250 PROBLEMS IN ELEMENTARY NUMBER THEORY

WACLAW SIERPIŃSKI

MODERN ANALYTIC AND COMPUTATIONAL METHODS IN SCIENCE AND MATHEMATICS

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250 Problems in Elementary Number Theory

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"250 Problems in Elementary Number Theory" presents problems and their solutions in five specific areas of this branch of mathematics: divisibility of numbers, relatively prime numbers, arithmetic progressions, prime and composite numbers, and Diophantic equations. There is, in addition, a section of miscellaneous problems.

Included are problems on several levels of difficulty—some are relatively easy, others rather complex, and a number so abstruse that they originally were the subject of scientific research and their solutions are of comparatively recent date. All of the solutions are given thoroughly and in detail; they contain information on possible generalizations of the given problem and further indicate unsolved problems associated with the given problem and solution.

This ancillary textbook is intended for everyone interested in number theory. It will be of especial value to instructors and students both as a textbook and a source of reference in mathematics study groups.

Modern Analytic and Computational Methods in Science and Mathematics

RICHARD BELLMAN, editor

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CONTENTS

PROBLEMS

I. DIVISIBILITY OF NUMBERS

1. Find all positive integers n such that n^2+1 is divisible by $n+1$.

2. Find all integers $x \neq 3$ such that $x-3|x^3-3$.

3. Prove that there exists infinitely many positive integers n such that $4n^2+1$ is divisible both by 5 and 13.

4. Prove that for positive integer n we have $169|3^{3n+3}-26n-27$.

5. Prove that $19|2^{2^{6k+2}}+3$ for $k = 0, 1, 2, \ldots$.

6. Prove the theorem, due to Kraïtchik, asserting that $13|2^{70}+3^{70}$.

7. Prove that $11 \cdot 31 \cdot 61|20^{15}-1$.

8. Prove that for positive integer m and $a > 1$ we have

$$\left(\frac{a^m-1}{a-1}, a-1\right) = (a-1, m).$$

9. Prove that for every positive integer n the number $3(1^5+2^5+ \ldots +n^5)$ is divisible by $1^3+2^3+ \ldots +n^3$.

10. Find all integers $n > 1$ such that $1^n+2^n+ \ldots +(n-1)^n$ is divisible by n.

11. For positive integer n, find which of the two numbers $a_n = 2^{2n+1}-2^{n+1}+1$ and $b_n = 2^{2n+1}+2^{n+1}+1$ is divisible by 5 and which is not.

12. Prove that for every positive integer n there exists a positive integer x such that each of the terms of the infinite sequence $x+1, x^x+1, x^{x^x}+1, \ldots$ is divisible by n.

13. Prove that there exists infinitely many positive integers n such that

for every even x none of the terms of the sequence x^x+1, $x^{x^x}+1$, $x^{x^{x^x}}+1, \ldots$ is divisible by n.

14. Prove that for positive integer n we have $n^2|(n+1)^n-1$.

15. Prove that for positive integer n we have $(2^n-1)^2|2^{(2^n-1)n}-1$.

16. Prove that there exist infinitely many positive integers n such that $n|2^n+1$; find all such prime numbers.

17*. Prove that for every positive integer $a > 1$ there exist infinitely many positive integers n such that $n|a^n+1$.

18*. Prove that there exist infinitely many positive integers n such that $n|2^n+2$.

19. Find all positive integers a for which $a^{10}+1$ is divisible by 10.

20*. Prove that there are no integers $n > 1$ for which $n|2^n-1$.

20a. Prove that there exist infinitely many positive integers n such that $n|2^n+1$.

21. Find all odd n such that $n|3^n+1$.

22. Find all positive integers n for which $3|n2^n+1$.

23. Prove that for every odd prime p there exist infinitely many positive integers n such that $p|n2^n+1$.

24. Prove that for every positive integer n there exist positive integers $x > n$ and y such that $x^x|y^y$ but $x \neq y$.

25*. Prove that for odd n we have $n|2^{n!}-1$.

26. Prove that the infinite sequence 2^n-3 $(n = 2, 3, 4, \ldots)$ contains infinitely many terms divisible by 5 and infinitely many terms divisible by 13, but contains no term divisible by $5 \cdot 13$.

27*. Find two least composite numbers n such that $n|2^n-2$ and $n|3^n-3$.

28*. Find the least positive integer n such that $n|2^n-2$ but $n \nmid 3^n-3$.

29. Find the least integer n such that $n \nmid 2^n-2$ but $n|3^n-3$.

30. For every positive integer a, find a composite number n such that $n|a^n-a$.

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* An asterisk attached to the number of a problem indicates that it is more difficult.

31. Prove that if for some integers a, b, c we have $9|a^3+b^3+c^3$, then at least one of the numbers a, b, c is divisible by 3.

32. Prove that if for positive integers a_k $(k = 1, 2, 3, 4, 5)$ we have $9|a_1^3+a_2^3+a_3^3+a_4^3+a_5^3$, then $3|a_1a_2a_3a_4a_5$.

33. Prove that if x, y, z are positive integers such that $(x, y) = 1$ and $x^2+y^2 = z^4$, then $7|xy$; show that the condition $(x, y) = 1$ is necessary.

34. Prove that if for integers a and b we have $7|a^2+b^2$, then $7|a$ and $7|b$.

35*. Prove that there exist infinitely many pairs of positive integers x, y such that

$$x(x+1)|y(y+1), \quad x\nmid y, \quad x+1\nmid y, \quad x\nmid y+1, \quad x+1\nmid y+1,$$

and find the least such pair.

36. For every positive integer $s \leqslant 25$ and for $s = 100$, find the least positive integer n_s with the sum of digits (in decimal system) equal to s, which is divisible by s.

37*. Prove that for every positive integer s there exists a positive integer n with the sum of digits (in decimal system) equal to s which is divisible by s.

38*. Prove that:

(a) every positive integer has at least as many divisors of the form $4k+1$ as divisors of the form $4k+3$;

(b) there exist infinitely many positive integers which have as many divisors of the form $4k+1$ as divisors of the form $4k+3$;

(c) there exist infinitely many positive integers which have more divisors of the form $4k+1$ than divisors of the form $4k+3$.

39. Prove that if a, b, c are any integers, and n is an integer > 3, then there exists an integer k such that none of the numbers $k+a$, $k+b$, $k+c$ is divisible by n.

40. Prove that for $F_n = 2^{2^n}+1$ we have $F_n|2^{F_n}-2$ $(n = 1, 2, \ldots)$.

II. RELATIVELY PRIME NUMBERS

41. Prove that for every integer k the numbers $2k+1$ and $9k+4$ are relatively prime, and for numbers $2k-1$ and $9k+4$ find their greatest common divisor as a function of k.

42. Prove that there exists an increasing infinite sequence of triangular numbers (i.e. numbers of the form $t_n = \frac{1}{2}n(n+1)$, $n = 1, 2, \ldots$) such that every two of them are relatively prime.

43. Prove that there exists an increasing infinite sequence of tetrahedral numbers (i.e. numbers of the form $T_n = \frac{1}{6}n(n+1)(n+2)$, $n = 1, 2, \ldots$), such that every two of them are relatively prime.

44. Prove that if a and b are different integers, then there exist infinitely many positive integers n such that $a+n$ and $b+n$ are relatively prime.

45*. Prove that if a, b, c are three different integers, then there exist infinitely many positive integers n such that $a+n$, $b+n$, $c+n$ are pairwise relatively prime.

46. Give an example of four different positive integers a, b, c, d such that there exists no positive integer n for which $a+n$, $b+n$, $c+n$, and $d+n$ are pairwise relatively prime.

47. Prove that every integer > 6 can be represented as a sum of two integers > 1 which are relatively prime.

48*. Prove that every integer > 17 can be represented as a sum of three integers > 1 which are pairwise relatively prime, and show that 17 does not have this property.

49*. Prove that for every positive integer m every even number $2k$ can be represented as a difference of two positive integers relatively prime to m.

50*. Prove that Fibonacci's sequence (defined by conditions $u_1 = u_2 = 1$, $u_{n+2} = u_n + u_{n+1}$, $n = 1, 2, \ldots$) contains an infinite increasing sequence such that every two terms of this sequence are relatively prime.

51*. Prove that $(n, 2^{2^n}+1) = 1$ for $n = 1, 2, \ldots$.

51a. Prove that there exist infinitely many positive integers n such that $(n, 2^n-1) > 1$, and find the least of them.

III. ARITHMETIC PROGRESSIONS

52. Prove that there exist arbitrarily long arithmetic progressions formed of different positive integers such that every two terms of these progressions are relatively prime.

53. Prove that for every positive integer k the set of all positive integers n whose number of positive integer divisors is divisible by k contains an infinite arithmetic progression.

54. Prove that there exist infinitely many triplets of positive integers x, y, z for which the numbers $x(x+1)$, $y(y+1)$, $z(z+1)$ form an increasing arithmetic progression.

55. Find all rectangular triangles with integer sides forming an arithmetic progression.

56. Find an increasing arithmetic progression with the least possible difference, formed of positive integers and containing no triangular number.

57. Give a necessary and sufficient condition for an arithmetic progression $ak+b$ $(k = 0, 1, 2, \ldots)$ with positive integer a and b to contain infinitely many squares of integers.

58*. Prove that there exist arbitrarily long arithmetic progressions formed of different positive integers, whose terms are powers of positive integers with integer exponents > 1.

59. Prove that there is no infinite arithmetic progression formed of different positive integers such that each term is a power of a positive integer with an integer exponent > 1.

60. Prove that there are no four consecutive positive integers such that each of them is a power of a positive integer with an integer exponent > 1.

61. Prove by elementary means that each increasing arithmetic progression of positive integers contains an arbitrarily long sequence of consecutive terms which are composite numbers.

62*. Prove by elementary means that if a and b are relatively prime positive integers, then for every positive integer m the arithmetic progression $ak+b$ $(k = 0, 1, 2, \ldots)$ contains infinitely many terms relatively prime to m.

63. Prove that for every positive integer s every increasing arithmetic progression of positive integers contains terms with arbitrary first s digits (in decimal system).

64. Find all increasing arithmetic progressions formed of three terms of the Fibonacci sequence (see Problem 50), and prove that there are no increasing arithmetic progressions formed of four terms of this sequence.

65*. Find an increasing arithmetic progression with the least difference formed of integers and containing no term of the Fibonacci sequence.

66*. Find a progression $ak+b$ $(k = 0, 1, 2, \dots)$, with positive integers a and b such that $(a, b) = 1$, which does not contain any term of Fibonacci sequence.

67. Prove that the arithmetic progression $ak+b$ $(k = 0, 1, 2, \dots)$ with positive integers a and b such that $(a, b) = 1$ contains infinitely many terms pairwise relatively prime.

68*. Prove that in each arithmetic progression $ak+b$ $(k = 0, 1, 2, \dots)$ with positive integers a and b there exist infinitely many terms with the same prime divisors.

69. From the theorem of Lejeune–Dirichlet, asserting that each arithmetic progression $ak+b$ $(k = 0, 1, 2, \dots)$ with relatively prime positive integers a and b contains infinitely many primes, deduce that for every such progression and every positive integer s there exist infinitely many terms which are products of s distinct primes.

70. Find all arithmetic progressions with difference 10 formed of more than two primes.

71. Find all arithmetic progressions with difference 100 formed of more than two primes.

72*. Find an increasing arithmetic progression with ten terms, formed of primes, with the least possible last term.

73. Give an example of an infinite increasing arithmetic progression formed of positive integers such that no term of this progression can be represented as a sum or a difference of two primes.

IV. PRIME AND COMPOSITE NUMBERS

74. Prove that for every even $n > 6$ there exist primes p and q such that $(n-p, n-q) = 1$.

75. Find all primes which can be represented both as sums and as differences of two primes.

76. Find three least positive integers n such that there are no primes between n and $n+10$, and three least positive integers m such that there are no primes between $10m$ and $10(m+1)$.

77. Prove that every prime of the form $4k+1$ is a hypotenuse of a rectangular triangle with integer sides.

78. Find four solutions of the equation $p^2+1 = q^2+r^2$ with primes p, q, and r.

79. Prove that the equation $p^2+q^2 = r^2+s^2+t^2$ has no solution with primes p, q, r, s, t.

80*. Find all prime solutions p, q, r of the equation $p(p+1)+q(q+1) = r(r+1)$.

81*. Find all primes p, q, and r such that the numbers $p(p+1)$, $q(q+1)$, $r(r+1)$ form an increasing arithmetic progression.

82. Find all positive integers n such that each of the numbers $n+1$, $n+3$, $n+7$, $n+9$, $n+13$, and $n+15$ is a prime.

83. Find five primes which are sums of two fourth powers of integers.

84. Prove that there exist infinitely many pairs of consecutive primes which are not twin primes.

85. Using the theorem of Lejeune–Dirichlet on arithmetic progressions, prove that there exist infinitely many primes which do not belong to any pair of twin primes.

86. Find five least positive integers for which n^2-1 is a product of three different primes.

87. Find five least positive integers n for which n^2+1 is a product of three different primes, and find a positive integer n for which n^2+1 is a product of three different odd primes.

88*. Prove that among each three consecutive integers > 7 at least one has at least two different prime divisors.

89. Find five least positive integers n such that each of the numbers n, $n+1$, $n+2$ is a product of two different primes. Prove that there are no four consecutive positive integers with this property. Show by an example that there exist four positive integers such that each of them has exactly two different prime divisors.

90. Prove that the theorem asserting that there exist only finitely many positive integers n such that both n and $n+1$ have only one prime divisor is equivalent to the theorem asserting that there exist only finitely many prime Mersenne numbers and finitely many prime Fermat numbers.

91. Find all numbers of the form 2^n-1 with positive integer n, not exceeding million, which are products of two primes, and prove that if n is even and > 4, then 2^n-1 is a product of at least three integers > 1.

92. Using Problem 47, prove that if p_k denotes the kth prime, then for $k \geqslant 3$ we have the inequality $p_{k+1}+p_{k+2} \leqslant p_1 p_2 \ldots p_k$.

93. For positive integer n, let q_n denote the least prime which is not a divisor of n. Using Problem 92, prove that the ratio q_n/n tends to zero as n increases to infinity.

94. Prove by elementary means that Chebyshev's theorem (asserting that for integer $n > 1$ there exists at least one prime between n and $2n$) implies that for every integer $n > 4$ between n and $2n$ there exists at least one number which is a product of two different primes, and that for integer > 15 between n and $2n$ there exists at least one number which is a product of three different primes.

95. Prove by elementary means that the Chebyshev theorem implies that for every positive integer s, for all sufficiently large n, between n and $2n$ there exists at least one number which is a product of s different primes.

96. Prove that the infinite sequence $1, 31, 331, 3331, \ldots$ contains infinitely many composite numbers, and find the least of them (to solve the second part of the problem, one can use the microfilm containing all primes up to one hundred millions [2]).

97. Find the least positive integer n for which $n^4+(n+1)^4$ is composite.

98. Show that there are infinitely many composite numbers of the form 10^n+3 ($n = 1, 2, 3, \ldots$).

99. Show that for integers $n > 1$ the number $\frac{1}{5}(2^{4n+2}+1)$ is composite.

100. Prove that the infinite sequence 2^n-1 ($n = 1, 2, \ldots$) contains arbitrarily long subsequences of consecutive terms consisting of composite numbers.

101. Show that the assertion that by changing only one decimal digit one can obtain a prime out of every positive integer is false.

102. Prove that the Chebyshev theorem T stating that for every integer $n > 1$ there is at least one prime between n and $2n$ is equivalent to the theorem T_1 asserting that for integers $n > 1$ the expansion of $n!$ into prime factors

contains at least one prime with exponent 1. The equivalence of T and T_1 means that each of these theorems implies the other.

103. Using the theorem asserting that for integers $n > 5$ between n and $2n$ there are at least two different primes (an elementary proof of this theorem can be found in W. Sierpiński [37, p. 137, Theorem 7]), prove that if n is an integer > 10, then in the expansion of $n!$ into prime factors there are at least two different primes appearing with exponent 1.

104. Using the theorem of Lejeune–Dirichlet on arithmetic progression, prove that for every positive integer n there exists a prime p such that each of the numbers $p-1$ and $p+1$ has at least n different positive integer divisors.

105. Find the least prime p for which each of the numbers $p-1$ and $p+1$ has at least three different prime divisors.

106*. Using the Lejeune-Dirichlet theorem on arithmetic progression, prove that for every positive integer n there exist infinitely many primes p such that each of the numbers $p-1$, $p+1$, $p+2$ has at least n different prime divisors.

107. Prove that for all positive integers n and s there exist arbitrarily long sequences of consecutive positive integers such that each of them has at least n different prime divisors, each of these divisors appearing in at least sth power.

108. Prove that for an odd $n > 1$ the numbers n and $n+2$ are primes if and only if $(n-1)!$ is not divisible by n and not divisible by $n+2$.

109. Using the theorem of Lejeune–Dirichlet on arithmetic progression, prove that for every positive integer m there exists a prime whose sum of decimal digits is $> m$.

110. Using the theorem of Lejeune–Dirichlet on arithmetic progression, prove that for every positive integer m there exist primes with at least m digits equal to zero.

111. Find all primes p such that the sum of all positive integer divisors of p^4 is equal to a square of an integer.

112. For every s, with $2 \leqslant s \leqslant 10$, find all primes for which the sum of all positive integer divisors is equal to the sth power of an integer.

113. Prove the theorem of Liouville, stating that the equation $(p-1)!+ +1 = p^m$ has no solution with prime $p > 5$ and positive integer m.

114. Prove that there exist infinitely many primes q such that for some positive integer $n < q$ we have $q|(n-1)!+1$.

115*. Prove that for every integer $k \neq 1$ there exist infinitely many positive integers n such that the number $2^{2^n}+k$ is composite.

116. Prove that there exist infinitely many odd numbers $k > 0$ such that all numbers $2^{2^n}+k$ $(n = 1, 2, \ldots)$ are composite.

117. Prove that all numbers $2^{2^{2n+1}}+3$, $2^{2^{4n+1}}+7$, $2^{2^{6n+2}}+13$, $2^{2^{10n+1}}+$ $+19$, and $2^{2^{6n+2}}+21$ are composite for $n = 1, 2, \ldots$.

118*. Prove that there exist infinitely many positive integers k such that all numbers $k \cdot 2^n+1$ $(n = 1, 2, \ldots)$ are composite.

119*. Using the solution of Problem 118*, prove the theorem, due to P. Erdös, that there exist infinitely many odd k such that every number 2^n+k is composite $(n = 1, 2, \ldots)$.

120. Prove that if k is a power of 2 with positive integer exponent, then for sufficiently large n all numbers $k \cdot 2^{2^n}+1$ are composite.

121. For every positive integer $k \leqslant 10$, find the least positive integer n for which $k \cdot 2^{2^n}+1$ is composite.

122. Find all positive integers $k \leqslant 10$ such that every number $k \cdot 2^{2^n}+1$ $(n = 1, 2, \ldots)$ is composite.

123. Prove that for integer $n > 1$ the numbers $\frac{1}{3}(2^{2^{n+1}}+2^{2^n}+1)$ are all composite.

124. Prove that there exist infinitely many composite numbers of the form $(2^{2n}+1)^2+2^2$.

125*. Prove that for every integer a with $1 < a \leqslant 100$ there exists at least one positive integer $n \leqslant 6$ such that $a^{2^n}+1$ is composite.

126. Prove by elementary means that there exist infinitely many odd numbers which are sums of three different primes, but are not sums of less than three primes.

127. Prove that there is no polynomial $f(x)$ with integer coefficients such that $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, and show that for every integer $m > 1$ there exists a polynomial $f(x)$ with rational coefficients such that $f(k) = p_k$ for $k = 1, 2, \ldots, m$, where p_k denotes the kth prime.

128*. From a particular case of the Lejeune–Dirichlet theorem, stating that the arithmetic progression $mk+1$ ($k = 1, 2, \ldots$) contains, for each positive integer m, infinitely many primes, deduce that for every positive integer n there exists a polynomial $f(x)$ with integer coefficients such that $f(1) < f(2) < \ldots < f(n)$ are primes.

129. Give an example of a reducible polynomial $f(x)$ (with integer coefficients) which for m different positive integer values of x would give m different primes.

130. Prove that if $f(x)$ is a polynomial of degree > 0 with integer coefficients, then the congruence $f(x) \equiv 0 \pmod{p}$ is solvable for infinitely many primes p.

131. Find all integers $k \geqslant 0$ for which the sequence $k+1, k+2, \ldots, k+10$ contains maximal number of primes.

132. Find all integers $k \geqslant 0$ for which the sequences $k+1, k+2, \ldots, k+100$ contains maximal number of primes.

133. Find all sequences of hundred consecutive positive integers which contain 25 primes.

134. Find all sequences of 21 consecutive positive integers containing 8 primes.

135. Find all numbers p such that all six numbers p, $p+2$, $p+6$, $p+8$, $p+12$, and $p+14$ are primes.

136. Prove that there exist infinitely many pairs of different positive integers m and n such that (1) m and n have the same prime divisors, and (2) $m+1$ and $n+1$ have the same prime divisors.

V. DIOPHANTINE EQUATIONS

137. Prove by elementary means that the equation $3x^2+7y^2+1 = 0$ has infinitely many solutions in positive integers x, y.

138. Find all integer solutions x, y of the equation $2x^3+xy-7 = 0$ and prove that this equation has infinitely many solutions in positive rationals x, y.

139. Prove by elementary means that the equation $(x-1)^2+(x+1)^2 = y^2+1$ has infinitely many solutions in positive integers x, y.

140. Prove that the equation $x(x+1) = 4y(y+1)$ has no solutions in positive integers x, y, but has infinitely many solutions in positive rationals x, y.

141*. Prove that if p is a prime and n is a positive integer, then the equation $x(x+1) = p^{2n}y(y+1)$ has no solutions in positive integers x, y.

142. For a given integer k, having an integer solution x, y of the equation $x^2-2y^2 = k$, find a solution in integers t, u of the equation $t^2-2u^2 = -k$.

143. Prove that the equation $x^2-Dy^2 = z^2$ has, for every integer D, infinitely many solutions in positive integers x, y, z.

144. Prove by elementary means that if D is any integer $\neq 0$, then the equation $x^2-Dy^2 = z^2$ has infinitely many solutions in positive integers x, y, z such that $(x, y) = 1$.

145. Prove that the equation $xy+x+y = 2^{32}$ has solutions in positive integers x, y, and there exists only one solution with $x \leqslant y$.

146. Prove that the equation $x^2-2y^2+8z = 3$ has no solutions in positive integers x, y, z.

147. Find all positive integer solutions x, y of the equation

$$y^2-x(x+1)(x+2)(x+3) = 1.$$

148. Find all rational solutions of the equation

$$x^2+y^2+z^2+x+y+z = 1.$$

149. Prove the theorem of Euler that the equation $4xy-x-y = z^2$ has no solutions in positive integers x, y, z, and prove that this equation has infinitely many solutions in negative integers x, y, z.

150. Prove by elementary means (without using the theory of Pell's equation) that if $D = m^2+1$, where m is a positive integer, then the equation $x^2+Dy^2 = 1$ has infinitely many solutions in positive integers x, y.

151*. Find all integer solutions x, y of the equation $y^2 = x^3+(x+4)^2$.

152. For every natural number m, find all solutions of the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = m$$

in relatively prime positive integers x, y, z.

153. Prove that the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 1$$

has no solutions in positive integers x, y, z.

154*. Prove that the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 2$$

has no solutions in positive integers x, y, z.

155. Find all solutions in positive integers x, y, z of the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 3.$$

156*. Prove that for $m = 1$ and $m = 2$, the equation $x^3+y^3+z^3 = mxyz$ has no solutions in positive integers x, y, z, and find all solutions in positive integers x, y, z of this equation for $m = 3$.

157. Prove that theorem T_1 asserting that there are no positive integers x, y, z for which $x/y+y/z = z/x$ is equivalent to theorem T_2 asserting that there are no solutions in positive integers u, v, w of the equation $u^3+v^3 = w^3$ (in the sense that T_1 and T_2 imply easily each other).

158*. Prove that there are no positive integer solutions x, y, z, t of the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x} = 1,$$

but there are infinitely many solutions of this equation in integers x, y, z, t (not necessarily positive).

159*. Prove that the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x} = m$$

has no solutions in positive integers x, y, z, t for $m = 2$ and $m = 3$, and find all its solutions in positive integers x, y, z, t for $m = 4$.

160. Find all solutions in positive integers x, y, z, t, with $x \leqslant y \leqslant z \leqslant t$, of the equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=1.$$

161. Prove that for every positive integer s the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s}=1$$

has a finite positive number of solutions in positive integers $x_1, x_2, \ldots, x_s$.

162*. Prove that for every integer $s > 2$ the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s}=1$$

has a solution $x_1, x_2, \ldots, x_s$ in increasing positive integers. Show that if l_s denotes the number of such solutions, then $l_{s+1} > l_s$ for $s = 3, 4, \ldots$.

163. Prove that if s is a positive integer $\neq 2$, then the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s}=1$$

has a solution in triangular numbers (i.e. numbers of the form $t_n = \frac{1}{2}n(n+1)$).

164. Find all solutions in positive integers x, y, z, t of the equation

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}=1.$$

165. Find all positive integers s for which the equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_s^2}=1$$

has at least one solution $x_1, x_2, \ldots, x_s$ in positive integers.

166. Represent the number $\frac{1}{2}$ as a sum of reciprocals of a finite number of squares of an increasing sequence of positive integers.

167*. Prove that for every positive integer m, for all sufficiently large s, the equation

$$\frac{1}{x_1^m}+\frac{1}{x_2^m}+\ldots+\frac{1}{x_s^m}=1$$

has at least one solution in positive integers $x_1, x_2, \ldots, x_s$.

168. Prove that for every positive integer s the equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_s^2}=\frac{1}{x_{s+1}^2}$$

has infinitely many solutions in positive integers $x_1, x_2, \ldots, x_s, x_{s+1}$.

169. Prove that for every integer $s \geqslant 3$ the equation

$$\frac{1}{x_1^3}+\frac{1}{x_2^3}+\ldots+\frac{1}{x_s^3}=\frac{1}{x_{s+1}^3}$$

has infinitely many solutions in positive integers $x_1, x_2, \ldots, x_s, x_{s+1}$.

170*. Find all integer solutions of the system of equations

$$x+y+z=3 \quad \text{and} \quad x^3+y^3+z^3=3.$$

171. Investigate, by elementary means, for which positive integers n the equation $3x+5y=n$ has at least one solution x, y in positive integers, and prove that the number of such solutions increases to infinity with n.

172. Find all solutions in positive integers n, x, y, z of the equation $n^x+n^y=n^z$.

173. Prove that for every system of positive integers m, n there exists a linear equation $ax+by=c$, where a, b, c are integers, such that the only solution in positive integers of this equation is $x=n, y=m$.

174. Prove that for every positive integer m there exists a linear equation $ax+by=c$ (with integer a, b, and c) which has exactly m solutions in positive integers x, y.

175. Prove that the equation $x^2+y^2+2xy-mx-my-m-1=0$, where m is a given positive integer, has exactly m solutions in positive integers x, y.

176. Find all solutions of the equation

$$x^3+(x+1)^3+(x+2)^3 = (x+3)^3$$

in integers x.

177. Prove that for every positive integer n the equation

$$(x+1)^3+(x+2)^3+ \dots +(x+n)^3 = y^3$$

has a solution in integers x, y.

178. Find all solutions of the equation

$$(x+1)^3+(x+2)^3+(x+3)^3+(x+4)^3 = (x+5)^3$$

in integers x.

179. Find all rational solutions x of the equation

$$(x+1)^3+(x+2)^3+(x+3)^3+(x+4)^3 = (x+10)^3.$$

180. Find two positive integer solutions x, y of the equation

$$y(y+1) = x(x+1)(x+2).$$

181. Prove that the equation $1+x^2+y^2 = z^2$ has infinitely many solutions in positive integers x, y, z.

182. Find all solutions in positive integers n, x, y, z, t of the equation $n^x+n^y+n^z = n^t$.

183. Find all solutions in positive integers x, y, z, t of the equation $4^x+4^y+4^z = 4^t$.

184. Find all solutions in positive integers m, n of the equation $2^m-3^n = 1$.

185. Find all solutions in positive integers m, n of the equation $3^n-2^m = 1$.

186. Find all solutions in positive integers x, y of the equation $2^x+1 = y^2$.

187. Find all solutions in positive integers x, y of the equation $2^x-1 = y^2$.

188. Prove that the system of equations $x^2+2y^2 = z^2$, $2x^2+y^2 = t^2$ has no solutions in positive integers x, y, z, t.

189. Using the identity

$$\big(2(3x+2y+1)+1\big)^2-2(4x+3y+2)^2 = (2x+1)^2-2y^2,$$

prove that the equation $x^2+(x+1)^2 = y^2$ has infinitely many solutions in positive integers x, y.

190. Using the identity

$$\big(2(7y+12x+6)\big)^2-3\big(2(4y+7x+3)+1\big)^2 = (2y)^2-3(2x+1)^2,$$

prove by elementary means that the equation $(x+1)^3-x^3 = y^2$ has infinitely many solutions x, y in positive integers.

191. Prove that the system of the equations $x^2+5y^2 = z^2$ and $5x^2+y^2 = t^2$ has no solutions in positive integers x, y, z, t.

192. Using Problem 34, prove that the system of two equations $x^2+6y^2 = z^2$, $6x^2+y^2 = t^2$ has no solutions in positive integers x, y, z, t.

192a. Prove that the system of two equations $x^2+7y^2 = z^2$, $7x^2+y^2 = t^2$ has no solutions in positive integers x, y, z, t.

193. Prove the theorem of V. A. Lebesgue that the equation $x^2-y^3 = 7$ has no integer solutions x, y.

194. Prove that if a positive integer c is odd, then the equation $x^2-y^3 = (2c)^3-1$ has no integer solutions x, y.

195. Prove that for positive integers k the equation $x^2+2^{2k}+1 = y^3$ has no solutions in positive integers x, y.

196. Solve the problem of A. Moessner of finding all solutions in positive integers x, y, z, t of the system of equations

$$x+y = zt, \qquad z+t = xy,$$

where $x \leqslant y$, $x \leqslant z \leqslant t$. Prove that this system has infinitely many integer solutions x, y, z, t.

197. Prove that for positive integers n the equation $x_1+x_2+ \dots +x_n = x_1x_2 \dots x_n$ has at least one solution in positive integers $x_1, x_2, \dots, x_n$.

198. For every given pair of positive integers a and n, find a method of determining all solutions of the equation $x^n-y^n = a$ in positive integers x, y.

199. Prove by elementary means that there exist infinitely many triangular numbers which are at the same time pentagonal (i.e. of the form $\frac{1}{2}k(3k-1)$, where k is a positive integer).

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200. If $f(x)$ is a polynomial with integer coefficients, and the equation $f(x) = 0$ has an integer solution, then obviously the congruence $f(x) \equiv 0 \pmod{p}$ has a solution for every prime modulus p. Using the equation of the first degree $ax+b = 0$, show that the converse is false.

201. Prove that if for integer a and b the congruence $ax+b \equiv 0 \pmod{m}$ has a solution for every positive integer modulus m, then the equation $ax+b = 0$ has an integer solution.

202. Prove that the congruence $6x^2+5x+1 \equiv 0 \pmod{m}$ has a solution for every positive integer modulus m, in spite of the fact that the equation $6x^2+5x+1 = 0$ has no integer solutions.

203. Prove that if k is odd and n is a positive integer, then $2^{n+2} | k^{2^n}-1$.

204. Prove that if an integer k can be represented in the form $k = x^2-2y^2$ for some positive integers x and y, then it can be represented in this form in infinitely many ways.

205. Prove that no number of the form $8k+3$ or $8k+5$, with integer k, can be represented in the form x^2-2y^2 with integers x and y.

206. Prove that there exist infinitely many positive integers of the form $8k+1$ $(k = 0, 1, 2, \ldots)$ which can be represented as x^2-2y^2 with positive integers x and y, and also infinitely many which cannot be so represented. Find the least number of the latter category.

207. Prove that the last decimal digit of every even perfect number is always 6 or 8.

208. Prove the theorem of N. Anning, asserting that if in the numerator and denominator of the fraction $\frac{101010101}{110010011}$, whose digits are written in an arbitrary integer scale $g > 1$, we replace the middle digit 1 by an arbitrary odd number of digits 1, the value of the fraction remains the same (that is, $\frac{101010101}{110010011} = \frac{10101110101}{11001110011} = \frac{1010111110101}{1100111110011} = \ldots$).

209*. Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.

210*. Prove that if k is any integer > 1 and c is an arbitrary digit in decimal system, then there exists a positive integer n such that the kth (counting from the end) digit of the decimal expansion of 2^n is c.

211. Prove that the four last digits of the numbers 5^n $(n = 1, 2, 3, \ldots)$ form a periodic sequence. Find the period, and determine whether it is pure.

212. Prove that for every s, the first s digits of the decimal expansion of positive integer may be arbitrary.

213. Prove that the sequence of last decimal digits of the numbers n^{n^n} $(n = 1, 2, 3, \ldots)$ is periodic; find the period and determine whether it is pure.

214. Prove that in every infinite decimal fraction there exist arbitrarily long sequences of consecutive digits which appear an infinite number of times in the expansion.

215. For every positive integer k, represent the number 3^{2k} as a sum of 3^k terms, which are consecutive positive integers.

216. Prove that for every integer $s > 1$ there exists a positive integer m_s such that for integer $n \geqslant m_s$ between n and $2n$ there is at least one sth power of an integer. Find least numbers m_s for $s = 2$ and $s = 3$.

217. Prove that there exist arbitrarily long sequences of consecutive positive integers, none of which is a power of an integer with an integer exponent > 1.

218. Find the general formula for the nth term of the infinite sequence u_n $(n = 1, 2, \ldots)$ defined by the conditions $u_1 = 1$, $u_2 = 3$, $u_{n+2} = 4u_{n+1} - 3u_n$ for $n = 1, 2, \ldots$.

219. Find the formula for the nth term of the infinite sequence defined by conditions $u_1 = a$, $u_2 = b$, $u_{n+2} = 2u_{n+1} - u_n$ for $n = 1, 2, \ldots$.

220. Find the formula for the nth term of the infinite sequence defined by conditions $u_1 = a$, $u_2 = b$, $u_{n+2} = -(u_n + 2u_{n+1})$ for $n = 1, 2, \ldots$. Investigate the particular cases $a = 1$, $b = -1$ and $a = 1$, $b = -2$.

221. Find the formula for the nth term of the infinite sequence defined by conditions $u_1 = a$, $u_2 = b$, $u_{n+2} = 2u_n + u_{n+1}$.

222. Find all integers $a \neq 0$ with the property $a^{a^n} = a$ for $n = 1, 2, \ldots$

223*. Give the method of finding all pairs of positive integers whose sum and product are both squares. Determine all such numbers $\leqslant 100$.

224. Find all triangular numbers which are sums of squares of two consecutive positive integers.

225*. Prove the theorem of V. E. Hogatt that every positive integer is a sum of distinct terms of Fibonacci sequence.

226. Prove that the terms u_n of Fibonacci sequence satisfy the relation

$$u_n^2 = u_{n-1}u_{n+1}+(-1)^{n-1} \quad \text{for} \quad n = 2, 3, \ldots .$$

227. Prove that every integer can be represented as a sum of five cubes of integers in infinitely many ways.

228. Prove that the number 3 can be represented as a sum of four cubes of integers different from 0 and 1 in infinitely many ways.

229. Prove by elementary means that there exist infinitely many positive integers which can be represented as sums of four squares of different integers in at least two ways, and that there exist infinitely many positive integers which can be represented in at least two ways as sums of four cubes of different positive integers.

230. Prove that for positive integers m, in each representation of the number $4^m \cdot 7$ as a sum of four squares of integers $\geqslant 0$, each of these numbers is $\geqslant 2^{m-1}$.

231. Find the least integer > 2 which is a sum of two squares of positive integers and a sum of two cubes of positive integers, and prove that there exist infinitely many positive integers which are sums of two squares and sums of two cubes of relatively prime positive integers.

232. Prove that for every positive integer s there exists an integer $n > 2$ such that for $k = 1, 2, \ldots, s$, n is a sum of two kth powers of positive integers.

233*. Prove that there exist infinitely many positive integers which cannot be represented as sums of two cubes of integers, but can be represented as sums of two cubes of positive rational numbers.

234*. Prove that there exist infinitely many positive integers which can be represented as differences of two cubes of positive integers, but cannot be represented as sums of such two cubes.

235*. Prove that for every integer $k > 1$, $k \neq 3$, there exist infinitely many positive integers which can be represented as differences of two kth powers of positive integers, but cannot be represented as sums of two kth powers of positive integers.

236*. Prove that for every integer $n > 1$ there exist infinitely many positive integers which can be represented as sums of two nth powers of positive integers, but cannot be represented as differences of two such nth powers.

237. Find the least integer $n > 1$ for which the sum of squares of consecutive numbers from 1 to n would be a square of an integer.

238. Let us call a number of the form a^b a *proper power* if a and b are integers > 1. Find all positive integers which are sums of a finite $\geqslant 1$ number of proper powers.

238a. Prove that every positive integer $n \leqslant 10$ different from 6 is a difference of two proper powers.

239. Prove that for every rectangular triangle with integer sides and for every positive integer n there exists a similar triangle such that each of its sides is a power of a positive integer with integer exponent $\geqslant n$.

240. Find all positive integers $n > 1$ for which $(n-1)!+1 = n^2$.

241. Prove that the product of two consecutive triangular numbers is never a square of an integer, but for every triangular number $t_n = \frac{1}{2}n(n+1)$ there exist infinitely many triangular numbers t_m, larger than it, such that $t_n t_m$ is a square.

242. Prove (without using the tables of logarithms) that the number $F_{1945} = 2^{2^{1945}}+1$ has more than 10^{582} digits, and find the number of digits of $5 \cdot 2^{1947}+1$ (which is, as it is well known, the least prime divisor of F_{1945}).

243. Find the number of decimal digits of the number $2^{11213}-1$ (this is the largest prime number known up to date).

244. Find the number of decimal digits of the number $2^{11212}(2^{11213}-1)$ (this is the largest known perfect number).

245. Prove that the number 3!!! written in decimal system has more than thousand digits, and find the number of zeros at the end of the expansion.

246*. Find integer $m > 1$ with the following property: there exists a polynomial $f(x)$ with integer coefficients such that for some integer x the value $f(x)$ gives remainder 0 upon dividing by m, for some integer x the value $f(x)$

gives remainder 1 upon dividing by m, and for all integer x, the value $f(x)$ gives remainder 0 or 1 upon dividing by m.

