

# Information Processing Equalities and the Information–Risk Bridge

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**Editor:** Aryeh Kontorovich

## Abstract

We introduce two new classes of measures of information for statistical experiments which generalise and subsume  $\varphi$ -divergences, integral probability metrics,  $\mathfrak{R}$ -distances (MMD), and  $(f, T)$  divergences between two or more distributions. This enables us to derive a simple geometrical relationship between measures of information and the Bayes risk of a statistical decision problem, thus extending the variational  $\varphi$ -divergence representation to multiple distributions in an entirely symmetric manner. The new families of divergence are closed under the action of Markov operators which yields an information processing *equality* which is a refinement and generalisation of the classical information processing *inequality*. This equality gives insight into the significance of the choice of the hypothesis class in classical risk minimization.

**Keywords:** information processing inequality,  $\varphi$ -divergence, MMD, Bayes risk, loss functions, Markov kernels, regularisation via noise.

## 1. Introduction

*A key word in statistics is information. . . But what is information? No other concept in statistics is more elusive in its meaning and less amenable to a generally agreed definition. — Debabrata Basu (1975, p. 1).*

Machine learning is information processing. But what “information” is meant? Choosing exactly how to measure information has become topical of late in machine learning, with methods such as GANs predicated on the notion of being unable to compute a likelihood function, but being able to measure an information distance between a target and synthesised distribution (Bińkowski et al., 2018). Commonly used measures include the the Shannon information/entropy of a single distribution and the Kullback-Leibler divergence or Variational divergence between two different distributions. Csiszár’s  $\varphi$ -entropies (Csiszár, 1967) and  $\varphi$ -divergences (Csiszár, 1963, 1967) subsume these and many other divergences, and satisfy the famous information processing inequality (Ziv and Zakai, 1973) which states that the amount of information can only decrease (or stay constant) as a result of “information processing.”

The present paper presents a new and general definition of information that subsumes many in the literature. The key novelty of the paper is the redefinition of classical measures

of information as expected values of the support function of particular convex sets. The advantage of this redefinition is that it provides a surprising insight into the classical information processing inequality, which can consequently be seen to be an *equality* albeit one with different measures of information on either side of the equality. The reformulation also enables an elegant proof of the 1:1 relationship between information and (Bayes) risk, showing in an unambiguous way that there can not be a sensible definition of information that does not take account of the use to which the information will be put.

The rest of the paper is organised as follows. In the remainder of the present section, we introduce the  $\varphi$ -divergence, summarise earlier work on extending it to several distributions, and sketch a philosophy of information which our main theoretical results formally justify and support. In §2 we present the necessary technical tools we use; §3 presents the general “unconstrained” information measures (with no restriction on the model class); §4 presents the bridge between information measures and the (unconstrained) Bayes risk; §5 presents the constrained measures of information (where there is a restriction on the model class), as well as the generalisation of the “bridge” to this case; §6 concludes. There are four appendices: Appendix A relates our definition of  $D$ -information to the classical variational representation of a (binary)  $\varphi$ -divergence. Appendix B shows how our measure of information is naturally viewed as an expected gauge function. Appendix C examines the different entropies induced by the  $\mathcal{F}$ -information, showing how they too implicitly have a model class hidden inside their definition. Finally, Appendix D summarises earlier attempts to generalise  $\varphi$ -divergences to take account of a model class<sup>1</sup>.

### 1.1 The $\varphi$ -divergence

Suppose  $\mu, \nu$  are two probability distributions, with  $\mu$  absolutely continuous with respect to  $\nu$  and let

$$\Phi \stackrel{\text{def}}{=} \{\varphi \mid \mathbb{R}_{>0} \rightarrow \mathbb{R}, \quad \varphi \text{ convex}, \quad \varphi(1) = 0\}. \quad (1)$$

For  $\varphi \in \Phi$ , the  $\varphi$ -divergence between  $\mu$  and  $\nu$  is defined as

$$I_\varphi(\mu, \nu) \stackrel{\text{def}}{=} \int \varphi\left(\frac{d\mu}{d\nu}\right) d\nu. \quad (2)$$

Popular examples of  $\varphi$ -divergences include the Kullback-Liebler divergence ( $\varphi(t) = t \log t$ ) and the Variational divergence ( $\varphi(t) = |t - 1|$ ) among others; see (Reid and Williamson, 2011).

There are two existing classes of extensions to binary  $\varphi$ -divergences — devising measures of information for more than two distributions, and restricting the implicit optimization in the variational form (see Appendix A) as a form of regularisation. We summarise work along the first of these lines in the next subsection, and the second in Appendix D after we have introduced the necessary concepts to make sense of these attempts.

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1. Part II of the present paper (Williamson, 2024), to appear in due course, will contain an explanation of the relationship between our information processing theorems and the traditional inequalities (usually couched in terms of mutual information); the derivation of classical information processing theorems for divergences (with the same measure of information on either side of the inequality) from the results in part I; relationships to measures of informativity of observation channels; and relationships to existing results connecting information and estimation theory.

## 1.2 Beyond Binary — “ $\varphi$ -divergences” for more than two distributions

Earlier attempts to extend  $\varphi$ -divergences beyond the case of two distributions include the  $\varphi$ -affinity between  $n > 2$  distinct distributions; this is also known as the Matusita affinity (Matusita, 1967, 1971), the  $f$ -dissimilarity (Györfi and Nemetz, 1975, 1978), the generalised  $\varphi$ -divergence (Ginebra, 2007) or (on which we build in the present paper)  $D$ -divergences (Gushchin, 2008). One could conceive of these as “ $n$ -way distances” (Warrens, 2010) but most of the intuition about distances does not carry across, and so we will not adopt such an interpretation, and in the body of the paper refer to the objects simply as “measures of information.”

Generalisations of *particular* divergences to several distributions include the *information radius* (Sibson, 1969)  $R(P_1, \dots, P_k) = \frac{1}{k} \sum_{i=1}^k \text{KL}(P_i, (P_1 + P_2 + \dots + P_k)/k)$  where  $\text{KL}(P, Q)$  is the Kullback-Leibler divergence and the *average divergence* (Sgarro, 1981)  $K(P_1, \dots, P_k) = \frac{1}{k(k-1)} \sum_{i=1}^k \sum_{j=1}^k \text{KL}(P_i, P_j)$ . Some other approaches to generalising  $\varphi$ -divergences to more than two distributions are summarised by Basseville, 2010.

The general multi-distribution divergence has been used in hypothesis testing (Menéndez et al., 2005; Zografos, 1998). Györfi and Nemetz (1975) bounded the minimal probability of error in terms of the  $f$ -affinity; see also (Glick, 1973; Toussaint, 1978). These results are analogous to surrogate regret bounds (Reid and Williamson, 2011, section 7.1) because there is in fact an *exact* relationship between  $I_\varphi$  and the Bayes risk of an associated multiclass classification problem; see §4. Multidistribution  $\varphi$ -divergences have also been used to extend rate-distortion theory (primarily as a technical means to get better bounds) (Zakai and Ziv, 1975) and to unify information theory with the second law of thermodynamics (Merhav, 2011). The *estimation* of these divergences has been studied by Morales, Pardo, and Zografos (1998). The connection to Bayes risk suggests alternate estimation schemes.

Going in the opposite direction, it is worth noting that the entropy of a *single* distribution can be viewed as the  $\varphi$ -divergence between the given distribution and a reference (or “uniform”) distribution (Torgersen, 1981); see Appendix C.

## 1.3 Information is as Information Does

In developing a philosophy of information, Adriaans and Benthem (2008, page 20) adopted the slogan “No information without transformation!” They asked “what does information *do* for each process?” We reverse this to: “what does each process *do* to information?”

We avoid an essentialist claim of “one true notion” of information, but do not feel it necessary to follow the example of Csiszár (1972) of eschewing the word “information” for the neologism “informativity.” We believe that the elements of our field need to prove their mettle by their relationships. Barry Mazur (2008, section 3) observed that “mathematical objects [are] determined by the network of relationships they enjoy with all the other objects of their species” and proposed to “subjugate the role of the mathematical object to the role of its network of relationships — or, a further extreme — simply replace the mathematical object by this network”.<sup>2</sup>

2. This perspective is sometimes described as “Grothendieck’s relative point of view” in mathematics, but the insight holds more generally: “We only understand something according to the transformations that can be performed on it” — Michel Serres (1974, page 9), quoted in (Sack, 2019, page 41).

One could argue that such systematic study of the elements and their fundamental transformations is essential to achieve the called for transition of machine learning from alchemy to a mature science (Rahimi, 2017). We make a small step in this direction, focusing upon the transformation that measures of information of an experiment undergo when the experiment is observed via a noisy observation channel. This is a return to roots, since the very notion of Shannon information was motivated by communication over noisy channels (Shannon, 1948, 1949), and that of the Kullback-Leibler divergence motivated by notions of sufficiency (Kullback and Leibler, 1951). That a sufficient statistic can be viewed as the output of a noisy observation channel is made precise in the general definition of sufficiency and approximate sufficiency due to LeCam (1964).

Our perspective is motivated by the largely forgotten conclusion of DeGroot (1962), that even if one is only seeking some vague sense of “information” in data, ultimately one will *use* this “information” through some act (else why bother?), and such acts incur a utility (or loss), which can be quantified<sup>3</sup>. Thus any *useful* notion of information needs to take account of utility. Our general notion of information of an experiment is consistent with DeGroot’s utilitarian premise; we suggest that it is the most general such concept consistent with the precepts of decision theory and statistical learning theory.

This philosophy is made precise by our results showing the equivalence of the measures of information (which subsume most of those in the literature) and the Bayes risk of a statistical decision problem. Significantly, this means that *the choice of a measure of information is equivalent to the choice of a loss function (plus, potentially, the choice of a convex model class)* — thus any notion of information subsumed by our general measures really encodes the use to which one envisages the information being put, as De Groot admonished 60 years ago.

## 2. Technical Background and Notation

For positive integer  $n$ , we write  $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ . Let  $e_i$  denote the  $i$ -th canonical unit vector, and  $1_n \stackrel{\text{def}}{=} (1, \dots, 1) \in \mathbb{R}^n$ . We use standard concepts of convex analysis<sup>4</sup>. Let  $\bar{\mathbb{R}} \stackrel{\text{def}}{=} [-\infty, \infty]$  and  $\bar{\mathbb{R}}_+ \stackrel{\text{def}}{=} [0, \infty]$ . Let  $f: X \rightarrow \bar{\mathbb{R}}$ . Its *domain*  $\text{dom } f \stackrel{\text{def}}{=} \{x \in X: f(x) \neq \infty\}$  and its *Legendre-Fenchel conjugate*,

$$f^*(x^*) \stackrel{\text{def}}{=} \sup_{x \in X} (\langle x^*, x \rangle - f(x)).$$

If  $f$  is proper, closed, and convex, it is equal to its biconjugate:  $f = (f^*)^*$ . The *epigraph* and *hypograph* of  $f$  are the sets

$$\text{epi } f \stackrel{\text{def}}{=} \{(x, t) \in \text{dom}(f) \times \mathbb{R} \mid t \geq f(x)\} \ \& \ \text{hyp } f \stackrel{\text{def}}{=} \{(x, t) \in \text{dom}(f) \times \mathbb{R} \mid t \leq f(x)\}.$$

The function  $f$  is *closed and convex* if and only if the set  $\text{epi}(f)$  (or equivalently  $\text{hyp}(-f)$ ) is also. The *subdifferential* of  $f$  at  $x \in X$  is the set

$$\partial f(x) \stackrel{\text{def}}{=} \{x^* \in X^* \mid \forall y \in \text{dom } f: f(y) - f(x) \geq \langle x^*, y - x \rangle\}.$$

3. Interestingly, DeGroot was motivated to extend the attempt of Lindley (1956) to quantify the “amount of information” in an experiment, but unlike Lindley, did not presume that this was necessarily Shannon information.

4. See (Aliprantis and Border, 2006; Bauschke and Combettes, 2011; Hiriart-Urruty and Lemaréchal, 2001; Penot, 2012; Rockafellar, 1970). Since notation in the literature varies, we spell out our choice in full.

The domain of the differential is the set  $\text{dom } \partial f \stackrel{\text{def}}{=} \{x \in X \mid \partial f(x) \neq \emptyset\}$ . A *selection* is a mapping  $\nabla f: \text{dom } \partial f \rightarrow X^*$  that satisfies  $\nabla f(x) \in \partial f(x)$  for all  $x \in \text{dom } \partial f$ , and it is commonly abbreviated to  $\nabla f \in \partial f$ . If  $\partial f$  is a singleton, then  $\partial f$  corresponds to the classical differential which we write  $Df$ .

For  $f: X \rightarrow \overline{\mathbb{R}}$  and  $\alpha \in \mathbb{R}$ , the  $\alpha$  below level set of  $f$  is

$$\text{lev}_{\leq \alpha}(f) \stackrel{\text{def}}{=} \{x \in X \mid f(x) \leq \alpha\}.$$

If  $f: \mathbb{R} \rightarrow \mathbb{R}_{>0}$  then its *perspective* is the function  $\check{f}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $\check{f}(x, y) = yf(x/y)$ . The perspective  $\check{f}$  is positively homogeneous and is convex whenever  $f$  is. Observe that  $\check{f}(x, 1) = f(x)$ . The *halfspace* with normal  $1_n$  (and zero offset) is  $H_{1_n}^{\leq} \stackrel{\text{def}}{=} \text{lev}_{\leq 0} \langle \cdot, 1_n \rangle$

We use  $\cdot$  for the *Hadamard product*: that is, if  $X \ni f, g$  is a function space then  $f \cdot g$  is the regular function product  $(f \cdot g)(\cdot) \stackrel{\text{def}}{=} f(\cdot)g(\cdot)$ ; if  $X$  has dimension  $n < \infty$  then element-wise vector product is written  $(f \cdot g) \stackrel{\text{def}}{=} (f_1g_1, \dots, f_ng_n)$ .

For  $S, T \subseteq X$  and  $x \in X$ ,  $S + x \stackrel{\text{def}}{=} \{s + x \mid s \in S\}$ , and  $S + T \stackrel{\text{def}}{=} \{S + t \mid t \in T\}$  (the *Minkowski sum*). For  $S \subseteq X$  we associate two functions: the *S support function*,

$$\sigma_S(x^*) \stackrel{\text{def}}{=} \sup_{x \in S} \langle x^*, x \rangle \quad (3)$$

and the *S indicator function*

$$\iota_S(x) \stackrel{\text{def}}{=} \infty \cdot \llbracket x \in S \rrbracket,$$

where  $\llbracket p \rrbracket = 1$  if  $p$  is true and 0 otherwise, and we adopt the convention that  $\infty \llbracket \text{true} \rrbracket = 0$ . If  $S$  is closed and convex then the support function is the Fenchel conjugate of the indicator function and vice versa. The *recession cone* of  $S$  is the set

$$\text{rec } S \stackrel{\text{def}}{=} \{d \in X \mid S + d = S\}.$$

If  $S$  is convex then  $\text{rec } S$  is convex. If  $X$  is finite dimensional and  $S$  is bounded then  $\text{rec } S = \{0\}$ . The *polar cone* of  $S$  is the set

$$S^* \stackrel{\text{def}}{=} \{x^* \in X^* \mid \forall s \in S : \langle x^*, s \rangle \leq 0\}.$$

The *dual cone* (negative polar cone) of  $S$  is the set

$$S^+ \stackrel{\text{def}}{=} \{x^* \in X^* \mid \forall s \in S : \langle x^*, s \rangle \geq 0\}. \quad (4)$$

The *convex hull* of  $S$  is the set

$$\text{co } S \stackrel{\text{def}}{=} \bigcap \{T \subseteq X \mid S \subseteq T, T \text{ is convex}\},$$

the *closed convex hull* of  $S$  is the set which we abbreviate as  $\text{cl co } S \stackrel{\text{def}}{=} \text{cl}(\text{co } S)$ .

For two measurable spaces  $(X, \Sigma_X)$  and  $(Y, \Sigma_Y)$  the notation  $f: (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$  means that  $f$  is a measurable function with respect to the respective  $\sigma$ -algebras, which it is often convenient to abbreviate to  $f: X \rightarrow (Y, \Sigma_Y)$ . The *Borel  $\sigma$ -algebra on a set  $X$*  with some topology is  $\mathfrak{B}(X)$ , and we write  $(X, \mathfrak{B}) \stackrel{\text{def}}{=} (X, \mathfrak{B}(X))$ . The set of proper, closed, convex and measurable sets  $S \in \Sigma_X$  is  $\mathcal{K}(X)$ . The subcollection of these that recess in directions at most  $T \subseteq X$  is

$$\mathcal{D}(S, T) \stackrel{\text{def}}{=} \{D \in \mathcal{K}(S) \mid \text{rec } D \subseteq T\}.$$

Let  $\mathfrak{P}(X)$  be the set of probability measures on a measurable space  $(X, \Sigma_X)$ . If  $X$  has dimension  $n < \infty$  this is isomorphic to the set of vectors  $\{p \in \mathbb{R}^n \mid p_i \geq 0, \sum_i p_i = 1\}$  and its relative interior  $\text{rint } \mathfrak{P}(X)$  is the subset of vectors for which  $p_i > 0$  for each  $i \in [n]$ . If  $f: X \rightarrow \mathbb{R}$  and  $\mu \in \Delta(X)$ , we write  $\mu f = \mu(f) := \int f d\mu$ . Conventionally a *Markov kernel* is a function  $M: Y \times \Sigma_X \rightarrow \mathbb{R}$  which is  $\Sigma_Y$ -measurable in its first argument and a probability measure over  $X$  in its second. We use the notation of Çinlar (2011) to more compactly write  $M: Y \rightsquigarrow X$ . When  $Y$  has dimension  $n < \infty$  we call a Markov kernel  $E: Y \rightsquigarrow X$  an *experiment*. It is convenient to stack the distributions  $E(1), \dots, E(n)$  induced by  $E$  into a vector of measures (one for each  $y \in Y$ ), the notation for which we overload:  $E \stackrel{\text{def}}{=} (E_1, \dots, E_n)$ . Note that while  $E$  is an experiment (Markov kernel),  $E_i$  ( $i \in [n]$ ) are measures. If  $\mu$  is a measure that dominates each  $E_i$ , then the vector of Radon-Nikodym derivatives with respect to  $\mu$  is

$$dE/d\mu \stackrel{\text{def}}{=} (dE_1/d\mu, \dots, dE_n/d\mu)$$

and as a function maps  $X \rightarrow \mathbb{R}_{\geq 0}^n$ .<sup>5</sup> An experiment  $E^{\text{tni}}: Y \rightsquigarrow X$  with  $(dE^{\text{tni}}/d\rho)(x) = c(x)1_n$ , for some  $c(x) > 0$  is a *totally noninformative experiment*. Conversely, an experiment  $E^{\text{ti}}: Y \rightsquigarrow X$  is a *totally informative experiment* if for all  $A \in \Sigma_X$ , for all  $i \neq j$ ,  $E_i^{\text{ti}}(A) > 0 \implies E_j^{\text{ti}}(A) = 0$  (Torgersen, 1991). When  $X = Y = [n]$ ,  $E: Y \rightsquigarrow X$  can be represented by an  $n \times n$  stochastic matrix.

For the following definitions, fix measurable spaces  $(\Omega_1, \Sigma_1)$  and  $(\Omega_2, \Sigma_2)$ . The *measurable functions*  $\Omega_1 \rightsquigarrow \Omega_2$  are  $\mathcal{L}_0(\Omega_1, \Omega_2)$  and  $\mathcal{L}_0(\Omega) \stackrel{\text{def}}{=} \mathcal{L}_0(\Omega, \mathbb{R})$  refers to the *real measurable functions*. The *signed measures* on  $\Omega$  are  $\mathfrak{M}(\Omega)$ , the subset of these which are *probability measures* is  $\mathfrak{P}(\Omega)$ . To a probability measure  $\mu \in \mathfrak{P}(\Omega)$  we associate the *expectation functional*

$$\mu: \mathcal{L}_0(\Omega_1) \rightarrow \mathbb{R}, \quad \mu f \stackrel{\text{def}}{=} \int \mu(dx) f(x).$$

There are two operators associated to and conventionally overloaded with  $E$ <sup>6</sup>:

$$\begin{aligned} E: \mathcal{L}_0(\Omega_2) &\rightarrow \mathcal{L}_0(\Omega_1) & E: \mathfrak{P}(\Omega_1) &\rightarrow \mathfrak{P}(\Omega_2) \\ E f(x_1) &\stackrel{\text{def}}{=} \int_{\Omega_2} E(x_1, dx_2) f(x_2), & \mu E(dx_2) &\stackrel{\text{def}}{=} \int_{\Omega_1} \mu(dx_1) E(x_1, dx_2). \end{aligned} \tag{5}$$

The definitions above make it convenient to chain experiments:

$$E_1 E_2(\omega_1, d\omega_3) \stackrel{\text{def}}{=} \int_{\Omega_2} E(\omega_1, d\omega_2) E(\omega_2, d\omega_3),$$

where  $E_1: \Omega_1 \rightsquigarrow \Omega_2$  and  $E_2: \Omega_2 \rightsquigarrow \Omega_3$ ; thus  $E_1 E_2: \Omega_1 \rightsquigarrow \Omega_3$ .

