A LOCAL View of the Polynomial Hierarchy

Fabian Reiter

LIGM, Université Gustave Eiffel

PODC 2024

Video on YouTube
youtu.be/lyxWOoVeqBU

Reverse Engineering a Theory

(A LOCAL view of the polynomial hierarchy)

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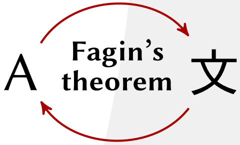
Two characterizations of NP



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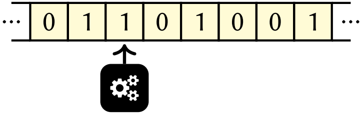
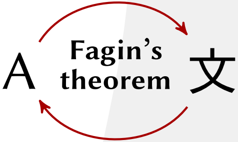


Nondet. polynomial-time
Turing machines



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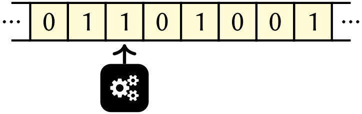
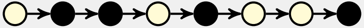
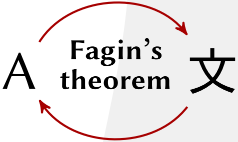




$\exists R\colon \mathit{Bijective}(R) \land \forall x, y\colon R(x, y) \rightarrow (\odot x \leftrightarrow \neg \odot y)$

Nondet. polynomial-time Turing machines



Existential fragment of second-order logic

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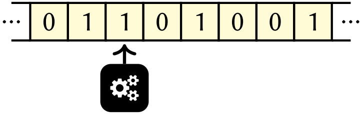

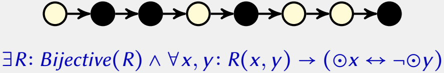

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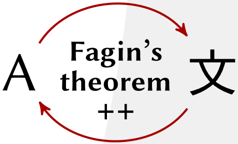

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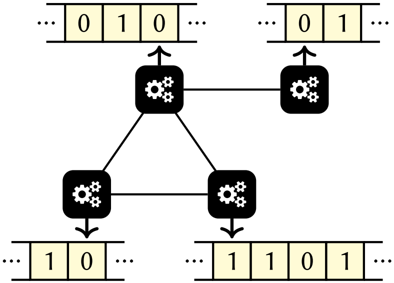
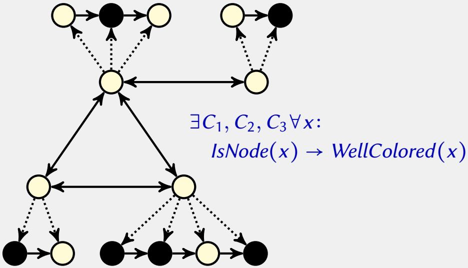
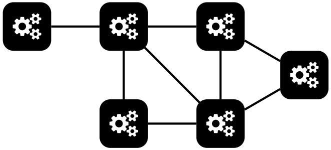




Model of computation



The LOCAL model

- Network of nodes with IDs & labels
- Same algorithm on all nodes
- Synchronous communication rounds

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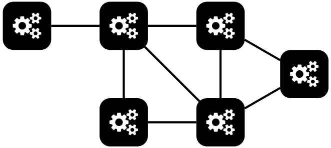

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- Constant number of rounds
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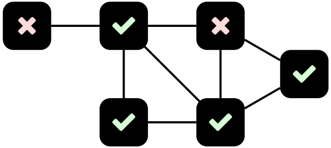

not Eulerian

(some nodes of odd degree)

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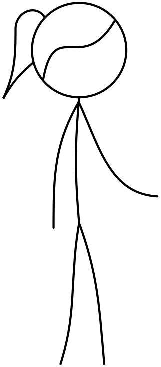
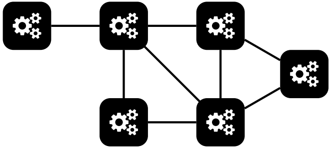


Eve (∃)



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- Certificates chosen by Eve

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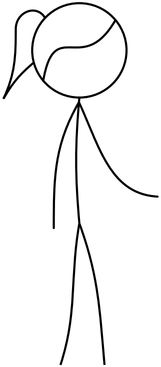

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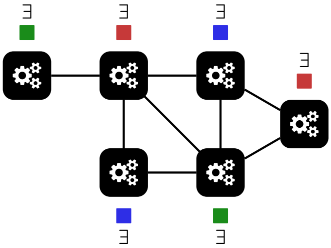

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3-colorable

(Eve can find a 3-coloring)

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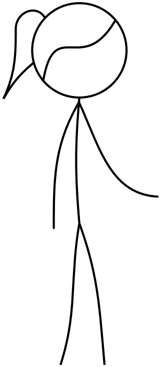

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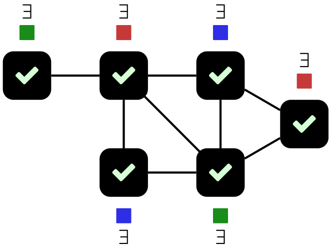