247. Find the expansion into arithmetic continued fraction of the number $\sqrt{D}$ where $D = ((4m^2+1)n+m)^2+4mn+1$, where m and n are positive integers.

248. Find all positive integers $\leqslant 30$ such that $\varphi(n) = d(n)$, where $\varphi(n)$ is the well-known Euler function, and $d(n)$ denotes the number of positive integer divisors of n.

249. Prove that for every positive integer g, each rational number $w > 1$ can be represented in the form

$$w = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)\cdots\left(1+\frac{1}{k+s}\right),$$

where k is an integer $> g$, and s is an integer $\geqslant 0$.

250*. Prove the theorem of P. Erdös and M. Surányi that every integer k can be represented in infinitely many ways in the form $k = \pm 1^2 \pm 2^2 \pm \pm \ldots \pm m^2$ for some positive integer m and some choice of signs $+$ or $-$.

SOLUTIONS

I. DIVISIBILITY OF NUMBERS

1. There is only one such positive integer: $n = 1$. In fact, $n^2+1 = n(n+1)-(n-1)$; thus, if $n+1|n^2+1$, then $n+1|n-1$ which for positive integer n is possible only if $n-1 = 0$, hence if $n = 1$.

2. Let $x-3 = t$. Thus, t is an integer $\neq 0$ such that $t|(t+3)^2-3$, which is equivalent to the condition $t|3^3-3$, or $t|24$. Therefore, it is necessary and sufficient for t to be an integer divisor of 24, hence t must be equal to one of the numbers ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24. For $x = t+3$ we obtain the values -21, -9, -5, -3, -1, 0, 1, 2, 4, 5, 6, 7, 9, 11, 15, and 27.

3. For instance, all numbers n in the arithmetic progression $65k+56$ $(k = 0, 1, 2, \ldots)$ have the desired property. Indeed, if $n = 65k+56$ with an integer $k \geqslant 0$, then $n \equiv 1 \pmod 5$ and $n \equiv 4 \pmod{13}$, hence $4n^2+1 \equiv 0 \pmod 5$ and $4n^2+1 \equiv 0 \pmod{13}$. Thus, $5|4n^2+1$ and $13|4n^2+1$.

4. We shall prove the assertion by induction. We have $169|3^6-26-27 = 676 = 4 \cdot 169$. Next, we have $3^{3(n+1)+3}-26(n+1)-27-(3^{3n+3}-26n-27) = 26(3^{3n+3}-1)$. However, $13|3^3-1$, hence $13|3^{3(n+1)}-1$, and $169|26(3^{3n+3}-1)$. The proof by induction follows immediately.

5. We have $2^6 = 64 \equiv 1 \pmod 9$, hence for $k = 0, 1, 2, \ldots$ we have also $2^{6k} \equiv 1 \pmod 9$. Therefore $2^{6k+2} \equiv 2^2 \pmod 9$, and since both sides are even, we get $2^{6k+2} \equiv 2^2 \pmod{18}$. It follows that $2^{6k+2} = 18t+2^2$, where t is an integer $\geqslant 0$. However, by Fermat's theorem, $2^{18} \equiv 1 \pmod{19}$, and therefore $2^{18t} \equiv 1 \pmod{19}$ for $t = 0, 1, 2, \ldots$. Thus $2^{2^{6k+2}} = 2^{18t+4} \equiv 2^4 \pmod{19}$; it follows that $2^{2^{6k+2}}+3 \equiv 2^4+3 \equiv 0 \pmod{19}$, which was to be proved.

6. By Fermat's theorem we have $2^{12} \equiv 1 \pmod{13}$, hence $2^{60} \equiv 1 \pmod{13}$, and since $2^5 \equiv 6 \pmod{13}$, which implies $2^{10} \equiv -3 \pmod{13}$,

we get $2^{70} \equiv -3 \pmod{13}$. On the other hand, $3^3 \equiv 1 \pmod{13}$, hence $3^{69} \equiv 1 \pmod{13}$ and $3^{70} \equiv 3 \pmod{13}$. Therefore $2^{70}+3^{70} \equiv 0 \pmod{13}$, or $13 | 2^{70}+3^{70}$, which was to be proved.

7. Obviously, it suffices to show that each of the primes 11, 31, and 61 divides $20^{15}-1$. We have $2^5 \equiv -1 \pmod{11}$, and $10 \equiv -1 \pmod{11}$, hence $10^5 \equiv -1 \pmod{11}$, which implies $20^5 \equiv 1 \pmod{11}$, and $20^{15} \equiv 1 \pmod{11}$. Thus $11 | 20^{15}-1$. Next, we have $20 \equiv -11 \pmod{31}$, hence $20^2 \equiv 121 \equiv -3 \pmod{31}$. Therefore $20^3 \equiv (-11)(-3) \equiv 33 \equiv 2 \pmod{31}$, which implies $20^{15} \equiv 2^5 \equiv 1 \pmod{31}$. Thus, $31 | 20^{15}-1$. Finally, we have $3^4 \equiv 20 \pmod{61}$, which implies $20^{15} \equiv 3^{60} \equiv 1 \pmod{61}$ (by Fermat's theorem); thus $61 | 20^{15}-1$.

8. Let $d = \left(\frac{a^m-1}{a-1}, a-1\right)$. In view of the identity

$$\frac{a^m-1}{a-1} = (a^{m-1}-1)+(a^{m-2}-1)+\ldots+(a-1)+m \tag{1}$$

and in view of the fact that $a-1 | a^k-1$ for $k = 0, 1, 2, \ldots$, we obtain $d | m$. Thus, if the numbers $a-1$ and m had a common divisor $\delta > d$, we would have, by (1), the relation $\delta \,\Big|\, \frac{a^m-1}{a-1}$ and the numbers $\frac{a^m-1}{a-1}$ and $a-1$ would have a common divisor $\delta > d$, which is impossible. It follows that d is the greatest common divisor of $a-1$ and m, which was to be proved.

9. For positive integer n, we have

$$1^3+2^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$$

(which follows by induction). By induction, we obtain also the identity

$$1^5+2^5+\ldots+n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

for all positive integer n. It follows from these formulas that

$$3(1^5+2^5+\ldots+n^5)/(1^3+2^3+\ldots+n^3) = 2n^2+2n-1,$$

which proves the desired property.

10. These are all odd numbers > 1. In fact, if n is odd and > 1, then the number $(n-1)/2$ is a positive integer, and for $k = 1, 2, \ldots, (n-1)/2$ we easily get

$$n|k^n+(n-k)^n \quad (\text{since } (-k)^n = -k^n);$$

thus $n|1^n+2^n+\ldots+(n-1)^n$.

On the other hand, if n is even, let 2^s be the highest power of 2 which divides n (thus, s is a positive integer). Since $2^s \geqslant s$, for even k we have $2^s|k^n$, and for odd k (the number of such k's in the sequence $1, 2, \ldots, n-1$ is $\frac{1}{2}n$) we have, by Euler's theorem, $k^{2^{s-1}} \equiv 1 (\text{mod } 2^s)$, hence $k^n \equiv 1 (\text{mod } 2^s)$ (since $2^{s-1}|n$). Therefore

$$1^n+3^n+\ldots+(n-3)^n+(n-1)^n \equiv \tfrac{1}{2}n \ (\text{mod } 2^s),$$

which implies

$$1^n+2^n+\ldots+(n-1)^n \equiv \tfrac{1}{2}n \ (\text{mod } 2^s),$$

in view of the fact that $2^n+4^n+\ldots+(n-2)^n \equiv 0 \ (\text{mod } 2^s)$. Now, if we had $n|1^n+2^n+\ldots+(n-1)^n$, then using the relation $2^s|n$ we would have $\frac{1}{2}n \equiv 0 \ (\text{mod } 2^s)$, hence $2^s|\frac{1}{2}n$ and $2^{s+1}|n$, contrary to the definition of s. Thus, for even n we have $n \nmid 1^n+2^n+\ldots+(n-1)^n$.

REMARK. It follows easily from Fermat's theorem that if n is a prime, then $n|1^{n-1}+2^{n-1}+\ldots+(n-1)^{n-1}+1$; we do not know any composite number satisfying this relation. G. Ginga conjectured that there is no such composite number and proved that there is no such composite number $n < 10^{1000}$.

11. Consider four cases:

(a) $n = 4k$, where k is a positive integer. Then

$$a_n = 2^{8k+1}-2^{4k+1}+1 \equiv 2-2+1 \equiv 1 \ (\text{mod } 5),$$

$$b_n = 2^{8k+1}+2^{4k+1}+1 \equiv 2+2+1 \equiv 0 \ (\text{mod } 5)$$

(since $2^4 \equiv 1 \ (\text{mod } 5)$, which implies $2^{4k} \equiv 2^{8k} \equiv 1 \ (\text{mod } 5)$).

(b) $n = 4k+1$, $k = 0, 1, 2, \ldots$. Then

$$a_n = 2^{8k+3}-2^{4k+2}+1 \equiv 8-4+1 \equiv 0 \ (\text{mod } 5),$$

$$b_n = 2^{8k+3}+2^{4k+2}+1 \equiv 8+4+1 \equiv 3 \ (\text{mod } 5).$$

(c) $n = 4k+2$, $k = 0, 1, 2, \ldots$. Then

$$a_n = 2^{8k+5}-2^{4k+3}+1 \equiv 2-8+1 \equiv 0 \pmod 5,$$

$$b_n = 2^{8k+5}+2^{4k+3}+1 \equiv 2+8+1 \equiv 1 \pmod 5.$$

(d) $n = 4k+3$, $k = 0, 1, 2, \ldots$. Then

$$a_n = 2^{8k+7}-2^{4k+4}+1 \equiv 8-1+1 \equiv 3 \pmod 5,$$

$$b_n = 2^{8k+7}+2^{4k+4}+1 \equiv 8+1+1 \equiv 0 \pmod 5.$$

Thus, the numbers a_n are divisible by 5 only for $n \equiv 1$ or $2 \pmod 4$, while the numbers b_n are divisible by 5 only for $n \equiv 0$ or $3 \pmod 4$. Thus one and only one of the numbers a_n and b_n is divisible by 5.

12. It is sufficient to take $x = 2n-1$. Then each of the numbers $x, x^x, x^{x^x}, \ldots$ is odd, and therefore $2n = x+1$ is a divisor of each of the terms of the infinite sequence $x+1$, x^x+1, $x^{x^x}+1, \ldots$.

13. For instance, all primes p of the form $4k+3$. In fact, for even x, each of the terms of the sequence $x, x^x, x^{x^x}, \ldots$ is even. If any of the terms of the sequence x^x+1, $x^{x^x}+1$, $x^{x^{x^x}}+1, \ldots$ were divisible by p, we would have for some positive integer m the relation $p|x^{2m}+1$, hence $(x^m)^2 \equiv -1 \pmod p$. However, -1 cannot be a quadratic residue for a prime modulus of the form $4k+3$.

14. From the binomial expansion

$$(1+n)^n = 1+\binom{n}{1}n+\binom{n}{2}n^2+\ldots+\binom{n}{n}n^n$$

it follows that for $n > 1$ (which can be assumed, in view of $1^2|2^1-1$), all terms starting from the third term contain n in the power with exponent $\geqslant 2$. The second term equals $\binom{n}{1}n = n^2$. Thus, $n^2|(1+n)^n-1$, which was to be proved.

15. By Problem 14, we have for positive integers m the relation $m^2|(m+1)^m-1$. For $m = 2^n-1$, we get, in view of $(m+1)^m = 2^{n(2^n-1)}$, the relation $(2^n-1)^2|2^{(2^n-1)n}-1$, which was to be proved.

16. We have $3|2^3+1$, and if for some positive integer m $3^m|2^{3^m}+1$, then $2^{3^m} = 3^m k-1$, where k is a positive integer. It follows that

$$2^{3^{m+1}} = (3^m k-1)^3 = 3^{3m}k^3-3^{2m+1}k^2+3^{m+1}k-1 = 3^{m+1}t-1,$$

where t is a positive integer. Thus, $3^{m+1}|2^{3^{m+1}}+1$, and by induction we get $3^m|2^{3^m}+1$ for $m = 1, 2, \ldots$. There are, however, other positive integers n satisfying the relation $n|2^n+1$. In fact, if for some positive integer n we have $n|2^n+1$, then also $2^n+1|2^{2^n+1}+1$. Indeed, if $2^n+1 = kn$, where k is an integer (obviously, odd), then $2^n+1|2^{kn}+1 = 2^{2^n+1}+1$. Thus, $9|2^9+1$ implies $513|2^{513}+1$.

Suppose now that n is a prime and $n|2^n+1$. By Fermat's theorem we have then $n|2^n-2$, which implies, in view of $n|2^n+1$, that $n|3$. Since n is a prime, we get $n = 3$. Indeed, $3|2^3+1$. Thus, there exists only one prime n such that $n|2^n+1$, namely $n = 3$.

17*. We shall prove first the following theorem due to O. Reutter (see [17]):

If a is a positive integer such that $a+1$ is not a power of 2 with integer exponent, then the relation $n|a^n+1$ has infinitely many solutions in positive integers.

If $a+1$ is not a power of 2 with integer exponent, then it must have a prime divisor $p > 2$. We have therefore $p|a+1$.

LEMMA. *If for some integer $k \geqslant 0$ we have*

$$p^{k+1}|a^{p^k}+1,$$

where a is an integer > 1, and p is an odd prime, then $p^{k+2}|a^{p^{k+1}}+1$.

PROOF OF THE LEMMA. Assume that for some integer $k \geqslant 0$ we have $p^{k+1}|a^{p^k}+1$. Writing $a^{p^k} = b$, we get $p^{k+1}|b+1$, hence $b \equiv -1 \pmod{p^{k+1}}$. Since p is odd, we obtain

$$a^{p^{k+1}}+1 = b^p+1 = (b+1)(b^{p-1}-b^{p-2}+\ldots-b+1), \qquad (1)$$

and (since $b \equiv -1 \pmod{p^{k+1}}$ which implies $b \equiv -1 \pmod{p}$) we get the relations $b^{2l} \equiv 1 \pmod{p}$ and $b^{2l-1} \equiv -1 \pmod{p}$ for $l = 1, 2, \ldots$. Therefore

$$b^{p-1}-b^{p-2}+\ldots-b+1 \equiv 1-1+1-\ldots+1 \equiv 0 \pmod{p},$$

which shows that the second term on the right-hand side of (1) is divisible by p. Since the first term is divisible by p^{k+1}, we get $p^{k+2}|a^{p^{k+1}}+1$, which proves the lemma.

The lemma implies by induction that if $p|a+1$, then $p^{k+1}|a^{p^k}+1$, and $p^k|a^{p^k}+1$ for $k = 1, 2, \ldots$. Thus there exist infinitely many positive integers n such that $n|a^n+1$, which proves the theorem of O. Reutter.

Since even positive integers satisfy the conditions of Reutter's theorem, it suffices to assume that a is an odd number > 1.

If a is odd, then $2|a^2+1$, and a^2 is of the form $8k+1$. Thus, $a^2+1 = 8k+ +2 = 2(4k+1)$ is a double odd number. We shall prove the following lemma:

LEMMA. *If a is odd > 1, the numbers s and a^s+1 are double odd numbers, and $s|a^s+1$, then there exists a positive integer $s_1 > s$ such that s_1 and $a^{s_1}+1$ are double odd numbers and $s_1|a^{s_1}+1$.*

PROOF. Since $s|a^s+1$ and both s and a^s+1 are double odd numbers, we have $a^s+1 = ms$, where m is odd. Thus $a^s+1|a^{ms}+1 = a^{a^s+1}+1$, hence $a^s+1|a^{a^s+1}+1$. Since a^s+1 is even, $a^{a^s+1}+1$ is a double odd number. For $s_1 = a^s+1$ we have therefore $s_1|a^{s_1}+1$, where s_1 and $a^{s_1}+1$ are double odd numbers. In view of the fact that $a > 1$, we have $s_1 > s$. This proves the truth of the lemma.

Since a is odd, we can put $s = 2$, which satisfies the conditions of the lemma. It follows immediately that there exist infinitely many positive integers n such that $n|a^n+1$, which was to be proved (see [35]).

18*. We shall prove that if n is even and such that $n|2^n+2$ and $n-1|2^n+1$ (which is true, for instance, for $n = 2$), then for the number $n_1 = 2^n+2$ we also have $n_1|2^{n_1}+2$ and $n_1-1|2^{n_1}+1$. In fact, if $n|2^n+2$ and n is even, then $2^n+2 = nk$, where k is odd, hence

$$2^n+1|2^{nk}+1 = 2^{2^n+2}+1$$

and for $n_1 = 2^n+2$ we have

$$n_1-1 = 2^n+1|2^{n_1}+1.$$

Next, we have $n-1|2^n+1$, which implies $2^n+1 = (n-1)m$, where m is odd. We obtain therefore $2^{n-1}+1|2^{(n-1)m}+1 = 2^{2^n+1}+1$, which yields $2^n+2|2^{2^n+2}+ +2$, or $n_1|2^{n_1}+2$.

Since $n_1 = 2^n+2 > n$, there are infinitely many even numbers n satisfying our conditions. Starting from $n = 2$, we get successively numbers $2, 6, 66, 2^{66}+2, \ldots$. However, C. Bindschedler noticed that this method does not lead to all numbers n for which $n|2^n+2$ since we have, for instance, $946|2^{946}+2$. See a solution to my problem 430 in *Elemente der Mathematik*, *18* (1963), p. 90, given by C. Bindschedler.

19. If a is a positive integer, and r denotes its remainder upon dividing by 10, then $a^{10}+1$ is divisible by 10 if and only if $r^{10}+1$ is divisible by 10. It suffices therefore to consider only numbers r equal to $0, 1, 2, \ldots, 9$, and for these numbers we easily check that only $3^{10}+1$ and $7^{10}+1$ are divisible by 10. Thus, all numbers a such that $a^{10}+1$ is divisible by 10 are of the form $10k+3$ and $10k+7$ for $k = 0, 1, 2, \ldots$.

20*. Suppose that there exist positive integers $n > 1$ such that $n|2^n-1$, and let n denote the smallest of them. By Euler's theorem, we have then $n|2^{\varphi(n)}-1$. However, the greatest common divisor of numbers 2^a-1 and 2^b-1 for positive integers a and b is the number 2^d-1, where $d = (a, b)$. For $a = n$ and $b = \varphi(n)$, $d = (n, \varphi(n))$, it follows that $n|2^d-1$. However, since $n > 1$, we have $2^d-1 > 1$, which implies $d > 1$ and $1 < d \leqslant \varphi(n) < n$, and $d|n|2^d-1$ contrary to the definition of n.

20a. Such are, for instance, all numbers of the form $n = 3^k$, where $k = 1, 2, \ldots$. We shall prove it by induction. We have $3|2^3+1$. If for some positive integer k we have $3^k|2^{3^k}+1$, then, in view of the identity

$$2^{3^{k+1}}+1 = (2^{3^k}+1)(2^{2\cdot 3^k}-2^{3^k}+1)$$

and in view of the remark that

$$2^{2\cdot 3^k}-2^{3^k}+1 = 2^{2\cdot 3^k}+2-(2^{3^k}+1), \quad \text{and} \quad 3|2^{2\cdot 3^k}+2$$

(since 4^{3^k} gives remainder 1 upon dividing by 3), the second term of the formula for $2^{3^{k+1}}+1$ is divisible by 3, which implies $3^{k+1}|2^{3^{k+1}}+1$.

21. There is only one such odd number n, namely $n = 1$. In fact, suppose that there exists an odd number $n > 1$ such that $n|3^n+1$. Thus we have $n|9^n-1$. Let n be a least positive integer > 1 such that $n|9^n-1$. In view of $n|9^{\varphi(n)}-1$, for $d = (n, \varphi(n))$ we shall have $n|9^d-1$. Moreover, $d > 1$ since if d were equal to 1, we would have $n|8$ which is impossible since n

is odd. Thus $1 < d \leqslant \varphi(n) < n$, and $d|n|9^d-1$, contrary to the definition of the number n. Thus there is no odd number $n > 1$ such that $n|3^n+1$.

22. Clearly, n cannot be divisible by 3. Thus n is of one of the forms $6k+1$, $6k+2$, $6k+4$, or $6k+5$ where $k = 0, 1, 2, \ldots$. If $n = 6k+1$, then, in view of $2^6 \equiv 1 \pmod 3$, we have $n2^n+1 \equiv (2^6)^k 2+1 \equiv 2+1 \equiv 0 \pmod 3$. Thus $3|n2^n+1$. If $n = 6k+2$, then $n\,2^n+1 \equiv 2(2^6)^k 2^2+1 \equiv 8+1 \equiv 0 \pmod 3$, hence $3|n2^n+1$.

If $n = 6k+4$, then $n2^n+1 \equiv 4(2^6)^k 2^4+1 \equiv 2^5+1 \equiv 2 \pmod 3$.

Finally, if $n = 6k+5$, then $n2^n+1 \equiv 5(2^6)^k 2^5+1 \equiv 2 \pmod 3$.

Therefore, the relation $3|n2^n+1$ holds if and only if n is of the form $6k+1$ or $6k+2$, $k = 0, 1, 2, \ldots$.

23. If p is an odd prime and $n = (p-1)(kp+1)$ where $k = 0, 1, 2, \ldots$, then $n = -1 \pmod p$ and $p-1|n$. By Fermat's theorem, it implies $2^n \equiv 1 \pmod p$, hence $n2^n+1 \equiv 0 \pmod p$.

REMARK. It follows from this problem that there exist infinitely many composite numbers of the form $n2^n+1$ where n is a positive integer. The numbers of this form are called *Cullen numbers*. It was proved that for $1 < n < 141$ all numbers of this form are composite, but for $n = 141$ the number $n2^n+1$ is prime. It is not known whether there exist infinitely many prime Cullen numbers.

24. Let n be a given positive integer, and let $k > 1$ be a positive integer such that $2^k > n$. Let p be a prime $> 2^{k-1}k$. Since $k > 1$, for $x = 2^k$, $y = 2p$ we have $x \nmid y$, and $x^x|y^y$, because $x^x = 2^{k2^k}$ and $y^y = (2p)^{2p}$, where $2p > 2^k k$. Thus, for instance, $4^4|10^{10}$, but $4 \nmid 10$, $8^8|12^{12}$ but $8 \nmid 12$, $9^9|21^{21}$ but $9 \nmid 21$.

25*. For positive integers n we have obviously $\varphi(n)|n!$. In fact, it is true for $n = 1$; if $n > 1$, and if $n = q_1^{a_1}q_2^{a_2}\ldots q_k^{a_k}$ is a decomposition of n into primes, where $q_1 < q_2 < \ldots < q_k$, then

$$\varphi(n) = q_1^{a_1-1}q_2^{a_2-1}\ldots q_k^{a_k-1}(q_1-1)\ldots(q_k-1)$$

and we have $q_1^{a_1-1}q_2^{a_2-1}\ldots q_k^{a_k-1}|n$, while $q_1-1 < q_k \leqslant n$, which implies that $q_k-1 < n$ and $q_1-1 < q_2-1 < \ldots < q_k-1$ are different positive integers smaller than n. Thus $(q_1-1)(q_2-1)\ldots(q_k-1)|(n-1)!$, and it follows that $\varphi(n)|(n-1)!n = n!$.

If n is odd, then (by Euler's theorem) $n|2^{\varphi(n)}-1|2^{n!}-1$, hence $n|2^{n!}-1$, which was to be proved.

26. By Fermat's theorem, we have $2^4 \equiv 1 \pmod 5$ and $2^{12} \equiv 1 \pmod{13}$.

Since $2^3 \equiv 3 \pmod 5$ and $2^4 \equiv 3 \pmod{13}$, we get $2^{4k+3} \equiv 3 \pmod 5$ and $2^{12k+4} \equiv 3 \pmod{13}$ for $k = 0, 1, 2, \ldots$. Therefore $5 | 2^{4k+3}-3$ and $13 | 2^{12k+4}-3$ for $k = 0, 1, 2, \ldots$.

Next, $2^6 \equiv -1 \pmod{65}$, which implies that $2^{12} \equiv 1 \pmod{65}$ and therefore $2^{n+12}-3 \equiv 2^n-3 \pmod{65}$, which shows that the sequence of remainders modulo 65 of the sequence 2^n-3 $(n = 2, 3, \ldots)$ is periodic with period 12. To prove that none of the numbers 2^n-3 $(n = 2, 3, \ldots)$ is divisible by 65 it is sufficient to check whether the numbers 2^n-3 for $n = 2, 3, \ldots, 13$ are divisible by 65. We find easily that the remainders upon dividing by 65 are 1, 5, 13, 29, 61, 60, 58, 54, 46, 30, 63, 64, and none of these remainders is zero.

27*. It is known (see, for instance, Sierpiński, [37, p. 215]) that the four smallest composite numbers n, such that $n | 2^n-2$, are 341, 561, 645, and 1105. For 341, we have $341 \nmid 3^{341}-3$ since, by Fermat's theorem, $3^{30} \equiv 1 \pmod{31}$, which implies $3^{330} \equiv 1 \pmod{31}$, hence $3^{341} \equiv 3^{11} \pmod{31}$. In view of $3^3 \equiv -4 \pmod{31}$, we get $3^9 \equiv -64 \equiv -2 \pmod{31}$, hence $3^{11} \equiv -18 \pmod{31}$. Therefore $3^{341}-3 \equiv 3^{11}-3 \equiv -21 \pmod{31}$, and $31 \nmid 3^{341}-3$, which implies $341 = 11 \cdot 31 \nmid 3^{341}-3$. On the other hand, $561 = 3 \cdot 11 \cdot 17 | 3^{561}-3$ since $11 | 3^{10}-1$ which implies $11 | 3^{390}-1$ and $11 | 3^{341}-3$, and also $17 | 3^{16}-1$ which implies $17 | 3^{16 \cdot 35}-1 = 3^{560}-1$. Thus $17 | 3^{561}-3$.

Thus, the least composite number n such that $n | 2^n-2$ and $n | 3^n-3$ is the number $n = 561$.

The number 645 is not a divisor of $3^{645}-3$ since $645 = 3 \cdot 5 \cdot 43$, while $3^{42} \equiv 1 \pmod{43}$ which implies $3^{42 \cdot 15} \equiv 1 \pmod{43}$. Thus $3^{630} \equiv 1 \pmod{43}$, and $3^{645} \equiv 3^{15} \pmod{43}$. Since $3^4 \equiv -5 \pmod{43}$, we have

$$3^6 \equiv -45 \equiv -2 \pmod{43}, \qquad 3^{12} \equiv 4 \pmod{43}, \qquad 3^{15} \equiv 108 \equiv 22 \pmod{43}.$$

Therefore $3^{645}-3 \equiv 19 \pmod{43}$, which implies $43 \nmid 3^{645}-3$.

On the other hand, we have $1105 | 3^{1105}-3$. Indeed, $1105 = 5 \cdot 13 \cdot 17$, $3^4 \equiv 1 \pmod 5$, and $3^{1104} \equiv 1 \pmod 5$, and $5 | 3^{1105}-3$. Next, $3^{12} \equiv 1 \pmod{13}$, $3^{1104} \equiv 1 \pmod{13}$ and $13 | 3^{1105}-3$. Finally, $3^{16} \equiv 1 \pmod{17}$, and since $1104 = 16 \cdot 69$, we get $3^{1104} \equiv 1 \pmod{17}$, which implies $17 | 3^{1105}-3$.

Thus, two smallest composite numbers for which $n | 2^n-2$ and $n | 3^n-3$ are 561 and 1105.

REMARK. We do not know whether there exist infinitely many composite numbers n for which $n | 2^n-2$ and $n | 3^n-3$. This assertion would follow from a conjecture of A. Schinzel concerning prime numbers ([22]). For prime numbers n, both relations $n | 2^n-2$ and $n | 3^n-3$ hold because of Fermat's theorem.

28*. In view of $n \nmid 3^n-3$ and Fermat's theorem, the number n must be composite, and the least composite n for which $n|2^n-2$ and $n \nmid 3^n-3$ is $n = 341$. In the solution to Problem 27 we proved that $341 \nmid 3^{341}-3$. Thus, the least number n such that $n|2^n-2$ and $n \nmid 3^n-3$ is $n = 341$.

REMARK. A. Rotkiewicz proved that there exist infinitely many positive integers n, both even and odd, such that $n|2^n-2$ and $n \nmid 3^n-3$.

29. Number $n = 6$ has the desired property. In fact, if $n \nmid 2^n-2$, then n must be composite. The least composite number is 4, but $4 \nmid 3^4-3 = 78$. Next composite number is 6, and we have $6 \nmid 2^6-2 = 62$, while $6|3^6-3$ since 3^6-3 is obviously even and divisible by 3.

REMARK. A. Rotkiewicz proved that there exist infinitely many composite numbers n, both even and odd, such that $n|3^n-3$ and $n \nmid 2^n-2$.

30. If a is composite, we may put $n = a$ since obviously $a|a^n-a$. If $a = 1$, we can put $n = 4$ since $4|1^4-1$. If a is a prime > 2, we may put $n = 2a$ since in this case a is odd, and the number $a^{2a}-a$ is even; thus, $a^{2a}-a$, being divisible by an odd number a and by 2, is divisible by $2a$.

It remains to consider the case $a = 2$. Here we can put $n = 341 = 11 \cdot 31$ since $341|2^{341}-2$; the last property can be proved as follows: we have $11|2^{10}-1 = 1023$, hence $11|2^{340}-1$, and $11|2^{341}-2$. Next, $31 = 2^5-1|2^{340}-1$, hence $31|2^{341}-2$. Thus the number $2^{341}-2$ is divisible by 11 and 31, hence also by their product 341.

REMARK. M. Cipolla proved that for every positive integer a there exist infinitely many composite numbers n such that $n|a^n-a$. (See [5].) We do not know, however, whether there exist infinitely many composite numbers n such that $n|a^n-a$ for every integer a. The least of such number is $561 = 3 \cdot 11 \cdot 17$. From a certain conjecture of A. Schinzel concerning prime numbers ([22]) it follows that there are infinitely many such composite numbers.

31. The cube of an integer which is not divisible by 3 gives remainder 1 or -1 upon dividing by 9. Thus, if none of the numbers a, b, c were divisible by 3, then the number $a^3+b^3+c^3$, upon dividing by 9, would give the remainder $\pm 1 \pm 1 \pm 1$ which is not divisible by 9 for any combination of signs $+$ and $-$. It follows that if $9|a^3+b^3+c^3$, then $3|abc$, which was to be proved.

32. The proof is analogous to the proof in Problem 31 since the number $\pm 1 \pm 1 \pm 1 \pm 1 \pm 1$ is not divisible by 9 for any combination of signs $+$ and $-$.

33. The condition $(x, y) = 1$ is necessary since, for instance, $15^2+20^2 = 5^4$, while $7 \nmid 15\cdot 20$. Now, if $(x, y) = 1$ and x, y, z are positive integers such that $x^2+y^2 = z^4$, then, as we know from the theory of Pythagorean equation, there exist integers m and n such that for instance $x = m^2-n^2$, $y = 2mn$, $z^2 = m^2+n^2$. Suppose that $7 \nmid y$; thus $7 \nmid m$ and $7 \nmid n$. It is easy to see that the square of an integer not divisible by 7 gives, upon dividing by 7, the remainders 1, 2 or 4. Since 1+2, 1+4, and 2+4 cannot be such remainders, neither they are divisible by 7, it follows from equation $z^2 = m^2+ +n^2$ that the numbers m and n must give the same remainders upon dividing by 7. Thus $7|x = m^2-n^2$.

34. The square of an integer not divisible by 7 gives upon dividing by 7 the remainder 1, 2, or 4, hence the sum of such squares gives the remainder 1, 2, 3, 4, 5, or 6. Thus, if a and b are integers such that $7|a^2+b^2$, then one of them, hence also the other, must be divisible by 7.

35*. The numbers $x = 36k+14, y = (12k+5)(18k+7)$, $k = 0, 1, 2, \ldots$, have the desired property.

In fact, we have obviously $x(x+1)|y(y+1)$ since

$$x(x+1) = 2\cdot 3(12k+5)(18k+7) = 6y,$$

while $6|y+1$.

The number y is not divisible by x since y is odd, while x is even. The number y is not divisible by $x+1$ since $3|x+1$, while $3 \nmid y$. The number $y+1$ is not divisible by x since $18k+7|x$ and $18k+7|y$, hence $18k+7 \nmid y+1$. Finally, the number $y+1$ is not divisible by $x+1$ since $12k+5|x+1$ and $12k+5|y$, hence $12k+5 \nmid y+1$.

For $k \pm 0$, we obtain $x = 14$, $y = 35$, and it is easy to show that there are no smaller numbers with the required property.

36. For $s < 10$, we have of course $n_s = s$. Next, studying successive multiples of s, we obtain $n_{10} = 190$, $n_{11} = 209$, $n_{12} = 48$, $n_{13} = 247$, $n_{14} = 266$, $n_{15} = 155$, $n_{16} = 448$, $n_{17} = 476$, $n_{18} = 198$, $n_{19} = 874$, $n_{20} = 9920$, $n_{21} = 399$, $n_{22} = 2398$, $n_{23} = 1679$, $n_{24} = 888$, $n_{25} = 4975$. Finally, we have $n_{100} = 19999999999900$. In fact, two last digits of every number divisible by 100 must be zero, and the sum of digits of every number smaller than 199999999999 is obviously smaller than 100. See Kaprekar [11].

37*. Let s be a positive integer, $s = 2^\alpha 5^\beta t$, where α and β are integers $\geqslant 0$, and t is a positive integer not divisible by 2 or 5. By Euler's theorem we have $10^{\varphi(t)} \equiv 1 \pmod t$. Let $n = 10^{\alpha+\beta}(10^{\varphi(t)}+10^{2\varphi(t)}+ \ldots +10^{s\varphi(t)})$.

We have $10^{\varphi(t)}+10^{2\varphi(t)}+ \ldots +10^{s\varphi(t)} \equiv s \equiv 0 \pmod{t}$ (since $t|s$), and in view of $2^{\alpha}5^{\beta}|10^{\alpha+\beta}$, the number n is divisible by s. On the other hand, it is clear that the sum of decimal digits of the number n equals s.

38*. (a) The theorem is obviously true if the number has no prime divisor of the form $4k+3$. Suppose that the theorem is true for all numbers, whose expansion into primes in first powers (hence not necessarily distinct) contains $s \geqslant 0$ primes of the form $4k+3$. Let n be a positive integer, whose expansion into primes in first powers (hence not necessarily distinct) contains $s+1$ prime factors of the form $4k+3$. Then we have $n = mq$, where the expansion of m into primes in first powers contains s factors of the form $4k+$ $+3$, and q is a prime of the form $4k+3$. Let g denote the number of integer divisors of m which are of the form $4k+1$ and let h denote the number of integer divisors of n which are of the form $4k+3$. By assumption (concerning s) we have $g \geqslant h$. Now, the integer divisors of the form $4k+1$ of mq are of course the divisors of the form $4k+1$ of m (the number of these divisors being equal to g), and also the products of integer divisors of the form $4k+$ $+3$ of m by the number q; the number of these divisors is h. Thus, the number mq will have $g+h$ integer divisors of the form $4k+1$. On the other hand, integer divisors of mq of the form $4k+3$ will be the integer divisors of the form $4k+3$ of m (the number of those divisors being h), and the products of the divisors of the form $4k+1$ of m by q (the number of those divisors is g). However, among the latter there may be divisors which are divisors of the form $4k+3$ of m. Thus the total number of integer divisors of the form $4k+3$ of mq is $\leqslant h+g$ (and, perhaps, $< h+g$). The theorem being true for every number mq, we obtain by induction (with respect to s) that the theorem is true for positive integer s.

(b) The number 3^{2n-1} $(n = 1, 2, \ldots)$ has as many integer divisors of the form $4k+1$ (namely, $1, 3^2, 3^4, \ldots, 3^{2n-2}$) as divisors of the form $4k+3$ (namely $3, 3^3, 3^5, \ldots, 3^{2n-1}$).

(c) The number 3^{2n} (where $n = 1, 2, \ldots$) has $n+1$ divisors of the form $4k+1$ (namely $1, 3^2, 3^4, \ldots, 3^{2n}$) and only n divisors of the form $4k+3$ (namely $3, 3^3, \ldots, 3^{2n-1}$). The number 5^n has all $n+1$ divisors of the form $4k+1$, and has no divisor of the form $4k+3$.

39. Let r_1, r_2, and r_3 be the remainders upon dividing the integers $-a$, $-b$, and $-c$ by n. Thus, r_1, r_2, and r_3 are integers from the sequence $0, 1, 2, \ldots$ $\ldots, n-1$, and since there is at most three different among the numbers r_1,

r_2, r_3, while $n > 3$, there exists a number r in this sequence such that $r \neq r_1$, $r \neq r_2$, and $r \neq r_3$. If we had $n|a+r$, then in view of $-a \equiv r_1 \pmod{n}$ we would have $n|r-r_1$. However, r and r_1 are integers $\geqslant 0$ and $< n$, and if their difference is divisible by n, then we must have $r = r_1$ contrary to the definition of r. In a similar way we show that $n \nmid b+r$ and $n \nmid c+r$. Thus, we can put $k = r$.

40. We easily show by induction that for positive integers n we have $2^n \geqslant n+1$, which implies that $2^{n+1}|2^{2^n}$ and $2^{2^{n+1}}-1|2^{2^{2^n}}-1$. Therefore $F_n = 2^{2^n}+1|2^{2^{n+1}}-1|2^{2^{2^n}}-1|2^{2^{2^{n+1}}}-2 = 2^{F_n}-2$, and $F_n|2^{F_n}-2$, which was to be proved.

REMARK. T. Banachiewicz suspected that this relation led P. Fermat to his conjecture that all numbers F_n $(n = 1, 2, \ldots)$ are primes. During Fermat's times it was thought that the so-called Chinese theorem is true, namely the theorem asserting that if an integer $m > 1$ satisfies the relation $m|2^m-2$, then m is a prime (it was checked for first several hundred integers). This breaks down, however, for $m = 341 = 11 \cdot 31$, which was not known then.

