Proving Tight Bounds on Univariate Expressions with Elementary Functions in Coq

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Agenda

1. **Motivation:** Formal proof of approximation errors
2. The **Coq** proof assistant: computation and proof reflection
3. **CoqInterval:** Methodology, Architecture, and Examples
4. **Related works:** Comparison with existing tools
5. Conclusion and perspectives

Accuracy of floating-point elementary functions

- elementary functions ($\exp$, $\cos$, etc.) are ubiquitous in today's software
- it is crucial that `libm`s (libraries of mathematical functions) document the accuracy of the computed values!
- the IEEE 754–2008 std for floating-point arithmetic gives recommendation on their accuracy

Example of correctness claim

- Proving the implementation of $\exp$ in CRlibm[1] relies on the claim:

$$\forall x \in \mathbb{R},\ |x| \le 355 \cdot 2^{-22} \implies \left| \frac{x + 0.5 \cdot x^2 + c_3 x^3 + c_4 x^4 - \exp x + 1}{\exp x - 1} \right| \le 2^{-62} \quad (1)$$

with $c_3 = 6004799504235417 \cdot 2^{-55}$ and $c_4 = 1501199876148417 \cdot 2^{-55}$.

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- Tedious and error-prone to prove by hand!

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- Attempt to verify (1) by plotting $f : x \mapsto \frac{x+0.5 \cdot x^2 + c_3 x^3 + c_4 x^4 - \exp x + 1}{\exp x - 1}$:

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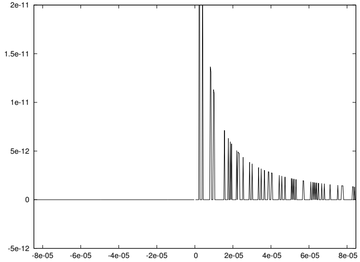

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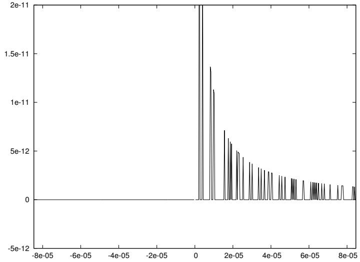

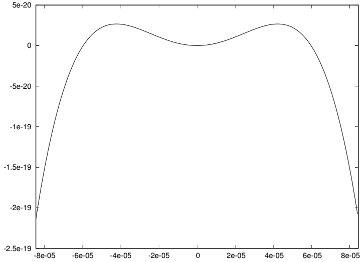

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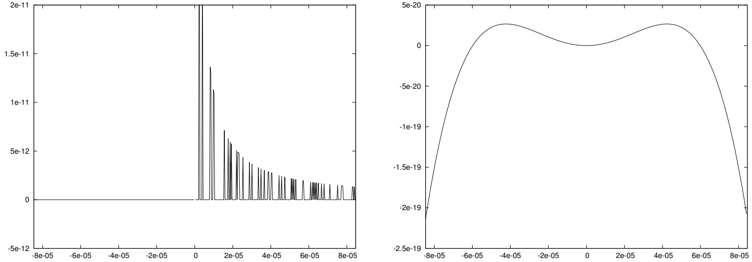

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⇝ Need to use dedicated tools, e.g. proof assistants, to verify statements like (1) that are critical for the correctness of `libm`s' implementations

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Coq, computation, and proof by reflection

Coq comes with a primitive notion of computation, called conversion.

Key feature of Coq's logic: the convertibility rule

In environment E, if $p : A$ and if A and B are convertible, then $p : B$.

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  - So we conclude by using reflexivity and the convertibility rule.

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- solutions: bisection, automatic differentiation. . . or Taylor Models:
  [N. Brisebarre, M. Joldeș, EMD, M. Mayero, J-M. Muller, I. Pașca, L. Rideau, and L. Théry (2012): Rigorous Polynomial Approximation Using Taylor Models in Coq]

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- `i_bisect_taylor` x d: do a bisection along variable x while computing degree-d univariate Taylor models

Bisection

- Idea: Split $\boldsymbol{x}$ into sub-intervals $\boldsymbol{x} = \boldsymbol{a} \cup \boldsymbol{b}$, so we get $f(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{a}) \cup \boldsymbol{f}(\boldsymbol{b})$ (which is a tighter inclusion than $f(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{x})$)
- Then: Iterate the process recursively on $\boldsymbol{a}$ and $\boldsymbol{b}$.