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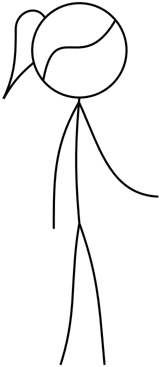

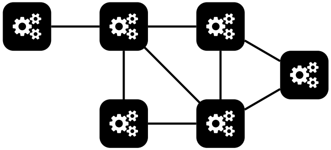

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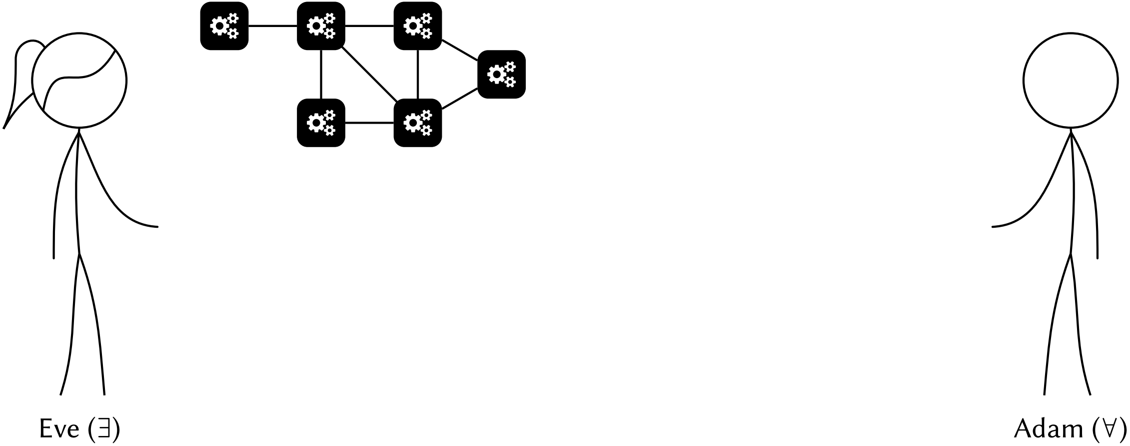

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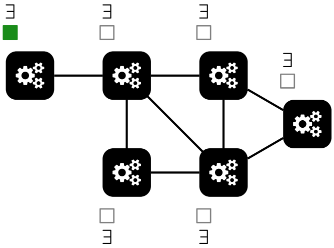

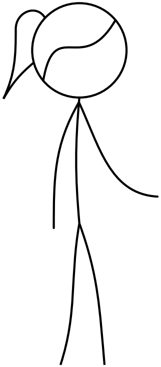

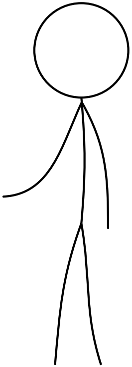

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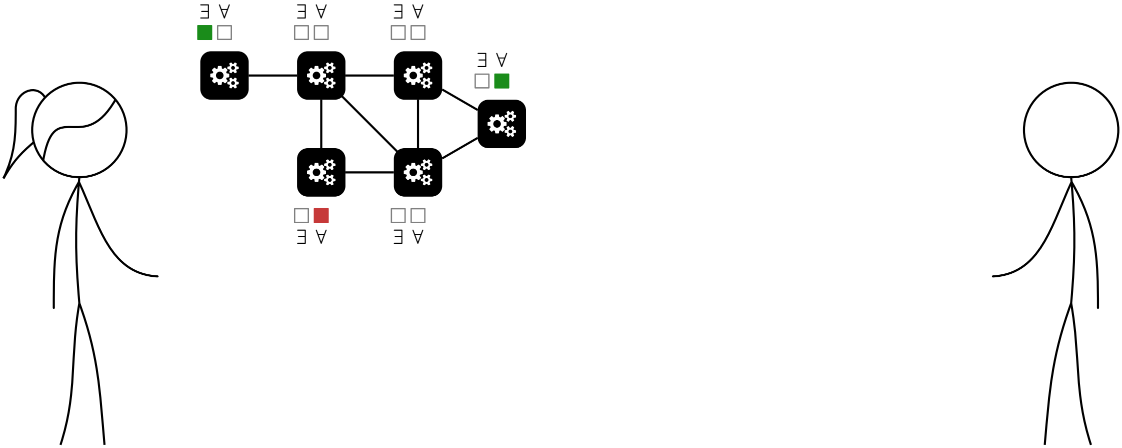
Adam (∀)

Alternation



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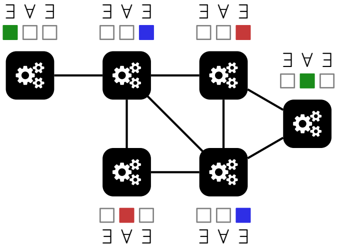

