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Stable and Envy-free Partitions in Hedonic Games

Nathanaël Barrot[1,2] and **Makoto Yokoo**[2,1]
[1]Riken, AIP Center
[2]Kyushu University
nathanaelbarrot@gmail.com, yokoo@inf.kyushu-u.ac.jp

Abstract

In this paper, we study coalition formation in hedonic games through the fairness criterion of envy-freeness. Since the grand coalition is always envy-free, we focus on the conjunction of envy-freeness with stability notions. We first show that, in symmetric and additively separable hedonic games, an individually stable and justified envy-free partition may not exist and deciding its existence is NP-complete. Then, we prove that the top responsiveness property guarantees the existence of a Pareto optimal, individually stable, and envy-free partition, but it is not sufficient for the conjunction of core stability and envy-freeness. Finally, under bottom responsiveness, we show that deciding the existence of an individually stable and envy-free partition is NP-complete, but a Pareto optimal and justified envy-free partition always exists.

1 Introduction

Coalition formation plays a major role in our social, economic, or politic life. In examples as diverse as academic research or labor unions, individuals (*agents*) perform activities in groups (*coalitions*) rather than on their own. In coalition formation with hedonic preferences, or *hedonic games*, each agent's preference over the set of coalitions only concerns the coalitions that she joins. The outcome of such game is a set of disjoint coalitions of all agents (*partition*). A natural question is which coalitions are expected to form based on agents' preferences. The main criterion to analyze this question is stability, for which many notions have been discussed in the literature (see handbook chapter Aziz and Savani [2016]).

Introduced by Foley [1967], *envy-freeness* is a notion of fairness which has a broad range of applicability. This notion has received considerable attention in resource allocation [Chevaleyre *et al.*, 2006]. Since envy-freeness can be trivially achieved (by not allocating any resources), the central issue in resource allocation is the study of envy-free outcomes that additionally satisfy efficiency or feasibility requirements. Contrary to other well-studied fairness notions, e.g. maxmin or proportionality, envy-freeness does not require to compare inter-personnal preferences, and thus, it can be meaningfully expressed in ordinal settings like hedonic games.

In hedonic games, a partition is said to be envy-free if no agent prefers another agent's coalition. Similarly to resource allocation, envy-freeness is always trivially achieved by the grand coalition and the partition of singletons. Since these two partitions are not always relevant, requiring only envy-freeness may not be restrictive enough. Hence, it makes sense to impose additional requirements, like stability or feasibility, in conjunction with envy-freeness. However, only a couple of works have done so in hedonic games, Aziz *et al.* [2013b] and Ueda [2018], following two different directions: The first paper considers envy-freeness in conjunction with stability and efficiency requirements. The second one restricts the set of feasible partitions and formulates *justified envy-freeness*, a stronger fairness notion motivated by matching theory.

Following the first direction, we investigate the conjunction of envy-freeness with stability or efficiency notions in three subclass of hedonic games where some level of stability is guaranteed: *additively separable* hedonic games (ASHG) which form a natural subclass, and *top* or *bottom responsive* hedonic games which encompass well-studied hedonic games with realistic interpretations, e.g., friend-enemy oriented hedonic games [Dimitrov *et al.*, 2006] or $\mathcal{B}$-hedonic games [Cechlárová and Romero-Medina, 2001].

1.1 Our Contribution

First, we propose a natural weakening of justified envy-freeness and explore the implications between stability and envy-freeness notions (Figure 1). In symmetric ASHGs, we strengthen results from Aziz *et al.* [2013b], by proving that an individually stable and justified envy-free partition may not exist and deciding its existence is NP-complete. We also show that a Pareto optimal and justified envy-free partition may not exist. In top responsive hedonic games, we propose the *Extended Top Covering Algorithm* which returns a Pareto optimal, individually stable, and envy-free partition. In bottom responsive hedonic games, we show that deciding the existence of an individually stable and envy-free partition is NP-complete, but a Pareto optimal and justified envy-free partition is guaranteed. Table 1 summarizes our results.

Note that the partition of singletons is always envy-free and individually rational. Hence, our paper settles the existence of partitions satisfying any conjunction of stability and envy-freeness notions from Figure 1, in the three class at hand.

	General	Symmetric/Mutual
ASHG	—	IS+JEV: NP-c (Prop.1, Th.1) PO+JEV: $\nexists$ (Prop.2)
TR	PO+IS+EF: P (Th.2) CS+EF: $\nexists$ (Prop.3)	SSNS+EF: $\exists$ (Th.3)
BR	PO+JEF: $\exists$ (Th.5)	—
SBR	—	IS+EF: NP-c (Prop.4, Th.4) CS/PO+WJEF: $\nexists$ (Prop.4/5)

Table 1: Summary of the results. TR, BR, and SBR refer to top, bottom, and strong bottom responsiveness. $\exists$ and $\nexists$ means that the desired partition *always* exists and *may not* exist. P and NP-c stand for *polynomial computation* and *NP-complete existence* problem.

1.2 Related Work

In hedonic games, initiated by Banerjee *et al.* [2001] and Bogomolnaia and Jackson [2002], the central question is to identify the conditions on preferences that guarantee stability. An important contribution is Aziz and Brandl [2012] which clarified the inclusions between well-studied stability concepts. In fractional hedonic games, introduced by Aziz *et al.* [2014], Brandl *et al.* [2015] showed that individually stable outcomes may not exist, Bilò *et al.* [2018] extensively studied Nash stability, and Monaco *et al.* [2018] explored Nash and core stability in a slightly modified setting. Concerning efficiency notions, Aziz *et al.* [2013a] proposed an algorihm computing Pareto optimal and individually rational partitions, and Elkind *et al.* [2016] introduced the price of Pareto optimality.