- Drawback: Proving something like $\forall x \in [0,1],\ |x - x| \le 2^{-40}$ with this technique alone yields a huge number of sub-intervals
- And it will not succeed in proving $\forall x \in [0,1],\ x - x = 0$.

- Advantage: Can be combined with other approaches to reduce the dependency effect (cf. `i_bisect_diff` and `i_bisect_taylor`)

Automatic differentiation

- Based on the interval version of Taylor-Lagrange's formula at order 0,

$$\forall x \in \boldsymbol{x},\ \exists \xi \in \boldsymbol{x}, \quad f(x) = f(x_0) + (x - x_0) \cdot f'(\xi),$$
$$\forall x \in \boldsymbol{x}, \quad f(x) \in \boldsymbol{f}([x_0, x_0]) + (\boldsymbol{x} - [x_0, x_0]) \cdot \boldsymbol{f'}(\boldsymbol{x}).$$

- Rely on automatic differentiation to compute $\boldsymbol{f'}(\boldsymbol{x})$
  - Work with pairs of intervals $(\underbrace{\boldsymbol{u}}_{\text{enclosure of } f(x)}, \underbrace{\boldsymbol{u'}}_{\text{enclosure of } f'(x)})$
  - Example of rule: $(\boldsymbol{u}, \boldsymbol{u'}) \times (\boldsymbol{v}, \boldsymbol{v'}) = (\boldsymbol{uv}, \boldsymbol{u'v} + \boldsymbol{uv'})$
- For the toy example $f(x) = x - x$ over $\boldsymbol{x} = [0, 1]$ (cf. previous slide), we get $\boldsymbol{f'}(\boldsymbol{x}) = [0, 0]$, so f is a constant function $f \equiv f(x_0) = 0$. QED.

CoqApprox: formally verified library of Taylor models

A Taylor model is a pair (polynom, error interval) and we will say that $(P, \boldsymbol{\Delta})$ represents a function f over $\boldsymbol{I}$ if we have $\forall x \in \boldsymbol{I},\ f(x) - P(x) \in \boldsymbol{\Delta}$

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1. For "basic functions", compute an enclosure of the Taylor–Lagrange remainder at order n;
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Within CoqApprox: verified computation of Taylor models for functions $\sqrt{\cdot}$, $\frac{1}{\sqrt{\cdot}}$, $x \mapsto x^n$ $(n \in \mathbb{Z})$, $\exp$, $\sin$, $\cos$, $\ln$, $\tan$, $\arctan$, as well as the operations $+$, $-$, $\times$, $\div$, $\circ$.

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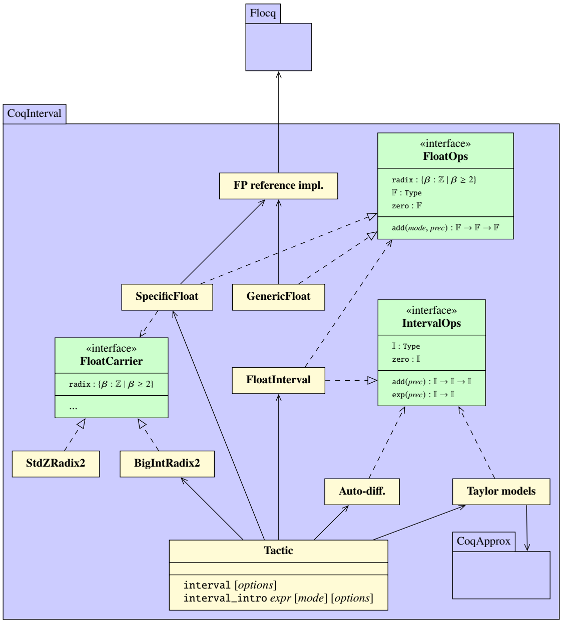
- Sharp bounds: thanks to the implemented algorithm called Zumkeller's technique, the approximation of basic functions leads to sharp bounds in practice (Idea: take advantage of the monotonicity of $R_n(f, \xi_0)(x) := f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(\xi_0)}{i!} \cdot (x - \xi_0)^i$ over $[\inf(\boldsymbol{x}), \xi_0]$ and over $[\xi_0, \sup(\boldsymbol{x})]$.)

