An Existence Theorem of Nash Equilibrium in Coq and Isabelle

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In a nutshell Definitions Paper-and-pencil results

A game theory result formalised in Coq and Isabelle

Le Roux has shown a result on two-player games: starting from a game with multiple outcomes, one can derive a game that maps those outcomes into just two possible outcomes, namely that player 1 wins or player 2 wins.

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A game theory result formalised in Coq and Isabelle

- Le Roux has shown a result on two-player games: starting from a game with multiple outcomes, one can derive a game that maps those outcomes into just two possible outcomes, namely that player 1 wins or player 2 wins.
- If the game is such that any way of deriving such a win-lose game leads to a game with a Nash equilibrium (and hence a pre-determined winner), then the original game also has a Nash equilibrium.
- We prove this result in Coq and Isabelle.

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Game forms

Definition

A game form is a tuple $\langle A, (S_a)_{a \in A}, O, v \rangle$ such that

- A is a non-empty set of players,
- $\prod_{a \in A} S_a$ is a non-empty Cartesian product of strategy profiles, where S_a represents the strategies available to player a,
- O is a non-empty set of possible outcomes,
- $v: \prod_{a \in A} S_a \to O$ is the outcome function.

Providing \prec_a , a binary preference relation over O for each player a, constitutes a game.

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Nash equilibrium

Definition

Let $g = \langle A, (S_a)_{a \in A}, O, v, (\prec_a)_{a \in A} \rangle$ be a game. A strategy profile s in $S := \prod_{a \in A} S_a$ is a Nash equilibrium if it makes every player a stable, *i.e.* $v(s) \not\prec_a v(s')$ for all $s' \in S$ that differ from s at most at the a-component.

$$NE(s) := \forall a \in A, \forall s' \in S, \quad (\forall b \in A \setminus \{a\}, s_b = s'_b) \implies v(s) \not\prec_a v(s')$$

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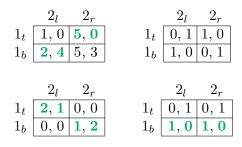
Theoretical part Isabelle Coq

Conclusion

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Four games

Nash equilibria



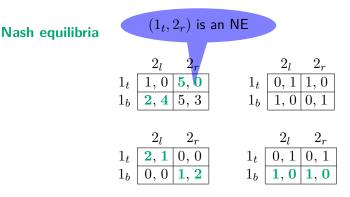
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Four games



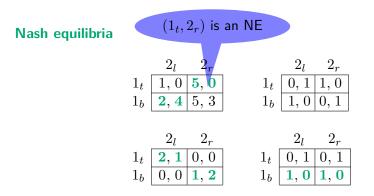
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Four games



We will concentrate on two-player games from now on.

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Win-lose games

Definition

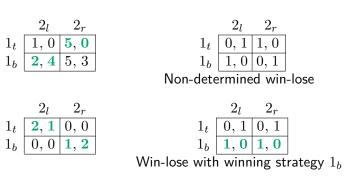
- A win-lose game is a game where $A = \{1, 2\}$ and $O = \{(1, 0), (0, 1)\}$ with preferences as expected . . .
- ▶ Winning strategy $s_1 \in S_1$ for Player 1: $v(s_1, s_2) = (1, 0)$ for all $s_2 \in S_2$. Analogous for Player 2.
- A win-lose game such that one player has a winning strategy is said to be determined.

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The four games again



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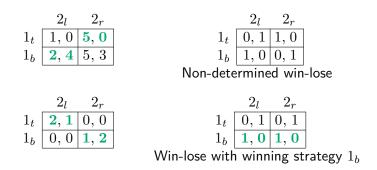
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We will concentrate on two-player games from now on.

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Derived games

Definition

Let $gf = \langle \{1,2\}, S_1, S_2, O, v \rangle$ be a two-player game form.

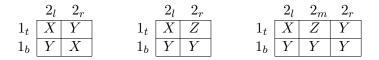
- ► For all $\prec_1, \prec_2 \subseteq O^2$ the game $\langle \{1, 2\}, S_1, S_2, O, v, \{\prec_1, \prec_2\} \rangle$ is said to be derived from *gf*.
- ▶ Let wl be a function from O to $\{(1,0), (0,1)\}$. The win-lose game $\langle S_1, S_2, wl \circ v \rangle$ is also said to be derived from gf.
- ▶ If all win-lose games derived from a game form are determined (*via* strategies in R₁, R₂), the game form is also said to be determined (*via* strategies in R₁ and R₂).
- ▶ Let $P \subseteq O$, and let $s_1 \in S_1$ such that $v(s_1, S_2) := \{v(s_1, s_2) \mid s_2 \in S_2\} \subseteq P$. The strategy s_1 is said to enforce P and exclude $O \setminus P$.

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Examples of derived games



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Lifting the preferences

The main theorem of this paper needs in the proof a lifting of preferences \prec to sets, i.e., we must define what it means for an agent to prefer a set of outcomes over another set.