It is common in the information theory literature to write  $X \rightarrow Y \rightarrow Z$  to denote random variables  $X$ ,  $Y$  and  $Z$  which form a *Markov chain*; that is,  $Z$  is independent of  $X$  when conditioned on  $Y$ . For our purposes however, it is more convenient to eschew the introduction of random variables, and to consider the kernels simply as mappings between spaces as defined above. Thus rather than writing a Markov chain in terms of the random variables  $X$ ,  $Y$  and  $Z$ ,  $X \xrightarrow{E_1} Y \xrightarrow{E_2} Z$ , we will write the ‘‘chain’’ as a string of experiments operating on spaces  $X$ ,  $Y$  and  $Z$  as  $X \xrightarrow{E_1} Y \rightsquigarrow Z$ .

5. While this overloading may appear overeager, it provides substantial simplification subsequently.

6. Note the postfix notation for action of  $E$  on probability measures.

### 3. Unconstrained Information Measures — $D$ -information

In this section we introduce the “unconstrained” information measure  $I_D(E)$ . The name is in contrast to the “constrained” family we introduce in §5. The unconstrained information measures subsume the classical  $\varphi$ -divergences and their  $n$ -ary generalisations (see §3.2).

#### 3.1 $D$ -information

For a set  $D \subseteq \mathbb{R}^n$  and an experiment  $E : [n] \rightsquigarrow \Omega$ , the  $D$ -information of  $E$  is

$$I_D(E) \stackrel{\text{def}}{=} \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \cdot \frac{dE_i}{d\rho} \right) d\rho, \quad (6)$$

where  $\rho \in \mathfrak{P}(\Omega)$  is a reference measure that dominates each of the  $(E_i)$ ,<sup>7</sup> and  $d = (d_1, \dots, d_n)$ . The definition above was first proposed by Gushchin (2008) and is analogous to the approach used by Williamson (2014) and Williamson and Cranko (2023) where loss functions are defined in terms of a convex set, and which forms the basis of the bridge in §4.

**Remark 1.** The choice of  $\rho$  is unimportant since (6) is invariant to reparameterisation:

$$\begin{aligned} \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \cdot \frac{dE_i}{d\rho_1} \right) d\rho_1 &= \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \cdot \frac{dE_i}{d\rho_1} \right) \frac{d\rho_1}{d\rho_2} d\rho_2 \\ &= \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \cdot \frac{dE_i}{d\rho_1} \frac{d\rho_1}{d\rho_2} \right) d\rho_2 \\ &= \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \cdot \frac{dE_i}{d\rho_2} \right) d\rho_2, \end{aligned}$$

for all dominating  $\rho_1, \rho_2$ .

**Remark 2.** The form of (6) indicates we can equivalently write

$$I_D(E) \stackrel{\text{def}}{=} \int \sigma_D \left( \frac{dE}{d\rho} \right) d\rho, \quad (7)$$

where  $dE/d\rho \stackrel{\text{def}}{=} (dE_1/d\rho, \dots, dE_n/d\rho)$  is the vector of Radon-Nikodym derivatives, and  $\sigma_D$  is the support function of  $D$  (3). This suggests, using standard polar duality results, that the  $D$ -information can be viewed as an expected gauge function, a perspective developed in Appendix B.

Observe that (6) places no requirements on the continuity of the distributions  $(E_y)_{y \in Y}$  with respect to one another. Thus the  $D$ -information is more than just a multi-distribution  $\varphi$ -divergence; when defined as the  $D$ -information, Theorem 4 guarantees that  $I_{\text{hyp}(-\varphi^*)}((E_1, E_2))$  agrees with  $I_\varphi(E_1, E_2)$  on all measures  $E_1, E_2$  with  $E_1 \ll E_2$ , and it is a natural extension

7. It is always easy to find such a  $\rho$ . For example one may take  $\rho \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i \in [n]} E_i$ .

to compare measures that don't have this absolute continuity condition<sup>8</sup>. Most existing generalisations of  $\varphi$ -divergences to  $n > 2$  (Garcia-Garcia and Williamson, 2012; Györfi and Nemetz, 1975, 1978; Keziou, 2015; Matusita, 1971) are subsumed by  $D$ -information; the one exception (Birrell, Dupuis, et al., 2022) is discussed in Appendix D.

### 3.2 From $\varphi$ -divergence to $D$ -information

Before proceeding with a more thorough study of (6) we justify its introduction as a generalisation of the  $\varphi$ -divergences. It is convenient to slightly refine our definition of  $\Phi$  as follows:

$$\bar{\Phi} \stackrel{\text{def}}{=} \{\varphi \mid \mathbb{R}_{>0} \rightarrow \mathbb{R}, \quad \varphi \text{ convex}, \quad \varphi(1) = 0, \quad \varphi \text{ lsc and } \mathbb{R}_{\geq 0} \subseteq \text{cl}(\text{dom } \varphi)\}.$$

This is a very mild refinement of  $\Phi$  and all  $\varphi$  used in the literature on  $\varphi$ -divergences are in fact contained in  $\bar{\Phi}$ . Observe that demanding  $\varphi$  be a proper function to  $\mathbb{R}$  defined on all of  $\mathbb{R}_{>0}$  implies that  $\mathbb{R}_{\geq 0} \subseteq \text{cl}(\text{dom } \varphi)$ . Assuming lower semi-continuity is a mere convenience since one can enforce it by taking closures, and, as we shall see, the information functionals will not change in this case since they are expressible in terms of support functions of the epigraph of functions related to  $\varphi$ , which remain invariant under taking closures of the sets concerned. In any case,  $\text{cl } \varphi$  and  $\varphi$  coincide on  $\text{rint } \text{dom } \varphi$  (Hiriart-Urruty and Lemaréchal, 2001, Proposition B.1.2.6). If we simply require that  $\varphi(x) < \infty$  for all  $x \in (0, \infty)$  then lower semicontinuity and the claim re domain follow as logical consequences.

Suppose  $\mu, \nu \in \mathfrak{P}(\Omega)$ , with a common dominating measure  $\rho$ . Choose  $\varphi \in \bar{\Phi}$ . Then the  $\varphi$ -divergence (2) has the following representation using the perspective function  $\check{\varphi}$ ,<sup>9</sup>

$$\begin{aligned} I_{\varphi}(\mu, \nu) &= \int \varphi\left(\frac{d\mu}{d\nu}\right) d\nu & (8) \\ &= \int \check{\varphi}\left(\frac{d\mu}{d\nu}, 1\right) \frac{d\nu}{d\rho} d\rho \\ &= \int \check{\varphi}\left(\frac{d\mu}{d\nu} \frac{d\nu}{d\rho}, \frac{d\nu}{d\rho}\right) d\rho \\ &= \int \check{\varphi}\left(\frac{d\mu}{d\rho}, \frac{d\nu}{d\rho}\right) d\rho. & (9) \end{aligned}$$

Equation (9) is symmetric in  $\mu$  and  $\nu$ , in contrast to (8), with any intrinsic asymmetry relegated to the choice of sublinear function  $\check{\varphi}$ . By the same argument used in Remark 1, the choice of  $\rho$  does not matter. Observe that upon substituting the definition of the perspective into (9) we obtain the formula

$$I_{\varphi}(\mu, \nu) = \int \frac{d\nu}{d\rho} \cdot \varphi\left(\frac{d\mu/d\rho}{d\nu/d\rho}\right) d\rho,$$

as also recently observed in (Agrawal and Horel, 2021, Remark 19), and which of course remains invariant to the the choice of  $\rho$ .

8. There are more general definitions of  $\varphi$ -divergences that hold in the general case; see e.g. (Liese and Miescke, 2007, p. 35). The approach we take further generalises to be applicable to comparisons of measures that are only finitely additive instead of countably additive, as explained by Gushchin (2008), whose work was a major inspiration for the present paper.

9. This observation is due to Gushchin (2008).



**Remark 3.** It is a common result in nonsmooth analysis (due to Hörmander (Penot, 2012, Corollary 1.81, p. 56)) that the mapping taking a set to its support function,  $D \mapsto \sigma_D$ , is an injection from the family of closed convex subsets to the set of positively homogeneous functions that are null at zero. Thus it is natural, as well as meaningful for our subsequent analysis, to parameterise (9) by a convex set as in (6) or (7). That is, given  $\varphi$ , we will work with the convex set  $D \in \mathcal{K}(\mathbb{R}^2)$  such that  $\check{\varphi} = \sigma_D$ ; an explicit formula for such a  $D$  in terms of  $\varphi$  is provided in Proposition 4 below.

Since we will be considering  $n$ -ary extensions of  $I_\varphi$ , it is convenient to number the measure arguments and stack them into a vector  $E \stackrel{\text{def}}{=} (E_1, E_2)$ , in which case the pair  $(E_1, E_2)$  may interpreted, equivalently, as a binary experiment  $[2] \rightsquigarrow \Omega$ .

**Proposition 4.** *Suppose  $E : [2] \rightsquigarrow \Omega$  satisfies  $E_1 \ll E_2$  and  $\varphi \in \bar{\Phi}$ . Let*

$$D_\varphi \stackrel{\text{def}}{=} \text{hyp}(-\varphi^*) \subseteq \mathbb{R}^2. \quad (10)$$

Then

$$I_\varphi(E_1, E_2) = I_{D_\varphi}((E_1, E_2)).$$

*Proof.* The assumptions on  $\varphi$  ensure that it is closed. We have

$$\begin{aligned} I_\varphi(E_1, E_2) &= \int \varphi \left( \frac{dE_1}{dE_2} \right) dE_2 \\ &= \int \text{cl} \check{\varphi} \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right) d\rho \end{aligned} \quad (11)$$

$$\begin{aligned} &= \int \sigma_{\text{hyp}(-\varphi^*)} \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right) d\rho \\ &= I_{\text{hyp}(-\varphi^*)}(E), \end{aligned} \quad (12)$$

where (11) holds because a closed convex function  $\varphi$  and the closure of its perspective satisfies  $\varphi = \text{cl} \check{\varphi}(\cdot, 1)$ , (12) follows from the the relationship  $\text{cl} \check{\varphi}(x, t) = \sigma_{\text{epi} \varphi^*}(x, -t)$  (Hiriart-Urruty and Lemaréchal, 2001, Proposition 1.2.1, p. 214), and it is easy to verify that  $\sigma_{\text{epi} \varphi^*}(x, -t) = \sigma_{\text{hyp}(-\varphi^*)}(x, t)$ .  $\square$

**Proposition 5.** *Suppose  $\varphi \in \bar{\Phi}$  then  $\text{hyp}(-\varphi^*) \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  and  $\sigma_D(1_2) = 0$ .*

*Proof.* The Fenchel conjugate is always closed and convex, thus  $\text{hyp}(-\varphi^*)$  is closed and convex. Let  $A$  be the linear operator that flips the sign of the last element a vector  $x \in \mathbb{R}^n$ ;  $Ax \stackrel{\text{def}}{=} (x_1, \dots, x_{n-1}, -x_n)$ . Then (Auslender and Teboulle, 2003, Proposition 2.1.11, p. 31) implies that

$$\text{hyp}(-\varphi^*) = A \text{epi}(\varphi^*) \quad \text{and} \quad \text{rec}(\text{hyp}(-\varphi^*)) = A \text{rec}(\text{epi} \varphi^*).$$

Auslender and Teboulle (2003, Theorem 2.5.4, p. 55) show  $\text{rec}(\text{epi} \varphi^*) = \text{epi}(\sigma_{\text{dom} \varphi})$ . Since  $\sigma_{\text{dom} \varphi}$  is 1-homogeneous and closed, its epigraph is a closed cone, thus it is equal to its recession cone (Auslender and Teboulle, 2003, Proposition 2.1.1, p. 26), that is,  $\text{rec}(\text{epi} \sigma_{\text{dom} \varphi}) = \text{epi}(\sigma_{\text{dom} \varphi})$ . By assumption  $\text{cl}(\text{dom} \varphi) \supseteq \mathbb{R}_{\geq 0}$ , thus  $\sigma_{\text{dom} \varphi} = \sigma_{\text{cl}(\text{dom} \varphi)} = \iota_{-(\text{cl}(\text{dom} \varphi))^+}$ ,

where, recall, the  $+$  denotes the dual cone (4). Since  $\varphi: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is presumed to be defined (finite) on the whole of  $\mathbb{R}_{\geq 0}$ ,  $\text{cl}(\text{dom } \varphi) \supseteq \mathbb{R}_{\geq 0}$  and thus we have  $\text{cl}(\text{dom } \varphi)^* \subseteq \mathbb{R}_{\geq 0}$ ; thus  $\text{epi}(\text{t}_{\mathbb{R}_{\geq 0}^n}) \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$ . This gives us

$$\text{rec}(\text{hyp}(-\varphi^*)) = A \text{rec}(\text{epi}(\sigma_{\text{dom } \varphi})) = A \text{epi}(\sigma_{\text{dom } \varphi}) = A(\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}) = \mathbb{R}_{\leq 0}^2,$$

which shows  $\text{hyp}(-\varphi^*) \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ . Finally we have that  $\sigma_D(1_2) = \check{\varphi}(1, 1) = \varphi(1) = 0$  by assumption on  $\varphi$ .  $\square$

We also have the following converse result:

**Proposition 6.** *Let  $D \in \mathcal{D}(\mathbb{R}^2, \mathbb{R}_{\leq}^2)$  with  $\sigma_D(1_n) = 0$ . Let  $\varphi_D(x) \stackrel{\text{def}}{=} \sigma_D((x, 1))$ . Then  $\varphi_D \in \bar{\Phi}$ .*

*Proof.* Support functions are convex and thus it is immediate that  $\varphi_D$  is too. We have  $\varphi_D(1) = \sigma_D((1, 1)) = 0$  by assumption. Since  $\text{rec } D = \mathbb{R}_{\leq 0}^2$  we have  $\text{dom } \sigma_D = \mathbb{R}_{\geq 0}^2$  and thus  $\text{dom } \varphi_D = [0, \infty)$ .  $\square$

Since Proposition 4 shows every  $\varphi$ -divergence corresponds to a  $D$ -information, it is natural then to ask which  $D$ -informations correspond to  $\varphi$ -divergences. Similarly to Proposition 4 we may obtain, from any  $D \subseteq \mathbb{R}^2$  a convex, lower semicontinuous function  $\varphi_D: \mathbb{R} \rightarrow \bar{\mathbb{R}}$  by the mapping  $D \mapsto \sigma_D(\cdot, 1)$ . Ensuring that this function is finite on  $\mathbb{R}_{\geq 0}$  and normalised appropriately to be consistent with (1) is more subtle.

**Proposition 7.** *Suppose  $D \subseteq \mathbb{R}^n$  is nonempty and closed convex. Then we have  $\sigma_D \geq 0$  if and only if  $0 \in D$ . Assume  $0 \in D$ , and let  $Z \stackrel{\text{def}}{=} \{c1_n \in \mathbb{R}^n \mid c > 0\}$ . Then  $\sigma_D$  is minimised with minimum value 0 along the  $Z$  ray if and only if  $D \subseteq \text{H}_{1_n}^{\leq}$ .*

*Proof.* The common result (Hiriart-Urruty and Lemaréchal, 2001, theorem C.3.3.1) that  $A \subseteq B$  if and only if  $\sigma_A \leq \sigma_B$  with  $A = \{0\}$  easily shows the first claim. For the remainder of the proof assume  $0 \in D$ . Whence

$$D \subseteq \text{H}_{1_n}^{\leq} \iff \forall d \in D : \langle d, 1_n \rangle \leq 0 \iff \sup_{d \in D} \langle d, 1_n \rangle \leq 0 \iff \sigma_D(1_n) \leq 0.$$

Since  $\sigma_D \geq 0$  (owing to the assumption  $0 \in D$ ), we must have  $\inf \sigma_D = \sigma_D(1_n) = 0$ . Positive homogeneity of  $\sigma_D$  implies this holds along the ray  $Z$  too.  $\square$

**Corollary 8.** *Suppose  $D \subseteq \mathbb{R}^n$  is nonempty and closed convex. Then  $\sigma_D$  is minimised along the  $Z$  ray if and only if  $D - \partial \sigma_D(1_n) \subseteq \text{H}_{1_n}^{\leq}$ .*

Thus the  $D$ -informations that correspond to a normalised  $\varphi$ -divergence (with  $\varphi$  strictly convex) are those strictly convex  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  which lie in the half space with outer normal vector  $1_n$  and pass through the origin at their boundary. This normalisation corresponds to the well known fact that  $\varphi$ -divergences are insensitive to affine offsets:

**Proposition 9.** *Suppose  $\varphi \in \bar{\Phi}$  and  $c \in \mathbb{R}$ . Let  $\varphi_c(x) \stackrel{\text{def}}{=} \varphi(x) + c(x - 1)$ . Then*

$$\text{I}_{\varphi_c} = \text{I}_{\varphi} \tag{13}$$

$$\varphi_c^*(x) = \varphi^*(x - c) + c \tag{14}$$

$$\check{\varphi}_c(s, t) = \check{\varphi}(s, t) + c(s - t) \quad (15)$$

$$\sigma_{D_{\varphi_c}}(s, t) = \sigma_{D_\varphi}(s, t) + c(s - t) \quad (16)$$

$$D_c = D + \{(c, -c)'\}. \quad (17)$$

Observe that transforming  $D$  to  $D_c$  corresponds to “sliding”  $D$  along the supporting hyperplane  $\{x \in \mathbb{R}^2 \mid \langle x, 1_2 \rangle = 0\}$ , i.e. the boundary of  $H_{1_2}^{\leq}$ .

*Proof.* Substituting  $\varphi_c$  into (2) gives (13); (Hiriart-Urruty and Lemaréchal, 2001, Proposition E.1.3.1 (i) and (vi)) gives (14); (15) follows by substitution into the definition of the perspective; (16) follows from (15) by the fact that the perspective of  $\varphi$  is the support function of  $D_\varphi$ , and (17) follows from the additivity of support functions under Minkowski sums (Schneider, 1993) and the support function of a singleton being a linear function (Hiriart-Urruty and Lemaréchal, 2001).  $\square$

**Lemma 10.** *Suppose  $D \subseteq \mathbb{R}^n$  and let  $E : [n] \rightsquigarrow \Omega$  be an experiment. Suppose  $p \in \mathbb{R}^n$  is such that  $\langle p, 1_n \rangle = 0$ . Let  $D_p \stackrel{\text{def}}{=} D + \{p\}$ . Then  $I_{D_p}(E) = I_D(E)$ .*

*Proof.* From (7) we have

$$\begin{aligned} I_{D_p}(E) &= \int \sigma_{D_p} \left( \frac{dE}{d\rho} \right) d\rho \\ &= \int \left( \sigma_D \left( \frac{dE}{d\rho} \right) + \left\langle p, \frac{dE}{d\rho} \right\rangle \right) d\rho \\ &= I_D(E) + \int \left\langle p, \frac{dE}{d\rho} \right\rangle d\rho \\ &= I_D(E) + \left\langle p, \int \frac{dE}{d\rho} d\rho \right\rangle \\ &= I_D(E) + \langle p, 1_n \rangle \\ &= I_D(E), \end{aligned}$$

where the second equality follows from additivity of support functions of Minkowski sums, and the fact that  $\sigma_{\{p\}}(x) = \langle p, x \rangle$ .  $\square$

A special case of this result is when  $n = 2$  and  $p = (c, -c)$  which corresponds to the situation of Proposition 9, showing that translating  $D$  in the manner of Lemma 10 corresponds to the classical result that an affine offset to  $\varphi$  does not change  $I_\varphi$ .

**Remark 11.** With this result it is clear that we can always canonically assume that for any  $D$  such that  $\sigma_D(1_n) = 0$ , we have  $0_n \in \text{bd } D$ . To see this, suppose  $0_n \notin \text{bd } D$ , and denote by  $s \stackrel{\text{def}}{=} D \sigma_D(1_n)$ , the support point of  $D$  in direction  $1_n$ . Then using  $v = -s$  in the above proposition to determine  $D_v$  ensures  $I_{D_v} = I_D$  and that  $0_n \in \text{bd } D_v$ . Requiring  $0_n \in \text{bd } D$  and  $\sigma_D(1_n) = 0$  corresponds, in the case that  $n = 2$ , to choosing the affine offset for  $\varphi$  such that  $\varphi$  is everywhere non-negative.

