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BLOCK DECOUPLING INVARIANTS: GEOMETRIC AND TRANSFER MATRIX CHARACTERIZATIONS

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In this paper we consider the block decoupling problem for linear time invariant systems in the general case i.e. the system transfer matrix is not supposed to be surjective. The aim of this work is twofold, first to introduce new lists of integers called "Block Essential Structures" of the system and then to provide several equivalent characterizations of these invariants within both transfer matrix and geometric approaches.

It turns out these integers represent precisely the minimal infinite structure achievable for the blocks of the decoupled system through precompensation. When the system is decouplable by dynamic state feedback the minimal infinite structure achievable is the same as previously.

1. INTRODUCTION

During the last twenty years a great deal of interest has been brought to the theory of decoupling for linear time invariant systems. Important steps in the development of the decoupling theory as well as references can be found in Falb and Wolovich [12], Morse and Wonham [16], Wonham [19], Hautus and Heymann [14]. In the recent years a great deal of interest has been devoted to the structural features of decoupling problems. Various lists of integers strongly related with the deep structure of the system have been introduced. In particular Descusse and Dion in [5] expressed the solvability condition of [12] in terms of infinite structure equalities. In [4] the essential orders characterizing the minimal infinite structure of the decoupled system are presented for the row by row decoupling problem.

In this paper we will focus our interest on the block decoupling problem. We are interested in obtaining the simplest possible decoupled systems. More precisely we

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look for decoupled systems possessing the least possible infinite structure. We consider a linear system whose transfer matrix $T(s)$ is partitioned in subsystems $T_i(s)$ according to a desidred output partition. We assume that the rank of $T(s)$ is equal to the sum of the ranks of the subsystems $T_i(s)$, then block decoupling by precompensation is always possible [14]. The aim of the paper is mainly to introduce new lists of integers called "the ith block essential orders" which represent precisely the minimal infinite structure achievable for the ith block of the decoupled system through precompensation. We will define the "block essential orders" as the union of the ith block essential orders. We will prove that the block essential orders are feedback invariants and will provide several equivalent characterizations of these invariants within both transfer matrix and geometric approaches.

The present work generalizes that one already done by Commault and Coworkers [4] where only the row decoupling problem was considered.

This paper is organized as follows: Section two is devoted to notations and basic concepts. In the third section we introduce the block decoupling problem and some preliminaries are given.. The new lists of invariants called block essential orders are introduced in section four. We give also several geometric and transfer matrix characterizations of these invariants. Its application to the block decoupling problem through precompensation, static or dynamic state feedback take place also in section four. We show that the block essential orders represent the least infinite structure achievable for the decoupled system. In the last section an illustrative example exhibits both geometric and transfer matrix points of view.

2. NOTATIONS AND PRELIMINARIES

In this paper we will deal with linear time invariant (C, A, B) systems described by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathscr{X} \sim \mathbb{R}^n$, $u \in \mathscr{U} \sim \mathbb{R}^m$ and $y \in \mathscr{Y} \sim \mathbb{R}^p$ with B monic and C epic.

Associated with (C, A, B) we will consider a k-partition of the output y into k nonempty subsets of components y_i, each of dimension $p_i(0 < p_i, \sum_{i=1}^{k} p_i = p)$. This partition induces a corresponding partition of C, written as $(C_1^{\mathrm{T}}, C_2^{\mathrm{T}}, \ldots, C_k^{\mathrm{T}})^{\mathrm{T}}$ with T denoting transposition. We will denote by C^i the matrix obtained from C by removing the ith row-block C_i.

Hereafter, we will use $\mathscr{V}_i^*$ the maximal (A, B)-invariant subspace contained in Ker C_i, also written sup $I(A, B, \operatorname{Ker} C_i)$, $\mathscr{T}_i^*$ that maximal one contained in Ker C^i. $\mathscr{R}_i^*$ will denote the maximal controllability subspace contained in $\mathscr{T}_i^*$. The classical tools of the geometric approach are derived from the following algorithms [19].

(A, B)-invariant subspace algorithm:

$$\mathscr{V}_i^0 = \mathscr{X}$$

$$\mathscr{V}_i^{\mu+1} = \operatorname{Ker} C_i \cap A^{-1}(\mathscr{B} + \mathscr{V}_i^{\mu})$$

Controllability subspace algorithm:

$$\mathscr{R}_i^0 = 0$$

$$\mathscr{R}_i^{\mu+1} = \mathscr{T}_i^* \cap (A\mathscr{R}_i^{\mu} + \mathscr{B}) \tag{2}$$

with $\mathscr{B}$ the image of B.

Define

$$p'_{\mu+1} = \dim(\mathscr{B} \cap \mathscr{V}^{\mu}/\mathscr{B} \cap \mathscr{V}^*)\,; \quad \mu \geqq 0 \tag{3}$$

where $\mathscr{V}^{\mu}$ and $\mathscr{V}^*$ are related to $\sup I(A, B, \operatorname{Ker} C)$.

The infinite zero structure of (C, A, B) is the list of integers $\{n_i\}$ defined by $n_i = \operatorname{card}\{p'_{\mu} \geqq i\}$. By the way, (C, A, B) has p'_1 infinite zeros the orders of which are n_i. Recall that $p'_1 = \operatorname{rank} T(s)$.

Let $\mathbf{R}(s)$ be the field of rational functions. A rational function $f(s) = n(s)/d(s)$ is said to be proper (resp. strictly proper) if $\deg(d(s)) \geqq \deg(n(s))$ (resp. $\deg(d(s)) > \deg(n(s))$) where $\deg(n(s))$ denotes the polynomial degree of $n(s)$.