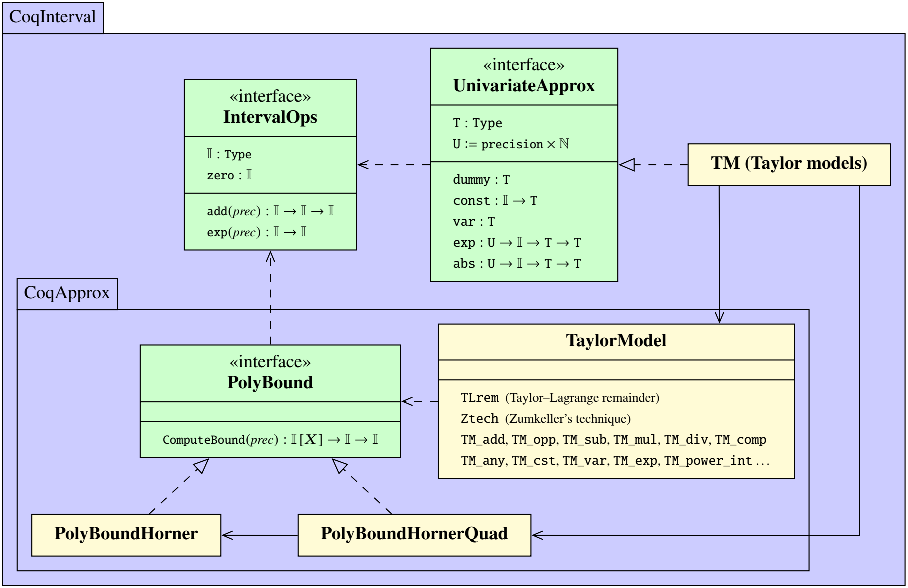


Caption:

A - - - - - - -> I
if the module A is parameterized by a module implementing I

C - - - - - - -▷ I
if the module C implements the interface I

M ———→ C
if the module M uses the module C


CoqInterval
«interface»
IntervalOps
𝕀 : Type
zero : 𝕀
add(prec) : 𝕀 → 𝕀 → 𝕀
exp(prec) : 𝕀 → 𝕀
«interface»
UnivariateApprox
T : Type
U := precision × ℕ
dummy : T
const : 𝕀 → T
var : T
exp : U → 𝕀 → T → T
abs : U → 𝕀 → T → T
TM (Taylor models)
CoqApprox
«interface»
PolyBound
ComputeBound(prec) : 𝕀[X] → 𝕀 → 𝕀
TaylorModel
TLrem (Taylor–Lagrange remainder)
Ztech (Zumkeller's technique)
TM_add, TM_opp, TM_sub, TM_mul, TM_div, TM_comp
TM_any, TM_cst, TM_var, TM_exp, TM_power_int ...
PolyBoundHorner
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- Depend on the `Coquelicot` library of real analysis
- Support for Coq 8.5 and MathComp 1.6

Overview of the CoqInterval library — Proof example #1

Example taken from [John Harrison (1997): Verifying the Accuracy of Polynomial Approximations in HOL]

```
Require Import Reals Interval_tactic.
Local Open Scope R_scope.
```

Theorem Harrison97 : $\forall x : \mathbb{R}, -\frac{10831}{1000000} \leq x \leq \frac{10831}{1000000} \implies$
$$\left|(e^x - 1) - \left(x + \frac{8388676}{2^{24}}x^2 + \frac{11184876}{2^{26}}x^3\right)\right| \leq \frac{23}{27} \times \frac{1}{2^{33}}.$$

