A Bi-level Formulation for Label Noise Learning with Spectral Cluster Discovery

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Abstract

Practically, we often face the dilemma that some of the examples for training a classifier are incorrectly labeled due to various subjective and objective factors. Although intensive efforts have been put to design classifiers that are robust to label noise, most of the previous methods have not fully utilized data distribution information. To address this issue, this paper introduces a bi-level learning paradigm termed "**S**pectral **C**luster **D**iscovery" (SCD) for combating with noisy labels. Namely, we simultaneously learn a robust classifier (*Learning* stage) by discovering the low-rank approximation to the ground-truth label matrix and learn an ideal affinity graph (*Clustering* stage). Specifically, we use the learned classifier to assign the examples with similar label to a mutual cluster. Based on the cluster membership, we use the learned affinity graph to explore the noisy examples based on the cluster membership. Both stages will reinforce each other iteratively. Experimental results on typical benchmark and real-world datasets verify the superiority of SCD to other label noise learning methods.

1 Introduction

Traditional supervised learning models such as Support Vector Machines and Deep Neural Networks usually require accurately labeled datasets for model training. However, labeling errors often occur in practice due to the human fatigue [Magoulas and Prentza, 1999], knowledge limitation [Gong *et al.*, 2017], or measurement error of instruments [Grubbs, 1973]. Therefore, it is crucial for us to design effective training algorithms which are robust to noisy labels.

Several methods for tackling label noise have been developed so far, which can be roughly divided into two types. The methods of first type rely on data cleansing technique. The early-staged methods basically consist of two steps, namely picking up the clean data or filtering out the noisy data, and then deploying the clean data to train the classifier. For instance, neighborhood relationship [Muhlenbach *et al.*, 2004] was utilized to filter out the noisy examples. In [Miranda *et al.*, 2009], different classifiers were employed to vote for the clean data. Recent data cleansing methods have focused on Deep Neural Networks. MentorNet [Jiang *et al.*, 2017] used a self-paced curriculum to select the clean set. Co-teaching [Han *et al.*, 2018b] adopted peer networks and used the "small-loss" and disagreement behaviors of network to choose the probably reliable examples for network updating. However, all the above approaches deploy various heuristic and ad-hoc criteria to decide the clean examples with correct labels, which lack theoretical guarantees.

Therefore, the methods belonging to the second type aim to train the classifiers that are robust to label noise directly, without the need of selecting the clean examples. For example, in [Natarajan *et al.*, 2013], an unbiased risk estimator was designed for establishing loss function, which made it possible to resist the influence of label noise and learn a robust classifier. In [Patrini *et al.*, 2016], the loss function was factorized into two parts, in which only one part was affected by noisy labels. Consequently, they slightly modified the vanilla Stochastic Gradient Descent (SGD) during parameter optimization to reduce the impact of label noise on model training. Besides, in [Gao *et al.*, 2016; Shi *et al.*, 2018; Gong *et al.*, 2019], a similar decomposition was proposed, where the label-dependent part was tackled by unbiasely estimating the labeled instance centroid. Nonetheless, these techniques can only deal with canonical binary classification problem and often require prior knowledge on noise rate or label flipping probability, which may be infeasible or inaccessible in real-world applications. In fact, several methods [Patrini *et al.*, 2017; Northcutt *et al.*, 2017; Han *et al.*, 2018a] was developed for prior knowledge estimation, but their performances are often not satisfactory in the presence of high-dimensional datasets [Han *et al.*, 2018b].

In addition, there are some other approaches for dealing with noisy labels. For instance, [Xia *et al.*, 2019] utilized Transition-revision to designed a deep-learning-based risk-consistent estimator to accurately tune the transition matrix; [Tanaka *et al.*, 2018] jointly trained the networks and estimated true labels, and [Wei *et al.*, 2019] employed the row-sparse residual matrix to capture the incorrectly labeled

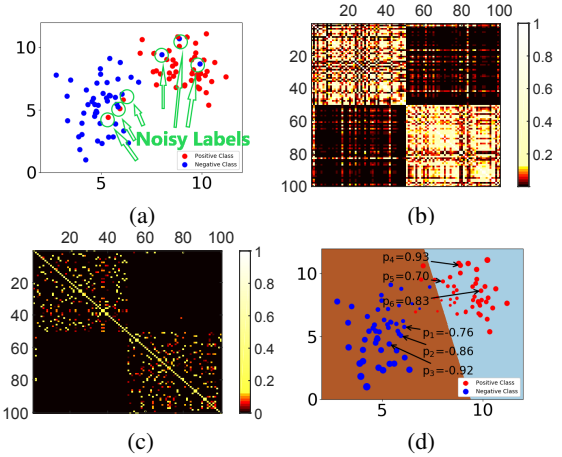


Figure 1: The explanation of our SCD model. Based on the dataset in (a), the affinity graph used in LNSI, computed by Gaussian kernel, is visualized in (b), where the data cluster distribution seems quite unclear, and it may damage the learning performance. Instead, in (c), our SCD establishes an ideal affinity graph with exactly two connected components which is the same as the amount of classes. In (d), the ground-truth labels of the six noisily labeled examples are discovered thanks to the exploited spectral clusters, and the pseudo-label prediction of each noisy data is presented. The point size in (d) indicates the absolute value of pseudo-label for every example.

examples. However, all the above approaches share an implicit deficiency that they have not fully utilized the data distribution information for label noise learning.

