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PARTIAL DECOUPLING OF NON-MINIMUM PHASE SYSTEMS BY CONSTANT STATE FEEDBACK*

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The decoupling of the input-output behaviour of linear multivariable systems generally requires the compensation of all invariant zeros, which causes unstability in the case of non-minimum phase systems. The paper presents a method for *partial and stable decoupling* with only *one* output affected by *several* inputs. All transmission-poles can be chosen arbitrarily.

1. PROBLEM STATEMENT

Consider an nth order linear time-invariant multivariable system

$$\dot{x}(t) = \boldsymbol{A}\,x(t) + \boldsymbol{B}\,\boldsymbol{u}(t)\,, \quad y(t) = \boldsymbol{C}\,x(t)\,, \tag{1.1}$$

with the $(n, 1)$-state vector $x(t)$, the $(m, 1)$-input vector $\boldsymbol{u}(t)$, the $(m, 1)$-output vector $y(t)$ and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ of conformal dimensions. *Input-output decoupling* is achieved if one can find a constant (m, n)-controller matrix $\boldsymbol{R}$ and a constant (m, m)-prefilter $\boldsymbol{F}$ such that via the state feedback law

$$\boldsymbol{u}(t) = -\boldsymbol{R}\,x(t) + \boldsymbol{F}\,w(t) \tag{1.2}$$

every output y_i, $i = 1, \ldots, m$, is only affected by the corresponding w_i. Hence the transfer-function matrix

$$\boldsymbol{G}_w(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{R})^{-1}\,\boldsymbol{B}\boldsymbol{F} \tag{1.3}$$

of the closed-loop system must be *diagonal*, i.e.

$$\boldsymbol{G}_w(s) = \mathrm{diag}\,[g_{11}(s), \ldots, g_{mm}(s)]\,. \tag{1.4}$$

Falb and Wolovich [2] first gave a solution to this problem, Roppenecker and Lohmann [7] achieved decoupling by the design method of "Complete Modal Synthesis". Systems with *invariant zeros* [5] in the right half of the complex plane cannot be *stabilized and decoupled* by these methods. For this class of *non-minimum phase*

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systems the approach presented in the following sections leads to a *partial and stable decoupling* of the form

$$G_w(s) = \begin{bmatrix} g_{11}(s) & & & & 0 \\ & \ddots & & & \\ g_{j1}(s) & \dots & g_{jj}(s) & \dots & g_{jm}(s) \\ & & & \ddots & \\ & & 0 & & g_{mm}(s) \end{bmatrix} . \tag{1.5}$$

With the transfer-function matrix (1.5) the partial decoupling is an advantage compared to the triangular of block decoupling [4, 9], where a greater or equal number of elements of $G_w(s)$ are non-zero.

2. COMPLETE DECOUPLING AND FUNDAMENTALS

The design method of *Complete Modal Synthesis* by Roppenecker [6] is based on the fact that every state-feedback controller R is related to a set of closed-loop eigenvalues λ_μ and *invariant parameter vectors* p_μ by the equation

$$R = [p_1, \dots, p_n] \cdot [v_1, \dots, v_n]^{-1} , \quad \text{where} \tag{2.1}$$

$$v_\mu = (A - \lambda_\mu I)^{-1} B p_\mu , \quad \mu = 1, \dots, n . \tag{2.2}$$

In order to determine the free parameters λ_μ and p_μ such that a diagonal closed-loop transfer-function matrix is achieved, we first apply the modal transformation

$$(A - BR) = V \Lambda V^{-1} , \tag{2.3}$$

with $V = [v_1, \dots, v_n]$ as the matrix of closed-loop eigenvectors and Λ as the diagonal matrix of closed loop eigenvalues, to eq. (1.3) resulting in

$$G_w(s) = CV(sI - \Lambda)^{-1} V^{-1} BF = \sum_{\mu=1}^{n} \frac{C v_\mu w_\mu^{\mathrm{T}} BF}{s - \lambda_\mu} . \tag{2.4}$$

The transposed vectors w_μ^{T} are the rows of V^{-1}. Now the elements $g_{ii}(s)$ in the desired diagonal transfer-function matrix (1.4) are set up as

$$g_{ii}(s) = \frac{\prod_{k=1}^{\delta_i} (-\lambda_{ik})}{(s - \lambda_{i1}) \cdot \ldots \cdot (s - \lambda_{i\delta})} , \quad i = 1, \dots, m . \tag{2.6}$$