**Proposition 12.** *Let  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ . Then  $I_D(E^{\text{tni}}) = 0$  for all totally non-informative experiments  $E^{\text{tni}}$  if and only if  $D \subseteq H_{1_n}^{\leq} = \{x \in \mathbb{R}^n \mid \langle x, 1_n \rangle \leq 0\}$ .*

*Proof.* Let  $D$  be such that for all totally non-informative experiment  $E^{\text{tni}}$ ,  $I_D(E^{\text{tni}}) = 0$ . This means that

$$\mathbb{E}_\mu [\sigma_D(c(x)1_n)] = \mathbb{E}_\mu [c(x)\sigma_D(1_n)] = \sigma_D(1_n) \int_X c(x)d\mu(x) = 0$$

for all measures  $\mu$ , and all functions  $c: \mathbb{R} \rightarrow \mathbb{R}_+$ . Hence,  $\sigma_D(1_n) = 0$ . By definition of the support function,  $\sigma_D(1_n) = 0$  means that the hyperplane  $\{x \mid \langle x, 1_n \rangle = 0\}$  supports  $D$  and thus  $D \subseteq \{x \mid \langle x, 1_n \rangle \leq 0\}$ . Conversely, if  $D \subseteq \{x \in \mathbb{R}^n \mid \langle x, 1_n \rangle \leq 0\}$ ,  $\forall d \in D, \langle d, 1_n \rangle \leq 0$ , and so  $\sigma_D(1_n) = 0$ , which gives  $I_D(E^{\text{tni}}) = 0$  for all totally non-informative experiments  $E^{\text{tni}}$ .  $\square$

In light of the above arguments, we define the class of such normalised  $D$  by<sup>10</sup>

$$\mathfrak{D}^n \stackrel{\text{def}}{=} \{D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n) \mid \sigma_D(1_n) = 0\}.$$

Observe that  $\sigma_D(1_n) = 0$  and  $\text{rec } D = \mathbb{R}_{\leq 0}^n$  together imply that  $D \subseteq \text{lev}_{\leq 0} \langle \cdot, 1_n \rangle$ . Given Remark 11, we could always restrict ourselves to

$$\mathfrak{D}_0^n \stackrel{\text{def}}{=} \{D \in \mathfrak{D}^n \mid 0_n \in \text{bd } D\}.$$

Although many of the results below hold for more general choices of  $D$ , one loses nothing (in terms of the expressive power of  $I_D$ ) in restricting  $D$  to  $\mathfrak{D}^n$  or indeed  $\mathfrak{D}_0^n$ . In the case where  $n = 2$ , so  $Y = \{1, 2\}$ , for  $D \in \mathfrak{D}^n$ , the corresponding function  $\varphi$  such that  $I_\varphi = I_D$  can be obtained as the mapping  $\varphi_D: x \mapsto \sigma_D((x, 1))$  and one is guaranteed that  $\varphi_D \in \bar{\Phi}$  (Proposition 6). Some further observations on the relationship between  $D$ -information and  $\varphi$ -information are given in Remark 26.

### 3.3 Properties of $D$ -information

Since  $D_1 \subseteq D_2 \iff \sigma_{D_1} \leq \sigma_{D_2}$ , (7) immediately gives that  $D_1 \subseteq D_2 \iff I_{D_1} \leq I_{D_2}$ . The  $D$ -information is insensitive to certain operations on  $D$ : taking closed convex hulls; and taking Minkowski sums with the negative orthant:

**Lemma 13.** *Suppose  $D \subseteq \mathbb{R}^n$  is closed and let  $E: [n] \rightsquigarrow \Omega$  be an experiment. Then*

$$I_D(E) = I_{\text{co}(D)}(E) = I_{\text{co}(D) + \mathbb{R}_{\leq 0}^n}(E).$$

*Proof.* Using some elementary properties of the support function (Auslender and Teboulle, 2003; Hiriart-Urruty and Lemaréchal, 2001)  $\sigma_D = \sigma_{\text{co } D}$ . Appealing to Definition 7 this shows the first equality. In order to prove the second we use the fact that  $\sigma_C = \iota_{C^+}$ , where  $C$  is a cone and  $C^+$  is its dual cone (4). Thus

$$\sigma_{\text{co}(D) + \mathbb{R}_{\leq 0}^n} = \sigma_{\text{co}(D)} + \sigma_{\mathbb{R}_{\leq 0}^n} = \sigma_{\text{co } D} + \iota_{\mathbb{R}_{\geq 0}^n},$$

where the last step is a consequence of  $(\mathbb{R}_{\geq 0}^n)^+ = \mathbb{R}_{\leq 0}^n$  (Hiriart-Urruty and Lemaréchal, 2001, p. 49). Since the function  $dE/d\rho$  maps into  $\mathbb{R}_{\geq 0}^n$ , appeal to the alternate definition (7) completes the proof.  $\square$

10. These sets are also called “comprehensive” (“downward” and convex); see (Martinez-Legaz, Rubinov, and Singer, 2002).

**Proposition 14.** *The  $D$ -information induces a quotient space on the closed convex sets  $A, B \subseteq \mathbb{R}^n$  where  $\text{rec } A \subseteq \mathbb{R}_{\leq 0}^n$  and  $\text{rec } B \subseteq \mathbb{R}_{\leq 0}^n$  via the equivalence relation*

$$A \sim_I B \iff I_A = I_B.$$

*This quotient space is isomorphic to  $\mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ .*

*Proof.* By hypothesis  $D \subseteq \mathbb{R}^n$  and  $\text{rec } D \subseteq \mathbb{R}_{\leq 0}^n$ . Thus  $\text{rec}(D + \mathbb{R}_{\leq 0}^n) = \text{rec } D + \mathbb{R}_{\leq 0}^n = \mathbb{R}_{\leq 0}^n$ ; the inclusion follows from (Auslender and Teboulle, 2003, Theorem 2.3.4, p. 39). Finally we note  $\overline{\text{co}}(D) \in \mathcal{K}(\mathbb{R}^n)$ , which, together with Lemma 13 completes the proof.  $\square$

**Remark 15.** Proposition 14 has a simple interpretation since for all bounded subsets  $D \subseteq \mathbb{R}^n$ ,  $\text{rec } D = \{0\}$ ; and thus the equivalence relation applies to these in addition to any set (unbounded) that recesses in directions  $R \subseteq \mathbb{R}_{\leq 0}^n$ . Thus  $\mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  is the natural parameter space for  $I_D$ .

Some  $\varphi$  divergences (e.g. variational) are always bounded, and others (e.g. Kullback-Leibler) are not. There is a simple characterisation of when  $I_D$  is guaranteed to be bounded:

**Proposition 16.** *Suppose  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ . Then*

$$\sup_{E: [n] \rightsquigarrow X} I_D(E) < \infty$$

*if and only if there exists some  $\alpha \in \mathbb{R}^n$  such that  $D \subset \mathbb{R}_{\leq 0}^n + \{\alpha\}$ .*

*Proof.* We first show that  $\sup_E I_D(E) < \infty$  if and only if  $\sigma_D(x) < \infty$  for all  $x \in \mathbb{R}_{\geq 0}^n$ . Recall  $I_D(E) = \int \sigma_D(dE/d\rho) d\rho$ . If there exists  $x^* \in \mathbb{R}_{\geq 0}^n$  such that  $\sigma_D(x^*) = \infty$ , then we can always choose  $E^*$  such that  $(dE^*/d\rho)(z) = cx^*$  for some  $c > 0$  for all  $z$ , and thus  $I_D(E^*) = \infty$ . Furthermore, if  $I_D(E^*) = \infty$  for some  $E^*$  then it must be the case that for at least one  $z$ , we have  $\sigma_D((dE^*/d\rho)(z)) = \infty$ . Conversely if  $\sigma_D(x) < \infty$  for all  $x$  then there is no way  $I_D(E)$  can be made infinite by choice of  $E$ . Furthermore, if  $I_D(E) < \infty$  for all  $E$  then  $I_D(E^*)$  for  $E^*$  such that  $(dE^*/d\rho)(z) = cx$  for arbitrary  $x \in \mathbb{R}_{\geq 0}^n$  and some constant  $c$  (recall  $\sigma_D$  is 1-homogeneous). It thus follows that  $\sigma_D(x)$  must not be infinite for all  $x$ .

We now show  $\sigma_D(x) < \infty$  for all  $x \in \mathbb{R}_{\geq 0}^n$  if and only if  $D \subset \mathbb{R}_{\leq 0}^n + \{\alpha\}$ . Suppose  $D \subset \mathbb{R}_{\leq 0}^n + \{\alpha\}$ . Then for all  $x \in \mathbb{R}_{\geq 0}^n$ ,  $\sigma_D(x) \leq \sigma_{\mathbb{R}_{\leq 0}^n}(x) + \sigma_{\{\alpha\}}(x) = 0 + \langle \alpha, x \rangle < \infty$ . Conversely, if  $\sigma_D(x) < \infty$  for all  $x \in \mathbb{R}_{\geq 0}^n$  then  $\sigma_D(e_i) < \infty$  for  $i \in [n]$  where  $e_i$  is the  $i$ th canonical basis vector. But  $\sigma_D(e_i) < \infty \Rightarrow \sup_{y \in D} \langle y, e_i \rangle < \infty \Rightarrow D \subset H_{i, \alpha_i}$  for some  $\alpha_i \in \mathbb{R}$ , where  $H_{i, \alpha_i} = \{x \mid \langle x, e_i \rangle \leq \alpha_i\}$  is the halfspace with normal  $e_i$  and offset  $\alpha_i$ . Since this holds for all  $i \in [n]$  we have that  $D \subset \bigcap_{i \in [n]} H_{i, \alpha_i} = \mathbb{R}_{\leq 0}^n + \{\alpha\}$ , with  $\alpha = (\alpha_1, \dots, \alpha_n)$ .  $\square$

**Remark 17.** We can express the Blackwell-Sherman-Stein theorem (Ginebra, 2007, section 3.2.2) in terms of  $I_D$ . Say one experiment  $E: [n] \rightsquigarrow X$  is *better than*  $F: [n] \rightsquigarrow X$ , and write  $E \succcurlyeq F$ , if there exists a Markov kernel  $T: X \rightsquigarrow X$  such that  $F = ET$ ; that is, experiment  $F$  can be obtained from experiment  $E$  by applying some corruption kernel  $T$ . The theorem states:

$$E \succcurlyeq F \iff \int f \left( \frac{dE}{d\rho} \right) d\rho \geq \int f \left( \frac{dF}{d\rho} \right) d\rho, \quad \forall f: \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}, f \text{ convex.} \quad (18)$$

(As usual, the choice of dominating measure  $\rho$  does not matter). We now argue that we can replace  $f$  by  $\sigma_D$  with  $D \in \mathfrak{D}^n$ . Since  $\frac{dE}{d\rho}(x), \frac{dF}{d\rho}(x) \in \mathbb{R}_{\geq 0}^n$  for all  $x$ , it suffices to ensure  $\text{dom } \sigma_D = \mathbb{R}_{\geq 0}^n$  which is guaranteed by the fact that  $\text{rec } D = \mathbb{R}_{\leq 0}^n$ . Since  $f$  appears on both sides of (18), an additive offset is cancelled, and thus we can always subtract  $\sigma_D(1_n)$  from both sides which is tantamount to assuming  $\sigma_D(1_n) = 0$ . Thus we can replace (18) by

$$E \succcurlyeq F \iff I_D(E) \geq I_D(F), \quad \forall D \in \mathfrak{D}^n.$$

That is,  $E$  is better than  $F$  if and only if, for all  $D$ , the  $D$ -information of  $E$  is greater than or equal to the  $D$ -information of  $F$ ; one cannot compare  $E$  and  $F$  in the absolute sense of  $\succcurlyeq$  by using only *one* measure of information.

## 4. The Bridge between Information and Risk

Having introduced the  $D$ -information, in this section we show its connection to the Bayes risk, and present the corresponding information processing equality.

### 4.1 $D$ -information and Bayes Risk

Classically, a *loss function* is a mapping  $\ell : \mathfrak{P}([n]) \times [n] \rightarrow \bar{\mathbb{R}}_+$ , where the quantity  $\ell(\mu, y)$  is to be interpreted as the penalty incurred when predicting  $\mu \in \mathfrak{P}([n])$  under the occurrence of the event  $y \in [n]$ . A loss function is said to be *proper* if the expected loss is minimised by predicting correctly, and *strictly proper* if it is minimised by predicting precisely<sup>11</sup>. That is, for all  $\mu \in \mathfrak{P}([n])$

$$\mu \in \underset{\nu \in \mathfrak{P}([n])}{\text{arginf}} \mathbb{E}_{Y \sim \mu}[\ell(\nu, Y)], \quad \text{and} \quad \{\mu\} = \underset{\nu \in \mathfrak{P}([n])}{\text{arginf}} \mathbb{E}_{Y \sim \mu}[\ell(\nu, Y)],$$

respectively. Considering a product space  $\Omega \times [n]$  and measures  $\mu \in \mathfrak{P}(\Omega \times [n])$ ,  $\nu \in \mathfrak{P}([n])$ , we introduce two classical quantities, the *Bayes risk*, and *conditional Bayes risk*:

$$\text{Bayes}_\ell(\mu) \stackrel{\text{def}}{=} \inf_{f \in \mathcal{L}_0(\Omega, \mathfrak{P}([n]))} \mathbb{E}_{(X, Y) \sim \mu}[\ell(f(X), Y)] \tag{19}$$

and

$$\text{CBayes}_\ell(\nu) \stackrel{\text{def}}{=} \inf_{\nu' \in \mathfrak{P}([n])} \mathbb{E}_{Y \sim \nu}[\ell(\nu', Y)].$$

These are related by

$$\text{Bayes}_\ell(\mu) = \mathbb{E}_{X \sim \mu_X}[\text{CBayes}_\ell(\mu_{Y|X})],$$

where  $\mu_X$  is the law of  $X$  and  $\mu_{Y|X}$  is the conditional distribution of  $Y$  given  $X$ .

We stack  $\ell$  into a vector over its second argument:  $\ell(\mu) \stackrel{\text{def}}{=} (\ell(\mu, 1), \dots, \ell(\mu, n))$ . The *superprediction set*<sup>12</sup> associated to  $\ell$  is

$$\text{spr}(\ell) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \exists \mu \in \mathfrak{P}([n]) : x - \ell(\mu) \in \mathbb{R}_{\geq 0}^n\}.$$

11. See (Buja, Stuetzle, and Shen, 2005; McCarthy, 1956; Reid and Williamson, 2010; Williamson, 2014; Williamson and Cranko, 2023) for further background and history of proper losses.

12. See (Cranko, 2021; Dawid, 2007; Kalnishkan, Vovk, and Vyugin, 2004; Williamson and Cranko, 2023) for uses of the superprediction set.

The superprediction set is the set of all “superpredictions” — points “north-east” of the image of the loss  $\ell(\mathfrak{P}([n]))$ :

$$\text{spr}(\ell) = \ell(\mathfrak{P}([n])) + \mathbb{R}_{\geq 0}^n. \quad (20)$$

There are relationships between properties of  $\ell$  and the geometry of  $\text{spr}(\ell)$ . For example:

1.  $\ell$  is proper only if  $\text{spr}(\ell)$  is convex when  $\ell$  is continuous (Cranko, 2021, Theorem 4.13).
2.  $\ell$  is  $\eta$ -mixable (Vovk, 1995) if  $e_\eta(\text{spr}(\ell))$  is convex, where for  $\eta > 0$ ,  $e_\eta: \mathbb{R}^n \ni x \mapsto (e^{-\eta x_1}, \dots, e^{-\eta x_n})$  and for  $S \subset \mathbb{R}^n$ ,  $e_\eta(S) = \{e_\eta(s) \mid s \in S\}$ . Equivalently, a loss  $\ell$  is mixable if and only if  $\text{spr}(\ell)$  “slides freely” in  $\text{spr}(\ell_{\log})$ , the superprediction set for log-loss (Pacheco and Williamson, 2023).

**Remark 18.** The conditional Bayes risk of  $\nu$  w.r.t. a loss  $\ell$  has the following representation using the support function and superprediction set:

$$\begin{aligned} \text{CBayes}_\ell(\nu) &= \inf_{\nu' \in \mathfrak{P}([n])} \mathbf{E}_{Y \sim \nu} [\ell(\nu', Y)] \\ &= \inf_{\nu' \in \mathfrak{P}([n])} \langle \ell(\nu'), \nu \rangle \\ &= \inf_{f \in \ell(\mathfrak{P}([n])) + \mathbb{R}_{\geq 0}^n} \langle f, \nu \rangle \\ &= - \sup_{f \in \ell(\mathfrak{P}([n])) + \mathbb{R}_{\geq 0}^n} \langle -f, \nu \rangle \\ &= - \sup_{f \in -(\ell(\mathfrak{P}([n])) + \mathbb{R}_{\geq 0}^n)} \langle f, \nu \rangle \\ &= -\sigma_{-\text{spr}(\ell)}(\nu). \end{aligned}$$

Since the Bayes risk and  $D$ -information can both be written in terms of a support function, it is unsurprising that there is a relationship between them, and in fact it is simple. In order to demonstrate this, we need some technical results first.

**Lemma 19** ((Rockafellar and Wets, 2004, Theorem 14.60)). *Suppose  $(\Omega, \Sigma)$  is a measurable space and let  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, \mathbb{R}^n)$  be decomposable relative to a sigma-finite measure  $\rho$  on  $\Sigma$ . Let  $\psi: \Omega \times \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  be a normal integrand<sup>13</sup>, and let  $I_\psi(f) \stackrel{\text{def}}{=} \int \psi(x, f(x)) \rho(dx)$ . If  $I_\psi \not\equiv \infty$  on  $\Omega$  then*

$$\inf_{f \in \mathcal{F}} I_\psi(f) = \int \left( \inf_{f \in \mathbb{R}^n} \psi(x, f) \right) \rho(dx). \quad (21)$$

**Lemma 20.** *Let  $(\Omega, \Sigma)$  be a measurable space, and  $\rho$ , a sigma-finite measure on  $\Sigma$ . Let  $D \subseteq \mathbb{R}^n$  be nonempty, closed and measurable. Let  $k: \Omega \times \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  be such that  $k(\cdot, d)$  is measurable for all  $d \in \mathbb{R}^n$  and  $-k(x, \cdot)$  is convex and lower semi-continuous for all  $x \in \Omega$ . Then*

$$\sup_{f \in \mathcal{L}_0(\Omega, D)} \int k(x, f(x)) \, d\rho(x) = \int \sup_{d \in D} k(x, d) \, d\rho(x).$$

13. The technical terms “normal integrand” and “decomposable” are defined by Rockafellar and Wets (2004, Definition 14.27, 14.59), to which we refer the reader for details.

*Proof.* In order to apply Lemma 19, let  $\psi(x, d) \stackrel{\text{def}}{=} \iota_D(x) - k(x, d)$ . Since  $\psi$  is the sum of the indicator function of a closed measurable set and an appropriately measurable, lower semicontinuous map, it is normal (Rockafellar and Wets, 2004, Proposition 14.39). The collection  $\mathcal{L}_0(\Omega, \mathbb{R}^n)$  is trivially decomposable. Therefore

$$\begin{aligned} \sup_{f \in \mathcal{L}_0(\Omega, D)} \int k(x, f(x)) \rho(\mathrm{d}x) &= \sup_{f \in \mathcal{L}_0(\Omega, \mathbb{R}^n)} \int (k(x, f(x)) - (\iota_D \circ f)(x)) \rho(\mathrm{d}x) \\ &\stackrel{(21)}{=} \int \sup_{d \in \mathbb{R}^n} (k(x, d) - \iota_D(d)) \rho(\mathrm{d}x) \\ &= \int \sup_{d \in D} k(x, d) \rho(\mathrm{d}x). \quad \square \end{aligned}$$

**Lemma 21.** *Suppose  $\Omega$  is a standard Borel space,  $\ell \in \mathcal{L}_0(\Omega \times [n], \mathbb{R})$ , then*

$$\forall \mu \in \mathfrak{P}(\Omega \times [n]) : \text{Bayes}_\ell(\mu) = \inf_{f \in \mathcal{L}_0(\Omega, \text{spr}(\ell))} \int f \, \mathrm{d}\mu.$$

*Proof.* Since  $\Omega$  is a standard Borel space, so is  $\Omega \times [n]$  and we have  $\mu = \mu_X \times \mu_{Y|X}$ . Let  $S \stackrel{\text{def}}{=} \ell(\mathfrak{P}([n]))$ . Then

$$\begin{aligned} \text{Bayes}_\ell(\mu) &= \inf_{f \in \mathcal{L}_0(\Omega, \mathfrak{P}([n]))} \mathbb{E}_{(X, Y) \sim \mu} [\ell(f(X), Y)] \\ &= \inf_{f \in \mathcal{L}_0(\Omega, \mathfrak{P}([n]))} \int \langle \ell(f(x)), \mu_{Y|X=x} \rangle \mu_X(\mathrm{d}x) \\ &= \inf_{f \in \mathcal{L}_0(\Omega, S)} \int \langle f(x), \mu_{Y|X=x} \rangle \mu_X(\mathrm{d}x) \\ &= \inf_{f \in \mathcal{L}_0(\Omega, S + \mathbb{R}_{\geq 0}^n)} \int \langle f(x), \mu_{Y|X=x} \rangle \mu_X(\mathrm{d}x) \\ &\stackrel{(20)}{=} \inf_{f \in \mathcal{L}_0(\Omega, \text{spr}(\ell))} \int \langle f(x), \mu_{Y|X=x} \rangle \mu_X(\mathrm{d}x) \\ &= \inf_{f \in \mathcal{L}_0(\Omega, \text{spr}(\ell))} \int f \, \mathrm{d}\mu. \quad \square \end{aligned}$$

**Proposition 22.** *Suppose  $E : [n] \rightsquigarrow \Omega$  is an experiment and  $D \subseteq \mathbb{R}^n$  is closed. Then*

$$I_D(E) = \sup_{f \in \mathcal{L}_0(\Omega, D)} \int \sum_{i \in [n]} f_i \, \mathrm{d}E_i. \quad (22)$$

*Proof.* The proposition follows from Lemma 20 with  $k(x, d) \stackrel{\text{def}}{=} \langle (\mathrm{d}E/\mathrm{d}\rho)(x), d \rangle$  and  $\mathrm{d}\rho \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i \in [n]} \mathrm{d}E_i$ , along with the observation that since  $D$  is closed, it is also Borel measurable.  $\square$

Let  $\pi \in \mathfrak{P}([n])$  be a prior distribution over  $[n]$  together with a binary experiment  $E : [n] \rightsquigarrow \Omega$ . Then there is a probability distribution  $\pi \times E \in \mathfrak{P}(\Omega \times [n])$  satisfying

$$(\pi \times E)(\mathrm{d}x, \mathrm{d}y) \stackrel{\text{def}}{=} \pi(\mathrm{d}y) E(y, \mathrm{d}x).$$

It will be convenient to write the prior  $\pi$  as a vector  $(\pi_1, \dots, \pi_n) \in \mathbb{R}^n$ .