II. RELATIVELY PRIME NUMBERS

41. Numbers $2k+1$ and $9k+4$ are relatively prime since $9(2k+1)-2(9k+4) = 1$. Since $9k+4 = 4(2k-1)+(k+8)$, while $2k-1 = 2(k+8)-17$, we have $(9k+4,\ 2k-1) = (2k-1,\ k+8) = (k+8,\ 17)$. If $k \equiv 9 \pmod{17}$, then $(k+8,\ 17) = 17$; in the contrary case, we have $17|k+8$, hence $(k+8,\ 17) = 1$. Thus, $(9k+4,\ 2k-1) = 17$ if $k \equiv 9 \pmod{17}$ and $(9k+4, 2k-1) = 1$ if $k \not\equiv 9 \pmod{17}$.

42. We show first that if for some positive integer m we have m triangular numbers $a_1 < a_2 < \ldots < a_m$ which are pairwise relatively prime, then there exists a triangular number $t > a_m$ such that $(t, a_1, a_2, \ldots, a_m) = 1$.

In fact, let $a = a_1 a_2 \ldots a_m$; the numbers $a+1$ and $2a+1$ are relatively prime to a. The number

$$a_{m+1} = t_{2a+1} = \frac{(2a+1)\,(2a+2)}{2} = (a+1)\,(2a+1)$$

is a triangular number $> a_m$; being relatively prime to a, it is relatively prime to every number $a_1, a_2, \ldots, a_m$.

It follows that if we have a finite increasing sequence of pairwise relatively prime triangular numbers, then we can always find a triangular number exceeding all of them and pairwise relatively prime to them. Taking always the least such number we form the infinite sequence

$$t_1 = 1,\ t_2 = 3,\ t_4 = 10,\ t_{13} = 91,\ t_{22} = 253,\ \ldots$$

of pairwise relatively prime triangular numbers.

43. We shall prove first that if for some positive integer m the tetrahedral numbers $a_1 < a_2 < \ldots < a_m$ are pairwise relatively prime, then there exists a tetrahedral number $T > a_m$ such that $(T, a_1, a_2, \ldots, a_m) = 1$. In fact, let $a = a_1 a_2 \ldots a_n$. Put $T = T_{6a+1} = (6a+1)(3a+1)(2a+1)$; clearly T is prime relatively to a, hence relatively to each of the numbers $a_1, \ldots, a_m$, and $T > a \geqslant a_m$.

Thus, we can define the required increasing sequence of pairwise relatively prime tetrahedral numbers by induction: take $T_1 = 1$ as the first term of the sequence, and, after having defined m first pairwise relatively prime tetrahedral numbers of this sequence, define the $m+1$st as the least tetrahedral number exceeding all first m terms, and being relatively prime to each of them. In this manner we obtain the infinite increasing sequence of pairwise relatively prime tetrahedral numbers

$$T_1 = 1,\ T_2 = 4,\ T_5 = 35,\ T_{17} = 969,\ \ldots .$$

44. Let a and b be two different integers. Assume for instance $a < b$, and let $n = (b-a)k+1-a$. For k sufficiently large, n will be positive integer. We have $a+n = (b-a)k+1$, $b+n = (b-a)(k+1)+1$, hence $a+n$ and $b+n$ will be positive integers. If we had $d|a+n$ and $d|b+n$, we would have $d|a-b$, and, in view of $d|a+n$, also $d|1$, which implies that $d = 1$. Thus, $(a+n, b+n) = 1$.

45*. If the integers a, b, c are distinct, then the number

$$h = (a-b)(a-c)(b-c)$$

is different from zero. In case $h \neq \pm 1$, let $q_1, \ldots, q_s$ denote all prime > 3 divisors of h.

If two or more among numbers a, b, c are even, put $r = 1$, otherwise put $r = 0$. Clearly, at least two of the numbers $a+r$, $b+r$, $c+r$ will be odd. If

a, b, c give three different remainders upon dividing by 3, put $r_0 = 0$. If two or more among a, b, c give the same remainder ϱ upon dividing by 3, put $r_0 = 1-\varrho$. Clearly, at least two of the numbers $a+r_0$, $b+r_0$, $c+r_0$ will be not divisible by 3.

Now, let i denote one of the numbers 1, 2, ..., s. In view of Problem 39 (and the fact that $q_i > 3$), there exists an integer r_i such that none of the numbers $a+r_i$, $b+r_i$, $c+r_i$ is divisible by q_i. According to the Chinese remainder theorem, there exist infinitely many positive integers n such that

$$n \equiv r \pmod{2}, \qquad n \equiv r_0 \pmod{3},$$

and

$$n \equiv r_i \pmod{q_i} \qquad \text{for } i = 1, 2, \ldots, s.$$

We shall show that the numbers $a+n$, $b+n$ and $c+n$ are pairwise relatively prime. Suppose, for instance, that $(a+n, b+n) > 1$. Then, there would exist a prime q such that $q|a+n$ and $q|b+n$, hence $q|a-b$, which implies $q|h$ and $h \neq \pm 1$. Since $n \equiv r \pmod{2}$ and at least two of the numbers $a+r$, $b+r$, $c+r$ are odd, at least two of the numbers $a+n$, $b+n$, $c+n$ are odd, and we cannot have $q = 2$. Next, since $n \equiv r_0 \pmod{3}$ and at least two of the numbers $a+r_0$, $b+r_0$, $c+r_0$ are not divisible by 3, at least two of the numbers $a+n$, $b+n$, $c+n$ are not divisible by 3, and we cannot have $q = 3$. Since $q|h$, in view of the definition of h, we have $q = q_i$ for a certain i from the sequence 1, 2, ..., s. However, in view of $n \equiv r_i \pmod{q_i}$, or $n \equiv r_i \pmod{q}$, and in view of the fact that none of the numbers $a+r_i$, $b+r_i$, $c+r_i$ is divisible by q_i, none of the numbers $a+n$, $b+n$, $c+n$ is divisible by $q_i = q$, contrary to the assumption that $q|a+n$ and $q|b+n$. Thus, we proved that $(a+n, b+n) = 1$. In a similar way we show that $(a+n, c+n) = 1$, and $(b+n, c+n) = 1$. Therefore the numbers $a+n$, $b+n$, and $c+n$ are pairwise relatively prime. Since there are infinitely many such numbers n, the proof is complete.

46. Such numbers are for instance $a = 1$, $b = 2$, $c = 3$, $d = 4$. In fact, for odd n, the numbers $a+n$ and $c+n$ are even, hence not relatively prime, and, for even n, the numbers $b+n$ and $d+n$ are even, hence not relatively prime.

47. If n is odd and > 6, then $n = 2+(n-2)$, where $n-2$ is odd and > 1, and we have $(2, n-2) = 1$.

The following proof for the case of even $n > 6$ is due to A. Mąkowski. If $n = 4k$, where k is an integer > 1 (since $n > 6$), then $n = (2k-1)+$

$+(2k+1)$, and $2k+1 > 2k-1 > 1$ (since $k > 1$). The numbers $2k-1$ and $2k+1$, as consecutive odd numbers, are relatively prime.

If $n = 4k+2$, where k is an integer > 1 (since $n > 6$), we have $n = (2k+3)+(2k-1)$, where $2k+3 > 2k-1 > 1$ (since $k > 1$). The numbers $2k+3$ and $2k-1$ are relatively prime since if $0 < d|2k+3$ and $d|2k-1$, then $d|(2k+3)-(2k-1)$ or $d|4$. Now, d as a divisor of an odd number must be odd, hence $d = 1$, and $(2k+3, 2k-1) = 1$.

48*. If n is even and > 8, then $n = 6k$, $n = 6k+2$ or $n = 6k+4$, and in the first two cases k is an integer > 1, and in the third case, k is a positive integer. The formulae

$$6k = 2+3+\big(6(k-1)+1\big), \quad 6k+2 = 3+4+\big(6(k-1)+1\big),$$
$$6k+4 = 2+3+(6k-1)$$

show easily that n is a sum of three pairwise relatively prime positive integers.

Suppose now that n is odd and > 17. We consider six cases: $n = 12k+1$, $n = 12k+3$, $n = 12k+5$, $n = 12k+7$, $n = 12k+9$, and $n = 12k+11$, where in the first three cases k is an integer > 1, and in the last three cases k is a positive integer. We have

$$12k-1 = \big(6(k-1)-1\big)+\big(6(k-1)+5\big)+9,$$

where the numbers $6(k-1)-1$, $6(k-1)+5$, and 9 are > 1 and relatively prime; in fact, the first two are not divisible by 3, and are relatively prime since $d|6(k-1)-1$ and $d|6(k-1)+5$ would imply $d|4$, while the numbers considered are odd.

If $n = 12k+3$, then we have $n = (6k-1)+(6k+1)+3$;
if $n = 12k+5$, then we have $n = (6k-5)+(6k+1)+9$;
if $n = 12k+7$, then we have $n = (6k+5)+(6k-1)+3$;
if $n = 12k+9$, then we have $n = (6k-1)+(6k+1)+9$;
if $n = 12k+11$, then we have $n = \big(6(k+1)-5\big)+\big(6(k+1)+1\big)+3$, and we easily check that in each case we have three terms > 1 and pairwise relatively prime.

The number 17 does not have the desired property since in the case $17 = a+b+c$, all three numbers a, b, c (as > 1 and pairwise relatively prime) would have to be odd and distinct. We have, however, $3+5+7 = 15 < 17$, $3+5+11 > 17$, and in case $3 < a < b < c$, we have $a \geqslant 5$, $b \geqslant 7$, $c \geqslant 9$, hence $a+b+c \geqslant 5+7+9 \geqslant 21 > 17$, which shows that 17 does not have the desired property.

49*. We shall present the proof based on an idea of A. Schinzel (see [19]). Let k denote a given positive integer and let m be the positive integer whose expansion into prime powers is $m = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$. Let $f(x) = x(x+2k)$ and let i denote one of the numbers $1, 2, \ldots, s$. We cannot have $q_i | x(x+2k)$ for all integer x since then for $x = 1$ we would have $q_i | 2k+1$, and for $x = -1$ we would have $q_i | 2k-1$, and $q_i | (2k+1)-(2k-1) = 2$, which is impossible in view of $q_i | 2k+1$ (and, in consequence, $q_i | 1$). Therefore there exists an integer x_i such that $q_i \nmid x_i(x_i+2k) = f(x_i)$. By the Chinese remainder theorem, there exists a positive integer x_0 such that $x_0 \equiv x_i \pmod{q_i}$ for $i = 1, 2, \ldots, s$, which yields $f(x_0) \equiv f(x_i) \not\equiv 0 \pmod{q_i}$ for $i = 1, 2, \ldots, s$. We have therefore $\big(f(x_0), q_i\big) = 1$ for $i = 1, 2, \ldots, s$, which (in view of the expansion of m into prime factors) gives $\big(f(x_0), m\big) = 1$, or $\big(x_0(x_0+2k), m\big) = 1$. Thus, if we put $a = x_0+2k$, $b = x_0$, we shall have $2k = a-b$, where $(a, m) = 1$, $(b, m) = 1$, which proves the theorem.

REMARK. Since adding arbitrary multiples of m to a and b does not change the fact that $2k = a-b$ and $(ab, m) = 1$, we proved that, for every m, every even number can be represented in infinitely many ways as a difference of positive integers relatively prime with m.

We do not know whether every even number is a difference of two primes. From a certain conjecture on prime numbers of A. Schinzel ([22]), it follows that every even number can be represented as a difference of two primes in infinitely many ways.

50*. We shall present the proof given by A. Rotkiewicz. If u_n is the nth term of the Fibonacci sequence, and if m and n are positive integers, then $(u_m, u_n) = u_{m,n}$ (see [27, p. 280, problem 5]). Since $u_1 = 1$, we see that if p_k denotes the kth successive prime, then every two terms of the increasing infinite sequence

$$u_{p_1}, u_{p_2}, u_{p_3}, \ldots$$

are relatively prime. Instead of p_κ we could take here $2^{2^k}+1$ since it is well known that $(2^{2^m}+1, 2^{2^n}+1) = 1$ for positive integers m and $n \neq m$.

51*. We know that every divisor > 1 of the number $F_n = 2^{2^n}+1$ $(n = 1, 2, \ldots)$ is of the form $2^{n+2}k+1$ where k is positive integer (see, for instance, [37, p. 343, Theorem 5]). Since for positive integers n and k we have $2^{n+2}k+1 \geqslant 2^{n+2}+1 > n$, all > 1 divisors of F_n must be $> n$, hence $(n, F_n) = 1$, which was to be proved.

51a. We easily check that $(n, 2^n-1) = 1$ for $n = 1, 2, 3, 4, 5$, while $(6, 2^6-1) = 3$. For $k = 1, 2, \ldots$, we have $3|2^6-1|2^{6k}-1$, hence $(6k, 2^{6k}-1) \geqslant 3$ for $k = 1, 2, \ldots$. The least such number n is equal to 6.

III. ARITHMETIC PROGRESSIONS

52. Let m be a given integer > 1. The numbers $m!k+1$ for $k = 1, 2, \ldots$ $\ldots, m$ are relatively prime since for positive integers k and l with $k < l \leqslant m$ if $d > 1$ were the common divisor of $m!k+1$ and $m!l+1$, we would have $d|l(m!k+1)-k(m!l+1) = l-k < m$, hence $1 < d < m$, and $d|m!$. This, in view of $d|m!k+1$, gives $d|1$, contrary to the assumption that $d > 1$.

53. The required property is satisfied, for instance, by all terms of the arithmetic progression $2^k t+2^{k-1}$ (where $t = 0, 1, 2, \ldots$) since in the expansion of $n = 2^k t+2^{k-1}$ into primes, the number 2 enters with the exponent $k-1$; from the well-known formula for the number of positive integer divisors it follows immediately that the number of positive integer divisors of the number n is divisible by k.

54. The required property holds for an arbitrary positive integer x and for $y = 5x+2$, $z = 7x+3$ since in this case the numbers

$$x(x+1) = x^2+x, \quad y(y+1) = 25x^2+25x+6, \quad z(z+1) = 49x^2+49x+12$$

form the arithmetic progression with the difference $24x^2+24x+6$.

REMARK. One can show that there are no four increasing positive integers x, y, z, t such that the numbers $x(x+1)$, $y(y+1)$, $z(z+1)$, and $t(t+1)$ form an arithmetic progression since then the numbers four times greater and increased by one, i.e. the numbers $(2x+1)^2$, $(2y+1)^2$, $(2z+1)^2$, and $(2t+1)^2$ would also form an arithmetic progression, contrary to the theorem of Fermat asserting that there are no four different squares of integers which form an arithmetic progression (the proof can be found in the book by W. Sierpiński [37, p. 74, theorem 8]).

55. If the sides of a rectangular triangle form an arithmetic progression, then we can denote them by $b-r$, b and $b+r$ where b and r are positive integers, and we have $(b-r)^2+b^2 = (b+r)^2$, hence $b = 4r$, which gives the

rectangular triangle with sides $3r$, $4r$, and $5r$, where r is an arbitrary positive integer. Thus, all rectangular triangles whose sides are integers forming an arithmetic progression are obtained by increasing integer number of times the triangle with sides 3, 4, 5.

56. Triangular numbers $t_n = \frac{1}{2}n(n+1)$ are odd for $n = 4u+1$ ($u = 0, 1, 2, \ldots$) and even for $4|n$. Thus both progressions with difference 2 contain infinitely many triangular numbers. On the other hand, the progression $3k+2$ ($k = 0, 1, 2, \ldots$) does not contain any triangular number since if $3|n$, then $3|t_n$; similarly, if $n = 3u+2$ for $u = 0, 1, 2, \ldots$, then $3|t_n$; finally, if $n = 3u+1$, where $u = 0, 1, 2, \ldots$, then $t_n = 9\dfrac{u(u+1)}{2}+1$, hence dividing by 3 yields the remainder 1.

57. It is necessary and sufficient for b to be a quadratic residue for modulus a. In fact, if for some positive integer x and some integer $k \geqslant 0$ we have $x^2 = ak+b$, then $x^2 \equiv b \pmod{a}$, and b is a quadratic residue for modulus a. Conversely, if b is a quadratic residue for modulus a, then there exist infinitely many positive integers x such that $x^2 \equiv b \pmod{a}$, hence $x^2 = ak+$ $+b$, where k is an integer, and consequently, is positive for sufficiently large x.

58*. We shall give the proof due to A. Schinzel. Let p_k denote the kth successive prime. Let s be an arbitrary positive integer and let $P = p_1p_2 \ldots p_s$. By the Chinese remainder theorem, for every positive integer $k \leqslant s$ there exists a positive integer a_k such that $a_k \equiv 0 \pmod{P/p_k}$, and $a_k \equiv -1 \pmod{p_k}$. Put $Q = 1^{a_1}2^{a_2}\ldots s^{a_s}$. The numbers kQ ($k = 1, 2, \ldots, s$) form an increasing arithmetic progression with s terms. By the definition of the numbers a_k ($k = 1, 2, \ldots, s$), we have $p_k|a_k+1$ and $p_k|a_n$ if $k \neq n$, where n is a positive integer $> s$. The numbers

$$Q_k = k^{(a_k+1)/p_k}\prod_{\substack{n=1\\ n\neq k}}^{s} n^{a_k/p_k}$$

are natural, and we easily see that $kQ = Q_k^{p_k}$ for $k = 1, 2, \ldots, s$, thus the numbers kQ ($k = 1, 2, \ldots, s$) are powers of integers with integer exponents > 1.

59. The desired theorem is clearly equivalent to the theorem that in every infinite increasing arithmetic sequence of integers there exists a term

which is not a power with integer exponent > 1 of any integer. Thus, let $ak+b$ ($k = 0, 1, 2, \ldots$) be an infinite arithmetic progression, where a and b are positive integers. There exists a prime $p > a+b$. Since $(a, p^2) = 1$, the equation $ax-p^2y = 1$ has a solution in positive integers x, y. Let $k = (p-b)x$; this will be a positive integer (since $p > b$), and we shall have $ak+b = p^2y(p-b)+p$. Thus the term $ak+b$ of our progression is not divisible by p^2, and therefore cannot be a power of a positive integer with integer exponent > 1.

60. Out of every four consecutive positive integers one must be of the form $4k+2$, where k is an integer $\geqslant 0$. No such number, as an even number which is not divisible by 4, can be a power of a positive integer with integer exponent > 1.

REMARK. A. Mąkowski proved that there are no three consecutive positive integers such that each of them is a power with integer exponent > 1 of a positive integer, but the proof is difficult (Khatri [13]). There exist, however, two consecutive numbers such that each of them is a power with integer exponent > 1 of a positive integer. Such numbers are, for instance, $2^3 = 8$, $3^2 = 9$. Catalan's problem whether there are any other pairs of such integers is open. S. Hyyrö [10] proved that in any such pair both bases are $> 10^{11}$.

61. The solution follows immediately from Problem 58, it can, however, be proved in a simpler way. Let $m > 1$ be an integer, and let q_i ($i = 1, 2, \ldots, m$) be primes such that $a < q_1 < q_2 < \ldots < q_m$. By the Chinese remainder theorem, there exists a positive integer x such that $ax \equiv -b-aj \pmod{q_j^2}$ for $j = 1, 2, \ldots, m$. Thus $q_j^2 | a(x+j)+b$ for $j = 1, 2, \ldots, m$. Thus m consecutive terms of the progression $ak+b$, namely the terms $a(x+j)+b$ for $j = 1, 2, \ldots, m$, are composite.

62*. We can assume, of course, that m is an integer > 1. Let P denote the product of all prime divisors of m which are the divisors od a, and let $P = 1$ if there are no such divisors. Let Q denote the product of all prime divisors of m which are divisors of b, and let $Q = 1$ if there are no such divisors. Since $(a, b) = 1$, we have $(P, Q) = 1$. Finally, let R denote the product of all prime divisors of m which do not divide a or b, and if there are no such divisors, let $R = 1$. Obviously, we have $(R, P) = 1$ and $(R, Q) = 1$. We shall show that $(aPR+b, m) = 1$. In fact, if it were not true, there would exist a prime p such that $p|m$, and $p|aPR+b$. If we had $p|P$, then $p|aPR+b$ would imply $p|b$, hence $p|Q$, contrary to the fact that (P, Q)

$= 1$. If we had $p|Q$, then we would have $p|b$, hence $p|aPR$, which is impossible since $(a, b) = 1$, $(b, P) = 1$, $(b, R) = 1$. Finally, if we had $p|R$, we would have $p|b$, hence $p|Q$, contrary to the fact that $(R, Q) = 1$. Thus, we proved that $(aPR+b, m) = 1$, and it follows that $\big(a(km+PR)+b, m\big) = 1$ for $k = 0, 1, \ldots$. Therefore our progression contains infinitely many terms relatively prime with m, which was to be proved.

63. Let b be the first term of our progression and let a be its difference; thus the numbers a and b are positive integers. Let x denote the remainder obtained from dividing b by a; we have therefore $b = at+r$ where t is an integer $\geqslant 0$, and r is an integer such that $0 \leqslant r < a$. Let s be an arbitrary positive integer, and let $c_1, c_2, \ldots, c_s$ with $c_1 \neq 0$ be an arbitrary sequence of decimal digits. Let N denote the s-digit integer, whose consecutive digits are $c_1, c_2, \ldots, c_s$.

Obviously there exists a positive integer n such that $10^n > 2a(t+1)$. The number $N10^n/a-t$ will be > 1.

Let k be the least positive integer greater than $N10^n/a-t$; thus, we shall have $k-1 \leqslant N10^n/a-t$, hence

$$k+1 \leqslant \frac{N10^n}{a} +2-t < \frac{(N+1)10^n}{a} -t$$

since $10^n > 2a$. We have therefore $N10^n < a(k+t) < ak+at+r = ak+b < a(k+t+1) < (N+1)10^n = N10^n+10^n$ and it follows that the first s digits of the number $ak+b$ are the same as the first s digits of the number N, i.e. the digits $c_1, c_2, \ldots, c_s$.

64. If the terms u_k, u_l, and u_m of the Fibonacci sequence form an arithmetic progression, then we must have $u_l > 1$, and therefore $l > 2$ (since $u_2 = 1$), and $m > 3$. Moreover, $u_m = u_l+(u_l-u_k)$, which implies that $u_m < u_l+u_l < u_l+u_{l+1} = u_{l+2}$. Thus $u_m < u_{l+2}$ and it follows that $u_m \leqslant u_{l+1}$. On the other hand, $u_m > u_l$, hence $u_m \geqslant u_{l+1}$, and we must have $u_m = u_{l+1}$. Therefore (since $l > 2$) we have $m = l+1$. We have thus $u_k = 2u_l-u_m = u_l-(u_{l+1}-u_l) = u_l-u_{l-1} = u_{l-2}$, which implies that $k = l-2$. Thus, if the terms u_k, u_l, and u_m of the Fibonacci sequence form an increasing arithmetic progression, we must have $l > 2$, $k = l-2$ and $m = l+1$. On the other hand, for any integer $l > 1$ the numbers u_{l-2}, u_l, and u_{l+1} form an arithmetic progression with the difference u_{l-1}. If n were an integer $> l+1$, we would have $n \geqslant l+2$, hence $u_n \geqslant u_{l+2}$. and $u_n-u_{l+1} \geqslant u_{l+2}-u_{l+1} = u_l > u_{l-1}$

(since $l > 2$). It follows that there are no four terms of the Fibonacci sequence which form an arithmetic progression.

65*. We know [27, p. 279, problem 3] that if m is a positive integer, then the remainders of dividing successive terms of the Fibonacci sequence by m form a periodic sequence with pure period. For $m = 2, 3, 4, 5, 6, 7$, the remainders upon dividing the terms of the Fibonacci sequence by m are respectively (we show here only first few remainders, and not all of them):

for $m = 2$: $1, 1, 0, \ldots,$
for $m = 3$: $1, 1, 2, 0, \ldots,$
for $m = 4$: $1, 1, 2, 3, 1, 0, \ldots,$
for $m = 5$: $1, 1, 2, 3, 0, 3, 3, 1, 4, \ldots,$
for $m = 6$: $1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, \ldots,$
for $m = 7$: $1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, \ldots.$

Since for every positive integer $m \leqslant 7$ all possible remainders modulo m appear in the above sequences, we see that each of the arithmetic progressions with the difference $m \leqslant 7$ contains infinitely many terms of the Fibonacci sequence.

We shall show now that the progression $8k+4$ ($k = 0, 1, 2, \ldots$) does not contain any term of the Fibonacci sequence.

Since $u_1 = u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$ for $n = 1, 2, \ldots,$ we easily see that the numbers $u_1, u_2, \ldots, u_{14}$ give the following remainders upon dividing by 8:

$$1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1.$$

It follows that $8|u_{13}-u_1$ and $8|u_{14}-u_2$. Thus, for $n = 1$ we have $8|u_{n+12}-u_n$ and $8|u_{n+13}-u_{n+1}$.

Suppose now that these two formulas hold for some positive integers n. We have then $8|u_{n+12}+u_{n+13}-(u_n+u_{n+1})$ or $8|u_{n+14}-u_{n+2}$ (since $8|u_{n+13}-u_{n+1}$). It follows by induction that $8|u_{n+12}-u_n$ for $n = 1, 2, \ldots,$ which shows that the sequence of consecutive remainders modulo 8 of the Fibonacci sequence is periodic and has a pure twelve-term period.

From the first fourteen remainders modulo 8 we see that these remainders may be only 0, 1, 2, 3, 5, and 7. Thus there are no remainders 4 or 6, which implies that the progressions $8k+4$ and $8k+6$ do not contain any term of the Fibonacci sequence. These are the progressions of integer terms with the desired property, and with the least possible difference.

66*. The progression $11k+4$ $(k = 0, 1, 2, \ldots)$ has the required property. As in the solution of Problem 65, we prove by an easy induction that $11 | u_{n+10} - u_n$ for $n = 1, 2, \ldots$. It follows that the sequence of remainders modulo 11 of the Fibonacci sequence is periodic, with period 10; we easily find this sequence to be $1, 1, 2, 3, 5, 8, 2, 10, 1, 0, \ldots$. The number 4 (and also $6, 7$, and 9) does not appear in this sequence.

67. Suppose that we have n terms of our progression

$$ak_1+b,\ ak_2+b,\ \ldots,\ ak_n+b,$$

which are pairwise relatively prime (for $n = 1$ we can put $k_1 = 1$). Let $m = (ak_1+b)(ak_2+b) \ldots (ak_n+b)$. From Problem 62* it follows that there exists a positive integer k_{n+1} such that $(ak_{n+1}+b, m) = 1$, hence $(ak_{n+1}+b, ak_i+b) = 1$ for $i = 1, 2, \ldots, n$. The numbers

$$ak_1+b,\ ak_2+b,\ \ldots,\ ak_n+b,\ ak_{n+1}+b$$

are therefore pairwise relatively prime. Thus we defined by induction the infinite sequence $k_1, k_2, \ldots$ such that the sequence ak_i+b $(i = 1, 2, \ldots)$ contains only relatively prime terms of the original arithmetic progression.

68*. Let $d = (a, a+b)$. Thus we have $a = da_1$, $a+b = dc$, where $(a_1, c) = 1$ and $c > 1$ (since $d \leqslant a$, while $dc = a+b > a$). In view of $(a_1, c) = 1$ and of Euler's theorem, we have $c^{\varphi(a_1)} \equiv 1 (\mathrm{mod}\ a_1)$, hence $c^{n\varphi(a_1)} \equiv 1\ (\mathrm{mod}\ a_1)$ for integer n. Therefore $c^{n\varphi(a_1)}-1 = t_n a_1$ with some positive integer t_n which (since $c > 1$ and n is arbitrarily large) can be made arbitrarily large. Moreover, we have

$$a(ct_n+1)+b = da_1ct_n+dc = dc^{n\varphi(a_1)+1}.$$

The term $a(ct_n+1)+b$ of our progression (which can be arbitrarily large) has therefore those and only those prime divisors which are the prime divisors of the number $dc > 1$. Thus, in our progression there exist infinitely many terms with the same set of prime divisors, which was to be proved (see Pólya [14]).

69. From the theorem of Lejeune-Dirichlet it immediately follows that the theorem is true for $s = 1$. Suppose now that the theorem is true for some positive integer s. Thus, if $(a, b) = 1$, then there exists a number k_0 such that $ak_0+b = q_1 q_2 \ldots q_s$, where $q_1 < q_2 < \ldots < q_s$ are

primes. By the theorem of Lejeune-Dirichlet, there exist infinitely many integers k such that $ak+1 = q$ is a prime $> q_s$. For $t = q_1 q_2 \dots q_s k + k_0$ we get

$$at+b = q_1 q_2 \dots q_s ak + ak_0 + b = q_1 q_2 \dots q_s(ak+1) = q_1 q_2 \dots q_s q.$$

Therefore the theorem is true for $s+1$. By induction it follows that the theorem is true for every positive integer s, which was to be proved.

70. If p is a prime, then one of the numbers $p, p+10$, and $p+20$ must be divisible by 3 (since $p+10 \equiv p+1 \pmod 3$ and $p+20 \equiv p+2 \pmod 3$, and out of every three consecutive integers, one must be divisible by 3). Thus, if all our numbers are primes, then one of them, hence the least, must be equal to 3 and we have $p = 3$, $p+10 = 13$, $p+20 = 23$. Therefore there exists only one arithmetic progression of difference 10 consisting of three primes, namely $3, 13, 23$. We show easily that there is no arithmetic sequence of difference 10 consisting of four (or more) primes since if $p, p+10$, $p+20$, and $p+30$ were primes, then, as we showed, we would have $p = 3$, while $p+30 = 33 = 3 \cdot 11$ is not a prime.

REMARK. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that there exist infinitely many primes p such that $p+10$ is also a prime, for instance 7 and 17, 13 and 23, 19 and 29, 31 and 41, 37 and 47, 61 and 71, 73 and 83, 79 and 89.

71. There are no such progressions since one of the numbers $p, p+100$, and $p+200$ must be divisible by 3, and if these numbers are primes, then $p = 3$. But in this case $p+200 = 203 = 7 \cdot 29$ is composite.

REMARK. In a similar way we show that there are no progressions with difference 1000 formed of three or more primes since $1003 = 17 \cdot 53$ is composite. On the other hand, from a conjecture of A. Schinzel ([22]) it follows that there are infinitely many primes p such that $p+1000$ is also a prime, such as 13 and 1013, 19 and 1019, 31 and 1031, 61 and 1061, 97 and 1097, 103 and 1103, 1039 and 2039.

72*. If the difference of our progression were odd, then every second term of our progression would be even, which is impossible if our progression is to be formed of ten primes. Thus, the difference must be even. If the first term were equal 2, then the next term would be even, and hence composite. Therefore the first term of our progression is an odd prime, and it follows that all terms must be odd primes. We shall use the following theorem due essentially to M. Cantor: *If n terms of an arithmetic progression are odd* *primes, then the difference of the progression is divisible by every prime* $< n$ (see, for instance, [37, p. 121, theorem 5]). It follows, for $n = 10$, that the difference of our progression must be divisible by 2, 3, 5, and 7, hence by 210. We shall try first to find an arithmetic progression with the difference 210 formed of 10 primes.

Since the number 210 (the difference of our progression) is divisible by 2, 3, 5, and 7, the first term cannot be equal to any of these primes. It cannot be equal to 11 since in this case the second term would be $221 = 13 \cdot 17$. Thus the first term of the progression is > 11, and none of the terms is divisible by 11. The remainder of 210 upon division by 11 equals 1. If the first term would give the remainder > 1 upon dividing by 11, then with every next term this remainder would increase by 1, and one of the terms of the sequence would be divisible by 11, which is impossible. Therefore the first term of the sequence must give the remainder 1 upon dividing by 11, and being odd, it must be of the form $22k+1$, where k is a positive integer. The successive primes of this form are $23, 67, 89, 199, \ldots$.

The first term cannot be 23 since then the sixth term would be $1073 = 29 \cdot 37$. If the first term were 67, then the fourth term would be $697 = 17 \cdot 41$. If the first term were 87, then the second term would be $229 = 13 \cdot 23$. If, however, the first term is 199, then we obtain a progression of ten successive primes

$$199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089.$$

Thus we found a progression with difference 210 formed of ten primes.

Suppose now that we have an increasing arithmetic progression formed of ten primes, with the difference r other that 210. Then r must be divisible by 210 (by the theorem of Cantor) and different from 210, hence $r \geqslant 420$. In this case, however, the second term of our progression would exceed 420, hence it would exceed the second term 409 of the progression which we found, and obviously, the next terms would also exceed the terms of the progression which we found. Thus, the progression with first term 199 and difference 210 is the ten-term increasing progression formed of primes with the least possible last term.

REMARK. The longest increasing arithmetic progression formed of primes known up to date is the progression of thirteen terms with the first term 4943 and difference 60060 found by W. N. Seredinsky from Moscow. From a conjecture of A. Schinzel concerning primes it follows that there exist infinitely many progressions with difference 30030 formed of thirteen primes (see [22, p. 191, $C_{1,4}$]). However, no such progression has been found as yet.

73. For instance, the progression $30k+7$ ($k = 1, 2, 3, \ldots$) has the required property. Indeed, if we had $30k+7 = p+q$, then in view of the fact that $30k+7$ is odd, one of the numbers p and q would be even, and equal to 2 as a prime. Suppose $q = 2$; then $p = 30k+5 = 5(6k+1)$ which is impossible if p is to be a prime. If $30k+7 = p-q$, where p and q are primes, then we would have $q = 2$ and $p = 30k+9 = 3(10k+3)$ which is also impossible.

REMARK. One can prove (but the proof is difficult) that there exist infinitely many even numbers which can be represented both as sums and as differences of two primes. From a certain conjecture of A. Schinzel concerning primes it follows that there exist infinitely many odd numbers which are both sums and differences of two primes. See Sierpiński [34].

IV. PRIME AND COMPOSITE NUMBERS

74. It suffices to take $p = 3$, $q = 5$. If n is even and > 6, then we have $n-1 \geqslant 6$, and $p < q < n-1$. The numbers $n-p = n-3$ and $n-q = n-5$ as consecutive odd numbers are relatively prime.

75. There is only one such prime, namely 5. In fact, suppose that the prime r can be represented both as a sum and as a difference of two primes. We must have obviously $r > 2$, hence r is an odd prime. Being both a sum and a difference of two primes, one of them must be even, hence equal 2. Thus we must have $r = p+2 = q-2$, where p and q are primes. In this case, however, p, $r = p+2$, and $q = r+2$ would be three consecutive odd primes, and there is only one such a triplet: 3, 5, and 7 (since out of every three consecutive odd numbers one must be divisible by 3). We have therefore $r = 5 = 3+2 = 7-2$.

76. $n = 113, 139, 181$; $m = 20, 51, 62$.

77. By the well-known Fermat theorem, every prime of the form $4k+1$ is a sum of squares of two positive integers (see, for instance, [37, p. 205, Theorem 9]). Thus, if p is a prime of the form $4k+1$, then we have $p = a^2+b^2$, where a and b are positive integers (of course different since p is odd). Assume, for instance, $a > b$. Then $p^2 = (a^2-b^2)^2+(2ab)^2$, hence p is a hypotenuse of a rectangular triangle whose two other sides are a^2-b^2 and $2ab$. We

have, for example, $5^2 = 3^2+4^2$, $13^2 = 5^2+12^2$, $17^2 = 15^2+8^2$, $29^2 = 21^2+$ $+20^2$.

78. $13^2+1 = 7^2+11^2$, $17^2+1 = 11^2+13^2$, $23^2+1 = 13^2+19^2$, 31^2+1 $= 11^2+29^2$.

REMARK. The identity $(5x+13)^2+1 = (3x+7)^2+(4x+1)^2$ shows that if $p = 5x+13$, $q = 3x+7$, and $r = 4x+1$ are primes, then $p^2+1 = q^2+r^2$. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that there are infinitely many such systems of primes.

79. Note first that if p, q, r, s, and t are primes and $q^2+q^2 = r^2+s^2+t^2$, then each of the numbers p and q must be different from each of the numbers r, s, and t. In fact, if we had, for instance, $p = r$, then we would also have $q^2 = s^2+t^2$ which is impossible since this equation cannot have solution in primes q, s, and t. Indeed, the numbers s and t could not be both odd nor could they be both even (since in this case we would have $q = 2$, which is impossible in view of the fact that the right-hand side is > 4). If we had $s = 2$, then the number 4 would be a difference of two squares of positive integers which is impossible.

If $p^2+q^2 = r^2+s^2+t^2$, then it is not possible that all numbers p, q, r, s, t are odd. If p is even, then $p = 2$, and the numbers q, r, s, t are odd. Since the square of an odd number gives the remainder 1 upon dividing by 8, the left-hand side would give the remainder 5, and the right-hand side would give the remainder 3, which is impossible. If both p and q are odd, then the left-hand side gives the remainder 2 upon dividing by 8, while on the right-hand side one (and only one) of the numbers must be even, for instance $r = 2$. Then, however, the right-hand side would give the remainder 6 upon dividing by 8, which is impossible.

80*. We present the solution found by A. Schinzel. There is only one solution, namely $p = q = 2$, $r = 3$. To see that, we shall find all solutions of the equation $p(p+1)+q(q+1) = n(n+1)$ where p and q are primes and n is a positive integer. Our equation yields

$$p(p+1) = n(n+1)-q(q+1) = (n-q)(n+q+1),$$

and we must have $n > q$. Since p is a prime, we have either $p|n-q$ or $p|n+q+1$. If $p|n-q$, then we have $p \leqslant n-q$, which implies $p(p+1) \leqslant (n-q)(n-q+1)$, and therefore $n+q+1 \leqslant n-q+1$, which is impossible. Thus we have $p|n+q+1$, which means that for some positive integer k

$$n+q+1 = kp, \quad \text{which implies} \quad p+1 = k(n-q). \tag{1}$$

If we had $k = 1$, then $n+q+1 = p$ and $p+1 = n-q$, which gives $p-q = n+1$ and $p+q = n+1$, which is impossible. Thus, $k > 1$. From (1) we easily obtain

$$2q = (n+q)-(n-q) = kp-1-(n-q)$$
$$= k[k(n-q)-1]-1-(n-q) = (k+1)[(k-1)(n-q)-1].$$

Since $k \geqslant 2$, we have $k+1 \geqslant 3$. The last equality, whose left-hand side has positive integer divisors $1, 2, q$, and $2q$ only, implies that either $k+1 = q$ or $k+1 = 2q$. If $k+1 = q$, then $(k-1)(n-q) = 3$, hence $(q-2)(n-q) = 3$. This leads to either $q-2 = 1, n-q = 3$, that is $q = 3, n = 6, k = q-1 = 2$, and, in view of (1), $p = 5$, or else, $q-2 = 3$, $n-q = 1$, which gives $q = 5$, $n = 6$, $k = 4$, and in view of (1), $p = 3$.