Denote by $\mathbf{R}_p(s)$ the ring of proper rational functions and $\mathbf{R}_p^{p\times m}(s)$ the set of proper rational $p \times m$ transfer matrices.

The units (invertible elements) of the ring $\mathbf{R}_p^{m\times m}(s)$ are called bicausal matrices and are characterized by the property that $B(s)$ is a bicausal matrix if and only if:

$$\det(\lim_{s\to\infty} B(s)) \neq 0$$

Definition 1. A full column rank proper rational matrix $V(s)$ is said to be a right bicausal matrix if there exists a proper rational matrix $W(s)$ such that $[V(s)\ W(s)]$ is a bicausal matrix.

The transfer matrix of system (1) is a $p \times m$ strictly proper rational matrix with null left and right static kernels, since B is monic and C is epic.

$$T(s) = C(sI - A)^{-1} B$$

For a $p \times m$ rational matrix $G(s)$, there exists Smith-McMillan factorizations at infinity which were studied in [17] and in [8]:

$$G(s) = B_1(s)\,\Lambda(s)\,B_2(s)$$

where $B_1(s)$, $B_2(s)$ are bicausal matrices and $\Lambda(s)$ is uniquely defined by:

$$\Lambda(s) = \begin{bmatrix} s^{-n_1} & & & \\ & \ddots & & 0 \\ & & s^{-n_r} & \\ & 0 & & 0 \end{bmatrix}$$

where r is the rank of $G(s)$. If n_i is negative, $-n_i$ is the order of a pole at infinity, whereas if n_i is positive, n_i is the order of a zero at infinity.

In the whole paper the infinite structure of $G(s)$ will be denoted by:

$$\Sigma_\infty(G(s)) = \{n_1, \ldots, n_r\} \tag{4}$$

we will also denote: $-\Sigma_\infty(G(s)) = \{-n_1, \ldots, -n_r\}$.

In case, when dealing with a realization (C, A, B) of a strictly proper transfer matrix $T(s)$, the infinite structure will be denoted either by $\Sigma_\infty(C, A, B)$ or by $\Sigma_\infty(T(s))$.

Another useful notation, in the spirit of compactness, will be $d_\infty(T(s))$ to denote the sum of infinite pole and zero orders of $T(s)$, this means:

$$d_\infty(T(s)) = \sum_{i=1}^{k} |n_i| \tag{5}$$

The equivalence between Morse's second list I_2 and the ordered list of the degrees of a minimal polynomial basis for the right kernel of the transfer matrix is well known [13], [18]. In particular the dimension of the largest controllability subspace $\mathscr{R}^*$ contained in Ker C of a minimal realization (A, B, C) of $T(s)$ is equal to the sum of the degrees of a minimal polynomial basis for the right kernel of $T(s)$. In fact we will use the dual formulation of this result namely:

$$\dim\left(\frac{\mathscr{X}}{\mathscr{S}^* + \mathscr{V}^*}\right) = \sigma(T(s)) \tag{6}$$

where $\sigma(T(s))$ denotes the sum of the degrees of a minimal polynomial basis of the left kernel of $T(s)$ and $\mathscr{S}^*$ is the minimal (C, A)-invariant subspace containing the image of B.

3. THE BLOCK DECOUPLING PROBLEM

Problem formulation

We will consider now the block decoupling problem. In order to avoid trivialities we will require the compensated system to be just as "output controllable" as the original system is.

We will say that the proper precompensator $C(s)$ is *admissible* if

$$\text{rank } T(s)\, C(s) = \text{rank } T(s)$$

This admissibility condition is equivalent to the preservation of the C^∞ controlled output trajectories, see [1].

Let $T(s)$ be a $p \times m$ proper rational matrix, partitioned in row-blocks relatively to a given list of positive integers $(p_1, \ldots, p_k)$, such that $\sum_{i=1}^{k} p_i = p$, in the following way:

$$T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_k(s) \end{bmatrix}; \quad \text{with} \quad T_i(s) \in \mathbf{R}_p^{p_i \times m}(s) \quad \text{for} \quad i = 1, \ldots, k$$

The system with transfer matrix $T(s)$ is said to be block decoupled relatively to the partition $\{p_i\}$ if there exist positive integers $m_1, \ldots, m_k$ satisfying $\sum_{i=1}^{k} m_i = m$, such that $T(s)$ has the block diagonal form:

$$T(s) = \begin{bmatrix} T_{11}(s) & 0 & \\ 0 & \ddots & \\ & & T_{kk}(s) \end{bmatrix} = \operatorname{diag}(T_{11}(s), \ldots, T_{kk}(s))$$

with $T_{ii}(s) \in \mathbf{R}_p^{p_i \times m_i}(s)$ for $i = 1, \ldots, k$

This means that each above defined input block influences only one output block. If one wants this influence to be effective the $T_{ii}(s)$ must be non null for each i, in this case the system is called non degenerate.

The decoupling problem can be formulated as follows: Is it possible to exhibit an admissible proper precompensator $C(s)$ such that $T(s)\,C(s)$ is block decoupled and non degenerate?

It is well known that the above defined decoupling problem is solvable by precompensation when the row blocks $T_i(s)$ are independent ($r = \sum_{i=1}^{k} r_i$, $r = \operatorname{rank} T(s)$ and $r_i = \operatorname{rank} T_i(s)$), Hautus and Heyman [14].