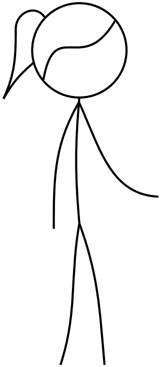

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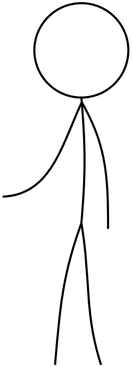

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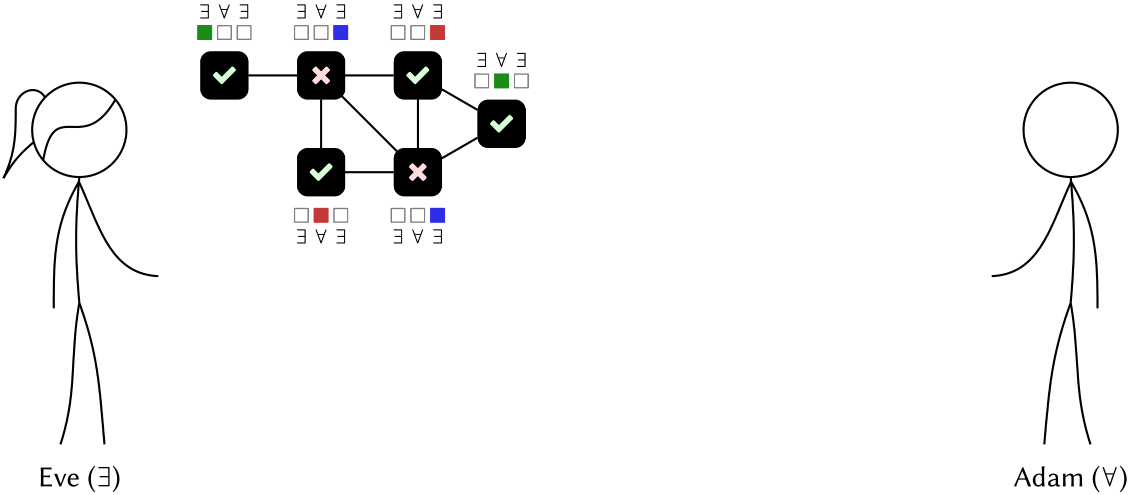

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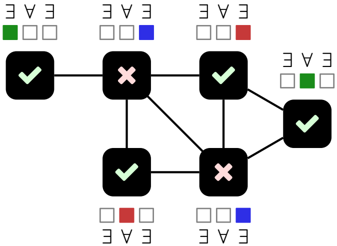
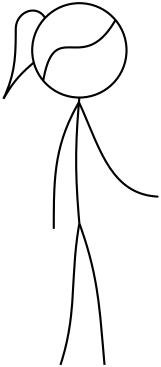


not 3-round 3-colorable

(Adam has a winning strategy)



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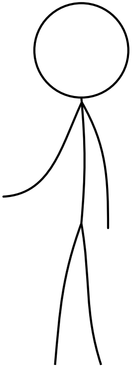

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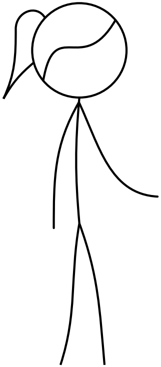

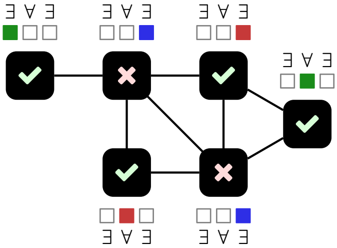

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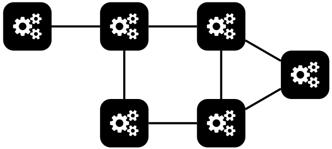

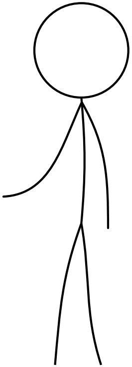

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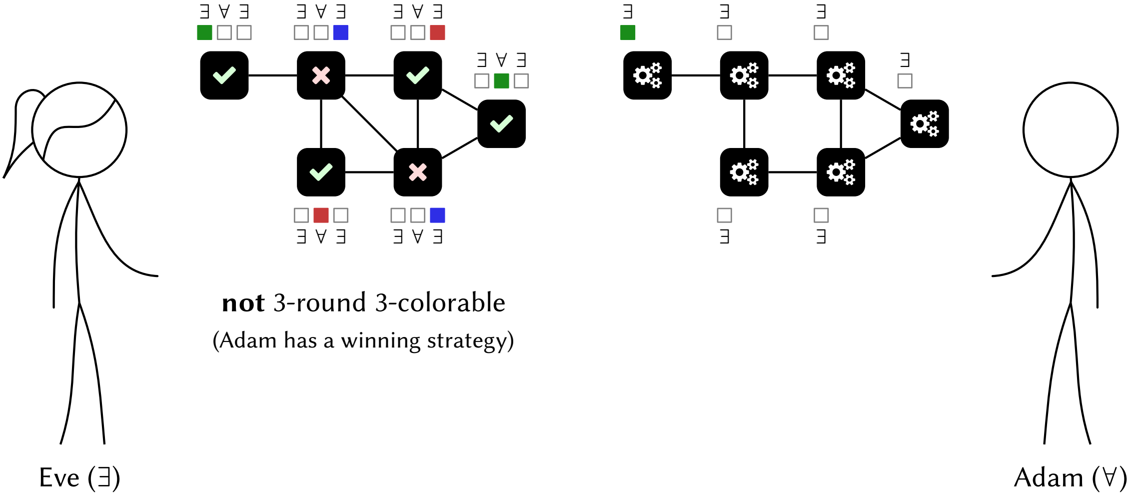

