

Approximating Fair Division on D -Claw-Free Graphs

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Abstract

We study the problem of fair allocation of indivisible goods that form a graph and the bundles that are distributed to agents are connected subgraphs of this graph. We focus on the maximin share and the proportional fairness criteria. It is well-known that allocations satisfying these criteria may not exist for many graphs including complete graphs and cycles. Therefore, it is natural to look for approximate allocations, i.e., allocations guaranteeing each agent a certain portion of the value that is satisfactory to her. In this paper we consider the class of graphs of goods which do not contain a star with $d + 1$ edges (where $d > 1$) as an induced subgraph. For this class of graphs we prove that there is an allocation assigning each agent a connected bundle of value at least $\frac{1}{d}$ of her maximin share. Moreover, for the same class of graphs of goods, we show a theorem which specifies what fraction of the proportional share can be guaranteed to each agent if the values of single goods for the agents are bounded by a given fraction of this share.

1 Introduction

The problem of fair allocation of indivisible goods is a fundamental problem in social choice theory [Brams and Taylor, 1996; Bouveret *et al.*, 2016]. We assume that we have a set of items that we call goods and a set of agents, each with her own utility function, which assigns some values to all subsets of goods (called *bundles*). The utility functions are commonly assumed to be additive and so we need to specify their values on individual goods only. We make this assumption in this paper, too. The objective is to assign the goods to the agents so that some fairness criterion is satisfied. Some of these criteria like proportionality or envy-freeness originate from the problem of fair allocation of divisible goods which is known as the cake cutting problem [Brams and Taylor, 1996; Procaccia 2016]. Unlike in the case of allocation of divisible goods where proportional and envy-free allocations always exist, for indivisible goods it is very easy to construct examples when no allocations satisfying these criteria exist.

[Budish, 2011] proposed a new fairness criterion for indivisible goods called the maximin share fairness criterion. It is

based on the well-known cut and choose protocol in the cake cutting problem. Each of n agents first finds a partition of the set of goods into n bundles such that the least-valued bundle is maximized. The maximin share for this agent is equal to the value of this least-valued bundle. An allocation satisfies the maximin criterion if each agent receives a set of goods of value not smaller than her maximin share. We call such allocations mms-allocations. There are examples which show that an mms-allocation may not exist [Procaccia and Wang, 2014; Kurokawa *et al.*, 2016]. Nevertheless such examples are quite intricate. To get around the difficulty of constructing mms-allocations, several researchers proposed to relax the maximin share criterion by requiring that the value of each agent's share in an allocation be at least equal to some positive fraction of the maximin share [Procaccia and Wang, 2014; Amanatidis *et al.*, 2017; Barman and Murthy, 2017; Garg *et al.*, 2019; Ghodsi *et al.*, 2018]. Currently, the strongest result of this kind shows that a $3/4$ -approximating mms-allocation always exists and can be found in polynomial-time [Garg and Taki, 2021].

In this paper we study the problems of proportional and maximin share allocations in the setting proposed by [Bouveret *et al.*, 2017]. In that setting the goods are vertices of an undirected graph (that we call the *graph of goods*) and the bundles to be assigned to the agents, as well as those in the definition of the maximin share, induce connected subgraphs of the graph of goods. This setting captures constraints that appear in some applications by excluding certain undesirable sets of goods. Typical examples of such applications include problems of consolidating land plots [King and Burton, 1982], or allocating rooms in a building to research groups [Bouveret *et al.*, 2017]. In these applications agents might require that the sets of goods (rooms or plots of land) allocated to them form a connected subgraph of the graph of goods describing the neighborhood relation between the goods. The original problem of fair allocation of indivisible goods is now a special case of this generalization when the graph of goods is complete.

[Bouveret *et al.*, 2017] showed several complexity and algorithmic results on envy-free, proportional and mms-allocations for different graphs of goods. Among others, they proved that if the graph of goods is a tree then an mms-allocation always exists and can be computed in polynomial time. Moreover, they gave an example demonstrating that

for some choice of utility functions of the agents an mms-allocation does not exist when the graph of goods is a cycle.

The case of cycles as graphs of goods was thoroughly studied by [Lonc and Truszczynski, 2020]. They proved that in this case it is possible to guarantee to each agent the fraction $\frac{\sqrt{5}-1}{2} \approx 0.62$ of her maximin share. They also found an example showing that if the graph of goods is a cycle then for some choice of the utility functions of the agents in any allocation some agent receives at most $\frac{3}{4}$ of her maximin share.

For arbitrary graphs of goods it is not known whether a positive fraction of the maximin share can be guaranteed. In this paper we prove that for a large class of graphs (including graphs with vertex degrees bounded by a constant) such guarantee indeed exists.

We focus on a wide class of graphs of goods called *d-claw-free graphs*. These are graphs that do not contain a *d*-edge star $K_{1,d}$ as an induced subgraph. Obviously, all graphs with maximum degree at most *d* are $(d + 1)$ -claw-free. Probably the most interesting case concerns 3-claw-free graphs for which the approximation ratios that we prove are the largest. The class of 3-claw-free graphs (also called just *claw-free graphs*) is a large and important class of graphs widely studied in graph theory [Faudree *et al.*, 1997; Chudnovsky and Seymour, 2010].

The problem of fair allocation for 3-claw-free graphs of goods is closely related to the following “edge variant” proposed by [Truszczynski, 2021]: Given a connected graph *G*, where edges represent goods, and a set of agents with utilities on edges, find a fair allocation of connected bundles of edges to agents. This problem is equivalent to the problem where vertices of a graph represent goods, considered for the graph of goods $L(G)$ being the line graph of *G*. (In the line graph $L(G)$ the edges of *G* are vertices and two vertices of $L(G)$ are adjacent if the corresponding edges in *G* have a common vertex.) It is well-known that line graphs are 3-claw-free, which provides additional motivation to consider fair allocations for 3-claw-free graphs of goods.