In this paper, we propose an effective learning paradigm called "**S**pectral **C**luster **D**iscovery" (SCD), which allows us to train a robust classifier even when the noise rate is relatively high (e.g., 60% noisy examples occur in the training set). Our idea stems from LNSI approach [Wei *et al.*, 2019] and the observation that the examples distributed within a connected cluster are likely to have same class label. Similar to LNSI, our SCD also handle label noise in a matrix recovery fashion (*Learning* stage). LNSI works on a fixed pre-constructed affinity graph (Figure 1 (b)) for noise identification, which lacks adaptability during the learning process. Thus, the distribution information given by the affinity graph may be misleading. In contrast, the graph in our method will be dynamically updated (*Clustering* stage) according to the gradually discovered data clusters. Namely, we use the learned classifier to assign the examples with similar labels into the same connected component in the affinity graph, where each connected component represents a class of examples. Naturally, we want the number of connected component to be exactly the same as the total amount of classes. Based on the well-learned graph (Figure 1 (c)), we can then train a classifier by discovering the noisy examples whose labels are different with those in the same cluster and re-annotate them according to the cluster membership (Figures 1 (c)(d)). Moreover, we alternate between the above two stages to make sure that both the classifier and affinity graph are optimal.

Consequently, we formulate our model as a bi-level optimization problem [Wang *et al.*, 2015], where the upper-level problem (*Learning* stage) is to optimize a classifier in the presence of label noise, and the lower-level problem (*Clustering* stage) is to exploit the spectral clusters for graph updating. The two problems are iteratively solved and are also benefited from each other, so that a robust classifier can be finally established. Theoretically, we prove that the generalization error of the induced classifier is upper bounded. Experimentally, we compare our proposed SCD with other state-of-the-art label noise learning algorithms on several popular datasets, and the results verify the superiority of SCD to other typical methods.

2 Problem Setup

Let $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2), \cdots, (\mathbf{x}_n, \mathbf{y}_n)$ be n training examples identically and independently (i.i.d) drawn from an underlying (noise-free) distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subseteq \mathbb{R}^d$ denotes the feature space of dimension d, and $\mathcal{Y} \subseteq \{0,1\}^c$ denotes the c-dimensional label space with c being the number of classes. Conventionally, we represent the class label of each data as a "1-of-c" indicator vector $\mathbf{y} = (y^1, y^2, \cdots, y^c)^\top$, where $y^k = 1$ if the corresponding example $\mathbf{x}$ belongs to the k-th class and $y^k = 0$ otherwise. Therefore, traditional supervised learning algorithm aims to design a classifier f': $\mathcal{X} \to \mathcal{Y}$ which is able to classify any unseen test example $\mathbf{x} \in \mathbb{R}^d$ to one of the c classes.

However, in label noise learning, the examples from clean distribution $\mathcal{D}$ are unavailable. Before being observed, random classification noise are injected into training examples and what we can obtain are the corrupted examples $(\mathbf{x}_1, \widetilde{\mathbf{y}}_1), (\mathbf{x}_2, \widetilde{\mathbf{y}}_2), \cdots, (\mathbf{x}_n, \widetilde{\mathbf{y}}_n)$ drawn from the noisy distribution $\widetilde{\mathcal{D}}$ over $\mathcal{X} \times \widetilde{\mathcal{Y}}$, where $\widetilde{\mathcal{Y}} \subseteq \{0,1\}^c$ denotes the noisy label space while $\mathcal{X} \subseteq \mathbb{R}^d$ remains the same, then our goal is to train a robust classifier f that can achieve accurate classification in the presence of noisily labeled training examples. In the rest of the paper, we denote a matrix $\mathbf{M}$ with bold capital letter and denote $\mathbf{M}_i$ as the i-th column of matrix $\mathbf{M}$.

3 Model Establishment

As is mentioned in the introduction, SCD alternates between learning a robust classifier in the presence of label noise (*i.e. Learning* stage) and discovering data clusters in the training set (*i.e. Clustering* stage) to aid noise correction, in which the latter is the core of SCD to boost the learning performance. Next we will detail our SCD algorithm.

Learning Stage. The function of *Learning* stage is to learn a classifier in the presence of label noise (Figures 1 (a)(d)), by optimizing the low-rank approximation of the ground-truth label martrix. Let $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \in \mathbb{R}^{d \times n}$ and $\widetilde{\mathbf{Y}} = (\widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \cdots, \widetilde{\mathbf{y}}_n) \in \mathbb{R}^{c \times n}$ denote the feature matrix and the noisy label matrix, respectively, and we decompose $\widetilde{\mathbf{Y}}$ as $\widetilde{\mathbf{Y}} = \mathbf{Y} + \mathbf{E} = \mathbf{W}^\top \mathbf{X} + \mathbf{E}$, in which the ground-truth label matrix $\mathbf{Y} \in \mathbb{R}^{c \times n}$ can be recovered by the projection matrix $\mathbf{W} \in \mathbb{R}^{d \times c}$ of the classifier f, and the error matrix $\mathbf{E} \in \mathbb{R}^{c \times n}$ measures the difference between the observed label matrix and the ground-truth label matrix. Moreover, it is reasonable to assume that the ground-truth label matrix can be well approximated by a low-dimensional subspace [Xu *et al.*, 2016], so we encourage the ground-truth label matrix $\mathbf{W}^\top \mathbf{X}$ to be low-rank by employing matrix factorization, *i.e.* $\mathbf{W} = \mathbf{U}\mathbf{V}$ [Xu *et al.*, 2016], where $\mathbf{U} \in \mathbb{R}^{d \times r}$ and