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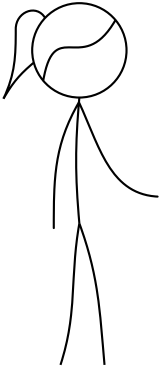

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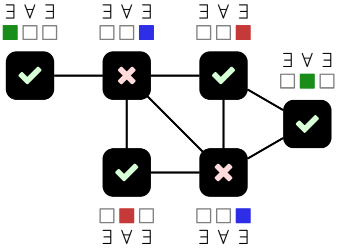

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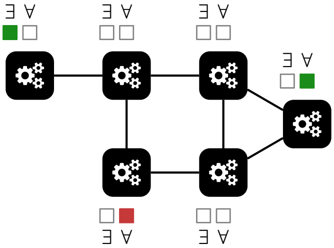

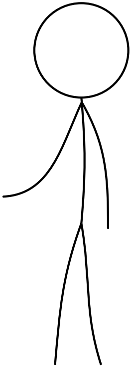

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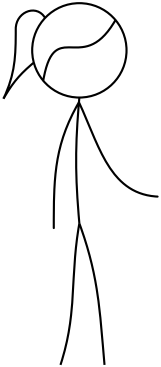

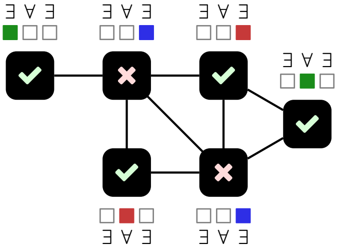

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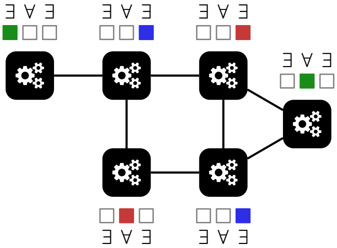

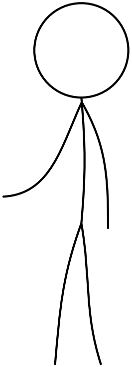

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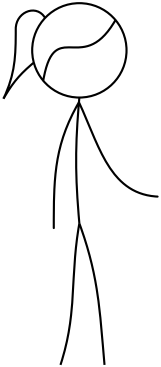

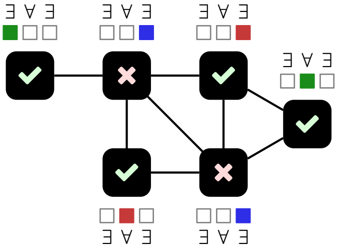

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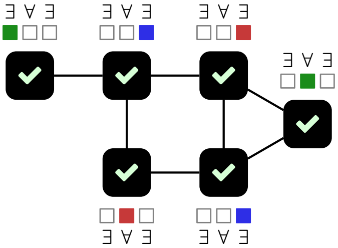

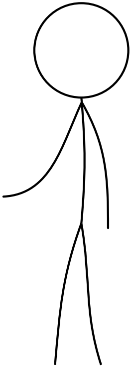

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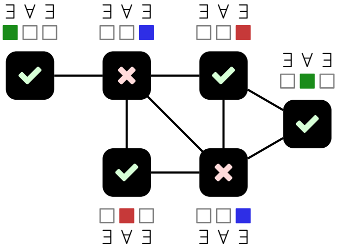

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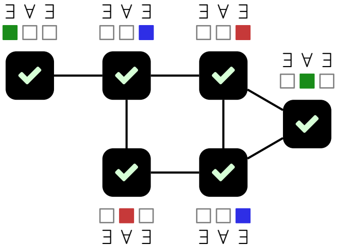

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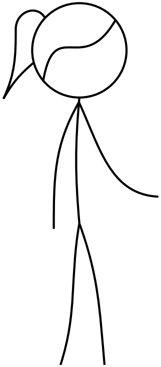

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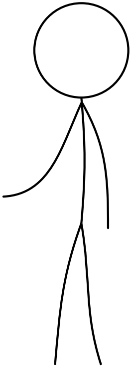