Among many complexity results, Ballester [2004] showed that deciding the existence of individually stable partitions is NP-complete, and Peters and Elkind [2015] developed a framework for proving NP-hardness of existence problems.

Envy-freeness has been extensively studied as a fairness criterion in resource allocation. In ordinal settings, Brams *et al.* [2003] analyzed conflicts between Pareto optimality and envy-freeness. When the preferences are incomplete, Bouveret *et al.* [2010] proposed possible and necessary envy-freeness. Justified envy-freeness is motivated in two-sided matching theory, particularly in the school choice setting, where Abdulkadiroğlu and Sönmez [2003] argued that it could lead to legal actions, and Fragiadakis *et al.* [2016] studied its compatibility with efficiency and strategy-proofness.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ denote the *set of agents* and $\mathcal{N}_i$ denote the set of all subsets of N that contain agent i. A *coalition* $X \subseteq N$ is a subset of agents and $\mathcal{X}_i$ denotes the set of all subsets of X that contain agent i. A *partition* π is a set of disjoint coalitions, containing all agents. Let $\pi(i)$ denote the coalition to which agent i belongs in π. A *hedonic game* (N, P) is defined by a set of agents N and a *preference profile* $P = (\succcurlyeq_i)_{i \in N}$. Each agent i has a *preference*, denoted $\succcurlyeq_i$, which is a weak order on the coalitions to which she belongs; let $\succ_i$ and $\sim_i$ respectively denote the strict preference and the indifference relation derived from $\succcurlyeq_i$.

Additively separable hedonic games (ASHG) form a natural class of hedonic game where each agent has a value for any other agent and the utility that an agent derives from a coalition is the sum of the values she has for its members.

Definition 1 (ASHG)**.** A hedonic game $(N, \succcurlyeq)$ is additively separable if for each agent $i \in N$ there exists a utility function $v_i : N \mapsto \mathbb{R}$ such that $v_i(i) = 0$ and for any two coalitions $S, T \in \mathcal{N}_i$, $S \succcurlyeq_i T \Leftrightarrow \sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j)$.

An ASHG is *symmetric* if any two agents, $i, j \in N$, associate the same value to each other, i.e., $v_i(j) = v_j(i)$.

Envy-freeness is a notion of fairness in which no agent has envy toward another agent. Informally, a partition is *envy-free* if no agent prefers another agent's coalition over his own.

Definition 2 (Envy-freeness (EF))**.** In partition π, agent i has envy toward an agent j with $\pi(i) \neq \pi(j)$, if $(\pi(j) \setminus \{j\}) \cup \{i\} \succ_i \pi(i)$. Partition π is envy-free if no agent has envy.

Justified envy-freeness is a stronger notion of fairness, formulated in hedonic games by Ueda [2018]. We propose a natural weakening of justified envy. Agent i has (weakly) justified envy toward agent j if i has envy toward j and each agent in j's coalition is (weakly) in favor of replacing j by i.

Definition 3 (Justified envy-freeness (JEF))**.** In partition π, i has justified envy toward j with $\pi(i) \neq \pi(j)$, if i has envy toward j and for all $j' \in \pi(j) \setminus \{j\}$, $(\pi(j) \setminus \{j\}) \cup \{i\} \succ_{j'} \pi(j)$. Partition π is justified envy-free if no agent has justified envy.

Definition 4 (Weakly justified envy-freeness (WJEF))**.** In partition π, agent i has weakly justified envy toward j with $\pi(i) \neq \pi(j)$, if i has envy toward j and for all $j' \in \pi(j) \setminus \{j\}$, $(\pi(j) \setminus \{j\}) \cup \{i\} \succcurlyeq_{j'} \pi(j)$. Partition π is weakly justified envy-free if no agent has weakly justified envy.

The following example illustrates these fairness notions.

Example 1. Consider a symmetric ASHG with four agents $\{1, 2, 3, 4\}$ and values $v_1(2) = v_2(3) = v_3(1) = 2$, $v_2(4) = v_3(4) = x$, and $v_1(4) = -7$. In this game, consider partition $\{\{1, 2, 3\}, \{4\}\}$. If $x = 1$ then agent 4 has envy toward agent 1, but not weakly justified envy. If $x = 2$ then 4 has weakly justified envy toward 1, but not justified envy. Finally, if $x = 3$ then 4 has justified envy toward 1.

By definition, envy-freeness implies weakly justified envy-freeness, which implies justified envy-freeness. Example 1 further shows that the opposite relations do not hold.

Below, we define stability concepts and Pareto optimality. First, we say that partition π' is *reachable* from partition π by a set of agents H, denoted $\pi \xrightarrow{H} \pi'$, if for all $i, j \in N \setminus H$, $i \neq j : \pi(i) = \pi(j) \Leftrightarrow \pi'(i) = \pi'(j)$.