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```
Proof.
intros x H.
interval with (i_bisect_diff x, i_prec 50, i_depth 16). (* 35s *)
Qed.
```

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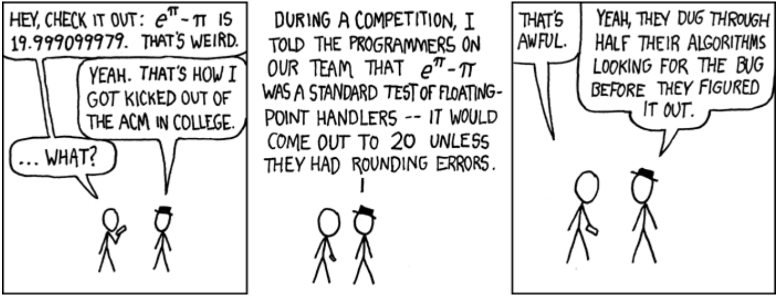
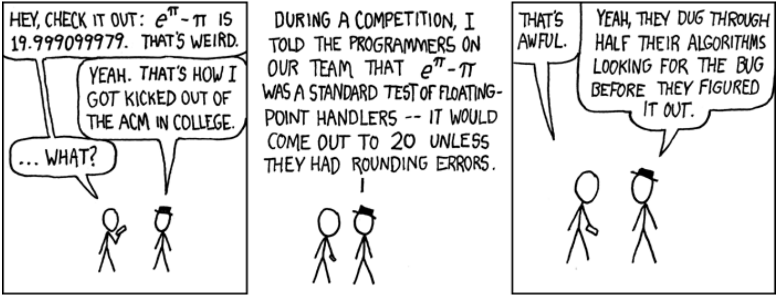
```
Proof.
intros x H.
interval with (i_bisect_taylor x 3, i_prec 50). (* 0.50s *)
Qed.
```

Overview of the CoqInterval library — Proof example #2



(xkcd.com/217)

Overview of the CoqInterval library — Proof example #2



(xkcd.com/217)

```
Require Import Reals Interval_tactic.
Local Open Scope R_scope.