On the other hand, if $k+1 = 2q$, then $(k-1)(n-k) = 2$, hence $2(q-1)(n-q) = 2$. This leads to $q-1 = 1$ and $n-q = 1$, or $q = 2, n = 3$, and, in view of (1), $p = 2$. Thus, for positive integers n, we have the following solutions in primes p and q: 1) $p = q = 2$, $n = 3$; 2) $p = 5$, $q = 3$, $n = 6$, and 3) $p = 3$, $q = 5$, $n = 6$. Only in the first solution all three numbers are primes.

REMARK. If we denote by $t_n = \frac{1}{2}n(n+1)$ the nth triangular number, then we can express the above theorem as follows: the equation $t_p+t_q = t_r$ has only one solution in prime numbers, namely $p = q = 2$, $r = 3$.

81*. Such numbers are, for instance, $p = 127$, $q = 3697$, $r = 5527$. It is easy to check (for instance, in the tables of prime numbers) that these numbers are primes, and that the numbers $p(p+1)$, $q(q+1)$, and $r(r+1)$ form an arithmetic progression. We shall present a method of finding such numbers.

From the identity

$$n(n+1)+(41n+20)(41n+21) = 2(29n+14)(29n+15)$$

it follows that for a positive integer n, the numbers

$$n(n+1), \quad (29n+14)(29n+15), \quad \text{and} \quad (41n+20)(41n+21)$$

form an arithmetic progression. If for some positive integer n the numbers n, $29n+14$, and $41n+20$ were all primes, we would have found a solution. Thus, we ought to take consecutive odd primes for n and check whether the numbers $29n+14$ and $41n+20$ are primes.

The least such number is $n = 127$ which leads to the above solution. We cannot claim, however, that in this manner we obtain all triplets of primes with the required properties.

REMARK. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that there exist infinitely many primes n such that the numbers $29n+14$ and $41n+20$ are also primes.

The above problem may be expressed as follows: find three triangular numbers with prime indices, which form an increasing arithmetic progression.

82. There is only one such positive integer, namely $n = 4$. In fact, for $n = 1$, the number $n+3 = 4$ is composite; for $n = 2$, the number $n+7 = 9$ is composite; for $n = 3$, the number $n+1 = 4$ is composite; and for $n > 4$, all our numbers exceed 5, and at least one of them is divisible by 5. The last property follows from the fact that the numbers $1, 3, 7, 9, 13$, and 15 give upon dividing by 5 the remainders $1, 3, 2, 4, 3$, and 0, hence all possible remainders. Thus, the numbers $n+1$, $n+3$, $n+7$, $n+9$, $n+13$, and $n+15$ give also all possible remainders upon dividing by 5; therefore at least one of them is divisible by 5, and as > 5, is composite. On the other hand, for $n = 4$ we get the prime numbers $5, 7, 11, 13, 17$, and 19.

REMARK. From a certain conjecture of A. Schinzel concerning the prime numbers ([22]) it follows that there exist infinitely many positive integers n such that each of the numbers $n+1$, $n+3$, $n+7$, $n+9$, and $n+13$ is a prime. This is, for instance, the case where $n = 4, 10, 100$. See also Sierpiński [33, p. 319, P_2].

83. $2 = 1^4+1^4$, $17 = 1^4+2^4$, $97 = 2^4+3^4$, $257 = 1^4+4^4$, $641 = 2^4+5^4$.

REMARK. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that there exist infinitely many primes which can be represented as sums of two fourth powers on positive integers, and, generally, for every positive integers n there exist infinitely many primes of the form $a^{2^n}+b^{2^1}$ where a and b are positive integers.

84. Let p_k denote the kth prime, and for positive integer n, let p_{k_n} be the largest prime $\leqslant 6n+1$. Since the numbers $6n+2 = 2(3n+1)$, $6n+3 = 3(2n+1)$, and $6n+4 = 2(3n+2)$ are composite, we have $p_{k_n+1} \geqslant 6n+5$, and $p_{k_n+1}-p_{k_n} \geqslant (6n+5)+(6n+1) = 4$, hence the primes p_{k_n} and p_{k_n+1} are not twin primes. Since $p_{k_n+1} \geqslant 6n+5$, and n can be arbitrary, there are infinitely many such numbers p_{k_n} and p_{k_n+1}. Note, however, that in the

pair p_{k_n}, p_{k_n+1} the number p_{k_n} may be the larger of a pair of twin primes, and p_{k_n+1} may be the smaller in another pair of twin primes. Thus, for $n = 1$, we get $p_{k_n} = 7$, which is the larger in the pair 5, 7 of twin primes, and $p_{k_n+1} = 11$, which is the smaller in the pair 11, 13. For $n = 2$, we get $p_{k_n} = 13$, which is the larger in the pair 11, 13, while $p_{k_n+1} = 17$, which is the smaller in the pair 17, 19. For $n = 17$, we get $p_{k_n} = 103 = 6 \cdot 17+1$, which is the larger in the pair 101, 103, while $p_{k_n+1} = 107$, which is the smaller in the pair 107, 109.

85. By the theorem of Lejeune–Dirichlet on arithmetic progressions, there exist infinitely many primes in the progression $15k+7$ ($k = 1, 2, \ldots$). None of these numbers can belong to a pair of twin primes since $(15k+7)+2 = 3(5k+3)$, and $(15k+7)-2 = 5(3k+1)$ are composite (since $k > 0$).

86. If for a positive integer n the number n^2-1 is a product of three different primes, then (in view of $2^2-1 = 3$) we have $n > 2$. Next, in view of the identity $n^2-1 = (n-1)(n+1)$, the number n must be even since otherwise the factors on the right-hand side would be even, and $2^2|n^2-1$. Moreover, the numbers $n-1$ and $n+1$ (which are both > 1 since $n > 2$) cannot be both composite since in this case n^2-1 could not be a product of three different primes. Thus, one of the numbers $n-1$ and $n+1$ must be a prime, and the other one must be a product of two primes. For $n = 4$, we get $n-1 = 3$, $n+1 = 5$, and this condition is not satisfied. Similarly, for $n = 6$, we get $n-1 = 5$, $n+1 = 7$; for $n = 8$, we have $n-1 = 7$, $n+1 = 9 = 3^2$. For $n = 10$, we have $n-1 = 3^2$, and for $n = 12$ we have $n-1 = 11, n+1 = 13$. For $n = 14$, we have $n-1 = 13, n+1 = 15 = 3 \cdot 5$. Thus, the least positive integer n for which n^2-1 is a product of three different primes is $n = 14$, for which $n^2-1 = 3 \cdot 5 \cdot 13$. Since $16^2-1 = 3 \cdot 5 \cdot 17$, we see that the next number which satisfies the required property is $n = 16$. Now, $18^2-1 = 17 \cdot 19$, $20^2-1 = 19 \cdot 21 = 3 \cdot 7 \cdot 19$, and the third such number is $n = 20$. Next, $22^2-1 = 3 \cdot 7 \cdot 23$, and the fourth of the required numbers is $n = 22$. Continuing in this way we find easily the fifth such number to be $n = 32$, for which $32^2-1 = 3 \cdot 11 \cdot 31$. Thus, the first five integers n for which n^2-1 is a product of three different primes are 14, 16, 20, 22, and 32.

Remark. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that there are infinitely many such numbers n. More generally, for every $s > 1$ there exist infinitely many positive integers n such that

n^2-1 is a product of s different primes. Obviously, for $s = 2$ the numbers $n-1$ and $n+1$ form then a pair of twin primes.

87. The five least positive integers n for which n^2+1 is a product of three different primes are $n = 13, 17, 21, 23$, and 27. We have $13^2+1 = 2\cdot 5\cdot 17$, $17^2+1 = 2\cdot 5\cdot 29$, $21^2+1 = 2\cdot 13\cdot 17$, $23^2+1 = 2\cdot 5\cdot 53$. For $n = 112$, we have $112^2+1 = 5\cdot 13\cdot 193$.

REMARK. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that for every s there exist infinitely many positive integers n such that n^2+1 is a product of s different primes.

88*. Suppose that each of the numbers n, $n+1$, and $n+2$, where $n > 7$, has only one prime divisor. None of these numbers is divisible by 6, which implies that n must be of the form $6k+1$, $6k+2$ or $6k+3$, where k is a positive integer.

If $n = 6k+1$, then the number $6k+2$, being even and having only one prime divisor, must be of the form 2^m; now, since $n > 7$ which implies $6k+2 = n+1 > 8$, the number m must be > 3. The number $n+2 = 6k+3$ is divisible by 3, and if it has only one prime divisor, it must be of the form 3^s. Since $6k+3 = n+3 > 9$, the number s must be > 2. Moreover, we have $3^s-2^m = 1$; this equation, however, has only two integer solutions, namely $s = m = 1$ and $s = 2$, $m = 3$ (see Problem 185).

If $n = 6k+2$, then $n = 2^m$ and $n+1 = 6k+3 = 3^s$ where $m > 2$ (since $n > 6$). We get $3^s-2^m = 1$, which is impossible for $m > 3$.

Finally, if $n = 6k+3$, then $n = 3^s$, $n+1 = 2^m$ and in view of $n > 7$ we get $s \geqslant 2$, $m > 3$, while the equation $2^m-3^s = 1$ has only one integer solution, namely $m = 2$, $s = 1$ (see Problem 184).

Thus, the assumption that for integer $n > 7$ none of the numbers n, $n+1$, and $n+2$ has more than one prime divisor led to a contradiction. On the other hand, for $n = 7$ we have $n+1 = 2^3$, $n+2 = 3^2$, and each of the numbers n, $n+1$, and $n+2$ has only one prime divisor.

89. $n = 33$ $(n = 3\cdot 11, n+1 = 2\cdot 17, n+2 = 5\cdot 7)$,
$n = 85$ $(n = 5\cdot 17, n+1 = 2\cdot 43, n+2 = 3\cdot 29)$,
$n = 93$ $(n = 3\cdot 31, n+1 = 2\cdot 47, n+2 = 5\cdot 19)$,
$n = 141$ $(n = 3\cdot 47, n+1 = 2\cdot 71, n+2 = 11\cdot 13)$,
$n = 201$ $(n = 3\cdot 67, n+1 = 2\cdot 101, n+2 = 7\cdot 29)$.

There are no four consecutive positive integers such that each of them is a product of two different primes since out of each four consecutive numbers one must be divisible by 4. An example of four consecutive positive integers such that each of them has exactly two different prime divisors are numbers $33 = 3\cdot 11$, $34 = 2\cdot 17$, $35 = 5\cdot 7$, $36 = 2^2\cdot 3^2$.

REMARK. We cannot prove that there exist infinitely many positive integers n such that each of the numbers n, $n+1$, and $n+2$ is a product of two different primes; this follows from a certain conjecture of A. Schinzel concerning primes. See [22, p. 197, consequence C_7].

90. Suppose that there exist infinitely many positive integers n such that both n and $n+1$ have only one prime divisor. We can assume $n > 1$, and since one of the numbers n and $n+1$ is even and the other odd, we must have for some odd prime p the relation $p^k-2^m = \pm 1$ where k and m are positive integers. Thus we get $p^k = 2^m\pm 1$. Since a Mersenne number > 1 cannot be equal to a power with exponent > 1 of any prime (see [37, p. 335, theorem 2]), in the case $p^k = 2^m-1$ we must have $k = 1$, and $2^m-1 = p$ is a Mersenne prime number.

On the other hand, if $p^k = 2^m+1$, then either $k = 1$, in which case $p = 2^m+1$ is either equal to 3 or (if $m > 1$) is a Fermat prime, or else we have $k > 1$ in which case we get $2^m = p^k-1 = (p-1)(p^{k-1}+p^{k-2}+ \ldots +1)$. Since the left-hand side is even, the number k must be even; thus $k = 2l$, where l is a positive integer, and we have $2^m = (p^l-1)(p^l+1)$. Therefore the numbers p^l-1 and p^l+1 are powers of 2 which differ by 2, which implies that $p^l-1 = 2$, $p^l+1 = 4$, hence $p^l = 3$, and $p = 3$, $2^m = 2\cdot 4 = 8$, and $m = 3$, which yields $3^2 = 2^3+1$.

We proved therefore that if for $n > 8$ the numbers n and $n+1$ have one prime divisor each, then either n is a Mersenne prime or $n+1$ is a Fermat prime.

Conversely, if $M_m = 2^m-1$ is a Mersenne prime, then the numbers M_m and $M_m+1 = 2^m$ have one prime divisor each. If $F_k = 2^{2^k}+1$ is a Fermat prime, then each of the numbers $F_k-1 = 2^{2^k}$ and F_k has one prime divisor each, which completes the proof of our theorem. See [26, p. 209].

REMARK. Up to date we know only 29 positive integers n such that n and $n+1$ have one prime divisor each. The least five are $n = 2, 3, 4, 7, 8$, and the largest of them is $n = 2^{11213}-1$.

91. We have $2^2-1 = 3$, $2^4-1 = 3\cdot 5$ and $2^{2k}-1 = (2^k-1)(2^k+1)$. If for $n = 2k > 4$ the number $2^{2k}-1$ were equal to the product of two primes, then the numbers 2^k-1 and 2^k+1 would have to be primes. Since these

numbers are consecutive odd numbers, we must have $2^k-1 \leqslant 5$, hence $k \leqslant 2$. In view of $k > 1$, we must have $k = 2$, contrary to the assumption that $k > 2$. Therefore the numbers 2^n-1 are, for n even and > 4, equal to the product of at least three natural numbers.

For odd n, we have $2^3-1 = 7$, $2^5-1 = 31$, $2^7-1 = 127$, $2^9-1 = 7\cdot 73$, $2^{11}-1 = 23\cdot 89$, $2^{13}-1 = 8191$, which is a prime, $2^{15}-1 = 7\cdot 31\cdot 151$, the numbers $2^{17}-1$ and $2^{19}-1$ are primes, and $2^{21}-1 = 7\cdot 127\cdot 337$, while $2^{23}-1$ is already $> 10^6$. Thus, all positive integers of the form 2^n-1 which are $< 10^6$ and which are equal to the product of two primes are $2^4-1 = 3\cdot 5$, $2^9-1 = 7\cdot 73$ and $2^{11}-1 = 23\cdot 89$.

REMARK. The Mersenne numbers exceeding million which are known to be the product of two different primes are numbers $M_n = 2^n-1$ for $n = 23, 37, 49, 67$, and 101. We do not know whether there are infinitely many such numbers.

92. Since $k \geqslant 3$, we have $p_1 p_2 \dots p_k \geqslant p_1 p_2 p_3 = 2\cdot 3\cdot 5 > 6$, and in view of Problem 47, we have $p_1 p_2 \dots p_k = a+b$ where a and b are > 1 and relatively prime, hence also prime with respect to the product $p_1 p_2 \dots p_k$. Since a and b are > 1, they have different prime divisors; let $p|a$, $q|b$, and suppose that $p < q$. Since $(p, p_1 p_2 \dots p_k) = 1$, we have $p \geqslant p_{k+1}$, and in view of $q > p$, also $q \geqslant p_{k+2}$. Therefore $p+q \leqslant a+b$ and we have $p_{k+1}+p_{k+2} \leqslant \leqslant p_1 p_2 \dots p_k$, which was to be proved.

93. Let m denote an arbitrary integer > 3, and let n be an integer such that $n > p_1 p_2 \dots p_m$. Then there exists an integer $k \geqslant m \geqslant 4$ such that

$$p_1 p_2 \dots p_k \leqslant n < p_1 p_2 \dots p_k p_{k+1}. \tag{1}$$

If we had $q_n \geqslant p_{k+1}+1 > p_{k+1}$, then in view of the definition of the number q_n, each of the numbers $p_1, p_2, \dots, p_{k+1}$ would be a divisor of n, hence $n \geqslant p_1 p_2 \dots p_{k+1}$ contrary to (1).

We have therefore $q_n < p_{k+1}+1 < p_k+p_{k+1}$ and, in view of $k \geqslant 4$ and Problem 92, we get $q_n \leqslant p_1 p_2 \dots p_{k-1}$ which gives, using (1), the relation

$$\frac{q_n}{n} < \frac{1}{p_k} \leqslant \frac{1}{k} \leqslant \frac{1}{m}.$$

We proved that for arbitrary $m > 3$ for $n > p_1 p_2 \dots p_m$ we have $q_n/n < 1/m$ which shows that the ratio q_n/n tends to zero as n tends to infinity, which was to be shown.

94. Let n be an integer > 4. We have either $n = 2k$ where $k > 2$ or

$n = 2k+1$ where $k > 1$. If $n = 2k$, where $k > 2$, then by Chebyshev's theorem there exists a prime p such that $k < p < 2k$, and we have $p > 2$ since $p > k > 2$. Thus $n = 2k < 2p < 4k = 2n$, and in view of $p > 2$, the number $2p$ is a product of two different primes, and $n < 2p < 2n$. If $n = 2k+1$ where $k \geqslant 2$, then by Chebyshev's theorem there exists a prime p such that $k < p < 2k$, hence $3 \leqslant k+1 \leqslant p < 2k$ and $n = 2k+1 < 2k+2 \leqslant 2p < 4k < 4k+2 = 2n$, and again we have $n < 2p < 2n$, while $2p$ is a product of two different primes.

Let now n be an integer > 15. If $n = 16, 17, \ldots, 29$, then the number $30 = 2\cdot 3\cdot 5$ lies between n and $2n$. We can therefore assume that $n \geqslant 30$. We have then $n = 6k+r$, where k is an integer $\geqslant 5$, and r is the remainder obtained from dividing n by 6, i.e. r satisfies the inequality $0 \leqslant r \leqslant 5$. By Chebyshev's theorem, there exists a prime p such that $k < p < 2k$, hence $p > 5$ and $k+1 \leqslant p < 2k$. It follows that $n = 6k+r < 6(k+1) \leqslant 2\cdot 3\cdot p < 12k \leqslant 2n$, hence $n < 2\cdot 3\cdot p < 2n$, and $2\cdot 3\cdot p$ is a product of three different primes.

95. Let p_k denote the kth consecutive prime, and let s be an arbitrary integer > 1. Let $n > p_1p_2\ldots p_s$. We shall show that between n and $2n$ there exists at least one positive integer which is a product of s different primes.

Let $n = kp_1p_2\ldots p_{s-1}+r$, where r is the remainder of dividing n by $p_1p_2\ldots p_{s-1}$, hence, in view of $n > p_1p_2\ldots p_s$, we have $k > p_s$ and $0 \leqslant r < p_1p_2\ldots p_{s-1}$. By the Chebyshev theorem, there exists a prime p such that $k < p < 2k$, hence $p > p_s$ and $k+1 \leqslant p < 2k$. It follows that $n = p_1p_2\ldots p_{s-1}k+r < p_1p_2\ldots p_{s-1}(k+1) \leqslant p_1p_2\ldots p_{s-1}p < 2p_1p_2\ldots p_{s-1}k \leqslant 2n$, hence $n < p_1p_2\ldots p_{s-1}p < 2n$. The number $p_1p_2\ldots p_{s-1}p$ is, in view of $p > p_s$, equal to the product of s different primes.

REMARK. An elementary proof of the Chebyshev theorem is given in Sierpiński [37, p. 137, theorem 8].

96. We easily check that the nth term of our sequence equals to $\frac{1}{3}(10^n-7)$. We have $10^2 \equiv 15 \equiv -2 \pmod{17}$, hence $10^8 \equiv 16 \equiv -1 \pmod{17}$. Thus $10^9 \equiv -10 \equiv 7 \pmod{17}$, and since $10^{16} \equiv 1 \pmod{17}$, we get $10^{16k+9} \equiv 7 \pmod{17}$ for $k = 0, 1, 2, \ldots$. Thus $17 | \frac{1}{3}(10^{16k+9}-7)$, and since the numbers $\frac{1}{3}(10^{16k+9}-7)$ for $k = 0, 1, 2, \ldots$ are $\geqslant \frac{1}{3}(10^9-7) > 17$, they are all composite.

As it was checked by A. Mąkowski in the tables of primes, the numbers $\frac{1}{3}(10^n-7)$ are primes for $n = 1, 2, 3, 4, 5, 6, 7$, and 8. The least composite number of this form is therefore $\frac{1}{3}(10^9-7) = 333333331$.

The problem arises whether there are other composite numbers of the considered form, besides those which we found. We have $10^2 \equiv 5 \pmod{19}$, hence $10^4 \equiv 25 \equiv 6 \pmod{19}$ and $10^{12} \equiv 6^3 \equiv 7 \pmod{19}$, and since $10^{18k} \equiv 1 \pmod{19}$, we get $19|\frac{1}{3}(10^{18k+12}-7)$ for $k = 0, 1, 2, \ldots$. Thus, for instance, the number $\frac{1}{3}(10^{12}-7)$ is composite. We do not know, however, whether there are other primes of this form besides the ones which were given above, and if so, whether there are infinitely many of them.

97. The number $n = 5$, since $1^4+2^4 = 17$, $2^4+3^4 = 97$, $3^4+4^4 = 337$, and $4^4+5^4 = 881$ are primes, while $5^4+6^4 = 1921 = 17\cdot 113$.

98. All numbers of the form $10^{6k+4}+3$, where $k = 0, 1, 2, \ldots$, are composite since they are divisible by 7. In fact, we have $10^4 \equiv 4 \pmod 7$, and by the Fermat theorem, $10^6 \equiv 1 \pmod 7$. Thus, for integer k we have $10^{6k+4}+3 \equiv 10^4+3 \equiv 4+3 \equiv 0 \pmod 7$.

REMARK. We do not know whether among numbers of the form 10^n+3 for $n = 1, 2, \ldots$ there exist infinitely many primes. Such numbers are prime for $n = 1$ and $n = 2$, but are composite for $n = 3$ and $n = 4$ (since $1003 = 17\cdot 59$ and $7|10^4+3$).

99. For integer n we have the identity

$$2^{4n+2}+1 = (2^{2n+1}-2^{n+1}+1)\,(2^{2n+1}+2^{n+1}+1). \tag{1}$$

Since $5|2^2+1|2^{4n+2}+1$ and for integer $n > 1$ we have $2^{2n+1}-2^{n+1}+1 = 2^{n+1}(2^n-1)+1 \geqslant 2^3\cdot 3+1 = 25$, it follows that at least one of the factors on the right-hand side is divisible by 5, and (for $n > 1$) upon division by 5 it gives the ratio exceeding one. Therefore $\frac{1}{5}(2^{4m+2}+1)$ is, for $n = 1, 2, \ldots$, equal to the product of two integers > 1, hence is composite.

100. Let m be an arbitrary integer > 1, and let $n = m!+k$, where $k = 2, 3, \ldots, m$. We have $k < m!+k$ and $k|m!+k$, hence $2^k-1 < 2^{m!+k}-1$ and $2^k-1|2^{m!+k}-1$. Thus the numbers $2^{m!+k}-1$ are composite for $k = 2, 3, \ldots, m$, hence for $m-1$ consecutive terms of the sequence 2^n-1.

101. In order to obtain a prime from number 200, one has to change its last digit into an odd number. We have, however, $3|201$, $7|203$, $5|205$, $3|207$, and $11|209$. Thus, by changing only one digit, one cannot obtain a prime from number 200.

REMARK. We do not know whether, by changing two digits, one can obtain a prime out of every number. On the other hand, it is easy to prove that there exist infinitely many positive integers n such that no change of its (decimal) digit would result in a prime. For instance, for $n = 2310k-210$ (where $k = 1, 2, \ldots$) we would have to change its last digit (that is, 0), (obviously, to one of the numbers 1, 3, 7 or 9) while it is easy to see that $11|n+1$, $3|n+3$, $7|n+7$ and $3|n+9$.

102. Suppose that the theorem T is true. Theorem T_1 is obviously true for $n = 2$ and $n = 3$. Assume that n is an integer > 3. If n is even, that is, $n = 2k$, then, in view of $n > 3$, we have $k > 1$ and by theorem T there exists a prime p such that $k < p < 2k$, hence $p < n < 2p$, and p divides only one factor in the product $n! = 1 \ldots n$. If $n = 2k+1$ where (in view of $n > 1$), k is an integer > 1, then by theorem T there exists a prime p such that $k < p < 2k < n$, hence $k+1 \leqslant p$ which implies $2k+1 < 2p$ and $p < n < 2p$. As before, p enters in the expansion of $n!$ into primes with exponent 1. We showed, therefore, that T implies T_1.

Suppose now that T_1 holds, and let n denote an integer > 1. By theorem T_1, there exists a prime p which enters in the expansion of $(2n)!$ into primes with the exponent 1. We have, therefore $p \leqslant 2n < 2p$ (since if we had $2p \leqslant 2n$, then in the product $(2n)! = 1 \cdot 2 \cdot \ldots \cdot (2n-1)2n$ we would have factors p and $2p$, and p would enter with the exponent $\geqslant 2$ contrary to theorem T_1). We have, therefore, $n < p < 2n$ (since, in view of $n > 1$, the equation $p = 2n$ is impossible for prime p). Thus, theorem T follows from theorem T_1, which shows that T and T_1 are equivalent.

103. In the expansion of $11!$ into primes, the primes 7 and 11 enter obviously with exponents 1. We may, therefore, assume that $n > 11$, which implies in the case $n = 2k$, as well as in the case $n = 2k+1$, that $k > 5$. By the theorem we are going to use, there exist two primes p, and $q > p$ such that $k < p < q < 2k$. We obtain therefore at least $p < q < n$, and $p \geqslant k+1$, which implies $2q > 2p > n$. Thus, both primes p and q enter in the expansion of $n!$ with exponent 1.

As regards the number $10!$, only the prime 7 enters its expansion with exponent 1.

104. Let n be a given positive integer. By the theorem of Lejeune-Dirichlet on arithmetic progressions, there exists a prime of the form $p = 6^n k + 2 \cdot 3^{2^{n-1}} - 1$, where k is a positive integer. It follows (in view of $2^{n-1} \geqslant n$ for positive integer n), that $3^n | p+1$, and the number $p+1$ has more than n different positive integer divisors (for instance, $1, 3, 3^2, \ldots, 3^n$). On the other hand, by Euler's theorem we have $3^{\varphi(2^n)} \equiv 1 \pmod{2^n}$, hence $2^n | 3^{2^{n-1}} - 1$,

which implies that $2^n | p-1$ and the number $p-1$ has more than n different positive integer divisors (for instance, $1, 2, 2^2, \ldots, 2^n$).

105. $p = 131$. We have here $p-1 = 2\cdot 5\cdot 13$ and $p+1 = 2^2\cdot 3\cdot 11$.

106*. Let n be a given positive integer, and let p_i denote the ith prime. By the Chinese remainder theorem, there exists a positive integer b such that $b \equiv 1 \pmod{p_1p_2 \ldots p_n}$, $b \equiv -1 \pmod{p_{n+1}p_{n+2} \ldots p_{2n}}$ and $b \equiv -2 \pmod{p_{2n+1}p_{2n+2} \ldots p_{3n}}$.

We have $(b, p_1p_2 \ldots p_{3n}) = 1$, and by the theorem of Dirichlet, there exists a positive integer k such that the number $p = p_1p_2 \ldots p_{3n}k+b$ is a prime. We shall have then

$$p_1p_2 \ldots p_n | p-1, \quad p_{n+1}p_{n+2} \ldots p_{2n} | p+1, \quad \text{and} \quad p_{2n+1}p_{2n+2} \ldots p_{3n} | p+2,$$

hence each of the numbers $p-1$, $p+1$, and $p+2$ has at least n different prime divisors.

107. Let p_k denote the kth prime. For positive integer n, s and m, write

$$a_j = p_{(j-1)n+1}^s p_{(j-1)n+2}^s \ldots p_{jn}^s \quad \text{for} \quad j = 1, 2, \ldots, m.$$

We have $(a_i, a_j) = 1$ for $1 \leqslant i < j \leqslant m$, and by the Chinese remainder theorem, there exists a positive integer x such that

$$x \equiv -j \pmod{a_j} \quad \text{for} \quad j = 1, 2, \ldots, m.$$

Thus we have $a_j | x+j$ for $j = 1, 2, \ldots, m$, which implies that each of the numbers $x+j$ ($j = 1, 2, \ldots, m$) has at least m different prime divisors, each of these divisors appearing in at least sth power. Therefore the sequence $x+1, x+2, \ldots, x+m$ satisfies the required conditions.

108. If for a positive integer m the number $m!$ is divisible by a prime p, then p must divide at least one of the factors in the product $m! = 1\cdot 2\cdot \ldots \cdot m$, hence we must have $p \leqslant m$. Thus, if $m!$ is divisible by an integer $n > m$, then n must be composite. It follows that if for some integer $n > 1$ the number $(n-1)!$ were divisible by n or $n+2$, then n or $n+2$ would be composite. Thus, the condition is necessary.

Suppose now that for an odd $n > 1$ the number $(n-1)!$ is not divisible by n or $n+2$. We shall show that the numbers n and $n+2$ are primes. It suffices to assume that $n \geqslant 7$ since for $n = 3$ and $n = 5$ the numbers n and $n+2$ are primes. If n were composite, we would have $n = ab$, where a and b are positive integers $< n$, hence $a \leqslant n-1$ and $b \leqslant n-1$. Thus a and b would appear as factors in the product $(n-1)! = 1\cdot 2\cdot \ldots \cdot(n-1)$. In case $a \neq b$, we would have $n = ab|(n-1)!$, contrary to the assumption. In case $a = b$, we would have $n = a^2$, and since n is odd and > 1, $a \geqslant 3$, which implies $n = a^2 \geqslant 3a > 2a$, hence $2a \leqslant n-1$. Thus, a and $2a$ are different factors in the product $(n-1)! = 1 \cdot 2 \cdot \ldots \cdot (n-1)$, hence $n = a^2|(n-1)!$ contrary to the assumption. Thus, n is a prime.

If the number $n+2$ were composite, we would have $n+2 = ab$, where a and b are integers > 1. Since n is odd, the numbers a and b are odd, and therefore $\geqslant 3$. Next, since $n \geqslant 7$, we have $a \leqslant \frac{1}{3}(n+2) \leqslant \frac{1}{2}(n-1)$, and we have $2a \leqslant n-1$. In a similar way we show that $2b \leqslant n-1$. If a and b are different, then they appear as different factors in the product $1 \cdot 2 \cdot \ldots \cdot (n-1) = (n-1)!$, which implies that $n+2 = ab|(n-1)!$, contrary to the assumption. If $a = b$, then a and $2b$ are different factors in the product $(n-1)!$, hence $n+2|2ab|(n-1)!$, again contrary to the assumption.

The condition of the theorem is therefore sufficient.

109. Let m be a given positive integer. We have $(10^m, 10^m-1) = 1$ and, by the theorem on arithmetic progression, there exists a positive integer k such that $p = 10^m k + 10^m - 1$ is a prime. Obviously, the last $m-1$ digits of this number are equal to 9, which implies that the sum of all digits of this number is m.

REMARK. A. Mąkowski noticed that the theorem remains valid for an arbitrary scale of notation $g > 1$; for the proof, it suffices to replace in the above proof the number 10 by g.

See Sierpiński [31], and Erdös [8].

We do not know if the sum of digits of a prime tends to infinity as the prime increases.

110. Let m be a given positive integer. Since $(10^{m+1}, 1) = 1$, by the theorem on arithmetic progression, there exists a positive integer k such that $p = 10^{m+1}k+1$ is a prime. The last m digits of the number p are, obviously, m zeros and one. Thus, the prime p in decimal system has at least m digits equal to zero, which was to be proved.

REMARK. We do not know whether for every positive integer m there exists a prime, which in decimal system has exactly m zeros. For $m = 1$, the least such prime is 101; for $m = 2$, it is 1009.

111. If p is a prime, then the sum of all positive integer divisors of p^4 equals $1+p+p^2+p^3+p^4$. If $1+p+p^2+p^3+p^4 = n^2$ where n is a positive integer, then we have obviously $(2p^2+p)^2 < (2n)^2 < (2p^2+p+2)^2$, and it

follows that we must have $(2n)^2 = (2p^2+p+1)^2$. Thus, $4n^2 = 4p^4+$ $+4p^3+5p^2+2p+1$, and since $4n^2 = 4(p^4+p^3+p^2+p+1)$, we have p^2-2p- $-3 = 0$, which implies $p|3$, hence $p = 3$. If fact, for $p = 3$ we obtain $1+3+$ $+3^2+3^3+3^4 = 11^2$. Thus, there exists only one prime p, namely $p = 3$, satisfying the conditions of problem.

112. A prime number p has only two positive integer divisors, namely 1 and p. Thus, if the sum of all positive integer divisors of a prime p is equal to the sth power of a positive integer n, then $1+p = n^s$, which implies

$$p = n^s-1 = (n-1)\,(n^{s-1}+n^{s-2}+\ldots+1).$$

We have $n > 1$, and for $s \geqslant 2$ the first factor of the product on the right-hand side is smaller than the second. We obtain therefore a representation of a prime p into a product of two positive integer factors, the first of which is smaller than the second. It follows that the first factor must be equal to 1, hence $n-1 = 1$, or $n = 2$, and $p = 2^s-1$. Thus, for every integer $s \geqslant 2$ there exists at most one prime satisfying the conditions of the problem, and such a prime exists if and only if the number 2^s-1 is a prime. For $s = 2$, we obtain the number 3; for $s = 3$, the number 7; for $s = 5$, the number 31; and for $s = 7$, the number 127. For $s = 4, 6, 8$, and 10, there are no such primes since the numbers $2^4-1 = 3\cdot 5$, $2^6-1 = 3^2\cdot 7$, $2^8-1 = 3\cdot 5\cdot 17$ and $2^{10}-1 = 3\cdot 11\cdot 31$ are composite.

113. For primes $p > 5$, we have

$$2 < \frac{p-1}{2} < p-1$$

which implies

$$(p-1)^2 = 2\frac{p-1}{2}(p-1)|(p-1)!.$$

If for a prime $p > 5$ and some positive integer m we had

$$(p-1)!+1 = p^m, \tag{1}$$

then we would have

$$(p-1)^2|p^m-1$$

and dividing both sides by $p-1$ we would get

$$p-1|p^{m-1}+p^{m-2}+\ldots+p+1. \tag{2}$$

However, $p-1|p^k-1$, hence $p^k \equiv 1 \pmod{p-1}$ for $k = 0, 1, 2, \ldots$, which implies that $p^{m-1}+p^{m-2}+\ldots+p+1 \equiv m \pmod{p-1}$, and in view of (2), we find $p-1|m$, hence $m \geqslant p-1$. We get, therefore,

$$p^m \geqslant p^{p-1} > (p-1)^{p-1} > (p-1)!,$$

hence $p^m > (p-1)!+1$, contrary to (1).

114. By the Liouville theorem (see Problem 113), if p is a prime > 5, then we cannot have $(p-1)!+1 = p^m$ for positive integer m. The odd number $(p-1)!+1 > 1$ has therefore an odd prime divisor $q \neq p$, and from $q|(p-1)!+1$ it follows that $q > p-1$, hence (in view of $q \neq p$), we have $q > p$. Since p can be arbitrarily large, there exist infinitely many primes q such that for some $p < q$ we have $q|(p-1)!+1$, which was to be proved.

115*. We shall give the proof of A. Schinzel. Let a denote an arbitrary positive integer, and let k be an integer $\neq 1$. Further, put $k-1 = 2^s h$, where 2^s is the highest power of 2 which divides $k-1$, and h is an odd number, positive or negative. Choose a positive integer m so that $2^{2^m} > a-k$, and let l denote an integer such that $l \geqslant s$, and $l \geqslant m$. If the number $2^{2^l}+k \geqslant 2^{2^m}+$ $+k > a$ were composite, we would have a composite number of the desired form, and $> a$. Suppose then that the number $p = 2^{2^l}+k$ is a prime. In view of $l \geqslant s$ and $k-1 = 2^s h$ we get $p-1 = 2^{2^l}+k-1 = 2^s h_1$, where h_1 is odd and > 0. By the Euler theorem we have $2^{\varphi(h_1)} \equiv 1 \pmod{h_1}$, hence also (in view of $p-1 = 2^s h_1$) $2^{s+\varphi(h_1)} \equiv 2^s \pmod{p-1}$. Since $l \geqslant s$, we get $2^{l+\varphi(h_1)}$ $\equiv 2^l \pmod{p-1}$. By the Fermat theorem we obtain

$$2^{2^{l+\varphi(h_1)}}+k \equiv 2^{2^l}+k \equiv 0 \pmod{p},$$

and, in view of $2^{l+\varphi(h_1)} > 2^l$ we get

$$2^{2^{l+\varphi(h_1)}}+k > 2^{2^l}+k = p.$$

Thus, the number $2^{2^{l+\varphi(h_1)}}+k$ is composite and $> a$ since $p = 2^{2^l}+k \geqslant 2^{2^m}+$ $+k > a$, which completes the proof. This proof fails for $k = 1$ since we do not know if there exist infinitely many composite Fermat numbers.

Let us note that the weaker version of the theorem, asserting that for every integer k there exists at least one integer n such that $2^{2^n}+k$ is composite, has been obtained in 1943 by J. Reiner as a special case of a rather complicated theorem; see [16]. To obtain this weaker version from our theorem it suffices to note that (for $k = 1$) the number $2^{2^5}+1$ is composite, namely divisible by 641.

116. For instance, all numbers $k = 6t-1$, where $t = 1, 2, \ldots$, are of this form since for every positive integer n the number 2^{2^n} gives the remainder 1 upon dividing by 3, hence the number $2^{2^n}+k = 2^{2^n}-1+6t$ is divisible by 3 and > 3, thus composite.

117. (a) For positive integer n, the number $2^{2n}-1$ is divisible by 3, hence the number $2^{2n+1}-2 = 2(2^{2n}-1)$ is divisible by 6, and we have $2^{2n+1} = 6k+$ $+2$ where k is a positive integer. It follows that

$$2^{2^{2n+1}}+3 = (2^6)^k \cdot 2^2+3 \equiv 2^2+3 \equiv 0 \pmod 7,$$

and $7|2^{2^{2n+1}}+3$ for $n = 1, 2, \ldots$. Since $2^{2^{2n+1}}+3 \geqslant 2^{2^2}+3 > 7$, the numbers of the form $2^{2^{2n+1}}+3$ are composite for $n = 1, 2, \ldots$.

(b) for positive integer n, we have $2^{4n}-1 = 16^n-1 \equiv 0 \pmod 5$, which implies that $10|2^{4n+1}-2$. Therefore $2^{4n+1} = 10k+2$ where k is a positive integer and $2^{2^{4n+1}}+7 \equiv (2^{10})^k \cdot 2^2+7 \equiv 2^2+7 \equiv 0 \pmod{11}$. Thus we have $11|2^{2^{4n+1}}+7$, and since $2^{2^{4n+1}}+7 \geqslant 2^{2^5}+7 > 11$, the numbers of the form $2^{2^{4n+1}}+7$ are all composite for $n = 1, 2, \ldots$.

