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CHARACTERIZATION OF α-ENTROPY WITH PREFERENCE

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A characterization of α-entropy with preference is provided. Further an attempt is made to characterize this measure by weakening the symmetry postulate.

1. INTRODUCTION

Let $\Delta_n = (p_1, p_2, \ldots, p_n)$, $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$ be a probability distribution associated with discrete random variable X assuming a finite number of values $X_1, X_2, \ldots, X_n$. Havrda and Charvát entropy of order α (cf. [2]) is defined as

$$I_n(\Delta_n) = \frac{\sum_{i=1}^{n} p_i^\alpha - 1}{2^{1-\alpha} - 1}, \quad \alpha \neq 1, \quad \alpha > 0. \tag{1.1}$$

In order to distinguish the events $X_1, X_2, \ldots, X_n$ of a goal directed experiment according to their importance with respect to the goal, Belis and Guiaşu [1] introduced a 'utility distribution' $(u_1, u_2, \ldots, u_n)$, where each $u_i > 0$ is the utility of an event with probability p_i. Then α-entropy with preference is given by

$$I_n(P, U) = \frac{\sum_{i=1}^{n} u_i p_i (p_i^{\alpha-1} - 1)}{2^{1-\alpha} - 1}, \quad \alpha \neq 1, \alpha > 0. \tag{1.2}$$

In this paper, we characterize this measure by axiomatic approach. Also the postulate of symmetry is replaced by a weaker postulate.

2. CHARACTERIZATION

Let

$$I_n\colon \Delta_n \times \mathbb{R}_+^n \to \mathbb{R}\,, \quad n \geqq 2$$

where

$$\Delta_n = (p_1, p_2, \ldots, p_n)\,, \quad 0 \leqq p_i \leqq 1\,, \quad \sum_{i=1}^{n} p_i = 1\,, \quad \mathbb{R}_+ = (0, \infty)\,,$$

satisfies the following postulates.

Postulate P 1 (Recursivity).

$$I_n(p_1, p_2, \ldots, p_n\,;\, u_1, u_2, \ldots, u_n) =$$
$$I_{n-1}(p_1 + p_2, \ldots, p_n;\, u_{1,2}, u_3, \ldots, u_n) +$$
$$+ (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, u_1, u_2\right)$$

where $u_{1,2} = (u_1 p_1 + u_2 p_2)/(p_1 + p_2)$, $p_1 + p_2 \in (0, 1]$.

Postulate P 2 (Symmetry). $I_n(p_1, p_2, \ldots, p_n;\, u_1, u_2, \ldots, u_n)$ is a pairwise symmetric function of its arguments.

Postulate P 3 (Differentiability). $f(p, u_1, u_2) = I_2(p, 1 - p;\, u_1, u_2)$ has continuous first order partial derivatives with respect to all the three variables.

Postulate P 4 (Normalization). $I_2(\frac{1}{2}, \frac{1}{2}; \lambda, \lambda) = \lambda$, $I_2(1, 0, u_1, u_2) = 0$; $\lambda > 0$.

Theorem 2.1. If $I_n\colon \Delta_n \times \mathbb{R}_+^n \to \mathbb{R}$, $(n = 2, 3, \ldots)$ satisfy Postulates P 1, P 2, P 3 and P 4, then

$$(2.1)\quad I_n(p_1, p_2, \ldots, p_n;\, u_1, u_2, \ldots, u_n) = \frac{\sum_{i=1}^{n} p_i u_i (p_i^{\alpha-1} - 1)}{2^{1-\alpha} - 1} \quad \alpha \neq 1\,, \quad \alpha > 0\,.$$

Proof. By symmetric Postulate P 2, we have

$$(2.2)\quad I_3(p_1, p_2, p_3;\, u_1, u_2, u_3) = I_3(p_2, p_3, p_1;\, u_2, u_3, u_1) =$$
$$= I_3(p_3, p_1, p_2;\, u_3, u_1, u_2)\,.$$

