Approximately Maximizing the Broker's Profit in a Two-sided Market

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Abstract

We study how to maximize the broker's (expected) profit in a two-sided market, where she buys items from a set of sellers and resells them to a set of buyers. Each seller has a single item to sell and holds a private value on her item, and each buyer has a valuation function over the bundles of the sellers' items. We consider the Bayesian setting where the agents' values/valuations are independently drawn from prior distributions, and aim at designing dominant-strategy incentive-compatible (DSIC) mechanisms that are approximately optimal.

Production-cost markets, where each item has a publicly-known cost to be produced, provide a platform for us to study two-sided markets. Briefly, we show how to covert a mechanism for production-cost markets into a mechanism for the broker, whenever the former satisfies *cost-monotonicity*. This reduction holds even when buyers have general combinatorial valuation functions. When the buyers' valuations are additive, we generalize an existing mechanism to production-cost markets in an approximation-preserving way. We then show that the resulting mechanism is cost-monotone and thus can be converted into an 8-approximation mechanism for two-sided markets.

1 Introduction

Two-sided markets are widely studied in economics [Myerson and Satterthwaite, 1983; McAfee, 1992; McAfee, 2008], where a number of buyers and a number of sellers are connected by an intermediary, such as antique markets, used-car markets, and pre-owned house markets. Here each seller has a single item to trade for money and holds a private value for her item, while each buyer's private information is a combinatorial valuation over the bundles of the sellers' items. A common feature in these situations is that the intermediary keeps the difference between the payments made by the buyers and the payments made to the sellers —that is, the intermediary's *profit*. We call such an intermediary a *broker*. The objective of the broker is to acquire the items from the sellers and resell them to the buyers to maximize her profit. The problem studied in our paper is to design the mechanism in the two-sided market that maximize the broker's profit. For convenience, we refer to the sub-market between the sellers and the broker the *seller-side market* and to the sub-market between the broker and the buyers the *buyer-side market*.

If the broker had all the items, then we would only have the buyer-side market, which is an auction where the broker tries to maximize her revenue. Auctions have been well studied in the literature following the seminal work of Myerson [Myerson, 1981]. In Section 1.2, we will briefly recall the most relevant literature on auctions. If the broker would keep the items, then we only have the seller-side market, which is a procurement game. Budget feasible procurement has been studied by many in the Algorithmic Game Theory literature [Singer, 2010; Dobzinski *et al.*, 2011; Chen *et al.*, 2011; Chan and Chen, 2014]. The broker wants to maximize her value for the items she buys, subject to a budget constraint.

Although auctions and procurements are closely related to the broker's problem, they cannot be dealt with separately in two-sided markets. Indeed, the difficulty of the broker's problem is to simultaneously and truthfully elicit both the sellers' and the buyers' valuations, so as to generate a good profit.

1.1 Main Results and Techniques

In this paper we assume the values of the sellers and buyers are independently distributed, and we study simple *dominant-strategy incentive compatible* (DSIC) mechanisms. To approximately maximize the (expected) profit of the broker, we first develop a reduction, through which we can directly convert mechanisms for *production-cost markets* into mechanisms for two-sided markets. In a production-cost market, the broker is able to produce all the items, each item has a cost to be produced and the costs are publicly known. Roughly speaking, we say a mechanism for production-cost markets is *cost-monotone* if, when the cost of an item increases, the likelihood that it is sold does not increase. We show that any cost-monotone mechanism for production-cost markets can be converted into a mechanism for two-sided markets via a black-box approach. This reduction holds for general combinatorial valuation functions of buyers.

Theorem 1 (Informal). *Any cost-monotone DSIC mechanism that is an α-approximation for production-cost markets, can be converted into a DSIC mechanism that is an α-approximation for two-sided markets.*

Next, we use cost-monotonicity as a guideline in constructing concrete mechanisms for two-sided markets. When the buyers have additive valuations, we generalize the duality framework of [Cai *et al.*, 2016] and the mechanism there to design a cost-monotone mechanism for production-cost markets. Following our reduction, we immediately obtain a mechanism for two-sided markets.

Theorem 3 (Informal). *When the buyers have additive valuations, there exists a DSIC mechanism for two-sided markets which is an 8-approximation to the optimal profit.*

1.2 Related Work

Bayesian auctions have been extensively studied since the seminal work of [Myerson, 1981]. For single-parameter settings, Myerson's mechanism is optimal. The problem becomes more complicated in multi-parameter settings [Hart and Nisan, 2017]. Although optimal Bayesian incentive-compatible (BIC) mechanisms have been characterized [Cai *et al.*, 2012b; Cai *et al.*, 2012a], they are too complex to be practical. Also, optimal DSIC mechanisms remain unknown. Thus simple DSIC mechanisms that are approximately optimal have been studied, such as [Kleinberg and Weinberg, 2012; Yao, 2015; Cai *et al.*, 2016]