(c) For positive integer n, we have $2^{6n} \equiv (2^6)^n \equiv 1 \pmod 7$, which implies $7|2^{6n}-1$ and $28|2^{6n+2}-2^2$. Thus, $2^{6n+2} = 28k+4$, where k is a positive integer. It follows that $2^{2^{6n+2}} = (2^{28})^k \cdot 2^4 \equiv 16 \pmod{29}$, that is, $29|2^{2^{6n+2}}+13$, and since $2^{2^{6n+2}}+13 \geqslant 2^{2^8}+13 > 29$, the numbers of the form $2^{2^{6n+2}}+13$ are composite for $n = 1, 2, \ldots$.

(d) For positive integer n, we have $(2^{10})^n \equiv 1 \pmod{11}$, which implies that $22|2^{10n+1}-2$ and $2^{10n+1} = 22k+2$, where k is a positive integer. It follows that $2^{2^{10n+1}} = (2^{22})^k \cdot 2^2 \equiv 4 \pmod{23}$, and $23|2^{2^{10n+1}}+19$. Since $2^{2^{10n+1}}+19 > 23$ for all $n = 1, 2, \ldots$, the numbers $2^{2^{10n+1}}+19$ are all composite.

(e) For positive integer n, we have $2^{6n} = (2^3)^{2n} \equiv (-1)^{2n} \equiv 1 \pmod 9$, hence $9|2^{6n}-1$, and $36|2^{6n+2}-2^2$, which implies that $2^{6n+2} = 36k+4$ for a positive integer k. It follows that $2^{2^{6n+2}} = (2^{36})^k \cdot 16 \equiv \pmod{37}$, hence $37|2^{2^{6n+2}}+21$, and since $2^{2^{6n+2}}+21 > 37$, for $n = 1, 2, \ldots$, the numbers $2^{2^{6n+2}}+21$ are composite for $n = 1, 2, \ldots$.

REMARK. We know of no integer k such that we could prove that among the numbers $2^{2^n}+k$ $(n = 1, 2, \ldots)$ there exist infinitely many primes.

118*. As we know, the numbers $F_n = 2^{2^n}+1$ are primes for $n = 0, 1, 2, 3, 4$, while the number $F_5 = 641p$, where p is a prime $> 2^{16}+1 = F_4$. We have also $(p, 2^{32}-1) = 1$ since $p|F_5$ implies $(p, F_5-2) = 1$. By the Chinese remainder theorem, there exist infinitely many positive integers k satisfying the congruences

$$k \equiv 1 \left(\bmod (2^{32}-1)641\right) \quad \text{and} \quad k \equiv -1 \pmod{p}. \tag{1}$$

We shall show that if k is an integer $> p$, satisfying congruences (1), then the numbers $k \cdot 2^n+1$ $(n = 1, 2, \ldots)$ are all composite.

The number n can be represented in the form $n = 2^m(2t+1)$, where m and t are integers $\geqslant 0$. Suppose first that m is one of the numbers $0, 1, 2, 3$, or 4. In view of (1) we shall have

$$k \cdot 2^n+1 \equiv 2^{2^m(2t+1)}+1 \pmod{2^{32}-1}, \tag{2}$$

and since for $m = 0, 1, 2, 3, 4$ we have $F_m|2^{32}-1$ and $F_m|2^{2^m(2t+1)}$, we obtain, in view of (2), that $F_m|k \cdot 2^n+1$. Since $k \cdot 2^n+1 > p > F_4$ (because $k > p$), the number $k \cdot 2^n+1$ is composite.

If $m = 5$, then by (1) we get $k \cdot 2^n+1 \equiv 2^{2^5(2t+1)}+1 \pmod{641}$ and since $641|F_5|2^{2^5(2t+1)}+1$ we get $641|k \cdot 2^n+1$ and the number $k \cdot 2^n+1 > p > 641$ is composite.

It remains to consider the case $m \geqslant 6$. In this case we have $2^6|n$, hence $n = 2^6h$ where h is a positive integer, and in view of (1) we have $k \cdot 2^n+1 \equiv -2^{2^6h}+1 \pmod{p}$; since $p|2^{2^5}+1|2^{2^6}-1|2^{2^6h}-1$, we obtain $p|k \cdot 2^n+1$. In view of the fact that $k \cdot 2^n+1 > k > p$, the number $k \cdot 2^n+1$ is composite.

Thus, the numbers $k \cdot 2^n+1$ are composite for $n = 1, 2, 3, \ldots$, which was to be proved. (See [30].)

REMARK. We do not know the smallest number k for which all numbers $k \cdot 2^n+1$ $(n = 1, 2, \ldots)$ are composite.

119*. Let us first note that in the proof of theorem in Problem 118* we could add to congruences (1) the congruence $k \equiv 1 \pmod{2}$, and this would result in the following theorem T: There exist infinitely many odd numbers $k > p$ such that each of the numbers $k \cdot 2^l+1$ $(l = 1, 2, \ldots)$ is divisible by at least one of the six primes

$$F_0, F_1, F_2, F_3, F_4, \text{ and } p \tag{3}$$

(where $p > F_4$). Let us denote by Q the product of the six numbers (3). Since these numbers are odd, we have $2^{\varphi(Q)} \equiv 1 \pmod{Q}$ and consequently, $2^{\varphi(Q)} \equiv 1 \pmod{q}$, where q denotes any of the numbers (3). Let n be an arbitrary positive integer. By theorem T $\big(\text{for } l = n(\varphi(Q)-1)\big)$, the number $k \cdot 2^{n(\varphi(Q)-1)}+1$ is divisible by at least one of the numbers (3), say by q. We shall have therefore $k \cdot 2^{n(\varphi(Q)-1)}+1 \equiv 0 \pmod{q}$ hence, multiplying by 2^n we obtain $k \cdot 2^{n\varphi(Q)}+2^n \equiv 0 \pmod{q}$, and since $2^{\varphi(Q)} \equiv 1 \pmod{q}$ and consequently, $2^{n\varphi(Q)} \equiv 1 \pmod{q}$, we get $k+2^n \equiv 0 \pmod{q}$; since $k > p$, we get $k > q$ and $k+2^n > q$; thus, the number $k+2^n$ is composite, and we showed that there exist infinitely many odd numbers k such that all numbers 2^n+k, $n = 1, 2, \ldots$, are composite.

120. Let $k = 2^m$ where m is a positive integer, and let $m = 2^s h$ where s is an integer $\geqslant 0$, and h is odd. We have $k \cdot 2^{2^n}+1 = 2^{2^s(2^{n-s}+h)}+1$, and for $n > s$ the number $2^{n-s}+h$ is an odd positive integer. Thus we get $2^{2^s}+1 | k \cdot 2^{2^n}+1$, and since $n > s$, we have $k \cdot 2^{2^n}+1 > 2^{2^s}+1$ and the numbers $k \cdot 2^{2^n}+1$ are composite for $n > s$ (they are divisible by $2^{2^s}+1$).

In particular, if k is a power of 2 with an odd exponent, then all numbers $k \cdot 2^{2^n}+1$ for $n = 1, 2, \ldots$ are divisible by 3.

121. For $k = 1$, $n = 5$ since the numbers $2^{2^n}+1$ are prime for $n = 1, 2, 3, 4$ while $641 | 2^{2^5}+1$ and $2^{2^5}+1$ is composite.

For $k = 2$, $n = 1$ since $3 | 2 \cdot 2^2+1$.

For $k = 3$, $n = 2$ since the number $3 \cdot 2^2+1$ is prime, while $7 | 3 \cdot 2^{2^2}+1 = 49$.

For $k = 4$, $n = 2$ since $4 \cdot 2^2+1 = 17$ is a prime, while $5 | 4 \cdot 2^{2^2}+1$.

For $k = 5$, $n = 1$ since $3 | 5 \cdot 2^2+1$.

For $k = 6$, $n = 1$ since $5 | 6 \cdot 2^2+1$.

For $k = 7$, $n = 3$ since $7 \cdot 2^2+1 = 29$ and $7 \cdot 2^{2^2}+1 = 113$ are primes, while $11 | 7 \cdot 2^{2^3}+1$.

For $k = 8$, $n = 1$ since $3 | 8 \cdot 2^2+1$.

For $k = 9$, $n = 2$ since $9 \cdot 2^2+1 = 37$ is prime, while $5 | 9 \cdot 2^{2^2}+1$.

For $k = 10$, $n = 2$ since $10 \cdot 2^2+1 = 41$ is a prime, while $7 | 10 \cdot 2^{2^2}+1$.

122. It follows from the solution of Problem 121 that the numbers $k = 1, 3, 4, 7, 9$, and 10 do not satisfy the requirements. The number 6 does not satisfy the requirement either since $6 \cdot 2^{2^2}+1 = 97$ is a prime.

On the other hand, the numbers $2\cdot 2^{2^2}+1$, $5\cdot 2^{2^2}+1$, and $8\cdot 2^{2^n}+1$ are all composite for $n=1,2,\ldots$ since they are divisible by 3 and exceed 3.

REMARK. If $k=3t+2$, where $t=0,1,2,\ldots$, then the numbers $k\cdot 2^{2^n}+1$ $(n=1,2,\ldots)$ are all divisible by 3 and composite.

123. The numbers $\frac{1}{3}(2^{2^{n+1}}+2^{2^n}+1)$ are positive integers for $n=1,2,\ldots$. If n is even, then $2^n\equiv 1 \pmod 3$, hence $2^n=3k+1$ for some positive integer k, and $2^{2^n}=(2^3)^k\cdot 2=8^k\cdot 2\equiv 2 \pmod 7$, which implies that $2^{2^{n+1}}=(2^{2^n})^2 \equiv 4 \pmod 7$. It follows that $2^{2^{n+1}}+2^{2^n}+1\equiv 4+2+1\equiv 0 \pmod 7$. If n is odd, then $2^n\equiv 2 \pmod 3$, hence $2^n=3k+2$ where k is an integer $\geqslant 0$. It follows that $2^{2^n}=2^{3k+2}=8^k\cdot 4\equiv 4 \pmod 7$, while $2^{2^{n+1}}=(2^{2^n})^2\equiv 4^2 \equiv 2 \pmod 7$. Thus, $2^{2^{n+1}}+2^{2^n}+1\equiv 2+4+1\equiv 0 \pmod 7$. Consequently, the numbers $\frac{1}{3}(2^{2^{n+1}}+2^{2^n}+1)$ are divisible by 7 for positive integer n, and since for $n>1$ they are $\geqslant \frac{1}{3}(2^{2^3}+2^{2^2}+1)=91>7$, they are composite for $n=2,3,\ldots$.

Compare with the theorem of Michael Stiffel from the XVIIth century; see *Elemente der Mathematik*, 18 (1963), p. 18.

124. For instance, all numbers of our sequence for n of the form $28k+1$ $(k=1,2,3,\ldots)$ have the desired property.

In fact, by the Fermat theorem, we have $2^{28}\equiv 1 \pmod{29}$, which implies, for $k=1,2,\ldots$, that $2^{2\cdot 28k}\equiv 1 \pmod{29}$, Thus, for $n=28k+1$ $(k=1,2,\ldots)$ we have $(2^{2n}+1)^2+2^2\equiv 25+4\equiv 0 \pmod{29}$, which means that $29|(2^{2n}+1)^2+2^2$. For $k=1,2,\ldots$ we have obviously $n=28k+1 \geqslant 29$, which implies $(2^{2n}+1)^2+2^2>29$. Thus, all numbers of the form $(2^{2n}+1)^2+2^2$ for $n=28k+1$, $k=1,2,\ldots$, are composite.

125*. If a is odd and >1, the numbers $a^{2^n}+1$, being even and >1, are composite (for $n=1,2,\ldots$); thus, we may assume that a is even. We have $641|2^{2^5}+1$, hence also $641|4^{2^4}+1$ and $641|16^{2^3}+1$. Next, we easily check that $17|2^{2^2}+1$, $17|4^2+1$, $17|6^{2^3}+1$, $17|8^{2^2}+1$, $17|10^{2^3}+1$, $17|12^{2^3}+1$, $17|14^{2^3}+1$, $17|20^{2^3}+1$, $\ldots$, $17|22^{2^3}+1$, $17|24^{2^3}+1$, $17|26^{2^2}+1$, $17|28^{2^3}+1$, $17|30^2+1$, $17|32^{2^2}+1$.

For instance, to check that $17|28^{2^3}+1$ we start from the congruence $28\equiv 11 \pmod{17}$, which implies $28^2\equiv 121\equiv 2 \pmod{17}$, which in turn yields $28^{2^3}\equiv 2^{2^2}\equiv -1 \pmod{17}$, and, consequently, $17|28^{2^3}+1$.

In view of these formulas, we obtain immediately for $k=0,1,2,\ldots$

$$17|(34k+2)^{2^2}+1, \quad 17|(34k+4)^2+1, \quad 17|(34k+6)^{2^3}+1,$$
$$17|(34k+8)^{2^2}+1, \quad 17|(34k+10)^{2^3}+1, \quad 17|(34k+12)^{2^3}+1,$$
$$17|(34k+14)^{2^3}+1, \quad 17|(34k+20)^{2^3}+1, \quad 17|(34k+22)^{2^3}+1,$$
$$17|(34k+24)^{2^3}+1, \quad 17|(34k+26)^{2^2}+1, \quad 17|(34k+28)^{2^3}+1,$$
$$17|(34k+30)^2+1, \quad 17|(34k+32)^{2^2}+1.$$

Using the fact that $5|18^2+1$ and $13|34^2+1$, we deduce that for every positive integer $a \leqslant 100$, except perhaps numbers 50, 52, 68, 84, and 86, there exists a positive integer $n \leqslant 5$ such that $a^{2^n}+1$ is composite. On the other hand, $50^2+1 = 2501 = 41 \cdot 61$, $5|52^2+1$, $5|68^2+1$, $257|84^{2^6}+1$ and $13|86^2+1$. Thus, for every positive integer $a \leqslant 100$ there exists a positive integer $n \leqslant 6$ such that $a^{2^n}+1$ is composite.

REMARK. A. Schinzel proved that for every positive integer a such that $1 < a < 2^{27}$ there exists a positive integer n such that $a^{2^n}+1$ is composite; see [20].

We do not know whether for every integer $a > 1$ there exists a positive integer n such that $a^{2^n}+1$ is composite; we cannot prove it, for instance, for the number $a = 2^{2^{1945}}$. On the other hand, we can prove that for $n = 2^{2^{1944}}$ the number a^2+1 is composite, and we even know its least prime divisor, namely $5 \cdot 2^{2^{1947}}+1$; see Sierpiński [37, p. 349, Section 6].

126. Each prime > 5 is obviously of the form $30k+r$ where k is an integer $\geqslant 0$, and r is one of the numbers 1, 7, 11, 13, 17, 19, 23 or 29. Since there exist infinitely many primes, for at least one of these eight values r there exist infinitely many primes of the form $30k+r$, where k is a positive integer. It is, therefore, sufficient to consider the following eight cases:

(1) There exist infinitely many primes of the form $30k+1$. Let p be one of them, and let $n = 7+19+p$; this will be an odd composite number since $n = 7+19+30k+1 = 3(10k+9)$. Thus, the number n is a sum of three different primes (since $p = 30k+1$ is different from 7 and 19), and n is not a sum of two primes since then one of them would have to be even, hence equal 2, and we would have $n = 30k+27 = q+2$, that is, $q = 5(6k+5)$, which is impossible.

(2) There exists infinitely many primes of the form $30k+7$. Let $p > 7$ be one of these primes, and let $n = 7+13+p$; n is odd and composite since $n = 30k+27 = 3(10k+9)$ and, in view of $p \geqslant 37$, n will be equal to a sum of three different primes. Since $n-2 = 30k+25 = 5(6k+5)$, we see that n satisfies the required conditions.

(3) There exist infinitely many primes of the form $30k+11$. Let $p > 11$ be one of them, and let $n = 11+13+p$; thus, n will be odd and equal to a sum of three different primes. Since $n = 30k+35 = 5(6k+7)$ and $n-2 = 3(30k+11)$, the number n satisfies the required conditions.

(4) There exist infinitely many primes of the form $30k+13$. Let p be one of them, and let $n = 3+11+p$; thus, n will be odd and will equal to the sum of three different primes. Since $n = 3(10k+9)$ and $n-2 = 5(6k+5)$, the number n satisfies the required conditions.

(5) There exist infinitely many primes of the form $p = 30k+17$. Let p be one of them, and put $n = 3+7+p$. Since $n = 3(10k+9)$ and $n-2 = 5(6k+5)$, the number n satisfies the required conditions.

(6) There exist infinitely many primes of the form $30k+19$. Let p be one of them, and let $n = 3+5+p$. As before, we deduce that n satisfies the required conditions.

(7) There exist infinitely many primes of the form $30k+23$. Let p be one of them, and let $n = 5+7+p$. Since $n = 5(6k+7)$ and $n-2 = 3(10k+11)$, the number n satisfies the required conditions.

(8) There exist infinitely many primes of the form $30k+29$. Let p be one of them, and let $n = 5+31+p$. Since $n = 5(6k+13)$ and $n-2 = 3(10k+21)$, the number n satisfies the required conditions.

The proof is complete. See [28].

127. If $f(x)$ were a polynomial with integer coefficients such that $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, then $g(x) = f(x)-2$ would be a polynomial with integer coefficients such that $g(1) = 0$, and we would have $g(x) = (x-1)h(x)$, where $h(x)$ is a polynomial with integer coefficients. Since $f(3) = 5$, we have $g(3) = f(3)-2 = 3$, which gives $2h(3) = 3$; this, however, is impossible since $h(3)$ is an integer.

Now let m be an integer > 1, and let

$$g_k(x) = \frac{(x-1)(x-2)\dots(x-m)}{x-k} \quad \text{for} \quad k = 1, 2, \dots, m.$$

Obviously, $g_k(x)$ is a polynomial with integer coefficients of the degree $m-1$, and such that $g_k(x) = 0$ for every positive integer $x \leqslant m$ different from k, while $g_k(k)$ will be an integer $\neq 0$. Let $f_k(x) = g_k(x)/g_k(k)$; obviously, $f_k(x)$ will be a polynomial of the order $m-1$ with rational coefficients such that $f_k(x) = 0$ for every positive integer $x \leqslant m$ different from k, while $f_k(k) = 1$.

Put

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \ldots + p_m f_m(x);$$

clearly, this polynomial will satisfy the required conditions: $f(x)$ has rational coefficients and $f(k) = p_k$ for $k = 1, 2, \ldots, m$.

128*. Proof due to J. Browkin. Let n be a given positive integer. For positive integer $k \leqslant n$, define by induction the positive integers t_k as follows: let $t_0 = 1$. Suppose that we have already defined for a positive integer $k \leqslant n$ the number t_{k-1}. According to the particular case of the theorem of Lejeune–Dirichlet, there exists a positive integer t_k such that the number $q_k = (k-1)!(n-k)!t_k+1$ is a prime, and, in case $k > 1$, it is greater than the number $(k-2)!(n-k+1)!t_{k-1}+1$ (where we put $0! = 1$). Thus, the numbers $q_1, q_2, \ldots, q_n$ will be primes, and $q_1 < q_2 < \ldots < q_n$. Let

$$f(x) = 1 + \sum_{j=1}^{n} (-1)^{n-j} \frac{(x-1)(x-2)\ldots(x-n)}{x-j} t_j.$$

Clearly, $f(x)$ will be a polynomial of the order $\leqslant n-1$ with integer coefficients, and we easily check that

$$f(k) = 1 + (k-1)!(n-k)!t_k = q_k.$$

129. As an example we may take, for instance, the polynomial

$$f(x) = [(x-p_1)(x-p_2)\ldots(x-p_m)+1]x,$$

where p_k denotes kth prime.

We shall have here $f(p_k) = p_k$ for $k = 1, 2, \ldots, m$.

130. If the constant term of the polynomial $f(x)$ with integer coefficients were equal 0, then we would have $f(0) = 0$ and the congruence $f(x) \equiv 0 \pmod{p}$ would be solvable for every modulus p. Thus, suppose that the constant term of the polynomial $f(x)$ equals a_0 and is not zero. Since $f(a_0 x) = a_0 f_1(x)$, where $f_1(x)$ is a polynomial with integer coefficients with the constant term equal to 1, it suffices to prove our theorem only for such polynomials.

Let n be a given positive integer. We have obviously $n!|f_1(n!)-1$, hence $f(n!) = n!k+1$, where k is an integer. The absolute value of the polynomial $f_1(x)$ (which is of the order > 0) increases over all bounds with x; for sufficiently large n we shall have therefore $|f(n!)| = |n!k+1| > 1$, and the number $n!k+1$ has a prime divisor p. In view of $p|n!k+1$ we must have $p > n$, and since $p|f_1(n!)$, the congruence $f_1(x) \equiv 0 \pmod{p}$ is solvable for a prime modulus $p > n$. Since n is arbitrary, we deduce that the congruence $f_1(x) \equiv 0 \pmod{p}$, and also the congruence $f(x) \equiv 0 \pmod{p}$ is solvable for infinitely many primes p.

131. There is only one such number, namely $k = 1$. Then the sequence

$$k+1,\ k+2,\ \ldots,\ k+10 \tag{1}$$

contains five primes: 2, 3, 5, 7, and 11. For $k = 0$ and $k = 2$, sequence (1) contains four primes. If $k \geqslant 3$, then sequence (1) does not contain number 3; as we know, out of each three consecutive odd numbers, one must be divisible by 3. It follows that sequence (1) contains at least one odd composite number. Besides that, sequence (1) contains five even numbers, hence (for $k \geqslant 2$) these numbers are composite. Thus, for $k \geqslant 3$, sequence (1) contains at least 6 composite numbers, and the numbers of primes cannot exceed 4.

REMARK. Sequence (1) contains four primes for $k = 0, 2, 10, 100, 190, 820$. We do not know whether there exist infinitely many such numbers k. From a certain conjecture of A. Schinzel concerning primes ([22]) it follows that the answer is positive.

132. There exists only one such number, namely $k = 1$. For this value the sequence

$$k+1,\ k+2,\ \ldots,\ k+100 \tag{1}$$

contains 26 primes. For $k = 0, 2, 3$ or 4, sequence (1) contains 25 primes. Thus, we may assume that $k \geqslant 5$. Sequence (1) contains 50 even numbers, which for $k > 1$ are all composite. Next, it contains also 50 successive odd numbers, and since every three consecutive odd numbers contain one divisible by 3, sequence (1) contains at least 16 numbers divisible by 3, which are all composite for $k > 2$.

Let us compute now the number of terms of sequence (1) which are divisible by 5, and neither by 3 or 2. All such numbers will be of the form

$30t+r$ where t is an integer $\geqslant 0$, and r is one of the numbers $5, 25$. Let us arrange these numbers in the infinite increasing sequence

$$5, 25, 35, 55, 65, 85, 95, 115, 125, 145, 155, 175, 185, \ldots \qquad (2)$$

and let u_n denote the nth term of this sequence. We easily check that $u_{n+6}-u_n < 100$ for $n = 1, 2, \ldots$. Let u_n denote the last term of this sequence which does not exceed k. We shall have $u_n \leqslant k < u_{n+1} < u_{n+6} < u_n+100 \leqslant k+100$, which shows that sequence (1) contains at least 6 terms of sequence (2), and, consequently, at least 6 terms divisible by 5, but not divisible by 2 or 3, hence composite for $k \geqslant 5$.

Finally, let us compute the number of terms of sequence (1) which are divisible by 7, but not by 2, 3 or 5. These will be the terms of the form $210t+r$ where t is an integer $\geqslant 0$, and r is one of the numbers $7, 49, 77, 91, 119, 133, 161, 203$. Let us arrange these numbers in the infinite increasing sequence

$$7, 49, 77, 91, 119, 133, 161, 203, 217, 259, 287, \ldots \qquad (3)$$

and let v_n denote the nth term of this sequence. We easily check that $v_{n+3}-v_n < 100$ for $n = 1, 2, \ldots$. Let v_n denote the last term of the sequence $v_1, v_2, \ldots$ which does not exceed k. We shall have $v_n \leqslant k < v_{n+1} < v_{n+3} < v_n+100 \leqslant k+100$, which shows that sequence (1) contains at least 3 terms of sequence (3), that is, at least three numbers divisible by 7, but not divisible by 2, 3 or 5. For $k \geqslant 7$, all these numbers will be composite.

It follows that for $k \geqslant 7$, sequence (1) contains at least $50+16+6+3 = 75$ composite numbers, hence at most 25 primes. For $k = 5$ and $k = 6$, sequence (1) contains the composite numbers v_2, v_3, and v_4. Thus, for $k > 1$, sequence (1) contains at most 25 primes.

133. There are only 6 such sequences, namely those starting from 1, 3, 4, 5, 10, and 11. The proof follows from the following lemma:

For $k > 11$, among the numbers $k, k+1, \ldots, k+99$ there is at least 76 numbers divisible by either 2, 3, 5, 7 or 11.

The proof of the lemma can be obtained if we write in the form of an increasing infinite sequence all numbers divisible by 2, 3, 5, 7 or 11. This sequence has the property that if a number r appears in it, then so does the number $r+2310$, and conversely (since $2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$). Thus, if $r_1, r_2, \ldots, r_s$ denote all positive integers $\leqslant 2310$ divisible by 2, 3, 5, 7 or 11, then all such numbers are contained in s arithmetic progressions

$2310t+r_i$, where $i = 1, \ldots, s$ and $t = 0, 1, 2, \ldots$. Thus, it suffices to write down all positive integers $\leqslant 2310+100$ divisible by 2, 3, 5, 7 or 11, and check that in each hundred of the numbers $k, k+1, \ldots, k+99$ for $1 \leqslant k \leqslant 2310$ there is at least 76 numbers of this sequence.

It would be more difficult to prove that there exists only a finite number of such positive integers k for which sequence (1) contains 24 primes. On the other hand, a certain conjecture concerning primes due to A. Schinzel ([22]) implies that there exist infinitely many numbers k such that sequence (1) contains 23 primes.

134. LEMMA. *Out of every* 21 *consecutive positive integers, at least* 14 *are divisible by one of the numbers* 2, 3, *or* 5.

PROOF. In every consecutive 21 positive integers we have at least 10 divisible by 2, and at least 10 consecutive odd numbers, out of which at least 3 are divisible by 3. Thus, it suffices to show that in every sequence of 21 consecutive positive integers there is at least one which is divisible by 5 but not by 2 or 3. Let r denote the remainder of the division of x by 30; we have then $x = 30t+r$, where t is an integer $\geqslant 0$, and $0 \leqslant r < 30$. If $r \leqslant 5$, then $x \leqslant 30t+5 \leqslant x+20$ and the number $30t+5$ is a term of the sequence $x, x+1, \ldots, x+20$ which is divisible by 5, but not by 2 or 3. If $5 < r \leqslant 25$, then $x \leqslant 30t+25 < x+20$ and the number $30t+25$ is a term of the sequence $x, x+1, \ldots, x+20$ which is divisible by 5 but not by 2 or 3. Finally, if $25 \leqslant r < 30$, then $x < 30t+35 < x+20$, and the number $30t+35$ is a term of the sequence $x, x+1, \ldots, x+20$ which is divisible by 5, but not by 2 or 3. This completes the proof of the lemma.

Our lemma implies immediately that out of each 21 consecutive positive integers exceeding 5 we have at least 14 composite numbers, hence at most 7 primes. For $x = 1, 2$, and 3, the sequence $x, x+1, \ldots, x+20$ contains 8 primes each, while for $x = 4$ and $x = 5$ this sequence contains 7 primes. Thus, the sequence $x, x+1, \ldots, x+20$ contains 8 primes for $x = 1, 2$, and 3.

135. There is only one such number, namely $p = 5$. We easily find that the required property does not hold for $p < 5$. For $p = 5$, we obtain primes 5, 7, 11, 13, 17, and 19. If $p > 5$ and $p = 5k$ with some positive integer k, then p is composite. If $p = 5k+1$, then $p+14$ is divisible by 5, hence composite. If $p = 5k+2$, then $p+8$ is divisible by 5, hence composite. If $p = 5k+3$, then $5|p+12$ and $p+12$ is composite. Finally, if $p = 5k+4$, then $5|p+6$, and $p+6$ is composite.

136. We easily find that for integer $k > 1$ such pairs are $m = 2^k - 2$ and $n = 2^k(2^k-2)$, for which $m+1 = 2^k-1$ and $n+1 = (2^k-1)^2$.

REMARK. P. Erdös posed a problem of existence of other such pairs; see [9, p. 126, problem 60]. A. Mąkowski has found a pair: $m = 75 = 3\cdot 5^2$, $n = 1215 = 5\cdot 3^5$, for which $m+1 = 2^2\cdot 19$, $n+1 = 2^6\cdot 19$.

V. DIOPHANTINE EQUATIONS

137. The identity

$$3(55a+84b)^2-7(36a+55b)^2 = 3a^2-7b^2$$

implies that if the integers $x = a$ and $y = b$ satisfy the equation $3x^2-7y^2+1 = 0$, then the same equation is satisfied by larger integers $x = 55a+84b$ and $y = 36a+55b$. Since the numbers $x = 3$ and $y = 2$ satisfy this equation, it has infinitely many solutions in positive integers x, y.

138. Since $x(2x^2+y) = 7$, the number x must be an integer divisor of number 7, that is, must be equal to one of the numbers 1, 7, -1, -7. Substituting there values to the equation, we obtain for y the values 5, -97, -9, -99. Thus, our equation has four solutions in integers, namely $(1, 5)$, $(7, -97)$, $(-1, -9)$, $(-7, -99)$.

Now let n denote an arbitrary integer > 5, and let $x = 7/n$, $y = n-98/n^2$. Since $n > 5$, we have $n \geqslant 6$, and x, y will be rational and positive; we easily check that they satisfy the equation $2x^3+xy-7 = 0$.

139. We easily see that if x and y satisfy the equation

$$(x-1)^2+(x+1)^2 = y^2+1, \tag{1}$$

then

$$(2y+3x-1)^2+(2y+3x+1)^2 = (3y+4x)^2+1.$$

Thus, for every positive integer solution x, y of equation (1), we obtain another solution $2y+3x$, $3y+4x$ in larger integers; since this equation has a solution $x = 2$, $y = 3$, it has infinitely many solutions in positive integers.

140. If for positive integers x and y we had $x(x+1) = 4y(y+1)$, then we would also have $3 = [2(2y+1)]^2-(2x+1)^2 = (4y-2x+1)(4y+2x+3)$, hence the number 3 would be divisible by a positive integer $4y+2x+3$ exceeding 3, which is impossible.

On the other hand, we easily see that for integer $n > 1$ and for

$$x = \frac{3^n - 3^{1-n} - 2}{4}, \qquad y = \frac{3^n + 3^{1-n} - 4}{8}$$

we have $x(x+1) = 4y(y+1)$. For instance, for $n = 2$ we get $x = 5/3$, $y = 2/3$. Our equation has infinitely many rational solutions x, y.

141*. Proof due to A. Schinzel. Let p be a prime, and let n be a positive integer; suppose that positive integers x and y satisfy the equation $x(x+1) = p^{2n}y(y+1)$, Since x and $x+1$ are relatively prime, we have either $p^{2n}|x$ or $p^{2n}|x+1$, and hence in each case, $x+1 \geqslant p^{2n}$. However, our equation is equivalent to the equation

$$p^{2n} - 1 = [p^n(2y+1) + (2x+1)]\,[p^n(2y+1) - (2x+1)].$$

Since the left-hand side, and the first factor on the right are both positive integers, the second factor on the right must also be a positive integer. It follows that $p^{2n} - 1 > 2x+1$, hence $p^{2n} > 2(x+1)$, which, in view of the previously found relation $x+1 \geqslant p^{2n}$, gives $p^{2n} > 2p^{2n}$, which is impossible.

142. In view of the identity $(x-2y)^2 - 2(x-y)^2 = -(x^2-2y^2)$, it suffices to put $t = x-2y$, $u = x-y$.

143. The proof follows immediately from the identity

$$(m^2+Dn^2)^2 - D(2mn)^2 = (m^2-Dn^2)^2.$$

It suffices to choose, for an arbitrary positive integer n, number m such that $m^2 > Dn^2$ and put

$$x = m^2+Dn^2, \quad y = 2mn, \quad z = m^2-Dn^2.$$

144. If D is odd, then for integer $k > 1$ the number $D+2^{2k-2}$ is odd, and we have $(D+2^{2k-2}, 2^k) = 1$; we easily find that

$$(D+2^{2k-2})^2 - D(2^k)^2 = (D-2^{2k-2})^2.$$

We can put $x = |D+2^{2k-2}|$, $y = 2^k$, $z = |D-2^{2k-2}|$. If D is even, then for every integer $y > 1$ we have $(\frac{1}{2}Dy^2+1, y) = 1$, and

$$(\tfrac{1}{2}Dy^2+1)^2 - Dy^2 = (\tfrac{1}{2}Dy^2-1)^2,$$

and we can put $x = |\frac{1}{2}Dy^2+1|$, $z = |\frac{1}{2}Dy^2-1|$.

145. Our equation is equivalent to the equation $2^{2^5}+1 = (x+1)(y+1)$. Since the Fermat number $F_5 = 2^{2^5}+1$ is equal to the product of two primes, the smaller being 641, we have only one solution of our equation in positive integers x, and $y \geqslant x$, where $x = 640$.

REMARK. It is interesting that we know of some equations of the second order with two unknowns that they have only one solution x and $y \geqslant x$, but (for purely technical reasons) we cannot find this solution. Such is, for instance, the case of equation $xy+x+y+2 = 2^{137}$. On the other hand, we do not know if the equation $xy+x+y = 2^{2^{17}}$ has a solution in positive integers x, y.

146. If y is even, then $x^2 = 3-8z+2y^2$ gives the remainder 3 upon division by 8, which is impossible. If y is odd, then $y = 2k+1$ where k is an integer, then $x^2 = 3-8z+8k^2+2$, which gives the remainder 5 upon division by 8, which again is impossible since the square of every odd number gives the remainder 1 upon division by 8.

147. Let x be an arbitrary positive integer. We easily check the identity $x(x+1)(x+2)(x+3)+1 = (x^2+3x+1)^2$, which, in view of our equation, implies $y = x^2+3x+1$. Thus, all solutions in positive integers x, y of our equations are: x—an arbitrary positive integer, and $y = x^2+3x+1$.

148. The equation $x^2+y^2+z^2+x+y+z = 1$ has no rational solutions since we easily see that it is equivalent to the equation

$$(2x+1)^2+(2y+1)^2+(2z+1)^2 = 7,$$

and the number 7 would have to be a sum of three squares of rational numbers. We shall show that it is impossible. In fact, if 7 were a sum of squares of three rational numbers, then, after muliplying by the common denominator, we would have

$$a^2+b^2+c^2 = 7m^2 \tag{1}$$

where a, b, and c are integers, and m is a positive integer. Then, there would exist the least positive integer m for which (1) has a solution in the integers a, b, c. If m were even, $m = 2n$, where n is a positive integer, then all three numbers a, b, c would be even, hence $a = 2a_1$, $b = 2b_1$, $c = 2c_1$ where a_1, b_1, c_1 are integers. Putting this into (1) we get, in view of $m^2 = 4n^2$

$$a_1^2+b_1^2+c_1^2 = 7n^2$$

where n is a positive integer $< m$, contrary to the assumption that m is the least positive integer for which $7m^2$ is a sum of squares of three integers.

Thus, m is odd, and m^2 gives the remainder 1 upon division by 8. Thus, the right-hand side of (1) gives the remainder 7 upon division by 8; we know, however, that no such number can be a sum of three squares of integers.

149. If positive integers x, y, z would satisfy the equation $4xy-x-y = z^2$, we would have $(4x-1)(4y-1) = (2z)^2+1$, and the positive integer $4x-1 \geqslant 3$ would have a prime divisor p of the form $4k+3$. We would, therefore, have $(2z)^2 \equiv -1 \pmod{p}$, and, in view of $p = 4k+3$, also $(2z)^{p-1} = (2z)^{2(2k+1)} \equiv -1 \pmod{p}$, contrary to the Fermat theorem.

On the other hand, let n denote an arbitrary positive integer, and let $x = -1$, $y = -5n^2-2n$, $z = -5n-1$. We easily check that the numbers x, y, and z satisfy the equation $4xy-x-y = z^2$.

150. We can easily check that for positive integers m and $D = m^2+1$ we have $(2m^2+1)^2-D(2m)^2 = 1$. If for positive integers x and y we have $x^2-Dy^2 = 1$, then, in view of the identity

$$(x^2+Dy^2)^2-D(2xy)^2 = (x^2-Dy^2)^2,$$

we also have $x_1^2-Dy_1^2 = 1$, where $x_1 = x^2+Dy^2$ and $y_1 = 2xy$ are positive integers greater than x and y.

If follows, for example, that the equation $x^2-Dy^2 = 1$ has infinitely many solutions in positive integers x, y for $D = 2, 5, 10, 17, 26, 37, 50, 65, 82$.

151*. The equation $y^2 = x^3+(x+4)^2$ has two obvious solutions: $x = 0$, $y = 4$ and $x = 0$, $y = -4$. We shall now give the proof, due to A. Schinzel, that this equation has no positive integer solutions x, y with $x \neq 0$ (see [29]).

Suppose that the positive integers $x \neq 0$ and y satisfy the equation. We have, therefore,

$$x^3 = (y-x-4)(y+x+4). \tag{1}$$

In view of (1) and $x \neq 0$, the integers $y-x-4$ and $y+x+4$ are $\neq 0$. Let

$$d = (y-x-4, y+x+4). \tag{2}$$

If d had an odd prime divisor p, then in view of (1) we would have $p|x$, and by $p|d$ and (2), we would have $p|y-x-4$ and $p|y+x+4$, hence $p|2y$. Since p is odd, it would follow that $p|y$ and $p|4$, which is impossible. Thus, d has no odd prime divisor, and must be equal to a power of 2 with an integer exponent $\geqslant 0$.

If we had $16|d$, then by (1) and (2) we would have $2^8|x^3$, which implies that $2^3|x$, and since $d|(y+x+4)-(y-x-4) = 2x+8$, we would have $16|8$, which is impossible. Thus, $16 \nmid d$.

