Aggregation of Continuous Preferences in One Dimension

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Abstract

We develop a general, formal model of social choice in which voters have continuous preferences over a one-dimensional space. Our model is parameterized by different restrictions that we introduce regarding the way voter preferences change in time as well as the optimization criteria (that correspond to a normative continuum of fairness definitions) desired from an aggregation method—that outputs a continuous, one-dimensional curve—given such inputs. We discuss the applicability of the model to different real-world situations and, as a first step towards an analysis of the different model realizations, we concentrate on identifying those cases that are computationally feasible to compute.

1 Introduction

Social choice theory offers algorithms that aggregate voter preferences and output various aggregated outputs, such as in single-winner elections [Sen, 1986], multiwinner elections [Faliszewski *et al.*, 2017], and participatory budgeting [Aziz and Shah, 2021]. These examples, however, are all of a discrete flavor and with a simple structure. In particular, both the output (the name of a single candidate, the names of the k committee members, the names of the projects to fund) as well as the input (i.e., voter preferences) are discrete.

Indeed, some social choice settings are continuous (most notably perhaps are randomized mechanisms and portioning methods [Airiau *et al.*, 2023]), however, there is a general need for devising aggregation algorithms that are able to output more structurally-involved, continuous outputs. Correspondingly, here we are interested in social choice settings in which both the inputs (i.e., voter preferences) as well as the output (the aggregated result) are continuous. Consider first the following example usecases.

- **Deciding on a monetary policy**: One of the important features of a monetary system is its policy, where one of the main goals of such a policy is to stabilize the economy using certain policy tools [Friedman, 1995]. These tools include setting the interest rates, adjusting the reserve requirements for banks, and continuously buying and selling various securities. Consider, say, a group of experts, each with its own idea on a certain issue such as the required change in the interest rate for the coming year: as each of these experts may have different preferences regarding the ideal monetary policy, those individual preferences may be aggregated to a single, agreeable policy.[1]
- **Optimizing production**: In industrial production environments, certain time-based estimations and decisions have to be made. Two of the most fundamental ones include estimating the future product demand—and, based on such estimations, decide the future product supply [Levinsohn and Petrin, 2003]. Correspondingly, consider a group of experts—that may also include certain data-intensive artificial experts—each with its own estimation and preferences, and consider the need for a method that aggregates these.
- **Controlling energy usage and consumption**: For many energy sources (e.g., electricity), the energy cost is not only affected by the sheer amount of consumed energy but also on how the consumption is spread in time [Johnson *et al.*, 2011]. In particular, as the cost of producing electricity increase dramatically in times of peak demand, many systems aim at so-called "flattening the curve" (of household electricity demand) [Barker *et al.*, 2012]. Correspondingly, consider an agent community (say, residents of some residential complex) where different agents may have different preferences regarding the consumption of energy in time. For sustainability reasons as well as for purely economical reasons, there is a need to aggregate those preferences to come to a single, agreeable energy consumption curve.
- **Collaborative forecasting**: Consider a set of sensors/machines/agents, each of which is producing a forecast for, say, next weeks' weather. These predictions may be aggregated to arrive to a single, agreeable forecast [Leutbecher and Palmer, 2008].
- **Space-based preferences**: The above scenarios relate to aggregating time-based preferences. There are, however, similar scenarios that involve aggregating space-

[1]In this context, we mention cryptoeconomical situations involving time-based bonding curves [Zargham *et al.*, 2020], Conviction Voting [Emmett, 2019], and Commitment Voting [Berg *et al.*, 2020].

based preferences. E.g., a group of agents wishing to jointly decide on a water policy [Perret, 2002].

More abstractly, in such applications as described above, we are interested in aggregation methods that take continuous, one-dimensional voter preferences and output a continuous, one-dimensional output. To the best of our knowledge, no solutions are currently available for such social choice settings.[2] Thus, motivated by such scenarios—and by the lack of appropriate aggregation algorithms for their instances—in this paper we develop a formal model for the aggregation of continuous preferences in one dimension and suggest several corresponding aggregation algorithms for such situations.

As our model is novel, we are mainly interested in understanding what is possible to compute in our model—i.e., what realizations of the model and which special cases of it have a mathematical structure that admits efficient aggregation algorithms. Generally speaking, we observe that, while the model is computationally intractable in its most generality, it admits efficient, exact algorithms for certain aggregation functions and certain restrictions on voter preferences.

In what follows, after reviewing related work (Section 2), we describe our parameterized formal model (Section 3). Then, we present our theoretical results (which are formally summarized in Section 3.2): a proof of the general computational intractability of the model is described in Section 4; algorithms for basic aggregation functions are described in Section 5; and efficient algorithms for linear inputs and output are described in Section 6. We end with a future-facing discussion.

2 Related Work

Our work has some relation with the relatively-new line of work on *perpetual social choice* [Bulteau *et al.*, 2021; Lackner, 2020; Lackner and Maly, 2021]. However, while the model of perpetual social choice deals with preferences that may change between timesteps, it is nevertheless a discrete model of preferences and aggregation. (Note that there is also work on perpetual fair division [Igarashi *et al.*, 2024].)

Another related work is that of Lodi *et al.* [2022] who consider a time-based model of decision making; while their model (and focus) is of a somewhat continuous flavour, it is significantly different than ours, in particular by their model of individual utilities. We also mention the work of Bredereck *et al.* [2022] regarding a sequence of committee elections that does not directly fit within the framework of perpetual social choice but is of a somewhat similar style of a repeated, discrete decision making flavor.