Two-sided markets are also called double auctions [McAfee, 1992], bilateral trading [Myerson and Satterthwaite, 1983] or market intermediation [Jain and Wilkens, 2012] in the literature. Maximizing the broker's profit is an important objective for two-sided market. The seminal paper [Myerson and Satterthwaite, 1983] characterized the optimal mechanism for one seller and one buyer, which is further generalized by [Deng *et al.*, 2014] to multiple single-parameter sellers and buyers. Unlike our work, [Deng *et al.*, 2014] studies the Bayesian Incentive Compatible (BIC) mechanisms. DSIC mechanisms are also studied in the literature, but only for some special cases: [Jain and Wilkens, 2012] studies the case of a single buyer and multiple sellers, [Balseiro *et al.*, 2019] studies the case of a single seller and multiple buyers, and [Gerstgrasser *et al.*, 2016] studies the optimal mechanism when the numbers of sellers and buyers are both constants. Although [Chan and Chen, 2016] studies two-sided markets with multiple buyers and multiple sellers, the dealer there has a fixed budget and their mechanism guarantees that the payment to sellers is within the budget. Before our work, it remained unknown how to design a (simple) DSIC mechanism that approximates the optimal profit in multi-parameter settings with a general number of sellers and buyers.

Finally, we briefly discuss the efficiency of two-sided markets, which is measured by *gain-from-trade* (GFT), i.e., the total value gained by the buyers minus the value contributed by the sellers. [McAfee, 1992] gave the first approximation mechanism for the one seller and one buyer case, and [Brustle *et al.*, 2017] gives approximation mechanisms for multiple buyers with unit demand valuations. Recently, [Segal-Halevi *et al.*, 2018a] and [Segal-Halevi *et al.*, 2018b] study the asymptotically efficient mechanisms instead of constant approximations. For maximizing *social welfare*, [Colini-Baldeschi *et al.*, 2016; Colini-Baldeschi *et al.*, 2017] provide constant-approximation mechanisms.

2 Preliminaries

A two-sided market includes a set M of m sellers, and a set N of n buyers. We consider the setting where each seller j has one item j to sell, so we may refer to items and sellers interchangeably. The total payment made by the buyers is the broker's *revenue*, and her *profit* is the revenue minus the total payment to the sellers.

Each buyer i has valuation $v_i^B : 2^M \to \mathbb{R}^+ \cup \{0\}$ with $v_i^B(\emptyset) = 0$. The function v_i^B is monotone: for any $T \subseteq S \subseteq M$, $v_i^B(T) \leq v_i^B(S)$. In our reduction between production-cost and two-sided markets, we consider combinatorial valuations and do not impose any restriction on v_i^B.

Each function v_i^B is independently drawn from a distribution D_i^B over the set of all possible valuation functions, with density function f_i^B and cumulative probability F_i^B. Let $D^B = \times_{i\in N} D_i^B$, $f^B = \times_{i\in N} f_i^B$ and $F^B = \times_{i\in N} F_i^B$. Each seller j's value on her item, $v_j^S \in \mathbb{R}^+ \cup \{0\}$, is independently drawn from a distribution D_j^S, with density function f_j^S and cumulative probability F_j^S. Let $D^S = \times_{j\in M} D_j^S$, $f^S = \times_{j\in M} f_j^S$ and $F^S = \times_{j\in M} F_j^S$. Let the supports of distributions D_i^B and D_j^S be T_i^B and T_j^S, respectively. T_i^B and T_j^S are called the *valuation spaces* of buyer i and seller j. Let $T^B = \times_{i\in N} T_i^B$ and $T^S = \times_{j\in M} T_j^S$. Finally, denote by $\mathcal{I} = (N, M, D^B, D^S)$ a two-sided market instance.

A mechanism $\mathcal{M}$ for two-sided markets is represented by (x^B, x^S, p^B, p^S). Given a valuation profile (v^B, v^S),

- $x^B(v^B, v^S) = (x_i^B(v^B, v^S))_{i\in N}$ is the allocation of the buyers, where $x_i^B(v^B, v^S) = (x_{iA}^B(v^B, v^S))_{A\subseteq M}$ with $x_{iA}^B(v^B, v^S) \in [0, 1]$, representing the probability that buyer i gets the item set A, under valuation profile v^B and v^S. Moreover, $\sum_A x_{iA}^B(v^B, v^S) = 1$.
- $x^S(v^B, v^S) = (x_j^S(v^B, v^S))_{j\in M}$ is the allocation of the sellers with $x_j^S(v^B, v^S) \in [0, 1]$, representing the probability that seller j's item is sold under (v^B, v^S).
- $p^B(v^B, v^S) = (p_i^B(v^B, v^S))_{i\in N}$ is the payment made by the buyers, where $p_i^B(v^B, v^S) \in \mathbb{R}^+ \cup \{0\}$.
- $p^S(v^B, v^S) = (p_j^S(v^B, v^S))_{j\in M}$ is the payment made to the sellers, where $p_j^S(v^B, v^S) \in \mathbb{R}^+ \cup \{0\}$.