We now present the relationship between  $D$ -information and the Bayes risk<sup>14</sup>.

**Theorem 23.** *Suppose  $\Omega$  is a standard Borel space,  $\ell \in \mathcal{L}_0(\mathfrak{P}([n]) \times [n], \mathbb{R})$ ,  $E : [n] \rightsquigarrow \Omega$ , and  $\pi \in \mathfrak{P}([n])$ . Let  $\pi \cdot \text{spr}(\ell) \stackrel{\text{def}}{=} \{(\pi \cdot f) \mid f \in \text{spr}(\ell)\}$  denote the Hadamard vector product  $\pi \cdot f = (\pi_1 f_1, \dots, \pi_n f_n)$  for each element of  $\text{spr}(\ell)$ . Then*

$$\text{Bayes}_\ell(\pi \times E) = -\mathbf{I}_{-\pi \cdot \text{spr}(\ell)}(E). \quad (23)$$

*Proof.* Equation (23) is obtained from Lemma 21 and Proposition 22 as follows:

$$\begin{aligned} \text{Bayes}_\ell(\pi \times E) &= \inf_{f \in \mathcal{L}_0(\Omega, \mathfrak{P}([n]))} \sum_{y \in [n]} \int \ell(f(x), y) \pi(y) E(y, dx) \\ &\stackrel{\text{L21}}{=} \inf_{f \in \mathcal{L}_0(\Omega, \text{spr}(\ell))} \int \sum_{y \in [n]} f_y(x) \pi(y) E_y(dx) \\ &= \inf_{f \in \mathcal{L}_0(\Omega, \pi \cdot \text{spr}(\ell))} \int \sum_{y \in [n]} f_y(x) E_y(dx). \\ &\stackrel{\text{P22}}{=} -\mathbf{I}_{-\pi \cdot \text{spr}(\ell)}(E). \quad \square \end{aligned}$$

**Remark 24.** The relationship  $D = -\pi \cdot \text{spr}(\ell)$  is a generalisation of that developed for  $\varphi$ -divergences ( $n = 2$ ) in (Reid and Williamson, 2011) as we now elucidate. Inverting the relationship we have  $\text{spr}(\ell) = -\left(\frac{1}{\pi}\right) \cdot D$ , where  $\frac{1}{\pi} \stackrel{\text{def}}{=} \left(\frac{1}{\pi_1}, \dots, \frac{1}{\pi_n}\right)$ . It is elementary (and also follows using  $r \cdot x = \text{diag}(r)x$  from (Hiriart-Urruty and Lemaréchal, 2001, Proposition C.3.3.3)) that

$$\sigma_{-\left(\frac{1}{\pi}\right) \cdot D}(x) = \sigma_D\left(-\left(\frac{1}{\pi}\right) \cdot x\right).$$

Setting  $D = \text{hyp}(-\varphi^*)$  we know from Proposition 4 that  $\sigma_D = \check{\varphi}$  and thus

$$\sigma_{\text{spr}(\ell)}(x) = \check{\varphi}\left(\frac{-x_1}{\pi_1}, \frac{-x_2}{\pi_2}\right) = \frac{-x_2}{\pi_2} \varphi\left(\frac{x_1}{x_2} \frac{\pi_2}{\pi_1}\right). \quad (24)$$

14. This theorem (sans the geometric insight) was presented by Garcia-Garcia and Williamson (2012), and restated in a related form by Duchi, Khosravi, and Ruan (2018). It both extends and simplifies the version for  $n = 2$  presented by Reid and Williamson (2011), which itself extended beyond the symmetric (margin loss) case the version due to Nguyen, Wainwright, and Jordan (2009), which first appeared in (Nguyen, Wainwright, and Jordan, 2005), and which in turn extended the observations of Österreicher and Vajda (1993) and (Gutenbrunner, 1990). The relationship with different entropies and scoring rules was also closely studied in (Grünwald and Dawid, 2004). Earlier attempts to connect measures of information to Bayes risks include Fano’s inequality (Fano, 1961, Section 9.2) (Polyanskiy and Wu, 2024), and the inequalities derived by Pérez (1967) and Toussaint (1974, 1977, 1978). Other precursors are the generalised entropies of Dupuis et al. (2014) defined in terms of a Neyman-Pearson hypothesis testing problem (and thus equivalent to generalised variational divergence). In the binary case, with Variational divergence and 0-1 loss, the bridge is classical (Devroye, Györfi, and Lugosi, 2013). There is now quite a literature on information-theoretic statistical inference based on divergences (Pardo, 2018); the bridge described in the present section suggests that such methods can be profitably viewed as a re-parametrisation of classical decision-theoretic methods based on expected losses. The relationship between measures of information and the Bayes risk was also observed in (Chatzikokolakis, Palamidessi, and Panangaden, 2008) for information security problems, and in (Alvim et al., 2012) for general information leakage problems.

The negative support function of the superprediction set  $\text{spr}(\ell)$  corresponds to the conditional Bayes risk  $\underline{L}^\pi$  in (Reid and Williamson, 2011, Theorem 9) (confer remark 18). Parametrising in the same manner with  $(x_1, x_2) = (\eta, 1 - \eta)$  and  $(\pi_1, \pi_2) = (\pi, 1 - \pi)$  and substituting into (24) we obtain

$$\sigma_{\left(\frac{-1}{\pi}, \frac{-1}{1-\pi}\right) \cdot D}((\eta, 1 - \eta)) = \underline{L}^\pi(\eta) = -\frac{1 - \eta}{1 - \pi} \varphi \left( \frac{1 - \pi}{\pi} \frac{\eta}{1 - \eta} \right),$$

consistent with (Reid and Williamson, 2011, Theorem 9). The loss  $\ell$  can be recovered from  $\sigma_{\text{spr}(\ell)}$  via the derivative:  $\ell = D \sigma_{\text{spr}(\ell)}$  (Williamson, 2014; Williamson and Cranko, 2023). Evaluating the partial derivatives we obtain explicit formulae for  $\ell_1$  and  $\ell_2$  in terms of  $\varphi$ :

$$\ell_1(x_1, x_2) = -\frac{1}{\pi_1} \varphi' \left( \frac{x_1 \pi_2}{x_2 \pi_1} \right) \quad \ell_2(x_1, x_2) = -\frac{1}{\pi_2} \varphi \left( \frac{x_1 \pi_2}{x_2 \pi_1} \right) + \frac{1}{\pi_1} \frac{x_1}{x_2} \varphi' \left( \frac{x_1 \pi_2}{x_2 \pi_1} \right),$$

which are 0-homogeneous in  $x$  as expected from Euler's homogeneous function theorem (see below).

## 4.2 The witness to the supremum in $I_D$

$I_D$  is defined via a supremum. There is insight to be had by examining the function that attains this. Let  $\nabla \sigma_D$  be a selection of  $\partial \sigma_D$ . Euler's homogeneous function theorem:

$$\forall x \in \text{dom } \partial \sigma_D : \sigma_D(x) = \langle \nabla \sigma_D(x), x \rangle, \quad (25)$$

and the 1-homogeneity of  $\sigma_D$  implies  $\partial \sigma_D$  is 0-homogeneous (so for any  $c > 0$ ,  $\partial \sigma_D(cx) = \partial \sigma_D(x)$ ). We can thus determine the argmax in (6):

**Proposition 25.** *Suppose  $E : [n] \rightsquigarrow \Omega$  is an experiment and  $D \subset \mathbb{R}^n$ . Let  $\rho$  be a measure that dominates each of the measures  $(E_y)_{y \in Y}$ . Then if  $\sigma_D$  is finite on  $\mathbb{R}_{>0}^n$  there exists a selection  $\nabla \sigma_D \in \partial \sigma_D$  over  $\mathbb{R}_{>0}^n$ , and*

$$I_D(E) = \sup_{p \in \mathcal{L}_0(X, \mathfrak{P}(Y))} \sum_{y \in [n]} \int_X (\nabla \sigma_D \circ p)_y dE_y = \sum_{y \in [n]} \int_X \left( \nabla \sigma_D \circ \frac{dE}{d\rho} \right)_y dE_y.$$

Note that the requirement is only that  $\sigma_D$  is finite on  $\mathbb{R}_{>0}^n$ , not on  $\mathbb{R}_{\geq 0}^n$  which would exclude standard unbounded information measures such as Kullback-Leibler divergence.

*Proof.* Since  $\mathbb{R}_{>0}^n \subseteq \text{rint}(\text{dom } \sigma_D) \subseteq \text{dom } \partial \sigma_D$ , by the Michael selection theorem (Aliprantis and Border, 2006, Theorem 17.66, p. 589) there exists a continuous selection  $\nabla \sigma_D$  mapping  $\mathbb{R}_{>0}^n \rightarrow D$ . From the definition of the support function

$$\sigma_D(x) = \sup_{d \in D} \langle x, d \rangle \geq \sup_{z \in \mathbb{R}_{>0}^n} \langle x, \nabla \sigma_D(z) \rangle.$$

By Euler's homogeneous function theorem, (25),  $\sigma_D(x) = \langle x, \nabla \sigma_D(x) \rangle$ , and consequently  $\sigma_D(x) = \sup_{f \in \mathbb{R}_{>0}^n} \langle x, \nabla \sigma_D(f) \rangle$  for all  $x \in \mathbb{R}_{>0}^n$ . Proposition 22 implies that

$$I_D(E) = \int \sigma_D \left( \frac{dE}{d\rho} \right) d\rho$$

$$\begin{aligned}
 &= \int \sup_{f \in \mathbb{R}_{>0}^n} \left\langle \frac{dE}{d\rho}(x), \nabla \sigma_D(f) \right\rangle d\rho(x) \\
 &= \int \sup_{d \in \nabla \sigma_D(\mathbb{R}_{>0}^n)} \left\langle \frac{dE}{d\rho}(x), d \right\rangle d\rho(x) \\
 &= \sup_{f \in \mathcal{L}_0(\Omega, \nabla \sigma_D(\mathbb{R}_{>0}^n))} \sum_{y \in [n]} \int f_y dE_y \\
 &= \sup_{f \in \mathcal{L}_0(\Omega, \nabla \sigma_D(\mathbb{R}_{>0}^n))} \sum_{y \in [n]} \int (\nabla \sigma_D \circ f)_y dE_y,
 \end{aligned}$$

where in the fourth equality we apply Lemma 20 with  $k = \langle \cdot, \cdot \rangle$ . This proves the first equality since  $\nabla \sigma_D$  is 0-homogeneous.

Euler's homogeneous function theorem implies

$$\begin{aligned}
 I_D(E) &= \int \sigma_D \left( \frac{dE}{d\rho} \right) d\rho \\
 &= \int \left\langle \frac{dE}{d\rho}, \nabla \sigma_D \circ \frac{dE}{d\rho} \right\rangle d\rho \\
 &= \sum_{y \in [n]} \int \left( \nabla \sigma_D \circ \frac{dE}{d\rho} \right)_y dE_y.
 \end{aligned}$$

This shows the second equality.  $\square$

**Remark 26.** It is instructive to evaluate the witness of the supremum in Proposition 25 in the case of  $Y = \{1, 2\}$  in terms of the  $\varphi$ -divergence parameterisation of  $I_D$ . With  $D_\varphi$  as in (10), we have  $\sigma_{D_\varphi} = \check{\varphi}$ . Assume  $\varphi$  is differentiable, so  $g_\varphi \stackrel{\text{def}}{=} D\sigma_{D_\varphi} = D\check{\varphi}$  exists and by direct calculation we obtain

$$g_\varphi(x, y) = \begin{pmatrix} \varphi'(x/y) \\ \varphi(x/y) - \varphi'(x/y)x/y \end{pmatrix},$$

with the witness is given by

$$g_\varphi \circ \frac{dE}{d\rho} = g_\varphi \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right) = \begin{pmatrix} \varphi'(dE_1/dE_2) \\ \varphi(dE_1/dE_2) - \varphi'(dE_1/dE_2)dE_1/dE_2 \end{pmatrix}$$

and thus

$$\begin{aligned}
 I_{D_\varphi}(E) &= E_\rho \left\langle \frac{dE}{d\rho}, g_\varphi \circ \frac{dE}{d\rho} \right\rangle \\
 &= \int \left[ \frac{dE_1}{d\rho} \varphi' \left( \frac{dE_1}{dE_2} \right) + \frac{dE_2}{d\rho} \left( \varphi \left( \frac{dE_1}{dE_2} \right) - \varphi' \left( \frac{dE_1}{dE_2} \right) \frac{dE_1}{dE_2} \right) \right] d\rho \\
 &= \int \varphi' \left( \frac{dE_1}{dE_2} \right) dE_1 + \int \varphi \left( \frac{dE_1}{dE_2} \right) dE_2 - \int \varphi' \left( \frac{dE_1}{dE_2} \right) dE_1 \\
 &= \int \varphi \left( \frac{dE_1}{dE_2} \right) dE_2,
 \end{aligned}$$

which is the classical form of the  $\varphi$ -divergence (2).

### 4.3 The Family of $D$ -informations

**Proposition 27.** *Suppose  $D \subseteq \mathbb{R}^n$  is convex. Then  $\text{cl}(\text{dom } \sigma_D) = \mathbb{R}_{\geq 0}^n$  if and only if  $\text{rec}(D) = \mathbb{R}_{\leq 0}^n$ .*

*Proof.* From (Auslender and Teboulle, 2003, Theorem 2.2.1 (c), p. 32) and the bipolar theorem we have

$$(\text{dom } \sigma_D)^* = \text{rec}(D) = \mathbb{R}_{\geq 0}^n \iff \text{cl}(\text{dom } \sigma_D) = \text{rec}(D)^* = \mathbb{R}_{\leq 0}^n. \quad \square$$

In Theorem 23 we observed an interesting connection between the Bayes risks associated to a risk minimisation and the negative  $D$ -information associated with its negative superprediction set. There is also a similar asymptotic characterisation of the superprediction sets of positive loss functions.

Proposition 28 is a special case of a much more general result stated for superprediction sets on general outcome spaces in (Cranko, 2021, Proposition 4.6 (a), p. 67). We include its short proof for completeness.

**Proposition 28.** *Let  $\ell : \mathfrak{P}([n]) \rightarrow \mathbb{R}_{\geq 0}^n$ . Then  $\text{rec}(\text{spr}(\ell)) = \mathbb{R}_{\geq 0}^n$ .*

*Proof.* We first use the property that  $A \subseteq B$  implies  $\text{rec}(A) \subseteq \text{rec}(B)$  and  $\ell(\mathfrak{P}([n])) \subseteq \mathbb{R}_{\geq 0}^n$  to obtain

$$\text{rec}(\text{spr}(\ell)) = \text{rec}(\ell(\mathfrak{P}([n])) + \mathbb{R}_{\geq 0}^n) \subseteq \text{rec}(\mathbb{R}_{\geq 0}^n + \mathbb{R}_{\geq 0}^n) = \mathbb{R}_{\geq 0}^n.$$

This shows  $\text{rec}(\text{spr}(\ell)) \subseteq \mathbb{R}_{\geq 0}^n$ . Next, using the associativity of the Minkowski sum

$$\mathbb{R}_{\geq 0}^n + \text{spr}(\ell) = \mathbb{R}_{\geq 0}^n + \mathfrak{P}([n]) + \mathbb{R}_{\geq 0}^n = \mathfrak{P}([n]) + \mathbb{R}_{\geq 0}^n = \text{spr}(\ell),$$

which shows  $\mathbb{R}_{\geq 0}^n \subseteq \text{rec}(\text{spr}(\ell))$ , and completes the proof.  $\square$

After observing that  $-\text{rec}(D) = \text{rec}(-D)$ , Propositions 27 and 28 yield another characterisation of the connection between the  $D$ -information and Bayes risks with nonnegative proper loss functions, this time in terms of the asymptotic geometry of these sets. Although it may seem coincidental that—despite very different origins and motivating definitions—the sets  $\text{spr}(\ell)$  and  $-D$  look very similar from afar, this relationship is not at all surprising when parameterising these functionals using a set, as we have done. The bilinearity of the expectation operator means that we are working with a pointwise infimum or supremum over linear forms, which means that, without loss of generality, we can replace the set by its closed convex hull. This explains the natural characterisation in terms of the support function (Remarks 2 and 18). Since both of these functionals operate on sets of probability measures, in order for them to be meaningful they should be sufficiently finite; this is the essence of the asymptotic characterisations in Propositions 27 and 28.

**Remark 29.** We have shown that the recession cone of  $\text{spr}(\ell)$  is such that the induced  $D$  has the right recession cone for  $D$  information, but what about normalisation? In the same way that there is some freedom in normalising  $D$ , we have freedom in normalising  $\ell$ . In previous work (Vernet, Williamson, and Reid, 2016; Williamson, 2014; Williamson and Cranko, 2023) we have normalised proper losses  $\ell$  such that  $\ell(e_i) = 0$  for  $i \in [n]$  (where  $e_i$  is the canonical

unit vector). This implies that  $\text{spr}(\ell) \subset \mathbb{R}_{\geq 0}^n$ . For the present paper it is more convenient to normalise such that

$$\ell(1_n/n) = 0_n \quad \text{and} \quad \sigma_{\text{spr}(\ell)}(1_n/n) = 0. \quad (26)$$

The first condition implies that  $0 \in \text{bd spr}(\ell)$  and the second that  $\text{spr}(\ell) \subseteq \text{lev}_{\leq 0} \langle \cdot, -1_n \rangle$ . The bridge from risks to information requires the specification of the prior  $\pi$  which can be seen to effectively scale  $\text{spr}(\ell)$  separately in each dimension. Of course the simplest case to consider is that  $\pi = 1_n/n$ , in which case it follows immediately that if  $\ell$  satisfies (26) then  $D := -\pi \cdot \text{spr}(\ell)$  satisfies  $\sigma_D(1_n) = 0 \Rightarrow D \subseteq \text{lev}_{\leq 0} \langle \cdot, 1_n \rangle$  and  $0 \in \text{bd } D$ , and consequently  $D \in \mathfrak{D}^n$ . Given an  $\ell$  that does not satisfy (26), it can be made to do so by translation and scaling. Thus the normalisation conditions we impose upon  $D$  can always be met by suitable adjustment of  $\ell$ . Adopting the normalisation in (26) means that for  $\pi = 1_n/n$ , the *statistical information* of DeGroot (1962) is simply the negative Bayes risk, because  $\sigma_{\text{spr}(\ell)}(1_n/n) = 0$  implies the “prior Bayes risk”  $\underline{\mathbb{L}}(\pi, M)$  is zero; see (Reid and Williamson, 2011, Sections 4.6 and 4.7).

**Remark 30.** The bridge result (Theorem 23) implies that any means by which multiple loss functions are combined, by combining their superprediction sets, provides an analogous combination scheme for information measures, by combining  $D_i \in \mathfrak{D}^n$ ,  $i \in [m]$ . The combination schemes in (Williamson and Cranko, 2023) based upon  $M$ -sums (Gardner, Hug, and Weil, 2013) suggest one can simply take  $M$ -sums of the  $D_i$ . This generalises the combination schemes proposed by Kús (2003) and Kús, Morales, and Vajda (2008).

#### 4.4 $D$ -Information Processing Equality

One of the most basic results in information theory is the information processing inequality (Cover and Thomas, 2012). It is often stated in terms of mutual information, but there is a version, which is equivalent, in terms of divergences (Polyanskiy and Wu, 2024). We defer until part II of the paper (Williamson, 2024) a detailed statement and examination of the connection between the two types, and indeed the connection with what we present below. Rather than an *inequality*, below we present an information processing *equality*, with, however, a different measure of information on either side of the equation. Also, the result below is for what in the machine learning community is called “label noise”. The traditional information processing inequality is for the situation of “attribute noise” and is treated in §5 below.

**Proposition 31.** *Suppose  $E : [n] \rightsquigarrow \Omega$ ,  $R : [n] \rightsquigarrow [n]$ , and  $D \subseteq \mathbb{R}^n$ . Then*

$$\mathbb{I}_D(RE) = \mathbb{I}_{R^\top D}(E), \quad (27)$$

where  $R^\top D \stackrel{\text{def}}{=} \{R^\top d \mid d \in D\}$  uses the representation of  $R$  as a matrix.