Definition 5 (Stability/Pareto optimality)**.** For a specific concept X, we say that partition π is X if there exists no X deviation, where an X deviation is defined as follows:

- *Individually rational (IR)*: agent i such that $\{i\} \succ_i \pi(i)$.
- *Individually stable (IS)*: agent i and coalition $X \in \pi \cup \{\emptyset\}$ such that $X \cup \{i\} \succ_i \pi(i)$ and for all $j \in X$, $X \cup \{i\} \succcurlyeq_j X$.
- *Strong individually stable (SIS)*: partition π' and set of agents H such that 1) $\pi \xrightarrow{H} \pi'$, 2) for all $i \in H$, $\pi'(i) \succ_i \pi(i)$, and 3) for all $i \in H$, for all $j \in \pi'(i)$, $\pi'(j) \succcurlyeq_j \pi(j)$.
- *Nash stable (NS)*: agent i and coalition $X \in \pi \cup \{\emptyset\}$ such

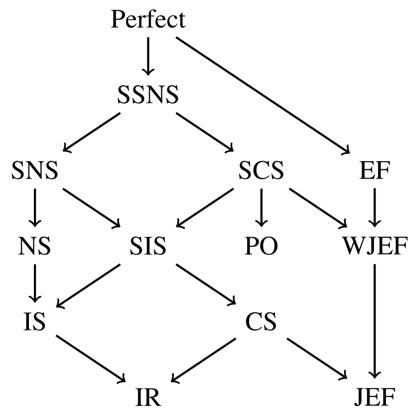

Figure 1: Implications between stability and envy-freeness notions.

that $X \cup \{i\} \succ_i \pi(i)$.
• *Strong Nash stable (SNS)*: partition π' and set of agents H such that 1) $\pi \xrightarrow{H} \pi'$ and 2) for all $i \in H$, $\pi'(i) \succ_i \pi(i)$.
• *Strict strong Nash stable (SSNS)*: partition π' and set of agents H such that 1) $\pi \xrightarrow{H} \pi'$, 2) for all $i \in H$, $\pi'(i) \succsim_i \pi(i)$, and 3) there exists $i \in H$ such that $\pi'(i) \succ_i \pi(i)$.
• *Core stable (CS)*: coalition $X \subseteq N$ such that for all $i \in X, X \succ_i \pi(i)$.
• *Strict core stable (SCS)*: coalition $X \subseteq N$ such that for all $i \in X, X \succsim_i \pi(i)$ and there exists $i \in X$ s.t. $X \succ_i \pi(i)$.
• *Pareto optimal (PO)*: partition π' such that for all $i \in N, \pi'(i) \succsim_i \pi(i)$ and there exists $i \in N$ s.t. $\pi'(i) \succ_i \pi(i)$.
• *Perfect*: coalition $X \subseteq N$ s.t. for some $i \in X, X \succ_i \pi(i)$.

Let us now describe the implications between envy-freeness and stability notions, summarized in Figure 1.

First, remark that the grand coalition is always envy-free but not always individually rational or Pareto optimal. Thus envy-freeness notions do not imply any of the stability concepts defined in this paper. Notice further that a perfect partition is envy-free, since no agent can improve. Moreover, an agent who has (weakly) justified envy forms a (strict) core deviation with the corresponding coalition. Thus, (strict) core stability implies (weakly) justified envy-freeness.
Finally, strict strong Nash stability does not imply envy-freeness; neither strong Nash stability nor Pareto optimality implies weakly justified envy-freeness; and Nash stability does not imply justified envy-freeness. Indeed, when x equals 1, 2, or 3 in Example 1, partition $\{\{1, 2, 3\}, \{4\}\}$ is respectively strict strong Nash stable, strong Nash stable and Pareto optimal, or Nash stable. Implications between stability notions are taken from Aziz and Brandl [2012].

We assume readers' familiarity with complexity class NP. In our proofs, we use the problem EXACTCOVERBY3SETS, which is NP-complete even when each $i \in Y$ appears in *at most three sets* $s \in S$ [Garey and Johnson, 2002].

Definition 6 (EXACTCOVERBY3SETS (X3C))**.** Consider a set $Y = \{1, \ldots, 3n\}$ and a collection $S = \{s_1, \ldots, s_m\}$ of subsets of Y such that for all $s \in S, |s| = 3$. Does there exist a subset $S' \subseteq S$ such that S' is a partition of Y?

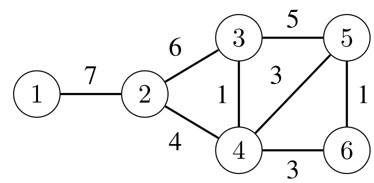

Figure 2: A symmetric ASHG with no IS and JEF partition.

3 Justified Envy in Symmetric ASHGs

In symmetric ASHGs, it is known that Nash (and thus individually) stable partitions always exist [Bogomolnaia and Jackson, 2002]. However, Aziz *et al.* [2013b] show that an individually stable and envy-free partition may not exist, and that deciding the existence of such partition is NP-complete. We strengthen both results with Proposition 1 and Theorem 1, showing that they even hold for justified envy-freeness.

Proposition 1. *In symmetric ASHGs, an individually stable and justified envy-free partition may not exist.*

The proof is based on the following counterexample.

Example 2. Consider the symmetric ASHG, illustrated by Figure 2, with six agents $\{1, \ldots, 6\}$ and values $v_1(2) = 7$, $v_2(3) = 6$, $v_2(4) = 4$, $v_3(4) = 1$, $v_3(5) = 5$, $v_4(5) = v_4(6) = 3$, and $v_5(6) = 1$. All other values are equal to -9.

Proof. Assume that there exists an individually stable and justified envy-free partition π in Example 2. First, if π contains a coalition where two agents have value -9 for each other, then π is not individually rational for at least one of these agents. Therefore, we limit the following study to coalitions containing no negative values, i.e., cliques in Figure 2.

Assume that coalition $\{1, 2\}$ belongs to π. If agent 3 is in coalition $\{3\}$ or coalition $\{3, 4\}$, then agent 5 individually deviates toward 3's coalition. Hence coalition $\{3, 4, 5\}$ belongs to π, but then agent 2 has justified envy toward 5.