A *feasible* mechanism $\mathcal{M}$ is such that

$$\sum_{A\ni j} \sum_{i\in N} x_{iA}^B(v^B, v^S) \leq x_j^S(v^B, v^S)$$

for any item $j \in M$ and any valuation profile (v^B, v^S). In principle, the above condition may allow a mechanism to sell an item that it didn't buy or to buy an item without selling it. However, these cases never happen in the mechanisms in this paper.[1] The expected profit $PFT(\mathcal{M}; \mathcal{I})$ of mechanism $\mathcal{M}$ for instance $\mathcal{I}$ is

$$\mathop{\mathbb{E}}_{v^S\sim D^S; v^B\sim D^B} \sum_{i\in N} p_i^B(v^B, v^S) - \sum_{j\in M} p_j^S(v^B, v^S).$$

[1]Note that our feasibility constraint only requires "feasible in expectation" which is weaker than ex post feasibility. All of our results still hold if we change the requirement to be ex post feasible.

The utilities of the agents are quasi-linear. That is, for each buyer i, for any valuation subprofile v_{-i}^B of the buyers and any valuation profile v^S of the sellers, when i reports her true valuation function v_i^B, her utility under mechanism $\mathcal{M}$ is

$$u_i^B(v_i^B;\mathcal{M},v_{-i}^B,v^S)=\sum_{A\subseteq M}x_{iA}^B(v^B,v^S)v_i^B(A)-p_i^B(v^B,v^S).$$

For each seller j, for any valuation subprofile v_{-j}^S and v^B, when j reports her true value v_j^S, her utility is

$$u_j^S(v_j^S;\mathcal{M},v^B,v_{-j}^S)=p_j^S(v^B,v^S)-v_j^Sx_j^S(v^B,v^S).$$

Mechanism $\mathcal{M}$ is *dominant-strategy incentive-compatible* (DSIC) if: (1) for any buyer i, v_{-i}^B, v^S, and v_i^B, $v_i'^B$,

$$\begin{aligned}&u_i^B(v_i^B;\mathcal{M},v^B,v_{-i}^S)\\ \geq\;&\sum_{A\subseteq M}x_{iA}^B(v_i'^B,v_{-i}^B,v^S)v_i^B(A)-p_i^B(v_i'^B,v_{-i}^B,v^S);\end{aligned}$$

and (2) for any seller j, v_{-j}^S, v^B and v_j^S, $v_j'^S$,

$$u_j^S(v_j^S;\mathcal{M},v_{-j}^S,v^B)\geq p_j^S(v^B,v_j'^S,v_{-j}^S)-v_j^Sx_j^S(v^B,v_j'^S,v_{-j}^S).$$

Mechanism $\mathcal{M}$ is *individually rational* (IR) if: (1) for any buyer i, v_i^B, v_{-i}^B and v^S, $u_i^B(v_i^B;\mathcal{M},v_{-i}^B,v^S)\geq 0$; and (2) for any seller j, v_j^S, v_{-j}^S and v^B, $u_j^S(v_j^S;\mathcal{M},v_{-j}^S,v^B)\geq 0$.

Mechanism $\mathcal{M}$ is *Bayesian incentive-compatible* (BIC) if (1) for any buyer i and valuation functions v_i^B, $v_i'^B$,

$$\begin{aligned}u_i^B(v_i^B;\mathcal{M})&=\mathbb{E}_{v_{-i}^B\sim D_{-i}^B;v^S\sim D^S}\,u_i^B(v_i^B;\mathcal{M},v_{-i}^B,v^S)\\ &\geq\mathbb{E}_{v_{-i}^B\sim D_{-i}^B;v^S\sim D^S}\Big[\textstyle\sum_{A\subseteq M}x_{iA}^B(v_i'^B,v_{-i}^B,v^S)v_i^B(A)\\ &\qquad -p_i^B(v_i'^B,v_{-i}^B,v^S)\Big];\end{aligned}$$

and (2) for any seller j and values v_j^S, $v_j'^S$,

$$\begin{aligned}u_j^S(v_j^S;\mathcal{M})&=\mathop{\mathbb{E}}_{v^B\sim D^B;v_{-j}^S\sim D_{-j}^S}u_j^S(v_j^S;\mathcal{M},v^B,v_{-j}^S)\geq\\ &\mathop{\mathbb{E}}_{v^B\sim D^B;v_{-j}^S\sim D_{-j}^S}\Big[p_j^S(v^B,v_j'^S,v_{-j}^S)-v_j^Sx_j^S(v^B,v_j'^S,v_{-j}^S)\Big].\end{aligned}$$

Mechanism $\mathcal{M}$ is *Bayesian individually rational* (BIR) if (1) for any buyer i and valuation function v_i^B, $u_i^B(v_i^B;\mathcal{M})\geq 0$; and (2) for any seller j and value v_j^S, $u_j^S(v_j^S;\mathcal{M})\geq 0$.

Finally, we denote by $OPT(\mathcal{I})$ the (expected) profit generated by the optimal DSIC mechanism for instance $\mathcal{I}$.