The “edge variant” described above has the following natural interpretation: Several air carriers (agents) want to offer service between some cities. Connections (goods) between some pairs of cities, to be distributed among carriers, are edges of a graph whose vertices represent cities. The goal for each carrier is to have its connections form a connected subgraph of this graph. The goal of the authority constructing an allocation is to cover all edges by distributing them to carriers according to some fairness criterion.

Our contribution Intuitively, for proportional allocations, the smaller the values of goods relative to the agent’s proportional share, the closer to the proportional allocation we can get. Our first main contribution quantifies this intuition for connected $(d + 1)$ -claw-free graphs of goods, where $d \geq 2$ (Theorem 1). More precisely, let S_i be the total value to agent *i* of all vertices of the graph of goods. For every $0 < \alpha \leq 1$ we find the largest value of $\beta = \beta(\alpha)$ such that for any connected $(d + 1)$ -claw-free graph of goods and any *n* agents for whom the values of single vertices are bounded by $\alpha \frac{S_i}{n}$, there exists an allocation which assigns every agent *i* a connected bundle of value at least $\beta \frac{S_i}{n}$.

For mms-allocations, we prove that for any $(d + 1)$ -claw-free graph of goods, any set of agents and $d \geq 2$, there exists an allocation that guarantees each agent a bundle of value at least $\frac{1}{d}$ of her maximin share (Theorem 2). In contrast to Theorem 1 for which we provide examples showing that the results are tight, we do not have a tightness result for Theorem 2.

Theorem 1 is a crucial ingredient of the proof of Theorem 2. We also use in this proof another result (Lemma 2) that allows us to reduce the problem of existence of an allocation assigning each agent a *c* fraction of her maximin share value when the agents are arbitrary, to the analogous problem when the agents are not arbitrary but their utility functions are bounded by the product of *c* and the maximin share value. Lemma 2, which is perhaps of an independent interest, applies not only to the class of $(d + 1)$ -claw-free graphs but to any hereditary class of graphs.

Proofs of Theorem 2 and Lemma 2 require an extension of the problem of approximation of mms-allocations to disconnected graphs of goods. Therefore, we formulate and prove these results in this more general setting. In this extension we still require that bundles are sets of vertices of connected subgraphs of the graph of goods. However, we do not require that all vertices of the graph of goods are distributed to agents.

Even though mms-allocations for disconnected graphs of goods play an auxiliary role in this paper, we believe that studying such allocations is a natural and interesting direction for future research.

Related work The subject of fair division of a graph into connected bundles received a considerable attention in recent years (see a survey paper by [Suksompong, 2021]). We have already mentioned the papers by [Bouveret *et al.*, 2017], which initiated research in this direction, and by [Lonc and Truszczynski, 2020] who examined the case when the graph of goods is a cycle. [Bei *et al.*, 2022] studied the concept of *price of connectivity* for a graph *G* which they defined as the worst-case ratio between the maximin share computed with the restriction that all bundles are connected in *G* and without this restriction. Their result most closely related to the problems considered in this paper [Bei *et al.*, 2022, Theorem 3.14] implies that for every graph of goods *G* there exists an allocation that guarantees each agent a bundle of value at least $\frac{1}{m-n+1}$ of her maximin share, where *n* is the number of agents, *m* is the number of vertices in *G* and $m \geq n$. This guarantee becomes, however, very weak when the size of the graph grows. Our Theorem 2 guarantees that, for a large class of graphs, a fraction of the maximin share independent on the size of the graph can be assigned to each agent. [Bilò *et al.*, 2022] and [Bei *et al.*, 2022] proved several results on some relaxations of envy-free allocations with connected bundles. [Suksompong, 2019] showed several approximation results on envy-freeness, proportionality and equitability fairness criteria when the graph of goods is a path. [Igarashi and Peters, 2019] studied the problem of allocation of connected bundles that are Pareto optimal. [Bouveret *et al.*, 2019] examined the problem of connected fair allocation of indivisible chores (i.e., items that yield disutility to the agents). [Greco and Scarcello, 2020] and [Deligkas *et al.*, 2021] studied com-

putational complexity issues pertaining to fair allocation on graphs.

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a set of agents and let V be a set of goods. We assume that the set of goods V is the vertex set of some undirected (and not necessarily connected) graph G called the *graph of goods*. Every subset of V inducing in G a connected subgraph is called a *G-bundle* (or just a *bundle* if the graph G is clear from the context).

For each agent $i \in N$ there is a *utility function* u_i which assigns non-negative real numbers to goods in V . We extend utility functions to subsets of V by assuming additivity, i.e., for a utility function u_i defined on V and for every $X \subseteq V$, we define $u_i(X) := \sum_{v \in X} u_i(v)$. We say that a good $v \in V$ (respectively, a set of goods $X \subseteq V$) is *of value x to agent i* if $u_i(v) = x$ (resp. $u_i(X) = x$).

In our setting by an *allocation* we mean an assignment of pairwise disjoint G -bundles to the agents in N . We do not assume here that every good of V is assigned to some agent. If it is then we call such an allocation *complete*. We represent allocations by sequences (A_1, \dots, A_n) , where each set $A_i \subseteq V$ is a G -bundle assigned to agent $i \in N$.