The degree δ_i of the denominator is called the *difference order* of the output y_i and is defined as

$$\delta_i = \min_{\mu = 1, \dots, m} \delta_{i\mu} \tag{2.6}$$

where $\delta_{i\mu}$ is the difference between the degrees of denominator and numerator of the element $g_{i\mu}$ of $G_w(s)$ from eq. (1.3) calculated with arbitrary R and arbitrary, regular F. The difference orders δ_i are invariant under the feedback law (1.2), hence the numerators in eq. (2.5) must be set up as constants. In the special form of (2.5) they avoid steady state error.

Comparing eq. (2.4) to eqns. (1.4) and (2.5) we get conditions that must be satisfied by the eigenvectors: If the eigenvalue λ_{ik} appears only in the ith element $g_{ii}(s)$ of $\boldsymbol{G}_w(s)$, then the corresponding closed loop eigenvector must satisfy

$$\boldsymbol{C}\boldsymbol{v}_{ik} = \boldsymbol{e}_i\,, \quad i = 1, \ldots, m\,, \quad k = 1, \ldots, \delta_i \tag{2.7}$$

(the indices of the $\boldsymbol{v}$ are adapted to those of the eigenvalues λ_{ik}). $\boldsymbol{e}_i$ denotes the ith unit-vector. Eq. (2.7) guarantees the strict connection of every eigenvalue λ_{ik} to *one row* of $\boldsymbol{G}_w(s)$ since every dyadic product $\boldsymbol{C}\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{F}$ in eq. (2.4) is an (m, m)-matrix in which only the ith row is unequal to $\boldsymbol{0}^{\mathrm{T}}$. Combining eqns. (2.7) and (2.2) to

$$\begin{bmatrix} \boldsymbol{A} - \lambda_{ik}\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_{ik} \\ -\boldsymbol{p}_{ik} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{e}_i \end{bmatrix}, \quad \begin{aligned} i &= 1, \ldots, m \\ k &= 1, \ldots, \delta_i \end{aligned} \tag{2.8}$$

we can calculate the vectors $\boldsymbol{v}_{ik}$ and $\boldsymbol{p}_{ik}$ if the eigenvalues λ_{ik} are prescribed.

By eq. (2.8) only the

$$\delta = \delta_1 + \delta_2 + \ldots + \delta_m \tag{2.9}$$

poles of the elements of $\boldsymbol{G}_w(s)$ are transformed to conditions on the closed loop eigenvectors. For the remaining $n - \delta$ eigenvalues (it is $n - \delta \geqq 0$, see [2] or [7]), which do *not* appear in $\boldsymbol{G}_w(s)$, it again follows from eq. (2.4) that

$$\boldsymbol{C}\boldsymbol{v}_\nu \boldsymbol{w}_\nu^{\mathrm{T}}\boldsymbol{B}\boldsymbol{F} = \boldsymbol{0}\,, \quad \nu = \delta + 1, \ldots, n\,. \tag{2.10}$$

Assuming controllability of the system[1], i.e. $\boldsymbol{w}_\nu^{\mathrm{T}}\boldsymbol{B}\boldsymbol{F} \neq \boldsymbol{0}^{\mathrm{T}}$, $\nu = 1, \ldots, n$, eq. (2.10) can only be satisfied if

$$\boldsymbol{C}\boldsymbol{v}_\nu = \boldsymbol{0}\,, \quad \nu = \delta + 1, \ldots, n\,. \tag{2.11}$$

Together with eq. (2.2) we get

$$\begin{bmatrix} \boldsymbol{A} - \lambda_\nu\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_\nu \\ -\boldsymbol{p}_\nu \end{bmatrix} = \boldsymbol{0}\,. \tag{2.12}$$

Non-null solutions $\boldsymbol{v}_\nu$, $\boldsymbol{p}_\nu$ of this equation exist if the eigenvalues λ_ν are chosen such that

$$\det \begin{bmatrix} \boldsymbol{A} - \lambda\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = 0\,. \tag{2.13}$$

Since the solutions λ of eq. (2.13) just define the *invariant zeros* of the system [5] we can state: Eq. (2.12) is solvable if the λ_ν are chosen equal to the invariant zeros of the system, whereas eq. (2.8) is solvable for any other choice of λ_{ik}.

