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ON EXACT NULL CONTROLLABILITY OF BLACK – SCHOLES EQUATION

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In this paper we discuss the exact null controllability of linear as well as nonlinear Black–Scholes equation when both the stock volatility and risk-free interest rate influence the stock price but they are not known with certainty while the control is distributed over a subdomain. The proof of the linear problem relies on a Carleman estimate and observability inequality for its own dual problem and that of the nonlinear one relies on the infinite dimensional Kakutani fixed point theorem with L^2 topology.

Keywords: Black–Scholes equation, volatility, controllability, observability, Carleman estimates

AMS Subject Classification: 93B05, 93C20, 45K05, 93E03

1. INTRODUCTION

1.1. Black–Scholes model

The partial differential equation derived by Black and Scholes [7] in their renowned work of 1973 to analyze the European option on a stock market that does not pay a dividend during the life of the option has been of enormous interest to financiers and mathematicians alike. They limited their analysis to conditions which made the problem simpler mathematically. The development of more sophisticated options and pricing models has resulted in many diverse mathematical and computational techniques being employed in the field.

A call option is the right to buy a security at a specified price (called the exercise or strike price) during a specified period of time. European options can only be exercised on the day of expiration of the option. American options can be exercised at any time up to and including the day of expiration of the option. With the following assumptions we shall give some outline of the derivation of Black–Scholes equation. The underlying stock pays no dividends during the life of the option. The price of the stock one period ahead has a log-normal distribution with mean μ and volatility σ, which are both constants over the life of the option. There exists a risk-free interest rate r which is constant over the life of the option. Individuals can borrow as well as lend at the risk-free interest rate.

The value of a European call option on a non dividend paying stock could depend upon a number of factors; the current price of the stock s, the exercise price x, the time until expiration t, the risk-free interest rate r, the volatility of the stock price σ and the expected rate of return μ on the stock. Let v be the price of the call option. The functional dependence can then be expressed as:

$$v = v(s, x, t, r, \sigma, \mu).$$

The analysis will reveal that the last variable μ plays no role in determining the option value for this case. We assume that the change in the underlying typical stock price $\mathrm{d}s$ follows a geometric Brownian motion

$$\mathrm{d}s = \mu s \mathrm{d}t + \sigma s \mathrm{d}w$$

where $\mathrm{d}w$ is Brownian. The value of the option matures is known. To determine its value at an earlier time we need to know how the value evolves as we go backward in time. By Ito's lemma for two variables we have

$$\mathrm{d}v = \Big(v_t + \mu s v_s + \frac{1}{2}\sigma^2 s^2 v_{ss}\Big)\mathrm{d}t + \sigma s v_s \mathrm{d}w.$$

Now consider a portfolio containing one return call whose value is v and h shares of the underlying stock. The value C of this portfolio is given as

$$C = hs - v.$$

The composition of this portfolio will vary from time-step to time-step. The change in value is then $\mathrm{d}C = h\mathrm{d}s - \mathrm{d}v$. If h is equal to v_s then

$$\mathrm{d}C = v_s \mathrm{d}s - \mathrm{d}v.$$

This means that the change in the value of the portfolio $\mathrm{d}C$ over the interval $\mathrm{d}t$ is

$$\mathrm{d}C = -\Big(v_t + \frac{1}{2}\sigma^2 s^2 v_{ss}\Big)\mathrm{d}t.$$

Note that when the terms $\mathrm{d}s, \mathrm{d}v$ are combined we find that those involving $\mathrm{d}w$ and μ cancel out. Thus C is independent of the random variable $\mathrm{d}w$. That is, it is a risk free portfolio. Also the value $\mathrm{d}C$ is independent of the expected rate of return μ, which is also the expected rate of growth of the stock price s. Since the value of the portfolio is independent of the random variable it should increase in value at the same rate as the risk free interest rate r. Therefore $\mathrm{d}C = rC\mathrm{d}t = r(v_s s - v)\mathrm{d}t$. For this to hold for all $\mathrm{d}t$ requires the following Black–Scholes partial differential equation

$$v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} + rsv_s - rv = 0.$$

This is the law of evolution of the value of the option. With the assumptions we made above, this equation holds whenever v has two derivatives with respect to s and one with respect to t.

Before entering into the discussion on controllability, we briefly describe some of the results available in the literature regarding the existence and uniqueness of a solution to this model based on different techniques. Amster et al. [3] studied a nonlinear partial differential equation by generalizing the Black–Scholes formula for an option pricing model with stochastic volatility by topological methods. Also they proved the existence of at least a solution of this stationary Dirichlet problem applying an upper and lower solutions method. Further, Amster et al. [4] proved the existence of a solution by generating a Black–Scholes formula with the initial-Dirichlet conditions namely

$$\begin{aligned} v_t + b\sigma^2 s^3 (v_{ss})^2 &+ \frac{1}{2}\tilde{\sigma}^2 s^2 v_{ss} + r(sv_s - v) = 0, \\ v(s,T) &= f(s), \quad \forall\, s \in (c,d), \\ v(c,t) &= f(c), \quad v(d,t) = f(d), \end{aligned}$$

for some $f \in C([c,d])$, and $\tilde{\sigma}$ depending on σ, t with the help of same upper and lower solutions method. Kangro and Nicolaides [19] used the partial differential equations approach for valuing European-style options. They solved the equations numerically by introducing an artificial boundary in order to make the computational domain bounded and also derived point wise bounds for the error caused by various boundary conditions imposed on the artificial boundary. Jodar et al. [18] solved the Black–Scholes equation

$$\begin{aligned} v_t + \sigma^2 s^2 v_{ss} + r(sv_s - v) &= 0, \quad 0 < s < \infty, \quad 0 \le t < T, \\ v(s,T) &= f(s), \end{aligned}$$

where T is the maturity date and σ, r are positive constants, by using the Mellin transform. Widdicks et al. [23] applied the singular perturbation technique to price European and American barrier options which leads to a significant simplification of the problem by reducing the number of parameters.

However, this Black–Scholes model can not be used in modelling the real world problem exactly. If the Black–Scholes holds, then the implied volatility of an option on a particular stock would be constant, even as the strike and maturity vary and roughly equal to the historic volatility. In practice, the volatility surface for a two dimensional graph of implied volatility against strike and maturity is not flat. In fact, in a typical market, the graph of strike against volatility for a mixed maturity is typically smile shaped. In addition to this, the volatility is a troublesome input. Whether a single numerical value or a deterministic function implied by market data or a stochastic probability (with an attendant change in the above Black–Scholes equation) should be chosen remains an open question and subject of intense debate and research. The risk-free interest rate is somewhat easier to pin down since it is closely linked to the spot rate, but for long-term options a term structure for r is appropriate which is also not known with certainty.

Regarding some generalization of the Black–Scholes equation, Ingber and Wilson [17] generalized the functional form of the diffusion of these systems and also considered multi factor models including stochastic volatility and they modelled these issues using a previous development of statistical mechanics of financial markets. In fact they considered a slight generalization of the above Black–Scholes equation in terms of the other variables using methods given in the standard text [24].

Nowadays the study of inverse problem in option pricing is used to caliberate the volatility or other unknown parameters with known price in order to increase the efficiency of the model. For example, the authors [8, 9] studied the inverse problem of determining the volatility by using Schauder type estimates and gave some numerical interpretations. In [12] the authors discussed the stability and convergence analysis of the inverse problem of identifying local volatility using Tikhonov regularization method. Moreover the stability of the multidimensional Black–Scholes model is studied in [22] recently using Carleman estimates. As the problem of identification of a suitable control parameter is closely related with the inverse problems, in this paper, we discuss the controllability and observability properties of the general Black–Scholes model defined by the equation (1).