A special case of two-sided markets is *production-cost markets*, where the broker can produce the items by himself and each item $j\in M$ has a publicly known production cost $c_j\in\mathbb{R}^+\cup\{0\}$. Therefore we do not need to consider the sellers' incentives. Letting $c=(c_j)_{j\in M}$, we use $\mathcal{I}^c=(N,M,D^B,c)$ to denote a production-cost market instance and $\mathcal{M}^c=(x^B,p^B)$ a production-cost market mechanism, where the input of x^B and p^B is the buyers' valuation profile. Then the broker's profit is the revenue minus the total production cost $PFT(\mathcal{M}^c;\mathcal{I}^c)$, which is

$$\mathop{\mathbb{E}}_{v^B\sim D^B}\sum_{i\in N}\left(p_i^B(v^B)-\sum_{A\subseteq M}\sum_{j\in A}x_{iA}^B(v^B)c_j\right).$$

Auctions are production-cost markets with cost 0. We use $\mathcal{I}^a=(N,M,D^B)$ to denote an auction instance and $\mathcal{M}^a=(x^B,p^B)$ an mechanism. The expected revenue is $PFT(\mathcal{M}^a;\mathcal{I}^a)=\mathbb{E}_{v^B\sim D^B}\sum_{i\in N}p_i^B(v^B)$. When there is no ambiguity, the superscript B is omitted in auctions and production-cost markets.

In Section 4, we will consider additive valuations for the buyers. In this case, for any buyer i, there exists a valuation vector $(v_{ij}^B)_{j\in M}$ such that $v_{ij}^B=v^B(\{j\})$ is i's value on each item j. Then, v_i^B is *additive* if $v_i^B(A)=\sum_{j\in A}v_{ij}^B$ for any $A\subseteq M$. To simplify the notation, in this case we use v_i^B to denote the vector $(v_{ij}^B)_{j\in M}$ instead of the corresponding function. Each v_{ij}^B is independently drawn from a distribution D_{ij}^B, and $D_i^B=\times_{j\in M}D_{ij}^B$. Finally, when buyers have additive valuations, their allocation is simplified as $x^B(v^B,v^S)=(x_i^B(v^B,v^S))_{i\in N}$, where $x_i^B(v^B,v^S)=(x_{ij}^B(v^B,v^S))_{j\in M}$ with $x_{ij}^B(v^B,v^S)\in[0,1]$, representing the probability that buyer i gets the item j, when the valuations are v^B and v^S.

3 A Reduction from Two-sided Markets to Production-cost Markets

Note that the sellers are single-parameter in the two-sided markets under consideration. Thus, each seller is truthful in a mechanism if and only if the selling probability of her item is non-increasing with respect to her value and the payment to her is the threshold payment, i.e., the highest value such that her item can still be sold. More precisely, for any single-value distribution D with density function f and cumulative probability F, if D is a seller's value distribution, then the virtual value function is $\phi^S(v)=v+\frac{F(v)}{f(v)}$. In addition, if D is not regular then ϕ^S is the ironed virtual value. Following [Myerson and Satterthwaite, 1983], for single-parameter sellers and any DSIC mechanism $\mathcal{M}=(x^S,x^B,p^S,p^B)$, the total payment to the sellers is the virtual social welfare of them, i.e.,

$$\mathop{\mathbb{E}}_{v^S\sim D^S}\sum_{j\in M}p_j^S(v^B,v^S)=\mathop{\mathbb{E}}_{v^S\sim D^S}\sum_{j\in M}\phi_j(v_j^S)x_j^S(v^B,v^S)\tag{1}$$

for any valuation profile v^B of the buyers.

We now show how to convert a mechanism for production-cost markets into a two-sided market's mechanism. The main idea is to use the sellers' virtual values in two-sided markets as costs, and run the mechanism for production-cost markets.

Definition 1. *A mechanism $\mathcal{M}^c=(x,p)$ for production-cost markets is* cost-monotone *if for any two instances $\mathcal{I}^c=(N,M,D^c,c)$ and $\mathcal{I}'^c=(N,M,D^c,c')$, where c and c' differ only at an item j and $c_j\leq c_j'$, for any buyers' valuation profile $v^c\sim D^c$, the probabilities of item j being sold under the two instances, $x_j=\sum_{i\in N}\sum_{A\ni j}x_{iA}(v^c;\mathcal{I}^c)$ and $x_j'=\sum_{i\in N}\sum_{A\ni j}x_{iA}(v^c;\mathcal{I}'^c)$, satisfy $x_j\geq x_j'$.*

Reduction. Let $\mathcal{I}=(N,M,D^S,D^B)$ be a two-sided market instance. For any valuation profile v^S of the sellers, denote by $\phi^S(v^S)=(\phi_j^S(v_j^S))_{j\in M}$ the sellers' virtual-value vector, and let $\mathcal{I}^c_{\phi^S(v^S)}=(N,M,D^B,\phi^S(v^S))$ be a production-cost market instance.

We first show that the optimal profit of the two-sided market is no more than the optimal profit generated by the corresponding production-cost markets in expectation.