We can now summarize the *steps of calculation of the controller matrix* $\boldsymbol{R}$: To every pole of the elements of the desired $\boldsymbol{G}_w(s)$ (eqns. (1.4), (2.5)) corresponding vectors $\boldsymbol{v}_{ik}$, $\boldsymbol{p}_{ik}$ are determined via eq. (2.8) which is solvable if all δ poles are chosen

[1] This assumption can be dropped without eqns. (2.8) and (2.12) loosing their sufficiency for decoupling.

unequal to the invariant zeros of the system. The remaining $n - \delta$ eigenvalues are chosen equal to the invariant zeros which ensures solvability of eq. (2.12). Necessarily the system must have *at least* $n - \delta$ zeros. The so found n pairs of vectors $\boldsymbol{v}$ and $\boldsymbol{p}$ determine the controller matrix $\boldsymbol{R}$ via eq. (2.1). It can be shown that the required inverse in eq. (2.1) exists if the system has *not more* than $n - \delta$ invariant zeros.

How is the precompensatoe $\boldsymbol{F}$ to be chosen? In the desired transfer functions $g_{ii}(s)$ of eq. (2.5) the numerators avoid steady state error; hence the precompensator must satisfy the well-known relation

$$\boldsymbol{F} = \lim_{s \to 0} [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{BR})^{-1} \boldsymbol{B}]^{-1} . \tag{2.14}$$

Actually, this choice of $\boldsymbol{F}$ ensures decoupling (for *all* s) as the following consideration shows: With $\boldsymbol{F}$ of eq. (2.14) all non-diagonal elements $g_{ik}(s)$, $i \neq k$, of $\boldsymbol{G}_w(s)$ disappear at $s = 0$. The controller $\boldsymbol{R}$ guarantees by eqns. (2.7), (2.11) δ_i poles in the elements $g_{ik}(s)$, $i = 1, \ldots, m$. Because the δ_i don't change under feedback, the numerators of the $g_{ik}(s)$ are of degree zero, i.e. they don't depend on s. Hence, to avoid steady state error, these numerators disappear and thus $g_{ik}(s) \equiv 0$, which ensures the desired decoupling.

Necessary condition for decouplability: The number of invariant zeros must be $n - \delta$. (2.15)

The equivalence of this condition to that one given by Falb, Wolovich [2] is shown in [7].

3. PARTIAL DECOUPLING

The decoupling of non-minimum phase systems results in *unstable* closed-loop eigenvalues since eq. (2.12) requires the choice of the λ_v *equal* to invariant zeros. Generally stability can only be achieved if these compensations and also the complete decoupling are renounced. The design steps for a *partially decoupling, but stabilizing controller* are derived in the following.

The diagonal elements $i \neq j$ of the desired $\boldsymbol{G}_w(s)$ in eq. (1.5) are chosen as in eq. (2.5), the functions $g_{j1}(s), \ldots, g_{jm}(s)$ are left undefined for the present. Comparing eq. (2.4) to eq. (1.5) we find for the poles of the $g_{ii}(s)$, $i \neq j$ (in correspondence to eq. (2.7)) the relations

$$\boldsymbol{Cv}_{ik} = \boldsymbol{e}_i + a_{ik}\boldsymbol{e}_j , \quad \begin{matrix} i = 1, \ldots, m , & i \neq j , \\ k = 1, \ldots, \delta_i . & \end{matrix} \tag{3.1}$$

The free parameters a_{ik} are introduced since the appearance of the poles λ_{ik}, $i \neq j$, in elements of the jth row of $\boldsymbol{G}_w(s)$ must be allowed explicitly. Together with eq. (2.2) we thus get for the vectors $\boldsymbol{v}_{ik}$ and $\boldsymbol{p}_{ik}$ the condition

$$\begin{bmatrix} \boldsymbol{A} - \lambda_{ik}\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_{ik} \\ -\boldsymbol{p}_{ik} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{e}_i \end{bmatrix} + a_{ik} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{e}_j \end{bmatrix} \quad \begin{matrix} i = 1, \ldots, m , & i \neq j , \\ k = 1, \ldots, \delta_i . & \end{matrix} \tag{3.2}$$

