kernel contraction.

Theorem 14 (Base Dependence to Contraction). *Given relations $\tilde{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\bar{\div}$ *holds, if $\tilde{\rightsquigarrow}$ is a base dependence, then $\div$ is a kernel contraction.*

To prove this theorem, we show one by one that the postulates of kernel contraction also hold, assuming Cond$\bar{\div}$ and postulates of base dependence hold.

6.3 From Contraction to Base Dependence

Assume the following are present: a kernel contraction operator $\div$, and Cond$\tilde{\rightsquigarrow}$. Theorem 15 states that, given the above assumptions, all axioms of base dependence $\tilde{\rightsquigarrow}$ relation are satisfied.

Theorem 15 (Contraction to Base Dependence). *Given relations $\tilde{\rightsquigarrow}$ and $\div$ for base B such that* Cond$\tilde{\rightsquigarrow}$ *holds, if $\div$ is a kernel contraction, then $\tilde{\rightsquigarrow}$ is a base dependence.*

For this theorem, we show one by one that the properties of base dependence hold, assuming Cond$\tilde{\rightsquigarrow}$ and axioms of kernel contraction.

6.4 Axiomatic Characterization

In order to provide an axiomatic characterization theorem, in common with the previous works (Remark 13), we assume that the given contraction operator satisfies inclusion.

Theorem 16 (Characterization). *Let the relations $\tilde{\rightsquigarrow}$ and $\div$ for base B be such that $\div$ satisfies* inclusion*, and that* Cond$\tilde{\rightsquigarrow}$ *holds. Then, $\div$ is a kernel contraction if and only if $\tilde{\rightsquigarrow}$ is a base dependence.*

To prove this characterization theorem, we show that in presence of inclusion, Cond$\tilde{\rightsquigarrow}$ entails Cond$\bar{\div}$. Thus, assuming inclusion and Cond$\tilde{\rightsquigarrow}$, based on Theorems 14 and 15, kernel contraction and base dependence are logically equivalent.

7 Discussion and Related Work

Many authors have shown interest in defining the concepts of dependence/relevance of formulas. Hansson and Wassermann [2002] classify the works of these authors into two groups. One group, which includes [Parikh, 1999; Chopra and Parikh, 2000; Makinson and Kourousias, 2006; Kourousias and Makinson, 2007; Makinson, 2007; Ji *et al.*, 2008; Suntisrivaraporn *et al.*, 2008; Ismail and Kasrin, 2010; Wu *et al.*, 2011; Perrussel *et al.*, 2011; Falappa *et al.*, 2011], captures dependence/relevance of formulas through syntactical means, including variable sharing and language splitting. The other group has focused on inferential dependency of formulas. In other words, how some formulas contribute to the inference of some other formulas. Examples of authors interested in this approach include [Fariñas del Cerro and Herzig, 1996; Hansson and Wassermann, 2002; Cuenca Grau *et al.*, 2007; Oveisi *et al.*, 2014], as well as the present work. Usually, syntactical approaches are simpler and more efficient computationally as opposed to inferential approaches. On the other hand, inferential approaches usually give a more accurate and tighter definition of dependence/relevance.

8 Conclusion

Gärdenfors' preservation criterion suggests a particularly interesting way of establishing a link between belief change and dependence. Such dependence relations can in turn be used as a theoretical benchmark against which to evaluate other approximate dependence or relevance relations. We have built on GPC and provided the most general formulation of it currently available (to the best of our knowledge). Basically, there are three corresponding pairs: (1) AGM contraction is a subclass of (2) saturated kernel contraction, which is a subclass of (3) kernel contraction; likewise, (1) FH's dependence is a subclass of (2) strong base dependence, which is a subclass of (3) base dependence. Our formalism connects the most general pair (3) above: kernel contraction and base dependence.

We have explored different conditions to construct base dependence relations using belief contraction operators. In doing so we have fully expanded the usage of an existing condition to construct base dependence, Cond$\tilde{\rightsquigarrow}$. We have also come up with new conditions, strong base dependence

$\mathsf{Cond}_{\hat{\rightsquigarrow}}$ and weak base dependence $\mathsf{Cond}_{\check{\rightsquigarrow}}$, and described their various relations to one another.

We also provide the means to study redundancy in light of weak base dependence. The fact that weak base dependence captures redundancy may be exploited for different purposes. For example, one may use weak base dependence to distinguish between redundant and informative formulas in a belief base.

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