A different line of work within social choice that relates to our work considers settings that have some continuous ingredients. For example, some papers consider a model of divisible participatory budgeting [Freeman *et al.*, 2021] (also referred to as portioning [Airiau *et al.*, 2023]), in which the aggregated output is a division of a joint budget (usually represented pictorially as a pie chart). Another example include probabilistic social choice [Brandt, 2017], in particular aggregation methods that are randomized and thus can be viewed as outputing a probability distribution that may be continuous. Yet a different flavor of continuity in social choice corresponds to the line of work on so-called *society graphs*: here, the electorate is treated as a continuous entity [Faliszewski *et al.*, 2022; Gonen *et al.*, 2023].

Somehow farther away is the vast literature on economic forecasting [Elliott and Timmermann, 2013]. In a way, our work can be applied on top of methods of traditional forecasting in that we offer a principled way of aggregating different forecasters. Note, however, that in our settings the ground truth (i.e., an ultimately perfect prediction) is not always the goal, as we concentrate on the collaborative aspects that relate to representation and equality in the decision-making mechanisms (as such, situations that have some subjective aspects perhaps fit better to our model). In this context, our work also has a similar structure to work on ensemble learning [Dong *et al.*, 2020] in which several machine learning models are aggregated to create one, hopefully better, model. Here, again, we differ also as we concentrate on a social choice perspective over such aggregation—indeed, the different q-norms we use correspond to a continuum of normative fairness guarantees, ranging from a utilitarian extreme (with $q = 1$) to an egalitarian extreme (with $q = \infty$). Another scientific field that relates to our work is decision theory [DeGroot, 2004] in which (sometimes continuous) values are to be decided in an uncertain environment.

3 Formal Model

We describe our general model. Our model consists of a continuous time axis T; we normalize the time axis so that $T = [0, 1]$, and we use $t \in T$ (i.e., $0 \leq t \leq 1$) to denote points in time. We consider a decision space $S = [0, 1]$ (corresponding to selecting a value between 0 and 1). By $V = \{v_1, \ldots, v_n\}$ we denote a set of voters, where each voter v_i provides her ideal point for each point in time: formally, $v \in V$ is a continuous function $v\colon T \to S$ where $v(t)$ is the ideal point of v at time t. A solution $W\colon T \to S$ is a continuous function as well.

Example 1. *Consider* $V = \{v_1, v_2, v_3\}$ *with* $v_1(t) = 0.5$, $v_2(t) = 0.75$, *and* $v_3(t) = t$. *For such voter preferences, a solution* W *may be, e.g.,* $W(t) = 0.5$.

Given voter preferences as above, we define a general framework for corresponding objective functions for our aggregation methods to pursue. First, we define the cost of a solution for a voter in a certain timepoint; then, the cost of a solution for a voter; and, then, the cost of a solution to the voter community (which we aim at minimizing):

- We define a measure on the decision space to define the *timepoint-cost* of voter v at time t with respect to some solution W, as:

$$\mathrm{cost}(v, t, W) = |v(t) - W(t)| \,.$$

- Based on the values of the timepoint-cost functions that we get for each $t \in T$, we define an *(aggregated)*

[2]Indeed, models developed by political economists, such as the spatial model of elections [Enelow and Hinich, 1984] concentrate on continuous preferences, however differ in that we concentrate on continuous functions as modeling both voter preferences and the output of the aggregation process.

cost function for a voter v with respect to some solution W: $\mathrm{cost}(v, W)$. We use several specific functions as $\mathrm{cost}(v, W)$; in particular, L^p norms—the continuous counterpart of p-norms, parameterized by $p \in [1, \infty]$:

$$\mathrm{cost}_p(v, W) = \left(\int_T \mathrm{cost}^p(v, t, W) dt \right)^{\frac{1}{p}} .$$

In particular, $\mathrm{cost}_\infty(v, W) = \max_{t \in T} \mathrm{cost}(v, t, W)$.

- Based on these (aggregated) cost values for each voter $v \in V$, we define a *total cost* function $\mathrm{cost}(W)$ (for V) that we wish to minimize. We use several specific functions as $\mathrm{cost}(W)$; in particular, q-norms (over the voters), parameterized by $q \in [1, \infty]$:

$$\mathrm{cost}_q(W) = \left(\sum_{v \in V} \mathrm{cost}^q(v, W) \right)^{\frac{1}{q}} .$$

In particular, $\mathrm{cost}_\infty(W) = \max_{v \in V} \mathrm{cost}(v, W)$.

When p and q are clear by the context, we shorten the notation and use $\mathrm{cost}(W)$ to denote the cost of the solution W.