Real eigenvalues λ_{ik} demand real a_{ik}, self-conjugate λ_{ik} demand the choice of self conjugate a_{ik}. Supposing the system to have one "unstable" zero we may only compensate the remaining $n - \delta - 1$ "stable" zeros by satisfying

$$\begin{bmatrix} A - \lambda_\nu I & B \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} v_\nu \\ -p_\nu \end{bmatrix} = 0 \,, \quad \nu = \delta + 2, \ldots, n \,, \tag{3.3}$$

where the λ_ν have to be chosen equal to these $n - \delta - 1$ zeros. By eqns. (3.2) and (3.3)

$$\delta_1 + \ldots + \delta_{j-1} + \delta_{j+1} + \ldots + \delta_m + n - \delta - 1 = n - \delta_j - 1 \tag{3.4}$$

vectors $\boldsymbol{v}$ and $\boldsymbol{p}$ are determined. The remaining $\delta_j + 1$ pairs of vectors $\boldsymbol{v}$, $\boldsymbol{p}$ must satisfy the relation

$$C v_{jk} = e_j \tag{3.5}$$

since all poles in rows $i \neq j$ of $G_w(s)$ are already considered by eq. (3.2). With eq. (2.2) this yields

$$\begin{bmatrix} A - \lambda_{jk} I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} v_{jk} \\ -p_{jk} \end{bmatrix} = \begin{bmatrix} 0 \\ e_j \end{bmatrix}, \quad k = 1, \ldots, \delta_j + 1 \,, \tag{3.6}$$

where the λ_{jk} can be chosen arbitrarily but unequal to all invariant zeros. With the solutions $\boldsymbol{v}$ and $\boldsymbol{p}$ of the n linear eqns. (3.2), (3.3), (3.6) the controller matrix $\boldsymbol{R}$ can be calculated from eq. (2.1). The precompensator $\boldsymbol{F}$ (eq. 2.14) guarantees the desired $G_w(s)$ since the proof of decoupling of the rows $i \neq j$ of $G_w(s)$ is possible in the same way as in the last section. For the existence of $\boldsymbol{R}$ the system must be stabilizable, i.e. must not have uncontrollable eigenvalues in the right half of the complex plane. $\boldsymbol{F}$ exists if there is no invariant zero equal to zero. Both conditions are satisfied by all systems appropriate to be controlled. Furtheron note the following condition when choosing $G_w(s)$.

4. CHOICE OF THE COUPLED CHANNEL

The choice of the coupled row j of $G_w(s)$ in eq. (1.5) to allow partial decoupling of a system with the "unstable" zero η is restricted:

Coupling can be prescribed optionally in one of the rows $j \in [1, \ldots, m]$ that satisfies

$$q^{\mathrm{T}} \cdot e_j \neq 0 \,, \tag{4.1}$$

where the vector q^{T} is defined via the solution of

$$[r^{\mathrm{T}}, q^{\mathrm{T}}] \cdot \begin{bmatrix} A - \eta I & B \\ C & 0 \end{bmatrix} = 0^{\mathrm{T}} \,. \tag{4.2}$$

It is $q^{\mathrm{T}} \neq 0^{\mathrm{T}}$ since the matrix in eq. (4.2) is singular (see eq. (2.13)) and the block $[A - \eta I, B]$ is, stabilizable systems assumed, of full rank[2]. Hence there always

[2] From the controllability criterion of Hautus follows: rank $[A - \eta I, B] = n$, see [3].

exists at least *one* $j \in [1, \ldots, m]$ satisfying eq. (4.1) and thus allowing partial decoupling. In order to proof the *necessity* of condition (4.1) we first multiple eq. (4.2) with a regular matrix:

$$[\boldsymbol{r}^T, \boldsymbol{q}^T] \begin{bmatrix} \boldsymbol{A} - \eta \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{R} & \boldsymbol{F} \end{bmatrix} = [\boldsymbol{r}^T, \boldsymbol{q}^T] \cdot \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BR} - \eta \boldsymbol{I} & \boldsymbol{BF} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}^T . \quad (4.3)$$

Again multiplying a suitable matrix we find an equation containing the transfer-function matrix $\boldsymbol{G}_w(\eta)$:

$$[\boldsymbol{r}^T, \boldsymbol{q}^T] \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BR} - \eta \boldsymbol{I} & \boldsymbol{BF} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{A} - \boldsymbol{BR} - \eta \boldsymbol{I})^{-1} \boldsymbol{BF} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} =$$

$$= [\boldsymbol{r}^T, \boldsymbol{q}^T] \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BR} - \eta \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{G}_w(\eta) \end{bmatrix} = \boldsymbol{0}^T . \quad (4.4)$$