1.2. Mathematical formulation

The price $v(s,t)$ of a financial option for buying or selling an asset of value s is generally found from the following generalized Black–Scholes equation with initial-Dirichlet boundary conditions

$$\left.\begin{aligned} \mathcal{L}v &= 1_\omega u(s,t) + f(s,t) && \text{in } \ Q = I \times (0,T) \\ v(a,t) &= v(b,t) = 0 && \text{on } \ \Sigma = \partial I \times (0,T) \\ v(s,0) &= v_0(s) && \text{in } I, \end{aligned}\right\} \tag{1}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{1}{2}\sigma^2(s,t)s^2\frac{\partial^2}{\partial s^2} - r(s,t)s\frac{\partial}{\partial s} + r(s,t),$$

and $I = (a,b)$ is a bounded interval in $\mathbb{R}^+$. Also since $0 \le t < T$, where t is the time to expiry, T is the time of expiry and $v(s,t)$ is the value of the option at time t if the price of the underlying stock at time t is s. The value of the option at the time that the option matures is known and if s_0 is the current price of the underlying stock, then the value $0 < s_0 \le s$ is known.

In the generalized Black–Scholes equation occur two functions, the stock volatility $\sigma(s,t) \in C^{2,1}(\bar{Q})$, of the evolution of the price s of the underlying asset with time t, and the risk-free interest rate $r(s,t) \in C^1(\bar{Q})$. Both the quantities influence the price $v(s,t)$ but they are not known with certainty. Though the choice for σ and r may be controversial, upper and lower bounds on the volatility and interest during the life of the option can often be imposed with reasonable certainty and suppose that we have the upper and lower bounds as follows:

$$\sigma_0(s,t) \le \sigma(s,t) \le \sigma_1(s,t) \quad \text{with} \quad \sigma_0(s,t) \ge c > 0, \tag{2}$$

and

$$r_0(s,t) \le r(s,t) \le r_1(s,t) \quad \text{with} \quad r_0(s,t) \ge c > 0. \tag{3}$$

Let $f(s,t)$ represents the nonnegative cash flow that is received to compensate the loss, if any, that happens due to exercising the option along with a control input $u(s,t)$, to control the option value $v(s,t)$, associated with equation (1), to be within the barrier limits.

Clearly, the functions $f \in L^2(Q)$ is given, while $u \in L^2(Q)$ is a control input and let $v_0 \in H_0^1(I)$ be arbitrary but fixed initial data. Moreover, 1_ω is the usual characteristic function

$$1_\omega = \begin{cases} 1 & \text{for} \quad s \in \omega \\ 0 & \text{for} \quad s \in I \backslash \omega, \end{cases}$$

where ω is the suitable open subset of I. We note here that for the observability problem the subdomain ω represents the region where measurements are made.

Definition. In the Black–Scholes model (1), an initial data $v_0 \in H_0^1(I)$ is exactly null controllable in time T if there is a control $u \in L^2(Q)$ so that its solution v satisfies $v \in C(0,T;L^2(I)) \cap L^2(0,T;H_0^1(I) \cap H^2(I))$ and $v(s,T) \equiv 0$.

Global exact null controllability at time T for (1) holds if any initial data $v_0 \in H_0^1(I)$ is exact null controllable in time T.

In general, null controllability is a stronger property of control to the trajectories, which guarantees that every state which is the value of the final time of a solution of the uncontrlled equation is reachable from any initial datum by means of suitable control. Note that the null controllability of a system is a very useful property from the applications point of view. Indeed, it permits us to reach in a finite amount of time a state that is more "natural" for the system. Thus for the model (1), establishing the null controllability result ensures that the efficiency and reliability of the model can be maintained.

In order to study the controllability of (1) we use the duality arguments [13, 15] and exactly it would be stated as *"The exact controllability of the linear system can be reduced to the observability estimate of its own dual problem"*. This is achieved by deriving a Carleman estimate corresponding to the linearized system. A basic Carleman inequality for elliptic operators can be found in [14]. Therefore the dual problem associated with (1) is given by

$$\left.\begin{aligned} &\mathcal{L}^* y = l(s,t) && \text{in } Q \\ &y(a,t) = y(b,t) = 0 && \text{on } \Sigma \\ &y(s,T) = y_T(s) && \text{in } I, \end{aligned}\right\} \tag{4}$$

where $l \in L^2(Q)$ and the operator $\mathcal{L}^*$ is the formal adjoint of the operator $\mathcal{L}$ and is given by

$$\mathcal{L}^* = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial s^2}(\sigma^2(s,t)s^2) - \frac{\partial}{\partial s}(r(s,t)s) - r(s,t).$$

Also, throughout this study, we use the following standard notations for the Sobolev spaces $H^m(I), H_0^m(I)$ and the L^p spaces on I and Q, $1 \le p \le \infty$, with the norm denoted $\|\cdot\|$ and we use $\langle\cdot\rangle$ for the inner product of $L^2(I)$, $|\cdot|$ to denote the usual norm in $\mathbb{R}^+$.

$$W_m^p(I) = \Big\{ w(x) : \|w\|_{W_m^p} = \Big(\sum_{|\alpha|\le m} \int_I |D^\alpha w|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} < \infty \Big\};$$

when $p = 2$, instead of W_m^2 we shall write $H^m(I)$.

$$\begin{aligned} |w|_2 &= \Big(\int_I |w(s)|^2 \,\mathrm{d}s \Big)^{\frac{1}{2}}, \\ H^1(0,T;L^2(I)) &= \Big\{ w \in L^2(0,T;L^2(I)) : \frac{\mathrm{d}w}{\mathrm{d}t} \in L^2(0,T;L^2(I)) \Big\}, \\ C^{2,1}(\bar{Q}) &= \Big\{ w(s,t) \mid w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial s}, \frac{\partial^2 w}{\partial s^2} \in C(\bar{Q}) \Big\}, \end{aligned}$$

where $D^\alpha w$, $\mathrm{d}w/\mathrm{d}t$ is taken in the sense of distributions. For more detailed definitions on these function spaces one can refer [1].

The rest of the paper is organized as follows: The main result of this paper is stated in Section 3.2 and then is proved *via* the Kakutani fixed point theorem and L^2 estimates for the control u obtained in Section 3.1. In fact estimate for this control is obtained by a Carleman-type estimate stated in Section 2 and observability inequality for the backward adjoint linear problem given in (4) and the corresponding controllability of the system (1) is proved in Section 3.1 with the help of Pontryagin's maximum principle. The technique we used to extend the exact controllability of the linearized system to the general nonlinear system is rather general (see, for example, [2, 5, 6, 11, 20]).

2. CARLEMAN ESTIMATES FOR BLACK–SCHOLES EQUATION

In order to obtain the global controllability for the linearized system, we need observability inequality for the backward adjoint system. We establish this inequality by deriving a Carleman-type estimate for the adjoint Black–Scholes equation of (1) and so much of this work will consist of deriving such an estimate. Though in the proof of this estimate we use some of the techniques adopted for general parabolic problem in [10, 16], one has to do some careful estimations for the more general form (4) with the coefficients stock volatility σ, interest rate r and stock price s.

Essentially, a Carleman estimate for the solutions of (4) is an *a priori* estimate which contains only the restriction of solution on $Q_\omega = \omega \times (0,T)$ in the right hand side, instead of the solution taken on the entire domain Q. But to get such an estimate, it is necessary to multiply the solution by some suitable weight functions.

In this way we need to introduce the following auxiliary functions to express the inequality in the desired form. Let $\omega_0 \Subset \omega$, where ω_0 is the suitable subdomain of ω and since I is bounded and connected, one may have the following lemma:

Lemma. Let $\omega_0 \Subset \omega$ be a suitable subdomain. Then there exists a function $\psi \in C^2(\bar{I})$ such that

$$\psi(s) > 0 \ \ \forall\, s \in I, \ \ \psi|_{\partial I} = 0, \ \ |\psi_s(s)| > 0 \ \forall\, s \in I \backslash \omega_0.$$

The lemma has been proven by the simple arguments in [13]. Next we introduce functions $\phi,\ \alpha : Q \to \mathbb{R}$ by the formulae

$$\phi(s,t) = \frac{e^{\lambda\psi(s)}}{\gamma(t)}, \qquad \alpha(s,t) = \frac{e^{\lambda\psi(s)} - e^{2\lambda\Psi}}{\gamma(t)}, \tag{5}$$

where

$$\gamma(t) = t(T-t), \quad \text{and} \quad \Psi = \|\psi(s)\|_{C(\bar{I}),}$$

where the parameter $\lambda > 1$ and the function ψ is defined in Lemma.