In this paper we focus our attention on the *simplest achievable decoupled systems*. More precisely we will give the minimal McMillan degree and the minimal infinite structure achievable for the blocks of the decoupled system.

4. THE BLOCK ESSENTIAL STRUCTURES

In the first part of this section we are concerned with some lists of integers strongly related with the infinite structure of the system under study. Such lists, called the ith block essential structures, are feedback invariant and represent, as we will see later, the minimal infinite structure achievable for the ith block of the decoupled system.

We are also interested in the minimal McMillan degree achievable for the blocks of the decoupled system. We will recall the geometric definition and some transfer matrix properties of the block decoupling invariants n_{ie}, which were introduced in [11].

4.1. Geometric and Transfer Matrix Characterizations

We will give first a geometric definition of the block essential structure of a system (C, A, B). Then we will give two suitable possible interpretations of the block essential structure in the transfer matrix framework.

Definition 2. Let (C, A, B) be the system (1) decomposed according to the output partition $(p_1, \ldots, p_k)$. Let F_i denote any matrix such that: $(A + BF_i)\,\mathscr{R}_i^* \subset \mathscr{R}_i^*$, where $\mathscr{R}_i^*$ is the maximal controllability subspace contained in Ker C^i and C^i is the matrix obtained from C by removing its ith row-block. Consider $\mathscr{B}_i := \mathscr{B} \cap \mathscr{R}_i^*$, B_i such that $\operatorname{Im}(B_i) = \mathscr{B}_i$ and $A_i := A + BF_i$.

We define *the ith block essential structure of* (C, A, B) relatively to $\{p_i\}$, denoted $\Sigma_{ei}(C, A, B)$, as the structure at infinity of the system (C_i, A_i, B_i), i.e.:

$$\Sigma_{ei}(C, A, B) := \Sigma_{\infty}(C_i, A_i, B_i) \quad \text{for} \quad i = 1, \ldots, k$$

The *block essential structure of* (C, A, B) relatively to $\{p_i\}$, denoted $\Sigma_e(C, A, B)$ is defined by:

$$\Sigma_e(C, A, B) := \bigcup_{i=1}^{k} \Sigma_{ei}(C, A, B)$$

In what follows we will consider the system (C, A, B) of transfer matrix $T(s) = C(sI - A)^{-1} B$, partitioned according to $(p_1, \ldots, p_k)$, that means:

$$T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_k(s) \end{bmatrix}; \quad \text{with} \quad T_i(s) = C_i(sI - A)^{-1} B$$

where C_i denotes the ith row-block of matrix C.

In the whole paper it will be assumed that:

$$\operatorname{rank} T(s) = \sum_{i=1}^{k} \operatorname{rank} T_i(s) \tag{7}$$

The notation $T^i(s)$ will be used to denote the matrix obtained from $T(s)$ by removing its ith row-block, in other words:

$$T^i(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_{i-1}(s) \\ T_{i+1}(s) \\ \vdots \\ T_k(s) \end{bmatrix}$$

In order to give a first transfer matrix interpretation of the block essential structure of (C, A, B), let us introduce the following:

Definition 3. Let $\vartheta(s)$ be a k-dimensional rational subspace and $M(s)$ a polynomial matrix which columns are a minimal polynomial basis for $\vartheta(s)$. Denote h_i the ith column degrees of $M(s)$. $N(s) := M(s) \operatorname{diag}(s^{-h_1}, \ldots, s^{-h_k})$ will be called a minimal polynomial basis in s^{-1} for $\vartheta(s)$.

Remark that a minimal basis in s^{-1} is a right bicausal matrix.

Theorem 1. The ith block essential structure $\Sigma_{ei}(C, A, B)$ of the system (C, A, B). which transfer matrix is $T(s)$, relatively to $\{p_i\}$ satisfies:

$$\Sigma_{ei}(C, A, B) = \Sigma_{\infty}(T_i(s)\, V_i(s)) \quad \text{for} \quad i = 1, \ldots, k$$

where $T_i(s)$ is the ith row-block of $T(s)$ and $V_i(s)$ is any right bicausal matrix which is a basis for Ker $T^i(s)$, with $i = 1, \dots, k$.

Notice that there always exists such a basis for Ker $T^i(s)$, e.g. consider a minimal polynomial basis in s^{-1} for Ker $T^i(s)$.

Proof. Prove first that $\Sigma_\infty(T_i(s)\, V_i(s))$ is independent of the choice of $V_i(s)$. Let $V_i(s)$ and $W_i(s)$ be two right bicausal matrices which are basis matrices for Ker $T^i(s)$. $V_i(s) = W_i(s)\, Q_i(s)$, with $Q_i(s)$ a nonsingular rational matrix; since that $V_i(s)$ and $W_i(s)$ are full column rank at infinity, then $Q_i(s)$ is a bicausal matrix. It follows that: $\Sigma_\infty(T_i(s)\, V_i(s)) = \Sigma_\infty(T_i(s)\, W_i(s))$.

By definition:

$$\Sigma_{ei}(C, A, B) := \Sigma_\infty(C_i, A + BF_i, BG_i)$$

where F_i is any matrix such that $(A + BF_i)\, \mathscr{R}_i^* \subset \mathscr{R}_i^*$ and G_i is such that Im $BG_i = \mathscr{B} \cap \mathscr{R}_i^*$.