Using Postulate P 1 in (2.2), we get the following functional equation

$$(2.3)\quad f(p_1 + p_2;\, u_{1,2}, u_3) + (p_1 + p_2)^\alpha f\left(\frac{p_1}{p_1 + p_2};\, u_1, u_2\right) =$$
$$= f(p_1, u_1, u_{2,3}) + (1 - p_1)^\alpha f\left(\frac{p_2}{1 - p_1}, u_2, u_3\right),$$
$$p_1 + p_2 \in (0, 1]\,, \quad p_1 \in [0, 1)$$

and

(2.4) $$f(p_1 + p_2; u_{1,2}, u_3) + (p_1 + p_2)^\alpha f\left(\frac{p_1}{p_1 + p_2}; u_1, u_2\right) =$$

$$= f(p_2, u_2, u_{1,3}) + (1 - p_2)^\alpha f\left(\frac{p_1}{1 - p_2}, u_1, u_3\right), \quad p_1 + p_2 \in (0, 1], \quad p_2 \in [0, 1).$$

Differentiating partially (2.3) and (2.4) with respect to p_2, and p_1 respectively,

(2.5) $$f_p(p_1 + p_2, u_{1,2}, u_3) + \frac{p_1(u_2 - u_1)}{(p_1 + p_2)^2} f_{u_{1,2}}(p_1 + p_2, u_{1,2}, u_3) +$$

$$+ \alpha(p_1 + p_2)^{\alpha-1} f\left(\frac{p_1}{p_1 + p_2}, u_1, u_2\right) - p_1(p_1 + p_2)^{\alpha-2} f_p\left(\frac{p_1}{p_1 + p_2}, u_1, u_2\right) =$$

$$= (1 - p_1)^{\alpha-1} f_p\left(\frac{p_2}{1 - p_1}, u_2, u_3\right)$$

(2.6) $$f_p(p_1 + p_2, u_{1,2}, u_3) - \frac{p_2(u_2, u_1)}{(p_1 + p_2)^2} f_{u_{1,2}}(p_1 + p_2, u_{1,2}, u_3) +$$

$$+ \alpha(p_1 + p_2)^{\alpha-1} f\left(\frac{p_1}{p_1 + p_2}, u_1, u_2\right) + p_2(p_1 + p_2)^{\alpha-2} f_p\left(\frac{p_1}{p_1 + p_2}, u_1, u_2\right) =$$

$$= (1 - p_2)^{\alpha-1} f_p\left(\frac{p_1}{1 - p_2}, u_1, u_3\right)$$

Setting $p_1 + p_2 = 1$, the equations (2.5) and (2.6) give

(2.7) $$f_p(p, u_1, u_2) = p_1^{\alpha-1} f_p(1, u_1, u_3) - (1 - p_1)^{\alpha-1} f_p(1, u_2, u_3) +$$
$$+ (u_2 - u_1) f_{u_{1,2}}(1, u_{1,2}, u_3).$$

Setting $f_u(1, u, u') = B$, $f_p(1, u, u') = A\alpha u$ in (2.7) and integrating with respect to p,

(2.8) $$f(p, u_1, u_2) = A[u_1 p^\alpha + u_2(1 - p)^\alpha] - B[u_1 p + u_2(1 - p)] + C(u_1, u_2).$$

Normalization Postulate P 4 gives

$$A = B = 1/(2^{1-\alpha} - 1), \quad C = 0.$$

Therefore

(2.9) $$f(p, u_1, u_2) = \frac{u_1 p(p^{\alpha-1} - 1) + u_2(1 - p)((1 - p)^{\alpha-1} - 1)}{2^{1-\alpha} - 1}.$$