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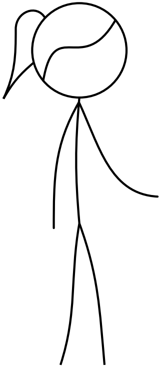

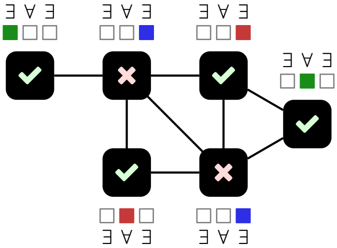

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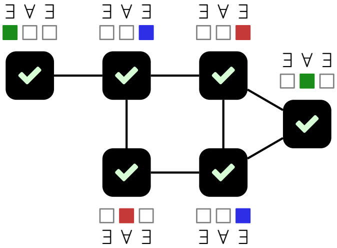

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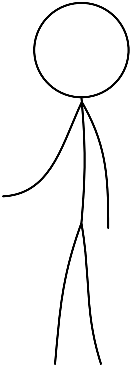

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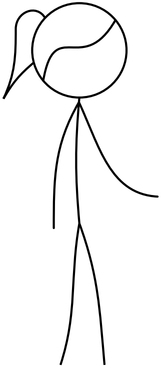
Adam (∀)

$\mathbf{\Sigma}_3$ $\exists\forall\exists$

$\mathbf{\Sigma}_2$ $\exists\forall$

$\mathbf{\Sigma}_1$ $\exists$

Alternation



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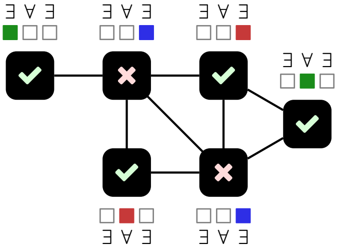

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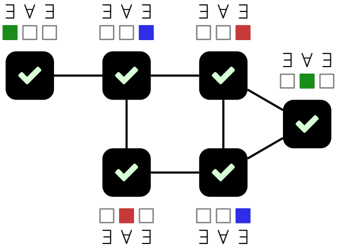

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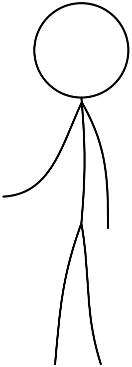

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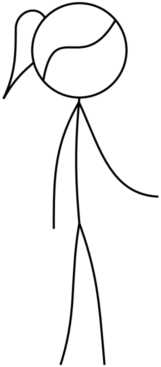
⟶ $\mathbf{\Sigma}_3$ ∃ ∀ ∃

$\mathbf{\Sigma}_2$ ∃ ∀

$\mathbf{\Sigma}_1$ ∃

Alternation



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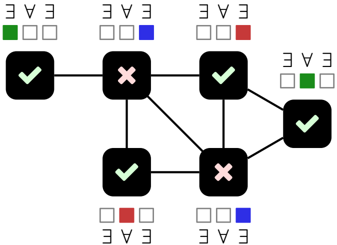

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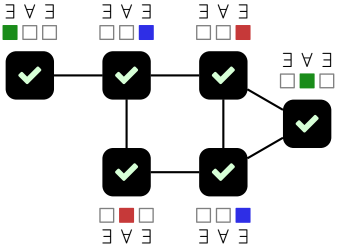
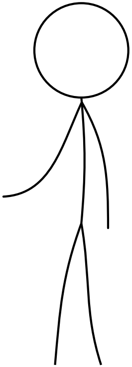


3-round 3-colorable

(Eve has a winning strategy)



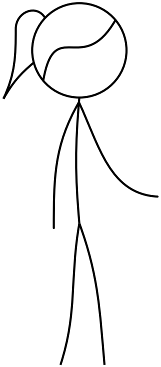
Adam (∀)

$\longrightarrow$ $\boldsymbol{\Sigma}_3$ ∃ ∀ ∃

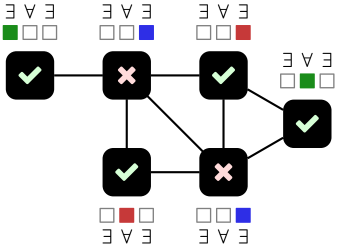
$\boldsymbol{\Sigma}_2$ ∃ ∀

$\longrightarrow$ $\boldsymbol{\Sigma}_1$ ∃

Alternation

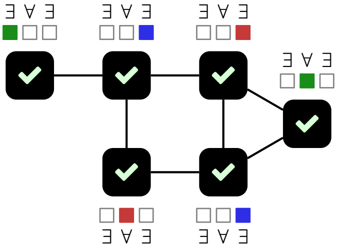


Eve (∃)



not 3-round 3-colorable

(Adam has a winning strategy)