We note that $\phi(s,t) \geq c > 0$ for all $(s,t) \in Q$ and $e^{\kappa\alpha}\phi^m \leq c < \infty$ for all $\kappa > 0,\ m \in \mathbb{R}$. Also we see that $\alpha < 0$ for the arbitrary parameter $\lambda > 0$. Therefore, α approaches $-\infty$ at $t = 0$ and $t = T$. This helps us to get the desired observability estimate.

Now we are ready to prove the Carleman estimate for the problem (4).

Theorem 1. Let ω be the open subset of I, the functions ϕ and α be defined as in (5) and let the assumptions on $\sigma \in C^{2,1}(\bar{Q}), r \in C^1(\bar{Q})$ defined in (2),(3) be fulfilled. Then there exists $\lambda_0 \geq 1$ such that, for an arbitrary $\lambda > \lambda_0$, there exists $\delta \geq \delta_0(\lambda) > 0$ satisfying the following inequality:

$$\begin{aligned}\int_Q \Big[\delta^{-1}\phi^{-1}(y_t^2(s,t) + s^4 y_{ss}^2(s,t)) + \delta\phi s^4 y_s^2(s,t) + s^4\delta^3\phi^3 y^2(s,t)\Big] e^{2\delta\alpha(s,t)}\, \mathrm{d}s\mathrm{d}t \\ \leq \ c(\lambda,\delta)\Big(\int_Q e^{2\delta\alpha(s,t)} l^2(s,t)\, \mathrm{d}s\mathrm{d}t + \int_{Q_\omega} e^{2\delta\alpha(s,t)}\phi^3 s^4 y^2(s,t)\, \mathrm{d}s\mathrm{d}t\Big),\end{aligned} \tag{6}$$

where y is the solution of the problem (4) with the Dirichlet boundary conditions, $Q_\omega = \omega \times (0,T)$ and the constant $c(\lambda,\delta) > 0$ is independent of y and l.

P r o o f. We shall start the proof by writing the problem (4) in the form

$$y_t + \frac{1}{2}\sigma^2 s^2 y_{ss} = -\frac{1}{2}(\sigma^2 s^2)_{ss} y - (\sigma^2 s^2)_s y_s + (rs)_s y + r(s y_s + y) + l. \tag{7}$$

Let us change the unknown variable by a simple transformation $y = e^{-\delta\alpha} z$; then the above problem can be written as

$$\begin{aligned} z_t + \frac{1}{2}\sigma^2 s^2 (z_{ss} - (\lambda^2\psi_s^2(\delta\phi - \delta^2\phi^2) - \delta\lambda\phi\psi_{ss})z) - \sigma^2 s^2 \delta\lambda\phi\psi_s z_s - \delta\alpha_t z \\ = \ -\frac{1}{2}(\sigma^2 s^2)_{ss} z - (\sigma^2 s^2)_s (z_s - \delta\lambda\phi\psi_s z) + rs(z_s - \delta\lambda\phi\psi_s z) \\ + ((rs)_s + r)z + e^{\delta\alpha} l \ \text{ in } \ Q, \\ z(a,t) = z(b,t) = 0 \quad \text{on} \ \ \Sigma, \\ z(s,0) = z(s,T) = 0. \end{aligned}$$

We rewrite this problem in the operator form as

$$z_t - B(t)z + X(t)z = R(t)z + e^{\delta\alpha} l, \tag{8}$$

where

$$\begin{aligned} B(t)z &= -\frac{1}{2}\sigma^2 s^2 (z_{ss} + (\delta\phi + \delta^2\phi^2)\lambda^2\psi_s^2 z) + \delta\alpha_t z, &(9)\\ X(t)z &= -\sigma^2 s^2 \delta\phi(\lambda\psi_s z_s + \lambda^2\psi_s^2 z), &(10)\\ R(t)z &= \frac{1}{2}\sigma^2 s^2 \delta\lambda\phi\psi_{ss} z - \frac{1}{2}(\sigma^2 s^2)_{ss} z - (\sigma^2 s^2)_s (z_s - \delta\lambda\phi\psi_s z)\\ &\quad + ((rs)_s + r(1 - s\delta\lambda\phi\psi_s))z + rsz_s. &(11) \end{aligned}$$

Consider

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_I (B(t)z)z\,\mathrm{d}s &= \int_I (B(t)z)z_t\,\mathrm{d}s + \int_I (B(t)z_t)z\,\mathrm{d}s + \int_I (B_t(t)z)z\,\mathrm{d}s\\ &= 2\int_I (B(t)z)(e^{\delta\alpha} l + R(t)z - X(t)z + B(t)z)\,\mathrm{d}s + \int_I (B_t(t)z)z\,\mathrm{d}s \end{aligned}$$

and integrating it on $(0, T)$ and using the boundary conditions, we get

$$\begin{aligned} 2\int_Q (B(t)z)^2\,\mathrm{d}s\mathrm{d}t + 2W(t) &= -2\int_Q (B(t)z)(e^{\delta\alpha} l + R(t)z)\,\mathrm{d}s\mathrm{d}t\\ &\quad - \int_Q (B_t(t)z)z\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{12}$$

where

$$\begin{aligned} W(t) = -\int_Q B(t)zX(t)z\,\mathrm{d}s\mathrm{d}t &= -\int_Q \Big[\frac{1}{2}\sigma^2 s^2 (z_{ss} + (\delta^2\phi^2 + \delta\phi)\psi_s^2\lambda^2 z) - \delta\alpha_t z\Big]\\ &\quad \times \Big[\sigma^2 s^2 \delta\phi(\lambda\psi_s z_s + \lambda^2\psi_s^2 z)\Big]\,\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{13}$$

Next we need to evaluate the terms in (12) one by one. From the definition of $B(t)$, we have

$$\begin{aligned} \int_Q (B_t(t)z)z\,\mathrm{d}s\mathrm{d}t &= \int_Q (s^2(\sigma_s\sigma_t + \sigma\sigma_{st}) + 2\sigma\sigma_t s)z_s z\,\mathrm{d}s\mathrm{d}t + \int_Q s^2\sigma\sigma_t z_s^2\,\mathrm{d}s\mathrm{d}t\\ &\quad - \int_Q \Big[\sigma\sigma_t(\delta^2\phi^2 + \delta\phi) + \frac{1}{2}\sigma^2(\delta^2\phi^2 + \delta\phi)_t\Big] s^2\lambda^2\psi_s^2 z^2\,\mathrm{d}s\mathrm{d}t + \int_Q \delta\alpha_{tt} z^2\,\mathrm{d}s\mathrm{d}t. \end{aligned}$$

If we set $\eta(\lambda) = e^{2\lambda\Psi}$, then for any $\lambda \geq 1, \delta \geq \eta(\lambda)$ and $s \geq s_0$, we have

$$\Big|\int_Q (B_t(t)z)z\,\mathrm{d}s\mathrm{d}t\Big| \leq c\Big(\int_Q s^2\delta\phi\lambda z_s^2\,\mathrm{d}s\,\mathrm{d}t + \int_Q s^2\delta^3\phi^3\lambda^3 z^2\,\mathrm{d}s\mathrm{d}t\Big), \tag{14}$$

since we used (here after we also use) the fact $|\phi_t| \leq c\phi^2$, $|\alpha_{tt}| \leq c\eta(\lambda)\phi^3$, $\phi^{-1} \leq (\frac{T}{2})^2$ where the constant c is independent of $(s, t) \in Q$, the parameters δ and λ

are bounded from below by constants, ψ is bounded from above (ψ is a continuous function with compact support in I) and the stock price $s \geq s_0 > 0$. Also recall that the volatility and the interest rate have upper and lower bounds. Further, throughout this proof, the generic constant c will vary step by step but it may depend on any one of the constant defined above.
For simplicity, let us introduce the notation

$$D(s,\lambda,\delta,z) = \int_Q s^2\delta\lambda\phi z_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q s^2\delta^3\lambda^3\phi^3 z^2 \,\mathrm{d}s\mathrm{d}t.$$

Next we shall estimate

$$\Big|\int_Q 2B(t)z(e^{\delta\alpha}l + R(t)z)\,\mathrm{d}s\mathrm{d}t\Big| \leq 2\|B(t)z\|^2_{L^2(Q)} + \|e^{\delta\alpha}l\|^2_{L^2(Q)} + \|R(t)z\|^2_{L^2(Q)}.$$