*Proof.* Identifying  $R$  with its representation as a stochastic matrix, and writing  $d^{E/d\rho}$  for the vector  $(d^{E_1/d\rho}, \dots, d^{E_n/d\rho})$  there is  $d^{(RE)_i/d\rho} = (R d^{E/d\rho})_i$  for  $i \in [n]$ , and

$$\mathbb{I}_D(E) = \int \sup_{d \in D} \left( \sum_{i \in [n]} d_i \frac{d^{(RE)_i}}{d\rho}(x) \right) \rho(dx)$$

$$\begin{aligned}
 &= \int \sup_{d \in D} \left\langle d, R \frac{dE}{d\rho}(x) \right\rangle \rho(dx) \\
 &= \int \sup_{d \in D} \left\langle R^\top d, \frac{dE}{d\rho}(x) \right\rangle \rho(dx) \\
 &= \int \sup_{d \in R^\top D} \left( \sum_{i \in [n]} d_i \frac{dE_i}{d\rho}(x) \right) \rho(dx) \\
 &= \mathbf{I}_{R^\top D}(E). \quad \square
 \end{aligned}$$

**Remark 32.** Observe that a  $n \times n$  permutation matrix  $P$  can be thought of as a Markov kernel  $P: [n] \rightsquigarrow [n]$ . Let  $PD := \{Pd \mid d \in D\}$  and say that  $D \subseteq \mathbb{R}^n$  is *permutation invariant* if for any such  $P$ ,  $PD = D$ . Then Proposition 31 implies for such  $D$  that  $\mathbf{I}_D(PE) = \mathbf{I}_{P^\top D}(E) = \mathbf{I}_D(E)$ , since  $P^\top$  is also a permutation matrix. Thus, in this situation,  $\mathbf{I}_D$  is permutation invariant. An equivalent, but less elegant, version of this observation was given in (Garcia-Garcia and Williamson, 2012). When  $n = 2$ , and  $D$  is parametrised by  $\varphi$  as in Theorem 4, this invariance corresponds to the requirement that for all  $x > 0$ ,  $\varphi(x) = \varphi^\diamond(x) = x\varphi(1/x)$  (the Csizár conjugate of  $\varphi$ ) which implies  $\mathbf{I}_\varphi(P, Q) = \mathbf{I}_\varphi(Q, P)$ .

**Example 33** (Label Noise). Let  $E: [n] \rightsquigarrow \Omega$  be an experiment and  $R: [n] \rightsquigarrow [n]$  a Markov kernel. We can thus form the product experiment  $RE$  as per the diagram

$$Y \xrightarrow{R} \tilde{Y} \xrightarrow{E} X.$$

This corresponds to “label noise” — that is noise in the observations of  $Y$ . Instead of learning from  $(X, Y)$  one only gets to observe  $(X, \tilde{Y})$  for some corrupted version  $\tilde{Y}$  of the true label  $Y$ . For example, when  $n = 2$ , and  $Y \in [2]$  one might have a label flip with probability  $p$ . This corresponds to  $R$  having the representation as the stochastic matrix

$$R \stackrel{\text{def}}{=} \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

Then for  $D \subseteq \mathbb{R}^n$ , with Proposition 31

$$\mathbf{I}_D(RE) = \mathbf{I}_{R^\top D}(E).$$

When  $n = 2$  one can translate the result of Proposition 31 to the language of  $\varphi$  divergences, in which form the result is less perspicuous than (27):

**Corollary 34.** Suppose  $\varphi \in \Phi$ ,  $R: [2] \rightsquigarrow [2]$  is the Markov Kernel parametrised as

$$R = \begin{bmatrix} r_1 & 1-r_1 \\ 1-r_2 & r_2 \end{bmatrix}, \quad \text{and} \quad \varphi_R(z) = ((1-r_2)z + r_2) \varphi \left( \frac{r_1 z + 1 - r_1}{(1-r_2)z + r_2} \right).$$

Then for all experiments  $E: [2] \rightsquigarrow \Omega$ ,  $\mathbf{I}_\varphi(RE) = \mathbf{I}_{\varphi_R}(E)$ .

*Proof.* The trick is to identify the perspective of  $\varphi$  with the support function of  $D$ :  $\check{\varphi}(x, y) = \sigma_D((x, y)^\top)$  for  $x, y, \in \mathbb{R}_{\geq 0}$ . The effect of the Markov kernel  $R$  on  $\varphi$  can be determined from

$$\check{\varphi}_R(x, y) = \sigma_D(R \cdot (x, y)^\top) = \check{\varphi}(R \cdot (x, y)^\top) = \check{\varphi}(r_1 x + (1-r_1)y, (1-r_2)x + r_2 y).$$

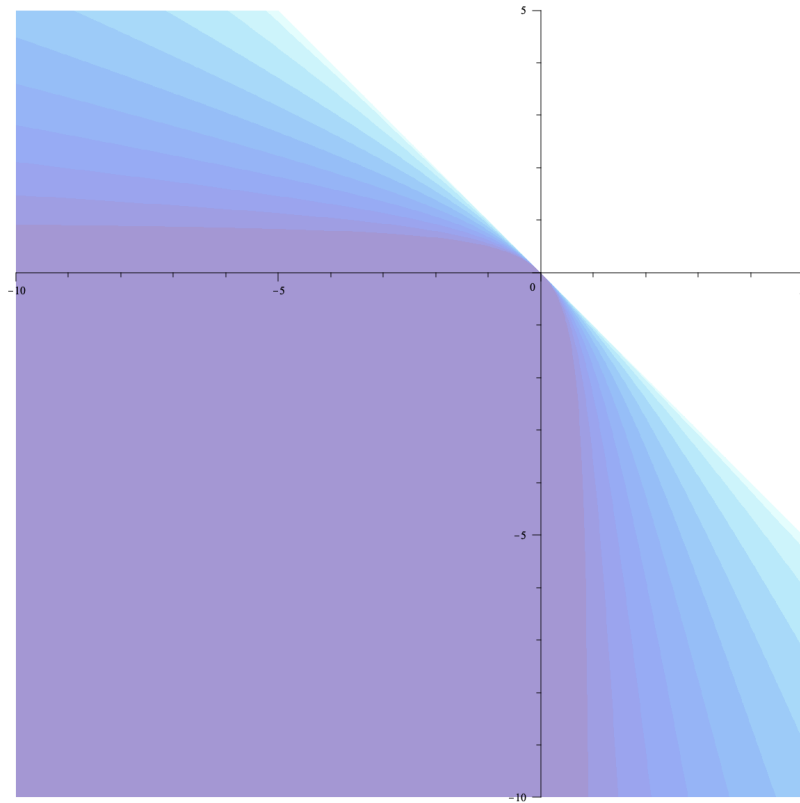


Figure 1: Effect of  $R_r$  on  $D_{\text{Hell}}$ . The plot shows  $R_r^* D_{\text{Hell}}$  (restricted to  $[-10, 5]^2$ ) for  $r$  ranging from 0.5 (very pale blue) to 1.0 (purple), which is of course just  $D_{\text{Hell}}$ .

Consequently  $I_\varphi(RE) = I_{\varphi_R}(E)$ , where

$$\varphi_R(z) = \check{\varphi}_R(z, 1) = \check{\varphi}(r_1 z + 1 - r_1, (1 - r_2)z + r_2) = ((1 - r_2)z + r_2) \varphi\left(\frac{r_1 z + 1 - r_1}{(1 - r_2)z + r_2}\right). \quad \square$$

**Remark 35.** Example 33 corresponds to previous work on loss correction, whereby learning with a given loss with noisy labels is equivalent to learning with a “corrected loss” with noiseless labels; see e.g. (Patrini et al., 2017; van Rooyen, Menon, and Williamson, 2015; van Rooyen and Williamson, 2018).

Figure 1 illustrates this case for the binary *symmetric* channel  $R_r = \begin{bmatrix} r & 1-r \\ 1-r & r \end{bmatrix}$  for  $D_{\text{Hell}}$  (corresponding to squared Hellinger divergence — see table 1 in Appendix A). Observe that (as motivated in Appendix B) if  $D^\circ = C$ , then  $(R^* D)^\circ = R^{-1} C$  (this follows simply by substitution into the definition of the polar).

## 5. Constrained Information Measures — $\mathcal{F}$ -information

As we saw in §4.1, Theorem 23 shows how the  $D$ -information can be connected to risk minimisation. In practice one can never actually attain the Bayes risk, because with access

only to a finite number samples rather than the exact underlying distribution, one needs to restrict the hypothesis class (Vapnik, 1998) in order to make the optimisation in (19) well-posed (when using empirical measures). It is helpful to consider the formula for the (unconstrained) Bayes risk in (19) repeated below for convenience.

$$\text{Bayes}_\ell(\mu) = \inf_{f \in \mathcal{L}_0(\Omega, \mathfrak{P}([n]))} \mathbb{E}_{(X,Y) \sim \mu}[\ell(f(X), Y)].$$

Embracing the above viewpoint, we modify this by optimising over  $\mathcal{H} \subsetneq \mathcal{L}_0(\Omega, \mathfrak{P}([n]))$ , and call this restriction the *constrained Bayes risk*:

$$\text{Bayes}_{\ell, \mathcal{H}}(\mu) \stackrel{\text{def}}{=} \inf_{h \in \mathcal{H}} \mathbb{E}_{(X,Y) \sim \mu}[\ell(h(X), Y)]. \quad (28)$$

There is a slight redundancy in (28) since the function class  $\mathcal{H}$  only appears via composition with the loss function  $\ell$ . When viewed in terms of information rather than risk, (28) is precisely the measure of information which we now introduce.

### 5.1 $\mathcal{F}$ -information

Recall the expression for  $I_D(E)$  in (22) (swapping the integral and the sum for convenience in what follows):

$$I_D(E) = \sup_{f \in \mathcal{L}_0(\Omega, D)} \sum_{i \in [n]} \int f_i \, dE_i.$$

If now we restrict the supremum to be over a set  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, D)$ , the  $\mathcal{F}$ -information of an experiment  $E : [n] \rightsquigarrow \Omega$  is<sup>15</sup>

$$I_{\mathcal{F}}(E) \stackrel{\text{def}}{=} \sup_{f \in \mathcal{F}} \sum_{i \in [n]} \int f_i \, dE_i. \quad (29)$$

With notation that is consistent with Theorem 23, from the definition of the constrained Bayes risk (28), a prior  $\pi \in \mathfrak{P}([n])$ , a loss function  $\ell$ , an experiment  $E$ , and a hypothesis class  $\mathcal{H} \subseteq \mathcal{L}_0(\Omega, \mathfrak{P}([n]))$ , the  $\mathcal{F}$ -information is related to the constrained Bayes risk by

$$\text{Bayes}_{\ell, \mathcal{H}}(\pi \times E) = -I_{-\pi \cdot \ell \circ \mathcal{H}}(E),$$

where  $\pi \cdot \ell \circ \mathcal{H} \stackrel{\text{def}}{=} \{x \mapsto (\pi_1 \ell(h(x), 1), \dots, \pi_n \ell(h(x), n)) \mid h \in \mathcal{H}\}$ , that is the composition of  $\mathcal{H}$  with  $\ell$ , and scaled by  $\pi$ . (This is proved below in Theorem 37.)

Consider the collection  $\mathcal{L}_0(X, D)$  of measurable mappings from  $X$  to some  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ . For  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  denote by  $\mathcal{C}_D \stackrel{\text{def}}{=} \{x \mapsto d \mid d \in D\}$  the *set of constant maps from  $X$  to  $D$* . We say that  $\mathcal{F}$  is *D-ranged* if  $\mathcal{C}_D \subseteq \mathcal{F} \subseteq \mathcal{L}_0(X, D)$ . If  $\mathcal{F} \subseteq \mathcal{L}_0(X, \mathbb{R}^n)$ , let

$$\mathcal{F}(X) \stackrel{\text{def}}{=} \bigcup_{f \in \mathcal{F}} f(X),$$

---

15. There are a number of precursors of  $I_{\mathcal{F}}$  which are summarised in Appendix D.



where  $f(X) \stackrel{\text{def}}{=} \{f(x) \mid x \in X\}$ . If  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  and  $\mathcal{F}$  is  $D$ -ranged, then  $\mathcal{F}(X) = D$  since 1)  $\mathcal{C}_D \subseteq \mathcal{F} \Rightarrow \mathcal{F}(X) \supseteq D$  and 2)  $\mathcal{F} \subseteq \mathcal{L}_0(X, D) \Rightarrow \mathcal{F}(X) \subseteq D$ .

Every  $D$ -ranged  $\mathcal{F}$  is a collection of appropriately measurable mappings from  $X$  into  $D$ , and every  $d \in D$  may be attained by  $f(x)$  for some  $f \in \mathcal{F}$  and  $x \in X$ . The maximal (by subset ordering)  $D$ -ranged  $\mathcal{F}$  is simply  $\mathcal{L}_0(X, D)$ , the set of *all* measurable mappings from  $X$  to  $D$ . Choosing smaller sets is equivalent to working with restricted hypothesis classes in normal statistical decision problems (an assertion we make precise below). The extra flexibility of working with such constrained function classes is necessary to capture the effects of attribute noise on information measures.

Suppose  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ ,  $\mathcal{F}$  is  $D$ -ranged and  $E : [n] \rightsquigarrow X$ . The  $\mathcal{F}$ -information of  $E$  is

$$\begin{aligned} I_{\mathcal{F}}(E) &\stackrel{\text{def}}{=} \sup_{f \in \mathcal{F}} \sum_{y \in [n]} \int f_y \, dE_y \\ &= \sup_{f \in \mathcal{F}} \sum_{y \in [n]} \int f_y \frac{dE_y}{d\rho} \, d\rho \\ &= \sup_{f \in \mathcal{F}} \int \sum_{y \in [n]} f_y \frac{dE_y}{d\rho} \, d\rho \\ &= \sup_{f \in \mathcal{F}} E_{\rho} \left\langle \frac{dE}{d\rho}, f \right\rangle, \end{aligned} \tag{30}$$

where  $\rho$  is an arbitrarily chosen reference measure. Suppose  $D \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$ . Define  $\mathcal{F}_D \stackrel{\text{def}}{=} \mathcal{L}_0(X, D)$ . Then it is apparent from (6) that  $I_{\mathcal{F}_D} = I_D$ ; for any other  $\mathcal{F} \subseteq \mathcal{L}_0(X, D)$  we obviously have  $I_{\mathcal{F}} \leq I_D$ , since the supremum is further restricted. If furthermore  $D$  is normalised (i.e.  $D \in \mathfrak{D}^n$ ) and  $\mathcal{F}$  is  $D$ -ranged, then  $0 \leq I_{\mathcal{F}}(E)$  as can be seen by considering  $\mathcal{F} = \mathcal{C}_D$  whence

$$I_{\mathcal{F}}(E) = \sup_{d \in D} \sum_{y \in [n]} \int d \, dE_y = \sup_{d \in D} \sum_{y \in [n]} d = \sup_{d \in D} \langle d, 1_n \rangle = \sigma_D(1_n) = 0.$$

Thus for  $D \in \mathfrak{D}^n$  and  $D$ -ranged  $\mathcal{F}$ ,

$$0 \leq I_{\mathcal{F}}(E) \leq I_D(E).$$

The  $\mathcal{F}$ -information is invariant under convex hulls and closure (Müller, 1997):

**Proposition 36.** *Suppose  $D \subset \mathbb{R}^n$ ,  $E : Y \rightsquigarrow X$ ,  $I_D(E) < \infty$  and  $\mathcal{F}$  is  $D$ -ranged. Then*

$$I_{\mathcal{F}}(E) = I_{\text{co } \mathcal{F}}(E) = I_{\overline{\text{co } \mathcal{F}}}(E).$$

*Proof.* Let  $\Delta = \{(\alpha_i)_i \mid 0 \leq \alpha_i \, \forall i \text{ and } \sum_i \alpha_i = 1\}$ . The convex hull of  $\mathcal{F}$  is

$$\text{co } \mathcal{F} = \left\{ \sum_{i \in \mathbb{N}} \alpha_i f^i \mid f^i \in \mathcal{F}, \forall i \in \mathbb{N} \text{ and } \alpha = (\alpha_i)_i \in \Delta \right\}.$$

Hence

$$I_{\text{co } \mathcal{F}}(E) = \sup_{\substack{f = \sum_i \alpha_i f^i \\ f^i \in \mathcal{F}, \alpha \in \Delta}} \sum_{y \in [n]} \int f_y \, dE_y$$

$$= \sup_{\alpha \in \Delta} \sup_{f^i \in \mathcal{F}} \sum_{y \in [n]} \int \sum_i \alpha_i f_y^i dE_y \quad (31)$$

$$= \sup_{\alpha \in \Delta} \sup_{f^i \in \mathcal{F}} \sum_i \alpha_i \sum_{y \in [n]} \int f_y^i dE_y \quad (32)$$

$$= \sup_{\alpha \in \Delta} \sum_i \alpha_i \sup_{f^i \in \mathcal{F}} \sum_{y \in [n]} \int f_y^i dE_y$$

$$= \sup_{\alpha \in \Delta} \sum_i \alpha_i I_{\mathcal{F}}(E)$$

$$= I_{\mathcal{F}}(E).$$

We need to justify the interchange of order of summation at (31)–(32). The reordering can only fail if there are two subsequences, one diverging to  $-\infty$  and one to  $+\infty$  which cancel each other out. But this is impossible because  $I_{\mathcal{F}}(E) \leq I_D(E) < \infty$  and thus there can be no terms that diverge to  $+\infty$  (even though it is possible that  $f_i(x) = -\infty$ , but such  $f$  would not be chosen by the supremum operation, and all the  $\alpha_i \in [0, 1]$ ). This proves the first equality.

We can now assume  $\mathcal{F}$  is convex. We need to show  $I_{\text{cl } \mathcal{F}}(E) = I_{\mathcal{F}}(E)$ . But  $I_{\mathcal{F}}(E) = \sup_{f \in \mathcal{F}} \Psi(f, E)$ , where  $\Psi(f, E) = \sum_{y \in [n]} \int f_y dE_y$  is bounded since  $I_{\mathcal{F}}(E) \leq I_D(E) < \infty$ . The function  $f \mapsto \Psi(f, E)$  is also linear and thus continuous for any  $E$ . The supremum of a continuous real-valued function over the closure of a set is equal to the supremum over the set, which proves the second equality.  $\square$

Thus there is no loss of generality in henceforth assuming that  $\mathcal{F}$  is closed and convex, as has been observed in the special case when  $n = 2$  and  $D = D_{\text{var}}$  (defined in Lemma 48) corresponding to “integral probability metrics” which are variants of variational divergence with a restricted function class (Müller, 1997). Equivalently, if  $\mathcal{F}$  was not closed and convex, one can take the closed convex hull and not change the value of  $I_{\mathcal{F}}(E)$  (nor indeed change the Rademacher complexity of  $\mathcal{F}$  (Bartlett and Mendelson, 2002)). Since convex function classes enable fast rates of convergence (Mendelson and Williamson, 2002; van Erven et al., 2015) and optimization is in principle simpler, this is an appealing restriction, and one which is receiving practical attention in the form of infinitely wide neural networks (Ergen and Pilanci, 2021). If  $\mathcal{F}$  is closed and convex then so is  $\mathcal{F}(X)$ .

## 5.2 The Bridge between $\mathcal{F}$ -Information and Constrained Bayes Risk

We now relate  $I_{\mathcal{F}}$  to the constrained Bayes risk (28) and to  $I_D$  (6):

**Theorem 37.** *Suppose  $\ell$  is a continuous proper loss,  $E : [n] \rightsquigarrow X$  an experiment,  $\pi \in \mathfrak{P}(Y)$  a prior distribution, and  $\mathcal{H} \subseteq \mathcal{L}_0(X, \mathfrak{P}([n]))$  an hypothesis class. Let  $\mathcal{F} \stackrel{\text{def}}{=} \text{co}(-\pi \cdot \ell \circ \mathcal{H})$ . Then*

$$\text{Bayes}_{\ell \circ \mathcal{H}}(\pi \times E) = -I_{\mathcal{F}}(E).$$

Furthermore  $\mathcal{F} \subseteq \mathcal{L}_0(X, D)$ , where  $D = -\pi \cdot \text{spr } \ell$ .

*Proof.* From the definition of the Bayes risk

$$\text{Bayes}_{\ell \circ \mathcal{H}}(\pi \times E) = \inf_{f \in \ell \circ \mathcal{H}} \sum_{y \in [n]} \int \pi_y f_y dE_y$$

$$\begin{aligned}
 &= - \sup_{f \in \ell \circ \mathcal{H}} \sum_{y \in [n]} \int -\pi_y f_y \, dE_y \\
 &= - \sup_{-\pi \cdot f \in \ell \circ \mathcal{H}} \sum_{y \in [n]} \int f_y \, dE_y \\
 &= - \sup_{f \in -\pi \cdot \ell \circ \mathcal{H}} \sum_{y \in [n]} \int f_y \, dE_y \\
 &= -I_{-\pi \cdot \ell \circ \mathcal{H}}(E) \tag{33} \\
 &= -I_{\mathcal{F}}(E). \tag{34}
 \end{aligned}$$

Whether one prefers the result (33) or (34) is a matter of taste; the equality of the two follows from Proposition 36. For the second part, we have  $\ell \circ \mathcal{H} \subset \text{spr } \ell$ . Thus  $\text{co}(\ell \circ \mathcal{H}(X)) \subset \text{co spr } \ell = \text{spr } \ell$ , since  $\text{spr } \ell$  is convex. Consequently,

$$\mathcal{F}(X) = \text{co}(-\pi \cdot \ell \circ \mathcal{H}(X)) = -\pi \cdot \text{co}(\ell \circ \mathcal{H}(X)) \subset -\pi \cdot \text{spr } \ell = D,$$

and thus  $\mathcal{F} \subseteq \mathcal{L}_0(X, D)$ . □

**Remark 38.** Observe that convexity of  $\mathcal{H}$  does *not* imply convexity of  $\ell \circ \mathcal{H}(X)$ , where  $\mathcal{H}(X) = \{h(x) \mid h \in \mathcal{H}, x \in X\}$ , but since  $\text{spr } \ell$  is convex, we do have that for  $h_0, h_1 \in \mathcal{H}$  and  $h_\alpha = (1 - \alpha)h_0 + \alpha h_1$  that  $\ell \circ h_\alpha \in \text{spr } \ell$  for all  $\alpha \in [0, 1]$ , and thus  $I_{-\pi \cdot \ell \circ \text{co } \mathcal{H}}(E) = I_{-\pi \cdot \ell \circ \mathcal{H}}(E)$ , and so convexity of  $\mathcal{H}$  does not “hurt.”