The inverse $(\boldsymbol{A} - \boldsymbol{BR} - \eta \boldsymbol{I})^{-1}$ exists, since η is not a closed-loop eigenvalue. From eq. (4.4) we have

$$\boldsymbol{q}^T \boldsymbol{G}_w(\eta) = \boldsymbol{0}^T . \quad (4.5)$$

which must be satisfied for *all obtainable* $\boldsymbol{G}_w(s)$. Putting the *desired* $\boldsymbol{G}_w(s)$ (eq. (1.5)) in eq. (4.5) and denoting the elements of $\boldsymbol{q}^T$ by $q_1, \ldots, q_m$, we can write

$$\boldsymbol{q}^T \boldsymbol{G}_w(\eta) =$$

$$= [q_1 \, g_{11}(\eta) + q_j \, g_{j1}(\eta), \ldots, q_j \, g_{jj}(\eta), \ldots, q_m \, g_{mm}(\eta) + q_j \, g_{jm}(\eta)] = \boldsymbol{0}^T . \quad (4.6)$$

Suppose now $\boldsymbol{G}_w(s)$ to injure condition (4.1), i.e. $q_j = 0$. Then, with a (always existing) $q_i \neq 0$, the ith element of $\boldsymbol{q}^T \boldsymbol{G}_w(\eta)$ reads $q_i \, g_{ii}(\eta)$, an expression which can never equal zero since $g_{ii}(\eta) \neq 0$, $i = 1, \ldots, m$, $i \neq j$ from eq. (2.5). Hence, with $q_j = 0$ and the transfer-function matrix (1.5), eq. (4.6) cannot be satisfied. Therefore, partially decoupling matrices $\boldsymbol{R}$ and $\boldsymbol{F}$ can only exist if the coupled channel in the desired $\boldsymbol{G}_w(s)$ is chosen such that $\boldsymbol{q}^T \boldsymbol{e}_j \neq 0$. The proof of the sufficiency of condition (4.1) can be found in [10].

The structure of the jth diagonal element, which has not been specified yet, can be derived from (4.6): We immediately find the relation $g_{jj}(\eta) = 0$ i.e. the invariant zero η appears in the partially decoupled matrix $\boldsymbol{G}_w(s)$ as zero of the jth diagonal element. This element reads

$$g_{jj}(s) = \frac{(s - \eta)}{(s - \lambda_{j1}) \cdot \ldots \cdot (s - \lambda_{j\delta_j + 1})} \frac{\prod_{\nu=1}^{\delta_j + 1} (-\lambda_{j\nu})}{(-\eta)} . \quad (4.7)$$

5. CHOICE OF THE REMAINING DESIGN PARAMETERS

Not only the arbitrarily prespecified eigenvalues but also the parameters a_{ik} determine the controller (see eq. (3.2)). A suitable choice of these a_{ik} demands a relation describing the influence of the a_{ik} on the elements $g_{ji}(s)$, $i \neq j$ of $\boldsymbol{G}_w(s)$.

If we assume $a_{ik} = 0$ the non-diagonal elements of $G_w(s)$ are

$$g_{ji}(s) = \frac{s \cdot f_{ji}}{(s - \lambda_{j1}) \cdot \ldots \cdot (s - \lambda_{j\delta_j+1})}, \quad \begin{array}{l} i = 1, \ldots, m, \\ i \neq j. \end{array} \tag{5.1}$$

In this case only the poles $\lambda_{j1}, \ldots, \lambda_{j\delta_j+1}$ appear, since all other eigenvalues are strictly connected to their rows $i \neq j$ (see Section 3). Because of the difference order δ_j the degree of the numerator is equal to one, hence f_{ji} is not depending on s. Evaluating eq. (4.6) element by element with regard to eq. (5.1) we find

$$f_{ji} = -\frac{1}{\eta}\frac{q_i}{q_j}\, g_{ii}(\eta) \cdot (\eta - \lambda_{j1}) \cdot \ldots \cdot (\eta - \lambda_{j\delta_j+1}) \tag{5.2}$$