Here we note that

$$\begin{aligned}\|R(t)z\|^2_{L^2(Q)} \quad \leq \quad & 8\Big(\frac{1}{4}\|\sigma^2s^2\delta\lambda\phi\psi_{ss}z\|^2_{L^2(Q)} + \frac{1}{4}\|(\sigma^2s^2)_{ss}z\|^2_{L^2(Q)} \\ & + \frac{1}{4}\|(\sigma^2s^2)_s z_s\|^2_{L^2(Q)} + \|(\sigma^2s^2)_s\delta\lambda\phi\psi_s z\|^2_{L^2(Q)} + \|(rs)_s z\|^2_{L^2(Q)} \\ & + \|rs\delta\lambda\phi\psi_s z\|^2_{L^2(Q)} + \|rz\|^2_{L^2(Q)} + \|rsz_s\|^2_{L^2(Q)}\Big).\end{aligned}$$

Since one can easily justify that the each term on the right hand side of the above inequality is bounded by $cD(s^2,\lambda,\delta,z)$ for any $\lambda \geq 1, \delta \geq \eta(\lambda)$ and $s \geq s_0$. Therefore,

$$\begin{aligned}\Big|\int_Q 2B(t)z(e^{\delta\alpha}l + R(t)z)\,\mathrm{d}s\mathrm{d}t\Big| \quad \leq \quad & 2\int_Q (B(t)z)^2\,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha}l^2\,\mathrm{d}s\mathrm{d}t \qquad (15) \\ & + cD(s^2,\lambda,\delta,z).\end{aligned}$$

Making use of the estimations (14),(15), we estimate (12) as

$$2W(t) \quad \leq \quad \int_Q e^{2\delta\alpha}l^2\,\mathrm{d}s\mathrm{d}t + cD(s^2,\lambda,\delta,z) \qquad (16)$$

for $\lambda \geq 1, \delta \geq \eta(\lambda)$ and $s \geq s_0$. Next we need to obtain the lower bound for $W(t)$ and so we shall estimate the L^2 integrals of (13) one by one. We observe by the calculation involving Green's theorem

$$\begin{aligned}& -\int_Q \frac{1}{2}\sigma^4s^4\delta\lambda^2\phi\psi_s^2 z_{ss}z\,\mathrm{d}s\mathrm{d}t \\ & \qquad = \frac{1}{2}\int_Q (\sigma^4s^4\delta\lambda^2\phi\psi_s^2)_s z_s z\,\mathrm{d}s\mathrm{d}t + \frac{1}{2}\int_Q \sigma^4s^4\delta\lambda^2\phi\psi_s^2 z_s^2\,\mathrm{d}s\mathrm{d}t; \qquad (17)\end{aligned}$$

but elementary computation shows that

$$\begin{aligned}\frac{1}{2}\int_Q (\sigma^4s^4\delta\lambda^2\phi\psi_s^2)_s z_s z\,\mathrm{d}s\mathrm{d}t \quad \geq \quad & -\frac{1}{4}\int_Q \sigma^4s^4\delta\lambda^2\phi\psi_s^2 z_s^2\,\mathrm{d}s\mathrm{d}t - c\Big(D(s^2,\lambda,\delta,z) \\ & + \int_Q s^4\delta^2\lambda^4\phi^3 z^2\,\mathrm{d}s\mathrm{d}t\Big) \qquad (18)\end{aligned}$$

and also note that

$$\begin{aligned} -\int_Q \frac{1}{2}\sigma^4 s^4 \delta\lambda\phi\psi_s z_{ss} z_s \,\mathrm{d}s\mathrm{d}t & \\ = & \ \frac{1}{4}\int_Q (\sigma^4 s^4\delta\lambda\phi\psi_s)_s z_s^2 \,\mathrm{d}s\mathrm{d}t - \frac{1}{4}\int_\Sigma \sigma^4 s^4 \delta\lambda\phi \frac{\mathrm{d}\psi}{\mathrm{d}\nu} z_s^2 \mathrm{d}\Sigma \\ \geq & \ -c\; D(s^2,\lambda,\delta,z), \end{aligned} \tag{19}$$

since we recall that $\psi = 0$ on ∂I and $\psi > 0$ in I, we have $\frac{\mathrm{d}\psi}{\mathrm{d}\nu} \leq 0$ and therefore

$$-\frac{1}{4}\int_\Sigma \sigma^4 s^4 \delta\lambda\phi \frac{\mathrm{d}\psi}{\mathrm{d}\nu} z_s^2 \mathrm{d}\Sigma \geq 0,$$

where ν is the outward unit normal to ∂I. Moreover,

$$\begin{aligned} -\int_Q \frac{1}{2}\sigma^4 s^4 \lambda^3(\delta^3\phi^3 + \delta^2\phi^2)\psi_s^3 z z_s \,\mathrm{d}s\mathrm{d}t & \\ \geq & \ \int_Q \frac{1}{4}\sigma^4 s^4\lambda^4\psi_s^4(3\delta^3\phi^3 + 2\delta^2\phi^2) z^2 \,\mathrm{d}s\mathrm{d}t - c\; D(s^2,\lambda,\delta,z). \end{aligned} \tag{20}$$

Immediately one can see that the addition of the terms

$$-\frac{1}{2}\int_Q \sigma^4 s^4 \lambda^4(\delta^3\phi^3 + \delta^2\phi^2)\psi_s^4 z^2 \,\mathrm{d}s\mathrm{d}t$$

of $W(t)$ to both sides of the inequality (20) further reduces its lower bounds. Furthermore, for any $\lambda \geq 1, \delta \geq \eta(\lambda)$, we obtain

$$\Big|\int_Q \sigma^2 s^2 \delta^2 \lambda^2 \phi\psi_s^2 \alpha_t z^2 \,\mathrm{d}s\mathrm{d}t\Big| \leq c\; D(s,\lambda,\delta,z) \tag{21}$$

and

$$\begin{aligned} \Big|\int_Q \sigma^2 s^2 \delta^2 \lambda\phi\psi_s \alpha (\ln\gamma^{-1}(t))_t z_s z \,\mathrm{d}s\mathrm{d}t\Big| & \\ = & \ \frac{1}{2}\Big|\int_Q (\sigma^2 s^2\delta^2\lambda\phi\psi_s\alpha)_s (\ln\gamma^{-1}(t))_t z^2 \,\mathrm{d}s\mathrm{d}t\Big| \\ \leq & \ c\; D(s,\lambda,\delta,z); \end{aligned} \tag{22}$$

here we note that $|\alpha_t| \leq c\eta(\lambda)\phi^2$ and $\alpha_t = \alpha(\ln\gamma^{-1}(t))_t$. Thus using the equations (17) – (22), we obtain

$$\begin{aligned} W(t) \ \geq & \ \frac{1}{4}\int_Q \sigma^4 s^4 \delta\lambda^2\phi\psi_s^2 z_s^2 \,\mathrm{d}s\mathrm{d}t + \frac{1}{4}\int_Q \sigma^4 s^4\delta^3\lambda^4\phi^3\psi_s^4 z^2 \,\mathrm{d}s\mathrm{d}t \\ & - c\big(\int_Q s^4\delta^2\lambda^4\phi^3 z^2 \,\mathrm{d}s\,\mathrm{d}t + D(s^2,\lambda,\delta,z)\big), \end{aligned} \tag{23}$$

for any $\lambda \geq 1, \delta \geq \eta(\lambda)$ and $s \geq s_0$. Hence the inequalities (16) and (23) imply the following key inequality

$$\int_Q \sigma^4 s^4 \delta \lambda^2 \phi \psi_s^2 z_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q \sigma^4 s^4 \delta^3 \lambda^4 \phi^3 \psi_s^4 z^2 \,\mathrm{d}s\mathrm{d}t \leq c\Big(\int_Q s^4 \delta^2 \lambda^4 \phi^2 z^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t + D(s^2, \lambda, \delta, z)\Big). \tag{24}$$