Example 2. *Let V and W be the same voter set and possible solution as in Example 1 and consider the following:*

- *For $p = 1$ and $q = 1$ we have* $\mathrm{cost}(W) = \mathrm{cost}_1(v_1, W) + \mathrm{cost}_1(v_2, W) + \mathrm{cost}_1(v_3, W) = 0 + 0.25 + 0.25 = 0.5$.
- *For $p = 1$ and $q = \infty$ we have* $\mathrm{cost}(W) = \max(\mathrm{cost}_1(v_1, W), \mathrm{cost}_1(v_2, W), \mathrm{cost}_1(v_3, W)) = \max(0, 0.25, 0.25) = 0.25$.
- *For $p = \infty$ and $q = 1$ we have* $\mathrm{cost}(W) = \mathrm{cost}_\infty(v_1, W) + \mathrm{cost}_\infty(v_2, W) + \mathrm{cost}_\infty(v_3, W) = 0 + 0.25 + 0.5 = 0.75$.
- *For $p = \infty$ and $q = \infty$ we have* $\mathrm{cost}(W) = \max(\mathrm{cost}_\infty(v_1, W), \mathrm{cost}_\infty(v_2, W), \mathrm{cost}_\infty(v_3, W)) = \max(0, 0.25, 0.5) = 0.5$.

Note that, for mathematical clarity, we have chosen an optimization-oriented exposition (in which the aggregation goal is to minimize some norms). Indeed, an equivalent exposition may use axiomatic properties—defining that an aggregation method is deemed "fair" if it satisfies the axiom of maximizing some norms. In particular the continuum of values for the q-norms corresponds to the normative tradeoff between the utilitarian extreme (with $q = 1$) and the egalitarian extreme (with $q = \infty$).

3.1 Notation

Let $\mathcal{C}: T \to S$ denotes a class of all continuous functions. By $\mathcal{L}: T \to S$, we define a class of linear functions. For a natural number n we use $[n] = \{1, 2, \ldots, n\}$.

Our CONTINUOUS VOTING model has four parameters:

1. the input functions' type: $\mathcal{T}_{\mathrm{input}} \subseteq \mathcal{C}$,
2. the output functions' type: $\mathcal{T}_{\mathrm{output}} \subseteq \mathcal{C}$,
3. the parameter p of the L^p norm that defines the aggregation of the timepoint-cost of a voter to its cost. We refer to this L^p as the *time-aggregation* method; and,
4. the parameter q of the q-norm that defines the objective function we wish to minimize. We refer to this q-norm as the *voter-aggregation* method.

Accordingly, for specific values of $\mathcal{T}_{\mathrm{input}}$, $\mathcal{T}_{\mathrm{output}}$, p, and q, we write $(\mathcal{T}_{\mathrm{input}}, \mathcal{T}_{\mathrm{output}}, p, q)$-CV to denote the corresponding computational problem, i.e., the problem of optimally aggregating input preferences from $\mathcal{T}_{\mathrm{input}}$ into a solution in $\mathcal{T}_{\mathrm{output}}$ where the minimization is over L^p norm time-aggregation and q-norm voter-aggregation. We note that depending on the specific values of $\mathcal{T}_{\mathrm{input}}$ (resp. $\mathcal{T}_{\mathrm{output}}$) an input (resp. output) may be defined as an oracle (e.g., in the case of general continuous functions) or by numerical values (e.g., a linear function may be defined by two numerical values). In this paper, the use of these approaches is clear from the context.

3.2 Summary of the Results

Throughout the paper we study the computational complexity of identifying optimal solutions for the different realizations of our formal model. These are our main results:

- **General intractability**: We establish, in Theorem 1, the computational intractability of our problem. Indeed, that is not surprising, albeit crucial as a starting point.
- **General continuous functions**: We then consider general continuous functions; in Theorem 2, we show that:
  - For any $\mathcal{T}_{\mathrm{input}} \subseteq \mathcal{C}, (\mathcal{T}_{\mathrm{input}}, \mathcal{C}, 1, 1)$-CV is polynomial-time solvable; i.e., for any input type and output type, as long as we use an L^1 time-aggregation function and summing voters' costs, we can provide an optimal solution oracle in polynomial time (and using at most one oracle access to every input function);
  - and, for any $\mathcal{T}_{\mathrm{input}} \subseteq \mathcal{C}, (\mathcal{T}_{\mathrm{input}}, \mathcal{C}, \infty, \infty)$-CV is polynomial-time solvable; i.e., similarly to the above, albeit with L^∞ time-aggregation function and taking maximum over voters' costs.
- **Linear functions**: We then go on to consider linear functions; we show that:
  - $(\mathcal{L}, \mathcal{L}, \infty, q)$-CV is ϵ-approximable in polynomial time (Theorem 3);
  - $(\mathcal{L}, \mathcal{L}, \infty, \infty)$-CV is polynomial time solvable (Theorem 4);
  - $(\mathcal{L}, \mathcal{L}, \infty, 1)$-CV is polynomial time solvable (Theorem 5); and
  - $(\mathcal{L}, \mathcal{L}, 1, q)$-CV is ϵ-approximable in polynomial time (Theorem 7).

Indeed, constraining the input and output functions may be a too-strong of a restriction for many of the applications of our model (recall the applications described in Section 1). Nevertheless, we wish to note that:

- scientifically, as we concentrate on the basic question of what is feasible to compute in our model, starting from certain restrictions is natural;
- mathematically, as the results of Section 6 demonstrate, even the linear case is highly challenging to analyze;

- algorithmically, using the positive algorithmic results for linear inputs may be a step towards the design of efficient algorithms for more involved inputs such as, e.g., piecewise linear functions. They can reasonably approximate general continuous functions, and thus are closer to the applications described in Section 1. It is, nevertheless, not trivial to get a reasonable approximation guarantee in general, as the loss on approximation depends both on the number of intervals as well as the volatility of the original functions over these intervals;
- practically, the results of Section 5 show that, for basic aggregation functions, also general continuous functions (which correspond to the applications described in Section 1) admit efficient aggregation. In particular, we show, using quite elegant results, that for $p = q = 1$ (which corresponds to utilitarian aggregation over "pragmatic" voters whose satisfaction is their average satisfaction over time) and for $p = q = \infty$ (which corresponds to egalitarian aggregation over "pessimstic" voters whose satisfaction is their worst satisfaction over time) efficient aggregation is possible.