### 5.3 $\mathcal{F}$ -Information Processing Equalities

The definition of  $\mathcal{F}$ -information (29) implies an immediate, similar result to Proposition 31.

**Proposition 39.** *Suppose  $E : [n] \rightsquigarrow \Omega$ ,  $R : [n] \rightsquigarrow [n]$ , and  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, \mathbb{R}^n)$ . Then*

$$I_{\mathcal{F}}(RE) = I_{R^\top \mathcal{F}}(E),$$

where  $R^\top \mathcal{F} \stackrel{\text{def}}{=} \{x \mapsto R^\top(f_1(x), \dots, f_n(x))^\top \mid f \in \mathcal{F}\}$ . Moreover, if  $D \subseteq \mathbb{R}^n$  and  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, D)$ , then  $R^\top \mathcal{F} \subseteq \mathcal{L}_0(\Omega, R^\top D)$ .

The additional generality of  $\mathcal{F}$ -information yields another kind of information processing equality, one which is more aligned with the traditional formulation of information processing inequalities. Rather than the “processing” being on the labels (the  $Y$  in usual terminology) as in Proposition 39:

$$[n] \xrightarrow{R} [n] \xrightarrow{E} \Omega,$$

it is applied to the output (the  $X$  of the experiment):

$$[n] \xrightarrow{E} \Omega \xrightarrow{S} \Omega.$$

For  $f \in \mathcal{F}$ , let  $Sf \stackrel{\text{def}}{=} (Sf_1, \dots, Sf_n)$ , with  $f = (f_1, \dots, f_n)$  (i.e.  $f_i$  are the partial functions of  $f$ ), and let  $S\mathcal{F} \stackrel{\text{def}}{=} \{Sf \mid f \in \mathcal{F}\}$  (that is, the application of  $S$  is component-wise and element-wise).

**Theorem 40.** *Suppose  $E : [n] \rightsquigarrow \Omega$ ,  $S : \Omega \rightsquigarrow \Omega$ , and  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, \mathbb{R}_{\geq 0}^n)$ . Then*

$$I_{\mathcal{F}}(ES) = I_{S\mathcal{F}}(E).$$

*Proof.* From the linearity of the integral, for each  $i \in [n]$ , and all  $f \in \mathcal{F}$

$$\begin{aligned} \int f_i(x)ES(i, dx) &= \int f_i(x) \int E(i, dx')S(x', dx) \\ &= \iint f_i(x)S(x', dx)E(i, dx') \\ &\stackrel{(5)}{=} \int (Sf_i)(x')E(i, dx'). \end{aligned} \tag{35}$$

In the final equality we apply Tonelli's theorem (Fubini's theorem for sign-definite integrands) to exchange the order of integration. We can do so since by assumption, all  $f \in \mathcal{F}$  are non-positive, and for all  $x'$  and  $i$  the measures  $S(x', \cdot)$  and  $E(i, \cdot)$  are probability measures and thus  $\sigma$ -finite. Thus

$$\begin{aligned} I_{\mathcal{F}}(ES) &= \sup_{f \in \mathcal{F}} \sum_{i \in [n]} \int f_i(x)(ES)_i(dx) \\ &\stackrel{(35)}{=} \sup_{f \in S\mathcal{F}} \sum_{i \in [n]} \int f_i dE_i. \\ &= I_{S\mathcal{F}}(E). \end{aligned} \quad \square$$

**Corollary 41.** *Suppose  $E : [n] \rightsquigarrow \Omega$ ,  $R : [n] \rightsquigarrow [n]$ , and  $S : \Omega \rightsquigarrow \Omega$ , and  $\mathcal{F}$  satisfies the conditions of Theorem 40. Then*

$$I_{\mathcal{F}}(RES) = I_{R^T S \mathcal{F}}(E) = I_{S R^T \mathcal{F}}(E). \tag{36}$$

*Proof.* Equation (36) is obtained by applying Proposition 39 and Theorem 40. The commutation of  $R$  and  $S$  is verified by applying Proposition 39 and Theorem 40 in alternating orders.  $\square$

**Remark 42.** More generally, the condition in Theorem 40 that  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, \mathbb{R}_{\geq 0})$  can be relaxed whenever there is  $(ES)f = E(Sf)$ . For example:

1.  $\mathcal{F} \subseteq \mathcal{L}_0(\Omega, \mathbb{R}_{\leq 0}^n)$ , using Tonelli's theorem with  $-f$ .
2.  $(\Omega, \lambda)$  is a sigma-finite measure space,  $S(x, \cdot) \ll \lambda$  is a Markov kernel, and  $\mathcal{F} \subseteq \mathcal{L}_1(\Omega, \mathbb{R}^n)$ . Then there is a measurable  $k \in \mathcal{L}_0(\Omega_1 \times \Omega_2, \mathbb{R}_{\geq 0})$  with  $k(x, y)\lambda(dy) = S(x, dy)$ , and

$$\forall \mu \in \mathfrak{P}(\Omega) : \iint |f(x)k(x, y)|\mu(dx)\lambda(dy) < \infty,$$

and using Lemma 43 below we can apply Fubini's theorem.

3.  $\mathcal{F} \subseteq C_b(\Omega, \mathbb{R}^n)$  (bounded continuous functions), then we have the dual pair  $\langle C_b(\Omega), \mathfrak{M}(\Omega) \rangle$  ( $\mathfrak{M}(\Omega)$  is the space of finitely additive signed measures on  $\Omega$ ) for which the interchange (35) is equivalent to the existence of an adjoint of the linear operator  $\mu \mapsto \mu S$ .

**Lemma 43.** *Suppose  $(\Omega_2, \lambda)$  is a measure space,  $S : \Omega_1 \rightsquigarrow \Omega_2$  is a Markov kernel, there is a measurable  $k \in \mathcal{L}_0(\Omega_1 \times \Omega_2, \mathbb{R}_{\geq 0})$  with  $k(x, y)\lambda(dy) = S(x, dy)$ . Then for  $f \in \mathcal{L}_1(\Omega_2, \mathbb{R})$*

$$\forall \mu \in \mathfrak{P}(\Omega_2) : \iint |f(y)k(x, y)|\lambda(dy)\mu(dx) < \infty.$$

*Proof.* Fix  $\mu \in \mathfrak{P}(\Omega_2)$ . Then

$$\begin{aligned} & \iint |f(y)k(x, y)|\mu(dx)\lambda(dy) \\ & \leq \underbrace{\left( \iint |f(y)|\mu(dx)\lambda(dy) \right)}_{\|f\|} \underbrace{\left( \iint k(x, y)\mu(dx)\lambda(dy) \right)}_1. \end{aligned}$$

The term in the second underbrace is 1 because, observing the integrand is nonnegative and  $S$  is a Markov kernel, we can apply Tonelli's theorem to obtain  $\int k(x, y)\mu(dx)\lambda(dy) = \int \mu(dx) = 1$ . By hypothesis  $\|f\| < \infty$ , which completes the proof.  $\square$

#### 5.4 The Information Processing Equality in terms of Constrained Bayes Risk

Let  $E : [n] \rightsquigarrow \Omega$  be an experiment and  $S : \Omega \rightsquigarrow \Omega$  be Markov kernel. We can thus form the product experiment  $ES$  as per the diagram  $[n] \xrightarrow{E} \Omega \xrightarrow{S} \Omega$ . This corresponds to ‘‘attribute noise’’ — that is noise in the observations of  $X$ . For example, instead of learning from  $(X, Y)$  one only gets to observe  $(X + N, Y)$  for some independent noise random variable  $N$ . More general (non-additive) corruptions are possible, but this additive one will be of particular interest.

We can express the information processing equality in terms of Bayes risks:

**Corollary 44.** *Suppose  $\ell : \Delta^n \rightarrow \mathbb{R}_{\geq 0}^n$  is a continuous proper loss,  $\pi \in \mathfrak{P}([n])$  a prior distribution on  $[n]$ , and  $\mathcal{H} \subseteq \mathcal{L}_0(X, \mathfrak{P}([n]))$  an hypothesis class. Then*

$$\text{Bayes}_{\ell \circ \mathcal{H}}(\pi \times ES) = \text{Bayes}_{S(\ell \circ \mathcal{H})}(\pi \times E).$$

*Proof.* Let  $\mathcal{F} = \text{co}(-\pi \cdot \ell \circ \mathcal{H})$ . By combining Theorem 37 with Theorem 40 we have

$$\text{Bayes}_{\ell \circ \mathcal{H}}(\pi \times ES) = -I_{\mathcal{F}}(ES) = -I_{S\mathcal{F}}(E) = \text{Bayes}_{S(\ell \circ \mathcal{H})}(\pi \times E).$$

The last equality is justified as follows. For any set  $A \subset \mathbb{R}^n$  and  $S : [n] \rightsquigarrow [n]$ , we have  $S \text{ co } A = \text{co}(SA)$  since

$$\begin{aligned} S \text{ co } A &= S \{ \sum_i \alpha_i a_i \mid a_i \in A, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \\ &= \{ \sum_i \alpha_i S a_i \mid a_i \in A, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \\ &= \{ \sum_i \alpha_i b_i \mid b_i \in SA, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \\ &= \text{co}(SA). \end{aligned}$$

Furthermore, for any  $v \in \mathbb{R}^n$ , any set  $C \subset \mathbb{R}^n$  and  $S : [n] \rightsquigarrow [n]$ , we have  $S(v \cdot C) = v \cdot SC$  since  $S(v \cdot C) = S\{v \cdot c \mid c \in C\} = \{(Sv) \cdot c \mid c \in C\} = \{v \cdot (Sc) \mid c \in C\} = \{v \cdot b \mid b \in SC\} = v \cdot SC$ . These two facts together imply  $S\mathcal{F} = S \text{ co}(-\pi \cdot \ell \circ \mathcal{H}) = \text{co}(S(-\pi \cdot \ell \circ \mathcal{H})) = \text{co}(-\pi \cdot S(\ell \circ \mathcal{H}))$ , and a second appeal to Theorem 37 concludes the proof.  $\square$

**Remark 45.** Kernel methods in machine learning (Schölkopf and Smola, 2001) are so named because of the kernel of the integral operator  $T_k: L_2 \rightarrow L_2$  given by

$$T_k f = \int k(\cdot, x) f(x) \rho(dx). \quad (37)$$

One can view the usual hypothesis class in kernel ML methods as the image of the unit ball under this operator (Williamson, Smola, and Schölkopf, 2001). But Markov kernels can also be written in a similar form. As Çinlar (2011, pages 38 and 46) observes, we can express a Markov kernel  $K: Y \rightsquigarrow X$  as

$$K(y, dx) = \rho(dx) k(y, x),$$

where  $k$  is known as a kernel density relative to the reference measure  $\rho$  and the operation of  $K$  on a function  $f: X \rightarrow \mathbb{R}$  can be written as

$$Kf(\cdot) = \int K(\cdot, dx) f(x) = \int k(\cdot, x) f(x) \rho(dx). \quad (38)$$

Comparing (37) and (38) we see that the Markov kernel performs a similar smoothing operation to  $T_k$ . When one takes account of Theorem 40, one concludes that the choice of a kernel in a kernel learning machine is in effect an hypothesis about the type of noise the observations will be affected by. For example, using a Gaussian translation invariant kernel is an inductive bias which implicitly assumes the  $X$  measurements are corrupted by additive Gaussian noise. (This last statement is perhaps misleading; we stress that it is  $\ell \circ \mathcal{H}$  which is smoothed by the kernel  $K$ , not  $\mathcal{H}$  itself. Understanding the effect of  $K$  directly on  $\mathcal{H}$  seems challenging.)

## 6. Conclusion

*There is no in-formation, only trans-formation.*  
— Bruno Latour<sup>16</sup>

Motivated by the epigram at the beginning of the paper, we have used Grothendieck’s “relative method”, whereby one understands an object, not by studying the object itself, but by studying its morphisms. We have seen that by construing information processing as a transformation on the *type* of information, rather than as a manipulation of the *amount* of some fixed type of information, one obtains new insights into the nature of information.

In doing so, we formulated a substantial generalisation of information, which subsumes existing measures, including  $\varphi$ -divergences and MMD. The  $D$  and  $\mathcal{F}$  informations also induce corresponding notions of entropy (see Appendix C). Their naturalness is manifest by the general bridge to Bayes risks and constrained Bayes risks. By working with the variational form in which we define them, we can readily determine the effect of noisy observations. We have shown that for both label noise and attribute noise, the effect of the noisy observations can be captured by a change of the measure of information used. This leads to information processing *equalities* instead of the traditional *inequalities* (which themselves are one of the basic results in information theory, underpinning the notion of statistical sufficiency).

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16. See (Latour, 2007, page 149) and (Lovink, 2004).

The new measures of information provide insight into the variational representation of  $\varphi$ -divergences, as well as a new interpretation of the choice of kernel in SVMs and MMD.

The bridge results offer a way to avoid duplicate analytical work: for example, one does not need to separately analyse the estimation properties of statistical divergences (Sreekumar and Goldfeld, 2022); one can simply convert to the equivalent statistical decision problem for which many results already exist. In light of the bridge result, one should hardly be surprised that the constrained variational representation of  $\varphi$ -divergence has generalization performance controlled by the Rademacher complexity of the discriminator set (Zhang et al., 2017).

The information processing equality for  $\mathcal{F}$ -information generalises an insight developed by Bishop (1995) that the addition of noise (in training) is equivalent to a form of regularization<sup>17</sup>. It goes beyond Bishop’s result in that it applies to any “noise” (not necessarily additive) and explains the effect of “adding” noise precisely in terms of the effect on the hypothesis class.

The information processing results in the paper differ from the classical ones in that they change the measure of information used. This is metaphorically changing the “ruler” used to measure information on either side of the noisy channel. The bridge between information and expected loss shows that there is no reason to expect there is a single canonical measure of information (as soon as one accepts there is no single canonical loss function).

Taken as a whole, the results show that at least for questions relating to prediction and learning, it makes no sense to talk of “the” information in one’s data. While it is widely accepted that different problems demand different loss functions, it is also often assumed that Shannon information is *the* only measure of “information” available<sup>18</sup>. For example Rauh et al. (2017) make much of the fact that although the worst Bayes risk (over all losses) of an experiment may be made worse after passing through a channel, particular measures of information may not be degraded at all. Given the bridge between risks and measures of information, this can be seen as simply a mistake about quantification; the Blackwell-Sherman-Stein (BSS) theorem (recall Remark 17) to which they appeal, is stated in terms of either all loss functions or all measures of information. Similarly, in much recent work in ML, the choice of a particular measure of information is taken to be essentially one of convenience, and not related to the underlying problem to be solved (in the way that one’s choice of loss function ideally is). The results of the paper show that choosing one’s measure of information is literally equivalent to choosing one’s loss function in a statistical decision problem, and thus is significant, consequential, and not a mere matter of convenience or convention.

Finally, we note that the work in the paper could be criticised in a few ways.

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17. Bishop’s result is not quite the whole story, as explained by An (1996). But the general conclusion is correct: adding noise to the input data encourages the learned model to be smoother than it would have been otherwise; confer (Grandvalet, Canu, and Boucheron, 1997)

18. This is a point well acknowledged by information theorists:

[T]he fact that entropy has been proved in a meaningful sense to be the unique correct information measure for the purposes of communication does not prove that it is either unique or a correct measure to use in some other field in which no issue of encoding or other changes in representation arises (Elias, 1983, page 500).

- The present paper is restricted to multiclass classification, where the label space is finite. One would ideally like to have an analogous theory for the uncountable case, which would allow one to apply the theory to problems such as regression. Some efforts along these lines have been made in (Cranko, 2021) (which however essentially only extends the related results of (Williamson and Cranko, 2023), and not the information processing theorems central to the present paper); and there is extant work on the geometry of scoring rules that does not restrict the outcome space as we have done (Dawid, 2007; Grünwald and Dawid, 2004); thus the hoped-for generalization would appear plausible.
- The results could only be directly “applied” if one knew the noise kernel that was corrupting the data. Indeed that is so, just like Shannon’s channel capacity theorem can only be applied if one knows the channel (and can actually compute its capacity). This is no different to most results in science: physics can compute the effect of various physical processes in terms of the parameters that describe them; it does not offer an automatic way of gleaning those parameters.
- The theory presumes knowledge of population distributions. In practice one usually only has access to samples, and thus an empirical distribution. So, it may be asked, what can you do with your theory then? There are, of course, results in the literature about such questions (for example (Goldfeld et al., 2020); (Polyanskiy and Wu, 2024, section 7.9)). But here the bridge result offers a neat resolution: the challenges of grappling with finite samples on the information side of the bridge are exactly those of grappling with the problem on the risk side, which is to say where almost all the existing work is done. Thus any insights one might glean regarding how to deal with finite samples on the risk side can be easily transported across the bridge to the information side, rather than repeating essentially the same arguments (Sreekumar and Goldfeld, 2022).
- The label noise case makes it clear that the loss function gets transformed, and indeed this underpins the “loss-correction” approach to label noise (van Rooyen and Williamson, 2018). Can this be extended? Indeed it can — an initial attempt is available in (Iacovissi, Lu, and Williamson, 2023).
- The present work offers a geometric theory explaining certain transformations of supervised learning problems. The viewpoint is essentially that of norms — as we showed (Appendix B), measures of information are expected gauge functions. Can one extend this “norm-centred” theory to a metric theory? Yes! A first attempt is (Mémoli, Vose, and Williamson, 2024).
- Finally, a reader interested in building practical ML systems may well complain there are no new methods, algorithms, experiments nor league table results, and so thus this paper is of no use. We agree with the facts, but not the conclusion. Surely, we argue, it is crucial to know what one is trying to do before one tries to actually do it. (Claude Shannon’s landmark paper (Shannon, 1948) could be similarly criticised; its long-term impact is immense.) But we certainly accept that the present paper is different in flavour to many in this journal; we offer it (and its cousins (Mémoli, Vose,



Divergence name	$\varphi(x), x \geq 0$	$\varphi^*(x), x \in \mathbb{R}$
Variational	$\varphi_{\text{Var}}(x) =  x - 1 $	$-\llbracket x < -1 \rrbracket + \llbracket -1 \leq x \leq 1 \rrbracket x + \llbracket x > 1 \rrbracket \infty$
Kullback-Leibler	$\varphi_{\text{KL}}(x) = x \log(x) - x + 1$	$e^x - 1$
Squared Hellinger	$\varphi_{\text{Hell}}(x) = (\sqrt{x} - 1)^2$	$\llbracket x \leq 1 \rrbracket \frac{x}{1-x} + \llbracket x > 1 \rrbracket \infty$
Chi-squared	$\varphi_{\chi^2}(x) = (x - 1)^2$	$\frac{1}{4}x^2 + x$
Jensen-Shannon	$\varphi_{\text{JS}}(x) = x \log x - (x + 1) \log \frac{x+1}{2}$	$-\llbracket x \leq \log 2 \rrbracket \log(2 - e^x) + \llbracket x > \log 2 \rrbracket \infty$
Triangular	$\varphi_{\text{Tri}}(x) = \frac{(x-1)^3}{x+1}$	$-\llbracket x \leq -3 \rrbracket + \llbracket 1 < x \rrbracket \infty$ $+ \llbracket -3 \leq x \leq 1 \rrbracket (\sqrt{1-x} - 1)(\sqrt{1-x} - 3)$

Table 1: Common  $\varphi$ -divergences and their associated  $\varphi^*$ ; drawn from (Terjék, 2021) which has further examples. In all cases,  $\varphi^*(0) = 0$ , and thus  $\text{hyp}(-\varphi^*) \in \mathfrak{D}^2$ . We adopt the convention that  $\llbracket \text{false} \rrbracket \cdot \infty = 0$ .

and Williamson, 2024; Williamson and Cranko, 2023)) as exemplars of “slow science” (Wikipedia, 2024) — a complement to the frantic pace of much existing ML research.

### Acknowledgements

RW’s contribution was funded in part by the Deutsche Forschungsgemeinschaft under Germany’s Excellence Strategy — EXC number 2064/1 — Project number 390727645. This work was presented (as “Data Processing Equalities”) at the Tokyo workshop on Deep Learning: Theory, Algorithms, and Applications (March 2018) (Sugiyama Lab, 2018), and at the Information Theory in Machine Learning workshop, NeurIPS December 2019 (ITML, 2019). A special case of the argument in Example 5.4 was developed by RW in April 2007 [sic] after a discussion with Arthur Gretton. An earlier version of the proof of Proposition 16 was developed by Etienne de Montbrun. Thanks to Zak Mhammedi, Aneesh Barthakur and Wenkai Xu for comments and corrections; to Kamalaruban Parmeswaran, Brendan van Rooyen, Nan Lu and Laura Iacovissi for discussions and questions; and to a referee for asking some good questions.

## Appendix A. The $\varphi$ -divergence and its Variational Representation

In this appendix we present some facts concerning the classical  $\varphi$ -divergences and its variational representation and their relationship to our  $D$ - and  $\mathcal{F}$ -informations.