Now we are in the position of expressing two integrals involving z_s^2, z^2 on the right hand side of the above inequality over the domain Q_{ω_0}. To do this we use the lower bound of σ and the fact that $\psi_s > 0 \;\forall\, s \in I\backslash\omega_0$ and $\psi|_{\partial I} = 0$ so that ψ_s also has a lower bound θ on $I\backslash\omega_0$ and hence on $Q\backslash Q_{\omega_0}$ (since ψ does not depend on t). Next choose the parameter λ such that $\lambda > \lambda_0 = c+1$ to have $\theta^4\lambda > c+1$ and $\theta^2\lambda > c+1$, where c is the constant defined in (24). In order to manage the other integral (with λ^4 on the right hand side) we choose $\delta \geq \delta_0(\lambda) = \max\big(\frac{c\lambda}{\theta^4\lambda - c}, \eta(\lambda)\big)$ to obtain $\lambda(\theta^4 - c) \geq c+1$. After this substitution we add the integrals $\int_{Q_{\omega_0}} s^4\delta^3\lambda^3\phi^3 z^2 \,\mathrm{d}s\mathrm{d}t$ and $\int_{Q_{\omega_0}} s^4 \delta\lambda\phi z_s^2 \,\mathrm{d}s\mathrm{d}t$ on both sides of the inequality (24) to arrive at (also see [21])

$$\int_Q s^4 \delta\lambda\phi z_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q s^4 \delta^3 \lambda^3 \phi^3 z^2 \,\mathrm{d}s\mathrm{d}t \leq c(\lambda)\Big(\int_{Q_{\omega_0}} s^4\delta^3\phi^3 z^2 \,\mathrm{d}s\mathrm{d}t + \int_{Q_{\omega_0}} s^4 \delta\phi z_s^2 \,\mathrm{d}s\mathrm{d}t\Big) + c\int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t, \tag{25}$$

for $\delta \geq \delta_0(\lambda), \lambda > \lambda_0, s \geq s_0$. Coming back to the original variable y by substituting $z = e^{\delta\alpha} y$, we have

$$\int_Q e^{2\delta\alpha} s^4 \delta\lambda\phi y_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha} s^4\delta^3\lambda^3\phi^3 y^2 \,\mathrm{d}s\mathrm{d}t \leq c(\lambda)\Big(\int_{Q_{\omega_0}} e^{2\delta\alpha} s^4\delta^3\phi^3 y^2 \,\mathrm{d}s\mathrm{d}t + \int_{Q_{\omega_0}} e^{2\delta\alpha} s^4 \delta\phi y_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t\Big) \tag{26}$$

for $\delta \geq \delta_0(\lambda), \lambda > \lambda_0, s \geq s_0$. Next we shall express the integral $\delta\phi y_s^2$ over Q_{ω_0} in a somehow larger domain Q_ω (since we remember that $\omega_0 \Subset \omega$). To this end, we choose $\chi \in C_0^\infty(I)$ such that $\chi \equiv 1$ in $\bar{\omega}_0$ and $\chi \equiv 0$ in $I\backslash\omega$. Multiplying the equation (7) by $e^{2\delta\alpha}\chi\delta\phi y$ and integrating over Q, we obtain after integration by parts

$$\begin{aligned} \frac{1}{2}\int_Q e^{2\delta\alpha}\sigma^2 s^2 \chi\delta\phi y_s^2 \,\mathrm{d}s\mathrm{d}t &= \frac{1}{2}\int_Q (e^{2\delta\alpha}\delta\chi\phi(\sigma^2 s^2)_{ss} - (e^{2\delta\alpha}\chi\delta\phi)_t) y^2 \,\mathrm{d}s\mathrm{d}t \\ &+ \int_Q ((\sigma^2 s^2)_s - rs) e^{2\delta\alpha}\delta\chi\phi y y_s \,\mathrm{d}s\mathrm{d}t - \frac{1}{2}\int_Q (e^{2\delta\alpha}\sigma^2 s^2\chi\delta\phi)_s y y_s \,\mathrm{d}s\mathrm{d}t \\ &- \int_Q e^{2\delta\alpha}(r + (rs)_s)\chi\delta\phi y^2 \,\mathrm{d}s\mathrm{d}t - \int_Q e^{2\delta\alpha}\chi\delta\phi l y \,\mathrm{d}s\mathrm{d}t. \end{aligned}$$

After some computation involving Cauchy's inequality ($ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}, a, b > 0$)

with $\varepsilon > 0$, we obtain

$$\int_Q e^{2\delta\alpha}\sigma^2 s^2 \chi\delta\phi y_s^2 \,\mathrm{d}s\mathrm{d}t \le c(\lambda)\Big(\int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t + \int_{Q_\omega} e^{2\delta\alpha}\delta^3\phi^3 s^2 y^2 \,\mathrm{d}s\mathrm{d}t\Big).$$

Therefore, the inequality (26) can be rewritten as

$$\begin{aligned}&\int_Q e^{2\delta\alpha} s^4\delta\phi y_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha} s^4\delta^3\phi^3 y^2 \,\mathrm{d}s\mathrm{d}t \\ &\quad\le \; c(\lambda)\Big(\int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t + \int_{Q_\omega} e^{2\delta\alpha}\delta^3\phi^3 s^4 y^2 \,\mathrm{d}s\mathrm{d}t\Big), \qquad (27)\end{aligned}$$

for $\delta \ge \delta_0(\lambda), \lambda > \lambda_0, s \ge s_0$. In order to complete this theorem we need to evaluate the integral $\int_Q e^{2\delta\alpha}(\delta\phi)^{-1}(y_t^2 + s^4 y_{ss}^2)\,\mathrm{d}s\mathrm{d}t$. Squaring the equation (7) on both sides and multiplying it with $e^{2\delta\alpha}\delta^{-1}\phi^{-1}$, we obtain

$$\begin{aligned}&\int_Q e^{2\delta\alpha}(\delta\phi)^{-1}\Big(y_t + \frac{1}{2}\sigma^2 s^2 y_{ss}\Big)^2 \mathrm{d}s\mathrm{d}t \\ &\quad\le \; c(\lambda)\Big(\int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha}\delta\phi s^4 y_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha}\delta^3 s^4\phi^3 y^2 \,\mathrm{d}s\mathrm{d}t\Big); \qquad (28)\end{aligned}$$

here we used the fact that $e^{\kappa\alpha}\phi^m\delta^\beta \le c < \infty$ for all $\kappa > 0,\ m \in \mathbb{R},\ \beta < 0$. At the same time we should note

$$\begin{aligned}\int_Q e^{2\delta\alpha}(\delta\phi)^{-1} y_t\sigma^2 s^2 y_{ss}\,\mathrm{d}s\mathrm{d}t \;&=\; \frac{1}{2}\int_Q (e^{2\delta\alpha}\delta^{-1}\phi^{-1}\sigma^2 s^2)_t y_s^2 \,\mathrm{d}s\mathrm{d}t \\ &\quad - \int_Q (e^{2\delta\alpha}\delta^{-1}\phi^{-1}\sigma^2 s^2)_s y_s y_t \,\mathrm{d}s\mathrm{d}t,\end{aligned}$$

and consequently, by the estimations $|\phi_t| \le c\phi^2,\ |\alpha_t| \le c\eta(\lambda)\phi^2$ and the Cauchy's inequality, we arrive at

$$\frac{1}{2}\int_Q (e^{2\delta\alpha}\delta^{-1}\phi^{-1}\sigma^2 s^2)_t y_s^2 \,\mathrm{d}s\mathrm{d}t \ge -c(\lambda)\int_Q e^{2\delta\alpha} s^2\phi y_s^2 \,\mathrm{d}s\mathrm{d}t$$

and

$$\begin{aligned}&-\int_Q (e^{2\delta\alpha}\delta^{-1}\phi^{-1}\sigma^2 s^2)_s y_s y_t \,\mathrm{d}s\mathrm{d}t \\ &\quad\ge\; -\frac{1}{2}\int_Q e^{2\delta\alpha}\delta^{-1}\phi^{-1} y_t^2 \,\mathrm{d}s\mathrm{d}t - c\int_Q e^{2\delta\alpha} s^4\delta\lambda^2\phi y_s^2 \,\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Evidently, one can rewrite the inequality (28) with the help of the lower bound of σ as

$$\begin{aligned}&\int_Q e^{2\delta\alpha}(\delta\phi)^{-1}(y_t^2 + s^4 y_{ss}^2)\,\mathrm{d}s\mathrm{d}t \le c(\lambda)\Big(\int_Q e^{2\delta\alpha} l^2 \,\mathrm{d}s\mathrm{d}t \\ &\quad + \int_Q e^{2\delta\alpha}\delta\phi s^4 y_s^2 \,\mathrm{d}s\mathrm{d}t + \int_Q e^{2\delta\alpha}\delta^3 s^4\phi^3 y^2 \,\mathrm{d}s\,\mathrm{d}t\Big) + c\int_Q e^{2\delta\alpha}\lambda^2\delta\phi s^4 y_s^2 \,\mathrm{d}s\mathrm{d}t, \qquad (29)\end{aligned}$$

for $\delta \ge \delta_0(\lambda),\ \lambda > \lambda_0, s \ge s_0$. Coupling the inequalities (27) and (29), we get the desired inequality (6). $\square$