3-round 3-colorable

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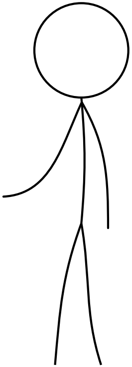
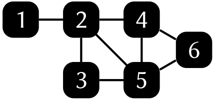


Adam (∀)

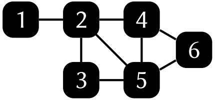
⟶	$\mathbf{\Sigma}_3$	∃ ∀ ∃	$\mathbf{\Pi}_3$	∀ ∃ ∀
	$\mathbf{\Sigma}_2$	∃ ∀	$\mathbf{\Pi}_2$	∀ ∃
⟶	$\mathbf{\Sigma}_1$	∃	$\mathbf{\Pi}_1$	∀

Related work



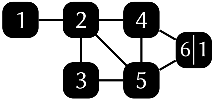
Feuilloley Fraigniaud Hirvonen (ICALP 2016)	Balliu D'Angelo Fraigniaud Olivetti (STACS 2017)	Aldema Tshuva Oshman (PODC 2022)	This work

Related work



	Feuilloley Fraigniaud Hirvonen (ICALP 2016)	Balliu D'Angelo Fraigniaud Olivetti (STACS 2017)	Aldema Tshuva Oshman (PODC 2022)	This work
ID uniqueness	global	global	global	

Related work



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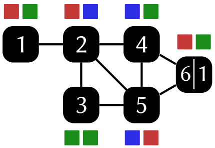
ID uniqueness	global	global	global	local

Related work



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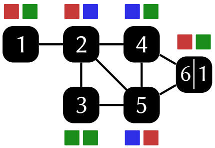
ID uniqueness	global	global	global	local
Certificate size	$O(\log n)$	unbounded	poly n	

n: number of nodes

Related work



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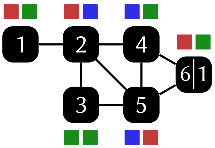
Certificate size	$O(\log n)$	unbounded	poly n	poly $\|N_r(v)\|$

n: number of nodes $\quad$ $|N_r(v)|$: size of node v's r-neighborhood

Related work



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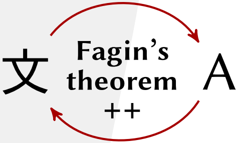
ID uniqueness	global	global	global	local
Certificate size	$O(\log n)$	unbounded	poly n	poly $\lvert N_r(v) \rvert$
Computation time	unbounded	unbounded	poly n	poly $\lvert N_r(v) \rvert$

n: number of nodes $\qquad$ $\lvert N_r(v) \rvert$: size of node v's r-neighborhood

Using logic and automata theory



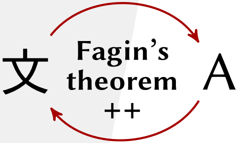
The LOCAL model

+ locally unique IDs
+ local-polynomial bounds

Using logic and automata theory

Monadic second-order logic (MSO)

- *Yields an infinite hierarchy on grids* [1].
- *Satisfies a locality property* [2].



The LOCAL model

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[1] Matz, Schweikardt, Thomas (2002)
[2] Giammarresi, Restivo, Seibert, Thomas (1996)

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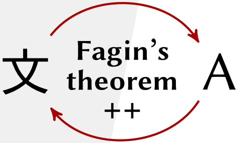


Finite-state automata

- *Satisfy a pumping lemma* [3].
- *Are equivalent to MSO on words* [4].

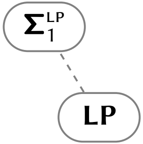
The LOCAL model

+ locally unique IDs
+ local-polynomial bounds

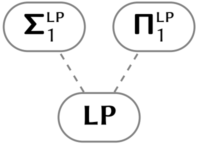
[1] Matz, Schweikardt, Thomas (2002)
[2] Giammarresi, Restivo, Seibert, Thomas (1996)
[3] Rabin, Scott (1959) & Bar-Hillel, Perles, Shamir (1961)
[4] Büchi (1960) & Elgot (1961) & Trakhtenbrot (1962)

The local-polynomial hierarchy

The local-polynomial hierarchy



The local-polynomial hierarchy



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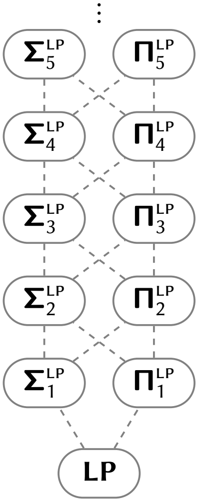

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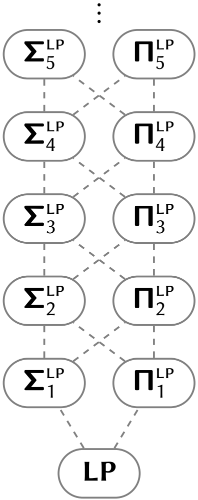
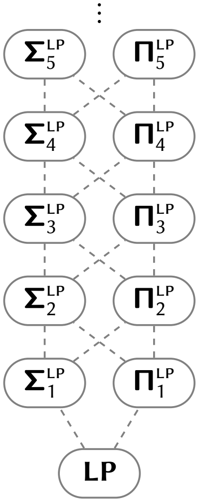