If we had $d = 2$, we would have $y-x-4 = 2m$, $y+x+4 = 2n$, where $(m, n) = 1$. In view of (1) and (2) we would have $2|x$, hence also $2|y$. But $2y = 2(m+n)$, hence $y = m+n$, and $2|m+n$; in view of $(m, n) = 1$, the numbers m and n must be both odd. Since $x^3 = 4mn$, we have $8 \nmid x^3$, which is impossible since $2|x$. Thus, $d \neq 2$.

If we had $d = 4$, then $y-x-4 = 4m$, $y+x+4 = 4n$, where $(m, n) = 1$. By (1), we would have $x^3 = 16mn$, hence $4|x$, which implies that $4|mn$; thus, since $(m, n) = 1$, one of the numbers m, n must be divisible by 4 and the other odd. However, since $4|x$ and $4 = d|x-y-4$, we have $4|y = 2(m+n)$, which is impossible. Therefore $d \neq 4$.

Since $16 \nmid d$, $d \neq 2$ and $d \neq 4$, and since d is a power of 2, it remains to consider two more cases: $d = 1$ and $d = 8$.

If $d = 1$, then from (1) and (2) it follows that the numbers $y-x-4$ and $y+x+4$ are cubes of integers; $y-x-4 = a^3$, $y+x+4 = b^3$, which implies, in view of (1), that $x = ab$ and $2x+8 = b^3-a^3$. We cannot have $a = b$ since then we would have $x = -4$ and the equation $y^2 = x^3+(x+4)^2$ would imply $y^2 = -4^3$, which is impossible. In view of $x = ab$ we have $2ab+8 = b^3-a^3 = (b-a)\big((b-a)^2+3ab\big)$. This implies that if $b-a = 1$, then $2ab+8 = 1+3ab$, hence $ab = 7$, and consequently $x = 7$, $y^2 = 7^3+11^2 = 464$, which is impossible since 464 is not a square. Thus, if we have $ab > 0$, then $b-a > 0$, and in view of $b-a \neq 1$, we get $b-a \geqslant 2$ and $2ab+8 > 6ab$. This implies $ab < 2$, hence $ab = 1$ and $a = b = 1$, which is impossible. If $ab < 0$, then either $a > 0$, $b < 0$, which leads to $a^3-b^3 = a^3+(-b)^3 \geqslant a^2+(-b)^2 \geqslant -2ab$, contrary to the fact that $a^3-b^3 = -2ab-8 < -2ab$, or else, $a < 0$, $b > 0$, which in view of $b^3 = a^3+2ab+8$ leads to $b^3 < 8$. Thus, $b = 1$, which gives in turn $a^3+2a+7 = 0$, which is impossible since this equation has no integer solutions. Thus, we must have $ab = 0$, and consequently $x = 0$, contrary to the assumption $x \neq 0$. We cannot have, therefore, $d = 1$, and we must have $d = 8$.

By (2) we have, therefore, $y-x-4 = 8m$, $y+x+4 = 8n$, where $(m, n) = 1$, and in view of (1) we find $x^3 = 64mn$. Thus, $(x/4)^3 = mn$, which implies, by $(m, n) = 1$, that the numbers m and n must be cubes of integers, say $m = a^3$, $n = b^3$. Thus $x/4 = ab$ and $2x+8 = 8(n-m) = 8(b^3-a^3)$, which leads to $ab+1 = b^3-a^3$. Clearly, we cannot have $a = b$, and we must have $|a-b| \geqslant 1$. If $ab > 0$, then $b > a$ and $b-a \geqslant 1$, and since $ab+1 = b^3-a^3$

$= (b-a)\,[(b-a)^2+3ab] > 3ab$, we get $2ab < 1$, contrary to the assumption $ab > 0$. Since $4ab = x \neq 0$, we have $ab < 0$. In view of $|b-a| \geqslant 1$ and $|b^3-a^3| = |b-a|\,|(b+a)^2-ab| \geqslant -ab$, and since we also have (in view of $ab < 0$) the relation $|ab+1| < |ab| = -ab$, the equation $ab+1 = b^3-a^3$ is impossible. This completes the proof of the fact that the equation $y^2 = x^3+(x+4)^2$ has no solution in integers $x \neq 0$ and y.

152. Our equation is equivalent to the equation $x^2z+y^2x+z^2y = mxyz$ in integers x, y, z different from 0, and pairwise relatively prime. It follows that $y|x^2z$, $z|y^2x$, and $x|z^2y$ and since $(x, y) = 1$, $(z, y) = 1$, which implies $(x^2z, y) = 1$, we get from $y|x^2z$ that $y = \pm 1$. In a similar way we find $z = \pm 1$, and $x = \pm 1$.

If all three numbers x, y, z are of the same sign, then our equation implies $1+1+1 = m$, hence $m = 3$. If two of them were positive and one negative, or two negative and one positive, then our equation would imply (in view of $x = \pm 1, y = \pm 1, z = \pm 1$) that m is negative, contrary to the assumption.

Thus, for positive integer m, the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = m$$

has integer solution x, y, z in pairwise relatively prime x, y, z only for $m = 3$, and in this case there are only two solutions: $x = y = z = 1$ and $x = y = z = -1$. For positive integer $m \neq 3$, our equation has no solution in integers x, y, z different from 0 and pairwise relatively prime.

153. We have

$$\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x} = 1,$$

hence the numbers (rational and positive) x/y, y/z, and z/x cannot be all < 1; if at least one of them is $\geqslant 1$, then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} > 1,$$

and the left-hand side cannot be $= 1$ for positive integers x, y, z.

REMARK. It is more difficult to prove that our equation has no solution in integers $\neq 0$, cf. Cassels [3], Sierpiński [2]; Cassels, Sansone [4].

154*. LEMMA. *If a, b, c are positive, real, and not all equal, then*

$$\left(\frac{a+b+c}{3}\right)^3 > abc. \tag{1}$$

PROOF. Suppose that the numbers a, b, c are positive and not all equal. Then there exist positive numbers u, v, and w, not all equal and such that $a = u^3$, $b = v^3$, and $c = w^3$. We have the identity

$$u^3+v^3+w^3-3uvw = \tfrac{1}{2}(u+v+w)\,[(u-v)^2+(v-w)^2+(w-u)^2].$$

Since not all numbers u, v, w are equal, the last factor is strictly positive, and we have

$$u^3+v^3+w^3 > 3uvw, \quad \text{hence} \quad \left(\frac{u^3+v^3+w^3}{3}\right)^3 > u^3v^3w^3,$$

which, in view of $u^3 = a$, $v^3 = b$, $w^3 = c$ gives (1), and completes the proof of the lemma.

Let now x, y, z be positive integers. It the numbers x/y, y/z, and z/x were all equal, then, being positive and their product being equal to 1, they would have to be all equal 1, and we would have

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 3 > 2.$$

Thus, not all numbers x, y, z are equal, and by the lemma we have

$$\left[\frac{1}{3}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\right]^3 > \frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x} = 1, \quad \text{hence} \quad \frac{x}{y}+\frac{y}{z}+\frac{z}{x} > 3.$$

Thus, the equation $x/y+y/z+z/x = 2$ is impossible in positive integers x, y, z.

155. Suppose that the positive integers x, y, z satisfy our equation. If not all three numbers x/y, y/z, and z/x are equal, then from the solution of Problem 154 it follows that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} > 3.$$

We must have, therefore, $x/y = y/z = z/x$, and our equation implies that each of these numbers is 1. Thus, $x = y = z$. In this case we have

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 1+1+1 = 3,$$

and our equation has infinitely many solution in positive integers x, y, z; all of them can be obtained by choosing an arbitrary positive integer for x and setting $x = y = z$.

REMARK. We do not know whether the equation $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 4$ has positive integer solutions x, y, z. On the other hand, the equation $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 5$ has a solution, for instance $x = 1$, $y = 2$, $z = 4$; also (as found by J. Browkin), the equation $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 6$ has a solution, for instance $x = 2$, $y = 12$, $z = 9$.

156*. As noticed by A. Schinzel, if for a given positive integer m the positive integers x, y, z satisfy the equation

$$x^3+y^3+z^3 = mxyz, \tag{1}$$

then we have

$$\frac{x^2y}{y^2z}+\frac{y^2z}{z^2x}+\frac{z^2x}{x^2y} = m. \tag{2}$$

Indeed, we have

$$\frac{x^2y}{y^2z} = \frac{x^3}{xyz}, \quad \frac{y^2z}{z^2x} = \frac{y^3}{xyz}, \quad \frac{z^2x}{x^2y} = \frac{z^3}{xyz},$$

and in view of (1) we get

$$\frac{x^3}{xyz}+\frac{y^3}{xyz}+\frac{z^3}{xyz} = m.$$

From Problems 153 and 154 it follows that for $m = 1$ and $m = 2$ equation (1) has no solution in the positive integers x, y, z, while Problem 155 implies that for $m = 3$ equation (1) has the only solution $x^2y = y^2z = z^2x = n$, where n is some positive integer. Then, however, $x^2y \cdot y^2z \cdot z^2x = n^3$, or $(xyz)^3 = n^3$, which implies $xyz = n$, and, in view of $x^2y = n$, we find $z/x = 1$, or $x = z$; on the other hand, in view of $y^2z = n$, we find $x/y = 1$, or $x = y$. Thus, we must have $x = y = z$. However, if $m = 3$, for any positive integer x, and $x = y = z$ we get a solution of equation (1). Thus, for $m = 3$ all solutions of equation (1) in positive integers are obtained by choosing as x an arbitrary positive integer, and setting $y = z = x$.

157. Suppose that theorem T_1 holds. If theorem T_2 were false, there would exist positive integers u, v, and w such that $u^3+v^3 = w^3$, and putting $x = u^2v$, $y = v^2w$, $z = w^2u$ we would have

$$\frac{x}{y}+\frac{y}{z} = \frac{u^2v}{v^2w}+\frac{v^2w}{w^2u} = \frac{u^2}{vw}+\frac{v^2}{wu} = \frac{u^3+v^3}{uvw} = \frac{w^3}{uvw} = \frac{z}{x}$$

contrary to theorem T_1. Thus, we proved that theorem T_1 implies theorem T_2 (this proof was found by A. Schinzel).

Suppose now that theorem T_1 is false. Then there exist positive integers x, y, z such that

$$\frac{x}{y}+\frac{y}{z} = \frac{z}{x}, \qquad \text{hence} \qquad x^2z+y^2x = z^2y.$$

Let $x^2z = a$, $y^2x = b$; we shall have then $z^2y = a+b$ and $ab(a+b) = (xyz)^3$. Let $d = (a, b)$; thus $a = da_1$, $b = db_1$ where $(a_1, b_1) = 1$. It follows that $a+b = d(a_1+b_1)$ and $a_1b_1(a_1+b_1)d^3 = (xyz)^3$. This implies that $d^3|(xyz)^3$, hence $d|xyz$ and $xyz = dt$, where t is a positive integer.

We have, therefore, $a_1b_1(a_1+b_1) = t^3$, and since a_1, b_1, and a_1+b_1 are pairwise relatively prime, it follows that $a_1 = u^3$, $b_1 = v^3$, $a_1+b_1 = w^3$, where u, v, and w are positive integers. Thus, $u^3+v^3 = w^3$, contrary to theorem T_2, which shows that theorem T_2 implies theorem T_1. Thus, T_1 and T_2 are equivalent, which was to be proved.

REMARK. One can prove by elementary means (though the proof is difficult) that theorem T_2 is true; thus, theorem T_1 is also true.

158*. If the numbers x, y, z, t are positive integers, then the numbers x/y, y/z, z/t, and t/x are rational and positive; their product equals 1, which implies that they cannot be all < 1. But if at least one of them is $\geqslant 1$, then their sum is > 1, and the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x} = 1$$

cannot hold. Thus, we proved that this equation has no solution in positive integers x, y, z, t.

We shall show now this equation has infinitely many solutions in integers $\neq 0$. It suffices to check that this equation is satisfied by numbers $x = -n^2$, $y = n^2(n^2-1)$, $z = (n^2-1)^2$, $t = -n(n^2-1)$, where n is an arbitrary integer > 1.

159*. LEMMA. *If a, b, c, d are positive and not all equal, then*

$$\left(\frac{a+b+c+d}{4}\right)^4 > abcd. \tag{1}$$

PROOF. Suppose that a, b, c, and d are positive, and that, for instance, $a \neq b$. We have then either $a+c \neq b+d$ or $a+d \neq b+c$ since if we had $a+c = b+d$ and $a+d = b+c$, then we would have $a-b = d-b = c-d$ and hence $a-b = 0$, contrary to the assumption $a \neq b$. If, for instance, $a+c \neq b+d$, let $u = a+c$, $v = b+d$; we have $u \neq v$, hence $(u-v)^2 > 0$, which gives $u^2+v^2 > 2uv$. Thus, $(u+v)^2 = u^2+v^2+2uv > 4uv$. It follows that $(a+b+c+d)^2 > 4(a+c)(b+d)$, and since $(a+c)^2 \geqslant 4ac$, $(b+d)^2 \geqslant 4bd$, we have

$$(a+b+c+d)^4 > 4^2(a+c)^2(b+d)^2 > 4^4abcd,$$

which gives inequality (1), and completes the proof of the lemma. Suppose now that for a positive integer m the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x} = m$$

has a solution in positive integers x, y, z, t. The product of these terms is equal 1. If all of them were equal 1, then $m = 4$. Thus, if m is a positive integer < 4, then not all four positive rational numbers x/y, y/z, z/t, and t/x are equal, and by the lemma, we have

$$\left[\frac{1}{4}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x}\right)\right]^4 > \frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{t}\cdot\frac{t}{x} = 1,$$

which implies

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x} > 4.$$

Thus, our equation has no positive integer solution x, y, z, t for positive integers $m < 4$, and for $m = 4$ it has only the solution in which all four numbers x/y, y/z, z/t, and t/x are equal, hence equal 1, which implies that $x = y = z = t$. Thus, for $m = 4$, our equation has infinitely many solutions in positive integers x, y, z, t, and they all are obtained by choosing arbitrary positive integer x, and putting $y = z = t = x$.

160. We must have $x \leqslant 4$ since for $x \geqslant 5$, in view of $x \leqslant y \leqslant z \leqslant t$, we would have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t} \leqslant \frac{4}{5} < 1.$$

Obviously, we also must have $x \geqslant 2$. Thus, it remains to consider only three cases, namely $x = 2$, 3, and 4.

First suppose that $x = 2$. In this case we have the equation

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{t} = \frac{1}{2}. \tag{1}$$

In view of $y \leqslant z \leqslant t$ we get $\frac{3}{y} \geqslant \frac{1}{2}$, which yields $y \leqslant 6$; on the other hand, we have, by (1), $\frac{1}{y} > \frac{1}{2}$, hence $y \geqslant 3$. Thus, we can have only $y = 3, 4, 5$, or 6.

If $y = 3$, we have $\frac{1}{6} = \frac{1}{z}+\frac{1}{t} \leqslant \frac{2}{z}$, which gives $z \leqslant 12$, and since $\frac{1}{6} > \frac{1}{z}$, the number z may assume only the values 7, 8, 9, 10, 11 or 12.

For $z = 7$, we have $\frac{1}{t} = \frac{1}{42}$, or $t = 42$, which gives the solution $x = 2$, $y = 3$, $z = 7$, $t = 42$ of our equation.

For $z = 8$, we have $\frac{1}{t} = \frac{1}{24}$, or $t = 24$, which gives the solution $x = 2$, $y = 3$, $z = 8$, $t = 24$ of our equation.

For $z = 9$, we have $\frac{1}{t} = \frac{1}{18}$, hence $t = 18$, which gives the solution $x = 2$, $y = 3$, $z = 9$, $t = 18$ of our equation.

For $z = 10$, we get $\frac{1}{t} = \frac{1}{15}$, or $t = 15$, which gives the solution $x = 2$, $y = 3$, $z = 10$, $t = 15$ of our equation.

For $z = 11$, we have $\frac{1}{t} = \frac{5}{66}$, which does not lead to integer value of t, and our equation has no integer solution.

For $z = 12$, we have $\frac{1}{t} = \frac{1}{12}$, or $t = 12$, which gives the solution $x = 2$, $y = 3$, $z = 12$, $t = 12$ of our equation.

If $y = 4$, we have $\frac{1}{4} = \frac{1}{z}+\frac{1}{t} \leqslant \frac{2}{z}$, hence $z \leqslant 8$, and since $\frac{1}{4} > \frac{1}{z}$, or $z > 4$, the number z may assume only values 5, 6, 7, or 8.

For $z = 5$, we have $\frac{1}{t} = \frac{1}{20}$, or $t = 20$, which gives the solution $x = 2$, $y = 4$, $z = 5$, $t = 20$ of our equation.

For $z = 6$, we have $\frac{1}{t} = \frac{1}{12}$, or $t = 12$, which gives the solution $x = 2$, $y = 4$, $z = 6$, $t = 12$ of our equation.

For $z = 7$, we have $\frac{1}{t} = \frac{3}{28}$ which does not lead to integer value of t, and our equation has no integer solution.

For $z = 8$, we have $\frac{1}{t} = \frac{1}{8}$, or $t = 8$, which gives the solution $x = 2$, $y = 4$, $z = 8$, $t = 8$ of our equation.

If $y = 5$, we have $\frac{3}{10} = \frac{1}{z}+\frac{1}{t} \leqslant \frac{2}{z}$, or $z \leqslant \frac{20}{3}$, that is, $z \leqslant 6$, hence $z \geqslant y = 5$, and we see that z may assume only values 5 or 6.

For $z = 5$, we have $\frac{1}{t} = \frac{1}{10}$, or $t = 10$, which gives the solution $x = 2$, $y = 5$, $z = 5$, $t = 10$ of our equation.

For $z = 6$, we have $\frac{1}{t} = \frac{2}{15}$, which does not lead to integer value of t, and our equation has no solution.

If $y = 6$, we have $\frac{1}{3} = \frac{1}{z}+\frac{1}{t} \leqslant \frac{2}{z}$ which gives $z \leqslant 6$, and since $z \geqslant y = 6$, we have $z = 6$, and consequently $t = 6$, which leads to the solution $x = 2$, $y = 6$, $z = 6$, $t = 6$.

We have completed the consideration of the case $x = 2$, showing that equation (1) has only 10 positive integer solutions y, z, t with $y \leqslant z \leqslant t$, namely 3, 7, 42; 3, 8, 24; 3, 9, 18; 3, 10, 15; 3, 12, 12; 4, 5, 20; 4, 6, 12; 4, 8, 8; 5, 5, 10, and 6, 6, 6.

Suppose now that $x = 3$. Then we have the equation

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{t} = \frac{2}{3}$$

and, by $y \leqslant z \leqslant t$, we get $\frac{3}{y} \geqslant \frac{2}{3}$ or $y \leqslant \frac{9}{2}$, which implies $y \leqslant 4$. Since $3 = x \leqslant y$, possible values for y are 3 and 4.

If $y=3$, then $\frac{1}{z}+\frac{1}{t}=\frac{1}{3}$, which implies $\frac{2}{z}\geqslant\frac{1}{3}$ or $z\leqslant 6$, and since $\frac{1}{z}<\frac{1}{3}$ or $z>3$, the possible values for z are only 4, 5 and 6.

For $z=4$, we have $t=12$, which gives the solution $x=3$, $y=3$, $z=4$, $t=12$ of our equation.

For $z=5$, we get $t=15/2$, which does not lead to a solution in integers x, y, z, t.

For $z=6$, we get $t=6$, which gives the solution $x=3$, $y=3$, $z=6$, $t=6$ of our equation.

If $y=4$, we have $\frac{1}{z}+\frac{1}{t}=\frac{5}{12}\leqslant\frac{2}{z}$, which implies $z\leqslant\frac{24}{5}<5$, and since $z\geqslant y=4$, we must have $z=4$, and consequently $t=6$, which gives the solution $x=3$, $y=4$, $z=4$, $t=6$ of our equation.

Suppose now that $x=4$. We have then the equation

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{3}{4},$$

which implies, by $y\leqslant z\leqslant t$, that $\frac{3}{4}\leqslant\frac{3}{4}$, or $y\leqslant 4$, and since $y\geqslant x=4$, we can have only $y=4$. This leads to $\frac{1}{z}+\frac{1}{t}=\frac{1}{2}\leqslant\frac{2}{z}$, or $z\leqslant 4$, and since $z\geqslant y=4$, we must have $z=4$. This in turn implies that $t=4$, and we obtain the solution $x=4$, $y=4$, $z=4$, $t=4$ of our equation.

We have thus exhausted all possible cases, which leads to the conclusion that our equation has 14 positive integer solutions x, y, z, t with $x\leqslant y\leqslant z\leqslant t$, namely 2, 3, 7, 42; 2, 3, 8, 24; 2, 3, 9, 18; 2, 3, 10, 15; 2, 3, 12, 12; 2, 4, 5, 24; 2, 4, 6, 12; 2, 4, 8, 8; 2, 5, 5, 10; 2, 6, 6, 6; 3, 3, 4, 12; 3, 3, 6, 6; 3, 4, 4, 6; and 4, 4, 4, 4.

REMARK. The equations considered occur in connection with the problem of covering the plane with regular polygons; see [25, p. 31 and following].

161. For every positive integer s our equation has at least one solution in positive integers, namely $x_1=x_2=\ldots=x_s=s$.

To prove that our equation has, for every positive integer s, only a finite number of solutions, we shall prove a more general theorem, asserting that for every rational w and every positive integer s the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s}=w$$

has a finite $\geqslant 0$ number of solutions in positive integers $x_1, x_2, \ldots, x_s$. The proof will proceed by induction with respect to s. The theorem is obvious for $s = 1$. Let now s be any positive integer, and suppose that the theorem is true for the number s. Suppose that the positive integers $x_1, x_2, \ldots, x_s, x_{s+1}$ satisfy the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s}+\frac{1}{x_{s+1}} = u, \tag{1}$$

where u is a given rational number, obviously positive. We may assume that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_s \leqslant x_{s+1}$. From (1) it follows that $(s+1)/x_1 \geqslant u$, which implies $x_1 \leqslant (s+1)/u$; thus, the number x_1 can assume only a finite number of positive integer values. Let us now take as x_1 any of these values; then the remaining s numbers $x_2, x_3, \ldots, x_s, s_{s+1}$ will satisfy the equation

$$\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_s}+\frac{1}{x_{s+1}} = u-\frac{1}{x_1} \tag{2}$$

where, for a given x_1, the right-hand side is rational. Consequently, by the inductive assumption of the truth of our theorem for the number s, it follows that this equation has a finite $\geqslant 0$ number of solutions in the positive integers, $x_2, x_3, \ldots, x_s, x_{s+1}$. Since x_1 can assume only a finite number of values, the theorem follows for the number $s+1$. This completes the proof.

162*. We easily check that for $s = 3$ we have a solution of our equation in increasing positive integers, namely $x_1 = 2$, $x_2 = 3$, $x_3 = 6$. If for some integer $s \geqslant 3$ the positive integers $x_1 < x_2 < \ldots < x_s$ satisfy our equation, then in view of $s \geqslant 3$ we have $x_1 > 1$ and $2 < 2x_1 < 2x_2 < \ldots < 2x_s$; thus the numbers $t_1 = 2$, $t_2 = 2x_1$, $t_3 = 2x_2$, $\ldots$, $t_s = 2x_{s-1}$, $t_{s+2} = 2x_s$ form an increasing sequence of positive integers and satisfy the equation

$$\frac{1}{t_1}+\frac{1}{t_2}+\ldots+\frac{1}{t_s}+\frac{1}{t_{s+1}} = 1. \tag{1}$$

In this manner we have l_s solutions of equation (1) in increasing positive integers $t_1, t_2, \ldots, t_s, t_{s+1}$, and consequently, $l_{s+1} \geqslant l_s$. Thus, for every integer $s \geqslant 3$, the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_s} = 1$$

has at least one solution in increasing positive integers $x_1, x_2, \ldots, x_s$.

For $s = 3$, the equation has only one solution in increasing positive integers since we must have $x_1 > 1$, hence $x_1 \geqslant 2$, and if we had $x_1 \geqslant 3$, we would have $x_2 \geqslant 4$, $x_3 \geqslant 5$, which is impossible since

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \leqslant \frac{1}{3}+\frac{1}{4}+\frac{1}{5} < 1.$$

We have, therefore, $x_2 = 3$, hence $x_3 = 6$, and consequently $l_3 = 1$. On the other hand, $l_4 > 1$ since the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4} = 1$$

has in positive integers the solutions 2, 3, 7, 42 and 2, 3, 8, 24 (and also other solutions).

We can, therefore, assume that $s \geqslant 4$. In this case the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+ \dots +\frac{1}{x_{s-1}} = 1$$

has at least one solution in increasing positive integers $x_1 < x_2 < \dots < x_{s-1}$, and then the numbers $t_1 = 2$, $t_2 = 3$, $t_3 = 6x_1$, $t_4 = 6x_2$, $\dots$, $t_{s+1} = 6x_{s-1}$ will be increasing positive integers, and will satisfy equation (1). This solution will be different than each of the l_s solutions obtained previously since there all numbers were even, while here the number 3 is odd. Thus, we have $l_{s+1} \geqslant l_s+1$, hence $l_{s+1} > l_s$ for $s \geqslant 3$, which was to be proved.

163. Let $t_n = n(n+1)/2$ denote the nth triangular number. We easily check that

$$\frac{1}{t_1} = 1, \quad \frac{1}{t_2}+\frac{1}{t_2}+\frac{1}{t_2} = 1, \quad \frac{1}{t_2}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_3} = 1.$$

Thus, it suffices to assume that s is an integer $\geqslant 5$. If s is odd, that is, $s = 2k-1$ where k is an integer $\geqslant 3$, then we have

$$\frac{1}{t_2}+\frac{1}{t_3}+ \dots +\frac{1}{t_{k-1}}+\frac{k+1}{t_k} = \frac{2}{2\cdot 3}+\frac{2}{3\cdot 4}+ \dots +\frac{2}{(k-1)k}+\frac{2}{k}$$

$$= 2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+ \dots +\left(\frac{1}{k-1}-\frac{1}{k}\right)\right]+\frac{2}{k} = 1,$$

and the left-hand side is the sum of reciprocals of $(k-2)+(k+1) = 2k-1 = s$ triangular numbers.

If s is even, that is, $s = 2k$ where k is an integer $\geqslant 3$, then we have, in case $k = 3$, $6/t_3 = 1$, while in case $k > 3$

$$\frac{2}{t_3}+\frac{1}{t_3}+\frac{1}{t_4}+\dots+\frac{1}{t_{k-1}}+\frac{k+1}{t_k}$$

$$= \frac{1}{3}+\frac{2}{3\cdot 4}+\frac{2}{4\cdot 5}+\dots+\frac{2}{(k-1)k}+\frac{2}{k} = 1,$$

and the left-hand side is a sum of reciprocals of $(k-1)+(k+1) = 2k = s$ triangular numbers.

164. Clearly, none of the positive integers x, y, z, t satisfying our equation can be $= 1$. None of them can be $\geqslant 3$, either, since if, for instance, $x \geqslant 3$, then by $y \geqslant 2$, $z \geqslant 2$, $t \geqslant 2$ we would have

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2} \leqslant \frac{1}{9}+\frac{3}{4} = \frac{31}{36} < 1,$$

which is impossible. Thus, we must have $x = y = z = t = 2$, which is the only solution of our equation in positive integers.

165. These are numbers 1, 4, and all integers $s \geqslant 6$.

For $s = 1$, we have an obvious solution $x_1 = 1$.

For $s = 2$ and $s = 3$, our equation has no solution in positive integers since these numbers would have to be > 1, hence $\geqslant 2$, while for such numbers x_1, x_2, x_3 we have

$$\frac{1}{x_1^2}+\frac{1}{x_2^2} \leqslant \frac{1}{4}+\frac{1}{4} < 1 \quad \text{and} \quad \frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2} \leqslant \frac{3}{4} < 1.$$

For $s = 4$, we have the solution $x_1 = x_2 = x_3 = x_4 = 2$.

For $s = 5$, our equation has no solution in positive integers. In fact, if the numbers $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant x_5$ would satisfy our equation, we would have $x_1 \geqslant 2$, and $x_1 < 3$, since in case $x_1 \geqslant 3$, we have

$$\frac{1}{x_1^2}+\dots+\frac{1}{x_5^2} \leqslant \frac{5}{9} < 1.$$

We must therefore have $x_1 = 2$, and consequently,

$$\frac{1}{x_2^2}+\frac{1}{x_3^2}+\frac{1}{x_4^2}=\frac{1}{x_5^2}=\frac{3}{4},$$

which implies $x_2 < 3$ since $4/9 < 3/4$. Thus $x_2 = 2$, which yields

$$\frac{1}{x_3^2}+\frac{1}{x_4^2}+\frac{1}{x_5^2}=\frac{1}{2}.$$

It follows that $x_3 < 3$ since $3/9 < 1/2$. Thus, $x_3 = 2$, which yields

$$\frac{1}{x_4^2}+\frac{1}{x_5^2}=\frac{1}{4},$$

which is impossible since $x_4 \geqslant 2$ and $x_5 \geqslant 2$.

For $s = 6$, our equation has a solution $x_1 = x_2 = x_3 = 2$, $x_4 = x_5 = 3$, $x_6 = 6$.

For $s = 7$, our equation has a solution $x_1 = x_2 = x_3 = 2$, $x_4 = x_5 = x_6 = x_7 = 4$.

For $s = 8$, our equation has a solution $x_1 = x_2 = x_3 = 2$, $x_4 = x_5 = 3$, $x_6 = 7$, $x_7 = 14$, $x_8 = 21$.

Suppose now that for some positive integer s the equation

$$\frac{1}{t_1^2}+\frac{1}{t_2^2}+\ldots+\frac{1}{t_s^2}=1$$

has a solution in positive integers $t_1, \ldots, t_s$. Since $1/t_s^2 = 4/(2t_s)^2$, the equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{s+3}^2}=1$$

has a solution in positive integers $x_1 = t_1$, $x_2 = t_2, \ldots, x_{s-1} = t_{s-1}$, $x_s = x_{s+1} = x_{s+2} = x_{s+3} = 2t_s$. Thus, if our equation is solvable in positive integers for some positive integer s, then it is also solvable for $s+3$, and since it is solvable for $s = 6$, 7, and 8, it is solvable for every integer $s \geqslant 6$ (and, in addition to that, for $s = 1$ and $s = 4$).

REMARK. One can prove that the rational number r can be represented as a sum of a finite number of reciprocals of squares of an increasing sequence of natural numbers if and only if either $0 < r < \frac{1}{6}\pi^2-1$ or $1 \leqslant r < \frac{1}{6}\pi^2$. See [36, theorem 5].

166. $\frac{1}{2}=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{12^2}+\frac{1}{14^2}+\frac{1}{21^2}+\frac{1}{36^2}+\frac{1}{45^2}+\frac{1}{60^2}$.
To check this equality one has to note that

$$\frac{1}{7^2}+\frac{1}{14^2}+\frac{1}{21^2}=\frac{1}{6} \quad \text{and} \quad \frac{1}{45^2}+\frac{1}{60^2}=\frac{1}{36^2},$$

and then reduce all fractions to the common denominator 36^2.

REMARK. I do not know whether the number 1/2 can be represented as a sum of less than twelve reciprocals of positive integers.

167*. Let m be a given positive integer. For $s = 2^m$, our equation has a solution in positive integers $x_1 = x_2 = \ldots = x_s = 2$.

Let now a be a given positive integer, and suppose that our equation is solvable in positive integers for the positive integer s. Thus, there exist positive integers $t_1, t_2, \ldots, t_s$ such that

$$\frac{1}{t_1^m}+\frac{1}{t_2^m}+\ldots+\frac{1}{t_s^m}=1,$$

and since $1/t_s^m = a^m/(at_s)^m$, for $x_1 = t_1$, $x_2 = t_2, \ldots, x_{s-1} = t_{s-1}$, $x_s = x_{s+1} = \ldots = x_{s+a^m-1} = at_s$ we shall have

$$\frac{1}{x_1^m}+\frac{1}{x_2^m}+\ldots+\frac{1}{x_{s+a^m-1}^m}=1.$$

Thus, if our equation is solvable in positive integers for a positive integer s, then it is also solvable in positive integers for $s+a^m-1$, and, more generally, for $s+(a^m-1)k$, where k is an arbitrary positive integer. Taking $a = 2$ and $a = 2^m-1$ we see that (for $s = 2^m$) our equation has a solution in positive integers for every $s = 2^m+(2^m-1)k+[(2^m-1)^m-1]l$ where k and l are arbitrary positive integers.

The numbers 2^m-1 and $(2^m-1)^m-1$ are obviously relatively prime. By the theorem, proved in Sierpiński [37, p. 29, Corollary 2], it follows that every sufficiently large positive integer is of the form $(2^m-1)k+ +[(2^m-1)^m-1]l$, where k and l are positive integers. This implies that every sufficiently large positive integer is also of the form $2^m+(2^m-1)k+ +[(2^m-1)^m-1]l$, hence for every such integer our equation is solvable in positive integers.

168. Clearly, it suffices to show that our equation has for every positive

integer s at least one solution in positive integers $x_1, x_2, \ldots, x_s$ since every such solution multiplied by a positive integer is again a solution.

For $s = 1$, we have an obvious solution $x_1 = x_2 = 1$.

For $s = 2$, we have the solution $\frac{1}{15^2}+\frac{1}{12^2} = \frac{1}{20^2}$.

Now let s be an arbitrary positive integer, and suppose that our equation has a solution in positive integers

$$\frac{1}{t_1^2}+\frac{1}{t_2^2}+\ldots+\frac{1}{t_s^2} = \frac{1}{t_{s+1}^2}.$$

Since

$$\frac{1}{(12t_s)^2} = \frac{1}{(15t_s)^2}+\frac{1}{(20t_s)^2},$$

the positive integers $x_i = 12t_i$ for $i = 1, \ldots, s-1$, $x_s = 15t_s$, $x_{s+1} = 20t_s$, $x_{s+2} = 12t_{s+1}$ satisfy the equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_s^2}+\frac{1}{x_{s+1}^2} = \frac{1}{x_{s+2}^2}$$

and the proof follows by induction.

169. It suffices to prove that for every integer $s \geqslant 3$ our equation has at least one solution in positive integers $x_1, x_2, \ldots, x_s, x_{s+1}$. For $s = 3$, it has the solution

$$\frac{1}{12^3}+\frac{1}{15^3}+\frac{1}{20^3} = \frac{1}{10^3}$$

(which can be obtained by dividing by 60^3 both sides of the equation 3^3+ $+4^3+5^3 = 6^3$), while for $s = 4$ we get the solution

$$\frac{1}{(5\cdot 7\cdot 13)^3}+\frac{1}{(5\cdot 12\cdot 13)^3}+\frac{1}{(7\cdot 12\cdot 13)^3}+\frac{1}{(5\cdot 7\cdot 12\cdot 13)^3} = \frac{1}{(5\cdot 7\cdot 12)^3}$$

(which follows from dividing by $(5\cdot 7\cdot 12\cdot 13)^3$ both sides of the equality $1^3+5^3+7^3+12^3 = 13^3$).

Now let s denote a given integer $\geqslant 3$ and suppose that our equation has a solution in positive integers for this value of s. Thus, there exist positive integers $t_1, t_2, \ldots, t_s, t_{s+1}$ such that

$$\frac{1}{t_1^3}+\frac{1}{t_2^3}+\ldots+\frac{1}{t_s^3} = \frac{1}{t_{s+1}^3}.$$

Putting $x_i = 10t_i$ for $i = 1, 2, \ldots, s-1$ and $x_s = 12t_s$, $x_{s+1} = 15t_s$, $x_{s+2} = 20t_s$, $x_{s+3} = 10t_{s+1}$, we obtain

$$\frac{1}{x_1^3}+\frac{1}{x_2^3}+\ldots+\frac{1}{x_{s+2}^3}=\frac{1}{x_{s+3}^3},$$

hence if our equation is solvable in positive integers for some s, it is also solvable for $s+2$. Since it is solvable for $s = 3$ and $s = 4$, we conclude that it is solvable for every $s \geqslant 3$, which was to be proved.

REMARK. One can prove by elementary means that for $s = 2$ our equation has no solution in positive integers but the proof is difficult.

170*. The solution found by A. Schinzel. We have the identity

$$(x+y+z)^3-(x^3+y^3+z^3) = 3(x+y)(x+z)(y+z). \tag{1}$$

Thus, if x, y, and z are integers such that $x+y+z = 3$ and $x^3+y^3+z^3 = 3$, then by (1) we get

$$8 = (x+y)(x+z)(y+z) = (3-x)(3-y)(3-z), \tag{2}$$

and in view of $x+y+z = 3$ we have

$$6 = (3-x)+(3-y)+(3-z). \tag{3}$$

The relation (3) implies that either all three numbers $3-x$, $3-y$, $3-z$ are even or only one of them is even. In the first case, in view of (2), all these numbers are equal to 2 in absolute value; thus, by (3), they are equal to 2, and then $x = y = z = 1$. In the second case, in view of (2), one of the numbers $3-x$, $3-y$, $3-z$ is equal to 8 in absolute value, and the remaining ones are equal to 1 in absolute value; thus, in view of (3), one of them is $= 8$, and the remaining ones are $= -1$. This yields $x = -5$, $y = z = 4$, or $x = y = 4$, $z = -5$, or, finally, $x = 4$, $y = -5$, $z = 4$.

Thus, our system of equations has only four integer solutions, namely $x, y, z = 1, 1, 1;\ -5, 4, 4;\ 4, -5, 4;\ 4, 4, -5$.

See Problem E 1355 from *The American Mathematical Monthly*, *69* (1962), 1009.

REMARK. We do not know whether the equation $x^3+y^3+z^3 = 3$ has other solutions in integers x, y, z besides the four given above.

171. Clearly we must have $n \geqslant 8$. If $n = 3k$, where k is an integer > 5, then for $x = k-5$, $y = 3$ we have $3x+5y = n$. If $n = 3k+1$ where

k is an integer > 3, then for $x = k-3$ and $y = 2$ we have $3x+5y = n$ Finally, if $n = 3k+2$, where k is an integer > 1, then for $x = k-1$, $y = 1$ we have $3x+5y = n$. It follows that our equation has at least one positive integer solution x, y for every $n > 15$. It remains to investigate the numbers 8, 9, 10, 12, and 15. For $n = 8$, we have the solution $x = 1$, $y = 1$. For $n = 9, 12$, and 15, our equation has no solution, since we would have $3|5y$, hence $3|y$ and $15|5y$, hence $n = 3x+5y > 5y \geqslant 15$. For $n = 10$, our equation has no solutions in positive integers either, since then we would have $5|3x$, hence $5|x$ and $15|3x$, hence $n = 3x+5y > 15$. Thus, our equation has at least one solution in positive integers x, y for all positive integers n except 1, 2, 3, 4, 5, 6, 7, 9, 10, 12, and 15.