Thus, coalition $\{1, 2\}$ does not belong to π, which implies that agent 1 is in a singleton coalition. If 2 is in coalition $\{2\}$, $\{2, 3\}$, or $\{2, 4\}$, then 2 individually deviates toward $\{1\}$. Hence, coalition $\{2, 3, 4\}$ belongs to π. Now, if agent 5 is in a singleton coalition then 5 deviates toward $\{6\}$, and thus $\{5, 6\}$ belongs to π. It leads to $\pi = \{\{1\}, \{2, 3, 4\}, \{5, 6\}\}$, but then 4 individually deviates toward $\{5, 6\}$. ☐

Note that Proposition 1 directly implies that the existence of an {IS, NS} and {JEF, WJEF} partition is not guaranteed. Moreover, Example 2 is minimal in the sense that an individually stable and justified envy-free partition always exists for five agents or less (we omit the case study proof). An interesting question is then the complexity of deciding whether an individually stable and jusified envy-free partition exists.

Theorem 1. *Given an ASHG, deciding whether an individually stable and justified envy-free partition exists is NP-complete, even with symmetric preferences.*

Proof. First, the problem is in NP since checking these properties is polynomial. Similarly to Aziz *et al.* [2013b], we show NP-hardness by reduction from X3C (see Definition 6).

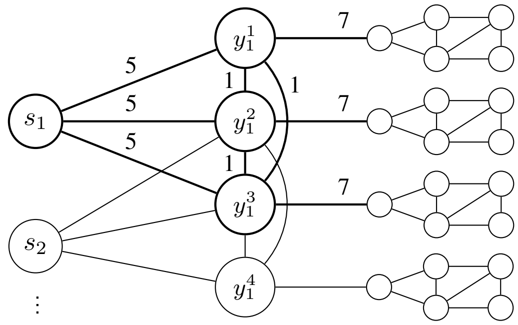

Figure 3: Symmetric ASHG obtained by reduction of an X3C instance with $s_1 = \{1, 2, 3\}$, $s_2 = \{2, 3, 4\}, \ldots$, with focus on s_1.

Consider (Y, S) an instance of X3C. From (Y, S), we construct a symmetric ASHG as follows, illustrated by Figure 3:

• For each $i \in Y$, we create six agents $(y_j^i)_{j=1,\ldots,6}$ that form an instance of Example 2.

• For each set $s = \{i, j, k\} \in S$, we create agent s such that $v_s(y_1^i) = v_s(y_1^j) = v_s(y_1^k) = 5$, and we also set $v_{y_1^i}(y_1^j) = v_{y_1^j}(y_1^k) = v_{y_1^k}(y_1^i) = 1$.

• All other values are equal to -11.

Before the main argument, notice that if an agent y_1^i is isolated from agents $(y_j^i)_{j=2,\ldots,6}$, then the (partial) partition $\{\{y_2^i, y_3^i\}, \{y_4^i, y_5^i, y_6^i\}\}$ is the only individually stable and justified envy-free partition for agents $(y_j^i)_{j=2,\ldots,6}$.

($\Rightarrow$) Assume that there exists a partition $S' \subseteq S$ of Y. Consider partition $\pi = \{\{s, y_1^i, y_1^j, y_1^k\} \mid s = \{i, j, k\} \in S'\} \cup \{\{y_2^i, y_3^i\}, \{y_4^i, y_5^i, y_6^i\}\} \mid i \in Y\} \cup \{\{s\} \mid s \in S \setminus S'\}$. We argue that π is individually stable and justified envy-free. First, for each $i \in Y$, agent y_1^i has value 7 in π. Thus, no agent y_1^i has incentive to individually deviate toward another coalition or has envy toward agent y_3^i. Hence, each y_1^i is isolated from agents $(y_j^i)_{j=2,\ldots,6}$, and, as argued above, each $\{\{y_2^i, y_3^i\}, \{y_4^i, y_5^i, y_6^i\}\}$ is individually stable and justified envy-free. Lastly, since for all $s \in S$ and $i \in Y \setminus s$, $v_s(y_1^i) = -11$, no agent $s \in S \setminus S'$ has envy toward any $s \in S'$.

($\Leftarrow$) Assume that there exists an individually stable and justified envy-free partition π. As shown in Example 2, if an agent y_1^i is in coalition $\{y_1^i\}$ or $\{y_1^i, y_2^i\}$, then π is not individually stable and justified envy-free. Hence each agent y_1^i is in coalition with agents from $S \cup \{y_1^j \mid j \in Y\}$, and thus, each coalition $\{\{y_2^i, y_3^i\}, \{y_4^i, y_5^i, y_6^i\}\}$ belongs to π.

Now, assume that for some $i \in Y$, $S \cap \pi(y_1^i) = \emptyset$. Since each $i \in Y$ appears in at most three sets $s \in S$, agent y_1^i has value at most 6 in $\pi(y_1^i)$, but then y_1^i has justified envy toward y_3^i. Hence for all $i \in Y$, $S \cap \pi(y_1^i) \neq \emptyset$. The only individually rational coalitions to which agent y_1^i belongs are $\{s, y_1^i\}$, $\{s, y_1^i, y_1^j\}$, and $\{s, y_1^i, y_1^j, y_1^k\}$, for some $s = \{i, j, k\} \in S$. However, if y_1^i is in coalition $\{s, y_1^i\}$ or $\{s, y_1^i, y_1^j\}$, then y_1^i has value at most 6 and justified envy toward agent y_3^i. Therefore, each y_1^i belongs to a coalition $\{s, y_1^i, y_1^j, y_1^k\}$ for some $s = \{i, j, k\} \in S$, which form a partition of Y. □

In addition, the reduction holds as it is for any conjunction of {IS, NS} and {JEF, WJEF}.