### A.1 Some examples of $D_\varphi$

When  $n = 2$ , we can compute some examples for classical  $\varphi$  divergences; see Table 1. Figure 2 illustrates  $D_\varphi$  and  $C_\varphi = (D_\varphi)^\circ$  for three different  $\varphi$  (for constructing such figures, it is helpful to use  $(-\text{hyp } \varphi^*)^\circ = \text{lev}_{\leq 1} \check{\varphi}$ ).

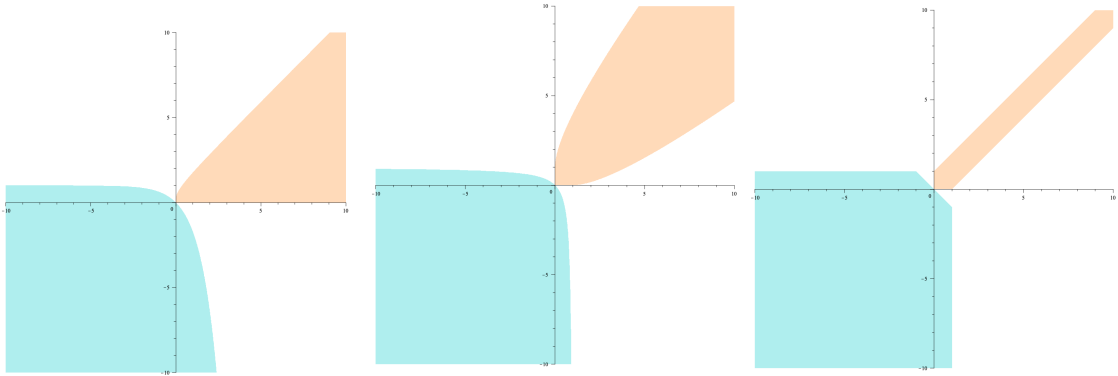


Figure 2:  $D_\varphi$  (Turquoise) and  $C_\varphi = (D_\varphi)^\circ$  (Peach) for  $\varphi_{\text{KL}}$ ,  $\varphi_{\text{Hell}}$  and  $\varphi_{\text{Var}}$  (left to right). Since in all cases  $D_\varphi$  and  $C_\varphi$  are unbounded sets, we have plotted their restriction to  $[-10, 10]^2$ . The purpose of including the polars  $(D_\varphi)^\circ$  is motivated in Appendix B .

## A.2 The variational representation of $\varphi$ -divergences

When  $n = 2$  ( $Y = [2]$ ) we can relate  $I_{\mathcal{F}}$  to the Csiszár divergence  $I_\varphi$ . For  $\varphi \in \Phi$  let

$$\mathcal{F}_\varphi \stackrel{\text{def}}{=} \mathcal{L}_0(X, \text{hyp}(-\varphi^*)).$$

Writing  $E = (E_1, E_2)$ , Proposition 4 and Theorem 37 immediately imply

$$I_\varphi(E_1, E_2) = I_{\mathcal{F}_\varphi}(E).$$

It is now instructive to relate  $I_{\mathcal{F}_\varphi}(E)$  to the variational representation of  $I_\varphi$  (Keziou, 2003)<sup>19</sup>, which is central to the concept of  $f$ -GANS (Nowozin, Cseke, and Tomioka, 2016).

**Proposition 46.** *Suppose  $\varphi \in \Phi$ ,  $Y = [2]$ , and  $E : Y \rightsquigarrow X$ . Then*

$$I_{\mathcal{F}_\varphi}(E) = \sup_{g \in \mathcal{L}_0(X, \mathbb{R})} (\mathbb{E}_{E_1} g - \mathbb{E}_{E_2} \varphi^* \circ g). \quad (39)$$

*Proof.* Using (30) we have

$$\begin{aligned} I_{\mathcal{F}_\varphi}(E) &= \sup_{f=(f_1, f_2) \in \mathcal{F}_\varphi} \mathbb{E}_\rho \left\langle \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right), (f_1, f_2) \right\rangle \\ &= \sup_{\{(f_1, f_2) \mid f_2 \leq -\varphi^* \circ f_1\}} \mathbb{E}_\rho \left\langle \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right), (f_1, f_2) \right\rangle \\ &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} \mathbb{E}_\rho \left\langle \left( \frac{dE_1}{d\rho}, \frac{dE_2}{d\rho} \right), (f_1, -\varphi^* \circ f_1) \right\rangle, \end{aligned}$$

19. Such representations have attracted some attention recently; for example (Agrawal and Horel, 2021, section 4.3) and (Birrell, Katsoulakis, and Pantazis, 2022; Ruderman et al., 2012; Terjék, 2021). A focus of these works is to develop restrictions on the class of functions one optimises over in order to aid their statistical estimation; the  $\mathcal{F}$ -information of the present paper can be seen to embrace a similar philosophy.

since  $\frac{dE_2}{d\rho} > 0$  and we can restrict the optimization to be such that  $f_2(x) = -\varphi^*(f_1(x))$  for all  $x \in X$ , and we exploited the fact that  $\mathcal{F}_\varphi$  is the set of *all* measurable functions mapping into  $\text{hyp}(-\varphi^*)$ ,

$$\begin{aligned}
 &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} \mathbb{E}_\rho \left( \frac{dE_1}{d\rho} f_1 - \frac{dE_2}{d\rho} \varphi^* \circ f_1 \right) \\
 &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} \int_X \left( f_1 \frac{dE_1}{d\rho} - \varphi^* \circ f_1 \frac{dE_2}{d\rho} \right) d\rho \\
 &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} \left( \int_X f_1 \frac{dE_1}{d\rho} d\rho - \int_X \varphi^*(f_1) \frac{dE_2}{d\rho} d\rho \right) \\
 &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} \left( \int_X f_1 dE_1 - \int_X \varphi^*(f_1) dE_2 \right) \\
 &= \sup_{f_1 \in \mathcal{L}_0(X, \mathbb{R})} (\mathbb{E}_{E_1} f_1 - \mathbb{E}_{E_2} \varphi^* \circ f_1). \quad \square
 \end{aligned}$$

It is apparent from the proof of the above that the asymmetry in the usual variational representation (39), whereby  $\varphi^*$  appears in only one of the terms, arises from the choice of  $E_2$  as the dominating measure and the parametrisation of  $D \in \mathfrak{D}^2$  by  $\varphi \in \Phi$ . Such a choice is problematic if  $E_2$  does not dominate  $E_1$ , leading to less elegant general definitions being necessary for  $I_\varphi$  (Liese and Vajda, 2006, 2008). The one advantage of (39) over (6) when  $Y = [2]$  is that the optimisation is over  $\mathbb{R}$ -valued functions rather than  $\mathbb{R}^2$ -valued functions. However, as seen in Section 3, the symmetric representation (6) has significant advantages in understanding the effect of the product of experiments (in the form of observation channels).

When  $\varphi = \varphi_{\text{var}} \stackrel{\text{def}}{=} t \mapsto |t - 1|$ ,  $I_{\varphi_{\text{var}}}$  is known as the *Variational divergence* which is examined in detail in §A.3. Finally, the form of (39) suggests the variant

$$I_{\mathcal{H}}(E) = \sup_{h \in \mathcal{H}} (\mathbb{E}_{E_1} h - \mathbb{E}_{E_2} \varphi^* \circ h),$$

where  $\mathcal{H} \subsetneq \mathcal{L}_0(X, \mathbb{R})$ . The functional  $I_{\mathcal{H}}$  is what is estimated in practice by virtue of choice of a suitable class over which to empirically optimise (39), often replacing  $E_i$  by their empirical approximations  $\hat{E}_i^m$ , where for  $A \in \Sigma_X$ ,  $\hat{E}_i^m(A) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{j \in [m]} \mathbb{1}[A \ni x_j]$ .

An alternate way of expressing the general form of  $I_{\mathcal{F}}(E)$  that is similar to the classical variational representation of a binary  $\varphi$ -divergence is given below. Let  $\mathcal{H} \subseteq \mathcal{L}_0(X, \mathfrak{P}([n]))$ . For  $D \in \mathfrak{D}^n$ , assume there is a measurable selection  $\nabla \sigma_D \in \partial \sigma_D$  (confer Proposition 25). We can thus write

$$I_{\nabla \sigma_D \circ \mathcal{H}}(E) = \sup_{h \in \mathcal{H}} \mathbb{E}_\rho \left\langle \frac{dE}{d\rho}, \nabla \sigma_D \circ h \right\rangle. \quad (40)$$

Observe that  $\nabla \sigma_D \circ \mathcal{H} = \{\nabla \sigma_D \circ h \mid h \in \mathcal{H}\} \subseteq \mathcal{L}_0(X, D)$ . This is a way to use classes of functions mapping to  $\mathbb{R}^n$  in an elegant manner to define a restricted version of  $I_D$ . Observe that (40) is symmetric in the appearance of  $\nabla \sigma_D$ , in a manner that (39) is not, but one needs to work with vector valued functions  $h : X \rightarrow \mathbb{R}^n$ . Given a function class  $\mathcal{R} \subseteq \mathcal{L}_0(X, \mathbb{R})$ , one could induce  $\mathcal{H}_{\mathcal{R}} \stackrel{\text{def}}{=} \{X \ni x \mapsto (r_1(x), \dots, r_n(x)) \mid r_i \in \mathcal{R} \forall i \in [n]\}$ , allowing us to define  $I_{\mathcal{R}, D}(E) \stackrel{\text{def}}{=} I_{\nabla \sigma_D \circ \mathcal{H}_{\mathcal{R}}}(E)$ .

### A.3 The Variational Divergence

The binary Variational divergence has  $\varphi_{\text{Var}}(u) = |u - 1|$  for  $u \in \mathbb{R}_{\geq 0}$ .

**Lemma 47.** *The Legendre-Fenchel conjugate of  $\varphi_{\text{Var}}$  is given by*

$$\varphi_{\text{Var}}^*(s) = \begin{cases} s & s \in [-1, +1] \\ +\infty & s \notin [-1, +1]. \end{cases}$$

*Proof.* We have

$$\begin{aligned} \varphi_{\text{Var}}^*(s) &= \sup_{u \geq 0} u \cdot s - \varphi_{\text{Var}}(u) \\ &= \sup_{u \geq 0} u \cdot s - \begin{cases} u - 1 & u \geq 1 \\ 1 - u & u \leq 1 \end{cases} \\ &= \sup_{u \geq 0} \begin{cases} u(s - 1) + 1 & u \geq 1 \\ u(s + 1) - 1 & u \leq 1. \end{cases} \end{aligned}$$

Suppose  $s > 1$ . Then the supremum is attained for  $u \geq 1$  and  $\varphi_{\text{Var}}^*(s) = \sup_{u \geq 0} uc + 1$  for  $c > 0$  which equals  $+\infty$ . Similarly if  $s < -1$ , the supremum is attained for  $u \leq 1$  and again  $\varphi_{\text{Var}}^*(s) = +\infty$ . Suppose  $s \in [-1, 1]$ ; we have

$$\begin{aligned} \varphi_{\text{Var}}^*(s) &= \max \left\{ \sup_{u \geq 1} u(s - 1) + 1, \sup_{u \in [0, 1]} u(s + 1) - 1 \right\} \\ &= \max \{ 1(s - 1) + 1, 1(s + 1) - 1 \} \\ &= s \quad s \in [-1, 1], \end{aligned}$$

which completes the proof.  $\square$

Now consider the evaluation of

$$I_{\mathcal{F}_{\varphi_{\text{Var}}}}(E) = \sup_{g: X \rightarrow \mathbb{R}} \mathbb{E}_{E_1} g - \mathbb{E}_{E_2} \varphi_{\text{Var}}^* \circ g.$$

If, for any  $x \in X$ ,  $g(x) \notin [-1, +1]$ , then the second term will be infinite which will push the whole value to  $-\infty$ . Thus the sup can never be attained if  $g$  takes on values outside of  $[-1, +1]$  (except on a  $E_2$ -negligible set). Hence we need only consider

$$I_{\mathcal{F}_{\varphi_{\text{Var}}}}(E) = \sup_{g: X \rightarrow [-1, 1]} \mathbb{E}_{E_1} g - \mathbb{E}_{E_2} g$$

Since the objective is linear, and the constraint set convex, the supremum is attained at the boundary and hence

$$\begin{aligned} I_{\mathcal{F}_{\varphi_{\text{Var}}}}(E) &= \sup_{g: X \rightarrow \{-1, 1\}} \mathbb{E}_{E_1} g - \mathbb{E}_{E_2} \varphi_{\text{Var}}^* \circ g \\ &= 2 \sup_{g: X \rightarrow \{0, 1\}} \mathbb{E}_{E_1} g - \mathbb{E}_{E_2} \varphi_{\text{Var}}^* \circ g \end{aligned} \quad (41)$$

$$\begin{aligned}
 &= 2 \sup_{A \in \Sigma_X} E_1(A) - E_2(A) \\
 &= 2 \sup_{A \in \Sigma_X} |E_1(A) - E_2(A)|,
 \end{aligned} \tag{42}$$

where the last step is shown in (Strasser, 1985). Observe that (41) can also be written as

$$2 \sup_{g: X \rightarrow [0,1]} E_{E_1} g - E_{E_2} \varphi_{\text{Var}}^* \circ g.$$

We now determine  $D_{\text{var}} \stackrel{\text{def}}{=} \text{hyp}(-\varphi_{\text{Var}}^*)$ .

**Lemma 48.** *Let  $H_{n,r}^- \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle n, x \rangle - r \leq 0\}$  denote the negative halfspace with normal vector  $n$  and offset  $r$ . The set  $D_{\text{var}}$  can be written*

$$D_{\text{var}} = H_{1_2,0}^- \cap H_{e_1,1}^- \cap H_{e_2,1}^-. \tag{43}$$

*Proof.* We have

$$\begin{aligned}
 \text{hyp}(-\varphi_{\text{Var}}^*) &= \{(x, y) \in \mathbb{R}^2 \mid y \leq -\varphi_{\text{Var}}^*(x)\} \\
 &= \left\{ (x, y) \in \mathbb{R}^2 \mid y \leq \begin{cases} -x & x \in [-1, +1] \\ -\infty & x \notin [-1, +1] \end{cases} \right\} \\
 &= \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, +1], y \leq -x\}.
 \end{aligned}$$

Now Lemma 13 implies  $I_D(E) = I_{\text{co}D}(E)$ , and hence we can take the convex hull of the above to obtain

$$\begin{aligned}
 D_{\text{var}} &= \{(x, y) \in \mathbb{R}^2 \mid y \leq 1\} \cap \{(x, y) \in \mathbb{R}^2 \mid y \leq -x\} \cap \{(x, y) \in \mathbb{R}^2 \mid x \leq 1\} \\
 &= \{(x, y) \in \mathbb{R}^2 \mid y \leq 1 \text{ and } x \leq 1 \text{ and } y \leq -x\}.
 \end{aligned}$$

We can thus more compactly write  $D_{\text{var}}$  as in (43).  $\square$

Lemma 48 suggests the following generalisation which we now take as a definition

$$D_{\text{var}}^{(n)} \stackrel{\text{def}}{=} H_{1_n,0}^- \cap \bigcap_{i \in [n]} H_{e_i,1}^-.$$

Observe that  $D_{\text{var}}^{(n)}$  is the maximal (by set inclusion) element of  $\mathcal{D}(\mathbb{R}^n, \mathbb{R}_{\leq 0}^n)$  satisfying the normalisation condition  $\bigvee_{i \in [n]} \sigma_D(e_i) = 1$ . We now compute  $\sigma_{D_{\text{var}}^{(n)}}$ .

**Lemma 49.** *The support function of  $D_{\text{var}}^{(n)}$  is given by*

$$\sigma_{D_{\text{var}}^{(n)}}(x) = \sum_{i \in [n]} x_i - n \bigwedge_{j \in [n]} x_j.$$

*Proof.* Note that  $D_{\text{var}}^{(n)}$  is a intersection of half spaces and thus its support function is the same as the support function of its extreme points, which is the union of the  $n$  vertices created. Denote the vertices  $v_j$  for  $j \in [n]$ . We have

$$v_j = H_{1_n,0}^- \cap \bigcap_{i \in [n] \setminus \{j\}} H_{e_i,1}^-$$

$$\begin{aligned}
 &= \{x \in \mathbb{R}^n \mid \langle x, e_i \rangle - 1 = 0 \ \forall i \neq j \text{ and } \langle x, 1_n \rangle = 0\} \\
 &= \left\{x \in \mathbb{R}^n \mid x_i = i \ \forall i \neq j \text{ and } \sum_{k \in [n]} x_k = 0\right\} \\
 &= \{x \in \mathbb{R}^n \mid x_i = 1 \ \forall i \neq j \text{ and } x_j = -(n-1)\} \\
 &= 1_n - ne_j.
 \end{aligned} \tag{44}$$

Thus for  $x \in \mathbb{R}_{\geq 0}^n$ , using (Hiriart-Urruty and Lemaréchal, 2001, Theorem C.3.3.2 (ii)) we have  $\sigma_{\overline{\text{co}}(\cup_{j \in [n]} \{v_j\})} = \sup_{j \in [n]} \sigma_{\{v_j\}}$  and hence

$$\begin{aligned}
 \sigma_{D_{\text{var}}^{(n)}}(x) &= \sup_{j \in [n]} \langle 1_n - ne_j, x \rangle \\
 &= \langle 1_n, x \rangle + n \sup_{j \in [n]} -x_j \\
 &= \sum_{i \in [n]} x_i - n \bigwedge_{j \in [n]} x_j.
 \end{aligned} \quad \square$$

We can now determine an explicit expression for  $I_{D_{\text{var}}^{(n)}}(E)$ . Let  $(\bar{X}_1, \dots, \bar{X}_n)$  be a measurable partition of  $X$  (i.e.  $\bar{X}_i$  are measurable for  $i \in [n]$ ) defined via

$$\bar{X}_i \stackrel{\text{def}}{=} \left\{x \in X \mid \min \left( \text{Argmin}_{j \in [n]} \frac{dE_j}{d\rho}(x) = i \right)\right\}.$$

(The additional min is to break ties.) It is immediate that this is indeed a partition of  $X$ , i.e.  $\bigcup_{k \in [n]} \bar{X}_k = X$  and  $\bar{X}_i \cap \bar{X}_j = \emptyset$  for  $i \neq j$ . Consequently

$$\begin{aligned}
 I_{D_{\text{var}}^{(n)}}(E) &= \int_X \sum_{i \in [n]} \frac{dE_i}{d\rho}(x) \rho(dx) - n \int_X \bigwedge_{j \in [n]} \frac{dE_j}{d\rho}(x) \rho(dx) \\
 &= \sum_{i \in [n]} \int_X dE_i - n \int_X \bigwedge_{j \in [n]} \frac{dE_j}{d\rho}(x) \rho(dx) \\
 &= n \left[ 1 - \sum_{k \in [n]} \int_{\bar{X}_k} \frac{dE_k}{d\rho}(x) \rho(dx) \right] \\
 &= n \left[ 1 - \sum_{k \in [n]} E_k(\bar{X}_k) \right],
 \end{aligned} \tag{45}$$

using the properties of the partition  $(\bar{X}_1, \dots, \bar{X}_n)$ .

Observe that choosing any other partition of  $X$  would result in a larger value of the second integral in (45) and thus a smaller value for the overall expression. Thus if  $\mathcal{P}_n(X)$  denotes the set of all measurable  $n$ -partitions of  $X$ , we can write

$$I_{D_{\text{var}}^{(n)}}(E) = \sup_{(X_1, \dots, X_n) \in \mathcal{P}_n(X)} n \left[ 1 - \sum_{k \in [n]} E_k(X_k) \right].$$

When  $Y = [2]$ , we obtain

$$\begin{aligned}
 I_{D_{\text{var}}^{(2)}}(E) &= \sup_{(X_1, X_2) \in \mathcal{P}_2(X)} 2[1 - (E_1(X_1) + E_2(X_2))] \\
 &= 2 \sup_{\substack{X_1 \subseteq X \\ X_1 \text{ measurable}}} [1 - (E_1(X_1) + E_2(X \setminus X_1))] \\
 &= 2 \sup_{\substack{X_1 \subseteq X \\ X_1 \text{ measurable}}} [1 - (E_1(X_1) + 1 - E_2(X_1))] \\
 &= 2 \sup_{\substack{X_1 \subseteq X \\ X_1 \text{ measurable}}} [E_1(X_1) - E_2(X_1)],
 \end{aligned}$$

which can be recognised as being equivalent to (42).

Finally we observe a special case of (27) for  $D = D_{\text{var}}^{(n)}$  when  $S$  takes the particular symmetric form  $S_\alpha$  where the  $j$ th column of  $S_\alpha^*$  is  $s_j^* = \alpha e_j + \frac{1-\alpha}{n} 1_n$ . When  $\alpha = 1$ , this is the identity matrix, and for  $\alpha \in [0, 1]$  it corresponds to the observation channel providing the correct label with probability  $\alpha$  and with probability  $1 - \alpha$  a label chosen at random from  $[n]$  is chosen (which could in fact be correct). The set  $S_\alpha^* D_{\text{var}}^{(n)}$  can be readily determined by exploiting the fact we need only determine its support function  $\sigma_{S_\alpha^* D_{\text{var}}^{(n)}}(x)$  for  $x \in \mathbb{R}_{\geq 0}^n$ . Thus we can exploit (44) and we need only compute (for  $j \in [n]$ )

$$\begin{aligned}
 S_\alpha^* v_j &= 1_n - n \left( \alpha e_j + \frac{1-\alpha}{n} 1_n \right) \\
 &= \alpha(1_n - n e_j) \\
 &= \alpha v_j.
 \end{aligned}$$

Thus  $\sigma_{S_\alpha^* D_{\text{var}}^{(n)}} = \alpha \sigma_D$  and so for any  $\alpha \in [0, 1]$  and any  $n$ , we have the homogeneous relationship

$$I_{D_{\text{var}}^{(n)}}(S_\alpha E) = \alpha I_{D_{\text{var}}^{(n)}}(E),$$

which we note has the *same* measure of information on either side of the equality (analogous to the typical strong data processing *inequalities* one finds in the literature).