Let now m be an arbitrary positive integer and let n be an integer $> 40m$. The equation $3x+5y = n$ has, therefore, a solution x_0, y_0, and at least one of these numbers must be $> 5m$ since in the case $x_0 \leqslant 5m$, $y_0 \leqslant 5m$ we would have $3x_0+5y_0 \leqslant 40m < n$. If $x_0 > 5m$, then for $k = 0, 1, 2, \ldots, m$ the numbers $x = x_0-5k$ and $y = y_0+3k$ are positive integers and satisfy the equation $3x+5y = 3x_0+5y_0 = n$. If $y_0 > 5m$, then for $k = 0, 1, 2, \ldots, m$ the numbers $x = x_0+5k$ and $y = y_0-3k$ are positive integers and satisfy the equation $3x+5y = n$. Thus, this equation has, for $n > 40m$, more than m solutions in positive integer x, y, which shows that the number of such solutions increases to infinity with n.

172. $n = 2$, $y = x$, $z = x+1$, where x is an arbitrary positive integer. In fact, for positive integers x we have $2^x+2^x = 2^{x+1}$. On the other hand, suppose that for positive integers n, x, y, and z we have $n^x+n^y = n^z$. We may assume that $x \leqslant y \leqslant z$. We cannot have $n = 1$, hence $n \geqslant 2$. We have $n^x = n^z-n^y = n^x(n^{z-x}-n^{y-x})$, which implies $n^{z-x}-n^{y-x} = 1$. If we had $y > x$, then we would have $n|1$, which is impossible. Thus, we must have $y = x$, hence $n^{z-x} = 2$, which yields $n = 2$, $z-x = 1$. We obtain, therefore, $n = 2$, $y = x$, $z = x+1$.

REMARK. The equation $n^x+n^y = n^z$ is obtained from the Fermat equation $x^n+y^n = z^n$ by reversing the roles of exponents and bases. See *Mem. Real. Acad. Sci. Art. Barcelona, 34* (1961), 17–25.

173. Let m and n be two given positive integers, and let a and b be two different primes $> m+n$. Put $c = am+bn$. The system $x = m$, $y = n$ satisfies obviously the equation $ax+by = c$. Suppose that there is some other system satisfying this equation, say x, y. We cannot have $x \geqslant m$, $y > n$, or $x > m$, $y \geqslant n$ since in this case we would have $ax+by > am+$

$+bn = c$. Thus, we must have either $x < m$ or $y < n$. If $x < m$, then $m-x$ is a positive integer $< m$, and in view of $ax+by = am+bn$ we have $by = a(m-x)+bn$, which implies that $b|a(m-x)$. Since a and b are different primes, it follows that $b|m-x$, which is impossible since by definition we have $m > b$. In a similar manner we prove that we cannot have $y < n$.

REMARK. It is easy to note that not for all two systems of positive integers there exists a linear equation $ax+by = c$, with integer a, b, and c, which has these two systems as the only positive integer solution. On the other hand, we can easily prove that there always exists such an equation of the second degree with integer coefficients.

174. For instance, the equation $x+y = m+1$, which has exactly m solutions in positive integers x, y, namely $x = k$, $y = m-k+1$, where $k = 1, \ldots, m$.

REMARK. It is known that there is no linear equation $ax+by = c$ which would have a finite and > 0 number of solutions in integers x, y.

175. For $f(x, y) = x^2+y^2+2xy-mx-my-m-1$, we have the identity $f(x, y) = (x+y-m-1)(x+y+1)$. Since for positive integers x and y we have $x+y+1 > 0$, we can have $f(x, y) = 0$ if and only if $x+y-m-1 = 0$; from the solution of Problem 174 it follows that this equation has exactly m solutions in positive integers x and y.

REMARK. The polynomial in two variables considered in this problem is reducible. One could ask whether for every positive integer m there exists an irreducible polynomial $F(x, y)$ of the second degree with integer coefficients, and such that the equation $F(x, y) = 0$ has exactly m solutions in positive integer x, y. One can prove that for every positive integer m there exists a positive integer a_m such that the equation $x^2+y^2 = a_m$ has exactly m positive integer solutions x, y. More precisely, one can prove it for $a_{2k-1} = 2 \cdot 5^{2k-2}$ and $a_{2k} = 5^{2k-1}$, where $k = 1, 2, \ldots$, but the proof is not easy.

Let us also remark that A. Schinzel proved that for every positive integer m there exists a polynomial of the second degree in the variables x, y, say $f(x, y)$, such that the equation $f(x, y) = 0$ has exactly m integer solutions. See [18].

176. Put $x = t+3$. Then our equation reduces to the equation

$$2t(t^2+3t+21) = 0,$$

which has only one solution in real numbers, namely $t = 0$. It follows that our equation has only one integer solution, namely $x = 3$.

REMARK. One can prove that all solutions of the equation

$$x^3+(x+r)^3+(x+2r)^3+\ldots+(x+(n-1)r)^3 = (x+nr)^3$$

in positive integers x, r, n are only $n = 3$, $x = 3r$ where r is an arbitrary positive integer.

177. If $n = 2k-1$, where k is a positive integer, then obviously, $x = -k$, $y = 0$ is a solution of our equation; if $n = 2k$, where j is a positive integer, then $x = -k$, $y = k$ is a solution of our equation.

REMARK. There are also other solutions, for instance for $n = 8$, $x = -3$, $y = 6$; for $n = 25$, $x = -11$, $y = 20$; for $n = 1000$, $x = 1333$, $y = 16830$.

178. In this equation the coefficients at x^3, x^2, and x, are divisible by 3, and the constant term is -25, which is not divisible by 3. It follows that our equation has no solutions in positive integers x.

179. Substituting $x = t+10$ we reduce our equation to the equation

$$3t(t^2+40t+230) = 0.$$

Since the equation $t^2+40t-230 = 0$ has no rational solutions, we must have $t = 0$, and our equation has only one solution in positive integers, namely $x = 10$.

180. $x = 1$, $y = 2$ (since $2\cdot 3 = 1\cdot 2\cdot 3$) and $x = 5$, $y = 14$ (since $14\cdot 15 = 5\cdot 6\cdot 7$).

REMARK. L. J. Mordell proved that our equation has no other solutions in positive integers.

181. The solution follows immediately from the identity

$$1+(2n)^2+(2n^2)^2 = (2n^2+1)^2 \quad \text{for} \quad n = 1, 2, \ldots .$$

REMARK. It is easy to prove that for every positive integer k the equation $k+x^2+y^2 = z^2$ has infinitely many solutions in positive integers x, y, z. It suffices to take as x an arbitrary integer $> |k|+1$, even if k is odd, and odd if k is even, and put

$$y = \frac{k+x^2-1}{2}, \quad z = \frac{k+x^2+1}{2}.$$

182. Suppose that positive integers n and $x \leqslant y \leqslant z$ satisfy the equation

$$n^x+n^y+n^z = n^t. \tag{1}$$

We cannot have $n = 1$. If $n = 2$, then from (1) we get $1+2^{y-x}+2^{z-x} = 2^{t-x}$, and we cannot have $y > x$. Thus we have $y = x$, and $2+2^{z-x} = 2^{t-x}$, which gives $z-x = 1$, hence $t-x = 2$. Thus, if $n = 2$, then we must have $y = x$, $z = x+1$, and $t = x+2$, while we easily check that for all positive integers x we have $2^x+2^x+2^{x+1} = 2^{x+2}$.

Suppose, next, that $n \geqslant 3$. In view of (1) we have $1+n^{y-x}+n^{z-x} = n^{t-x}$, and since $n > 2$, we must have $y = x$ and $z = x$. Thus $3 = n^{t-x}$, which implies $n = 3$ and $t-x = 1$. Therefore, if $n > 2$, we must have $n = 3$, $x = y = z$, $t = x+1$. We easily check that for every positive integer x we have $3^x+3^x+3^x = 3^{x+1}$.

Thus, all solutions of equation (1) in positive integers n, x, y, z, t with $x \leqslant y \leqslant z$ are $n = 2$, $y = x$, $z = x+1$, $t = x+2$, or $n = 3$, $y = x$, $z = x$, $t = x+1$, where x is an arbitrary positive integer.

183. From the solution of Problem 182 it follows that the equation $4^x+4^y+4^z = 4^t$ has no solutions in positive integers. Let us note that this equation is obtained from the equation $x^4+y^4+z^4 = t^4$ by reversing the role of bases and exponents. As regards the last equation, it is not known whether it has positive integer solutions x, y, z, t or not, as was conjectured by Euler.

184. This equation has only one solution in positive integers, namely $m = 2$, $n = 1$. In fact, since $3^2 \equiv 1 \pmod 8$, we have for positive integers k the relation $3^{2k}+1 \equiv 2 \pmod 8$ and $3^{2k-1}+1 \equiv 4 \pmod 8$, which shows that for a positive integer n the number 3^n+1 is not divisible by 8, hence is not divisible by 2^m for integers $m \geqslant 3$. Thus, if for positive integers m and n we have $2^m-3^n = 1$, then we must have $m \leqslant 2$, hence either $2-3^n = 1$, which is impossible, or $2^2-3^n = 1$, which gives $m = 2$, $n = 1$.

185. This equation has only two solutions in positive integers, namely $n = m = 1$ and $n = 2$, $m = 3$. In fact, if n is odd and > 1, then $n = 2k+1$, where k is a positive integer, and in view of $3^2 \equiv 1 \pmod 4$ we have $3^{2k+1} \equiv 3 \pmod 4$, which yields $2^m = 3^n-1 = 3^{2k+1}-1 \equiv 2 \pmod 4$. This implies that $m \leqslant 1$ or $m = 1$, and in view of $3^n-2^m = 1$ we have also $n = 1$. If n is even, $n = 2k$ for some positive integer k, then we have $2^m = 3^{2k}-1 = (3^k-1)(3^k+1)$. Two successive even numbers 3^k-1 and 3^k+1 are, therefore, powers of the number 2, which implies that these numbers are 2 and 4, which gives $k = 1$, hence $n = 2$. This yields the solution $n = 2$, $m = 3$.

186. If for positive integer x and y we have $2^x+1 = y^2$, then $(y-1)(y+1)$

$= 2^x$, hence $y > 1$, and $y-1 = 2^k$, $y+1 = 2^l$, where k is an integer $\geqslant 0$, and l is an integer $> k$. Moreover, $k+l = x$. It follows that $2^l - 2^k = 2$, which shows that $k > 0$ and, in view of $k < l$, we have $2^k | 2$. Consequently, $k \leqslant 1$, and since $k \geqslant 0$, we obtain $k = 1$. Thus, $2^l = 2^k+2 = 4$, which yields $l = 2$. We have, therefore, $x = k+l = 1+2 = 3$, hence $y^2 = 2^3+1 = 9$, and $y = 3$. The equation $2^x+1 = y^2$ has, therefore, only one solution in positive integers, namely $x = y = 3$.

187. This equation has only one such solution, namely $x = y = 1$ since in case $x > 1$ the number 2^x-1 is of the form $4k-1$, where k is a positive integer, and no square of an integer is of this form since upon division by 4 it gives the remainder either 0 or 1.

188. Suppose that our system has positive integer solution x, y, z, t. We may assume that $(x, y) = 1$ since in the case $(x, y) = d > 1$ we could divide both sides of our equations by d^2. Thus, at least one of the numbers x, y is odd. It is impossible that both are odd since in this case the left-hand sides of our equations would give remainder 3 upon dividing by 4, which is impossible, the right-hand sides being squares. However, if for instance x is even, then y cannot be odd since in this case the left-hand side of the first equation would give the remainder 2 upon dividing by 4, which is impossible since it is a square. Thus, both numbers x and y are even, contrary to the assumption that $(x, y) = 1$.

189. Our equation is equivalent to the equation $(2x+1)^2-2y^2 = -1$, which has a solution in positive integers, namely $x = 3$, $y = 5$. Our identity implies that if positive integers x and y satisfy the equation, then greater numbers $x_1 = 3x+2y+1$ and $y_1 = 4x+3y+2$ also satisfy this equation. It follows that this equation has infinitely many solutions in positive integers x and y. For $x = 3$, $y = 5$, we obtain in this manner $x_1 = 20$, $y_1 = 29$.

190. Our equation is satisfied for $x = 7$, $y = 13$. This equation is equivalent to the equation $3x^2+3x+1 = y^2$, which in turn is equivalent to $4y^2 = 12x^2+12x+4 = 3(2x+1)^2+1$, thus, to the equation $(2y)^2-3(2x+1)^2 = 1$. This implies that if x and y satisfy this equation, then greater numbers $x_1 = 4y+7x+3$ and $y_1 = 7y+12x+6$ also satisfy this equation. It follows that the equation considered has infinitely many positive integer solutions x, y. For instance, for $x = 7$, $y = 13$ we obtain $x_1 = 104$, $y_1 = 181$.

191. Proof (according to J. Browkin). If our system had a solution in positive integers x, y, z, t, then it would also have a solution with $(x, y) = 1$.

Adding our equations we obtain $6(x^2+y^2) = z^2+t^2$, which implies that $3|z^2+t^2$. Since a square of an integer which is not divisible by 3 gives the remainder 1 upon dividing by 3, it is impossible that both numbers z and t are not divisible by 3. Since, however, $3|z^2+t^2$, if one of the numbers z, t is divisible by 3, so must be the other. Thus, both z and t are divisible by 3, which implies that the right-hand side of the equation $6(x^2+y^2) = z^2+t^2$ is divisible by 9, and $3|x^2+y^2$, which, as we know, shows that both x and y are divisible by 3, contrary to the assumption that $(x, y) = 1$.

192. Our equations imply that $7(x^2+y^2) = z^2+t^2$. We have, therefore, $7|z^2+t^2$, hence, by Problem 34, we have $7|z$ and $7|t$. Thus, $49|7(x^2+y^2)$, which implies $7|x^2+y^2$, which again implies that $7|x$ and $7|y$. Thus, our system cannot have solutions with $(x, y) = 1$, which, of course, is impossible if it has at least one positive integer solution x, y, z, t, In fact, if $(x, y) = d > 1$, we would have $d|z$ and $d|t$, and it would suffice to divide each of the numbers x, y, z, t by d.

192a. It has, for instance, a solution $x = 3$, $y = 1$, $z = 4$, $t = 8$.

193. If y were even, then x^2 would be of the form $8k+7$, which is impossible. If y were odd, then we would have $x^2+1 = y^3+2^3 = (y+2)[(y-1)^2+3]$ and, in view of $y = 2k+1$, we would have $(2k)^2+3|(x^2+1)$. Since the left-hand side has a prime divisor of the form $4t+3$, the number x^2+1 would have a prime divisor of the same form, which is impossible (in view of $(x, 1) = 1$).

194. We have

$$x^2+1 = (2c)^3+y^3 = (y+2c)(y^2-2cy+4c^2)$$
$$= (y+2c)\big((y-c)^2+3c^2\big).$$

Since $c^2 \equiv 1 \pmod 8$, we have $3c^2 \equiv 3 \pmod 8$ and if y is odd, then $y-c$ is even and $(y-c)^2+3c^2$ is of the form $4k+3$; thus, it has a prime divisor of this form, which is at the same time a divisor of x^2+1, which is impossible. If y were even, then we would have $x^2 = y^3+(2c)^3-1 \equiv -1 \pmod 8$, which is impossible. It follows that there exist infinitely many positive integers which are not of the form x^2-y^3, where x and y are integers.

195. Suppose that x is odd. Then y is of the form $y^3 \equiv 0 \pmod 8$, hence $y^3-1 \equiv 7 \pmod 8$, and $x^2+(2^k)^2$ would have a prime divisor of the form $4k+3$, which is impossible, being a sum of two squares of relatively prime numbers. Thus, x is even. Let $x = 2^\alpha z$, where α is a positive integer. If $\alpha = k$,

then $2^{2k}(z^2+1) = y^3-1 = (y-1)(y^2+y+1)$, hence y must be odd and $y-1$ cannot be of the form $4k+3$. Thus, $y \equiv 1 \pmod 4$, and $y^2+y+1 \equiv 3 \pmod 4$, which is impossible. If $\alpha < k$, then $2^{2\alpha}\big((2^{k-\alpha})^2+z^2\big) = (y-1)(y^2+y+1)$, and, in view of the fact that z is odd, we proceed as above. Finally, if $\alpha > k$, then $2^{2k}\big((2^{\alpha-k}z)^2+1\big) = (y-1)(y^2+y+1)$, and we proceed as above. In particular, if $k = 1$, we see that the equation $y^3-x^2 = 5$ has no positive integer solutions x, y.

See: L. Aubry in Dickson, [7, p. 538].

196. Suppose first that $x = 1$, Then we have the equations $1+y = zt$, and $z+t = y$, which imply $zt = z+t+1$. It follows that $z \neq 1$ (since $z = 1$ would give $t = t+2$, which is impossible). If $z = 2$, then $t = 3$, hence by $y = z+t$, we get $y = 5$, which yields the solution $x = 1$, $y = 5$, $z = 2$, $t = 3$. If $z \geqslant 3$, then $t \geqslant z \geqslant 3$ and we have $z = z_1+2$, $t = t_1+2$, where $z_1 \geqslant 1$, $t_1 \geqslant 1$. It follows that $zt = (z_1+2)(t_1+2) = z_1t_1+2z_1+2t_1+$ $+4 \geqslant z_1+t_1+7 = z+t+3$, contrary to the fact that (in view of $x = 1$) we have $zt = z+t+1$.

Suppose now that $x = 2$. We then have $z \geqslant x = 2$. If $z = 2$, then $2+y$ $= 2t$, $2+t = 2y$, which implies $y = t = 2$. We would therefore have x $= y = z = t = 2$ which is a solution of our system. If $z > 2$, then, in view of $t \geqslant z$, we have $t > 2$ and we may put $z = z_1+2$, $t = t_1+2$, which implies $zt = (z_1+2)(t_1+2) = z_1t_1+2z_1+2t_1+4 \geqslant z_1+t_1+7 = z+t+3$. However, since $x = 2$, we have $2+y = zt$, $z+t = 2y$, which yields $zt = \frac{1}{2}(z+t)+2$. Thus, $\frac{1}{2}(z+t)+2 \geqslant z+t+3$, which leads to $z+t+2 \leqslant 0$, which is impossible.

Suppose now that $x > 2$, hence $x \geqslant 3$, and $z \geqslant x \geqslant 3$, $t \geqslant z \geqslant 3$. We can put $z = z_1+2$, $t = t_1+2$, where $z_1 \geqslant 1$ and $t_1 \geqslant 1$. It follows that $zt = (z_1+2)(t_1+2) = z_1t_1+2z_1+2t_1+4 \geqslant z_1+t_1+9 = z+t+5$. Similarly, since $x \geqslant 3$, we have $y \geqslant x \geqslant 3$, $xy \geqslant x+y+5$. We have, however, $z+t$ $= xy$, which implies $zt \geqslant z+t+5 = xy+5 \geqslant x+y+10 = zt+10$, which is impossible.

Thus, our system has only two solutions in positive integers x, y, z, t with $x \leqslant y$ and $x \leqslant z \leqslant t$, namely $x = 1$, $y = 5$, $z = 2$, $t = 3$ and $x = y = z$ $= t = 2$.

As regards solutions of our system in integers, there are infinitely many such solutions. J. Browkin noticed that such solutions are $x = z = 0$, $t = y$, with arbitrary y, while A. Mąkowski noticed that the solutions of our system are $x = t = -1$, y arbitrary, $z = 1-y$.

197. For $n = 1$, x may be arbitrary. For $n = 2$, we have $x_1 = x_2 = 2$.

For $n > 2$, $x_1 = x_2 = \dots = x_{n-2} = 1$, $x_{n-1} = 1$, $x_n = n$ is a solution. There are, however, other solutions, for instance for $n = 5$, we have $x_1 = x_2 = x_3 = 1$, $x_4 = x_5 = 3$. Thus, we can say that for every positive integer n there exist n positive integers such that their sum equals to their product.

198. If n is odd and > 1, then

$$a = x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1}),$$

and in view of $a > 0$, we must have $x - y \geqslant 1$, hence $x^{n-1} + y^{n-1} \leqslant a$. Thus, $x < \sqrt[n-1]{a}$ and $y < \sqrt[n-1]{a}$ and a finite number of checking will suffice. If $n = 1$, then all positive integer solutions of the equation $a = x - y$ are: y arbitrary, $x = a + y$.

If $n = 2k$, where k is a positive integer, then

$$a = x^n - y^n = x^{2k} - y^{2k} = (x^k - y^k)(x^k + y^k),$$

where $x^k - y^k \geqslant 1$, hence $x^k + y^k \leqslant a$. It follows that $x < \sqrt[k]{a}$, $y < \sqrt[k]{a}$, and a finite number of checking will suffice.

199. In order for a triangular number $t_x = x(x+1)/2$ to be pentagonal, it is necessary and sufficient that for every x there exists a positive integer y such that

$$y(3y-1) = x(x+1). \tag{1}$$

It suffices, therefore, to show that equation (1) has infinitely many positive integer solutions x, y.

We easily check that

$$(4x+7y+1)(12x+21y+2) - (7x+12y+1)(7x+12y+2) = y(3y-1) - x(x+1),$$

and it immediately follows that if positive integers x, y satisfy (1), then the greater integers

$$x_1 = 7x + 12y + 1, \quad y_1 = 4x + 7y + 1 \tag{2}$$

satisfy the equation $y_1(3y_1 - 1) = x_1(x_1 + 1)$. Since the numbers $x = y = 1$ satisfy (1), it follows that this equation has infinitely many positive integer solutions x, y. The solution $x = 1 = 1$ gives by (2) the solution $x_1 = 20$, $y_1 = 12$, which in turn leads to $x_2 = 285$, $y_2 = 165$, and so on.

MISCELLANEA

200. The equation $4x+2=0$ has obviously no integer roots. On the other hand, the congruence $4x+2 \equiv 0 \pmod{p}$ is solvable for every prime modulus p. For the modulus equal 2, it is, of course, solvable identically while if p is an odd prime, $p=2k+1$, where k is a positive integer, it has the solution $x=k$.

201. Put $m=a$; if the congruence $ax+b \equiv 0 \pmod{m}$ has a solution, then $a|b$, hence $b=ak$, where k is an integer, and the equation $ax+b=0$ has the integer root $x=-k$.

202. We have identically $6x^2+5x+1=(3x+1)(2x+1)$, which implies that the equation $6x^2+5x+1=0$ has no integer solutions. Let m be an arbitrary positive integer. We have then $m=2^{\alpha}m_1$, where α is an integer $\geqslant 0$, and m_1 is odd. Since $(2^{\alpha}, m_1)=1$, there exists a positive integer x such that $3x \equiv -1 \pmod{2^{\alpha}}$, and $2x \equiv -1 \pmod{m_1}$, which yields $m=2^{\alpha}m_1 | (3x+1)(2x+1)$, and consequently, $6x^2+5x+1 \equiv 0 \pmod{m}$.

203. This it true for $n=1$ since a square of an odd number gives the remainder 1 upon dividing by 8. Suppose that the assertion holds for a positive integer n. Then for odd k: $k^{2^n}=2^{n+2}t+1$, where t is an integer. It follows that $k^{2^{n+1}}=(2^{n+2}t+1)^2=2^{2n+4}t^2+2^{n+3}t+1=2^{n+3}(2^{n+1}t^2+t)+1$, which implies that $2^{n+3}|k^{2^{n+1}}-1$. The proof follows by induction.

204. The proof follows immediately from the identity

$$(3x+4y)^2-(2x+3y)^2=x^2-2y^2,$$

and from the remark that for positive integers x and y we have $3x+4y>x$, and $2x+3y>y$.

205. If for some integers x and y the number x^2-2y^2 is odd, then x must be odd, hence $x^2 \equiv 1 \pmod 8$. In the case where y is even, we have $2y^2 \equiv 0 \pmod 8$, and in the case where it is odd, we have $2y^2 \equiv 2 \pmod 8$. Thus, in case of x^2-2y^2 being odd, we have $x^2-2y^2 \equiv \pm 1 \pmod 8$, which shows that for integers x and y the number x^2-2y^2 cannot be of the form $8k+3$ or $8k+5$, where k is an integer.

206. It can be seen quite readily that for every positive integer n, the number $(2n+1)^2-2\cdot 2^2$ is of the form $8k+1$, where k is an integer $\geqslant 0$. Next, we have $1=3^2-2\cdot 2^2$, $9=9^2-2\cdot 6^2$, $17=5^2-2\cdot 2^2$, $25=15^2-2\cdot 10^2$, while the number 33 cannot be represented in the form x^2-2y^2, where x

and y are positive integers. We shall prove, more generally, that no number of the form $72t+33$, where $t = 0, 1, 2, \ldots$, can be represented in the form x^2-2y^2 with integers x and y. In fact, suppose that $72t+33 = x^2-2y^2$ where t, x and y are integers. The left-hand side is divisible by 3, but is not divisible by 9. It follows that none of the numbers x, y is divisible by 3 since if $3|x$, we would have $3|y$ and the right-hand side would be divisible by 9, which is impossible. Thus, the numbers x and y are not divisible by 3, hence x^2 and y^2 give remainder 1 upon dividing by 3; thus, the number x^2-2y^2 gives the remainder 2 upon dividing by 3, which is impossible since the left-hand side is divisible by 3.

Thus, there exist infinitely many positive integers of the form $8k+1$ (where $k = 1, 2, \ldots$) which are not of the form x^2-2y^2, where x and y are integers, and the least such number is $33 = 8\cdot 4+1$.

207. The even perfect numbers are, as it is well known, of the form $2^{p-1}(2^p-1)$, where p and 2^p-1 are primes (see, for instance, Sierpiński [37, p. 172, corollary]). For $p = 2$ we have the number 6. If $p > 2$, then p is a prime of the form $4k+1$ or $4k+3$. If $p = 4k+1$, then $2^{p-1} = 2^{4k} = 16^k$, and the last digit of 2^{p-1} is obviously 6, while $2^p-1 = 2^{4k+1}-1 = 2\cdot 16^k-1$ and the last digit is obviously 1. Thus, the last digit of the product $2^{p-1}(2^p-1)$ is 6. If $p = 4k+3$, then the number $2^{p-1} = 2^{4k+2} = 4\cdot 16^k$ has the last digit 4, while the last digit of 2^p is 8, hence the last digit of the number 2^p-1 is 7, and, consequently, the number $2^{p-1}(2^p-1)$ (as the product of two numbers, one with the last digit 4, and the other with the last digit 7) has the last digit 8.

This completes the proof.

REMARK. One could prove (but the proof is more difficult) that if the last digit of a perfect number is 8, then the last but one digit is 2.

208. The value of our fraction, in the scale g, is

$$\frac{1+g^2+g^4+g^6+g^8}{1+g+g^4+g^7+g^8}$$

and we have to prove that for every positive integer k, this fraction is equal to the fraction

$$\frac{1+g^2+g^4+g^5+\ \ldots\ +g^{2k+2}+g^{2k+4}+g^{2k+6}}{1+g\ +g^4+g^5+\ \ldots\ +g^{2k+2}+g^{2k+5}+g^{2k+6}}. \qquad (1)$$

The assertion can be shown by checking that the products of the numerator of each of these fractions by the denominator of the other are the same. See P. Anning [1].

REMARK. J. Browkin noticed that for positive integer k we have the identity

$$1+g^2+g^4+g^5+\dots+g^{2k+2}+g^{2k+4}+g^{2k+6}$$
$$=(1-g+g^2-g^3+g^4)(1+g+g^2+\dots+g^{2k+2}),$$

and

$$1+g+g^4+g^5+\dots+g^{2k+2}+g^{2k+5}+g^{2k+6}$$
$$=(1-g^2+g^4)(1+g+g^2+\dots+g^{2k+2}),$$

which implies that the fraction (1) is, for $k = 1, 2, \dots$, equal to the fraction

$$\frac{1-g+g^2-g^3+g^4}{1-g^2+g^4},$$

hence its value is independent of k.

209*. A. Schinzel proved a more general theorem, namely the theorem asserting that if g is a positive integer, even, and not divisible by 10, then the sum of decimal digits of g^n increases to infinity with n. We shall present his proof.

Let us define an infinite sequence of integers a_i $(i = 0, 1, 2, \dots)$ as follows: put $a_0 = 0$, and for $k = 0, 1, 2, \dots$, let a_{k+1} denote the smallest positive integer such that $2^{a_{k+1}} > 10^{a_k}$ (thus, we shall have $a_1 = 1$, $a_2 = 4$, $a_3 = 14$, and so forth). Clearly, $a_1 < a_2 < a_3 < \dots$.

We shall prove that if for some positive integer k we have $n \geqslant a_k$, then the sum of digits of g^n is $\geqslant k$.

Let c_j denote the digit of the decimal expansion of g^n standing at 10^j. Since g is even, we have $2^n|g^n$, and since $n \geqslant a_k$, we have, for $i = 1, 2, \dots$ $\dots, k-1$, the relation $2^{a_i}|g^n$. Moreover, since $2^{a_i}|10^{a_i}$, we have

$$2^{a_i}|c_{a_i-1}10^{a_i-1}+\dots+c_0.$$

If for $a_{i-1} \leqslant j < a_i$ all digits c_j were equal zero, we would have

$$2^{a_i}|c_{a_{i-1}-1}10^{a_{i-1}-1}+\dots+c_0,$$

and, in view of $c_0 \neq 0$, also

$$2^{a_i} \leqslant c_{a_{i-1}-1}10^{a_{i-1}-1}+\dots+c_0 < 10^{a_i-1}.$$

This implies $2^{a_i} < 10^{a_{i-1}}$, contrary to the definition of a_i. Thus, at least one of the digits c_j, where $a_{i-1} \leqslant j < a_i$, is different from zero. Since this is true for $i = 1, 2, \ldots, k$, at least k digits of g^n are different from zero. For sufficiently large n (for $n \geqslant a_k$), the sum of decimal digits of g^n is not smaller than an arbitrarily given number k. This shows that the sum of decimal digits of g^n increases to infinity together with n, which was to be proved.

A. Schinzel noted that in a similar way one can prove that if g is an odd positive integer divisible by 5, then the sum of decimal digits of g^n increases to infinity with n.

In particular, from the theorem proved above it follows (for $g = 2$) that the sum of decimal digits of 2^n increases to infinity with n. It does not mean, however, that the increase is monotone: we have, for instance, the sum of digits of 2^3 equal 8, while the sum of digits of 2^4 equal 7, and the sum of digits of 2^5 equal 5. Next, the sum of digits of 2^9 is 8, while that of 2^{10} is 7. Similarly, the sum of digits of 2^{16} is 25, while that of 2^{17} is 14.

210*. Proof due to A. Schinzel. Let k be a given integer > 1, and let c be an arbitrary fixed digit of decimal system. Since $k > 1$, we easily prove (for instance, by induction) that $10^{k-1} > 2 \cdot 2^k$. Let t denote the least integer such that $t \geqslant c \cdot 10^{k-1}/2^k$; we shall have, therefore,

$$t < c\frac{10^{k-1}}{2^k}+1, \quad \text{and} \quad t+1 < c\frac{10^{k-1}}{2^k}+2.$$

At least one of integers t and $t+1$ is not divisible by 5; denote this number by u. We shall have

$$c\frac{10^{k-1}}{2^k} \leqslant u < c\frac{10^{k-1}}{2^k}+2$$

and since $2 \cdot 2^k < 10^{k-1}$, we shall have, for $l = 2^k u$, the relation

$$c \cdot 10^{k-1} \leqslant l < (c+1)10^{k-1}, \tag{1}$$

which shows that the number $l = 2^k u$ has k digits, the first of which (hence the kth from the end) is c (this digit can be zero).

In view of $l = 2^k u$ we have $2^k | l$, and by the definition of u it follows that $5 | u$, hence $(l, 5) = 1$.

As we know, the number 2 is a primitive root for the modulus 5^k (see, for instance, W. Sierpiński [24, p. 246, lemma]). Since $(l, 5) = 1$, there exists an integer $n \geqslant k$ such that $2^n \equiv l \pmod{5^k}$. Since $2^k | l$ and $2^k | 2^n$, we have also

$2^n \equiv l \pmod{2^k}$, and consequently $2^n \equiv l \pmod{10^k}$, which shows that the k last digits of the number l coincide with the corresponding digits of 2^n. It follows that the kth from the end digit of the number 2^n is c, which was to be proved.

REMARK. The last four digits of powers of 2 cannot be of the form $111c$ with $c = 2, 4, 6$ or 8 since none of the numbers 1112, 1114, 1116 and 1118 is divisible by 16.

In the paper quoted above I proved (p. 249), that the third and second from the end digits of 2^n (where $n = 3, 4, \ldots$) can be arbitrary. I proved also that if m is an arbitrary positive integer and k is the number of its digits, then there exists a positive integer n such that k first digits of the number 2^n are the same as the digits of m.

211. For integer $n \geqslant 4$, we have $5^{n+4}-5^n = 5^n(5^4-1) = 5^n \cdot 16 \cdot 39$, hence $5^{n+4} \equiv 5^n \pmod{10000}$, and it follows that the last four digits of the sequence 5^n $(n = 4, 5, \ldots)$ form a four-term period. The period is 0625, 3125, 5625, 8125. This period is not pure since the numbers 5, $5^2 = 25$, $5^3 = 125$ do not belong to it.

212. Let s be a given positive integer, and let $c_1, c_2, \ldots, c_s$ be an arbitrary sequence of s decimal digits. Let $m = (c_1 c_2 \ldots c_s)_{10}$ be a number with s digits equal respectively to $c_1, c_2, \ldots, c_s$. Let us choose a positive integer k such that $2\sqrt{m} < 10^{k-1}$ and let $n = [10^k\sqrt{m}]+1$, where $[x]$ denotes the greatest integer $\leqslant x$. We have $10^k\sqrt{m} < n \leqslant 10^k\sqrt{m}+1$, which implies that

$$10^{2k}m < n^2 \leqslant 10^{2k}m+2\cdot 10^k\sqrt{m}+1 < 10^{2k}m+10^{2k-1}+1 < 10^{2k}m+10^{2k},$$

and consequently

$$10^{2k}m < n^2 < 10^{2k}m+(10^{2k}-1);$$

it follows that

$$(c_1 c_2 \ldots c_s 00 \ldots 0)_{10} < n^2 < (c_1 c_2 \ldots c_s 999 \ldots 9)_{10},$$

where the number of zeros and the number of nines is $2k$. It follows that the first s digits of n^2 are $c_1, c_2, \ldots, c_s$.

213. If n is a positive integer, then $n^{n+20}-n^n = n^n(n^{20}-1)$ is divisible by 4. In fact, if n is even, then $4|n^n$, and if n is odd, then n^{10} is odd, hence its square n^{20} gives the remainder 1 upon dividing by 8. Thus, $8|n^{20}-1$. For positive integer n, the number $n^{n+20}-n^n$, hence also the number $(n+20)^{n+20}-$

$-n^n$, is always divisible by 4. On the other hand, if a and b are positive integers such that $a > b$ and $4|a-b$, then for positive integer n we have $5|n^a-n^b$. Indeed, we have $a = b+4k$, where k is a positive integer, hence

$$n^a-n^b = n^b(n^{4k}-1).$$

If $5|n$, then the first factor on the right is divisible by 5; if $5\nmid n$, then by the Fermat theorem we have $n^4 \equiv 1 \pmod 5$, which implies $n^{4k} \equiv 1 \pmod 5$, and the second factor on the right-hand side of our equality is divisible by 5. We proved, therefore, that if a and b are positive integers, $a > b$, and $4|a-b$, then for positive integers n we have $5|n^a-n^b$, and of course, we must also have $5|(n+20)^a-n^b$. In particular, for $a = (n+20)^{n+20}$ and $b = n^n$, we have, as shown above, $4|a-b$, hence $5|(n+20)^{(n+20)^{n+20}}-n^{n^n}$. Since the right-hand side is always even (as n and $n+20$ are either both even or both odd), we have, for positive integers n, the relation

$$10|(n+20)^{(n+20)^{n+20}}-n^{n^n},$$

which shows that the numbers $(n+20)^{(n+20)^{n+20}}$ and n^{n^n} have the same last digit. The sequence of last digits of numbers n^{n^n} $(n = 1, 2, \dots)$ is therefore periodic; the period is pure, and consists of at most 20 terms. It is easy to see that the period consists of exactly 20 terms, equal to

$$1, 6, 7, 6, 5, 6, 3, 4, 9, 0, 1, 6, 3, 6, 5, 6, 3, 4, 9, 0.$$

214. Let m be an arbitrary positive integer. Let us partition the digits of the given infinite decimal fraction into blocks of m digits each; we shall have infinitely many such blocks. On the other hand, there are 10^m different sequences formed of m digits; this number being finite, we conclude that at least one of them must be repeated an infinite number of times.

REMARK. For irrational numbers $\sqrt{2}$, π or e, we do not even know which digit will be repeated in the decimal expansion an infinite number of times; it is easy to show that for each of these numbers there exist at least two such digits.

215. If $3^{2k} = (n+1)+(n+2)+ \dots +(n+3^k)$, then we have $3^{2k} = 3^kn+ +\frac{1}{2}3^k(3^k+1)$, which gives $n = \frac{1}{2}(3^k-1)$. Thus, the number 3^{2k} is a sum of 3^k terms, equal to consecutive positive integers, the least of them being $n+1 = \frac{1}{2}(3^k+1)$. We have, for example, for $k = 1$, 2 and 3: $3^2 = 2+3+4$, $3^4 = 5+6+ \dots +13$, $3^6 = 14+15+ \dots +40$. See Khatri [12].

216. As we know, if a and b are real numbers such that $b-a > 1$, then between a and b there is at least one integer; in fact, such an integer equals

for instance $[a]+1$, where $[x]$ denotes the greatest integer not exceeding x. Indeed, we have $a < [a]+1 \leqslant a+1 < b$ (since $b-a > 1$).