Any family $\{P_1, \dots, P_n\}$ of pairwise disjoint G -bundles is called a (G, n) -*packing*. If, in addition, $\bigcup_{i=1}^n P_i = V$ then such a (G, n) -packing is called a (G, n) -*split* of G . For an agent with utility function u we define the *maximin share* as

$$mms^{(n)}(G, u) := \max_{\{P_1, \dots, P_n\}} \min_{j=1, \dots, n} u(P_j),$$

where the maximum is computed over all (G, n) -splits $\{P_1, \dots, P_n\}$ of G . We call a (G, n) -split for which the maximum is attained an *mms-split*.

If the graph of goods G is disconnected and has more than n components then no (G, n) -split exists so the maximin share is undefined. However, if we replace in the definition of the maximin share (G, n) -splits by (G, n) -packings then we get a parameter which is defined for every (not necessarily connected) graph G . Formally, for an agent with utility function u , the *packing maximin share* is:

$$pmmms^{(n)}(G, u) := \max_{\{P_1, \dots, P_n\}} \min_{j=1, \dots, n} u(P_j),$$

where the maximum is computed over all (G, n) -packings $\{P_1, \dots, P_n\}$ of G .¹ We call a (G, n) -packing for which the maximum is attained an *mms-packing*. Obviously, $pmmms^{(n)}(G, u) \geq mms^{(n)}(G, u)$ for every graph for which $mms^{(n)}(G, u)$ is defined.

An allocation (A_1, \dots, A_n) is an *mms-allocation* (resp. *packing mms-allocation*) if

$$u_i(A_i) \geq mms^{(n)}(G, u_i)$$

$$\text{(respectively, } u_i(A_i) \geq pmmms^{(n)}(G, u_i)\text{),}$$

for every agent $i \in N$. For any $c > 0$, an allocation (A_1, \dots, A_n) is a *c-mms allocation* (resp. *c-pmms allocation*) if

$$u_i(A_i) \geq c \cdot mms^{(n)}(G, u_i)$$

$$\text{(respectively, } u_i(A_i) \geq c \cdot pmmms^{(n)}(G, u_i)\text{),}$$

for every agent $i \in N$ with utility function u_i .

For a *connected* graph of goods, we call an allocation (A_1, \dots, A_n) *proportional* (resp. *c-proportional*) if

$$u_i(A_i) \geq \frac{u_i(V)}{n} \quad \text{(respectively, } u_i(A_i) \geq c \cdot \frac{u_i(V)}{n}\text{),}$$

for every agent $i \in N$ with utility function u_i .

Every c -proportional allocation is both a c -mms and a c -pmms allocation. Indeed, let $\{P_1, \dots, P_n\}$ be an mms-packing for a connected graph G and agent i . Then,

$$\begin{aligned} u_i(V) &\geq \sum_{j=1}^n u_i(P_j) \geq n \cdot pmmms^{(n)}(G, u_i) \\ &\geq n \cdot mms^{(n)}(G, u_i). \end{aligned} \quad (1)$$

Let G be a *connected* graph of goods and let $c > 0$. An agent $i \in N$ with utility function u_i is *c-proportionally bounded* (resp. *c-mms bounded*) if

$$u_i(v) \leq c \cdot \frac{u_i(V)}{n} \quad \text{(resp., } u_i(v) < c \cdot mms^{(n)}(G, u_i)\text{),}$$

for every vertex $v \in V(G)$.

Clearly, by the inequalities (1), every c -mms bounded agent is c -proportionally bounded.

To illustrate some of the concepts introduced above, let us consider the graph, say G , in Figure 1. The graph defines the set of goods $V = \{v_1, \dots, v_6\}$ and their adjacency relation. The table in the figure shows two utility functions u_1 and u_2 of two different agents on the set of goods. By checking all $(G, 2)$ -splits one can easily show that the family $\{\{v_1, v_2\}, \{v_3, v_4, v_5, v_6\}\}$ is a $(G, 2)$ -mms split for the utility function u_1 with its bundles having values 8 and 10, respectively. Thus, $mms^{(2)}(G, u_1) = 8$. Similarly, $mms^{(2)}(G, u_2) = 3$, so the sequence $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$ is an mms-allocation because $u_1(\{v_1, v_2, v_3\}) = 12 \geq 8$ and $u_2(\{v_4, v_5, v_6\}) = 3$.

Since $u_1(V) = 18$ and $u_2(V) = 10$, the sequence $(\{v_4, v_5, v_6\}, \{v_1, v_2, v_3\})$ is a $\frac{2}{3}$ -proportional allocation because $u_1(\{v_4, v_5, v_6\}) = 6 \geq \frac{2}{3} \cdot \frac{18}{2}$ and $u_2(\{v_1, v_2, v_3\}) =$

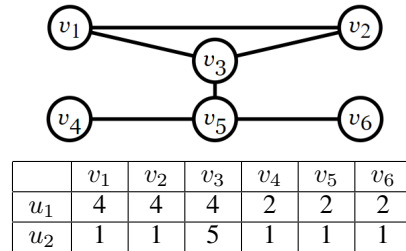


Figure 1: A graph of goods and two utility functions.

¹The abbreviation PMMS has already been used in the literature for ‘pairwise maximin share fairness’ (e.g., see [Amanatidis *et al.*, 2022]). These two meanings of this abbreviation should not be confused.

$7 \geq \frac{2}{3} \cdot \frac{10}{2}$. The agent 1 is $\frac{1}{2}$ -proportionally bounded because the largest value of a single vertex for this agent is $4 \leq \frac{1}{2} \cdot \frac{18}{2}$.