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Lifting the preferences

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$$\forall A, B \subseteq S, \quad A \prec^{\mathcal{P}} B := \exists a \in A \backslash B, \forall b \in B \backslash A, a \prec b$$

Rest of the construction then required \prec to be a strict linear order.

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Rest of the construction then required \prec to be a strict linear order. Contribution of this work (on the paper-and-pencil front): Using an alternative lifting that does not require \prec to be linear.

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Finitary equilibrium transfer

Theorem

Let $\langle \{1,2\}, S_1, S_2, O, v, \{\prec_1, \prec_2\} \rangle$ be a two-player game where O is finite and let us assume the following:

- 1. The game form is determined via strategies in R_1 and R_2 .
- 2. Both preferences \prec_1 and \prec_2 are strict partial orders.

Then the game $\langle\{1,2\},S_1,S_2,O,v,\{\prec_1,\prec_2\}\rangle$ has a Nash equilibrium in $R_1\times R_2.$

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Finitary equilibrium transfer: Proof sketch

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Proof sketch:

1. Let M be the $\prec_1^{\mathcal{P}}$ -greatest subset of O that Player 1 can enforce using strategy s_1 .

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Finitary equilibrium transfer: Proof sketch

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- 2. Let m be \prec_2 -maximal in M, and let $M' := (M \setminus \{m\}) \cup u(m)$. One can see that $M \prec_1^{\mathcal{P}} M'$.

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- 2. Let m be \prec_2 -maximal in M, and let $M' := (M \setminus \{m\}) \cup u(m)$. One can see that $M \prec_1^{\mathcal{P}} M'$.
- 3. Player 1 cannot enforce M'. So Player 2 can enforce $O \setminus M'$ using strategy s_1 . It turns out that $v(s_1, s_2) = \{m\}$ and that (s_1, s_2) is a Nash equilibrium.

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Nash equilibrium in Coq & Isabelle12/24



- Standard Isabelle/HOL in ISAR proof style without any special libraries
- Restriction to two players!

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- Standard Isabelle/HOL in ISAR proof style without any special libraries
- Restriction to two players!
- Around 1100 lines of proof code.
- ► Many lines for technicalities concerning the lifting of ≺, e.g., showing that the lifted order is transitive (160 lines).

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Games forms and games

There is nothing to define about strategies and the outputs: they are simply type parameters.

type_synonym ('0, 'S1, 'S2) game_form = "('S1 * 'S2) \Rightarrow '0"

type_synonym ('0,'S1,'S2) game = "('0 \Rightarrow '0 \Rightarrow bool) * ('0 \Rightarrow '0 \Rightarrow bool) * (('0,'S1,'S2) game_form)"

Functions pref1, pref2, and form extract each of the three components of a game g.

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Nash equilibrium and determined game

definition

isNash :: "(('0,'S1,'S2) game) ⇒ 'S1 ⇒ 'S2 ⇒ bool"
where "isNash g s1 s2 =
 ((∀ s1'. ¬(pref1 g) ((form g) (s1,s2)) ((form g) (s1',s2))) ∧
 (∀ s2'. ¬(pref2 g) ((form g) (s1,s2)) ((form g) (s1,s2'))))"

definition

```
determined :: "((bool, 'S1, 'S2) game) \Rightarrow ('S1 set) \Rightarrow ('S2 set)

\Rightarrow bool"

where "determined g R1 R2 =

((\exists s1 \in R1. \forall s2. (form g) (s1, s2) = True) \lor
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 $(\exists s2 \in \mathbb{R}2. \forall s1. (form g) (s1,s2) = False))"$

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Derived win-lose game and determined game form

definition

Note the simplified outcome type!

definition

determinedForm :: "(('0,'S1,'S2) game_form) \Rightarrow ('S1 set) \Rightarrow ('S2 set) \Rightarrow bool" where "determinedForm gf R1 R2 = (\forall Ou. determined (derivedWLGame gf Ou) R1 R2)"

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Main result

theorem equilibrium_transfer_finite : assumes finite0 : "finite (range (form g))" and trans1 : " \land a b c. (pref1 g) a b \Longrightarrow (pref1 g) b c \implies (pref1 g) a c" and irref1 : " \land a. \neg (pref1 g) a a" and trans2 : " \land a b c. (pref2 g) a b \Longrightarrow (pref2 g) b c \implies (pref2 g) a c" and irref2 : " \land a. \neg (pref2 g) a a" and det : "determinedForm (form g) R1 R2" shows " \exists s1 \in R1. \exists s2 \in R2. isNash g s1 s2"

153 lines of proof but uses various lemmas.

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Overview of the formal setup in Coq

Formalization choice: provide game-theoretic definitions (game form, Nash eq...) that are as general as possible before instantiating them to two-player games.

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Overview of the formal setup in Coq

- Formalization choice: provide game-theoretic definitions (game form, Nash eq...) that are as general as possible before instantiating them to two-player games.
- ► The entire formalization has around 1300 lines of Coq code.
- ► 270 lines of Coq code are devoted to prove all properties of the lifting of ≺.

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- ► The entire formalization has around 1300 lines of Coq code.
- ► 270 lines of Coq code are devoted to prove all properties of the lifting of ≺.
- Main dependency: SSReflect and MathComp
 - especially using theories fintype, finfun, finset, and bigop
 - \rightsquigarrow comprehensive formalization of finite sets
 - \rightsquigarrow facilities to reason about discrete objects in a "classical" way

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Summary of the main definitions (1/3)