and obviously

$$g_{ji}(s) \equiv 0 \quad \text{if} \quad q_i = 0\,. \tag{5.3}$$

In words: if an element q_i of q^T equals zero, $g_{ji}(s) \equiv 0$ can be achieved by choosing $a_{ik} = 0$, $k = 1, \ldots, \delta_i$. If q^T contains only *one* element unequal to zero, *complete decoupling* can be achieved by choosing $a_{ik} = 0$, $i = 1, \ldots, m$, $i \neq j$, $k = 1, \ldots, \delta_i$. In this *special case* the transfer-function matrix (1.4) for *complete* decoupling can be modified: the element $g_{jj}(s)$ is now set up in the form of eq. (4.7) and the design described in Section 3 guarantees complete decoupling. Figuratively speaking the influence of η is restricted to the output y_j ("non-interconnecting zero", [4]) and allows complete decoupling. With this result we can now formulate a

Necessary and sufficient condition for complete and stable decouplability: Every vector q^T (from 4.2) belonging to an invariant zero in the right half of the complex plane must not contain more than one non-zero element. Furtheron condition (2.15) must be satisfied. (5.4)

The criterion is equivalent to that one given by Cremer in [1].

But back to the general, partially decoupled case with $a_{ik} \neq 0$, where the formula for the non-diagonal elements of $G_w(s)$ reads (proof in [10]):

$$g_{ji}(s, a_{ik}) = g_{ji}(s) + h_{ji}(s, a_{ik}) \tag{5.5}$$

with $g_{ji}(s)$ from (5.1), (with f_{ji} from 5.2) and

$$h_{ji}(s, a_{ik}) = \sum_{k=1}^{\delta_i} a_{ik} \frac{r_{ik}}{\lambda_{ik}} \frac{\prod_{\nu=1}^{\delta_j+1} (\lambda_{ik} - \lambda_{j\nu})}{(\lambda_{ik} - \eta)} \frac{s(s - \eta)}{(s - \lambda_{ik}) \prod_{\nu=1}^{\delta_j+1} (s - \lambda_{j\nu})}\,. \tag{5.6}$$

The r_{ik} denote the residues of $g_{ii}(s)$, defined by

$$g_{ii}(s) = \sum_{k=1}^{\delta_i} \frac{r_{ik}}{s - \lambda_{ik}}, \quad i = 1, \ldots, m, \quad i \neq j\,. \tag{5.7}$$

With the explicit expression (5.5) for the non-diagonal element of $G_w(s)$ it is possible to minimize the coupling influence of the $g_{ji}(s)$ by suitable choice of the parameters

a_{ik}. For example one can minimize the energy function

$$J = \int_0^\infty d_{ji}^2(t)\, dt\,, \tag{5.8}$$

where $d_{ji}(t)$ is the response of $g_{ji}(s)$ to a unit step function. Alternatively one can try to minimize the numerator degrees of the $g_{ji}(s)$, causing low transmission of high frequencies.

6. EXAMPLE

Consider the system

$$A = \begin{bmatrix} 0{\cdot}0 & {\cdot}9945 & {\cdot}1044 & 0{\cdot}0 \\ 0{\cdot}0 & -1{\cdot}5250 & {\cdot}0678 & -30{\cdot}0200 \\ 0{\cdot}0 & -{\cdot}0166 & -{\cdot}1502 & 5{\cdot}1590 \\ {\cdot}035 & {\cdot}0698 & -{\cdot}9992 & -{\cdot}0903 \end{bmatrix},$$

$$B = \begin{bmatrix} 0{\cdot}0 & 0{\cdot}0 \\ 11{\cdot}5100 & 5{\cdot}241 \\ {\cdot}1894 & -1{\cdot}968 \\ -{\cdot}0030 & {\cdot}135 \end{bmatrix}. \quad C^{\mathrm{T}} = \begin{bmatrix} 1{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 1{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 \end{bmatrix}$$

given in [8] with the invariant zero $\eta = 0{\cdot}2771$. From eq. (4.2) we calculate $q^{\mathrm{T}} = [-0{\cdot}2731, 1]$. The plant is decouplable since the system order 4 decreased by the difference order 3 equals the number of invariant zeros. Condition (5.4) for stable decoupability is injured, thus only partial *but stable* decoupling can be achieved. By criterion (4.1) both channels can be prescribed for coupling. With regard to the difference orders $\delta_1 = 2$, $\delta_2 = 1$ we choose the transfer-function matrix $G_w(s)$ with coupling in channel 2:

$$G_w(s) = \begin{bmatrix} \dfrac{20}{(s+4-2\mathrm{j})(s+4+2\mathrm{j})} & 0 \\ g_{21}(s) & \dfrac{-18{\cdot}04(s-0{\cdot}2771)}{(s+2-\mathrm{j})(s+2+\mathrm{j})} \end{bmatrix};$$

the poles are oriented on those given by Sogaard-Andersen [8]. From eq. (5.4) we get

$$g_{21}(s) = \\ = s\frac{s^2(3{\cdot}26y - 1{\cdot}98x + 5{\cdot}47) + s(16{\cdot}1y - 0{\cdot}845x + 43{\cdot}8) + (0{\cdot}39x - 4{\cdot}72y + 109{\cdot}4)}{(s^2+8s+20)(s^2+4s+5)},$$

where $x = \operatorname{Re} a_{11} = \operatorname{Re} a_{12}$ and $y = \operatorname{Im} a_{11} = -\operatorname{Im} a_{12}$. $a_{11} = a_{12} = 0$ yields (design A):

$$g_{21}(s) = \frac{5{\cdot}469s}{s^2+4s+5}.$$

$a_{12} = -1{\cdot}88 - 2{\cdot}81\mathrm{j}\,, \quad a_{12} = -1{\cdot}88 + 2{\cdot}81\mathrm{j}$ yields

$$g_{21}(s) = \frac{122s}{(s^2 + 8s + 20)(s^2 + 4s + 5)},$$

i.e. minimal order of the numerator (design B). Controller and precompensator are in this case

$$\boldsymbol{R} = \begin{bmatrix} 42{\cdot}30 & 8{\cdot}08 & 1{\cdot}66 & -2{\cdot}62 \\ -92{\cdot}89 & -17{\cdot}23 & -3{\cdot}62 & {\cdot}16 \end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} 42{\cdot}46 & 36{\cdot}59 \\ -93{\cdot}26 & -83{\cdot}80 \end{bmatrix}.$$

The choice $a_{11} = -3{\cdot}01 - 2{\cdot}71\mathrm{j}$, $a_{12} = -3{\cdot}01 + 2{\cdot}71\mathrm{j}$ minimizes the cost function (5.8) and yields (design (C)

$$g_{21}(s) = \frac{2{\cdot}59s^3 + 2{\cdot}63s^2 + 121s}{(s^2 + 8s + 20)(s^2 + 4s + 5)}.$$

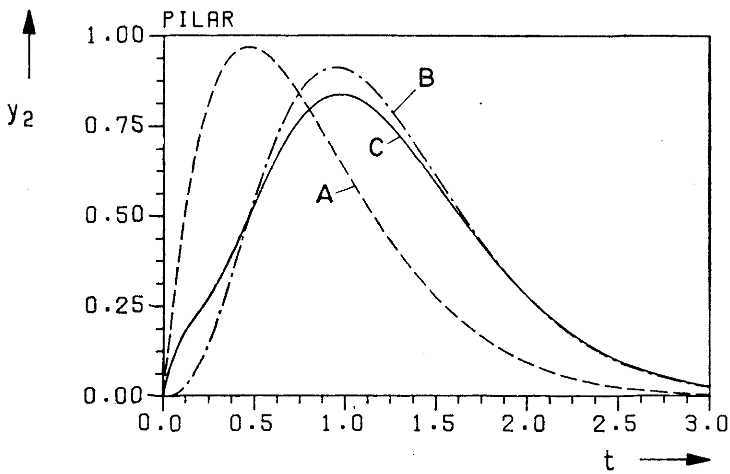

Fig. 1. Step response $y_2(t) = g_{21}(t) * \sigma(t)$ for the design A, B, and C.

Figure 1 shows the time-response of the non-diagonal element $g_{21}(s)$ to a unit-step function $\sigma(t)$.

7. CONCLUSIONS

The introduced method allows the partial stable decoupling of non-minimum phase systems having $n - \delta$ invariant zeros. Systems with *several* "unstable" zeros can be treated by extending the design steps of Section 3. Again the coupled channels have to be determined following Section 4 (self conjugate zeros cause coupling in only *one* channel). Also note that the design of partially decoupling controllers can be appropriate in cases where a complete and stable decoupling (following Section 2) requires high efforts in $u(t)$.

The design steps of Section 3 even allow a partial decoupling of plants having *less* than $n - \delta$ invariant zeros. Details can be found in [10].

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