By successive application of P 1 we get

(2.10) $$I_n(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n) =$$

$$= \sum_{k=2}^{n} (p_1 + p_2 + \dots + p_k)^\alpha I_2\left(\frac{p_1 + p_2 + \dots + p_{k-1}}{p_1 + p_2 + \dots + p_k}, \frac{p_k}{p_1 + \dots + p_k}, u_{1,2,\dots,k-1}, u_k\right) =$$

$$= \sum_{k=2}^{n} (p_1 + p_2 + \dots + p_k)^\alpha f\left(\frac{p_1 + p_2 + \dots + p_{k-1}}{p_1 + p_2 + \dots + p_k}, u_{1,2,\dots,k-1}, u_k\right) =$$

$$= \sum_{k=2}^{n} q_k^\alpha f\left(\frac{q_{k-1}}{q_k}, u_{1,2,\dots,k-1}, u_k\right)$$

where $q_k = p_1 + p_2 + \dots p_k$, $u_{1,2,\dots,k} = (q_{k-1} u_{1,2,\dots,k-1} + p_k u_k)/q_k$.

(2.9) and (2.10) gives

$$I_n(p_1, p_2, \dots, p_n; u_1; u_2, \dots, u_n) =$$

$$= \frac{1}{2^{1-\alpha} - 1}\left[\sum_{k=2}^{n} q_k^\alpha \left[u_{1,2,\dots,k-1}\left(\frac{q_{k-1}}{q_k}\right)\left[\left(\frac{q_{k-1}}{q_k}\right)^{\alpha-1} - 1\right] + u_k \frac{p_k}{q_k}\left[\left(\frac{p_k}{q_k}\right)^{\alpha-1} - 1\right]\right]\right] =$$

$$= \frac{1}{2^{1-\alpha} - 1} \sum_{k=2}^{n} \left[u_{1,2,\dots,k-1}\, q_{k-1}^\alpha + u_k p_k^\alpha - q_k^{\alpha-1}\left[u_{1,2,\dots,k-1}\, q_{k-1} + u_k p_k\right]\right] =$$

$$= \frac{\sum_{i=2}^{n} u_i p_i (p_i^{\alpha-1} - 1)}{2^{1-\alpha} - 1}$$

This completes the proof of the theorem.

3. CHARACTERIZATION WITHOUT SYMMETRY

In this section, we characterize α-entropy with preference without assuming symmetry. We replace the postulate of symmetry with a weaker postulate.

Definition 3.1. Let n be a positive integer greater than 1 and A, B be two non-empty subset of $\mathbb{R}$. A function f_n: $A^n \times B^n \to \mathbb{R}$ is said to be n-cyclic over its domain if it is invariant under one cyclic shift, that is

$$f_n(x_1, x_2, \dots, x_{n-1}, x_n; y_1, y_2, \dots, y_n) =$$

$$= f_n(x_n, x_1, x_2, \dots, x_{n-1}; y_n, y_1, \dots, y_{n-1}), \quad n \geqq 2$$

$$(x_1, x_2, \dots, x_n) \in A^n, \quad (y_1, y_2, \dots, y_n) \in B^n.$$

Definition 3.2. A sequence of functions $\{f_n\}_{n \geqq 2}$ where f_n: $\Delta_n \times \mathbb{R}_+^n \to \mathbb{R}$, is called left expansible if

$$f_n(p_1, p_2, \dots, p_n, u_1, u_2, \dots, u_n) =$$

$$= f_{n+1}(0, p_1, p_2, \dots, p_n, u, u_1, \dots, u_n) \quad u > 0$$

where $\mathbb{R}_+ = (0, \infty)$.

Postulate P 5 (Cyclic Symmetry). $I_n(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n)$ is an n-cyclic symmetric function of its arguments.