Finally, Pareto optimal partitions always exist by definition. However, we show that Pareto optimal and justified envy-free partitions may not exist in symmetric ASHGs.

Proposition 2. *In symmetric ASHGs, a Pareto optimal and justified envy-free partition may not exist.*

We omit the argument of the proof based on Example 3.

Example 3. Consider the symmetric ASHG with nine agents $\{1, \ldots, 9\}$ and values $v_1(2) = 8$, $v_2(3) = 7$, $v_2(4) = 4$, $v_3(4) = 2$, $v_3(5) = 6$, $v_4(5) = 3$, $v_4(6) = 4$, $v_5(6) = 2$, $v_5(7) = v_6(7) = 1$, and $v_3(8) = v_5(8) = 1$. All other values are equal to -10.

4 Top Responsiveness and Envy-freeness

Top responsiveness is a condition on agent's preference, introduced in Alcalde and Revilla [2004]. Intuitively, it requires that an agent's preference over two coalitions is responsive to her best subsets (*choice set*) in each coalition. Given agent $i \in N$ and coalition $X \in \mathcal{N}_i$, let $Ch(i, X)$ denote the *choice set* of i in X, i.e., $Ch(i, X) = \{Y \in \mathcal{X}_i : \forall\, Y' \in \mathcal{X}_i, Y \succcurlyeq_i Y'\}$.

A hedonic game $(N, \succcurlyeq)$ satisfies *top responsiveness* if for all $i \in N$, the three following conditions are satisfied:

(1) for all $X \in \mathcal{N}_i$, $|Ch(i, X)| = 1$,
and let $ch(i, X)$ denote the unique element of $Ch(i, X)$,

(2) for all $X, Y \in \mathcal{N}_i$, $ch(i, X) \succ_i ch(i, Y) \Rightarrow X \succ_i Y$,

(3) for all $X, Y \in \mathcal{N}_i$, $[ch(i, X) = ch(i, Y)$ and $X \subsetneq Y] \Rightarrow X \succ_i Y$.

Further, a top responsive hedonic game is *mutual* if for all $i, j \in N$ and $X \in \mathcal{X}_i \cap \mathcal{X}_j$, $j \in ch(i, X) \Leftrightarrow i \in ch(j, X)$.

Dimitrov and Sung [2007] prove that top responsiveness guarantees the existence of the strict core, relying on the Top Covering Algorithm, which outputs a strict core stable partition. Hence, we ask whether top responsiveness guarantees the existence of a (strict) core stable and envy-free partition.

Proposition 3. *Top responsiveness does not guarantee the existence of a core stable and envy-free partition.*

Example 4. Consider four agents $\{1, 2, 3, 4\}$ and the following preferences, which can be extended to be top responsive:

$1 : \{1\} \succ_1 \ldots$
$2 : \{1, 2, 3\} \succ_2 \{1, 2, 3, 4\} \succ_2 \{1, 2\} \sim_2 \{2, 3\} \succ_2 \ldots$
$3 : \{2, 3\} \succ_3 \{1, 2, 3\} \sim_3 \{2, 3, 4\} \succ_3 \ldots$
$4 : \{3, 4\} \succ_4 \{1, 3, 4\} \sim_4 \{2, 3, 4\} \succ_4 \ldots$

Proof. In this example, the only core stable partition is $\{\{1\}, \{2, 3\}, \{4\}\}$, however, agent 4 has envy toward 2. □

Proposition 3 directly holds for the conjunction of envy-freeness and {SIS, SCS}. Besides, top responsiveness does not guarantee Nash stable partitions [Dimitrov and Sung, 2006]. The remaining questions concern the conjunction of envy-freeness with individual stability or Pareto optimality.

Theorem 2. *In top responsive hedonic games, a Pareto optimal, individually stable, and envy-free partition always exists, and can be computed in polynomial time.*

Algorithm 1 Extended Top Covering Algorithm

Input: Top responsive hedonic game $(N, \succcurlyeq)$
Output: π, a partition of N

$R^1 := N, \pi := \emptyset$
for $k = 1$ to $|N|$ **do**
  Select agent $i \in R^k$ such that for all $j \in R^k$ $|CC(i, R^k)| \leq |CC(j, R^k)|$
  $\pi^k := CC(i, R^k)$
  if $|CC(i, R^k)| \geq 2$ **then**
    while $\exists\, j \in R^k \setminus \pi^k : ch(j, R^k) \cap \pi^k \neq \emptyset$ **do**
      $\pi^k := \pi^k \cup CC(j, R^k)$
    end while
  end if
  $\pi := \pi \cup \{\pi^k\}, R^{k+1} := R^k \setminus \pi^k$
  if $R^{k+1} = \emptyset$ **then**
    return π
  end if
end for
return π

Proposition 3 implies that the Top Covering Algorithm does not return an envy-free partition. We propose the *Extended Top Covering Algorithm*, which returns a Pareto optimal, individually stable, and envy-free partition, when applied to a top responsive hedonic game. To define it, we introduce additional notations taken from Aziz and Brandl [2012].

For a coalition $X \subseteq N$, let $\backsim_X$ denote the binary relation on $X \times X$ such that $i \backsim_X j$ if and only if $j \in ch(i, X)$. We define the *connected component* of i with respect to the binary relation $\backsim_X$, denoted $CC(i, X)$, as the set:

$$\{k \in X \mid \exists\, j_1, \ldots, j_l \in X \mid i \backsim_X j_1 \backsim_X \cdots \backsim_X j_l \backsim_X k\}.$$

Informally, $CC(i, X)$ represents the set of agents in X that are "reachable" from i by an iterative use of $ch(\cdot, X)$. We remark that for all $i \in N$ and $X \in \mathcal{N}_i$, the following inclusions hold: $ch(i, X) \subseteq CC(i, X) \subseteq X$.