4 General Intractability

As expected, our model is generally computationally intractable. Below, we show that already a specific model realization is NP-hard.

In particular, let $\mathcal{F}_z \subseteq \mathcal{C}$ be a family of continuous functions that are divisible into z equal intervals in time (each interval of length $1/z$) such that, in each interval, the function is either constant 0 or constant 1. Therefore, every function from $\mathcal{F}_z$ is represented by z bits. Continuity is obtained by defining $f(i/z) = 0.5$ for every $f \in \mathcal{F}_z$ and $i \in \{0, 1, \dots z\}$ and connecting the intervals using two linear segments (of gradients z^3 and $-z^3$). Formally, we connect points $(p_i^-, f(p_i^-))$ and $(p_i^+, f(p_i^+))$ via a point $(i/z, 0.5)$, where $p_i^- = i/z - 0.5/z^3$ and $p_i^+ = i/z + 0.5/z^3$.

Theorem 1. $(\mathcal{F}_z, \mathcal{F}_z, 1, \infty)$-CV *is NP-hard.*

We prove NP-hardness of $(\mathcal{F}_z, \mathcal{F}_z, 1, \infty)$-CV by considering its decision variant where in the input we are additionally given a number $r \in \mathbb{R}_{\geq 0}$. The question to decide is whether the input $(\mathcal{F}_z, \mathcal{F}_z, 1, \infty)$-CV instance has an objective value smaller or equal to r. We construct a polynomial-time reduction from the CLOSEST STRING problem, which is NP-hard even on the binary alphabet [Frances and Litman, 1997; Bulteau *et al.*, 2014]. For space constraints, the proof is deferred to the full version of the paper.

5 Basic Cases

We observe that for cases when $p = q = 1$ or $p = q = \infty$, if an output function is not restricted, i.e., it can be any continuous function, then we can provide an optimal solution oracle in polynomial time. Note that this holds for every class of input functions.

Theorem 2. *For every* $\mathcal{T}_{\text{input}} \subseteq \mathcal{C}$*, we can provide oracles for* $(\mathcal{T}_{\text{input}}, \mathcal{C}, 1, 1)$-CV *and* $(\mathcal{T}_{\text{input}}, \mathcal{C}, \infty, \infty)$-CV *in polynomial time and using at most one oracle access to every input function.*

Proof. We describe the solution for these two cases separately.

The case of $(\mathcal{T}_{\text{input}}, \mathcal{C}, 1, 1)$**-CV:** for this case, a solution W is, intuitively, the median of votes at time t. In the case of even number of voters, we interpolate between the two middle votes. Formally, for $t \in T$, we order $\{v_1(t), v_2(t), \dots, v_n(t)\}$ in a non-decreasing way (this requires one oracle access to every input function). We denote this ordered sequence as V_t. We define an optimal solution W as follows:

$$W(t) = \begin{cases} V_t\left(\frac{n+1}{2}\right) & n \text{ is odd,} \\ \frac{1}{2}\Big(V_t\left(\frac{n}{2}\right) + V_t\left(\frac{n}{2}+1\right)\Big) & n \text{ is even.} \end{cases}$$

The case of $(\mathcal{T}_{\text{input}}, \mathcal{C}, \infty, \infty)$**-CV:** for this case, an optimal solution W is the mid-range of an interval spanned between a minimum and a maximum vote at time t. Formally:

$$W(t) = \frac{1}{2}\left(\max_{v \in V} v(t) - \min_{v \in V} v(t)\right).$$

□

Example 3. *Consider the setting of Example 1. Then, following the algorithms described in Theorem 2 we have that:*

- *For* $p = q = 1$*, an optimal solution is:*

$$W(t) = \begin{cases} 1/2 & \textit{for} \quad t \in [0, 0.5), \\ t & \textit{for} \quad t \in [0.5, 0.75), \\ 3/4 & \textit{for} \quad t \in [0.75, 1). \end{cases}$$

- *For* $p = q = \infty$*, an optimal solution is:*

$$W(t) = \begin{cases} t/2 + 3/8 & \textit{for} \quad t \in [0, 0.5), \\ 5/8 & \textit{for} \quad t \in [0.5, 0.75), \\ t/2 + 1/4 & \textit{for} \quad t \in [0.75, 1). \end{cases}$$

6 Linear Inputs and a Linear Output

In this section, we assume that voters provide linear functions, and our goal is to obtain a linear function. Formally, for voter v_i, $i \in [n]$, we let the function $v_i \colon [0,1] \to \mathbb{R}$ be defined by $v_i(t) := a_i t + b_i$, for $a_i, b_i \in \mathbb{R}$. The solution $W \colon [0,1] \to \mathbb{R}$ is defined by $W(t) := at + b$, for $a, b \in \mathbb{R}$.

We first consider the case with $p = \infty$ and then the case with $p = 1$. Below we show that we can solve the case of $p = \infty$, for any q, to any ϵ-accuracy, in polynomial time. We also show that in two special cases, we can solve this case exactly in polynomial time.