The transfer matrix of the system $(C_i, A + BF_i, BG_i)$, denoted $T_{F_i}(s)$ can be written as follows:

$$T_{F_i}(s) = T_i(s)\, V_i(s)$$

with:

$$V_i(s) = (I - F_i(sI - A)^{-1} B)^{-1} G_i \quad i = 1, \dots, k$$

$V_i(s)$ is clearly a right bicausal matrix, we will show now that $V_i(s)$ is a basis for the null space of $T^i(s)$. First, observe that $\mathscr{R}_i^*$ is contained in Ker C^i, for all $i \neq j$, this means that $T^j(s)\, V_i(s) = 0$ for all $i \neq j$. We will show now that rank $(V_i(s)) = \dim(\text{Ker } T^i(s))$, which will end the proof.

For this we have:

$$\operatorname{rank}(T^i(s)) = r - r_i = \dim\left(\frac{\mathscr{B}}{\mathscr{B} \cap \mathscr{R}_i^*}\right) = m - \dim(\mathscr{B} \cap \mathscr{R}_i^*)$$

then: $\dim(\mathscr{B} \cap \mathscr{R}_i^*) = m - r + r_i$.

From the definition of G_i and $V_i(s)$:

$$\dim(\mathscr{B} \cap \mathscr{R}_i^*) = \operatorname{rank} G_i = \operatorname{rank} V_i(s)$$

On the other hand, because of the independence between the blocks $T_i(s)$, $\dim(\text{Ker } T^i(s)) = m - r + r_i$. Then:

$$\operatorname{rank} V_i(s) = \dim(\text{Ker } T^i(s)) \quad \text{for} \quad i = 1, \dots, k$$

which ends the proof. □

Give now a technical result which will be used to provide a transfer matrix characterization of $\Sigma_{ei}(C, A, B)$.

Lemma 1. Let $T(s)$ be a $p \times m$ proper rational matrix partitioned in row-blocks relatively to $(p_1, \dots, p_k)$, such that $\sum_{i=1}^{k} p_i = p$. Let r_i denote the rank of $T_i(s)$, for

$i = 1, \ldots, k$. If $T(s)$ is such that:

$$\operatorname{rank} T(s) = \sum_{i=1}^{k} r_i$$

then $T(s)$ can be decomposed as:

$$T(s) = B(s)\, \tilde{I}\, \tilde{T}(s)$$

where $\tilde{T}(s)$ is an $r \times m$ full row rank proper rational matrix partitioned in row-blocks relatively to $\{r_i\}$. $B(s) = \operatorname{diag}(B_1(s), \ldots, B_k(s))$, $B_i(s)$ is a $p_i \times p_i$ bicausal matrix and $\tilde{I} = \operatorname{diag}(\tilde{I}_1, \ldots, \tilde{I}_k)$, $\tilde{I}_i = \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}$; I_{r_i} is the $r_i \times r_i$ identity matrix.

The proof is quite similar to that of Lemma 1 of [11]. It can be performed by replacing the Smith-McMillan factorization of $T_i(s)$ by a Smith-McMillan factorization at infinity of $T_i(s)$.

Now, we are able to give another interesting characterization of the block essential structure $\Sigma_{ei}(C, A, B)$, in the rational transfer matrix framework. In fact this is a non trivial generalization of [4] concerning the essential orders for the row by row case.

Theorem 2. Let (C, A, B) be the system (1) of transfer matrix $T(s)$, decomposed according to the output partition $(p_1, \ldots, p_k)$. Assume that $\operatorname{rank} T(s) = \sum_{i=1}^{k} r_i$, where $r_i = \operatorname{rank} T_i(s)$ and let $T(s)$ be decomposed as in Lemma 1, i.e.: $T(s) = B(s)\, \tilde{I}\, \tilde{T}(s)$.

Let $\tilde{T}(s)$ be factorized as follows:

$$\tilde{T}(s) = [R(s) \quad 0]\, B'(s)$$

where $R(s)$ is an $r \times r$ strictly proper rational matrix and $B'(s)$ is an $m \times m$ bicausal matrix. Consider $R^{-1}(s)$ decomposed in the following way:

$$R^{-1}(s) = [\bar{R}_1(s), \ldots, \bar{R}_k(s)] \; ; \quad \bar{R}_i(s) \in \mathbf{R}^{r \times r_i}(s)$$

Then the ith block essential structure of (C, A, B) relatively to $\{p_i\}$, $\Sigma_{ei}(C, A, B)$, satisfies:

$$\Sigma_{ei}(C, A, B) = -\Sigma_{\infty}(\bar{R}_i(s)) \quad \text{for} \quad i = 1, \ldots, k$$

Proof. Using Theorem 1, it is equivalent to prove:

$$\Sigma_{\infty}(T_i(s)\, V_i(s)) = -\Sigma_{\infty}(\bar{R}_i(s)) \quad \text{for} \quad i = 1, \ldots, k$$

where $V_i(s)$ is any right bicausal matrix which is a basis for Ker $T^i(s)$. In the proof we choose $V_i(s)$ a polynomial basis in s^{-1} for Ker $T^i(s)$. So, we begin by rewriting the ith row-block of $T(s)$ in Lemma 1 as follows:

$$T_i(s) = B_i(s) \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix} \tilde{T}_i(s)$$

where $B_i(s)$ is a $p_i \times p_i$ bicausal matrix. $\tilde{T}_i(s)$ is an $r_i \times m$ full row-rank proper rational matrix and where I_{r_i} is the $r_i \times r_i$ identity matrix. A possible solution is to consider the Smith-McMillan factorization at infinity of $T_i(s)$.