$\mathbf{V} \in \mathbb{R}^{r\times c}$ (r controls the maximal rank of $\mathbf{W}^\top\mathbf{X}$). Besides, we hope that the number of non-zero columns of error matrix $\mathbf{E}$ is small since the label noise in training examples is usually sparse [Wei *et al.*, 2019], so we minimize the $\ell_{2,1}$ norm $\|\mathbf{E}\|_{2,1} = \sum_{i=1}^{n} \|\mathbf{E}_i\|_2$. Moreover, to fulfill the assumption that similar examples should have similar labels, we further introduce the graph Laplacian regularizer. Define an affinity matrix $\mathbf{S} \in \mathbb{R}^{n\times n}$, with the (i,j)-th entry (*i.e.* s_{ij}) computed by $s_{ij} = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|^2/(2\sigma^2))$ (σ is Gaussian kernel width), and s_{ij} measures the similarity between $\mathbf{x}_i$ and $\mathbf{x}_j$. The graph Laplacian matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$ can then be computed based upon $\mathbf{S}$ as $\mathbf{L} = \mathbf{D} - (\mathbf{S}^\top + \mathbf{S})/2$ where $\mathbf{D} \in \mathbb{R}^{n\times n}$ is a diagonal degree matrix with the i-th diagonal element defined by $d_{ii} = \sum_j s_{ij}$. Therefore, the *Learning* stage is formulated as

$$\begin{aligned}\min_{\mathbf{U},\mathbf{V},\mathbf{E}} \|\widetilde{\mathbf{Y}} - (\mathbf{U}\mathbf{V})^\top\mathbf{X} - \mathbf{E}\|_{\mathrm{F}}^2 + \lambda_1\|\mathbf{E}\|_{2,1} \\ +\lambda_2(\|\mathbf{U}\|_{\mathrm{F}}^2 + \|\mathbf{V}\|_{\mathrm{F}}^2) + \lambda_3\mathrm{Tr}(((\mathbf{U}\mathbf{V})^\top\mathbf{X})\mathbf{L}((\mathbf{U}\mathbf{V})^\top\mathbf{X})^\top),\end{aligned} \tag{1}$$

where "$\mathrm{Tr}(\cdot)$" computes the trace of the corresponding matrix, λ_1, λ_2 and λ_3 are nonnegative trade-off parameters. However, this formulation depends on a fixed pre-constructed affinity matrix $\mathbf{S}$ for noise identification, which can be inaccurate and also lacks adaptability during the learning process.

Clustering Stage. The function of *Clustering* stage is to aid the *Learning* stage. To this end, this stage builds an ideal block diagonal affinity graph which is able to help the *Learning* stage to discover and correct noisy examples. Specifically, if the label of an example is different from its companion in the same connected component, then its label are probably wrong. Thus, we re-annotate this examples based on its cluster membership, so that the robust classifier can be trained (Figures 1 (c)(d)).

In order to learn such an affinity matrix, we strictly constrain the rank of the graph Laplacian matrix to be $n - c$ to make sure that the affinity graph will contain exactly c connected components [Nie *et al.*, 2016]. Meanwhile, we assume that the examples with similar labels should have larger s_{ij} than those with dissimilar ones. To avoid trivial solution, the sum of each column of $\mathbf{S}$ is constrained to be 1 (*i.e.* $\mathbf{S}_i^\top\mathbf{1} = 1$, where $\mathbf{1}$ denotes the all-one column vector). Further, the affinity matrix should be nonnegative, so the *Clustering* stage is formulated as

$$\begin{aligned}\min_{\mathbf{S}} \|\mathbf{S}\|_{\mathrm{F}}^2 + \mu\mathrm{Tr}(((\mathbf{U}\mathbf{V})^\top\mathbf{X})\mathbf{L}_s((\mathbf{U}\mathbf{V})^\top\mathbf{X})^\top), \\ \text{s.t. } \mathbf{S}_i^\top\mathbf{1} = 1, s_{ij} \geq 0, \mathrm{rank}(\mathbf{L}_s) = n - c,\end{aligned} \tag{2}$$

where $\mathbf{L}_s$ denotes the graph Laplacian matrix computed by a learned affinity matrix $\mathbf{S}$ instead of a pre-constructed one, μ is a nonnegative trade-off parameter, and $\|\mathbf{S}\|_{\mathrm{F}}^2$ is minimized to avoid over-fitting. By solving problem (2), we will establish an ideal affinity graph that can benefit the *Learning* stage, so we apply the learned affinity matrix $\mathbf{S}$ to problem (1), by substituting $\mathbf{L}$ by $\mathbf{L}_s$. After that, we can obtain the cluster membership for each example according to the ground-truth label matrix.

However, if we do not have access to the true labels of all training examples before we build such an affinity graph, the result may not be satisfactory. That being said, we may assign the examples with noisy label to the wrong connected component because we are misguided by that inaccurate information. To address this issue, we utilize the *Learning* stage to help the *Clustering* stage. Namely, we integrate the *Clustering* stage (*i.e.* problem (2)) to the *Learning* stage (*i.e.* problem (1) by which we can dynamically build an adaptive affinity matrix and learn a robust classifier according to the gradually explored data clusters (Figures 1 (b)(c)).