We need some structural lemmas first. In particular, we start with a lemma, which shows that the cost function of a single voter i, denoted by $\text{cost}(v_i, W)$, is a *piecewise linear convex function*, i.e., it can be written as the maximum of a finite number of affine linear functions.

Lemma 1. *In the* $(\mathcal{L}, \mathcal{L}, \infty, q)$ *case, we have*

$$\text{cost}(v_i, W) = \max\{b - b_i,\ -b + b_i,\ (a + b) - (a_i + b_i),\\ -(a + b) + (a_i + b_i)\}.$$

Proof. By definition, $\mathrm{cost}(v_i, W)$ is the following function of $(a, b) \in \mathbb{R}^2$:

$$\mathrm{cost}(v_i, W) := \max_{t\in[0,1]} \{|(at + b) - (a_i t + b_i)|\}.$$

Since the function $(at+b)-(a_it+b_i)$ is linear, we can write $\mathrm{cost}(v_i, W)$ in the form

$$\begin{aligned}\mathrm{cost}(v_i, W) &= \max_{t\in\{0,1\}} \{|(at + b) - (a_i t + b_i)|\}\\ &= \max\{|b - b_i|,\ |(a + b) - (a_i + b_i)|\}\\ &= \max\{b - b_i,\ -b + b_i,\\ &\quad (a + b) - (a_i + b_i),\ -(a + b) + (a_i + b_i)\}.\end{aligned}$$

This finishes the proof. □

The next lemma shows that, whenever the functions $\mathrm{cost}(v_i, W)$ are convex, then so is our objective function, which is the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\mathrm{cost}(W) := \left(\sum_{i=1}^{n} \mathrm{cost}(v_i, W)^p\right)^{\frac{1}{p}}.$$

Lemma 2. *Assume that the functions* $\mathrm{cost}(v_i, W)$*, for* $i \in \{1, 2, \dots, n\}$*, are convex. Then, the function*

$$\mathrm{cost}(W) := \left(\sum_{i=1}^{n} \mathrm{cost}(v_i, W)^p\right)^{\frac{1}{p}}$$

is convex, for every $p \geq 1$.

Proof. Consider the function $h\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(x) := \left(\sum_{i=1}^{n} \max\{x_i, 0\}^p\right)^{\frac{1}{p}}.$$

This function is convex and nondecreasing. Since the functions $\mathrm{cost}(v_i, W)$, for $i \in [n]$, are convex, we conclude that

$$h(\mathrm{cost}(v_1, W), \dots, \mathrm{cost}(v_n, W))$$

is a convex function of W (see "Vector composition" in Section 3.2.4 in [Boyd and Vandenberghe, 2014]). Since $\mathrm{cost}(v_i, W)$ is nonnegative for each $i \in [n]$, we have that

$$\mathrm{cost}(W) = h(\mathrm{cost}(v_1, W), \dots, \mathrm{cost}(v_n, W)),$$

so our conclusion is that $\mathrm{cost}(W)$ is convex. □

We are now ready to prove the main results of this section.

Theorem 3. *The* $(\mathcal{L}, \mathcal{L}, \infty, q)$ *case can be solved to any* ϵ*-accuracy in polynomial time.*

Proof. Our problem can then be cast as the following optimization problem:

$$\begin{aligned}\min \quad & \mathrm{cost}(W)\\ \text{s.t.} \quad & 0 \leq b \leq 1\\ & 0 \leq a + b \leq 1,\end{aligned}$$

where $\mathrm{cost}(W)$ is the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\mathrm{cost}(W) := \left(\sum_{i=1}^{n} \mathrm{cost}(v_i, W)^p\right)^{\frac{1}{p}}.$$

From Lemma 1, we know that $\mathrm{cost}(v_i, W)$ is a piecewise linear convex function, for every $i \in [n]$. From Lemma 2, we know that $\mathrm{cost}(W)$ is convex as well, for every $p \geq 1$.

In this optimization problem, we have two variables, a and b, the objective function $\mathrm{cost}(W)$ is convex, and the feasible region is a quadrilateral sandwiched between balls with center $(0, 0.5)$ and radii 0.3 and 1.2. Theorem 4.3.13 in [Grötschel *et al.*, 1988] implies that this optimization problem can be solved to any ϵ-accuracy in polynomial time, and we refer to the statement of this result for details. □

Next, we show that two important special cases can be solved exactly in polynomial time via linear programming.

Theorem 4. *The* $(\mathcal{L}, \mathcal{L}, \infty, \infty)$ *case is solvable in polynomial time.*

Proof. Our problem can then be cast as the following optimization problem:

$$\begin{aligned}\min \quad & \mathrm{cost}(W)\\ \text{s.t.} \quad & 0 \leq b \leq 1\\ & 0 \leq a + b \leq 1,\end{aligned}$$

where $\mathrm{cost}(W)$ is the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\mathrm{cost}(W) := \max_{i\in[n]} \{\mathrm{cost}(v_i, W)\}.$$