3. CONTROLLABILITY RESULTS

3.1. Controllability of linear Black–Scholes equation

In this section we study the global exact null controllability of the linear Black–Scholes equation

$$\left.\begin{aligned} &\mathcal{L}v = 1_\omega u(s,t) + f(s,t) && \text{in } Q \\ &v(a,t) = v(b,t) = 0 && \text{on } \Sigma \\ &v(s,0) = v_0(s) && \text{in } I, \end{aligned}\right\} \tag{30}$$

where the operator $\mathcal{L}$ is defined in (1). In the following theorem we shall use the notation for some weighted L^2 spaces as the space $L^2(Q, e^{-2\delta\alpha}\phi^{-3})$ of all equivalence classes of measurable functions $f : Q \to \mathbb{R}$ for which $e^{-\delta\alpha}\phi^{-\frac{3}{2}} f \in L^2(Q)$, that is, it satisfies

$$\int_Q e^{-2\delta\alpha}\phi^{-3} f^2 \,\mathrm{d}s\mathrm{d}t < \infty.$$

Theorem 2. Let I and ω be as in the statement of Theorem 1 and let the conditions (2),(3) on $\sigma \in C^{2,1}(\bar{Q}), r \in C^1(\bar{Q})$ be fulfilled. Then there are $\delta \geq \delta_0(\lambda)$, $\lambda > \lambda_0$ such that for any $f \in L^2(Q, e^{-2\delta\alpha}s^{-4}\phi^{-3})$, $v_0 \in H_0^1(I)$ there exists $(u,v) \in L^2(Q) \times L^2(0,T; H_0^1(I) \cap H^2(I)) \cap H^1(0,T; L^2(I))$ which satisfies the equation (30) and the final condition

$$v(s,T) \equiv 0, \text{ a.e. } s \in I$$

and which has the following decay at $t = T$:

$$u \in L^2(Q, e^{-2\delta\alpha}s^{-4}\phi^{-3}). \tag{31}$$

Before going into the proof of this theorem we shall obtain an *a priori* estimate of unique solution $v \in L^2(0,T; H_0^1(I) \cap H^2(I)) \cap H^1(0,T; L^2(I))$ of (30) for a given $v_0 \in H_0^1(I)$ in the usual way. First multiplying (30) by v and integrating over $Q_t = I \times (0,t)$, for some $t \in (0,T)$, we arrive, after some calculation involving Green's theorem at

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v(t)|^2_{L^2(I)} + \frac{1}{2}\int_I \sigma^2 s^2 v_s^2 \,\mathrm{d}s + \frac{1}{2}\int_I r v^2 \,\mathrm{d}s \\ &\quad\leq \frac{1}{2}\int_I (1 - (rs)_s) v^2 \,\mathrm{d}s + \int_I (1_\omega u^2 + f^2) \,\mathrm{d}s - \int_I \frac{1}{2}(\sigma^2 s^2)_s v v_s \,\mathrm{d}s. \end{aligned}$$

Since one could easily see that

$$\left|\frac{1}{2}\int_I (\sigma^2 s^2)_s v v_s \,\mathrm{d}s\right| \leq \frac{1}{4\eta}\int_I (\sigma\sigma_s s + \sigma^2)^2 v^2 \,\mathrm{d}s + \eta\int_I s^2 v_s^2 \,\mathrm{d}s.$$

The differential version of Gronwall's inequality gives

$$|v(t)|^2_{L^2(I)} + \int_{Q_t} s^2 v_s^2 \,\mathrm{d}s\,\mathrm{d}\tau + \int_{Q_t} v^2 \,\mathrm{d}s\mathrm{d}\tau \leq c\Big(|v_0(s)|^2_{L^2(I)} + \int_{Q_t} (1_\omega u^2 + f^2)\,\mathrm{d}s\mathrm{d}\tau\Big). \tag{32}$$

Here we recall that $\sigma \geq \sigma_0 > 0, r \geq r_0 > 0$ and $s \geq s_0$ and we have chosen the parameter η such that $\eta \leq \inf_{(s,t)\in Q} \sigma_0$. Further, multiplying (30) by $v_t - v_{ss}$ and integrating on $I \times (0,t)$, we get

$$\begin{aligned}
&\int_{Q_t} v_t^2 \,\mathrm{d}s\mathrm{d}\tau + \frac{1}{2}\int_{Q_t} \sigma^2 s^2 v_{ss}^2 \,\mathrm{d}s\mathrm{d}\tau + c\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_I v_s^2 \,\mathrm{d}s\,\mathrm{d}\tau \\
&\leq \int_{Q_t} \Big(rs(v_s v_t - v_s v_{ss}) + r(v v_{ss} - v v_t)\Big)\,\mathrm{d}s\mathrm{d}\tau + \frac{1}{4}\int_{Q_t} v_t^2 \,\mathrm{d}s\mathrm{d}\tau \\
&\quad + \eta\int_{Q_t} s^2 v_{ss}^2 \,\mathrm{d}s\mathrm{d}\tau - \int_{Q_t} (\sigma^2 s^2)_s v_t v_s \,\mathrm{d}s\mathrm{d}\tau + c(\eta)\int_{Q_t} (1_\omega u^2 + f^2)\,\mathrm{d}s\mathrm{d}\tau.
\end{aligned}$$

Using Green's theorem, we further obtain that

$$\begin{aligned}
&\int_{Q_t} rs(v_s v_t - v_s v_{ss})\,\mathrm{d}s\mathrm{d}\tau + \int_{Q_t} r(v v_{ss} - v v_t)\,\mathrm{d}s\mathrm{d}\tau \\
&\qquad \leq c\Big(\int_{Q_t} s^2 v_s^2 \,\mathrm{d}s\mathrm{d}\tau + \int_{Q_t} v^2 \,\mathrm{d}s\mathrm{d}\tau\Big) + \frac{1}{4}\int_{Q_t} v_t^2 \,\mathrm{d}s\mathrm{d}\tau.
\end{aligned}$$

Making use of the preceding estimate and (32), we have (for any η as above)

$$\|v(t)\|^2_{H_0^1(I)} + \int_Q v_t^2 \,\mathrm{d}s\mathrm{d}\tau + \int_Q s^2 v_{ss}^2 \,\mathrm{d}s\mathrm{d}\tau \leq c\Big(|v_0(s)|^2_{L^2(I)} + \|v_0(s)\|^2_{H_0^1(I)} + \int_Q 1_\omega u^2 \,\mathrm{d}s\mathrm{d}\tau + \int_Q f^2 \,\mathrm{d}s\mathrm{d}\tau\Big). \tag{33}$$

Now we are ready prove Theorem 2.