Lemma 1. *For any two-sided market instance* $\mathcal{I} = (N, M, D^B, D^S)$*,* $OPT(\mathcal{I}) \leq \mathbb{E}_{v^S \sim D^S} OPT(\mathcal{I}^c_{\phi^S(v^S)})$*.*

Proof. It suffices to show that for any DSIC mechanism $\mathcal{M} = (x^S, x^B, p^S, p^B)$ for two-sided markets, there exists a DSIC mechanism $\mathcal{M}^c$ for production-cost markets such that $PFT(\mathcal{M}; \mathcal{I}) \leq \mathbb{E}_{v^S \sim D^S} PFT(\mathcal{M}^c; \mathcal{I}^c_{\phi^S(v^S)})$. Indeed, this would imply $PFT(\mathcal{M}; \mathcal{I}) \leq \mathbb{E}_{v^S \sim D^S} OPT(\mathcal{I}^c_{\phi^S(v^S)})$ for any $\mathcal{M}$, and thus $OPT(\mathcal{I}) \leq \mathbb{E}_{v^S \sim D^S} OPT(\mathcal{I}^c_{\phi^S(v^S)})$.

Given $\mathcal{M}$ and $\mathcal{I}$, we define mechanism $\mathcal{M}^c = (x^c, p^c)$ as follows. For any instance $\mathcal{I}^c_{\phi^S(v^S)}$, $\mathcal{M}^c$ first computes v^S, the (randomized) *pre-image* of $\phi^S(v^S)$ with respect to D^S. In particular, if for some seller j, the (ironed) virtual value $\phi^S_j(v^S_j)$ corresponds to a value interval in the support of D^S_j, then v^S_j is randomly sampled from D^S_j conditional on it belongs to this interval.

For any reported valuation profile v^B and buyer $i \in N$,

$$x^c_{iA}(v^B) = x^B_{iA}(v^B, v^S)$$

for any $A \subseteq M$, and

$$p^c_i(v^B) = p^B_i(v^B, v^S).$$

It is easy to see that, given any v^S and v^B_{-i}, for any true valuation v^B_i, buyer i has the same utility in $\mathcal{M}^c$ and $\mathcal{M}$ by reporting the same v'^B_i. Thus $\mathcal{M}^c$ is DSIC whenever $\mathcal{M}$ is DSIC. Next, we lower-bound the profit of $\mathcal{M}^c$ for each instance $\mathcal{I}^c_{\phi^S(v^S)}$.

$$\begin{aligned}
& PFT(\mathcal{M}^c; \mathcal{I}^c_{\phi^S(v^S)}) \\
= \; & \mathop{\mathbb{E}}_{v^B \sim D^B} \sum_{i \in N} \left(p^c_i(v^B) - \sum_{A \subseteq M} x^c_{iA}(v^B) \sum_{j \in A} \phi^S_j(v^S_j) \right) \\
= \; & \mathop{\mathbb{E}}_{v^B \sim D^B} \mathop{\mathbb{E}}_{v^S \sim D^S | \phi^S(v^S)} \left(\sum_{i \in N} p^B_i(v^B, v^S) \right. \\
& \left. - \sum_{j \in M} \sum_{i \in N} \sum_{A \ni j} x^B_{iA}(v^B, v^S) \phi^S_j(v^S_j) \right) \\
\geq \; & \mathop{\mathbb{E}}_{v^B \sim D^B} \mathop{\mathbb{E}}_{v^S \sim D^S | \phi^S(v^S)} \left(\sum_{i \in N} p^B_i(v^B, v^S) \right. \\
& \left. - \sum_{j \in M} \phi^S_j(v^S_j) x^S_j(v^B, v^S) \right)
\end{aligned}$$

The inequality above is because any feasible mechanism should satisfy $\sum_{i \in N} \sum_{A \ni j} x^B_{iA}(v^B, v^S) \leq x^S_j(v^B, v^S)$ for any $j \in M$ and any valuation profiles v^B, v^S. Thus,

$$\begin{aligned}
& \mathop{\mathbb{E}}_{v^S \sim D^S} PFT(\mathcal{M}^c; \mathcal{I}^c_{\phi^S(v^S)}) \\
= \; & \mathop{\mathbb{E}}_{\phi^S(v^S) \sim \phi^S(D^S)} PFT(\mathcal{M}^c; \mathcal{I}^c_{\phi^S(v^S)}) \\
\geq \; & \mathop{\mathbb{E}}_{v^B \sim D^B} \mathop{\mathbb{E}}_{v^S \sim D^S} \left(\sum_{i \in N} p^B_i(v^B, v^S) \right. \\
& \left. - \sum_{j \in M} \phi_j(v^S_j) x^S_j(v^B, v^S) \right) \\
= \; & \mathop{\mathbb{E}}_{v^S \sim D^S, v^B \sim D^B} \left(\sum_{i \in N} p^B_i(v^B, v^S) - \sum_{j \in M} p^S_j(v^B, v^S) \right) \\
= \; & PFT(\mathcal{M}, \mathcal{I}),
\end{aligned}$$

as desired. Here $\phi^S(D^S)$ is the distribution of virtual values induced by D^S, and the second equality is by Equation 1. ☐

In the following, we show that if a mechanism for production-cost markets is cost-monotone, then it can be converted into a mechanism for two-sided markets.