In addition, since that subspaces Ker $T^i(s)$ and Ker $\tilde{T}^i(s)$ are identical, and since $B_i(s)$ is bicausal we have:

$$\Sigma_\infty(T_i(s)\, V_i(s)) = \Sigma_\infty(\tilde{T}_i(s)\, V_i(s)) \tag{8}$$

Therefore it is sufficient to perform the proof for: $\tilde{T}(s) = \begin{bmatrix} \tilde{T}_1(s) \\ \vdots \\ \tilde{T}_k(s) \end{bmatrix}$ which is a full row-rank rational matrix, because rank $T(s) = r_1 + r_2 + \ldots + r_k$.

Let $R_i(s)$ denote the ith row-block of the matrix $R(s)$ partitioned respectively to $\{r_i\}$. Recall that $R^i(s)$ is the $(r - r_i) \times r$ matrix obtained by removing the ith row-block $R_i(s)$ from $R(s)$. From $V_i(s)$ we will first construct a polynomial basis in s^{-1}, $U_i(s)$, for Ker $R^i(s)$. Then we will prove that $\Sigma_\infty(T_i(s)\, V_i(s)) = \Sigma_\infty(R_i(s)\, U_i(s))$. From this, the result will follow, basically because the infinite structures $-\Sigma_\infty(R^{-1}(s))$, $\Sigma_\infty(R(s))$ and $\Sigma_\infty(T(s))$ are identical.

Define $W_i(s) = B'(s)\, V_i(s)$, which forms a basis of Ker $[R^i(s)\ 0]$. We can write:

$$\tilde{T}(s)\,[W_1(s), \ldots, W_k(s)] = \begin{bmatrix} R_1(s) & 0 \\ \vdots & \\ R_k(s) & 0 \end{bmatrix} [W_1(s), \ldots, W_k(s)] =$$
$$= \operatorname{diag}[D_1(s), \ldots, D_k(s)] \tag{9}$$

where $W_i(s)$ is an $m \times (m - r + r_i)$ proper rational matrix. Notice that, since $V_i(s)$ is a right bicausal matrix and $B'(s)$ is a bicausal matrix, $W_i(s)$ is also a right bicausal matrix.

From $W_i(s)$ we can obtain a basis, denoted $U_i(s)$, for Ker $R^i(s)$. This can be made by choosing the first r rows of $W_i(s)$, i.e., $W_i'(s) := [I_r\ 0]\, W_i(s)$, where I_r is the $r \times r$ identity matrix. Since dim (Ker $R^i(s)$) is r_i and since $W_i(s)$ possesses $m - r + r_i$ columns independent at infinity we can take r_i independent columns at infinity from $W_i'(s)$ to constitute basis $U_i(s)$. Therefore $W_i'(s)$ can be factorized as follows:

$$W_i'(s) = [U_i(s), Z_i(s)]\, P_i$$

where P_i is an $(m - r + r_i)$ permutation matrix. Because $U_i(s)$ is a basis of span $(W_i'(s))$, we have:

$$Z_i(s) = U_i(s)\, \Omega_i(s)$$

we can write: $Z_i(s) = [U_i(s), Y_i(s)] \begin{bmatrix} \Omega_i(s) \\ 0 \end{bmatrix}$ where $Y_i(s)$ is chosen in such a way that $[U_i(s), Y_i(s)]$ is a bicausal matrix. Since $Z_i(s)$ is proper then $\Omega_i(s)$ is also proper.

We can then construct a bicausal matrix $B_i'(s)$ such that:

$$[U_i(s), Z_i(s)] = [U_i(s), 0] \begin{bmatrix} I_{r_i} & \Omega_i(s) \\ 0 & I_{m-r} \end{bmatrix} = [U_i(s), 0]\, B_i'(s)$$

it follows that:

$$[I_r\ 0]\, W_i(s) = W_i'(s) = [U_i(s)\ 0]\, B_i''(s) \tag{10}$$

where $B_i''(s) = B_i'(s)\, P_i$ is bicausal. With this in mind, write the following expression:

$$\begin{bmatrix} R_1(s) \\ \vdots \\ R_k(s) \end{bmatrix} [U_1(s), \ldots, U_k(s)] = \operatorname{diag}[E_1(s), \ldots, E_k(s)] \tag{11}$$

where $E_i(s)$ is an $r_i \times r_i$ proper rational matrix. By construction $U_i(s)$ constitutes a right bicausal matrix.

Consider $R^{-1}(s)$ partitioned in column-blocks relatively to $\{r_i\}$ as follows:

$$R^{-1}(s) = [\bar{R}_1(s), \ldots, \bar{R}_k(s)]$$

so, we have:

$$[U_1(s), \ldots, U_k(s)] = [\bar{R}_1(s), \ldots, \bar{R}_k(s)] \operatorname{diag}[E_1(s), \ldots, E_k(s)]$$

Since $r = \sum_{i=1}^{k} r_i$, and $[U_1(s), \ldots, U_k(s)]$ is non singular, then $E_i(s)$ are non singular matrices. This allows us to rewrite the last expression in the following way:

$$[U_1(s), \ldots, U_k(s)] \operatorname{diag}[E_1^{-1}(s), \ldots, E_k^{-1}(s)] = [\bar{R}_1(s), \ldots, \bar{R}_k(s)]$$

then, for each block we have:

$$[U_i(s)\; X_i(s)] \begin{bmatrix} E_i^{-1}(s) \\ 0 \end{bmatrix} = \bar{R}_i(s)\,; \quad \text{for} \quad i = 1, \ldots, k$$

where $X_i(s)$ is any $r \times (r - r_i)$ rational matrix. Since $U_i(s)$ is a right bicausal matrix, we choose $X_i(s)$ in order to make $[U_i(s)\; X_i(s)]$ a bicausal matrix. Therefore the infinite structure of $[U_i(s)\; X_i(s)] \begin{bmatrix} E_i^{-1}(s) \\ 0 \end{bmatrix}$ and $E_i^{-1}(s)$ are equal.