To achieve this effect, our SCD paradigm is finally formulated as a bi-level optimization problem as

$$\begin{aligned}&\min_{\mathbf{U},\mathbf{V},\mathbf{E}} \|\widetilde{\mathbf{Y}} - (\mathbf{U}\mathbf{V})^\top\mathbf{X} - \mathbf{E}\|_{\mathrm{F}}^2 + \lambda_1\|\mathbf{E}\|_{2,1} \\ &+ \lambda_2(\|\mathbf{U}\|_{\mathrm{F}}^2 + \|\mathbf{V}\|_{\mathrm{F}}^2) \\ &\underbrace{+\lambda_3\mathrm{Tr}\left(((\mathbf{U}\mathbf{V})^\top\mathbf{X})^\top\mathbf{L}_s((\mathbf{U}\mathbf{V})^\top\mathbf{X})\right)}_{\text{Upper-level problem: } \textit{Learning} \text{ stage}}, \\ &\underbrace{\text{s.t. } \mathbf{L}_s = \operatorname*{argmin}_{\substack{\mathrm{rank}(\mathbf{L}_s)=n-c,\\ \mathbf{S}_i^\top\mathbf{1}=1, s_{ij}\geq 0}} \|\mathbf{S}\|_{\mathrm{F}}^2 + \mu\mathrm{Tr}\left(((\mathbf{U}\mathbf{V})^\top\mathbf{X})^\top\mathbf{L}_s((\mathbf{U}\mathbf{V})^\top\mathbf{X})\right).}_{\text{Lower-level poblem: } \textit{Clustering} \text{ stage}}\end{aligned} \tag{3}$$

The upper-level problem (which is directly related to label noise learning) can guide the lower-level problem (which boosts label noise learning) to gradually explore the true data clusters, meanwhile the lower-level problem is helpful for the upper-level problem to progressively discover the noisy examples whose labels are different from those in the same connected component, and train the robust classifier. Consequently, these two procedures can be benefited from each other and dynamically work together to acquire good learning result. Besides, according to the Weierstrass Theorem in [Patriksson, 2008], it can be easily verified that our bi-level model (3) satisfies the conditions which are sufficient for a bi-level problem to have an optimal solution.

4 Optimization

In this section, we present the detailed solution for optimizing the SCD parameters in the bi-level formulation (3), which alternatively optimizes between the upper-level problem and the lower-level problem mentioned above.

4.1 Upper-level Problem Optimization

The upper-level problem (*i.e.* *Learning* stage) can be iteratively solved by sequentially updating $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{E}$ by solving their respect subproblem. After taking the derivative of subproblem regarding $\mathbf{U}$ $w.r.t.$ $\mathbf{U}$, and denoting $(\cdot)^\dagger$ as the pseudo-inverse of the corresponding matrix, we can update $\mathbf{U}$ by solving the following Sylvestor equation [Bartels and Stewart, 1972]

$$\begin{aligned}(\mathbf{X}\mathbf{X}^\top + \lambda_3\mathbf{X}\mathbf{L}_s\mathbf{X}^\top)\mathbf{U} + \mathbf{U}(\lambda_2(\mathbf{V}\mathbf{V}^\top)^\dagger) \\ = \mathbf{X}(\widetilde{\mathbf{Y}} - \mathbf{E})^\top\mathbf{V}^\top(\mathbf{V}\mathbf{V}^\top)^\dagger,\end{aligned} \tag{4}$$

By denoting $\mathbf{I}$ as the identity matrix with proper size throughout this paper, then we have

$$\mathbf{V} = (\mathbf{U}^\top\mathbf{X}\mathbf{X}^\top\mathbf{U} + \lambda_2\mathbf{I} + \lambda_3\mathbf{U}^\top\mathbf{X}\mathbf{L}_s\mathbf{X}^\top\mathbf{U})^\dagger\mathbf{U}^\top\mathbf{X}(\widetilde{\mathbf{Y}} - \mathbf{E})^\top, \tag{5}$$

According to [Gong *et al.*, 2017], by defining $\mathbf{Q} = (\mathbf{U}\mathbf{V})^\top\mathbf{X} - \widetilde{\mathbf{Y}}$, the closed-form solution of $\mathbf{E}$ is given by

$$\mathbf{E}_i = \begin{cases} \frac{\|\mathbf{Q}_i\|_2 - \lambda_2}{\|\mathbf{Q}_i\|_2}\mathbf{Q}_i, & \text{if } \|\mathbf{Q}_i\|_2 > \lambda_2 \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

4.2 Lower-level Problem Optimization

The lower-level problem (*i.e. Clustering* stage) defined in Eq. (2) is difficult to solve due to the rank constraint, so necessary ways should be found to relax it. According to [Nie *et al.*, 2016], given a large enough ρ, the lower-level problem is equivalent to the following problem:

$$\min_{\mathbf{S},\mathbf{F}} \|\mathbf{S}\|_{\mathrm{F}}^2+\mu\mathrm{Tr}(((\mathbf{U}\mathbf{V})^\top\mathbf{X})\mathbf{L}_s((\mathbf{U}\mathbf{V})^\top\mathbf{X})^\top)+\rho\mathrm{Tr}(\mathbf{F}^\top\mathbf{L}_s\mathbf{F}) \quad (7)$$

$$\text{s.t. } \mathbf{S}_i^\top\mathbf{1}=1, s_{ij}\geq 0, \mathbf{F}\in\mathbb{R}^{n\times c}, \mathbf{F}^\top\mathbf{F}=\mathbf{I}.$$

When $\mathbf{S}$ is fixed, the problem (7) becomes:

$$\min_{\mathbf{F}} \mathrm{Tr}(\mathbf{F}^\top\mathbf{L}_s\mathbf{F}), \quad \text{s.t. } \mathbf{F}\in\mathbb{R}^{n\times c}, \mathbf{F}^\top\mathbf{F}=\mathbf{I}. \quad (8)$$