Proof of Theorem 2. Let us start with the optimal control problem associated with (30)

$$J(u,v) = \int_Q e^{-2\delta\alpha} s^{-4}\phi^{-3} u^2 \,\mathrm{d}s\mathrm{d}t + \frac{1}{\varepsilon}\int_I v^2(s,T)\,\mathrm{d}s \;\longrightarrow\; \inf$$

for over all $u \in L^2(Q)$ and v satisfying (30). Here δ and λ are chosen as in Theorem 1. Then by Pontryagin's maximum principle, this problem has a unique solution $(u_\varepsilon, v_\varepsilon)$. We shall show that $(u_\varepsilon, v_\varepsilon)$ really converges (as a subsequence of $\{\varepsilon\}$) in a certain topology as $\varepsilon \to 0$. Then, the limit (u,v) will be proved to be a solution of the control problem (30). To attain this end, we need to obtain suitable estimate for $(u_\varepsilon, v_\varepsilon)$. The maximum principle also states that $(u_\varepsilon, v_\varepsilon)$ satisfies the following necessary conditions for optimality:

$$u_\varepsilon = 1_\omega e^{2\delta\alpha}\phi^3 s^4 y_\varepsilon \quad \text{a.e. in } \ Q, \tag{34}$$

where y_ε is the solution of the following adjoint system

$$\left.\begin{aligned} &(y_\varepsilon)_t + \tfrac{1}{2}\big(\sigma^2 s^2 y_\varepsilon\big)_{ss} - (rs)_s y_\varepsilon - r y_\varepsilon = 0 && \text{in } \ Q \\ &y_\varepsilon(a,t) = y_\varepsilon(b,t) = 0 && \text{on } \Sigma \\ &y_\varepsilon(s,T) = -\tfrac{1}{\varepsilon}\ v_\varepsilon(s,T) && \text{in } \ I. \end{aligned}\right\} \tag{35}$$

Multiplying (35) by v_ε and (30) by y_ε (after replacing (u,v) by $(u_\varepsilon, v_\varepsilon)$) and adding the two resulting equations together and integrating on Q we get

$$\begin{aligned} &\int_{Q_\omega} s^4 e^{2\delta\alpha}\phi^3 y_\varepsilon^2(s,t)\,\mathrm{d}s\mathrm{d}t + \frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s \\ &\quad = -\int_I v_0(s) y_\varepsilon(s,0)\,\mathrm{d}s - \int_Q f y_\varepsilon(s,t)\,\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{36}$$

Here we need to estimate the term $\int_I y_\varepsilon^2(s,0)\,\mathrm{d}s$ so-called observability estimate for the adjoint problem (4), that is, an estimate for the initial state on I by means of the states taken on ω at all the subsequent moments. For this, we scalarly multiply (35) by y_ε and integrate on I (for notation simplicity we use y instead of y_ε) to get

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I y^2\,\mathrm{d}s + \frac{1}{2}\int_I \sigma^2 s^2 y_s^2\,\mathrm{d}s = \frac{1}{2}\int_I (2rs - (\sigma^2 s^2)_s) y y_s\,\mathrm{d}s - \int_I r y^2\,\mathrm{d}s.$$

Notice that

$$\int_I (2rs - (\sigma^2 s^2)_s) y y_s\,\mathrm{d}s \le \frac{1}{\eta}\int_I (r^2 + (\sigma^2 + \sigma\sigma_s s)^2) y^2\,\mathrm{d}s + \eta\int_I s^2 y_s^2\,\mathrm{d}s,$$

and so we choose $\eta \le \inf_{(s,t)\in Q}\sigma_0$ to have

$$-\frac{\mathrm{d}}{\mathrm{d}t}|y|_{L^2(I)}^2 \le c|y|_{L^2(I)}^2 \quad \text{a.e. } t\in(0,T),$$

whence it follows that

$$\int_I y^2(s,0)\,\mathrm{d}s \le e^{cT}\int_I y^2\,\mathrm{d}s \quad \text{for } t\in(0,T).$$

Now we fix t_1 and t_2 such that $0 < t_1 < t_2 < T$ and $\widetilde{\alpha}(t) = \max_{s\in\bar{I}}\{-\alpha(s,t)\}$. Integrating the above inequality in (t_1,t_2) and taking $\phi^{-3} \le (\frac{T}{2})^6$ into account, we obtain

$$\int_{t_1}^{t_2}\int_I e^{-2\delta\widetilde{\alpha}(t)} y^2(s,0)\,\mathrm{d}s\mathrm{d}t \le c\int_{t_1}^{t_2}\int_I e^{2\delta\alpha}\phi^3 y^2\,\mathrm{d}s\mathrm{d}t.$$

Noting $\inf_{t\in(t_1,t_2)}\{e^{-2\delta\widetilde{\alpha}(t)}\} \geq \breve{c} > 0$, we obtain that

$$\int_I y^2(s,0)\,\mathrm{d}s \leq c\int_Q e^{2\delta\alpha}\phi^3 y^2\,\mathrm{d}s\mathrm{d}t. \tag{37}$$

Next, by the simple application of Hölder's inequality and Theorem 1, we arrive at, for every $\varepsilon > 0$,

$$\begin{aligned}\Big|\int_Q f y_\varepsilon\,\mathrm{d}s\mathrm{d}t\Big| &\leq \int_Q |f y_\varepsilon|\,\mathrm{d}s\mathrm{d}t\\ &\leq \Big(\int_Q e^{2\delta\alpha}s^4\phi^3 y_\varepsilon^2\,\mathrm{d}s\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_Q e^{-2\delta\alpha}s^{-4}\phi^{-3}f^2\,\mathrm{d}s\mathrm{d}t\Big)^{\frac{1}{2}}\\ &\leq c(\lambda,\delta)\Big(\int_{Q_\omega} e^{2\delta\alpha}s^4\phi^3 y_\varepsilon^2\,\mathrm{d}s\mathrm{d}t+\frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s\Big)^{\frac{1}{2}}\|e^{-\delta\alpha}s^{-2}\phi^{-\frac{3}{2}}f\|_{L^2(Q)}. \end{aligned}\tag{38}$$

Also, by the observability inequality (37) and Theorem 1, we eventually obtain

$$\begin{aligned}\Big|\int_I v_0(s)y_\varepsilon(s,0)\,\mathrm{d}s\Big| &\leq \Big(\int_I y_\varepsilon^2(s,0)\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_I v_0^2(s)\,\mathrm{d}s\Big)^{\frac{1}{2}}\\ &\leq c(\lambda,\delta)\Big(\int_{Q_\omega} e^{2\delta\alpha}s^4\phi^3 y_\varepsilon^2\,\mathrm{d}s\mathrm{d}t+\frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s\Big)^{\frac{1}{2}}|v_0(s)|_{L^2(I)}. \end{aligned}\tag{39}$$

Consequently, through the inequalities (38),(39) for every $\varepsilon > 0$, we have

$$\begin{aligned}&\int_{Q_\omega} e^{2\delta\alpha}s^4\phi^3 y_\varepsilon^2\,\mathrm{d}s\mathrm{d}t+\frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s\\ &\qquad\leq c(\lambda,\delta)\Big(\int_{Q_\omega} e^{2\delta\alpha}s^4\phi^3 y_\varepsilon^2\,\mathrm{d}s\mathrm{d}t+\frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s\Big)^{\frac{1}{2}}\\ &\qquad\quad\times\big(|v_0(s)|_{L^2(I)}+\|e^{-\delta\alpha}s^{-2}\phi^{-\frac{3}{2}}f\|_{L^2(Q)}\big)\end{aligned}$$

and so by the definition of the control u_ε we get the desired estimate

$$\begin{aligned}&\int_Q e^{-2\delta\alpha}s^{-4}\phi^{-3}u_\varepsilon^2\,\mathrm{d}s\mathrm{d}t+\frac{1}{\varepsilon}\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s\\ &\qquad\leq c(\lambda,\delta)\Big(|v_0(s)|^2_{L^2(I)}+\int_Q e^{-2\delta\alpha}s^{-4}\phi^{-3}f^2\,\mathrm{d}s\mathrm{d}t\Big),\end{aligned}\tag{40}$$

where the constant $c(\lambda,\delta) > 0$ is independent of $\varepsilon > 0$ and which is somehow greater than the constant defined in Theorem 1 and the preceding inequality. The inequality (40), clearly gives

$$\int_I v_\varepsilon^2(s,T)\,\mathrm{d}s \leq c\,\varepsilon \tag{41}$$

for some positive constant c independent of ε. By the estimates (32), (33) together with (40) there exists a pair $(u_\varepsilon, v_\varepsilon)\in L^2(Q)\times L^2(0,T;H_0^1(I)\cap H^2(I))\cap H^1(0,T;L^2(I))$

such that, on a subsequence of $\{\varepsilon\}$ (also denoted by same $\{\varepsilon\}$) we have the following convergences:

$$\begin{aligned} u_\varepsilon &\to u \text{ weakly in } L^2(Q),\\ v_\varepsilon &\to v \text{ weakly in } L^2(0,T;H_0^1(I)\cap H^2(I))\cap H^1(0,T;L^2(I)). \end{aligned}$$