Lemma 2. *Given any DSIC cost-monotone mechanism* $\mathcal{M}^c$ *for production-cost markets, there exists a DSIC mechanism* $\mathcal{M}$ *for two-sided markets such that*

$$PFT(\mathcal{M}; \mathcal{I}) = \mathbb{E}_{v^S \sim D^S} PFT(\mathcal{M}^c; \mathcal{I}^c_{\phi^S(v^S)}).$$

Proof. Given mechanism $\mathcal{M}^c = (x^c, p^c)$, the mechanism $\mathcal{M} = (x^S, x^B, p^S, p^B)$ is defined as follows: $\mathcal{M}$ first collects v^B and v^S reported by the buyers and the sellers, and then run $\mathcal{M}^c$ on the production-cost instance $\mathcal{I}^c_{\phi^S(v^S)} = (N, M, D^B, \phi^S(v^S))$ to obtain $x^c(v^B)$ and $p^c(v^B)$. Then for each buyer i, let

$$x^B_{iA}(v^B, v^S) = x^c_{iA}(v^B)$$

for any $A \subseteq M$ and

$$p^B_i(v^B, v^S) = p^c_i(v^B).$$

For each seller j, let

$$x^S_j(v^B, v^S) = \sum_{i \in N} \sum_{A \ni j} x^c_{iA}(v^S, v^B)$$

and let $p^S_j(v^B, v^S)$ be the threshold payment for j: namely, the highest reported value of seller j such that the probability that item j is bought by the broker is $x^S_j(v^B, v^S)$.

We claim that $\mathcal{M}$ is DSIC. First, the buyers will truthfully report their valuations because $\mathcal{M}^c$ is DSIC and each buyer has the same allocation and payment in $\mathcal{M}$ and $\mathcal{M}^c$. For the sellers, since $\mathcal{M}^c$ is cost-monotone and each (ironed) virtual value function ϕ^S_j is non-decreasing in v^S_j, the allocation x^S_j is non-increasing in v^S_j. As the payments to the sellers are the threshold payments, the sellers are truthful as well.

Next we show that

$$
\begin{aligned}
& PFT(\mathcal{M},\mathcal{I}) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S, v^B\sim D^B}\left(\sum_{i\in N} p_i^B(v^B,v^S) - \sum_{j\in M} p_j^S(v^B,v^S)\right) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S, v^B\sim D^B}\left(\sum_{i\in N} p_i^B(v^B,v^S) \right.\\
& \left. - \sum_{j\in M} x_j^S(v^B,v^S)\phi_j(v_j^S)\right) \\
= & \mathop{\mathbb{E}}_{v^B\sim D^B}\mathop{\mathbb{E}}_{\phi^S(v^S)\sim\phi^S(D^S)}\left(\sum_{i\in N} p_i^c(v^B) \right.\\
& \left. - \sum_{j\in M}\sum_{i\in N}\sum_{A\ni j} x_{iA}^c(v^B)\phi_j(v_j^S)\right) \\
= & \mathop{\mathbb{E}}_{\phi^S(v^S)\sim\phi^S(D^S)}\mathop{\mathbb{E}}_{v^B\sim D^B}\sum_{i\in N}\left(p_i^c(v^B) \right.\\
& \left. - \sum_{A\subseteq M}\sum_{j\in A} x_{iA}^c(v^B)\phi_j(v_j^S)\right) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}).
\end{aligned}
$$

Thus Lemma 2 holds. □

Combining Lemmas 1 and 2, we get our first main result.

Theorem 1. *Given any DSIC mechanism $\mathcal{M}^c$ for production-cost markets, if $\mathcal{M}^c$ is cost-monotone and is an α-approximation to the optimal profit, then there exists a DSIC mechanism $\mathcal{M}$ for two-sided markets that is an α-approximation to the optimal profit.*

Proof. Mechanism $\mathcal{M}$ is defined as in Lemma 2. For any two-sided market instance $\mathcal{I}$,

$$
\begin{aligned}
PFT(\mathcal{M};\mathcal{I}) &= \mathop{\mathbb{E}}_{v^S\sim D^S} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}) \\
&\geq \frac{1}{\alpha}\mathop{\mathbb{E}}_{v^S\sim D^S} OPT(\mathcal{I}^c_{\phi^S(v^S)}) \geq \frac{1}{\alpha} OPT(\mathcal{I}),
\end{aligned}
$$

where the equality is by Lemma 2 and the last inequality is by Lemma 1. □

4 A Mechanism for Two-sided Markets with Additive Valuations

4.1 Broker's Profit in Production-cost Markets

We first design a mechanism $\mathcal{M}_A$ for production-cost markets which is an 8-approximation of the optimal profit. Our mechanism is inspired by the mechanism in [Yao, 2015] and the duality framework in [Cai *et al.*, 2016] for auctions. In particular, with probability $\frac{3}{4}$, $\mathcal{M}_A$ runs the mechanism of [Myerson and Satterthwaite, 1983] for two-sided markets for each item separately, denoted by $\mathcal{M}_{IT}$. The mechanism of [Myerson and Satterthwaite, 1983] is for a single buyer and a single seller, but can be generalized to multiple buyers and a single seller as shown in [Deng *et al.*, 2014]. Furthermore, $\mathcal{M}_A$ generalizes the bundling VCG mechanism of [Yao, 2015] to production-cost markets (denoted by $\mathcal{M}_{BVCG}$) and runs it with probability $\frac{1}{4}$.