The solution of the above optimization problem can be easily obtained by setting the columns of $\mathbf{F}$ equal to the c smallest eigenvectors of $\mathbf{L}_s$ [Nie *et al.*, 2016]. When $\mathbf{F}$ is fixed, with $\mathbf{F}_k$ being the k-th column of $\mathbf{F}$, the columns of $\mathbf{S}$ can be separately updated in parallel by solving

$$\min_{\substack{\mathbf{S}_i^\top\mathbf{1}=1\\ s_{ij}\geq 0}} \sum_{j=1}^{n} s_{ij}^2+\mu\sum_{j=1}^{n}\|(\mathbf{U}\mathbf{V})^\top\mathbf{x}_i-(\mathbf{U}\mathbf{V})^\top\mathbf{x}_j\|_2^2 s_{ij} \quad (9)$$

$$+\rho\sum_{j=1}^{n}\|\mathbf{F}_i-\mathbf{F}_j\|_2^2 s_{ij}.$$

By denoting $w_{ij}=\|(\mathbf{U}\mathbf{V})^\top\mathbf{x}_i-(\mathbf{U}\mathbf{V})\mathbf{x}_j\|_2^2$ and $b_{ij}=\|\mathbf{F}_i-\mathbf{F}_j\|_2^2$ as the j-th element of $\mathbf{W}_i$ and $\mathbf{B}_i$, and also denoting $\mathbf{V}_i=-\frac{\mu}{2}\mathbf{W}_i-\frac{\rho}{2}\mathbf{B}_i$, then based on [Huang *et al.*, 2015], s_{ij} has a closed-form solution as

$$s_{ij}^*=(z_{ij}-q_i^*)_+, \quad (10)$$

where $\mathbf{Z}_i=\mathbf{V}_i-\frac{\mathbf{1}\mathbf{1}^\top}{n}\mathbf{V}_i+\frac{1}{n}\mathbf{1}$ and q_i^* can be obtained by solving $f(q_i)=\frac{1}{n}\sum_{j=1}^{n}(q_i^*-u_{ij})_+-q_i=0$ with Newton method.

After that, we update the graph Laplacian matrix via $\mathbf{L}_s=\mathbf{D}-(\mathbf{S}^\top+\mathbf{S})/2$, for the next iteration. After the algorithm converges, we can obtain the parameter of robust classifier by $\mathbf{W}^*=\mathbf{U}^*\mathbf{V}^*$ and label y of any given example $\mathbf{x}$ can be computed by $y=\mathrm{argmax}_i(\mathbf{W}^{*\top}\mathbf{x})_i$. The complete algorithm is summarized in Algorithm 1, from which we see that our SCD model (Eq. (3)) can be easily trained.

For time complexity, in line 8 of Algorithm 1, the Sylvester equation for optimizing $\mathbf{U}$ is solved by Bartels-Stewart algorithm that takes $\mathcal{O}((max\{d,c\})^3)$ complexity. Next, we update $\mathbf{V}$ by computing the inverse of a $c\times c$ matrix in line 9, and the time complexity is $\mathcal{O}(c^3)$. The computation of the ℓ_2 norm of every column of $\mathbf{E}$ in line 10 requires $\mathcal{O}(nc)$ complexity. Moreover, in line 14, we search for q_i^* for every column $\mathbf{S}_i$ of $\mathbf{S}$ by Newton method, of which the time complexity is $\mathcal{O}(nlogn)$. Finally, computing the eigenvectors of $\mathbf{F}$ takes $\mathcal{O}(n^3)$ complexity. Therefore, the total complexity of the proposed model is $\mathcal{O}\left(T_1([max\{d,c\}]^3+c^3+nc)+T_2(nlogn+n^3)\right)$, by assuming that the upper-level and lower-level optimizations are iterated for T_1 and T_2 times, respectively. Since SCD often converges in a few iterations, the time complexity is acceptable.

5 Theoretical Analysis

This section analyzes the generalizability of our proposed SCD. SCD aims to learn an decision function $f_\theta:\mathcal{X}\to\widetilde{\mathcal{Y}}$ which is controlled by $\theta=(\mathbf{W},\mathbf{E})$. Let $\Theta=\{(\mathbf{W},\mathbf{E}):\mathrm{rank}(\mathbf{W})\leq k, \|\mathbf{W}\|_{\mathrm{F}}\leq \mathrm{A}, \|\mathbf{E}\|_{2,1}\leq\mathcal{E}_{2,1}\}$ and $\mathcal{F}_\Theta$ be the feasible solution set of θ and the feasible solution set of f_θ, respectively. In order to find the optimal f_θ^*, SCD minimizes the empirical ℓ-risk as

$$f_\theta^*=\operatorname*{argmin}_{f_\theta\in\mathcal{F}_\Theta}\widetilde{R}_\ell(f)=\operatorname*{argmin}_{(\mathbf{W},\mathbf{E})\in\Theta}\frac{1}{n}\sum_{i=1}^{n}\ell(\mathbf{W}^\top\mathbf{x}_i+\mathbf{E}_i,\widetilde{\mathbf{y}}_i). \quad (11)$$

Let $R_\ell(f)=\mathbb{E}_{(\mathbf{x},\mathbf{y})}[\widetilde{R}_\ell(f)]$ be the expected ℓ-risk, then our goal is to show that the empirical ℓ-risk will converge to the expected ℓ-risk when n is sufficiently large. As is shown in [Bartlett and Mendelson, 2002], the Rademacher complexity is an useful tool for analyzing the bound of the generalization error of SCD, therefore we present the definition of Rademacher complexity as follows. Denote

$$\mathcal{R}_n(\mathcal{F}_\Theta)=\mathbb{E}[\mathcal{R}(\mathcal{F}_\Theta)] \quad (12)$$

as the Rademacher complexity of the function class $\mathcal{F}$ on $\mathcal{X}$ and

$$\mathcal{R}(\mathcal{F}_\Theta)=\mathbb{E}_{\mathbf{x},\sigma}\left[\sup_{f\in\mathcal{F}_\Theta}\frac{1}{n}\sum_{i=1}^{n}\sigma_i f(\mathbf{x}_\mathrm{i})\right] \quad (13)$$

as the empirical Rademacher complexity on the training set, where $\sigma_i\in\{-1,1\}(i=1,2,\cdots,n)$ are independent uniform Rademacher random variables, then the Rademacher complexity of our model can be written as [Xu *et al.*, 2016]

$$\mathcal{R}_n(\mathcal{F}_\Theta)=\frac{1}{n}\mathbb{E}_{\mathbf{x},\sigma}\left[\sup_{(\mathbf{W},\mathbf{E})\in\Theta}\sum_{i=1}^{n}\sigma_i(\mathbf{W}^\top\mathbf{x}_i+\mathbf{E}_i)\right]. \quad (14)$$