```
Variables (Agt : Type)(Strat : Agt \rightarrow Type)(Outc : Type).
Definition strategy := \forall a : Agt, Strat a. (* dep. type *)
Record game_form := GameForm
{ preform :> strategy \rightarrow Outc ;
  eq_strategy : (* extensionality property *) }.
Record game := Game
{ form :> game_form ;
  prefs : Agt \rightarrow Outc \rightarrow Outc \rightarrow bool }.
Definition is_NE (g : game) (strat : strategy) : Prop :=
  \forall a : Agt, \forall strat' : strategy,
  (\forall b : Agt, a \neq b \rightarrow strat b = strat' b) \rightarrow
  \neg prefs g a (g strat) (g strat').
Definition ex NE (g : game) : Type :=
  {strat : strategy | is NE g strat}.
```

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```

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Summary of the main definitions (2/3)

```
Inductive player := player1 | player2.
Definition game form 2 := game form player. (* instantiation *)
Definition game 2 := game player.
Inductive winlose outc := win1 | win2.
Definition winlose_prefs (a: player) (o1 o2 : winlose_outc) :=
  match a, o1, o2 with
  | player1, win2, win1 \Rightarrow true
  | player2, win1, win2 \Rightarrow true
  | , , \Rightarrow false
  end.
Definition derivedWLGame :
  \forall Outc Strat, (Outc \rightarrow winlose_outc) \rightarrow
  game_form_2 Outc Strat \rightarrow game_2 winlose_outc Strat.
```

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Definition winlose_prefs (a: player) (o1 o2 : winlose_outc) :=
  match a, o1, o2 with
  | player1, win2, win1 \Rightarrow true
                                         player1 prefers win1
  | player2, win1, win2 \Rightarrow true
  | _, _, _ \Rightarrow false
  end.
Definition derivedWLGame :
  \forall Outc Strat, (Outc \rightarrow winlose_outc) \rightarrow
  game_form_2 Outc Strat \rightarrow game_2 winlose_outc Strat.
```

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Summary of the main definitions (3/3)

Variables (Strat : player → Type)(Outc : Type).
Definition preferred_outc (a : player) : winlose_outc :=
 if a is player1 then win1 else win2.
Definition win_strat (v : game_form_2 winlose_outc Strat)
(a : player) (sa : Strat a) :=
 ∀s : strategy Strat, s a = sa → v s = preferred_outc a.
Definition determined (v : game_form_2 winlose_outc Strat) :=
 {a : player & {sa : Strat a | win_strat v a sa}}.
Definition determined_form (v : game_form_2 Outc Strat) :=
 ∀wl : Outc → winlose_outc, determined (derivedWLGame wl v).

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The formalized theorem in a nutshell

```
Theorem finite_equilibrium_transfer :
 ∀ (Strat : player → Type) (_ : strategy player Strat)
 (Outc : finType) (g : game_2 Outc Strat)
 (Strat_R : player → Type)
 (incl : ∀a : player, Strat_R a → Strat a),
 StrictOrder (prefs g player1) →
 StrictOrder (prefs g player2) →
 determined_form_via incl (form g) →
 ex_NE_via incl g.
```

The formalized theorem in a nutshell

```
Theorem finite_equilibrium_transfer :
 ∀ (Strat : player → Type) (_ : strategy player Strat)
 (Outc : finType) (g : game_2 Outc Strat)
 (Strat_R : player → Type)
 (incl : ∀a : player, Strat_R a → Strat a),
 StrictOrder (prefs g player1) →
 StrictOrder (prefs g player2) →
 determined_form_via incl (form g) →
 ex_NE_via incl g.
```

- Focus on a finite set of outcomes
- Proved for arbitrary strategy spaces
- Axiom-free proof in Coq

Summary

- A dual formalization of a game-theoretic theorem in Coq and Isabelle.
- Involves key concepts such as game forms and determinacy.
- Mutual insemination between theory (paper-and-pencil proofs) & practice (formal proof) & between the 2 proof assistants.
- +lsar classical logic eases the proofs
 - more readable scripts thanks to structured, declarative proofs
- +Coq dependent types helpful to set general definitions even if EM is not available, we can work in decidable fragments or make decidability hypotheses explicit

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Perspectives

- ▶ feed our theorem (which transforms determinacy into ∃ of NE) with the positional determinacy of parity games ~→ Isabelle
- ▶ prove the full result by Le Roux (requires transfinite induction) → easier in Coq than in Isabelle
- aim: provide a wider game theory formal library