Connection to classical complexity:

$$\Sigma_\ell^{P} = \Sigma_\ell^{LP}\big|_{\text{NODE}} \qquad \Pi_\ell^{P} = \Pi_\ell^{LP}\big|_{\text{NODE}}$$

The local-polynomial hierarchy



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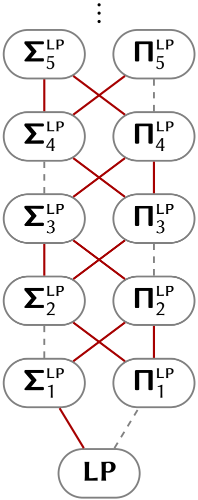
In particular:

$$\mathbf{P} = \mathbf{LP}\big|_{\text{NODE}} \qquad \mathbf{NP} = \Sigma_1^{LP}\big|_{\text{NODE}}$$

The local-polynomial hierarchy



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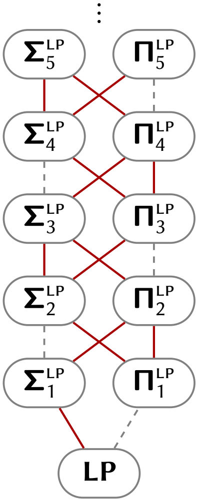
THEOREM: — Strict inclusions

The local-polynomial hierarchy



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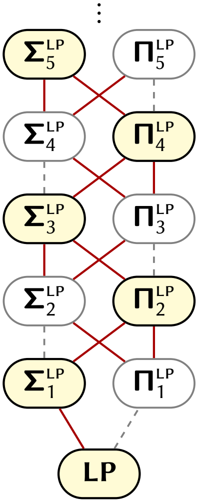
THEOREM: — Strict inclusions
- - - Equalities iff **P** = **NP**

The local-polynomial hierarchy



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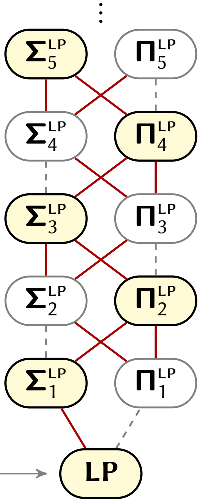

EULERIAN →
LP-complete

Connection to classical complexity:

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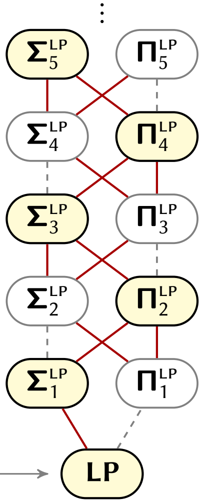
The local-polynomial hierarchy



3-COLORABLE → Σ_1^{LP}
Σ_1^{LP}-complete

EULERIAN → LP
LP-complete

Connection to classical complexity:

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THEOREM:
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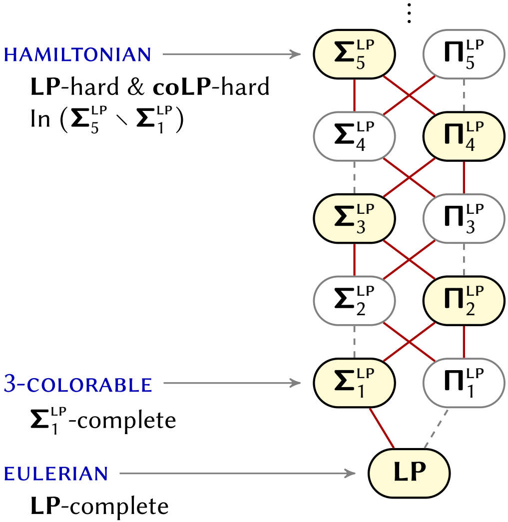
The local-polynomial hierarchy



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THEOREM: — Strict inclusions
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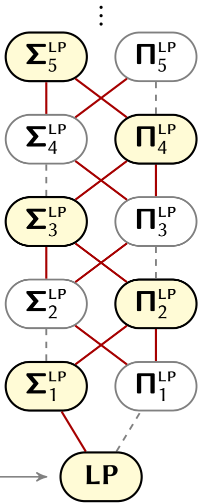
The local-polynomial hierarchy

PRIME-NB-NODES



HAMILTONIAN → Σ_5^{LP}

LP-hard & **coLP**-hard

In $(\Sigma_5^{LP} \setminus \Sigma_1^{LP})$

3-COLORABLE → Σ_1^{LP}

Σ_1^{LP}-complete

EULERIAN → **LP**

LP-complete

Connection to classical complexity:

$\Sigma_\ell^{P} = \Sigma_\ell^{LP}|_{\text{NODE}}$ $\qquad$ $\Pi_\ell^{P} = \Pi_\ell^{LP}|_{\text{NODE}}$

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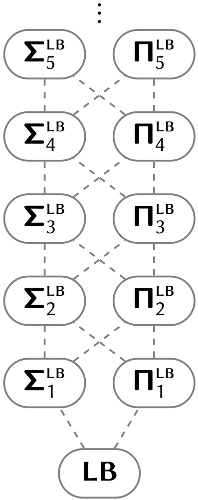
THEOREM: —— Strict inclusions
- - - Equalities iff **P** = **NP**

The local-*bounded* hierarchy



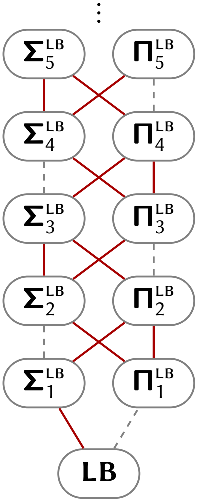
LP-hierarchy ⟶ **LB**-hierarchy

polynomial bounds — arbitrary **b**ounds

The local-*bounded* hierarchy

LP-hierarchy ⟶ **LB**-hierarchy

polynomial bounds   arbitrary **b**ounds



THEOREM: — Strict inclusions

The local-*bounded* hierarchy

LP-hierarchy ⟶ **LB**-hierarchy

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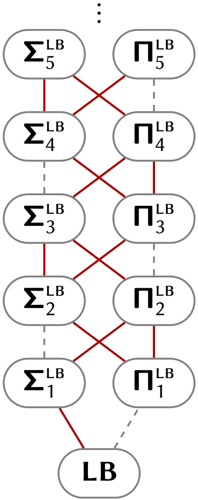
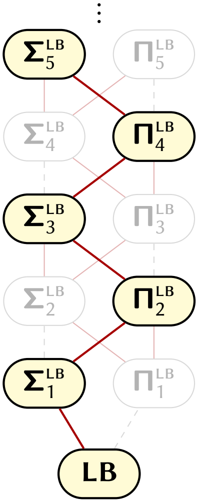


THEOREM: — Strict inclusions
- - - Equalities

The local-*bounded* hierarchy



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polynomial bounds — arbitrary **b**ounds

THEOREM: — Strict inclusions
- - - Equalities

The local-*bounded* hierarchy

HAMILTONIAN →
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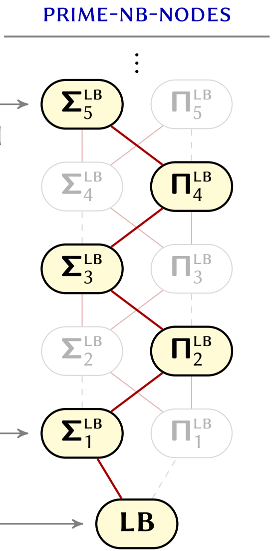
In $(\Sigma_5^{LB} \setminus \Sigma_1^{LB})$

3-COLORABLE →
Σ_1^{LB}-complete

EULERIAN →
LB-complete



LP-hierarchy ⟶ **LB**-hierarchy
polynomial bounds   arbitrary **b**ounds

THEOREM: — Strict inclusions
--- Equalities

The local-*bounded* hierarchy

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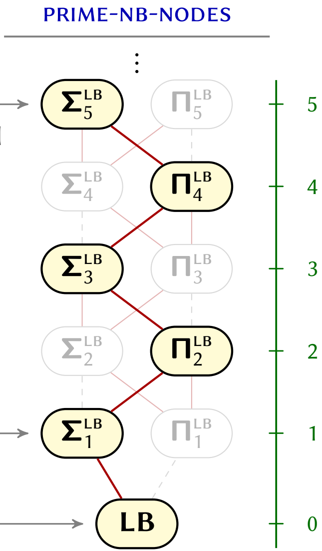

THEOREM: — Strict inclusions
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A measure of locality?

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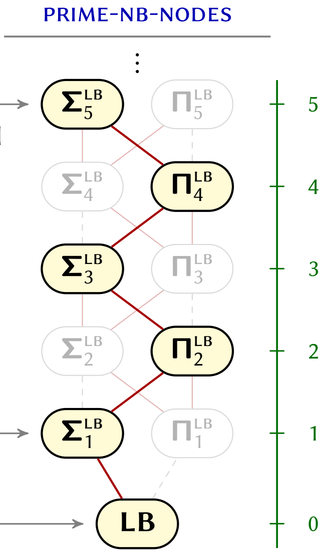

THEOREM: — Strict inclusions
- - - Equalities

We lose one thing:
Fagin's theorem

A measure of locality?

Building a theory

Image source: www.rhb.ch

Building a theory

Image source: www.rhb.ch