By defining

$$\mathcal{R}_n^{(1)}(\mathcal{F}_\Theta)=\frac{1}{n}\mathbb{E}_{\mathbf{x},\sigma}\left[\sup_{(\mathbf{W},\mathbf{E})\in\Theta}\sum_{i=1}^{n}\sigma_i\mathbf{W}^\top\mathbf{x}_i\right], \quad (15)$$

Algorithm 1 The algorithm for solving SCD

1: **Input:** Feature matrix $\mathbf{X}$, corrupted label matrix $\widetilde{\mathbf{Y}}$, trade-off parameters λ_1, λ_2, λ_3, μ;
2: Randomly generate an affinity matrix $\mathbf{S}$.
3: Initialize $\mathbf{U}=\mathbf{O}$, $\mathbf{V}=\mathbf{O}$, $\mathbf{E}=\mathbf{O}$; Set $\rho=1000$;
4: Set $iter=0$
5: **repeat**
6: // *Learning* stage
7: **repeat**
8: Update $\mathbf{U}$ via Eq. (4);
9: Update $\mathbf{V}$ via Eq. (5);
10: Update $\mathbf{E}$ via Eq. (6);
11: **until** *Learning* stage has converged;
12: // *Clustering* stage
13: **repeat**
14: Update $\mathbf{S}$ via Eq. (10);
15: Update $\mathbf{F}$ via solving Eq. (8);
16: **until** *Clustering* stage has converged;
17: // Update graph Laplacian matrix
18: $\mathbf{L}_s=\mathbf{D}-(\mathbf{S}^\top+\mathbf{S})/2$;
19: $iter:=iter+1$;
20: **until** SCD has converged.
21: **Output:** Optimal parameter of robust classifier $\mathbf{W}^*=\mathbf{U}^*\mathbf{V}^*$.

and

$$\mathcal{R}_n^{(2)}(\mathcal{F}_\Theta) = \frac{1}{n}\mathbb{E}_{\mathbf{x},\sigma}\left[\sup_{(\mathbf{W},\mathbf{E})\in\Theta}\sum_{i=1}^{n}\sigma_i\mathbf{E}_i\right], \tag{16}$$

we have

$$\mathcal{R}_n(\mathcal{F}_\Theta) = \mathcal{R}_n^{(1)}(\mathcal{F}_\Theta) + \mathcal{R}_n^{(2)}(\mathcal{F}_\Theta), \tag{17}$$

so we should analyze $\mathcal{R}_n^{(1)}(\mathcal{F}_\Theta)$ and $\mathcal{R}_n^{(2)}(\mathcal{F}_\Theta)$, respectively.

Given the optimal $\mathbf{S}^*$, and for any $\mathbf{S}$ which is not optimal but meets the constraints in the lower-level problem, we can prove that

$$\|\mathbf{S}^*\|_{\mathrm{F}}^2+\mu\mathrm{Tr}\left((\mathbf{W}^\top\mathbf{X})\mathbf{L}_s^*(\mathbf{W}^\top\mathbf{X})^\top\right)\leq n+2\mu c(n-c), \tag{18}$$

which leads to

$$\mathrm{Tr}\left((\mathbf{X}\mathbf{W})^\top\mathbf{L}_s^*(\mathbf{X}\mathbf{W})\right) < \frac{1}{\mu}\left(n+2\mu(n-c)\right). \tag{19}$$

According to [Wei *et al.*, 2019], we know that

$$\mathrm{Tr}((\mathbf{W}^\top\mathbf{X})\mathbf{L}_s^*(\mathbf{W}^\top\mathbf{X})^\top) \geq \lambda_{\min}(\mathbf{X}\mathbf{L}_s^*\mathbf{X}^\top)\|\mathbf{W}\|_{\mathrm{F}}^2, \tag{20}$$

where $\lambda_{\min}(\cdot)$ denotes the minimal eigenvalue of the corresponding matrix. Combining Eq. (19) and Eq. (20), and denoting $\mathcal{W}_F = \sqrt{\frac{1}{\mu}(n+2\mu(n-c))/\lambda_{\min}(\mathbf{X}\mathbf{L}_s^*\mathbf{X})^\top}$, we have

$$\|\mathbf{W}\|_{\mathrm{F}} < \mathcal{W}_F. \tag{21}$$

By defining $\bar{\mathbf{x}} = \sum_{i=1}^n \sigma_i\mathbf{x}_i$, and arranging c copies of $\bar{\mathbf{x}}$ as a $d\times c$ matrix $\bar{\mathbf{X}} = (\bar{\mathbf{x}},\bar{\mathbf{x}},\cdots,\bar{\mathbf{x}})$, we have

$$\mathcal{R}_n^{(1)}(\mathcal{F}_\Theta) = \frac{1}{n}\mathbb{E}_{\mathbf{x},\sigma}\left[\sup_{\mathbf{W}\in\Theta_\mathbf{W}}\left\langle\mathbf{W},\bar{\mathbf{X}}\right\rangle\right]. \tag{22}$$

Then we have the following lemma:

Lemma 1. *([Xu* et al.*, 2016]) Let $\Theta_\mathbf{W} = \{\mathbf{W} : \mathrm{rank}(\mathbf{W}) \leq k, \|\mathbf{W}\|_F \leq \mathcal{W}_F\}$, $\|\bar{\mathbf{X}}\|_{\mathrm{F}} \leq \mathcal{X}_F$ and $\|\mathbf{x}_i\|_2 \leq \mathcal{X}_2$, then the Rademacher complexity $\mathcal{R}_n^{(1)}(\mathcal{F}_\Theta)$ is upper bounded by:*

$$\mathcal{R}_n^{(1)}(\mathcal{F}_\Theta) \leq \sqrt{\frac{kc}{n}}\mathcal{W}_F\mathcal{X}_2. \tag{23}$$

Now we introduce another useful lemma.