In this paper we use standard graph theoretic definitions and notation. In particular, we write $V(G)$ for the set of vertices of a graph G . A subgraph H of a graph G is *induced* by a set of vertices $X \subseteq V(G)$ if $V(H) = X$ and a pair of vertices $x, y \in V(H)$ is an edge in H if and only if xy is an edge in G . For any $Y \subseteq V(G)$ we denote by $G - Y$ the subgraph of G induced by the set of vertices $V(G) - Y$. If Y has only one vertex, say x , then we write $G - x$ instead of $G - \{x\}$. A subgraph H of a graph G is *spanning* if $V(H) = V(G)$.

By $K_{1,m}$ we denote a star with m edges. We say that a graph G is *m-claw-free* if G does not contain a copy of $K_{1,m}$ as an induced subgraph. A family of graphs \mathcal{G} is *hereditary* if any induced subgraph of a graph in \mathcal{G} is in \mathcal{G} , too.

For connected graphs of goods the relationship between pmms-allocations and mms-allocations is very close. The proposition below explains this relationship.

Proposition 1. *Let G be a connected graph of goods.*

- (i) *For any utility function u defined on $V(G)$ and a positive integer n , $\text{pmms}^{(n)}(G, u) = \text{mms}^{(n)}(G, u)$.*
- (ii) *For any $c > 0$ and any collection of $n \geq 1$ agents a complete c -mms allocation exists if and only if a c -pmms allocation exists.*
- (iii) *For any utility function u defined on $V(G)$ and positive integers m, n such that $m \leq n$, $\text{pmms}^{(m)}(G, u) \geq \text{pmms}^{(n)}(G, u)$. \square*

Simple proofs of parts (i) and (ii) base on an observation that for a connected graph G , every (G, n) -packing extends to a (G, n) -split. Part (iii) follows from the fact that by removing any $n - m$ bundles from an mms-packing for G and n agents we get a (G, m) -packing with the value of every bundle at least $\text{pmms}^{(n)}(G, u)$.

3 Approximate Proportional Allocations

In this section we will show the following result.

Theorem 1. *Let $d \geq 2$ be an integer and $0 < \alpha \leq 1$. Define*

$$\beta = \beta(\alpha) = \begin{cases} \frac{1-\alpha}{d-1} & \text{if } d \geq 3 \text{ or } d = 2 \text{ and } \alpha \geq \frac{1}{2}; \\ \frac{1}{2} & \text{if } d = 2 \text{ and } \alpha < \frac{1}{2}. \end{cases} \quad (2)$$

For every connected $(d + 1)$ -claw-free graph of goods and every set of α -proportionally bounded agents there exists a β -proportional allocation.

Before we prove this theorem let us recall some basic definitions and facts from graph theory. A *cut vertex* of a connected graph is any vertex whose removal makes the graph disconnected. A connected graph is *biconnected* if it has no cut vertices. A *block* of a graph G is a maximal biconnected subgraph of G . It is easy to observe [Bondy and Murty, 2008, Proposition 5.3] that each two blocks of a graph G have at most one vertex in common and this vertex, if it exists, is a cut vertex of G .

Following the terminology introduced in [Bondy and Murty, 2008], for a connected graph G , we define $B(G)$ to be a bipartite graph with bipartition (B, C) , where B is the

set of blocks of G and C is the set of cut vertices of G . A block B and a cut vertex v are adjacent in $B(G)$ if B contains v . It is easy to observe that the graph $B(G)$ is a tree. The blocks corresponding to the leaves of the tree $B(G)$ are called *end blocks* of G . If G is not biconnected then it has at least two end blocks. Clearly, each end block contains exactly one cut vertex of G .

Every tree with more than one vertex contains a vertex whose all neighbors except for possibly one, are leaves. If G is not biconnected then $B(G)$ has more than one vertex so it contains such a vertex, say v . Since leaves in $B(G)$ are end blocks in G , the vertex v is a cut vertex of G such that all blocks containing v except for possibly one, are end blocks. We call such a cut vertex of G *terminal*.

A *bipolar ordering* of a graph G is an ordering v_1, \dots, v_m of all vertices of G such that for each $i \in \{1, \dots, m-1\}$ both the graph induced in G by the set of vertices $\{v_1, \dots, v_i\}$ and the graph induced in G by the set of vertices $\{v_{i+1}, \dots, v_m\}$ are connected. This concept as well as the concepts of a block and a block decomposition were already used before in the context of fair division [Bei et al., 2022; Bilò et al., 2022].

In the proof of Theorem 1 we apply the following lemma. A somewhat stronger version of this lemma, formulated in a different terminology, was proved by [Lempel et al., 1967].

Lemma 1. *If G is a biconnected graph then for every vertex v of G there exists a bipolar ordering starting with v . \square*

We are ready to prove the main result of this section.

Proof of Theorem 1. We will prove this theorem here for $\frac{1}{d} \leq \alpha \leq 1$. For this range of α , we have $\beta = \frac{1-\alpha}{d-1}$ for every integer $d \geq 2$. The proof for $0 < \alpha < \frac{1}{d}$ is similar but differs by some details.

Our assumption that $\alpha \geq \frac{1}{d}$ implies that $0 \leq \beta \leq \frac{1}{d}$. In particular, $\beta \leq \alpha$.

The theorem is obviously true for one agent so we assume that there are at least two agents.

Suppose the theorem is false and let $n \geq 2$ be the smallest number of agents such that for some connected $(d + 1)$ -claw-free graph G of goods and a set $N = \{1, \dots, n\}$ of α -proportionally bounded agents, a β -proportional allocation does not exist. Assume additionally that the graph G has the smallest number of vertices among such graphs.

For every agent $i \in N$, let u_i be her utility function. To simplify notation, we define $S_i := u_i(V(G))$.