The Extended Top Covering Algorithm is formally defined as Algorithm 1. Let us describe its application on Example 4:

Step 1 $R^1 = \{1, 2, 3, 4\}$: $CC(1, R^1) = \{1\}$, $CC(2, R^1) = CC(3, R^1) = \{1, 2, 3\}$, and $CC(4, R^1) = \{1, 2, 3, 4\}$. Agent 1 is selected and $\pi^1 = CC(1, R^1)$.

Step 2 $R^2 = \{2, 3, 4\}$: $CC(2, R^2) = CC(3, R^2) = \{2, 3\}$, and $CC(4, R^2) = \{2, 3, 4\}$. Agent 2 is selected and $\pi^2 = CC(2, R^2) \cup CC(4, R^2) = \{2, 3, 4\}$.
Then, $R^3 = \emptyset$ and the outcome is $\pi = \{\{1\}, \{2, 3, 4\}\}$.

Let us state directly that Algorithm 1 runs in polynomial time, returns a partition of N, and at any step k, for all $i \in \pi^k$: (i) $ch(i, R^k) \subseteq \pi^k$, and (ii) for all step k' such that $k' < k$ and $|\pi^{k'}| \geq 2$, $ch(i, R^{k'}) \cap \pi^{k'} = \emptyset$. To prove Theorem 2, we use the three following lemmas for which we omit the proofs.

Lemma 1. *Consider the k^{th} step of Algorithm 1, then for all $i \in \pi^k, \pi^k \succcurlyeq_i \{i\}$.*

Lemma 2. *Consider the k^{th} step of Algorithm 1. Let $X \subseteq R^k$ such that $\pi^k \subsetneq X$, then for all $i \in \pi^k, \pi^k \succ_i X$.*

Lemma 3. *Consider the k^{th} step of Algorithm 1. Let $X \subseteq R^k$ such that $\pi^k \neq X$ and $\pi^k \cap X = \{i\}$, then $\pi^k \succ_i X$.*

Proof of Theorem 2. Consider a top responsive hedonic game $(N, \succcurlyeq)$ and let π denote the outcome of Algorithm 1.

[IS] Toward a contradiction, assume first that agent $i \in \pi^k$ individually deviates toward $\pi^{k'}$, i.e., $\pi^{k'} \cup \{i\} \succ_i \pi^k$ and for all $j \in \pi^{k'}, \pi^{k'} \cup \{i\} \succcurlyeq_j \pi^{k'}$. Remark first that Lemma 1 implies that $\pi^k \succcurlyeq_i \{i\}$, and thus $\pi^{k'} \neq \emptyset$.

• If $k < k'$, then $\pi^{k'} \subseteq R^k$ and $\pi^k \cap (\pi^{k'} \cup \{i\}) = \{i\}$. By Lemma 3 [for $\pi^k \leftarrow \pi^k$, $X \leftarrow \pi^{k'} \cup \{i\}$], we get that $\pi^k \succ_i \pi^{k'} \cup \{i\}$, which is a contradiction.

• If $k' < k$, then $\pi^{k'} \subsetneq \pi^{k'} \cup \{i\} \subseteq R^{k'}$. By Lemma 2 [for $\pi^k \leftarrow \pi^{k'}$, $X \leftarrow \pi^{k'} \cup \{i\}$], we get that for all $j \in \pi^{k'}$, $\pi^{k'} \succ_j \pi^{k'} \cup \{i\}$, which is a contradiction.

[PO] Assume now that π is not Pareto optimal, i.e., there exists partition π' such that for all $i \in N$, $\pi'(i) \succcurlyeq_i \pi(i)$ and there exists $i \in N$ such that $\pi'(i) \succ_i \pi(i)$. Consider the set $A = \{i \in N \mid \pi(i) \neq \pi'(i)\}$, i.e., the set of agents who are in different coalitions in π and π'. Let k denote the first step of Algorithm 1 such that $\pi^k \cap A \neq \emptyset$, and consider agent $i \in \pi^k$.

• If $\pi^k \subseteq \pi'(i)$, then by Lemma 2 [for $\pi^k \leftarrow \pi^k$, $X \leftarrow \pi'(i)$], we get that $\pi^k \succ_i \pi'(i)$, which is a contradiction.

• If $\pi^k \nsubseteq \pi'(i)$, then there exists $j \in \pi^k$ (maybe $j = i$) such that $ch(j, R^k) \nsubseteq \pi'(j)$. In addition, by definition of step k, $\pi'(j) \subseteq R^k$, and thus $ch(j, R^k) \succ_j ch(j, \pi'(j))$. Furthermore, since $ch(j, R^k) \subseteq \pi^k$, $ch(j, \pi^k) = ch(j, R^k)$. Hence, $ch(j, \pi^k) \succ_j ch(j, \pi'(j))$, and top responsiveness implies $\pi^k \succ_j \pi'(j)$, which is a contradiction.

[EF] Finally, assume that agent $i \in \pi^k$ has envy toward agent $j \in \pi^{k'}$, i.e., $(\pi^{k'} \setminus \{j\}) \cup \{i\} \succ_i \pi^k$. First, Lemma 1 implies $\pi^k \succcurlyeq_i \{i\}$, and thus $(\pi^{k'} \setminus \{j\}) \neq \emptyset$.