From the latter, we have:

$$\Sigma_\infty(E_i(s)) = -\Sigma_\infty(\bar{R}_i(s))\,; \quad \text{for} \quad i = 1, \ldots, k$$

putting together expressions (9), (10) and (11), we obtain:

$$D_i(s)\;\; B_i''^{-1}(s) = [E_i(s)\;\; 0]\,; \quad \text{for} \quad i = 1, \ldots, k$$

Finally, since $B_i''(s)$ is bicausal we get:

$$\Sigma_\infty(D_i(s)) = -\Sigma_\infty(\bar{R}_i(s)) \quad \text{for} \quad i = 1, \ldots, k$$

which by (8) and (9) ends the proof. □

Remark. In the proof of the above theorem the choice $V_i(s)$ minimal basis in s^{-1} for Ker $T_i(s)$ is not actually necessary. Any right bicausal matrix being a basis for Ker $T^i(s)$ would do the job.

Corollary 1. With the above notations, the ith block essential structure $\Sigma_{ei}(C, A, B)$ is feedback invariant.

Proof. Immediate from Theorem 2 and the fact that any state feedback control action can always be represented by a bicausal precompensator [14]. □

In order to study the minimal McMillan degree achievable for the blocks of the decoupled system, let us recall the following "geometric" definition [11].

Definition 4. Consider the linear system (C, A, B), with B monic and C epic, and an output partition $(p_1, \ldots, p_k)$, such that $\sum_{i=1}^{k} p_i = p$. Let $\mathscr{V}^*$ (resp. $\mathscr{T}_i^*$) denote the largest (A, B)-invariant subspace in $\operatorname{Ker} C$ (resp. $\operatorname{Ker} C^i$). The *block decoupling invariants* of $T(s)$, denoted n_{ie}. are defined as follows:

$$n_{ie} = \dim(\mathscr{T}_i^*/\mathscr{V}^*) \quad \text{for} \quad i = 1, \ldots, k\,.$$

The above definition gives rise to the following transfer matrix characterizations of the block decoupling invariants n_{ie}.

Theorem 3. Let $T(s)$ be a $p \times m$ strictly proper rational matrix of null left and right static kernels decomposed in row-blocks according to a given partition $(p_1, \ldots, p_k)$ and (C, A, B) a realization of $T(s)$. Assume that rank $T(s) = \sum_{i=1}^{k} r_i$, where r_i denotes the rank of $T_i(s)$. Then the block decoupling invariants n_{ie} of $T(s)$ satisfy:

(i) $n_{ie} = \sum_{j=1}^{r_i} \mu_{ij} + \sigma(T_i(s))$
where $\{\mu_{i1}, \ldots, \mu_{ir_i}\}$ is the ith block essential structure $\Sigma_{ei}(C, A, B)$.

(ii) $n_{ie} = d_\infty(T(s)) - d_\infty(T^i(s)) + \sigma(T_i(s))$
where $T_i(s)$ denotes the ith row-block of $T(s)$, $T^i(s)$ denotes the matrix obtained from $T(s)$ by removing the ith row-block and $\sigma(T_i(s))$ denotes the sum of the degrees of a minimal polynomial basis for the left kernel of $T_i(s)$.

Roughly speaking the idea of the proof of (ii) is as follows. Let $\mathscr{X}$ be the state space of (C, A, B), making invariant $\mathscr{V}^*$, the state space dimension of the reduced system is $\dim(\mathscr{X}/\mathscr{V}^*) = d_\infty(T(s)) + \sigma(T(s))$. This comes from the well known equivalences between Morse's list I_3 and the infinite structure of (C, A, B) and between Morse's list I_4 and $\sigma(T(s))$ (see [15], [18] and [8]). Analogously, making invariant $\mathscr{T}_i^*$, we obtain $\dim(\mathscr{X}/\mathscr{T}_i^*) = d_\infty(T^i(s)) + \sigma(T^i(s))$. The proof is finished noting that $\sigma(T_i(s)) = \sigma(T(s)) - \sigma(T^i(s))$, because of the row block independence hypothesis, i.e., rank $T(s) = r_1 + r_2 + \ldots + r_k$.

In [11], it is shown that $d_\infty(T(s)) - d_\infty(T^i(s)) = d_\infty(\bar{R}_i(s))$, which implies (i), using Theorem 2 of this paper.

4.2. Application to the Block Decoupling Problem

We are now ready to give our results above the minimal infinite structure and the minimal McMillan degree achievable for the blocks of a decoupled system.

Theorem 4. Let $T(s)$ be a $p \times m$ strictly proper rational matrix of null left and right static kernels, decomposed in row-blocks according to a given partition

$(p_1, \ldots, p_k)$ and (C, A, B) a realization of $T(s)$. If the system $T(s)$ can be decoupled relatively to $(p_1, \ldots, p_k)$ by an admissible decoupling precompensator, then we have the following:

(i) The minimal McMillan degree achievable for the ith block of the decoupled system is given by the block decoupling invariant n_{ie}.

(ii) The minimal infinite structure of the ith block of the decoupled system is given by the ith block essential structure $\Sigma_{ei}(T(s))$.