Let B be the set of blocks of G and let C be the set of its cut vertices. If G is not biconnected then it contains a terminal cut vertex, say v . Let B_0, B_1, \dots, B_s be the blocks in G containing v indexed so that B_1, \dots, B_s are end blocks (B_0 may but does not have to be an end block). Obviously, v is the only cut vertex contained in the blocks B_1, \dots, B_s . Let $B = V(B_1) \cup \dots \cup V(B_s)$. Clearly, $s + 1 \leq d$ because otherwise v is the center of an induced star $K_{1,d+1}$, a contradiction as G is $(d + 1)$ -claw-free. Moreover, by the definition of v , the graph $G - B$ is connected.

We consider three cases.

Case 1. G is biconnected or G is not biconnected but for some terminal cut vertex v , some end block B_r containing v and some agent i , we have $u_i(V(B_r - v)) > \beta \cdot \frac{S_i}{n}$.

If G is biconnected then we define $H := G$, i is an arbitrary agent and v is an arbitrary vertex of G . If G is not biconnected then we define $H := B_r$. In the former case, $u_i(V(H-v)) = S_i - u_i(v) \geq S_i - \alpha \frac{S_i}{n} = (n - \alpha) \frac{S_i}{n} > (1 - \alpha) \frac{S_i}{n} = \beta(d-1) \frac{S_i}{n} \geq \beta \frac{S_i}{n}$ because the agents are α -proportionally bounded and $d \geq 2$.

By Lemma 1, there exists a bipolar ordering v_1, v_2, \dots, v_m , where $m = |V(H)|$, of vertices of H such that $v_1 = v$.

Let $\ell > 1$ be the largest t such that there is an agent, say $k \in N$, for whom the value of the set $\{v_t, v_{t+1}, \dots, v_m\}$ is at least $\beta \frac{S_k}{n}$. Clearly, ℓ is well-defined because the value for the agent i of the set $\{v_2, v_3, \dots, v_m\} = V(H-v)$ is larger than $\beta \frac{S_i}{n}$. By the definition of a bipolar ordering, the graph induced in G by the set of vertices $L = \{v_\ell, v_{\ell+1}, \dots, v_m\}$ is connected. We assign this set to the agent k .

By the definition of ℓ the value of the set $\{v_{\ell+1}v_{\ell+2}, \dots, v_m\}$ for each agent $j \in N$ is smaller than $\beta \frac{S_j}{n}$. Since $u_j(v_\ell) \leq \alpha \frac{S_j}{n}$ for every agent j , we have $u_j(L) < (\beta + \alpha) \frac{S_j}{n} = (1 - \beta(d-2)) \frac{S_j}{n} \leq \frac{S_j}{n}$ as $d \geq 2$. By the definition of a bipolar ordering, the graph $G - L$ is connected.

Let $G' = G - L$ and $N' = N - \{k\}$. For every agent $j \in N'$ the total value of all vertices of G' is $S'_j = S_j - u_j(L) > S_j - \frac{S_j}{n} = \frac{(n-1)S_j}{n}$, so for each vertex $x \in V(G')$, we have $u_j(x) \leq \alpha \frac{S_j}{n} \leq \alpha \frac{S'_j}{n-1}$. Thus, the utility functions u_j restricted to the set of vertices of G' are α -proportionally bounded for each agent $j \in N'$. By minimality of n there exists a β -proportional allocation for the graph of goods G' and the set of agents N' . In this allocation every agent $j \in N'$ receives a bundle of goods of value at least $\beta \frac{S'_j}{n-1} \geq \beta \frac{S_j}{n}$. This allocation together with the set L assigned to the agent k defines a β -proportional allocation of the vertices of G to the agents of N , a contradiction which completes the proof in this case.

Case 2. G is not biconnected and for each agent $i \in N$, $u_i(B) \leq \alpha \frac{S_i}{n}$.

Let G' be the graph obtained from G by removing the vertices of $B - \{v\}$. Clearly, the graph G' is connected. For each agent $i \in N$ we define a new utility function u'_i on the set $V(G')$ as follows: $u'_i(v) := u_i(B) \leq \alpha \frac{S_i}{n}$ and $u'_i(x) := u_i(x)$ for all $x \neq v$.

By the minimality of G , there is a β -proportional allocation (A_1, \dots, A_n) for the graph of goods G' and the agents of N with the utility functions u'_i which assigns the bundle A_i to agent i , for each $i \in N$. We add to the bundle, say A_j , containing v the vertices of $B - \{v\}$. This way we get a β -proportional allocation for the graph of goods G and the agents of N with the utility functions u_i , a contradiction with the definition of G .

Case 3. G is not biconnected and there is an agent $k \in N$ with $u_k(B) > \alpha \frac{S_k}{n}$.

Clearly, $u_k(B) > \beta \frac{S_k}{n}$ because $\alpha \geq \beta$.

We can assume that for every agent $i \in N$ and every $r \in \{1, \dots, s\}$, we have $u_i(V(B_r - v)) \leq \beta \frac{S_i}{n}$ because otherwise

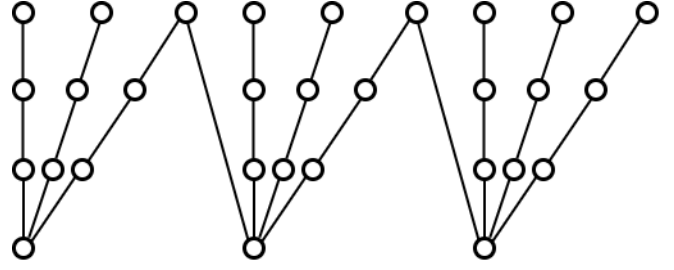


Figure 2: The graph G for $d = 4$, $p = 4$ and $k = 3$.

the condition in Case 1 is satisfied.