From Lemma 1, we obtain that $\mathrm{cost}(W)$ can be written as the maximum of the $4n$ affine linear functions $b - b_i$, $-b + b_i$, $(a + b) - (a_i + b_i)$, $-(a + b) + (a_i + b_i)$, for $i \in [n]$. The above optimization problem can then be cast as the following linear programming problem (see Section 1.3 in [Bertsimas and Tsitsiklis, 1997]):

$$\begin{aligned}\min \quad & z & \\ \text{s.t.} \quad & z \geq b - b_i & \forall i \in [n]\\ & z \geq -b + b_i & \forall i \in [n]\\ & z \geq (a + b) - (a_i + b_i) & \forall i \in [n]\\ & z \geq -(a + b) + (a_i + b_i) & \forall i \in [n]\\ & 0 \leq b \leq 1 & \\ & 0 \leq a + b \leq 1. &\end{aligned}$$

It is well-known that linear programming problems can be solved in polynomial time via the ellipsoid algorithm or interior point methods [Bertsimas and Tsitsiklis, 1997]. □

Theorem 5. *The* $(\mathcal{L}, \mathcal{L}, \infty, 1)$ *case is solvable in polynomial time.*

For space constraints, the proof is deferred to the full version of the paper. We go on to consider $p = 1$. Below, we show that we can solve the case with $p = 1$, for any q, to any ϵ-accuracy, in polynomial time.

To prove this result, we use Theorem 5.5 in [Bauschke *et al.*, 2016]. Thus, we now state this result and introduce the required notation[3]. In the following, X is an Euclidean space and I is a nonempty finite set.

Definition 1 (Compatible system of sets [Bauschke *et al.*, 2016])**.** *Let $\mathcal{A} := \{A_i\}_{i\in I}$ be a system of convex subsets of X, and let $A := \bigcup_{i\in I} A_i$. We say that $\mathcal{A}$ is a* compatible system of sets *if*

$$\left.\begin{matrix} i \in I \\ j \in I \\ i \neq j \end{matrix}\right\} \quad \Rightarrow \quad \operatorname{cl} A_i \cap \operatorname{cl} A_j \cap \operatorname{ri} A = A_i \cap A_j \cap \operatorname{ri} A, \tag{1}$$

where cl *is the closure and* ri *is the relative interior of a set.*

Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$. The *domain* of f is $\mathrm{D}_f := \{x \in X \mid f(x) < +\infty\}$. f is said to be *proper* if $\mathrm{D}_f \neq \emptyset$. In the following, $\mathcal{F} := \{f_i\}_{i\in I}$ is a system of proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$ and $f := \min_{i\in I} f_i$ is the piecewise-defined function associated with $\mathcal{F}$. For $x \in X$, $I(x) := \{i \in I \mid x \in \mathrm{D}_{f_i}\}$ is the active index set function.

Definition 2 (Compatible system of functions [Bauschke *et al.*, 2016])**.** *We say that $\mathcal{F}$ is a* compatible system of functions *if, for every $i \in I$, $f_i|_{\mathrm{D}_{f_i}}$ is continuous and*

$$\left.\begin{matrix} i \in I \\ j \in I \\ i \neq j \\ \mathrm{D}_{f_i} \cap \mathrm{D}_{f_j} \neq \emptyset \end{matrix}\right\} \Rightarrow f_i|_{\mathrm{D}_{f_i}\cap \mathrm{D}_{f_j}} \equiv f_j|_{\mathrm{D}_{f_i}\cap \mathrm{D}_{f_j}}. \tag{2}$$

Theorem 6 (Theorem 5.5 in [Bauschke *et al.*, 2016])**.** *Assume that $\mathcal{F}$ is a compatible system of functions, that each f_i is differentiable on $\operatorname{int} \mathrm{D}_{f_i} \neq \emptyset$, and that the following hold:*

(a) $\mathrm{D}_f = \bigcup_{i\in I} \mathrm{D}_{f_i}$ *is convex and at least 2-dimensional.*

(b) $\{\mathrm{D}_{f_i}\}_{i\in I}$ *is a compatible system of sets.*

(c) There exists a finite subset E of X such that

$$\left.\begin{matrix} x \in (\operatorname{int} \mathrm{D}_f) \setminus E \\ \{i, j\} \subseteq I(x) \end{matrix}\right\} \Rightarrow \lim_{\substack{z\to x \\ z\in \operatorname{int} \mathrm{D}_{f_i}}} \nabla f_i(z) = \lim_{\substack{z\to x \\ z\in \operatorname{int} \mathrm{D}_{f_j}}} \nabla f_j(z) \quad \textit{exists.}$$

Then f is convex.

Theorem 7. *The $(\mathcal{L}, \mathcal{L}, 1, q)$ case can be solved to any ϵ-accuracy in polynomial time.*

Proof. First, we obtain the cost function of a single voter i, denoted by $\mathrm{cost}(v_i, W)$. From simple geometric arguments, we write $\mathrm{cost}(v_i, W)$ as a function of $(a, b) \in \mathbb{R}^2$ as follows:

$$\mathrm{cost}(v_i, W) = \begin{cases} \mathrm{quad}(a, b) & \text{if } b < b_i,\ a + b > a_i + b_i \\ \mathrm{lin}(a, b) & \text{if } b < b_i,\ a + b \leq a_i + b_i \\ -\mathrm{quad}(a, b) & \text{if } b \geq b_i,\ a + b < a_i + b_i \\ -\mathrm{lin}(a, b) & \text{if } b \geq b_i,\ a + b \geq a_i + b_i, \end{cases}$$

where

$$\mathrm{quad}(a, b) = \frac{(b - b_i)^2 + (a + b - a_i - b_i)^2}{2(a - a_i)},$$
$$\mathrm{lin}(a, b) = (-a - 2b + a_i + 2b_i)/2.$$

[3]For further background, we refer the reader to [Bauschke and Combettes, 2011; Mordukhovich and Nam, 2013].