Notice that when rank $T(s) = \sum_{i=1}^{k} r_i$, where r_i denotes the rank of $T_i(s)$, block decoupling via precompensation is always possible, [14].

Proof. (i) was proved in [11].
To prove (ii), consider $T(s)$ decomposed as in Lemma 1:

$$T(s) = B(s)\, \tilde{I}\, \tilde{T}(s)$$

and let $T(s)$ be factorized as:

$$\tilde{T}(s) = [R(s)\;\; 0]\, B'(s)$$

where $R(s)$ is an $r \times r$ full rank strictly proper rational matrix and $B'(s)$ is an $m \times m$ bicausal matrix.

Now partition $R^{-1}(s)$ as follows:

$$R^{-1}(s) = [\bar{R}_1(s), \ldots, \bar{R}_k(s)]\;; \quad \bar{R}_i(s) \in R^{r \times r_i}(s)$$

Consider now a Smith-McMillan factorization of $\bar{R}_i(s)$ written as:

$$\bar{R}_i(s) = B_{i1}(s) \begin{bmatrix} \Delta_i(s) \\ 0 \end{bmatrix} B_{i2}(s) \tag{12}$$

$$\Delta_i(s) = \operatorname{diag}(\mu_{i1}, \ldots, \mu_{ir_i}) \,.$$

Let us consider the following compensator $C(s)$:

$$C(s) := B'^{-1}(s) \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} \tag{13}$$

where $Y(s) := R^{-1}(s) \operatorname{diag}(B_{12}^{-1}(s)\, \Delta_1^{-1}(s), \ldots, B_{k2}^{-1}(s)\, \Delta_k^{-1}(s))$ and $X(s)$ any $(m - r) \times r$ proper rational matrix. Clearly $C(s)$ is an admissible precompensator.

The decoupled system is:

$$T(s)\, C(s) = B(s)\, \tilde{I} \operatorname{diag}(B_{12}^{-1}(s)\, \Delta_1^{-1}(s), \ldots, B_{k2}^{-1}(s)\, \Delta_k^{-1}(s))$$

where the infinite zero structure of the ith block is equal to the infinite pole structure of $\bar{R}_i(s)$ which is, from Theorem 2, $\Sigma_{ei}(T(s)) = \{\mu_{i1}, \ldots, \mu_{ir_i}\}$.

In [11] it is shown that the least value for the sum of the infinite zero orders for the ith block of the decoupled system is $n_{ie} - \sigma(T_i(s))$. Then, from (i) of Theorem 3 the result follows. □

Comments. It is proved in [2], that when the number of inputs is sufficiently large the system is decouplable by dynamic state feedback.

In this case the minimal infinite structure achievable when decoupling is possible is $\Sigma_e(C, A, B)$.

In the same vein it is possible to take into account stability requirements, the minimal McMillan degree achievable for the blocks of the decoupled system (with stability) is given in [3]. Some results generalizing those of this paper incorporating stability requirements are given without proofs in [10].

5. AN ILLUSTRATING EXAMPLE

Consider the linear system (C, A, B):

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} x$$

which transfer matrix is:

$$T(s) = \begin{bmatrix} s^{-1} & 0 & 0 & s^{-2} \\ 0 & s^{-1} & 0 & 0 \\ s^{-1} & s^{-1} & s^{-2} & s^{-2} \end{bmatrix}$$

We will show later that this system is not row by row decouplable, but it is possible to decouple $T(s)$ in blocks according to the output partition $(2, 1)$.

In this way, we will first compute the block essential structure of (C, A, B) relatively to $(2, 1)$ to illustrate, in a second time, the transfer matrix characterizations of $\Sigma_{ei}(C, A, B)$ given in this work. We will finish the example by computing a block decoupling precompensator which will exhibit the minimal infinite structure achievable for the block decoupled system.

Begin by decomposing the matrix C of (C, A, B) according to the partition $(2, 1)$ as follows:

$$C^1 = [1\ 1\ 1\ 0\ 0]\,; \quad C^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The maximal controllability subspaces $\mathscr{R}_i^*$ contained in Ker C^i, for $i = 1, 2$, are respectively:

$$\mathscr{R}_1^* = \text{span}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathscr{R}_2^* = \text{span}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By definition the ith block essential structure of (C, A, B) is equal to the infinite structure of the subsystem $(C_i, A + BF_i, B_i)$, where:

F_i is such that: $(A + BF_i)\,\mathscr{R}_i^* \subset \mathscr{R}_i^*$

G_i is such that: $\operatorname{Im} BG_i = \mathscr{B} \cap \mathscr{R}_i^*$

and B_i such that $\operatorname{Im}(B_i) = \mathscr{B}_i$, for $i = 1, 2$. In this case we have that:

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfy the required properties. Then computing the infinite structures of systems $(C_i, A + BF_i, B_i)$ we get the following ith block essential structure of (C, A, B) relatively to (2,1):

$$\Sigma_{e1}(C, A, B) = \Sigma_\infty(C_1, A + BF_1, BG_1) = \{1, 2\}$$

$$\Sigma_{e2}(C, A, B) = \Sigma_\infty(C_2, A + BF_2, BG_2) = \{2\}$$

which means that $(C_1, A + BF_1, BG_1)$ has two infinite zeros of orders $\{1, 2\}$ and $(C_2, A + BF_2, BG_2)$ has one infinite zero of order $\{2\}$.