Lemma 2. *([Wei* et al.*, 2019]) Let $\Theta_\mathbf{E} = \{\mathbf{E} : \|\mathbf{E}\|_{2,1} \leq \mathcal{E}_{2,1}\}$, $\mathcal{R}_n^{(2)}(\mathcal{F}_\Theta)$ is upper bounded by*

$$\mathcal{R}_n^{(2)}(\mathcal{F}_\Theta) \leq \mathcal{E}_{2,1}\sqrt{\frac{3ln(c)}{nc}}. \tag{24}$$

Based on Lemma 1 and Lemma 2, we can easily derive the bound of the expected ℓ-risk, namely

Theorem 1. *Let ℓ be the loss function bounded by $\mathcal{B}$ with Lipschitz constant L_ℓ, and δ be a constant where $0<\delta<1$. Then with probability at least $1-\delta$, we have*

$$\max_{f\in\mathcal{F}_\Theta}|R_l(f) - \widetilde{R}_l(f)| \tag{25}$$

$$\leq 2L_l\left(\sqrt{\frac{kc}{n}}\mathcal{W}_F\mathcal{X}_2 + \mathcal{E}_{2,1}\sqrt{\frac{3ln(c)}{nc}}\right) + \mathcal{B}\sqrt{\frac{ln(1/\delta)}{2nc}}. \tag{26}$$

This inequality can be easily derived based on Lemma 1 and Lemma 2. Theorem 1 indicates that the expected loss is upper bounded. Specifically, $\mathcal{W}_F$ and $\mathcal{E}_{2,1}$ are related to the learned affinity matrix and the error matrix, respectively. If the minimal eigenvalue of $\mathbf{X}\mathbf{L}_s^*\mathbf{X}^\top$ is large or the noise rate is small (*i.e.* $\mathcal{E}_{2,1}$ is small), the upper bound of the expected ℓ-risk in the righthand side of Eq. (26) will be reduced.

Dataset	#exapmles	#classes	#features
Zoo	101	7	16
Seed	210	3	7
Haberman	306	2	3
Ionosphere	351	2	34
German	1000	2	24

Table 1: An overview of the adopted UCI benchmark datasets

6 Experiment

In this section, we compare SCD with several representative methods on a number of real-world collections, and also study the parametric sensitivity of SCD.

6.1 Experiments on UCI Benchmark Dataset

We compare SCD with four baseline algorithms on six UCI benchmark datasets including *Zoo*, *Seed*, *Haberman*, *Ionosphere*, and *German*, whose attributes are summarized in Table 1. Note that 80% of the examples in each dataset are randomly chosen to establish the training set, and the remaining 20% examples in each dataset are served as test set. To incorporate different noise rate to the training sets, we randomly pick up 0%, 20%, 40%, and 60% examples from the training sets and inject symmetric noise [Patrini *et al.*, 2016] to these selected examples. Such contamination and partition are conducted five times, so the accuracies are the mean values of the outputs of five independent trials.

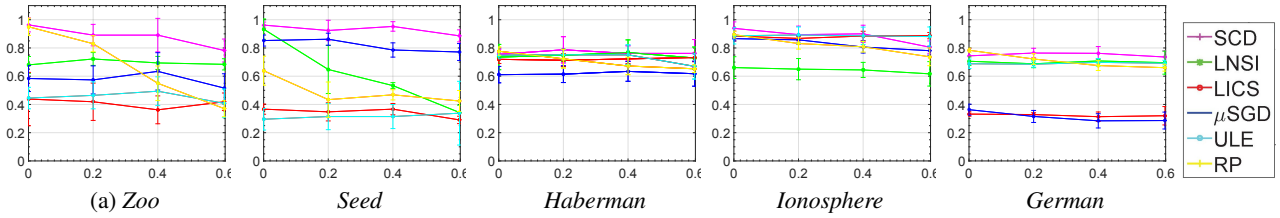
The baselines include: 1) Unbiased Logistic Estimator (ULE) [Natarajan *et al.*, 2013], 2) μ Stochastic Gradient Descent (μSGD) [Patrini *et al.*, 2016], 3) Labeled Instance Centroid Smoothing (LICS) [Gao *et al.*, 2016], 4) Rank Pruning (RP) [Northcutt *et al.*, 2017], and 5) Label Noise handling via Side Information (LNSI) [Wei *et al.*, 2019]. Note that the first three approaches are designed for binary classification tasks, so we use the one-vs-rest strategy to apply them to multi-class cases. For fair comparison, the prior knowledge such as noise rate is provided for all methods, and LNSI is implemented on a 10-NN graph with Gaussian kernel width $\sigma_k = 0.5$.

The classification accuracies of all compared methods on the test set are presented in Figure 2. We observe that SCD yields better performance than other baselines in most cases. Especially, when the noise rate increases, the accuracies of most baselines decrease, but SCD still performs robustly.