Let s be an integer > 1, and let

$$\mu_s = \frac{1}{(\sqrt[s]{2}-1)^s};$$

this number will be real and positive. Thus, for integer $n > \mu_s$, we shall have

$$n > \frac{1}{(\sqrt[s]{2}-1)^s}, \quad \text{hence} \quad \sqrt[s]{n} > \frac{1}{\sqrt[s]{2}-1} \quad \text{and} \quad \sqrt[s]{n}(\sqrt[s]{2}-1) > 1$$

which implies that

$$\sqrt[s]{2n} - \sqrt[s]{n} = \sqrt[s]{n}(\sqrt[s]{2}-1) > 1.$$

Thus, there exists a positive integer k such that $\sqrt[s]{n} < k \sqrt[s]{n} < k < \sqrt[s]{2n}$, which yields $n < k^s < 2n$. As m_s we may take number $[\mu_s]+1$.

For $s = 2$, we have $[\mu_2] = 5$, and already between 5 and 10 there lies a square number, namely 3^2, while between 4 and 8 there is no square number. Thus, the least m_2 is 5. Similarly, we easily compute that the least number m_3 is 33.

217. Let m be an arbitrary positive integer. By the Chinese remainder theorem, there exists a positive integer x such that

$$x \equiv p_i - i + 1 \pmod{p_i^2} \quad \text{for} \quad i = 1, 2, .., m, \tag{1}$$

where p_i denotes the ith prime. The sequence of m consecutive integers $x, x+ +1, \ldots, x+m-1$ has the desired property since by (1), for $i = 1, 2, \ldots, m$ we have $x+i-1 = p_i^2 k_i + p_i$, where k_i is an integer. This number will therefore be divisible by p_i but not divisible by p_i^2, hence $x+i-1$ cannot be a power with exponent > 1 of any positive integer.

218. $u_n = 3^{n-1}$ for $n = 1, 2, \ldots$. Easy proof by induction.

219. $u_n = (2-n)a+(n-1)b$ for $n = 1, 2, \ldots$. Easy proof by induction.

220. $u_n = (-1)^n[(n-2)a+(n-1)b]$ for $n = 1, 2, \ldots$. We easily check that the formula holds for $n = 1$ and $n = 2$. Assuming that for some n the formula is valid for u_n and u_{n+1}, we easily check, using the fact that $u_{n+2} = -(u_n+2u_{n+1})$, that the formula is valid for u_{n+2}. Thus, the proof follows by induction.

In particular, if $a = 1$, $b = -1$, we obtain $u_n = (-1)^{n+1}$, and for $a = 1$, $b = -2$, we obtain $u_n = (-1)^{n+1}n$.

221. $u_n = \frac{3}{4}[3^{n-2}+(-1)^{n-1}]a+\frac{1}{4}[3^{n-1}+(-1)^n]b$ for $n = 1, 2, \ldots$. Proof by induction.

222. There are only two such integers, namely $a = 1$ and $a = -1$. We easily check that both these numbers satisfy the desired condition. From this condition for $n = 1$ it follows that $a^a = a$. Thus, if a were an integer $\geqslant 2$, we would have $a^a \geqslant a^2 > a$, which is impossible. If we had $a \leqslant -2$, we would also have $|a^a| = 1/|a|^{|a|} < 1$, which again is impossible since $a^a = a$ and $a \leqslant -2$ imply $|a^a| = |a| > 2$.

223*. Let a and b be arbitrary positive integers, and let c^2 denote the greatest square divisor of a^2+b^2, that is, $a^2+b^2 = kc^2$. Let $x = a^2k$, $y = b^2k$; we have $x+y = a^2k+b^2k = (a^2+b^2)k = (kc)^2$ while $xy = (abk)^2$.

We shall show that all pairs of positive integers, whose sum and product are squares, can be obtained in this manner for suitably chosen a and b.

Suppose that $x+y = z^2$, $xy = t^2$, where z and t are positive integers. Let $d = (x, y)$ and let c_1 denote the greatest square divisor of d; we have, therefore, $d = kc_1^2$, where k is a positive integer, not divisible by any square of an integer > 1. We have $x = dx_1$, $y = dy_1$ where $(x_1, y_1) = 1$ and from $x+y = z^2$ it follows that $(x_1+y_1)d = z^2$. Thus, $d = kc_1^2|z^2$, and since k is not divisible by any square of an integer > 1, we find that $kc_1|z$, which implies that $z = kc_1z_1$, where z_1 is a positive integer. It follows that $(x_1+y_1)d = x+ +y = z^2 = k^2c_1^2z_1^2 = kdz_1^2$, which implies that $x_1+y_1 = kz_1^2$ and $x_1y_1 = t^2/d^2$. Since $(x_1, y_1) = 1$, it follows that the numbers x_1 and y_1 are squares, that is, $x_1 = a_1^2$, $y_1 = b_1^2$. Since $x = dx_1 = k(c_1a_1)^2$, $y = dy_1 = k(c_1b_1)^2$, putting $a = c_1a_1$, $b = c_1b_1$ we get $x = ka^2$, $y = kb^2$, and $a^2+b^2 = (c_1a_1)^2+(c_1b_1)^2 = c_1^2(x_1+y_1) = k(c_1z_1)^2$; putting $c_1z_1 = c$, we get $a^2+b^2 = kc^2$; since the number k is not divisible by any square of an integer $r > 1$, the number c^2 is the greatest square divisor of a^2+b^2.

All pairs of positive integers $\leqslant 100$ whose both sum and product are squares are 2, 2; 5, 20; 8, 8; 10, 90; 18, 18; 20, 80; 9, 16; 32, 32; 50, 50; 72, 72; 2, 98; 98, 98; 36, 64.

224. There is only one such number, namely 10. In fact, if $(2x-1)^2+ +(2x+1)^2 = \frac{1}{2}y(y+1)$, then $(2x+1)^2-(8x)^2 = 17$, and the number 17 has only one representation as the difference of two squares of integers, namely $17 = 9^2-8^2$. This yields $2y+1 = 9$, hence $y = 4$ and $t_y = \frac{1}{2}y(y+1) = 10$.

225*. We shall prove by induction with respect to n that the theorem of Hogatt holds for every positive integer $\leqslant u_n$. It is true for $n = 1$ since $u_1 = 1$, and for $n = 2$ since $u_2 = 1$. Let now n be an integer > 2, and suppose that every positive integer $\leqslant u_n$ is a sum of different terms of Fibonacci sequence. Let k denote an integer such that $u_n < k \leqslant u_{n+1}$. If we had $k-u_n > u_{n-1}$, we would have $u_{n+1} \geqslant k > u_{n-1}+u_n = u_{n+1}$, which is impossible. We have, therefore, $0 < k-u_n \leqslant u_{n-1}$. The positive integer $k-u_n$ is, by induction, equal to a sum of different terms of Fibonacci sequence, and in view of $k-u_n \leqslant u_{n-1} < u_n$, the number u_n does not appear in the representation. It follows that $k = (k-u_n)+u_n$ is a sum of different terms of Fibonacci sequence, which completes the proof of Hogatt theorem.

We have $1 = u_1$, $2 = u_3$, $3 = u_4 = u_1+u_3$, $4 = u_1+u_4$, $5 = u_5 = u_3+u_4$, $6 = u_1+u_5$, $7 = u_3+u_5$, $8 = u_6 = u_4+u_5$, $9 = u_1+u_6$, $10 = u_3+u_6$.

226. We shall proceed by induction. Our formula is valid for $n = 2$ since $1^2 = 1\cdot 2+(-1)$. Suppose that our formula holds for an integer $n \geqslant 2$. We have, therefore, $u_n^2 = u_{n-1}u_{n+1}+(-1)^{n-1}$. It follows that

$$u_{n+1}^2-u_nu_{n+2} = u_{n+1}^2-u_n(u_n+u_{n+1})$$

$$= u_{n+1}(u_{n+1}-u_n)-u_n^2 = u_{n+1}u_{n-1}-u_n^2 = (-1)^n$$

which proves the formula for $n+1$.

227. Let us notice first that from the identity

$$6t = (t+1)^2+(t-1)^3+(-t)^3+(-t)^3$$

it follows that every integer dividible by 6 is a sum of four cubes of integers.

Since for every integer k and positive integer n, for $r = 0, 1, 2, 3, 4, 5$ each of the numbers $6k+r-(6n+r)^3$ is divisible by 6 (as $6|r^3-r$ for integer r), it follows that every integer can be in infinitely many ways represented as a sum of five cubes of integers.

REMARK. It is conjectured (and this conjecture was checked for all positive integers < 1000) that every integer can be represented in infinitely many ways as a sum of four cubes of integers; see Schinzel, Sierpiński [21] and Demjanenko [6].

228. The solution follows immediately from the identity

$$3 = (4+24n^3)^3+(4-24n^3)^3+(-24n^3)^3+(-5)^3 \quad \text{for } n = 1, 2, 3, \ldots .$$

229. The proof follows immediately from the following two identities valid for integer $t > 8$:

$$(t-8)^2+(t-1)^2+(t+1)^2+(t+8)^2 = (t-7)^2+(t-4)^2+(t+4)^2+(t+7)^2$$

and

$$(t-8)^3+(t-1)^3+(t+1)^3+(t+8)^3 = (t-7)^3+(t-4)^3+(t+4)^3+(t+7)^3.$$

230. Suppose that for some positive integers m we have $4^m\cdot 7 = a^2+b^2+c^2+d^2$, where at least one of the numbers a, b, c, d, say a, is $\geqslant 0$ and $< 2^{m-1}$. We cannot have $a = 0$ since in this case $4^m\cdot 7$ would be a sum of three squares of integers, which is impossible (see, for instance, W. Sierpiński, [37, p. 363, Theorem 3]). We have therefore $m > 1$ and $a = 2^k(2t-1)$, where k is a non-negative integer $\leqslant m-2$, and t is a positive integer. It follows that

$$4^m\cdot 7-[2^k(2t-1)]^2 = 4^k[4^{m-k}\cdot 7-(8u+1)] = 4^k(8v+7),$$

where u and v are integer (since $k \leqslant m-2$, which implies that $m-k \geqslant 2$), and we have $4^k(8v+7) = b^2+c^2+d^2$, which is impossible.

REMARK. One can easily prove that the number $4^m\cdot 7$ (where m is a positive integer) has at least one representation as a sum of four squares of integers since

$$4^m\cdot 7 = (2^m)^2+(2^m)^2+(2^m)^2+(2^{m+1})^2.$$

231. We easily check that the first six integers > 2, which are sums of two cubes of positive integers are $1^3+2^3 = 9$, $2^3+2^3 = 16$, $1^3+3^3 = 28$, $2^3+3^3 = 35$, $3^3+3^3 = 54$, $1^3+4^3 = 65$. None of the numbers 9, 16, 28, 35, and 54 is a sum of two squares of integers, while $65 = 1^2+8^2$. Thus, the least integer > 2 which is a sum of two squares of integers and a sum of two cubes of positive integers is 65.

To show that there exists infinitely many positive integers which are sums of two squares and sums of two cubes of two relatively prime positive integers, it suffices to note that for positive integer k we have

$$1+2^{6k} = 1^2+(2^{3k})^2 = 1^3+(2^{2k})^3.$$

232. For instance, the number $1+2^{s!}$ has this property since $k|s!$ for $k = 1, \ldots, s$. Of course, instead of $s!$ we could take the number $[1, 2, \ldots, s]$.

233*. For instance, all numbers of the form $6\cdot 8^n$ ($n = 0, 1, 2, \ldots$) have the desired property. In fact, no such number is a sum of cubes of two positive integers, as in the case of even n, this number gives the remainder 6 upon division by 9, while in case of odd n, it gives the remainder 3 (since $8 \equiv -1 \pmod{9}$). On the other hand, every cube of an integer gives the remainder 0, 1, or -1 upon dividing by 9, hence a sum of two cubes can give only the remainder 0, 1, -1, 2 or -2, and it cannot give the remainder 3 or 6 (nor 4 or 5).

On the other hand, we easily check that $6 = (17/21)^3+(37/21)^3$, which gives

$$6\cdot 8^n = \left(\frac{17\cdot 2^n}{21}\right)^3 + \left(\frac{37\cdot 2^n}{21}\right)^3.$$

Thus, the numbers $6\cdot 8^n$ $(n = 0, 1, \ldots)$ are cubes of two positive rational numbers.

234*. Proof due to A. Schinzel.

For instance, all numbers of the form $7\cdot 8^n$ $(n = 0, 1, 2, \ldots)$ have the desired property. In fact, on one hand we have $7\cdot 8^n = (2^{n+1})^3-(2^n)^3$ for $n = 0, 1, 2, \ldots$; on the other hand we shall prove that none of the numbers $7\cdot 8^n$ $(n = 0, 1, 2, \ldots)$ is a sum of two cubes of positive integers. We easily check that the assertion is true for $n = 0$ and $n = 1$. Suppose now that there exists a positive integer n such that $7\cdot 8^n$ is a sum of two cubes of positive integers, and let n be the least of such numbers; we have, therefore, $n \geqslant 2$, and

$$7\cdot 8^n = x^3+y^3 = (x+y)(x^2-xy+y^2),$$

where x and y are positive integers. Since the left-hand side is even, x and y are either both even or both odd. If they are both odd, x^2-xy+y^2 is odd, and as the left-hand side has only odd divisors 1 and 7, we must have either $x^2-xy+y^2 = 1$ or $x^2-xy+y^2 = 7$. In the first case we would have x^3+ $+y^3 = x+y$, and since x and y are positive integers, this implies that $x = y = 1$, hence $7\cdot 8^n = 2$, which is impossible. If $x^2-xy+y^2 = 7$, then

$$(2x-y)^2+3y^2 = (2y-x)^2+3x^2 = 28,$$

which yields $3x^2 \leqslant 28$ and $3y^2 \leqslant 28$, hence $x \leqslant 3$, $y \leqslant 3$. Thus, $x^3+y^3 \leqslant 54$, which is impossible, as $x^3+y^3 = 7\cdot 8^n \geqslant 7\cdot 8^2$.

Thus, x and y are both even, $x = 2x_1$, $y = 2y_1$, where x_1 and y_1 are positive integers, and in view of $7\cdot 8^n = x^3+y^3$ we have $7\cdot 8^{n-1} = x_1^3+y_1^3$, contrary to the definition of the number n.

We proved, therefore, that the numbers $7 \cdot 8^n$ ($n = 0, 1, 2, \ldots$) have the desired property.

REMARK. It has been proved that there exist infinitely many positive integers n not divisible by any cube of an integer > 1, such that they cannot be represented as sums of two cubes of rational numbers, but the proof is difficult. Such numbers $\leqslant 50$ are 3, 4, 5, 10, 11, 14, 18, 21, 23, 25, 29, 36, 38, 39, 41, 44, 45, 46, 47. The number 22 is a sum of two cubes of rational numbers, but with large denominators:

$$22 = \left(\frac{17299}{9954}\right)^3 + \left(\frac{25469}{9954}\right)^3.$$

See [23, p. 301, and tables on pp. 354 and 357].

235*. Proof due to A. Schinzel.

Numbers of the form $(2^k-1)\cdot 2^{nk}$ for $n = 0, 1, 2, \ldots$ have the desired property. We obviously have $(2^k-1)2^{nk} = (2^{n+1})^k - 2^{nk}$ and it remains to show that the equation

$$(2^k-1)2^{nk} = u^k+v^k \tag{1}$$

has no positive integer solutions u and v. This is true for $n = 0$ since

$$1^k+1^k < 2^k-1 < 2^k+1^k.$$

Suppose that there exist positive integers n for which equation (1) has a solution in positive integers u and v, and let n be the least of such numbers. If u and v were both even, $u = 2u_1$, $v = 2v_1$, we would have, by (1):

$$(2^k-1)2^{(n-1)k} = u_1^k+v_1^k$$

contrary to the definition of the number n. Since the left-hand side of (1) is even, both numbers u and v have to be odd.

Suppose that k is odd and > 3. From the formula

$$\frac{u^k+v^k}{u+v} = u^{k-1} - u^{k-2}v + u^{k-3}v^2 - \ldots + v^{k-1}$$

where the right-hand side contains k terms, all of them odd, it follows that the left-hand side is odd; since this number is a divisor of $(2^k-1)2^{nk}$, we must have

$$\frac{u^k+v^k}{u+v} \leqslant 2^k-1.$$

We may assume that $u \geqslant v$, which implies

$$\frac{u^k+v^k}{u+v} \geqslant v^{k-1},$$

and consequently $v^{k-1} < 2^k$; thus $v < 2^{k/(k-1)} < 3$ (because $k > 3$). Since k is odd, we have $v = 1$, and

$$\frac{u^k+v^k}{u+v} = \frac{u^k+1}{u+1} \geqslant u^{k-2}(u-1) > (u-1)^{k-1}.$$

It follows that $(u-1)^{k-1} < 2^k$, which yields $u-1 < 3$, hence, in view of the fact that u is odd, $u = 1$ or $u = 3$. The relation $u = 1$ is impossible since we would then have $u^k+v^k = 2$, contrary to (1). The relation $u = 3$ is impossible, too, since it would yield

$$\frac{u^k+v^k}{u+v} = \frac{3^k+1}{4}$$

which is $> 2^k-1$ (for $k > 3$).

Suppose now that k is an even positive integer. Since u and v are odd, the number u^k+v^k gives the remainder 2 upon division by 4, which is impossible since the left-hand side of (1) is divisible by 4. This completes the proof.

236*. Proof due to A. Rotkiewicz.

If $2|n$, then for positive integers k and l the number $(2k+1)^n+(2l+1)^n$ is a sum of two nth powers of positive integers; as a number of the form $4t+2$, it is not a difference of two squares; since $2|n$, it is not a difference of two nth powers of positive integers, either. On the other hand, if $2 \nmid n$, then the numbers $(2^n+1)2^{nk} = (2^{k+1})^n+(2^k)^n$, where $k = 0, 1, 2, \ldots$, are not the differences of two nth powers of positive integers. In fact, if we had $(2^n+1)2^{nk} = x^n-y^n$ for positive integer x and y with $x > y$, then the numbers $x_1 = x/(x, y)$ and $y_1 = y/(x, y)$ would be positive integers, and being relatively prime, could not be both even. It easily follows that $2 \nmid (x_1^n-y_1^n)/(x_1-y_1)$ and since

$$(2^n+1)2^{nk} = (x, y)^n(x_1-y_1)\frac{x_1^n-y_1^n}{x_1-y_1},$$

we must have

$$\left.\frac{x_1^n-y_1^n}{x_1-y_1}\,\right|\, 2^n+1,$$

which implies that

$$\frac{x_1^n-y_1^n}{x_1-y_1} \leqslant 2^n+1.$$

We have, however,

$$\frac{x_1^n-y_1^n}{x_1-y_1} > x_1^{n-1} \geqslant 3^{n-1}$$

(since we cannot have $x_1 = 2$, for then we would have $y_1 = 1$, and $2^n-1|2^n+1$, which is impossible). We would therefore have $3^{n-1} < 2^n+1$, which is impossible for $n \geqslant 3$.

237. We shall use the well-known formula

$$1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We have to find the least integer $n > 1$ such that $n(n+1)(2n+1) = 6m^2$ where m is a positive integer. We shall distinguish six cases:

1. $n = 6k$, where k is a positive integer. Our equation takes on the form

$$k(6k+1)(12k+1) = m^2.$$

The factors on the left-hand side are pairwise relatively prime, hence they all must be squares. If $k = 1$, then $6k+1$ is not a square. The next square after 1 is 4. If $k = 4$, we have $6k+1 = 5^2$, $12k+1 = 7^2$, and consequently, for $n = 6k = 24$ the sum $1^2+2^2+\ldots+24^2$ is a square of a positive integer, 70.

2. $n = 6k+1$, where k is a positive integer. In this case we have

$$(6k+1)(3k+1)(2k+1) = m^2,$$

and each of the numbers $2k+1$, $3k+1$, and $6k+1$ (which are pairwise relatively prime) must be a square. The least k for which the number $2k+1$ is a square is $k = 4$; in this case, however, we have $n = 6k+1 > 24$.

3. $n = 6k+2$, where k is an integer $\geqslant 0$. We have in this case

$$(3k+1)(2k+1)(12k+5) = m^2,$$

and the numbers $3k+1$, $2k+1$, and $12k+5$ (as pairwise relatively prime) must be squares. If we had $k = 0$, the number $12k+5$ would not be a square. On the other hand, for positive integer k, we have, as before, $k \geqslant 4$, hence $n = 6k+2 > 24$.

4. $n = 6k+3$, where k is an integer $\geqslant 0$. In this case we have

$$(2k+1)(3k+2)(12k+7) = m^2;$$

we easily see that the numbers $2k+1$, $3k+2$, and $12k+7$ are pairwise relatively prime, hence they must be squares. We cannot have $k = 0, 1, 2$ or 3 since in this case the number $3k+2$ would not be a square. We have, therefore, $k \geqslant 4$, which implies $n = 6k+3 > 24$.

5. $n = 6k+4$, where k is an integer $\geqslant 0$. We have in this case

$$(3k+2)(6k+5)(4k+3) = m^2,$$

where the numbers $3k+2$, $6k+5$, and $4k+3$ are pairwise relatively prime, hence they must be squares. We cannot have $k = 0, 1, 2, 3$ since then the number $3k+2$ would not be a square. We have, therefore, $k \geqslant 4$, and consequently $n = 6k+4 > 24$.

6. $n = 6k+5$, where k is an integer $\geqslant 0$. We have in this case

$$(6k+5)(k+1)(12k+11) = m^2,$$

and the numbers $6k+5$, $k+1$ and $12k+11$ are pairwise relatively prime, hence they all must be squares. We cannot have $k = 0, 1, 2, 3$ since in this case the number $6k+5$ would not be a square. We have, therefore, $k \geqslant 4$, and $n = 6k+5 > 24$.

We proved, therefore, that the least integer $n > 1$ for which $1^2 + 2^2 + \ldots + n^2$ is a square is $n = 24$.

REMARK. It is rather difficult to show that $n = 24$ is the only positive integer for which $1^2+2^2+\ldots+n^2$ is a square. On the other hand, the sum $1^3+2^3+\ldots+n^3$ is a square for every positive integer n, but one can prove that it is not a cube of a positive integer for any n.

238. All positive integers except

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 19, 23.$$

It is easy show that none of the above thirteen numbers is a sum of a finite number of proper powers (these are successively equal to $2^2, 2^3, 3^2, 2^4 = 4^2, 5^2, 3^3, 2^5, 6^2, \ldots$).

Now let n be a positive integer different from any of the above thirteen numbers.

If $n = 4k$, where k is a positive integer, then the number n is a sum of k numbers 2^2.

If $n = 4k+1$, then, in view of $n \neq 1$ and $n \neq 5$, we can assume that $k \geqslant 2$; then $n = 4k+1 = 3^2+4(k-2)$, where $k-2$ is an integer $\geqslant 0$. If $k = 2$, then $n = 3^2$, while if $k > 2$, then $n = 3^2+2^2+\ldots+2^2$, where the number of terms equal to 2^2 is $k-2$.

If $n = 4k+2$, then since n is different from numbers 6, 10, and 14, we have $k \geqslant 4$ and $n = 4k+2 = 3^2+3^2+4(k-4)$. Again it follows that the number n has the desired property.

Finally, if $n = 4k+3$, then since $n \neq 3, 7, 11, 15, 19$, and 23, we have $k \geqslant 6$ and $n = 3^2+3^2+3^2+4(k-6)$, which again implies that n has the desired property.

238a. We have $1 = 3^2-2^3$, $2 = 3^3-5^2$, $3 = 2^7-5^3$, $4 = 5^3-11^2 = 2^3-2^2$, $5 = 3^2-2^2$, $7 = 2^7-11^2$, $8 = 2^4-2^3$, $9 = 5^2-4^2$, $10 = 13^3-3^7$.

REMARK. We do not know whether the number 6 is a difference of two proper powers. It has been conjectured that every positive integer has a finite $\geqslant 0$ number of representations as the difference of two proper powers.

239. If $a^2+b^2 = c^2$, where a, b, and c are positive integers, then multiplying both sides of this equality by the number

$$a^{2(4n^2-1)}b^{4n(2n+1)(n-1)}c^{4n^2(2n-1)}$$

we obtain

$$[(a^{2n}b^{(2n+1)(n-1)}c^{n(2n-1)})^{2n}]^2+[(a^{2n+1}b^{2n^2-1}c^{2n^2})^{2n-1}]^2 = [(a^{2n-1}b^{2n(n-1)}c^{2n^2-2n+1})^{2n+1}]^2.$$

240. There is only one such positive integer, namely $n = 5$. We easily check that this number satisfies the equation $(n-1)!+1 = n^2$, and we also check that the numbers $n = 2, 3$, and 4 do not satisfy this equation. For $n = 6$ we obtain $n^2 > 6n-4$ and we show by induction that the same inequality holds for every integer $n \geqslant 6$. If n is an integer $\geqslant 6$, then

$$(n-1)!+1 > 2(n-1)(n-2) = 2(n^2-3n+2) > n^2$$

since $n^2 > 6n-4$. Thus, we cannot have $(n-1)!+1 = n^2$ for integer $n \geqslant 6$.

REMARK. We know only two positive integers $n > 5$ such that

$$n^2 | (n-1)!+1,$$

namely numbers 13 and 563, and we do not know whether there are more such numbers, or whether there are finitely many of them. We know that every such number must be a prime.

Let us also note that for $n = 5, 6$, and 8 the numbers $(n-1)!+1$ are squares (of numbers 5, 11, and 71 respectively), and we do not know whether there are any other such numbers.

241. If for some integer $n > 1$ we had $t_{n-1}t_n = m^2$ where m is a positive integer, we would have $(n^2-1)n^2 = (2m)^2$, and since n^2-1 and n^2 are relatively prime, each of them would have to be a square, which is impossible since there are no two squares of integers, whose difference would be equal to one.

Let now n be a given positive integer. The equation $x^2-n(n+1)y^2 = 1$ has infinitely many solutions in positive integers x and y. In fact, one of these solutions is $x = 2n+1$ and $y = 2$, while if for some positive integers x and y we have $x^2-n(n+1)y^2 = 1$, then also

$$[(2n+1)x+2n(n+1)y]^2-n(n+1)[2x+(2n+1)y]^2 = 1.$$

If x and y are positive integers such that $x^2-n(n+1)y^2 = 1$, then

$$t_n t_{2t_n y^2} = t_n t_n y^2(2t_n y^2+1) = t_n^2 y^2 x^2 = (t_n yx)^2.$$

For instance, for $n = 2$ we get $t_3 t_{24} = 30^2$, $t_3 t_{2400} = (3 \cdot 20 \cdot 49)^2$, and so on.

242. We have $2^{10} = 1024 > 10^3$. It follows that

$$2^{1945} = 2^5(2^{10})^{194} > 10 \cdot 10^{3\cdot 194} = 10^{583}.$$

Thus,

$$2^{2^{1945}} > 2^{10^{583}} = (2^{10})^{10^{582}} > 10^{3\cdot 10^{582}},$$

and the number of digits of the last number is greater than 10^{582}.

The number $5 \cdot 2^{1947}+1$ has obviously the same number of digits as the number $5 \cdot 2^{1947} = 10 \cdot 2^{1946}$, and since the decimal logarithm of 2 equals $\log_{10} 2 = 0{,}30103 \ldots$, we have

$$2^{1946} = 10^{1946 \log_{10} 2} = 10^{585{,}8\ldots},$$

and it follows that our number has 586 digits.

REMARK. The number F_{1945} is the greatest known composite Fermat number.

243. The number $2^{11213}-1$ has (in decimal system) the same number of digits as 2^{11213}, as it differs only by one from the latter. Thus, it suffices to compute the number of decimal digits of 2^{11213}.

If a positive integer n is of the form $n = 10^x$ where x is real (of course, $x \geqslant 0$), then, denoting by $[x]$ the greatest integer $\leqslant x$, we have $10^x \leqslant n < 10^{[x]+1}$, and it follows that the number n has $[x]+1$ decimal digits. We have $2^{11213} = 10^{11213\log_{10}2}$, and since $\log_{10}2 = 0{,}30103\ldots$, we have $3375 < 11213\log_{10}2 < 3376$. Thus, the number 2^{11213} (hence also $2^{11213}-1$) has 3376 decimal digits.

244. We have $2^{11212}(2^{11213}-1) = 2^{22425}-2^{11212}$. We compute first the number of digits of the number 2^{22425}. Since $22425 \log_{10}2 = 22425 \cdot 0{,}30103\ldots = 6750{,}597\ldots$, we obtain (see the solution of Problem 243) the result that the number 2^{22425} has 6751 digits, and we have $2^{22425} = 10^{6750} \cdot 10^{0,597}$. Since $10^{0,597\ldots} > 10^{1/2} > 3$, we get $10^{6751} > 2^{22425} > 3 \cdot 10^{6750}$, which shows that the first digit of 2^{22425} is $\geqslant 3$. Thus, if we subtract from the number 2^{22425} the number 2^{11213}, which has smaller number of digits, we do not change the number of digits of the latter. Consequently, the number $2^{11212}(2^{11213}-1)$ has 6751 digits.

245. We have $3! = 6, 3!! = 6! = 720, 3!!! = 720! > 99!100^{621} > 10^{1242}$. Thus, the number $3!!!$ has more than thousand digits.

By the well-know theorem (see, for instance, Sierpiński [37, p. 131, Theorem 6]), if m is a positive integer and p is a prime, then the largest power of p dividing $m!$ is

$$\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\left[\frac{m}{p^3}\right]+\ldots$$

where $[x]$ denotes the greatest integer $\leqslant x$. It follows that the largest power of 5 which divides $3!!! = 720!$ is

$$\left[\frac{720}{5}\right]+\left[\frac{720}{25}\right]+\left[\frac{720}{125}\right]+\left[\frac{720}{625}\right] = 144+28+5+1 = 178,$$

while the largest power of 2 dividing $720!$ is still greater (since already $\left[\frac{720}{2}\right] = 360$). It follows that the number $3!!!$ has 178 zeros at the end of its decimal expansion.

246*. The solution found by A. Schinzel.

For positive integers m which are powers of primes (with positive integer exponents), and only for such numbers. In fact, if $m = p^k$, where p is a prime and k is a positive integer, then for $f(x) = x^{\varphi(p^k)}$, in case $p \nmid x$, by the Euler theorem, we have $f(x) \equiv 1 \pmod{p^k}$, while in case $p|x$, in view of $\varphi(p^k) \geqslant p^{k-1} \geqslant k$ (which can be easily shown by induction), we have $p^k|x^k$, and consequently, $p^k|x^{\varphi(p^k)}$. Thus, $f(x) \equiv 0 \pmod{p^k}$.

If m is an integer > 1, and m is not a power of a prime, then m has at least two different prime divisors, p and $q \neq p$. Suppose that $f(x)$ is a polynomial with integer coefficients, and that there exist integers x_1 and x_2 such that $f(x_1) \equiv 0 \pmod{m}$, while $f(x_2) \equiv 1 \pmod{m}$. We shall have, therefore, also (in view of $p|m$ and $q|m$) the relations $f(x_1) \equiv 0 \pmod{p}$ and $f(x_1) \equiv 1 \pmod{q}$. Since p and q are different primes, by the Chinese remainder theorem there exists an integer x_0 such that $x_0 \equiv x_1 \pmod{p}$ and $x_0 \equiv x_2 \pmod{q}$ It follows that $f(x_0) \equiv f(x_1) \equiv 0 \pmod{p}$ and $f(x_0) \equiv f(x_2) \equiv 1 \pmod{q}$. The first of these congruences implies that we cannot have $f(x_0) \equiv 1 \pmod{m}$. Similarly, the second congruence implies that we cannot have $f(x_0) \equiv 0 \pmod{m}$. Consequently, $f(x_0)$ does not give the remainder 0 upon dividing by m, nor does it give the remainder 1. Thus, if m is not a power of a prime, then there is no polynomial $f(x)$ with integer coefficients which would satisfy the required conditions.

247. We easily see that

$$D < [(4m^2+1)n+m+1]^2,$$

hence the integral part of the number $\sqrt{D}$ equals to $a_0 = (4m^2+1)n+m$, which implies that $D-a_0^2 = 4mn+1$ and

$$x_1 = \frac{1}{\sqrt{D}-a_0} = \frac{\sqrt{D}+a_0}{D-a_0}.$$

Since a_0 is the integral part of the number $\sqrt{D}$, we have $a_0 < \sqrt{D} < a_0+1$, which yields $2a_0 < \sqrt{D}+a_0 < 2a_0+1$ and since $a_0 = (4mn+1)m+n$, we find

$$2m+\frac{2n}{4mn+1} < \frac{\sqrt{D}+a_0}{D-a_0^2} < 2m+\frac{2n+1}{4mn+1};$$

since $(2n+1)/(4mn+1) \leqslant 1$, we see that the integral part of the number $x_1 = (\sqrt{D}+a_0)/(D-a_0^2)$ is equal to $a_1 = 2m$. We have, therefore

$$x_1 = a_1+1/x_2, \quad \text{and} \quad x_2 = 1/(x_1-a_1).$$

On the other hand,

$$x_1-a_1=\frac{\sqrt{D}+a_0}{4mn+1}-2m=\frac{\sqrt{D}-[(4mn+1)m-n]}{4mn+1},$$

and consequently,

$$x_2=\frac{(4mn+1)\,[\sqrt{D}+(4mn+1)m-n]}{D-[(4mn+1)m-n]^2}.$$

We easily check that

$$D=[(4mn+1)m-n]^2+(4mn+1)^2,$$

which yields

$$x^2=\frac{\sqrt{D}+(4mn+1)m-n}{4mn+1}$$

and since $a_0<\sqrt{D}<a_0+1$, or

$$(4mn+1)m+n<\sqrt{D}<(4mn+1)m+n+1,$$

we get

$$2m<x_2<2m+\frac{1}{4mn+1}.$$

Consequently, the integral part of x_2 equals $a_2=2m$. We have, therefore, $x_2=a_2+1/x_3$, which gives $x_3=1/(x_2-a_2)$. However,

$$x_2-a_2=\frac{\sqrt{D}+(4mn+1)m-n}{4mn+1}-2m=\frac{\sqrt{D}-(4mn+1)m-n}{4mn+1}.$$

Consequently, we have

$$x_3=\frac{(4mn+1)\,[\sqrt{D}+(4mn+1)m+n]}{D-[(4mn+1)m+n]^2}=\sqrt{D}+(4mn+1)m+n=\sqrt{D}+a_0$$

which implies that the integral part of x_3 is $2a_0$, and that the number $\sqrt{D}$ has the expansion into the arithmetic continued fraction with the three-term period, formed of numbers $2m$, $2m$ and $2a_0$.

REMARK. One can show that all positive integers D for which the expansion of $\sqrt{D}$ into arithmetic continued fraction has a three-term period are just the above considered numbers D. See Sierpiński [32].

248. Computing the values of functions $\varphi(n)$ and $d(n)$ for $n \leqslant 30$ from the well-known formulae for these functions, i.e. if $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_s^{\alpha_s}$, then

$$\varphi(n) = q_1^{\alpha_1-1}(q_1-1) \dots q_s^{\alpha_s-1}(q_s-1),$$

$$d(n) = (\alpha_1+1) \dots (\alpha_s+1),$$

we easily see that the only values $n \leqslant 30$ for which $\varphi(n) = d(n)$ are $n = 1, 3, 8, 10, 18, 24$, and 30. We have here $\varphi(1) = d(1) = 1$, $\varphi(3) = d(3) = 2$, $\varphi(8) = d(8) = 4$, $\varphi(10) = d(10) = 4$, $\varphi(18) = d(18) = 6$, $\varphi(24) = d(24) = 8$, $\varphi(30) = d(30) = 8$.

REMARK. It was proved that there are no other solutions of the equation $\varphi(n) = d(n)$ in positive integers n. It can be shown that for $n > 30$ we have $\varphi(n) > d(n)$; see Pólya and Szegö [15, Section VIII, problem 45].

249. We easily check that for positive integer k and integer $s \geqslant 0$ we have

$$\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) \dots \left(1+\frac{1}{k+s}\right) = 1+\frac{s+1}{k}. \tag{1}$$

A positive rational number $w-1$ can be always represented in the form $w-1 = m/n$ where m and n are positive integers (not necessarily relatively prime) and $n > g$. It suffices to take $k = n$ and $s = m-1$; then the right-hand side of (1) will be equal to w. In this way we obtain the desired decomposition for the number w.

250*. We shall first prove that every integer $k \geqslant 0$ can be in at least one way represented in the form

$$k = \pm 1^2 \pm 2^2 \pm \dots \pm m^2 \tag{1}$$

where m is a positive integer, and the signs + and − are suitably chosen. The assertion holds for the number 0 since $0 = 1^2+2^2-3^2+4^2-5^2-6^2+7^2$. It is also true for the numbers 1, 2, and 3 since $1 = 1^2$, $2 = -1^2-2^2-3^2+4^2$, $3 = -1^2+2^2$, $4 = -1^2-2^2+3^2$.

Now, it suffices to prove that our theorem is true for every positive integer k, and since it is true for numbers 0, 1, 2, and 3, it suffices to prove that if the theorem is true for an integer $k \geqslant 0$, it is also true for the number $k+4$.

Suppose, then, that the theorem is true for the number k; thus, there exists a positive integer m such that with the suitable choice of signs $+$ and $-$ we have relation (1). Since we have

$$(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2 = 4, \tag{2}$$

it follows from (1) that

$$k+4 = \pm 1^2 \pm 2^2 \pm \ldots \pm m^2+(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2,$$

that is, our theorem holds for the number $k+4$. Thus, it is true for every integer.

It follows from (2) that for every positive integer m we have

$$(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2-(m+5)^2+$$
$$+(m+6)^2+(m+7)^2-(m+8)^2 = 0.$$

Thus, in (1) we can replace the number m by $m+8$, hence also by $m+16$, and so on. This shows that every integer k can be in infinitely many ways represented in the form (1), which was to be proved.

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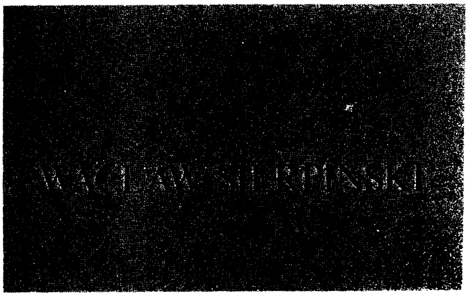

The late Waclaw Sierpiński, a member of the Polish Academy of Sciences, was Professor of Mathematics at the University of Warsaw from 1919 to 1960. He received the degree of Ph.D. from the University of Warsaw in 1906 and served as Professor of Mathematics at the University of Lwów from 1910 to 1919.

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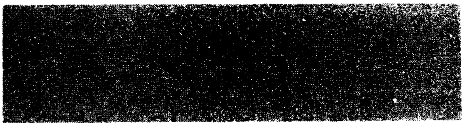

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