Claim 1. *The function $\mathrm{cost}(v_i, W)$ is a convex function from $\mathbb{R}^2$ to $\mathbb{R}$.*

Proof of Claim. To prove this claim, we use Theorem 6 (Theorem 5.5 in [Bauschke *et al.*, 2016]). Let $X := \mathbb{R}^2$ and $I := \{1, 2, 3, 4\}$. We define the following four functions f_i, for $i \in I$:

$$f_1(a, b) := \begin{cases} 0 & \text{if } (a, b) = (a_i, b_i) \\ \mathrm{quad}(a, b) & \text{if } (a, b) \neq (a_i, b_i),\ b \leq b_i, \\ & a + b \geq a_i + b_i \\ +\infty & \text{otherwise,} \end{cases}$$

$$f_2(a, b) := \begin{cases} \mathrm{lin}(a, b) & \text{if } b \leq b_i,\ a + b \leq a_i + b_i \\ +\infty & \text{otherwise,} \end{cases}$$

$$f_3(a, b) := \begin{cases} 0 & \text{if } (a, b) = (a_i, b_i) \\ -\mathrm{quad}(a, b) & \text{if } (a, b) \neq (a_i, b_i),\ b \geq b_i, \\ & a + b \leq a_i + b_i \\ +\infty & \text{otherwise,} \end{cases}$$

$$f_4(a, b) := \begin{cases} -\mathrm{lin}(a, b) & \text{if } b \geq b_i,\ a + b \geq a_i + b_i \\ +\infty & \text{otherwise.} \end{cases}$$

The domains of the above four functions D_{f_i}, for $i \in I$, are:

$$\begin{aligned} \mathrm{D}_{f_1} &:= \{x \in \mathbb{R}^2 \mid f_1(x) < +\infty\} \\ &= \{x \in \mathbb{R}^2 \mid b \leq b_i,\ a + b \geq a_i + b_i\}, \\ \mathrm{D}_{f_2} &:= \{x \in \mathbb{R}^2 \mid f_2(x) < +\infty\} \\ &= \{x \in \mathbb{R}^2 \mid b \leq b_i,\ a + b \leq a_i + b_i\}, \\ \mathrm{D}_{f_3} &:= \{x \in \mathbb{R}^2 \mid f_3(x) < +\infty\} \\ &= \{x \in \mathbb{R}^2 \mid b \geq b_i,\ a + b \leq a_i + b_i\}, \\ \mathrm{D}_{f_4} &:= \{x \in \mathbb{R}^2 \mid f_4(x) < +\infty\} \\ &= \{x \in \mathbb{R}^2 \mid b \geq b_i,\ a + b \geq a_i + b_i\}. \end{aligned}$$

We now check that $\mathcal{F} := \{f_i\}_{i\in I}$ is a compatible system of functions. First, for every $i \in I$, $f_i|_{\mathrm{D}_{f_i}}$ is continuous. Thus, we only need to check that condition (2) holds. For $(i, j) \in \{(1, 3), (2, 4)\}$, we have $\mathrm{D}_{f_i} \cap \mathrm{D}_{f_j} = \{(a_i, b_i)\}$ and

$$f_i(a_i, b_i) = f_j(a_i, b_i) = 0.$$

For $(i, j) = (1, 2)$, we have $\mathrm{D}_{f_1} \cap \mathrm{D}_{f_2} = \{x \in \mathbb{R}^2 \mid b \leq b_i,\ a + b = a_i + b_i\}$, and for every (a, b) in this set,

$$f_1(a, b) = f_2(a, b) = (b_i - b)/2.$$

For $(i, j) = (2, 3)$, we have $\mathrm{D}_{f_2} \cap \mathrm{D}_{f_3} = \{x \in \mathbb{R}^2 \mid b = b_i,\ a + b \leq a_i + b_i\}$, and for every (a, b) in this set,

$$f_2(a, b) = f_3(a, b) = (a_i - a)/2.$$

For $(i, j) = (3, 4)$, we have $\mathrm{D}_{f_3} \cap \mathrm{D}_{f_4} = \{x \in \mathbb{R}^2 \mid b \geq b_i,\ a + b = a_i + b_i\}$, and for every (a, b) in this set,

$$f_3(a, b) = f_4(a, b) = (b - b_i)/2.$$

For $(i, j) = (4, 1)$, we have $\mathrm{D}_{f_4} \cap \mathrm{D}_{f_1} = \{x \in \mathbb{R}^2 \mid b = b_i,\ a + b \geq a_i + b_i\}$, and for every (a, b) in this set,

$$f_4(a, b) = f_1(a, b) = (a - a_i)/2.$$

This completes the proof that $\mathcal{F} := \{f_i\}_{i\in I}$ is a compatible system of functions.