For $n = 2$, Postulate P 2 and P 5 are equivalent. It is obvious that P 2 implies P 5, but the converse is not true. For example consider the following:

Example 1. Let $\phi_n: \Delta_n \times \mathbb{R}_+^n \to \mathbb{R}$, $n = 2, 3, 4, \ldots$, be defined as

$$\phi_n(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) =$$
$$= \sum_{i=1}^{n-1} (u_i p_i + u_{i+1} p_{i+1})(p_i - p_{i+1}) + (u_n p_n + u_1 p_1)(p_n - p_1).$$

Then $\{\phi_n\}_{n \geqq 2}$ is an n-cyclic sequence of functions, but functions ϕ_n are not symmetric for $n > 2$. Hence Postulate P 5 is weaker than P 2.

Lemma 3.1. Postulates P 1, P 4 and P 5 ($n \geqq 2$) imply left expansibility, i.e.,

$$I_n(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) = \tag{3.1}$$
$$= I_{n+1}(0, p_1, p_2, \ldots, p_n; u, u_1, u_2, \ldots, u_n) \quad \text{for} \quad u > 0.$$

Proof. Let p_k be the last non-zero component of $(0, p_1, \ldots, p_n)$. Then the right side of equation (3.1) is

$$I_{n+1}(0, p_1, p_2, \ldots p_k, 0, 0, \ldots 0; u, u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n) = \tag{3.2}$$
$$\stackrel{\text{P5}}{=} I_{n+1}(p_k, 0, 0, \ldots, 0, p_1, \ldots, p_{k-1}; u_k, u_{k+1}, \ldots, u_n, u, u_1, u_2, \ldots, u_{k-1}) =$$
$$\stackrel{\text{P1}}{=} I_n(p_k, 0, \ldots, 0, p_1, \ldots, p_{k-1}; u_k, u_{k+2}, u_{k+3}, \ldots, u_n, u, u_1, \ldots, u_{k-1}) +$$
$$+ (p_k)^\alpha I_2(1, 0, u_k, u_{k+1}) =$$
$$\stackrel{\text{P4}}{=} I_n(p_k, 0, \ldots, p_1, \ldots, p_{k-1}; u_k, u_{k+2}, \ldots, u_n, u, u_1, \ldots, u_{k-1}) =$$
$$\stackrel{\text{Inductively}}{=} I_k(p_k, p_1, \ldots, p_{k-1}, u_k, u_1, u_2, \ldots, u_{k-1}).$$

Similarly, the left hand side of (3.1) can be proved to be equal to (3.2). This completes the proof.

Lemma 3.2. Postulates P 1, P 4 and P 5 imply P 2.

Proof. For $n = 2$, the result is obvious. For $n = 3$ and one of the p_i is zero, then the result follows from Lemma 3.1, Postulate P 5 and symmetry of I_2. If all $p_i > 0$, $i = 1, 2, 3$, then Postulate P 1 and symmetry of I_2 give

$$I_3(p_1, p_2, p_3; u_1, u_2, u_3) = I_3(p_2, p_1, p_3; u_2, u_1, u_3). \tag{3.3}$$

Postulate P 5 and equation (3.3) imply symmetry of I_3.

Now, we prove that I_n, $n \geqq 4$ are also symmetric functions of their arguments. We complete the proof by induction on n. The result follows immediately from Lemma 3.1, Postulate P 5 and induction hypothesis if one of p_i is zero. Hence it is enough to prove the following for $p_i > 0$ for each i

$$I_n(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) = I_n(p_2, p_1, p_3, \ldots, p_n; u_2, u_1, u_3, \ldots, u_n) \tag{3.4}$$

$$I_n(p_1, p_2, p_3, \ldots, p_n; u_1, u_2, u_3, \ldots, u_n) = \tag{3.5}$$
$$= I_n(p_1, p_2, p_{k(3)}, p_{k(4)}, \ldots, p_{k(n)}; u_1, u_2, u_{k(3)}, \ldots, u_{k(n)})$$

$$I_n(p_1, p_2, p_3, p_4, \ldots, p_n; u_1, u_2, u_3, u_4, \ldots, u_n) = \tag{3.6}$$
$$= I_n(p_3, p_2, p_1, p_4, \ldots, p_n; u_3, u_2, u_1, u_4, \ldots, u_n)$$

where $\{k(3), k(4), \ldots, k(n)\}$ is an arbitrary permutation of $\{3, 4, \ldots, n\}$.

