Formal proofs and certified computation in Coq

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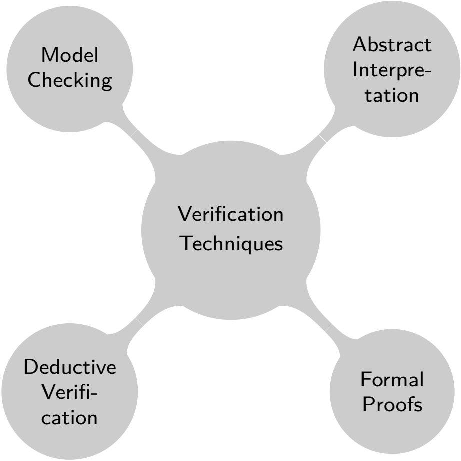
Formal Methods

- Gather
  - a set of mathematically-based techniques
  - designed to specify and verify computer systems



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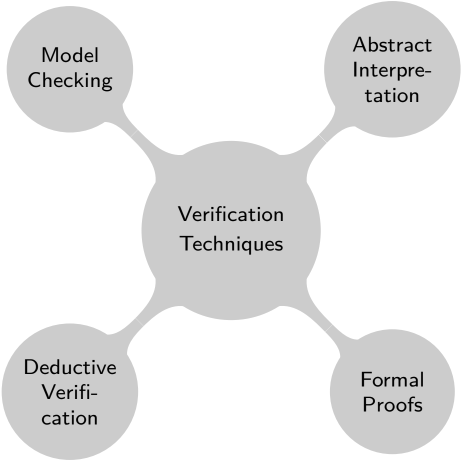

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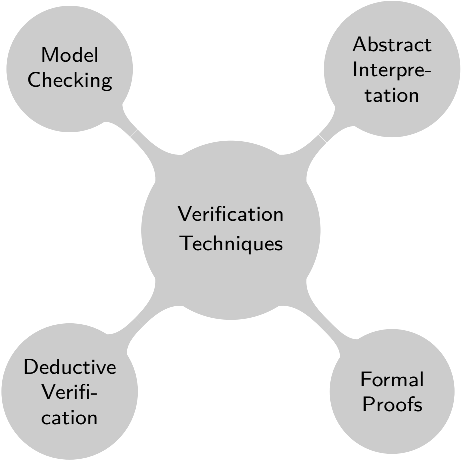
- For instance
  - for the Paris Métro Line 14
  - at Intel, AMD, ...



Formal Proofs

- needs a proof assistant (= proof checker (= theorem prover))
  - specify algorithms and theorems
  - develop proofs interactively
  - check proofs
  - but also perform computations, develop automatic tactics...
- various tools: ACL2, Agda, Coq, HOL Light, Isabelle, Mizar, PVS...

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- various tools: ACL2, Agda, Coq, HOL Light, Isabelle, Mizar, PVS...
- main criteria to classify them:
  - the kind of underlying logic (FOL/HOL, classical/intuitionistic...)
  - the presence of a proof kernel (De Bruijn's criterion)
  - the degree of automation
  - the availability of large libraries of formalized results
- see also [Freek Wiedijk (2006): The Seventeen Provers of the World]

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   - → this is an informal question
   - → well surveyable: check that the formalized definitions indeed correspond to the usual mathematical ones
     (no need to dive into proof details: they're fully handled by the checker)

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- Coq has been awarded the 2013 ACM Software System Award, and the 2013 SIGPLAN Programming Languages Software Award.

Recap the role of Coq's kernel

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Key feature of Coq's logic: the convertibility rule

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- We compute and get $0 = 0$, which trivially holds ($\mathrm{refl} : 0 = 0$)
- As $0 = 0$ and $4 \ominus 8 = 0$ are convertible, we also have $\mathrm{refl} : 4 \ominus 8 = 0$, hence the result.

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Approaches to certify computation with a Proof Assistant

Borrowing [G. Barthe, G. Ruys, H. Barendregt, 1995]'s terminology

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deductive verification: annotate the (imperative) program code and use dedicated tools, such as Frama-C/Jessie/Why3, to generate proof obligations (to be discharged by automated provers or proof assistants as back-ends)

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- gathers support results on derivability, Riemann integral (both defined with dependent types) and reference functions

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  - tactics to automate proofs on derivatives

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  - by construction, all functions overs the constructive reals are continuous
    $\rightsquigarrow$ hinders the applicability to proofs in standard/numerical analysis

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- $\leadsto$ MathComp: comprehensive library of algebra, based on SSReflect

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- [C. Cohen, M. Dénès, A. Mörtberg (2013): Refinements for Free!]