Next, we see that each f_i is indeed differentiable on $\operatorname{int} \mathrm{D}_{f_i}$. In fact, we have

$$\nabla \operatorname{quad}(a,b) = \left(-\frac{(b-b_i)^2 + (a-a_i+b-b_i)^2}{2(a-a_i)^2} + \frac{a-a_i+b-b_i}{a-a_i} , \frac{2(b-b_i)+2(a-a_i+b-b_i)}{2(a-a_i)} \right)$$

and

$$\nabla \operatorname{lin}(a,b) = (-0.5, -1)\,.$$

We have $\mathrm{D}_f = \bigcup_{i\in I} \mathrm{D}_{f_i} = \mathbb{R}^2$, which is convex and 2-dimensional. Thus, condition (a) in the statement of Theorem 6 is satisfied.

Next, we observe that condition (b) in the statement of Theorem 6 is satisfied, i.e., that $\{\mathrm{D}_{f_i}\}_{i\in I}$ is a compatible system of sets. This is simply because each D_{f_i} is closed and convex.

Consider now condition (c) in the statement of Theorem 6 with $E := \{(a_i, b_i)\}$. Due to this definition of E, it suffices to check the condition for the following pairs (i,j): $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$. For $(i,j) = (1,2)$, for every $(\bar{a},\bar{b}) \in \{x \in \mathbb{R}^2 \mid b < b_i,\ a+b = a_i + b_i\}$, we have

$$\lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_1}}} \nabla f_1(a,b) = \lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_2}}} \nabla f_2(a,b) = (-0.5,-1)\,.$$

For $(i,j) = (2,3)$, for every $(\bar{a},\bar{b}) \in \{x \in \mathbb{R}^2 \mid b = b_i,\ a + b < a_i + b_i\}$, we have

$$\lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_2}}} \nabla f_2(a,b) = \lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_3}}} \nabla f_3(a,b) = (-0.5,-1)\,.$$

For $(i,j) = (3,4)$, for every $(\bar{a},\bar{b}) \in \{x \in \mathbb{R}^2 \mid b > b_i,\ a + b = a_i + b_i\}$, we have

$$\lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_3}}} \nabla f_3(a,b) = \lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_4}}} \nabla f_4(a,b) = (0.5,1)\,.$$

For $(i,j) = (4,1)$, for every $(\bar{a},\bar{b}) \in \{x \in \mathbb{R}^2 \mid b = b_i,\ a + b > a_i + b_i\}$, we have

$$\lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_4}}} \nabla f_4(a,b) = \lim_{\substack{(a,b)\to(\bar{a},\bar{b})\\ (a,b)\in \operatorname{int} \mathrm{D}_{f_1}}} \nabla f_1(a,b) = (0.5,1)\,.$$

This completes the proof that condition (c) in the statement of Theorem 6 is satisfied.

We can now apply Theorem 6 and obtain that the function $f := \min_{i\in I} f_i$ is convex. This concludes the proof of the claim because we have $f \equiv \operatorname{cost}(v_i, W)$. $\diamond$

Our problem can then be cast as the following optimization problem:

$$\begin{aligned} \min \quad & \operatorname{cost}(W) \\ \text{s.t.} \quad & 0 \le b \le 1 \\ & 0 \le a + b \le 1, \end{aligned}$$

where $\operatorname{cost}(W)$ is the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\operatorname{cost}(W) := \left(\sum_{i=1}^{n} \operatorname{cost}(v_i, W)^p \right)^{\frac{1}{p}}.$$

From Claim 1, we know that $\operatorname{cost}(v_i, W)$ is convex, for every $i \in \{1, 2, \ldots, n\}$. From Lemma 2, we know that $\operatorname{cost}(W)$ is convex as well, for every $p \ge 1$.

In this optimization problem, we have two variables, a and b, the objective function $\operatorname{cost}(W)$ is convex, and the feasible region is a quadrilateral sandwiched between balls with center $(0, 0.5)$ and radii 0.3 and 1.2. Theorem 4.3.13 in [Grötschel *et al.*, 1988] implies that this optimization problem can be solved to any ϵ-accuracy in polynomial time, and we refer to the statement of this result for details. $\square$

7 Discussion

We have proposed a model of social choice that evolves around fair aggregation of continuous preferences in one dimension. We have discussed several of its usecases and positioned it within the literature on computational social choice.

By employing techniques from continuous optimization, we were able to show that—even though the corresponding optimization problem is generally computationally intractable—there are special cases for which efficient exact and approximate algorithms exist.

Here we discuss several avenues for future research:

- **Proportionality:** Here we considered one form of proportionality w.r.t. different values of the L^p norms we use. Considering other forms of proportionality, including formulating corresponding axiomatic properties is a natural future research direction. E.g., a possible adaptation of Proportional Justified Representation [Fernández *et al.*, 2017] may be that each group of voters that is sufficiently-large and for which the average pairwise distance is not too large shall have some upper-bounded average distance to the aggregated output.

- **Stability:** In some applications it is natural to aim at some form of stability with respect to the intensity and frequency of the changes in the aggregated output. In the context of our model, we may thus require that the derivative of the aggregated output is upper bounded by some predefined value. Studying such domain restrictions for our model is thus of interest.

- **Model variations:** Other possibilities for modeling our setting may be with different elicitation (e.g., letting voters not only specify their ideal point for each point in time, but more involved preferences); different utility functions (e.g., corresponding to arbitrary metric on $[0,1]$); and different aggregated output (e.g., returning several functions and not only one, i.e., a *committee of functions*). Furthermore, a corresponding online model (in which voter preferences change as a result of the partial aggregated values) is an interesting and well-motivated model to study.

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