## Appendix B. $D$ -Information as an Expected Gauge Function

Classical binary information “divergences” are sometimes supposed to be “like” a distance (a metric). In this appendix we show that there is an element of truth in this supposition. Metrics (as a formal notion of “distance”) are often (not always) induced by norms, and norms are particular examples of convex gauge functions (Minkowski functionals). In this appendix we show that it follows almost immediately from our definition of  $D$ -information that it is indeed an *expected* gauge function, albeit one where the associated “unit ball” of the gauge is neither symmetric nor compact. The restriction of  $D \in \mathfrak{D}$  allows an insightful representation of  $I_D$  making use of the classical polar duality of closed convex sets containing the origin.

The *conic hull* of a set  $C \subset \mathbb{R}^n$  is  $\text{pos } C \stackrel{\text{def}}{=} (0, \infty) \cdot C$ . Given  $C \in \mathcal{K}(\mathbb{R}^n)$ , the *polar* of  $C$  is defined by

$$C^\circ \stackrel{\text{def}}{=} \{x^* \mid \forall x \in C, \langle x, x^* \rangle \leq 1\}.$$

We will make use of the following from (Rockafellar, 1970, Theorem 14.6):

**Proposition 50.** *Suppose  $C, C^\circ \in \mathcal{K}(\mathbb{R}^n)$  are a polar pair both containing the origin. Then  $(\text{rec } C)^\circ = \text{cl pos } C^\circ$ .*

Given  $C \in \mathcal{K}(\mathbb{R}^n)$ , the *gauge* of  $C$  is defined by

$$\gamma_C(x) \stackrel{\text{def}}{=} \inf\{\mu \geq 0 \mid x \in \mu C\}.$$

Obviously given the gauge  $\gamma_C$  one can recover  $C$  via  $C = \text{lev}_{\leq 1} \gamma_C = \{x \mid \gamma_C(x) \leq 1\}$ . (If  $C$  is symmetric about the origin, then  $\gamma_C$  is a norm.) Let

$$\mathfrak{C}_0^n \stackrel{\text{def}}{=} \{C \in \mathcal{K}(\mathbb{R}^n) \mid \text{pos } C \subseteq \mathbb{R}_{\geq 0}^n, 0 \in \text{bd } C, \text{pos } 1_n \subseteq C\}.$$

**Lemma 51.** *If  $D \in \mathfrak{D}_0^n$  then  $D^\circ \in \mathfrak{C}_0^n$ .*

*Proof.* If  $D$  is convex then so is  $D^\circ$ . By Proposition 50, since  $0 \in D$ ,  $\text{rec } D$  is the largest cone contained in  $D$  and  $(\text{rec } D)^\circ = \text{cl pos } C$  is the smallest cone containing  $C$ . Thus when  $\text{rec } D = \{x \in \mathbb{R}^n \mid \langle x, 1_n \rangle \leq 1\}$ ,  $\text{cl pos } C = \{\alpha 1_n \mid \alpha \geq 0\}$ . Regardless of the choice of  $D$ , we always have  $0 \in \text{bd } D^\circ$ . The final condition in the definition of  $\mathfrak{C}_0^n$  follows since  $D \subseteq C \Rightarrow D^\circ \supseteq C^\circ$ , and  $(\text{lev}_{\leq 0} \langle \cdot, 1_n \rangle)^\circ = \text{pos } 1_n$ .  $\square$

Gauges and support functions are dual to each other in the polar sense (Hiriart-Urruty and Lemaréchal, 2001, Corollary C.3.2.5):

**Lemma 52.** *Suppose  $C \in \mathcal{K}(\mathbb{R}^n)$ , then  $\sigma_C = \gamma_{C^\circ}$ .*

**Proposition 53.** *For any  $D \in \mathfrak{D}^n$ ,  $D^\circ \in \mathfrak{C}^n$  and for any  $E: [n] \rightsquigarrow X$ , and any reference measure  $\rho$ ,*

$$I_D(E) = \int_X \gamma_{D^\circ} \left( \frac{dE}{d\rho} \right) d\rho. \quad (46)$$

*Proof.* The first claim is just lemma 51. The second claim follows by applying Lemma 52 pointwise.  $\square$

Expressing  $I_D$  as an average of a gauge function as in (46) justifies the oft made claim that divergence are “like” distances in some sense; the fact that  $D^\circ$  is not symmetric is why it is merely “like”. One can see that  $I_D$  is “gauging” the average degree to which the vector  $\frac{dE}{d\rho}(x)$  is “close” to one of the canonical basis vectors  $e_i$ ,  $i \in [n]$  since for  $D^\circ \in \mathfrak{C}_0^n$ ,  $\gamma_{D^\circ}(e_i) > 0$  in that case. Conversely, since  $\text{cl pos } D^\circ = \{\alpha 1_n \mid \alpha \geq 0\}$ , we always have  $\gamma_{D^\circ}(1_n) = \sigma_D(1_n) = 0$ , corresponding to situations where  $\frac{dE}{d\rho}(x) = 1_n$ , and consequently it being impossible to distinguish between the outcomes of the experiment at that  $x$  — in other words a complete absence of “information.” Some example of polars of  $D$  illustrated in Figure 3.



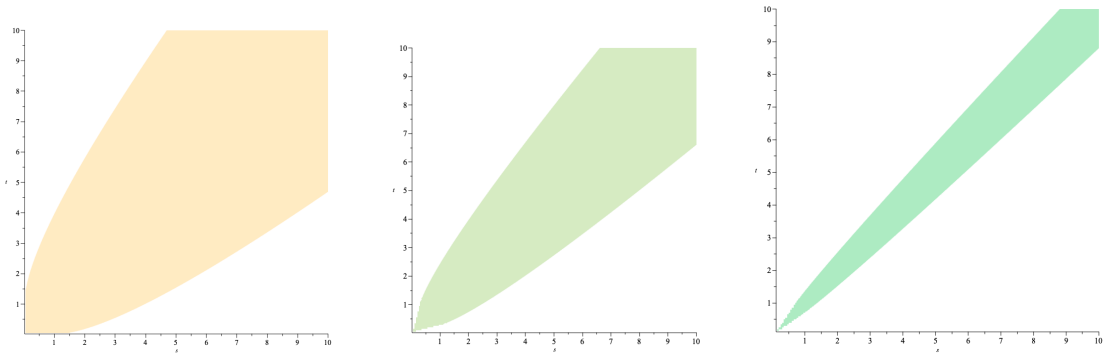


Figure 3: The corresponding polars for  $R_r D_{\text{Hell}}$  for  $r = 1, 0.8, 0.6$  (restricted to  $[0, 10]^2$ ) corresponding to the set-up as in Figure 1. Observe that for any  $E: [2] \rightsquigarrow X$ , as  $r \downarrow 0.5$ , the composition  $R_r E$  approaches the totally non-informative experiment  $E^{\text{tni}}$ , and  $I_{R_r^* D}$  approaches what we might (oxymoronically) call the totally noninformative information measure  $I^{\text{tni}} = I_{D^{\text{tni}}}$ , where  $D^{\text{tni}} = \{x \in \mathbb{R}^n \mid \langle x, 1_n \rangle \leq 0\}$  and  $C^{\text{tni}} = D^{\circ}_{\text{tni}} = \{\alpha 1_n \mid \alpha \geq 0\}$ . The name is justified since  $I^{\text{tni}}(E) = 0$  for all experiments  $E$ .

### Appendix C. Unconstrained and Constrained Entropies

Historically, the notion of the *entropy* of a single distribution (or random variable) preceded measures of information between two or more distributions (or random variables)<sup>20</sup>. There is a large literature on different notions of entropy, starting with (Shannon, 1948), with  $\varphi$ -entropies (analogous to  $\varphi$ -divergences) specifically considered in (Ben-Bassat, 1978; Csiszár, 1972; Daróczy, 1970). In this appendix we recall how the entropy of a distribution  $\mu \in \mathfrak{P}(X)$  can be defined on the basis of comparison against a “uniform” measure  $\nu \in \mathfrak{P}(X)$  cf. (Naudts, 2008; Torgersen, 1981). Traditionally this comparison measure is taken for granted as being Lebesgue measure, but we shall see it is an arbitrary choice and the choice matters<sup>21</sup>.

Given  $\mu \in \mathfrak{P}(X)$ , define the experiment  $E_\mu^\nu: [2] \rightsquigarrow X$  via

$$E_\mu^\nu(1, \cdot) \stackrel{\text{def}}{=} \mu(\cdot) \quad \text{and} \quad E_\mu^\nu(2, \cdot) \stackrel{\text{def}}{=} \nu(\cdot).$$

The measure  $\mu$  is that which we are interested in (we wish to compute its “entropy”); the measure  $\nu$  is a choice we make regarding what to compare it against. Often  $\nu = \lambda$ , Lebesgue measure. The *unconstrained entropy* can be defined as follows. For  $D \in \mathcal{D}(\mathbb{R}^2, \mathbb{R}_{\leq 0}^2)$ , the

20. In classical thermodynamics, entropy has been taken to be the fundamental notion, with relative entropy (i.e. KL-divergence) as subsidiary. However recent work has shown that one can develop classical thermodynamics starting from relative entropy, with a number of advantages (Floerchinger and Haas, 2020). They conclude by speculating that it could be beneficial, for the foundations of thermodynamics, “to think more often in terms of *distinguishability* instead of *missing information*” (Floerchinger and Haas, 2020, page 11); confer (Ben-Naim, 2008) which argued that “missing information” was a better viewpoint than the classical “degree of uncertainty” usually invoked to explain the intuition of physical entropy.

21. This idea that unary properties are intrinsically relative to some implicit reference has been developed for the notion of Lorenz curves (Buscemi and Gour, 2017), themselves related to ROC curves (Schechtman and Schechtman, 2019) which are intimately related to certain families of  $I_D$  (Reid and Williamson, 2011, §6.1).

$D$ -entropy of  $\mu$  relative to  $v$  is

$$H_D^v(\mu) \stackrel{\text{def}}{=} I_D(E_\mu^v).$$

Define  $D_\varphi$  via (10) and write  $H_\varphi^v(\mu) \stackrel{\text{def}}{=} H_{D_\varphi}^v(\mu)$ , the usual definition of  $\varphi$ -entropy when  $v$  is chosen to be “uniform” over the support of  $\mu$ .<sup>22</sup> Choose  $\rho$  as usual to be absolutely continuous with respect to  $\mu$  and  $v$ . Then using Proposition 22 we have  $H_\varphi^v(\mu) = \int_X \sigma_{D_\varphi}(\mathrm{d}\mu/\mathrm{d}\rho, \mathrm{d}v/\mathrm{d}\rho) \mathrm{d}\rho = \int_X \check{\varphi}(\mathrm{d}\mu/\mathrm{d}\rho, \mathrm{d}v/\mathrm{d}\rho) \mathrm{d}\rho = \int_X \mathrm{d}\mu/\mathrm{d}\rho \varphi(\mathrm{d}\mu/\mathrm{d}\rho, \mathrm{d}\rho/\mathrm{d}v) \mathrm{d}\rho = \int_X \varphi(\mathrm{d}\mu/\mathrm{d}v) \mathrm{d}\mu = E_\mu \varphi(\mathrm{d}\mu/\mathrm{d}v)$ . As usual, the choice of reference measure  $\rho$  does not matter. But the choice of comparison measure  $v$  does matter since  $H_\varphi^v(\mu) = I_\varphi(E_\mu^v)$  which clearly depends upon the choice of  $v$ .

This perspective offers an insight into why the entropy is difficult to estimate: one is implicitly attempting to determine the Bayes risk for a statistical decision problem where the two class conditional distributions are the given  $\mu$  and the reference (uniform) measure  $v$  using a loss  $\ell$  induced by  $\varphi$  as in Remark 24. This insight also offers an effective approach to estimating the entropy as we now explain.

The *constrained entropy* of  $\mu$  relative to  $v$  is defined similarly,

$$H_{\mathcal{F}}^v(\mu) \stackrel{\text{def}}{=} I_{\mathcal{F}}(E_\mu^v),$$

and simply amounts to regularising the  $\varphi$ -entropy (where  $\mathcal{F}(X) \subseteq D_\varphi$ ). This immediately suggests ways to estimate the entropy of a random variable defined on  $X$  (especially when  $X$  is high dimensional): use the bridge between  $\mathcal{F}$ -information and the  $\mathcal{H}$ -constrained Bayes risk and simply exploit the wide range of extant methods for solving binary class-probability estimation problems. That is given a random sample  $\{x_1, \dots, x_m\}$  drawn iid from  $\mu$ , estimate the entropy from the empirical measure  $\mu^m(A) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{i \in [m]} \mathbb{1}[x_i \in A]$  via  $H_{\mathcal{F}}^v(\mu^m) = I_{\mathcal{F}}(\mu^m, v)$ . The estimate is regularised by the choice of  $\mathcal{F}$ . Observe that one can immediately define a generalised mutual information using  $I_{\mathcal{F}}$  when  $n = 2$ : given two random variables  $Z$  and  $Y$  defined on  $X$  with joint distribution  $\mu_{ZY}$  and marginal distributions  $\mu_Z$  and  $\mu_Y$ , define the experiment  $E^{\text{MI}}: [2] \rightsquigarrow X$  via  $E^{\text{MI}}(1, \cdot) \stackrel{\text{def}}{=} \mu_{ZY}(\cdot)$  and  $E^{\text{MI}}(2, \cdot) \stackrel{\text{def}}{=} (\mu_Z \times \mu_Y)(\cdot)$ , and then define the  $\mathcal{F}$ -Mutual Information between  $Z$  and  $Y$  as

$$\text{MI}_{\mathcal{F}}(Z; Y) \stackrel{\text{def}}{=} I_{\mathcal{F}}(E^{\text{MI}}).$$

While this seems more complex than the usual notion of mutual information, we observe that this is what is typically computed in practice since one cannot ever find the Bayes optimal hypothesis implicit in the definition of the usual mutual information, but rather only optimises over a restricted model class.

Given that entropy can be reduced to binary divergences relative to an (arbitrarily chosen!) uniform measure, and further given the multitude of binary divergences that make decision-theoretic sense, axiomatic arguments for a single preferred entropy are less compelling, notwithstanding their mathematical elegance (Baez, Fritz, and Leinster, 2011).

One can apply Proposition 40 to  $\mathcal{F}$ -entropies where a given distribution  $\mu$  is pushed through a Markov kernel  $T$  to give  $\mu T$ . Since  $H_{\mathcal{F}}^v(\mu) = I_{\mathcal{F}}(E_\mu^v)$ , we have  $I_{\mathcal{F}}(E_\mu^v T) = I_{T^* \mathcal{F}}(E_\mu^v)$  and hence

$$H_{\mathcal{F}}^v(\mu T) = H_{T^* \mathcal{F}}^v(\mu).$$

22. This is not a new idea; see (Chafai, 2004, page 329).

## Appendix D. Precursors of $\mathcal{F}$ -Information

There are several precursors<sup>23</sup> to our notion of  $\mathcal{F}$ -information, including  $\mathfrak{N}$ -information (rediscovered as MMD), Integral Probability Metrics, Moreau-Yosida  $\varphi$ -divergences and  $(f, I)$ -Divergences, and in this Appendix we briefly summarise them.

The idea that one can view a model class as being the result of a rich class being “pushed through” a restrictive channel (what the information processing equality does in effect) was central to the calculations of covering numbers by Williamson, Smola, and Scholköpfung (2001).

As can be seen from (42) in Appendix A, the classical binary variational divergence of  $E: \{1, 2\} \rightsquigarrow X$  can be written as  $I_{\text{Var}}(E) = \sup_{f: X \rightarrow ([0,1], \mathfrak{B})} E_1 f - E_2 f$ . When the supremum is restricted to be over  $\mathcal{F}$ , a proper subset of  $\{f: X \rightarrow ([0,1], \mathfrak{B})\}$ , these are known as *integral probability metrics* (IPMs) (Müller, 1997) or *probability metrics with  $\zeta$ -structure* (Zolotarev, 1983), and extend the Variational divergence by restricting the class of functions which are optimised over in its variational representation; see A.2. Special cases of this include the Wasserstein distance (Villani, 2009).

When  $\mathcal{F}$  is the unit ball of a reproducing kernel Hilbert space, these are known as  $\mathfrak{N}$ -distances and were developed by Bakšajev (2010), Klebanov (2003, 2005), and Zinger, Kakosyan, and Klebanov (1992); (see Rachev et al., 2013, Chapters 21–26, for a recent review). The  $\mathfrak{N}$ -distances were rediscovered in the machine learning community as “Maximum Mean Discrepancy” (MMD) by Smola et al. (2007), Sriperumbudur et al. (2010), and Vangeepuram (2010). Muandet et al. (2017) presented a recent review (ignoring some prior work however).

The classical IPMs are a way of constraining the function class one optimises over in the variational representation of variational divergence. One can similarly restrict the class of functions in the variational representation of an arbitrary  $\varphi$ -divergence as was suggested by Reid and Williamson (2011, page 796), who proposed considering  $I_{\varphi, \mathcal{F}}(P, Q) \stackrel{\text{def}}{=} \sup_{\rho \in \mathcal{F}} (E_P \rho - E_Q \varphi^*(\rho))$ , explored the particular case for  $\varphi(t) = |t - 1|$  and  $\mathcal{F}$  being the unit ball in a reproducing kernel Hilbert space (Reid and Williamson, 2011, Appendix H), and posed the question of its relationship to a constrained Bayes risk also using the function class  $\mathcal{F}$  (Reid and Williamson, 2011, page 799) (which is answered by the present paper). Xu et al. (2020) proposed a generalization of Shannon Mutual information by restricting the class of functions optimised over in a variational representation, motivated slightly differently to the  $\mathcal{F}$ -information of the present paper — they motivated their definition on computational grounds, and observed as a consequence the estimation performance improves. (Note the brief discussion of  $\mathcal{F}$ -mutual information in Appendix C.) Terjék (2021) regularised the optimisation for binary  $\varphi$  divergences with a Wasserstein regulariser. More generally, Birrell, Dupuis, et al. (2022) considered a larger range of  $\mathcal{F}$  for arbitrary  $\varphi$ . However, they necessarily only considered the binary  $\varphi$ -divergence, and because they used the classical variational representation in terms of the Legendre-Fenchel conjugate of  $\varphi$ , their formulas become quite complex compared to the development in the present paper. A recent comparison of IPMs and  $\varphi$ -divergence (Agrawal and Horel, 2021) appears to mix up two things: a comparison of loss functions, combined with a question of the approximation power of a model class.

The  $\mathcal{F}$ -information is also related to ideas developed in distributionally robust optimisation, where existing divergences  $d(P, Q)$  are “smoothed.” Three examples are the “Gaussian

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23. As we should well expect: “far from being odd or curious or remarkable, the pattern of independent multiple discoveries in science is in principle the dominant pattern” (Merton, 1961, page 477).

smoothed sliced Wasserstein distance” of Rakotomamonjy et al. (2021) and the “smoothed divergence” defined either as  $d^\epsilon(P, Q) = \max_{\rho \in B_\epsilon(P)} d(\rho, Q)$  by Van Der Meer, Ng, and Wehner (2017), or (closest in spirit to the present paper, especially the information processing equality)  $d^\epsilon(P, Q) = d(P * K_\epsilon, Q * K_\epsilon)$ , where the  $*$  denotes convolution and  $K_\epsilon$  is a scaled kernel (of ‘width’  $\epsilon$ ) as defined by Manole and Ramdas (2023, page 25), Goldfeld et al. (2020) (motivated by effects of additive noise) and Nietert, Goldfeld, and Kato (2021).

The Wasserstein distance is related to “smoothed entropies” (Van Der Meer, Ng, and Wehner, 2017, equation 8), the idea of which is to use  $d^\epsilon(P, Q) = \max_{P' \in B_\epsilon(P)} d(P', Q)$ , where in their case,  $d$  is the Renyi divergence (related to, but different from  $\varphi$ -divergences), and the  $\epsilon$ -ball  $B_\epsilon(P)$  is relative to the trace distance.

There are links between IPMs and distributional robustness motivated by understanding  $f$ -GANs (Nowozin, Cseke, and Tomioka, 2016). Husain, Nock, and Williamson (2019) showed that “restricted  $f$ -GAN objectives are lower bounds to Wasserstein autoencoder.” Subsequently Husain (2020, Theorem 1) showed how the distributionally robust objective can be expressed via regularisation:  $\sup_{Q \in B_{\epsilon, F}} \int h dQ = \int h dP + \Lambda_{F, \epsilon}(h)$ ; see also (Song and Ermon, 2020).

Finally we mention the perspective of Birrell, Katsoulakis, and Pantazis (2022) who refined the variational representation of  $\varphi$ -divergences in a complementary way: instead of restricting the function class over which the objective is optimised, they tweaked the form of the objective function in a manner that the argmax remains the same, but the objective differs otherwise.

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