Thus, for every agent i

$$\begin{aligned} u_i(B) &= u_i(v) + \sum_{r=1}^s u_i(V(B_r - v)) \leq \alpha \frac{S_i}{n} + s \cdot \beta \frac{S_i}{n} \\ &\leq \alpha \frac{S_i}{n} + (d-1) \beta \frac{S_i}{n} = \frac{S_i}{n} \end{aligned}$$

because $s + 1 \leq d$.

We assign the set B to the agent k .

Let G' be the graph obtained from G by removing the set of vertices B and let $N' = N - \{k\}$. By the definition of B , the graph G' is connected. For every agent $i \in N'$ the total value of all vertices of G' is $S'_i = S_i - u_i(B) \geq S_i - \frac{S_i}{n} = \frac{(n-1)S_i}{n}$ so for each vertex $x \in G'$, we have $u_i(x) \leq \alpha \frac{S_i}{n} \leq \alpha \frac{S'_i}{n-1}$. By the minimality of n , there exists a β -proportional allocation for the graph of goods G' assigning each agent $i \in N' = N - \{k\}$ a bundle of value at least $\beta \frac{S'_i}{n-1} \geq \beta \frac{S_i}{n}$. This allocation together with the set B assigned to agent k defines a β -proportional allocation for the graph of goods G and the agents of N , a contradiction with the definition of G . \square

Theorem 1 is sharp. More precisely, for every $d \geq 2$, $0 < \alpha \leq 1$ and $\varepsilon > 0$, there is an example of a $(d+1)$ -claw-free graph $G = G(d, \alpha, \varepsilon)$ and a set of α -proportionally bounded agents such that there is no $(\beta + \varepsilon)$ -proportional allocation, where β is defined by the equality (2). The example described below shows this sharpness when $\beta = \frac{1-\alpha}{d-1}$, i.e., for $d \geq 3$ and $0 < \alpha \leq 1$ and for $d = 2$ and $\frac{1}{2} \leq \alpha \leq 1$. We also have an example demonstrating sharpness of Theorem 1 for $d = 2$ and $\alpha < \frac{1}{2}$ but we do not present it here.

We construct the graph G in the following way. Let T be a tree consisting of $d-1$ paths, each with $p = \lceil (\beta + \varepsilon/2)/\alpha \rceil + 1$ vertices, sharing one common vertex which is an end of each path. We call this vertex *the center* of T . We define G to be the tree obtained from $k \geq \max\left(\frac{2}{(d-1)\varepsilon}, \frac{2(\beta+\varepsilon)}{\varepsilon}\right)$ disjoint copies T_1, \dots, T_k of the tree T by joining with an edge the center of T_{i+1} with one of the leaves of T_i , for each $i = 1, \dots, k-1$ (see Figure 2). Clearly, the graph G is connected and $(d+1)$ -claw-free.

There are $n = k + 1$ agents with the same utility function assigning the value α to the k vertices which are centers of the trees T_i and the value $\frac{\beta+\varepsilon/2}{p-1} = \frac{\beta+\varepsilon/2}{\lceil (\beta+\varepsilon/2)/\alpha \rceil} \leq \alpha$ to the remaining $k(p-1)(d-1)$ vertices of G . The total value of all vertices of G is equal to $S = k\alpha + k(d-1)(\beta + \varepsilon/2) =$

$k(\alpha + (d-1)\beta) + k(d-1)\varepsilon/2 = k + k(d-1)\varepsilon/2 \geq k$ for each agent. By the inequality $k \geq \frac{2}{(d-1)\varepsilon}$, it follows that $\frac{\beta+\varepsilon/2}{p-1} \leq \alpha = \alpha \frac{k+1}{n} \leq \alpha \frac{k+k(d-1)\varepsilon/2}{n} = \alpha \frac{S}{n}$ so the agents are α -proportionally bounded.

To show that there is no $(\beta + \varepsilon)$ -proportional allocation for our n agents and the graph G we observe that the total value of every connected subgraph of G which does not contain the center of any tree T_i is at most $\beta + \varepsilon/2$. Since there are $k = n - 1$ centers only, in any allocation of bundles of G to n agents there is an agent who does not receive the center of any tree T_i . Thus, this agent gets a bundle of value at most $\beta + \varepsilon/2$. Since $n = k + 1 > \frac{2(\beta+\varepsilon)}{\varepsilon}$ and $S \geq k$, we have $\beta + \varepsilon/2 < (\beta + \varepsilon)(1 - \frac{1}{n}) = (\beta + \varepsilon)\frac{k}{n} \leq (\beta + \varepsilon)\frac{S}{n}$. Thus, no $(\beta + \varepsilon)$ -proportional allocation exists.

We will now extract from Theorem 1 several corollaries, including the result that will be of use when dealing with mms-allocations in Section 4.

Applying Theorem 1 for $\alpha = \frac{1}{d}$ we get the following statement.

Corollary 1. *Let $d \geq 2$ be an integer. For every connected $(d+1)$ -claw-free graph of goods and every set of $\frac{1}{d}$ -proportionally bounded agents there exists a $\frac{1}{d}$ -proportional allocation. \square*

It follows easily from Theorem 1 that Corollary 1 is best possible in the sense that there is no constant $c > \frac{1}{d}$ such that for every connected $(d+1)$ -claw-free graph of goods and any set of c -proportionally bounded agents there exists a c -proportional allocation.

Theorem 1 can be generalized in the following way.