Mechanism 1 $\mathcal{M}_{BVCG}$ for Production-Cost Markets

1: Collect the valuation profile v from the buyers.
2: For any buyer i and item j, let $P_{ij}(v_{-i}) = \max_{i'\neq i} v_{i'j}$ and $\beta_{ij}(v_{-i}) = \max\{P_{ij}(v_{-i}), c_j\}$.
3: For any buyer i, set the reserve price for item j to be $\beta_{ij}(v_{-i})$. Set the entry fee $e_i(v_{-i})$ to be the median of the random variable $\sum_{j\in M}(t_{ij}-\beta_{ij}(v_{-i}))^+$, where $t_i = (t_{ij})_{j\in M}\sim D_i$ and $x^+ = \max\{x,0\}$ for any $x\in\mathbb{R}$.
4: Each buyer i is considered to accept her entry fee if and only if $\sum_{j\in M}(v_{ij}-\beta_{ij}(v_{-i}))^+ \geq e_i(v_{-i})$.
5: If a buyer i accepts her entry fee, then she gets the set of items j with $v_{ij}\geq\beta_{ij}(v_{-i})$, and her price is $e_i(v_{-i}) + \sum_{j:v_{ij}\geq\beta_{ij}(v_{-i})}\beta_{ij}(v_{-i})$. If i does not accept her entry fee, then she gets no item and pays 0.

Essentially, Mechanism $\mathcal{M}_{IT}$ runs a second-price auction on the buyers' virtual values, with a reserve price which is the production cost of the item. As shown in [Myerson and Satterthwaite, 1983; Deng *et al.*, 2014], this mechanism is optimal for the broker's profit when the buyers have single-parameter valuations. Mechanism $\mathcal{M}_{BVCG}$ is well studied in auctions [Yao, 2015; Cai *et al.*, 2016], and we describe it in Mechanism 1 for production-cost markets $\mathcal{I}^c = (N,M,D,c)$. Essentially, it is a VCG mechanism with per-item reserve prices and per-agent entry fees.

It is not hard to see that both $\mathcal{M}_{IT}$ and $\mathcal{M}_{BVCG}$ are DSIC and IR. Indeed, the mechanism of [Myerson and Satterthwaite, 1983] is DSIC and IR, $\mathcal{M}_{IT}$ directly applies it to each item, and the buyers have additive valuations across the items. Moreover, $\mathcal{M}_{BVCG}$ is DSIC and IR with respect to any reserve prices β_{ij} that do not depend on v_{ij}, and Mechanism 1 simply incorporates the production costs into reserve prices.

In Theorem 2 we use $\mathcal{M}_A$ to upper-bound the optimal profit for any production-cost instance $\mathcal{I}^c = (N,M,D,c)$, with proof provided in the online appendix [Chen *et al.*, 2019]. In fact, the proof is similar to the one in [Cai *et al.*, 2016] with modifications to incorporate the production costs into consideration. Note that [Brustle *et al.*, 2017] also adapts the framework of [Cai *et al.*, 2016] to the 2-sided market, but their goal is to maximize the gain from trade and the buyers have unit-demand valuations.

Theorem 2. *When the buyers have additive valuations, Mechanism $\mathcal{M}_A$ is DSIC and is an 8-approximation to the optimal profit for production-cost markets.*

4.2 Converting $\mathcal{M}_A$ to Two-sided Markets

Next we prove the cost-monotonicity for Mechanism $\mathcal{M}_A$. But first, we start with Mechanism $\mathcal{M}_{IT}$.

Lemma 3. *$\mathcal{M}_{IT}$ is cost-monotone.*

Proof. For any two production-cost instances $\mathcal{I}^c = (N,M,D,c)$ and $\mathcal{I}'^c = (N,M,D,c')$, where there exists an item $j \in M$ such that $c'_j > c_j$ and $c'_{j'} = c_{j'}$ for any $j' \neq j$, we show that in Mechanism $\mathcal{M}_{IT}$, when buyers' valuation profile is $v \sim D$, if item j is not sold in $\mathcal{I}^c$, then item j is not sold in $\mathcal{I}'^c$. Since all buyers' valuation functions are additive and $\mathcal{M}_{IT}$ sells each item individually, the result of selling one item does not effect any other item. In the mechanism of [Myerson and Satterthwaite, 1983], given the reported valuation profile v, the potential winner of item j is the buyer who has highest virtual value on it, denoted by $i_j = \arg\max_{i \in N} v_{ij}$. If her virtual value $\phi_{i_j j}(v_{i_j j})$ is at least the cost of item j, buyer i_j takes item j. Otherwise, item j is kept unsold. Therefore, if item j is not sold in $\mathcal{I}^c$, then $\phi_{i_j j}(v_{i_j j}) - c_j < 0$ which implies $\phi_{i_j j}(v_{i_j j}) - c'_j < 0$ and item j cannot be sold in $\mathcal{I}'^c$. Thus $\mathcal{M}_{IT}$ satisfies cost-monotonicity. □

Next we show $\mathcal{M}_{BVCG}$ is cost-monotone. Since we need to apply $\mathcal{M}_{BVCG}$ to different instances with different cost vectors c and c', we explicitly write $\beta_{ij}(v_{-i}, c_j)$ and $e_i(v_{-i}, c)$ in Steps 2 and 3 of Mechanism 1.