First we prove (3.4)

$$I_n(p_1, p_2, p_3, \ldots, p_n;\ u_1, u_2, u_3, \ldots, u_n) =$$

$$\overset{\text{P1}}{=} I_{n-1}(p_1 + p_2, p_3, \ldots, p_n;\ u_{1,2}, u_3, \ldots, u_n) +$$

$$+ (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, u_1, u_2\right) =$$

$$\overset{\text{Symmetry of } I_2}{=} I_{n-1}(p_1 + p_2, p_3, \ldots, p_n; u_{1,2}, u_3, \ldots, u_n) +$$

$$+ (p_1 + p_2)^\alpha \left(\frac{p_2}{p_1 + p_2}, \frac{p_1}{p_1 + p_2}, u_2, u_1\right) =$$

$$\overset{\text{P1}}{=} I_n(p_2, p_1, p_3, \ldots, p_n;\ u_2, u_1, u_3, \ldots, u_n) .$$

Now we prove (3.5)

$$I_n(p_1, p_2, \ldots, p_n;\ u_1, u_2, \ldots, u_n) \overset{\text{P1}}{=} I_{n-1}(p_1 + p_2, p_3, \ldots, p_n;\ u_{1,2}, u_3, \ldots, u_n) +$$

$$+ (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, u_1, u_2\right) =$$

$$\overset{\text{by the induction hypothesis}}{=} I_{n-1}(p_1 + p_2, p_{k(3)}, \ldots, p_{k(n)};\ u_{1,2}, u_{k(3)}, \ldots, u_{k(n)}) +$$

$$+ (p_1 + p_2)^\alpha I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, u_1, u_2\right) =$$

$$\overset{\text{P1}}{=} I_n(p_1, p_2, p_{k(3)}, \ldots, p_{k(n)};\ u_1, u_2, u_{k(3)}, \ldots, u_{k(n)}) ,$$

where $\{k(3), k(4), \ldots, k(n)\}$ is an arbitrary permutation of $\{3, 4, \ldots, n\}$.

Finally we prove (3.6)

$$I_n(p_1, p_2, p_3, p_4, \ldots, p_n;\ u_1, u_2, u_3, u_4, \ldots, u_n) =$$

$$\overset{\text{P5}}{=} I_n(p_{n-1}, p_n, p_1, p_2, \ldots, p_{n-2}, u_{n-1}, u_n, u_1, \ldots, u_{n-2}) =$$

$$\overset{\text{Eq. (3.5)}}{=} I_n(p_{n-1}, p_n, p_3, p_2, p_1, p_4, \ldots, p_{n-2};\ u_{n-1}, u_n, u_3, u_2, u_1, u_4, \ldots, u_{n-2}) =$$

$$\overset{\text{P5}}{=} I_n(p_3, p_2, p_1, p_4, \ldots, p_n;\ u_3, u_2, u_1, u_3, \ldots, u_n) .$$

This completes the proof of the lemma.

Theorem 3.1. If $I_n\colon \Delta_n \times \mathbb{R}_+^n \to \mathbb{R}$, $(n = 2, 3, \ldots)$ satisfy Postulates P 1, P 3, P 4 and P 5, then

(3.7)

$$I_n(p_1, p_2, \ldots, p_n;\ u_1, u_2, \ldots, u_n) = \frac{1}{2^{1-\alpha} - 1} \sum_{i=1}^{n} u_i p_i (p_i^{\alpha-1} - 1) \quad \alpha > 0,\ \alpha \neq 1 .$$

The proof of the theorem follows from Lemma 3.2 and Theorem 2.1.

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