6.2 Experiment on Real-world Datasets

We use *ISOLET* dataset and *CIFAR-10* dataset to demonstrate the superiority of SCD in dealing with different kinds of practical problems. First, we address a speech recognition task based on the ISOLET dataset which contains 150 subjects that are required to pronounce each letter in the alphabet (*i.e.* from"A" to "Z") twice. The dataset consists of 7797 examples with 617 dimensions. The way for partitioning the training set and test set is the same as previous experiments, and we also vary the noise rate from 0% to 60%. From Table 2, it can be observed that SCD achieves better classification result than other baselines, and is more robust than other methods when the noise rate increases.

Second, we use *CIFAR-10* to validate the ability of SCD on processing image data, which contains 60000 natural images



(a) *Zoo* *Seed* *Haberman* *Ionosphere* *German*

Figure 2: Experimental results on five UCI benchmark datasets. The subfigures (a) $\sim$ (e) represent the results on *Zoo*, *Seed*, *Haberman*, *Ionosphere*, and *German* datasets, respectively.

Method	0%	20%	40%	60%
ULE	0.823±0.104✓	0.747±0.060✓	0.618±0.079✓	0.417±0.078✓
μSGD	0.796±0.020✓	0.744±0.021✓	0.470±0.047✓	0.475±0.037✓
LICS	0.883±0.022✓	0.758±0.048✓	0.615±0.060✓	0.442±0.052✓
RP	**0.965±0.005**	0.838±0.007✓	0.716±0.011✓	0.557±0.001✓
LNSI	0.891±0.006	0.823±0.006✓	0.821±0.012✓	0.805±0.019✓
SCD	0.914±0.003	**0.917±0.006**	**0.911±0.016**	**0.884±0.015**

Table 2: The results of all the compared methods on *ISOLET* dataset. The classification accuracies (mean±std) under different levels of label noise are presented. The best record under each label noise level is marked in bold and ✓indicates that SCD is significantly better than the corresponding method (paired t-test with 95% confidence level).

Method	0%	20%	40%	60%
ULE	0.823±0.104✓	0.747±0.060 ✓	0.618±0.079✓	0.417±0.078 ✓
μSGD	0.743±0.005 ✓	0.741±0.009 ✓	0.724±0.010 ✓	0.716±0.001✓
RP	0.870±0.007	0.769±0.005 ✓	0.644±0.004✓	0.475±0.007✓
LNSI	0.853±0.003	0.849±0.004	0.837±0.004	**0.776±0.003**
SCD	**0.874±0.003**	**0.866±0.004**	**0.842±0.006**	0.773±0.010

Table 3: The results of all the compared methods on *CIFAR-10* dataset. The classification accuracies (mean±std) under different levels of label noise are presented. The best record under each label noise level is marked in bold and ✓indicates that SCD is significantly better than the corresponding method (paired t-test with 95% confidence level). LICS is not involved as it is not scalable to *CIFAR-10* dataset.

across 10 classes. In our experiment, we randomly pick up 30000 image examples form *CIFAR-10* across different classes. The resolution of each image is $32\times32\times3$. Other experimental settings are the same as before. The classification accuracies of all the compared approaches under different noise rates are shown in Table 3. It can be observed that SCD outperforms other methods and is robust with the increase of the noise rate. Note that LICS is not involved as it is not scalable to *CIFAR-10* dataset.

6.3 Parametric Sensitivity

Note that the objective function Eq. (3) in our method contains four trade-off parameters λ_1, λ_2, λ_3 and μ that should be manually tuned. Therefore, it is necessary to discuss whether the choices of them will significantly influence the performance of SCD. To this end, we examine the classification accuracy of SCD on test set at two different noise rates (20% and 60%) via changing one of λ_1, λ_2, λ_3 and μ, and meanwhile fixing the others to the optimal constant values under different datasets and different noise rates (*i.e.* $\lambda_1^* = 10^{-3}$, $\lambda_2^* = 10^1$, $\lambda_3^* = 10^1$ and $\mu^* = 10^{-3}$ for both noise rate 20% and 60%).

The *ISOLET* dataset is used for the parametric sensitivity analysis. Table 4 shows the variation of accuracies *w.r.t.* the four trade-off parameters on *ISOLET* dataset.

As is shown in Table 4, SCD is robust to the variations of all the trade-off parameters in a wide range, so they can be easily tuned for practical use. Besides, we learn that λ_1 and μ are preferred to be small; while λ_2 and λ_3 are suggested to choose a relatively large number.

λ_1	10^{-3}	10^{-2}	10^{-1}	10^{0}
20%	0.912	0.914	0.926	0.919
60%	0.863	0.911	0.875	0.879

λ_2	10^{-3}	10^{-2}	10^{-1}	10^{0}
20%	0.804	0.917	0.917	0.911
60%	0.601	0.911	0.881	0.876

λ_3	10^{-3}	10^{-2}	10^{-1}	10^{0}
20%	0.915	0.917	0.912	0.914
60%	0.869	0.911	0.870	0.845

μ	10^{-3}	10^{-2}	10^{-1}	10^{0}
20%	0.916	0.917	0.919	0.911
60%	0.875	0.911	0.883	0.885

Table 4: Analysis of the parametric sensitivity of SCD.

7 Conclusion

In this paper, we propose a novel bi-level paradigm to solve the label inaccuracy problem. Specifically, we utilize the distribution information of the dataset by learning an adaptive affinity graph which is ideally block diagonal, and integrate it to the learning of the robust classifier. Moreover, the learned classifier can help the learning of the affinity graph, in which way this two stages mutually complement to each other. We devise the optimization algorithm for the proposed SCD. Experimental results on several typical datasets reveal that SCD generally obtains higher classification accuracies than existing state-of-the-art label noise learning methods.

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