Let us give now the first matrix characterization of $\Sigma_{ei}(C, A, B)$. For this, compute as in the proof of Theorem 1, the following matrices:

$$V_i(s) = (I - F_i(sI - A)^{-1} B)^{-1} G_i \quad \text{for} \quad i = 1, 2,$$

giving in this case:

$$V_1(s) = \begin{bmatrix} 1 & 0 & -s^{-1} \\ -1 & -s^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad V_2(s) = \begin{bmatrix} 0 & -s^{-1} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a simple calculation shows that:

$$T_1(s)\,V_1(s) = \begin{bmatrix} s^{-1} & 0 & 0 \\ -s^{-1} & -s^{-2} & 0 \end{bmatrix}; \quad T_2(s)\,V_1(s) = 0$$

$$T_2(s)\,V_2(s) = [s^{-2}\ 0]\,; \quad T_1(s)\,V_2(s) = 0$$

where clearly $V_i(s)$ is a right bicausal basis for Ker $T^i(s)$ and then:

$$\Sigma_{e1}(C, A, B) = \Sigma_\infty(T_1(s)\,V_1(s)) = \{1, 2\}$$

$$\Sigma_{e2}(C, A, B) = \Sigma_\infty(T_2(s)\,V_2(s)) = \{2\}$$

and stated in Theorem 1.

For illustrating Theorem 2, let us now factorize $T(s)$ as follows:

$$T(s) = [R(s)\ 0]\,B(s)$$

where $B(s)$ is a bicausal matrix and $R(s)$ is equal to:

$$R(s) = \begin{bmatrix} s^{-1} & 0 & 0 \\ 0 & s^{-1} & 0 \\ s^{-1} & s^{-1} & s^{-2} \end{bmatrix}; \quad \text{with} \quad R^{-1}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ -s^2 & -s^2 & s^2 \end{bmatrix}$$

and let $R^{-1}(s) = [\bar{R}_1(s), \bar{R}_2(s)]$ be partioned in column-blocks according to (2, 1). $\bar{R}_1(s)$ has two infinite poles of orders $\{1, 2\}$ and $\bar{R}_2(s)$ has one infinite pole of order $\{2\}$, this confirms us that:

$$\Sigma_{e1}(C, A, B) = \Sigma_\infty(\bar{R}_1(s)) = \{1, 2\} \quad \Sigma_{e2}(C, A, B) = \Sigma_\infty(\bar{R}_2(s)) = \{2\}$$

as stated in Theorem 2.

Compensator Construction

Below we will construct a block decoupling precompensator $C(s)$ which will exhibit the minimal infinite structure achievable for the decoupled system. In fact $T(s)$ is not row by row decouplable, since condition $m \geqq 2p - k$ does not hold [9], where $m = 4$ is the number of inputs, $p = 3$ is the number of outputs and $k = 1$ is the column rank at infinity of $R^{-1}(s)$. However it is block decouplable according to partition (2, 1), since condition $m \geqq 2p - k^*$ is satisfied, where $k^* = 2$ is the maximal column rank at infinity that can be obtained from the column blocks of $R^{-1}(s)$ [2].

To illustrate the compensator construction of Theorem 3, let us consider the following Smith-McMillan factorizations at infinity of $\bar{R}_1(s)$ and of $\bar{R}_2(s)$.

$$\bar{R}_1(s) = B_1(s)\begin{bmatrix} \Delta_1(s) \\ 0 \end{bmatrix} B_1'(s) = \begin{bmatrix} s^{-1} & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\bar{R}_2(s) = B_2(s)\begin{bmatrix} \Delta_2(s) \\ 0 \end{bmatrix} B_2'(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, from (13) given in the proof of Theorem 4 we have that a block decoupling precompensator is given by:

$$C(s) = B^{-1}(s)\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix}$$

$$Y(s) = R^{-1}(s)\,\mathrm{diag}\,(B_1^{-1}(s)\,\Delta_1^{-1}(s),\ B_2^{-1}(s)\,\Delta_2^{-1}(s))$$

and $X(s)$ is any $(m - r) \times r$ rational matrix. In this case we obtain:

$$Y(s) = \begin{bmatrix} s^{-1} & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \quad X(s) = [0\ 0\ 1]$$

yielding a compensator $C(s)$:

$$C(s) = B^{-1}(s)\begin{bmatrix} s^{-1} & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The decoupled system is finally:

$$T(s)\,C(s) = \begin{bmatrix} s^{-2} & -s^{-1} & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & s^{-2} \end{bmatrix} = \mathrm{diag}\,(D_1(s),\ D_2(s))$$

As a matter of fact we have:

$$\Sigma_\infty(D_1(s)) = \{1, 2\}\ ; \quad \Sigma_\infty(D_2(s)) = \{2\}$$

or $D_1(s)$ has two infinite zeros of orders $\{1, 2\}$ and $D_2(s)$ has an infinite zero of order $\{2\}$, which verifies:

$$\Sigma_{ei}(T(s)) = \Sigma_\infty(D_i(s)) \quad \text{for} \quad i = 1, 2\,.$$

Then $D_1(s)$ and $D_2(s)$ possess the minimal infinite structure achievable for the decoupled system.

To finish let us remark that the system (C, A, B) is block decouplable by non regular static state feedback. Actually we can verify that the following static state feedback:

$$F = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad G = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is equivalent to the precompensator $C(s)$ given above. In fact:

$$(I - F(sI - A)^{-1}\,B)^{-1}\,G = \begin{bmatrix} 1 & 0 & -s^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} G = C(s)$$

Then the block decoupling by non regular static state feedback (F, G) provides us with the minimal infinite structure.

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