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- solutions: bisection, automatic differentiation. . . or Taylor Models:
  [N. Brisebarre, M. Joldeș, EMD, M. Mayero, J-M. Muller, I. Pașca, L. Rideau, and L. Théry (2012): Rigorous Polynomial Approximation Using Taylor Models in Coq]

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Overview of the CoqInterval library — Proof example #1

Example taken from [John Harrison (1997): Verifying the Accuracy of Polynomial Approximations in HOL]

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Require Import Reals Interval_tactic.
Local Open Scope R_scope.
```

`Theorem Harrison97 :` $\forall x : \mathbb{R}, -\frac{10831}{1000000} \leqslant x \leqslant \frac{10831}{1000000} \implies$

$$\left|(e^x - 1) - \left(x + \frac{8388676}{2^{24}}x^2 + \frac{11184876}{2^{26}}x^3\right)\right| \leqslant \frac{23}{27} \times \frac{1}{2^{33}}.$$

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Local Open Scope R_scope.
```
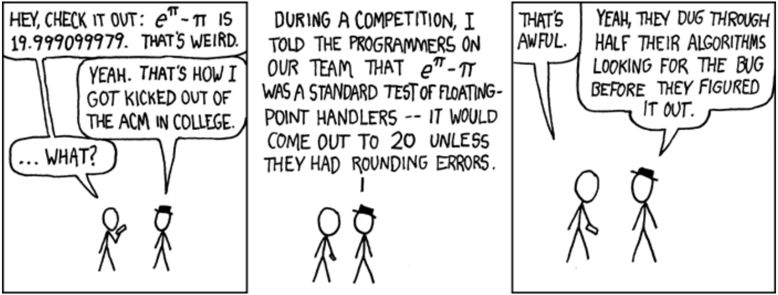

Theorem Harrison97 : $\forall x : \mathbb{R}, -\frac{10831}{1000000} \leqslant x \leqslant \frac{10831}{1000000} \implies$

$$\left|(e^x - 1) - \left(x + \frac{8388676}{2^{24}}x^2 + \frac{11184876}{2^{26}}x^3\right)\right| \leqslant \frac{23}{27} \times \frac{1}{2^{33}}.$$

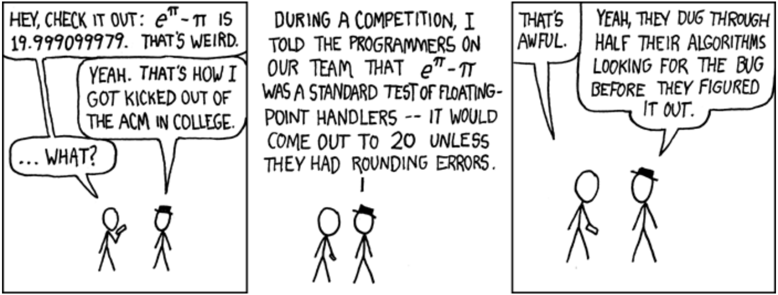
```
Proof.
intros x H.
interval with (i_bisect_taylor x 3, i_prec 50). (* in 0.5s *)
Qed.
```

Overview of the CoqInterval library — Proof example #2



(xkcd.com/217)

Overview of the CoqInterval library — Proof example #2



(xkcd.com/217)

```
Lemma xkcd217 : 19 999 099 979/10^9 < e^π − π < 19 999 099 980/10^9.
Proof.
split; interval with (i_prec 40). (* in 0.15s *)
Qed.
```

Related works on formalized game theory

- [René Vestergaard (2005): A constructive approach to sequential Nash equilibria]

$\leadsto$ proof, formalized in Coq, that all non-cooperative, sequential games have a Nash equilibrium

- [Stéphane Le Roux' PhD thesis, 2008]

$\leadsto$ generalizes and formalizes in Coq the notions of strategic game and Nash equilibrium (notably, not requiring payoffs to be real numbers)

- [Evgeny Dantsin, Jan-Georg Smaus, Sergei Soloviev (2012): Algorithms in Games Evolving in Time: Winning Strategies Based on Testing]

$\leadsto$ formalizes in Isabelle/HOL sufficient conditions for the computability of a winning strategy function (for two-player games evolving in time)

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Thank you for your attention!