Lemma xkcd217 : exp PI - PI <> 20.
Proof.
interval. (* 0.05s *)
Qed.
```

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- approximation problems: CRlibm's $\exp$ ($|x| \geq 2^{-20}$), a Remez of $\sqrt{\cdot}$, a degree-5 approx of $\arctan$, Earth's radius of curvature, Tang's $\exp$

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Forge: https://gforge.inria.fr/scm/browser.php?group_id=6316&extra=bench-ineqs

Experimental Results (univariate approximation problems)

Problems	CoqInterval 2.0	Sollya	MetiTarski	NLCertify (not verified)	NLCertify (verified polys)	PVS/interval	HOL Light/ verify_ineq	PVS/Bernstein	HOL Light/ REAL_SOS
crlibm_exp	0.83⋆	0.02	Failed	-	-	Failed	-	-	-
remez_sqrt	0.45	0.02	0.05	15.28⋆	Timeout	Failed	3.60⋆	-	-
abs_err_atan	0.45	0.01	0.07	Failed	Failed	Timeout	2.36⋆	-	-
rel_err_geo	3.10	2.24	Timeout	Timeout	Timeout	Failed	229.54⋆	-	-
harrison97	0.42	0.01	0.10	-	-	Failed	-	-	-
cos_cos_d2	0.71	0.05	Timeout	Timeout	Timeout	20.64	5.82⋆	-	-
cos_cos_d3	0.79	0.05	Timeout	Timeout	Timeout	48.87	6.28⋆	-	-
cos_cos_d4	0.91	0.06	Timeout	Timeout	Timeout	Timeout	8.83⋆	-	-
cos_cos_d5	1.44	0.06	Timeout	Timeout	Timeout	Timeout	15.70⋆	-	-
cos_cos_d6	1.54	0.07	Timeout	Timeout	Timeout	Timeout	20.92⋆	-	-
cos_cos_d7	2.21	0.07	Timeout	Timeout	Timeout	Timeout	41.88⋆	-	-
cos_cos_d8	2.79	0.08	Timeout	Timeout	Timeout	Timeout	87.78⋆	-	-

Experimental Results (MetiTarski 1/2)

Problems	CoqInterval 2.0	Sollya	MetiTarski	NLCertify (not verified)	NLCertify (verified polys)	PVS/interval	HOL Light/ verify_ineq	PVS/Bernstein	HOL Light/ REAL_SOS
MT1	0.53	-	0.13	-	-	Failed	-	-	-
MT2	1.56	-	0.06	9.99⋆	Timeout	Failed	-	-	-
MT3	0.18	-	0.18	-	-	1.14	-	-	-
MT4	0.23	-	0.17	1.31⋆	18.95⋆	1.19	-	-	-
MT5	0.11⋆	-	0.05	-	-	1.24	-	-	-
MT6	0.15⋆	-	0.07	-	-	1.23	-	-	-
MT7	0.04	-	0.04	-	-	0.69	-	-	-
MT8	0.33	-	0.15	-	-	Timeout	-	-	-
MT9	0.52	-	0.46	-	-	Timeout	-	-	-
MT10	0.19	-	0.04	0.96	14.86	Failed	-	-	-
MT11	0.10	-	0.22	0.40	6.73	1.72	-	-	-
MT12	2.84	-	0.07	Timeout	Timeout	Timeout	-	-	-
MT13	0.98	-	0.07	11.82	137.91	Failed	-	-	-
MT14	0.07	-	0.06	-	-	0.89	-	-	-
MT15	0.15	-	0.07	-	-	0.98	-	-	-

Experimental Results (MT 2/2 + multivariate problems)

Problems	CoqInterval 2.0	Sollya	MetiTarski	NLCertify (not verified)	NLCertify (verified polys)	PVS/interval	HOL Light/ verify_ineq	PVS/Bernstein	HOL Light/ REAL_SOS
MT16	0.13	-	0.02	$0.58^\star$	$8.23^\star$	3.23	$0.57^\star$	-	-
MT17	0.11	-	0.06	0.22	4.06	1.27	0.23	-	-
MT18	0.16	-	0.02	0.21	2.46	0.69	0.75	-	-
MT19	0.52	-	Failed	5.09	74.55	Failed	1.92	-	-
MT20	3.09	-	0.05	2.63	44.21	Timeout	15.54	-	-
MT21	0.33	-	0.38	3.69	51.94	Failed	1.37	-	-
MT22	0.69	-	0.06	Timeout	Timeout	Failed	113.74	-	-
MT23	1.17	-	0.12	Failed	Failed	Failed	86.90	-	-
MT24	0.10	-	0.36	0.17	2.38	Failed	0.24	-	-
MT25	0.29	-	0.17	-	-	1.78	-	-	-
RD	0.25	-	0.02	1.88	66.01	1.67	0.48	3.26	Timeout
adaptiveLV	0.16	-	0.04	0.23	3.18	1.00	1.26	4.02	3.78
butcher	0.42	-	0.05	0.73	11.08	19.99	2.21	18.23	Timeout
magnetism	0.17	-	0.05	1.35	20.60	Timeout	313.75	Timeout	0.24

Conclusion

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- All computations performed in Coq's logic, using interval arithmetic
- Implements bisection, automatic differentiation and Taylor models techniques to reduce the dependency effect
- Regarding performance, CoqInterval is competitive w.r.t the state-of-the-art inequality provers

Some future directions

- Bottleneck: Horner evaluation.
  Formalize alternative schemes that are amenable to formal methods?
- Certifying algorithms: Check polynomial approximations for special functions, by using certificates generated by Sollya?
- Reals/Coquelicot/CoqInterval/. . . : Increase automation for developing formal libraries of elementary functions more easily
- Symbolic-numeric methods: CoqInterval has been used and extended by Thomas Sibut-Pinote, Assia Mahboubi and Guillaume Melquiond to formally verify approximations of definite integrals $\rightsquigarrow$ ultimate goal to formally verify numerical solutions of differential equations.

Thanks for your attention!

Homepage: http://coq-interval.gforge.inria.fr/

Ref: http://www.irit.fr/publis/ACADIE/CoqInterval-JAR.pdf