Corollary 2. *Let $d \geq 2$ be an integer and $0 < \alpha \leq 1$. Moreover, let β be defined by the equality (2). If a graph G has a $(d+1)$ -claw-free connected spanning subgraph then, for any set of α -proportionally bounded agents, there exists a β -proportional allocation for the graph of goods G .*

Proof. This corollary follows immediately from Theorem 1 and the observation that for any $\beta \geq 0$ a β -proportional allocation for a spanning connected subgraph of G is a β -proportional allocation for the graph G . \square

In Corollary 2 the largest value of the approximation factor is achieved for $d = 2$ and it is equal to $\frac{1}{2}$. Since a path is obviously 3-claw-free, we have the following corollary which can also be easily proved directly.

Corollary 3. *If the graph of goods G has a Hamilton path then for any set of $\frac{1}{2}$ -proportionally bounded agents there exists a $\frac{1}{2}$ -proportional allocation. \square*

4 Approximate Mms-Allocations

The following theorem is the main result of this section.

Theorem 2. *Let $d \geq 2$ be an integer. For every connected $(d+1)$ -claw-free graph of goods and any set of agents there exists a complete $\frac{1}{d}$ -mms allocation.*

In fact, we will prove a stronger version of this theorem for graphs which do not have to be connected.

Theorem 2'. *Let $d \geq 2$ be an integer. For every $(d+1)$ -claw-free graph of goods and any set of agents there exists a $\frac{1}{d}$ -pmms allocation.*

Theorem 2 follows immediately from Theorem 2' by Proposition 1 (i) and (ii).

We first prove a general lemma which reduces the problem of the existence of a c -pmms allocation for arbitrary agents to the existence of a c -mms allocation for c -mms bounded agents, if the graphs of goods belong to a hereditary class of graphs. We will apply this lemma in the proof of Theorem 2' for the class of $(d+1)$ -claw-free graphs which is, obviously, hereditary. However, it is worth noting that there are many other important hereditary classes of graphs (e.g., the class of all graphs, the class of planar graphs, the class of bipartite graphs, the class of perfect graphs).

Lemma 2. *Let \mathcal{G} be a fixed hereditary class of graphs and let $c > 0$ be a fixed real. If there exists a c -mms allocation for any connected graph in \mathcal{G} and any set of c -mms bounded agents then there exists a c -pmms allocation for any graph of goods in \mathcal{G} and any set of agents.*

Proof. Consider an arbitrary graph of goods $G \in \mathcal{G}$ and an arbitrary set $N = \{1, \dots, n\}$ of agents. Let, for each agent $i \in N$, u_i be the utility function of i and let Π_i be an mms-packing for this agent. To simplify notation we define $pmms_i := pmms^{(n)}(G, u_i)$ for every agent $i \in N$. We need to construct an allocation assigning each agent i a bundle of value at least $c \cdot pmms_i$ to this agent.

We define $X := \{x_1, \dots, x_\ell\}$ to be a maximal with respect to inclusion (and possibly empty) set of pairwise different vertices of G such that every vertex x_j is of value at least $c \cdot pmms_{i_j}$ to some agent i_j and the agents i_1, \dots, i_ℓ are pairwise different. Every agent i_j receives the vertex x_j in our allocation.

It remains to show that there is an allocation assigning to each agent $i \in N' = N - \{i_1, \dots, i_\ell\}$ a bundle inducing a connected subgraph of $G - X$ which is of value at least $c \cdot pmms_i$ to this agent. By the maximality of X , for every vertex $v \in V(G - X)$ and every agent $i \in N'$,

$$u_i(v) < c \cdot pmms_i. \quad (3)$$

Since $|X| = \ell$, for every agent $i \in N'$ there are at least $n - \ell$ bundles in Π_i which do not intersect X .

Let G_1, \dots, G_s be the components of $G - X$. Let Π_i^j be the set of bundles in Π_i that are contained in $V(G_j)$ and let $f(i, j)$ be the number of such bundles. Clearly, Π_i^j is a $(G_j, f(i, j))$ -packing. Moreover, $\sum_{j=1}^s f(i, j) \geq n - \ell$ for each agent $i \in N'$. We observe that, by Proposition 1 (i) and the fact that the value of each of $f(i, j)$ bundles in Π_i^j is at least $pmms_i$ for agent i , we have

$$mms^{(f(i,j))}(G_j, u_i) = pmms^{(f(i,j))}(G_j, u_i) \geq pmms_i \quad (4)$$

for every agent $i \in N'$ and every $j \in \{1, \dots, s\}$.

Claim. Let $I_j \subseteq N'$ be a set of agents such that $f(i, j) \geq |I_j|$ for each agent $i \in I_j$. Then, there is a c -mms allocation for the graph of goods G_j and the set of agents I_j .

Proof of the Claim. We observe that the agents of I_j are c -mms bounded with respect to the graph of goods G_j and the

Algorithm 1 *allocate*($N', G - X, c$)

```

1   $T := N'$ ;
2  for  $j = 1, 2, \dots, s$  do
3      sort the agents  $i \in T$  non increasingly according to
        the key  $f(i, j): i_1^j, i_2^j, \dots, i_{|T|}^j$ ;
4       $k_j :=$  the largest  $p$  such that  $f(i_p^j, j) \geq p$ ;
5       $I_j := \{i_1^j, i_2^j, \dots, i_{k_j}^j\}$ ;
6      agents in  $I_j$  distribute the vertices of  $V(G_j)$  among
        themselves according to the allocation  $A(I_j, G_j)$ ;
7       $T := T - I_j$ ;
8      if  $T = \emptyset$  then
9          return
    
```

utility functions restricted to $V(G_j)$. Indeed, applying in turn the inequalities (3), (4) and Proposition 1 (iii), for every vertex $v \in V(G_j)$ we get

$$\begin{aligned} u_i(v) &< c \cdot pmms_i \leq c \cdot mms^{(f(i,j))}(G_j, u_i) \\ &\leq c \cdot mms^{(|I_j|)}(G_j, u_i). \end{aligned}$$

Since \mathcal{G} is a hereditary class of graphs, $G_j \in \mathcal{G}$. Thus, by our lemma assumption, there is a c -mms allocation for the graph of goods G_j and the set of agents I_j . ■

We denote by $A(I_j, G_j)$ the allocation whose existence is guaranteed by the Claim.