• If $k < k'$, then $(\pi^{k'} \setminus \{j\}) \subseteq R^k$ and $\pi^k \cap ((\pi^{k'} \setminus \{j\}) \cup \{i\}) = \{i\}$. By Lemma 3 [for $\pi^k \leftarrow \pi^k$, $X \leftarrow (\pi^{k'} \setminus \{j\}) \cup \{i\}$], we get that $\pi^k \succ_i (\pi^{k'} \setminus \{j\}) \cup \{i\}$, a contradiction.

• If $k' < k$, we start by showing that for all step k'' such that $k' \leq k''$, $|\pi^{k''}| \geq 2$. First, since $\pi^{k'} \setminus \{j\} \neq \emptyset$, $|\pi^{k'}| \geq 2$. By contradiction, consider the first step k^* such that $k' < k^*$ and $|\pi^{k^*}| = 1$, and denote by l the agent such that $\pi^{k^*} = \{l\}$. For all k'' such that $k' \leq k'' < k^*$, $|\pi^{k''}| \geq 2$, and thus $ch(l, R^{k''}) \cap \pi^{k''} = \emptyset$. Hence, $ch(l, R^{k''}) = ch(l, R^{k''} \setminus \pi^{k''})$ holds. In total, we obtain $ch(l, R^{k'}) = ch(l, R^{k'} \setminus \bigcup_{k' \leq k'' < k^*} \pi^{k''}) = ch(l, R^{k^*}) = \{l\}$. Hence, $CC(l, R^{k'}) = \{l\}$, which contradicts $|\pi^{k'}| \geq 2$.

Therefore, for all k'' such that $k' \leq k'' < k$, $|\pi^{k''}| \geq 2$, and then $ch(i, R^{k''}) \cap \pi^{k''} = \emptyset$. It implies that $ch(i, R^{k''}) = ch(i, R^{k''} \setminus \pi^{k''})$, and in total we obtain $ch(i, R^{k'}) = ch(i, R^{k'} \setminus \bigcup_{k' \leq k'' < k} \pi^{k''}) = ch(i, R^k)$. Moreover, $ch(i, R^k) \subseteq \pi^k$, and then $ch(i, \pi^k) = ch(i, R^k)$. Thus $ch(i, \pi^k) = ch(i, R^{k'})$. Also, since $ch(i, R^{k'}) \cap \pi^{k'} = \emptyset$, $ch(i, R^{k'}) \cap (\pi^{k'} \setminus \{j\}) = \emptyset$ also holds. By definition of $ch(i, R^{k'})$, it implies $ch(i, R^{k'}) \succ_i ch(i, (\pi^{k'} \setminus \{j\}) \cup \{i\})$. Hence we obtain $ch(i, \pi^k) \succ_i ch(i, (\pi^{k'} \setminus \{j\}) \cup \{i\})$, and then top responsiveness implies $\pi^k \succ_i (\pi^{k'} \setminus \{j\}) \cup \{i\}$, which is a contradiction. □

Finally, when a top responsive hedonic game satisfies mutuality, Aziz and Brandl [2012] show that the outcome of the Top Covering Algorithm is strict strong Nash stable. We further show that its outcome is envy-free.

Theorem 3. *Top responsiveness and mutuality guarantee the existence of strict strong Nash stable and envy-free partitions.*

Proof. Consider a top responsive and mutual hedonic game $(N, \succcurlyeq)$ and let π be the outcome of the Top Cover Algorithm. By Theorem 3 and Lemma 1 in Aziz and Brandl [2012] respectively, π is strict strong Nash stable, and for all $i \in N$, $ch(i,N) \subseteq \pi(i)$. Toward a contradiction, assume that agent i has envy toward agent j in π, i.e., $(\pi(j) \setminus \{j\}) \cup \{i\} \succ_i \pi(i)$.

First assume $\pi(i) = \{i\}$. It implies $(\pi(j) \setminus \{j\}) \neq \emptyset$, and then $\{i\} \subsetneq (\pi(j) \setminus \{j\}) \cup \{i\}$. Since $ch(i,N) \subseteq \pi(i)$, it holds that $ch(i,N) = \{i\}$. Hence, by definition of $ch(i,N)$, $\{i\} = ch(i,\{i\}) = ch(i,(\pi(j) \setminus \{j\}) \cup \{i\})$. With $\{i\} \subsetneq (\pi(j) \setminus \{j\}) \cup \{i\}$, top responsiveness implies $\{i\} = \pi(i) \succ_i (\pi(j) \setminus \{j\}) \cup \{i\}$, which is contradiction.

Thus $\pi(i) \neq \{i\}$. Since π is individually rational, it implies $ch(i,N) \neq \{i\}$, and thus $ch(i,N) \nsubseteq (\pi(j) \setminus \{j\}) \cup \{i\}$. By definition of $ch(i,N)$, for all $S \in \mathcal{N}_i$ and $S \neq ch(i,N)$, $ch(i,N) \succ_i S$. Hence, for all $S \subseteq (\pi(j) \setminus \{j\}) \cup \{i\}$, $ch(i,N) \succ_i S$, and thus $ch(i,N) \succ_i ch(i,(\pi(j) \setminus \{j\}) \cup \{i\})$. Moreover, since $ch(i,N) \subseteq \pi(i)$, we obtain $ch(i,\pi(i)) = ch(i,N)$, and thus $ch(i,\pi(i)) \succ_i ch(i,(\pi(j) \setminus \{j\}) \cup \{i\})$. Hence, top responsiveness implies $\pi(i) \succ_i (\pi(j) \setminus \{j\}) \cup \{i\}$, which is a contradiction. □