Lemma 4. *$\mathcal{M}_{BVCG}$ is cost-monotone.*

Proof. Similarly, for any two production-cost instances $\mathcal{I}^c = (N, M, D, c)$ and $\mathcal{I}'^c = (N, M, D, c')$, where there exists an item $j \in M$ such that $c'_j > c_j$ and $c'_{j'} = c_{j'}$ for any $j' \neq j$, we show that in Mechanism $\mathcal{M}_{BVCG}$, when buyers' valuation profile is $v \sim D$, if item j is sold in $\mathcal{I}'^c$, then item j is also sold in $\mathcal{I}^c$.

In Mechanism $\mathcal{M}_{BVCG}$, given the valuation profile v, the potential winner of item j is the buyer who has highest value on item j, denoted by $i_j = \arg\max_{i \in N} v_{ij}$. When the cost vector is c, item j is sold to i_j if and only if i_j accepts the entry fee $e_{i_j}(v_{-i}, c)$ and $v_{i_j} - \beta_{i_j j}(v_{-i_j}, c_j) > 0$. Otherwise item j is unsold. Note that given different production cost c', the potential winner of item j remains unchanged.

Note that the entry fee $e_i(v_{-i}, c)$ is selected such that the probability that buyer i accepts it is exactly $\frac{1}{2}$. Let $d_j = \beta_{i_j j}(v_{-i_j}, c'_j) - \beta_{i_j j}(v_{-i_j}, c_j)$ be the increase of the item reserve in $\mathcal{M}_{BVCG}$ for buyer i_j. Then

$$(v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ + d_j \geq (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j))^+. \quad (2)$$

Indeed the equality holds in Inequality 2 if $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0$.

Let $e'_{i_j} = e_{i_j}(v_{-i_j}, c) - d_j$. When the cost vector is c, buyer i_j's utility is

$$u_{i_j} = \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ - e_{i_j}(v_{-i_j}, c).$$

When the cost is c', if the entry fee is e'_{i_j}, buyer i_j's utility is

$$\begin{aligned}
&u'_{i_j}(e'_{i_j}) = \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e'_{i_j} \\
&= \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ - e'_{i_j} \\
&= \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ \\
&\qquad - e_{i_j}(v_{-i_j}, c) + d_j \\
&\geq \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j))^+ \\
&\qquad - e_{i_j}(v_{-i_j}, c) \\
&= u_{i_j}.
\end{aligned}$$

The inequality above is by Inequality 2. Moreover, if $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0$, then

$$u'_{i_j}(e'_{i_j}) = u_{i_j}. \quad (3)$$

That is for any valuation profile v_{i_j}, buyer i_j's utility under the entry fee e'_{i_j} and the cost vector c' is at least her utility under the entry fee $e_{i_j}(v_{-i_j}, c)$ and the cost vector c. Therefore,

$$\Pr_{t_{i_j} \sim D_{i_j}} \left[\sum_{j \in M} (t_{i_j j} - \beta_{i_j j}(v_{-i_j}))^+ \geq e'_{i_j}\right] \geq \frac{1}{2}.$$

Since the real entry fee $e_{i_j}(v_{-i_j}, c')$ is selected to be the median of the random variable $\sum_{j \in M} (t_{i_j j} - \beta_{i_j j}(v_{-i_j}))^+$, we have $e_{i_j}(v_{-i_j}, c') \geq e'_{i_j}$.

Now if under cost vector c', item j is sold, then $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0$, and

$$\sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e_{i_j}(v_{-i_j}, c') \geq 0.$$

Thus under cost vector c, we have $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j) \geq 0$, and by Equation 3,

$$\begin{aligned}
u_{i_j} &= u'_{i_j}(e'_{i_j}) = \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e'_{i_j} \\
&\geq \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e_{i_j}(v_{-i_j}, c') \geq 0.
\end{aligned}$$

Therefore, under cost vector c, item j is also sold. That is, $\mathcal{M}_{BVCG}$ is cost-monotone. □

By randomly selecting from $\mathcal{M}_{IT}$ and $\mathcal{M}_{BVCG}$, Mechanism $\mathcal{M}_A$ is still cost-monotone. Therefore, by Theorem 1, $\mathcal{M}_A$ can be converted into a mechanism for two-sided markets, which is again an 8-approximation to the optimal profit.

Theorem 3. *When the buyers have additive valuations, there exists a DSIC mechanism that is an 8-approximation to the optimal profit for two-sided markets.*

5 Conclusion and Open Problems

In this paper, we proved a general reduction from production-cost markets to two-sided markets, and provided a simple constant-approximation DSIC mechanism for the broker's optimal profit when the buyers have additive valuations. How to design DSIC mechanisms under other valuations of the buyers (e.g. unit-demand or sub-additive) is still open, and it would be interesting to understand the role of production-cost markets in those scenarios. Another interesting direction is to study the situation when the sellers have multiple items.

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