We shall prove that the algorithm *allocate* shown in the figure Algorithm 1 produces the required c -pmms allocation of the vertices of $G - X$ to the agents of N' . In the j th pass of the loop of this algorithm bundles of vertices of the component G_j of $G - X$ are distributed to agents. First, we sort (line 3) the agents i which have not got their bundles yet non increasingly according to the number $f(i, j)$ of bundles in their mms-packings Π_i which are contained in G_j . We denote the ℓ th agent in this sorting by i_ℓ^j . Next, we define the set I_j which consists of k_j initial agents in the sorting (line 5), where k_j is the largest p such that $f(i_p^j, j) \geq p$ (line 4). We observe that, by the definition of k_j , $f(i, j) \geq k_j = |I_j|$ for every agent $i \in I_j$. Thus, it follows from the Claim that there is an allocation $A(I_j, G_j)$ assigning to each agent $i \in I_j$ a bundle of value at least

$$\begin{aligned} c \cdot mms^{(k_j)}(G_j, u_i) &= c \cdot pmms^{(k_j)}(G_j, u_i) \\ &\geq c \cdot pmms^{(f(i,j))}(G_j, u_i) \geq c \cdot pmms_i. \end{aligned}$$

We applied here, in turn, Proposition 1 (i), (iii) and the inequality (4). We distribute the vertices of G_j to agents of I_j according to the allocation $A(I_j, G_j)$ (line 6) and the agents of I_j quit the game (line 7).

It remains to show that when the algorithm stops, the set T is empty, i.e., all agents received their bundles. Suppose otherwise. Let i be an agent who is still in the set T when the algorithm stops. Let T_j be the set T at start of the j th pass of the loop of our algorithm. By the definition of k_j , for all agents $t \in T_j$, which are not included in the set I_j (so do not receive a bundle) in the j th pass of the loop, we have $f(t, j) < k_j + 1$, i.e., $f(t, j) \leq k_j$. Since the agent i was not

allocated a bundle in any pass of the loop, we have $f(i, j) \leq k_j$ for all j 's. Clearly, $\sum_{j=1}^s k_j$ is the total number of agents who received their bundles when the algorithm stops. Hence,

$$n - \ell \leq \sum_{j=1}^s f(i, j) \leq \sum_{j=1}^s k_j < n - \ell,$$

a contradiction. Thus, all agents received their bundles when the algorithm stops. □

Theorem 2' follows now from Lemma 2 and Corollary 1.

Proof of Theorem 2'. Let \mathcal{G} be the class of $(d + 1)$ -claw-free graphs. Since \mathcal{G} is hereditary, by Lemma 2, it suffices to show our theorem for connected $(d + 1)$ -claw-free graphs of goods G and $\frac{1}{d}$ -mms bounded agents.

By Corollary 1, for every connected $(d + 1)$ -claw-free graph of goods G and any set of $\frac{1}{d}$ -proportionally bounded agents, there exists a $\frac{1}{d}$ -proportional allocation. Since c -mms bounded agents are c -proportionally bounded for any $c > 0$, a $\frac{1}{d}$ -proportional allocation exists for $\frac{1}{d}$ -mms bounded agents and the graph of goods G . Our theorem follows now from the fact that a c -proportional allocation is a c -mms allocation, for any $c > 0$. □

5 Final Remarks and Open Problems

As we mentioned in the Introduction, we do not know if the approximation ratio $\frac{1}{d}$ in Theorem 2 can be improved, even in the case of 3-claw-free graphs, i.e., for $d = 2$.

Problem 1. *What is the largest constant c such that for each 3-claw-free graph of goods and any set of agents there exists a c -mms allocation?*

It was shown [Lonc and Truszczynski, 2020, p. 639] that if the graph of goods is a cycle with 12 vertices (which is obviously 3-claw-free) then for some 6 agents no c -mms allocation exists for any $c > \frac{3}{4}$. This statement and Theorem 2 imply that the constant c in Problem 1 satisfies the inequalities $\frac{1}{2} \leq c \leq \frac{3}{4}$.

A similar question as the one in Problem 1 can be asked for arbitrary graphs. In this case, however, we do not even know whether $c > 0$.

Problem 2. *Is there an absolute constant $c > 0$ such that for each graph of goods and any set of agents there exists a c -mms allocation?*

We do not know any example of a graph and a collection of agents such that for this graph and these agents no $\frac{3}{4}$ -mms allocation exists. Thus, it is conceivable that the constant c in Problem 2 can be as large as $\frac{3}{4}$.

Finally, there are some computational complexity issues related to the subject of this paper. While the proof of Theorem 1 can be easily turned into a polynomial time algorithm constructing a β -proportional allocation for α -proportionally bounded agents, it is not the case for the proof of Theorem 2. In the latter proof we use the value of packing maximin share which is NP-hard to compute even for complete graphs [Bouveret and Lemaître, 2016]. Finding a polynomial time algorithm for constructing a $\frac{1}{d}$ -mms allocation for $(d + 1)$ -claw-free graphs of goods will be a subject of our future research.

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