5 Bottom Responsiveness and Envy-freeness

In opposition to top responsiveness, *bottom responsiveness* [Suzuki and Sung, 2010] is based on the notion of *avoid set*. Given agent $i \in N$ and coalition $X \in \mathcal{N}_i$, the avoid set of i in X is $Av(i,X) = \{Y \in \mathcal{X}_i : \forall\, Y' \in \mathcal{X}_i, Y' \succcurlyeq_i Y\}$. In other words, $Av(i,X)$ denotes the worst subsets in X for agent i. A hedonic game $(N, \succcurlyeq)$ satisfies *bottom responsiveness* if for all $i \in N$, the two following conditions are satisfied:

(1) for all $X, Y \in \mathcal{N}_i$, [for all $X' \in Av(i,X)$ and $Y' \in Av(i,Y)$, $X' \succ_i Y'$] $\Rightarrow X \succ_i Y$,

(2) for all $X, Y \in \mathcal{N}_i$, [$Av(i,X) \cap Av(i,Y) \neq \emptyset$ and $|X| \geq |Y|$] $\Rightarrow X \succcurlyeq_i Y$.

A bottom responsive hedonic game is *strongly bottom responsive* if for all $i \in N$ and $X \in \mathcal{X}_i$, $|Av(i,X)| = 1$, and it satisfies *mutuality* if for all $i, j \in N$ and $X \in \mathcal{X}_i \cap \mathcal{X}_j$, $j \in Av(i,X) \Leftrightarrow i \in Av(j,X)$.

Aziz and Brandl [2012] show that bottom responsiveness guarantees a strong individually stable partition, and that strong bottom responsiveness together with mutuality guarantee a strong Nash stable partition. Yet, we show that even the latter one does not guarantee the existence of a weakly justified envy-free and individually or core stable partition.

Proposition 4. *In a strong bottom responsive hedonic game, a weakly justified envy-free and individually or core stable partition may not exist, even with mutual preferences.*

Example 5. Consider three agents $\{1,2,3\}$ with preferences:

$$1 : \{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\}$$
$$2 : \{1,2,3\} \succ_2 \{1,2\} \sim_2 \{2,3\} \succ_2 \{2\}$$
$$3 : \{2,3\} \succ_3 \{3\} \succ_3 \{1,2,3\} \succ_3 \{1,3\}$$

Proof. In Example 5, partitions $\{\{1,2,3\}\}$ and $\{\{1,3\},\{2\}\}$ are not individually rational for agents 1 and 3. Also, partition $\{\{1\},\{2\},\{3\}\}$ is not core or individually stable, since $\{1,2\}$ is a deviation. For partitions $\{\{1,2\},\{3\}\}$ and $\{\{1\},\{2,3\}\}$, agent 3 has weakly justified envy toward agent 1 in the former, and 1 has weakly justified envy toward 3 in the latter. Preferences are strong bottom responsive and mutual. □

Proposition 4 directly holds for the conjunction of {WJEF, EF} and {NS, SIS, SNS}. An interesting question is then the complexity of deciding the existence of such partitions.

Theorem 4. *Given a strong bottom responsive hedonic game, deciding whether an individually stable and envy-free partition exists is NP-complete, even with mutual preferences.*

Proof sketch. We show NP-hardness by reduction from X3C.

Consider (Y,S) an instance of X3C with $Y = \{1,\ldots,3n\}$ and $S = \{s_1,\ldots,s_m\}$. We construct a symmetric and strong bottom responsive ASHG, as follows:

• For each $i \in Y$, we create two agents $\{y_1^i, y_2^i\}$ and a "clique" of seven agents K_7^i where for all $l,l' \in K_7^i$, $v_l(l') = 1$. We set for all $l \in K_7^i$, $v_l(y_1^i) = v_l(y_2^i) = 1$.

• For each set $s = \{i,j,k\} \in S$, we create a "clique" of five agents K_5^s where for all $l,l' \in K_5^s$, $v_l(l') = 1$. We set for all $l \in K_5^s$, $v_l(y_1^i) = v_l(y_1^j) = v_l(y_1^k) = 1$, and also $v_{y_1^i}(y_1^j) = v_{y_1^j}(y_1^k) = v_{y_1^k}(y_1^i) = 1$.

• All other values are equal to $-|N| = -(3n \cdot 9 + m \cdot 5)$.

The constructed ASHG is both strong bottom responsive and mutual (indeed, it is a symmetric *enemy aversion hedonic game* [Dimitrov *et al.*, 2006]). Notice that, in individually stable partitions, agents in each "clique" $(K_7^i)_{i \in Y}$ and $(K_5^s)_{s \in S}$ are in the same coalition. The main argument is then similar to the proof of Theorem 1 and we omit it. □

The reduction also holds for the conjunction of {IS, NS} and {WJEF, EF}. A direct corollary is that deciding the existence of an individually stable and envy-free partition is NP-complete in symmetric ASHGs with *only two different values*, and thus it strengthens Theorem 13 in Aziz *et al.* [2013b],

Finally, we state two results on envy-freeness and Pareto optimality under bottom responsiveness, omitting the proofs.

Proposition 5. *In a strong bottom responsive hedonic game, a Pareto optimal and weakly justified envy-free partition may not exist, even with mutual preferences.*

Theorem 5. *In a bottom responsive hedonic game, a Pareto optimal and justified envy-free partition always exists.*

6 Conclusion

We settled the existence of partitions satisfying the conjunction of stability and envy-freeness requirements in three important class of hedonic games. Our main positive result, the Extended Top Cover Algorithm, may not adapt well to weaker notion of top responsiveness. In ASHGs and bottom responsive hedonic games, we show that deciding the existence of an individually stable and envy-free partition is NP-complete. Future works include the study of complexity issues related to core stability and Pareto optimality, leading probably to NP^{NP} difficult problems.

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