So letting $\varepsilon \to 0$ in (30) where (u,v) are replaced by $(u_\varepsilon, v_\varepsilon)$, we obtain that (u,v) satisfies (30). Moreover by (41) and Fatou's lemma, we have

$$\int_I v^2(s,T)\,\mathrm{d}s \le \liminf_{\varepsilon\to 0} \int_I v_\varepsilon^2(s,T)\,\mathrm{d}s = 0$$

and so $v(s,T) \equiv 0$ a.e. $s \in I$. Passing to the limit $\varepsilon \to 0$ in (40), we obtain

$$\begin{aligned} \int_Q e^{-2\delta\alpha} s^{-4}\phi^{-3}u^2\,\mathrm{d}s\mathrm{d}t &\le \liminf_{\varepsilon\to 0} \int_I e^{-2\delta\alpha} s^{-4}\phi^{-3}u_\varepsilon^2\,\mathrm{d}s\mathrm{d}t\\ &\le \liminf_{\varepsilon\to 0} c(\lambda,\delta)\Big(|v_0(s)|^2_{L^2(I)} + \int_Q e^{-2\delta\alpha}s^{-4}\phi^{-3}f^2\,\mathrm{d}s\mathrm{d}t\Big) \le c, \end{aligned} \tag{42}$$

where c is a positive constant independent of ε. This convergence clearly shows that $u \in L^2(Q, e^{-2\delta\alpha}s^{-4}\phi^{-3})$ and this completes the proof of the Theorem 2. □

3.2. Controllability of nonlinear Black–Scholes equation

Consider the problem of exact controllability of general nonlinear Black–Scholes equation

$$\left.\begin{aligned} &\mathcal{L}v + g(s,t,v) = 1_\omega u(s,t) + f(s,t) && \text{in } Q\\ &v(a,t) = v(b,t) = 0 && \text{on } \Sigma\\ &v(s,0) = v_0(s) && \text{in } I, \end{aligned}\right\} \tag{43}$$

where the operator $\mathcal{L}$ is defined in (1). We impose the following assumptions on the function $g : I \times (0,T)\times\mathbb{R} \to \mathbb{R}$. It satisfies $g(s,t,0) = 0\ \forall\,(s,t) \in Q$ and global Lipschitz condition, that is, there exists a constant $M > 0$ such that

$$|g(s,t,\zeta_1) - g(s,t,\zeta_2)| \le M|\zeta_1 - \zeta_2| \quad \forall\,(s,t,\zeta) \in Q\times\mathbb{R}. \tag{44}$$

Now we are ready to state the main theorem of this section.

Theorem 3. Assume that $\sigma \in C^{2,1}(\bar{Q})$, $r \in C^1(\bar{Q})$, which satisfy (2),(3) and g satisfies $g(t,s,0) = 0$ and (44) a.e. $(s,t) \in Q$, $\zeta \in \mathbb{R}$. Then for each $v_0 \in H_0^1(I)$ there are $u \in L^2(Q)$ and $v \in L^2(0,T;H_0^1(I)\cap H^2(I))\cap H^1(0,T;L^2(I))$ satisfying (43) such that $v(s,T) \equiv 0$ a.e. in I.

Proof. The argument is standard and will be frequently used in the sequel. Consider the following family of problems of exact controllability

$$\left.\begin{aligned} &\mathcal{L}v + g_\varepsilon(s,t,v) - g_\varepsilon(s,t,0) = 1_\omega u(s,t) + f(s,t) && \text{in } Q\\ &v(a,t) = v(b,t) = 0 && \text{on } \Sigma\\ &v(s,0) = v_0(s) && \text{in } I, \end{aligned}\right\} \tag{45}$$

where the averaging factor of g_ε for $\varepsilon > 0$ is defined by

$$g_\varepsilon(s,t,v) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \rho\Big(\frac{|\tau - v|}{\varepsilon}\Big) g(s,t,\tau)\,\mathrm{d}\tau,$$

since ρ is a smooth function with compact support in the unit ball, that is, $\rho(z) \geq 0$ $\forall\, z \in \mathbb{R},\ \rho(z) = \rho(|z|)$ satisfies $\mathrm{supp}(\rho) \subset \{z : |z| \leq 1\}$ and $\int_{\mathbb{R}} \rho \mathrm{d}z \equiv 1$. Here one can see that

$$(g_\varepsilon(s,t,\zeta) - g_\varepsilon(s,t,0))|_{\zeta=0} = 0 \quad \forall\,(s,t) \in Q. \tag{46}$$

Furthermore, by the Lipschitz continuity of g, we have

$$|g_\varepsilon(s,t,\zeta_1) - g_\varepsilon(s,t,\zeta_2)| \leq \frac{M}{\varepsilon}\int_{\mathbb{R}} \rho\Big(\frac{|\tau|}{\varepsilon}\Big)\,\mathrm{d}\tau|\zeta_1 - \zeta_2| \ \ \forall\,(s,t) \in Q. \tag{47}$$

At the same time we infer, by the equations (46) and (47), that

$$g_\varepsilon(s,t,\zeta) - g_\varepsilon(s,t,0) = \widehat{g}_\varepsilon(s,t,\zeta)\zeta$$

and $|\widehat{g}_\varepsilon(s,t,\zeta)\zeta| \leq M|\zeta| \ \ \forall\,(s,t,\zeta) \in Q \times \mathbb{R}$. Now we consider the set

$$K = \{\zeta \in L^2(Q) : \|\zeta\|_{L^2(Q)} \leq \eta\},$$

the linear system

$$\left.\begin{aligned} &Lv + \widehat{g}_\varepsilon(s,t,\zeta)v = 1_\varepsilon u(s,t) + f(s,t) && \text{in } Q\\ &v(a,t) = v(b,t) = 0 && \text{on } \Sigma\\ &v(s,0) = v_0(s) && \text{in } I, \end{aligned}\right\} \tag{48}$$

and the set valued mapping $\Phi : K \to 2^K$ such that

$$\begin{aligned}\Phi(\zeta) \;=\; &\big\{v : v \in L^2(0,T;H_0^1(I) \cap H^2(I)) \cap H^1(0,T;L^2(I)),\ v(s,T) \equiv 0,\\ &\quad \text{and } \exists\ u \in L^2(Q) \text{ satisfying (42) such that } (u,v) \text{ satisfies (30)}\big\}.\end{aligned}$$

Clearly, by Theorem 2, $\Phi(\zeta)$ is closed and $\Phi(\zeta) \neq 0$ for each $\zeta \in K$ and has convex values in $L^2(Q)$. Now we are ready to apply Kakutani's fixed point theorem to show that Φ has a fixed point in K with respect to $L^2(Q)$ topology. Since from the estimates (33) and (42) we note that

$$\begin{aligned}&\|v(t)\|^2_{H_0^1(I)} + \int_Q v_t^2\,\mathrm{d}s\mathrm{d}t + \int_Q v_{ss}^2\,\mathrm{d}s\mathrm{d}t\\ &\quad\leq\ c(\lambda,\delta)\Big(|v_0(s)|^2_{L^2(I)} + \|v_0(s)\|^2_{H_0^1(I)} + \int_Q e^{-2\delta\alpha}s^{-4}\phi^{-3}f^2\,\mathrm{d}s\mathrm{d}t\Big)\end{aligned}$$

and by the Sobolev imbeddings

$$R.H.S. \leq c(\lambda,\delta)\Big(\|v_0(s)\|^2_{H_0^1(I)} + \|e^{-\delta\alpha}s^{-2}\phi^{-\frac{3}{2}}f\|^2_{L^2(Q)}\Big). \tag{49}$$

Therefore $\Phi(K) \subset K$ and by the preceding estimate, it follows via Aubin's compactness theorem that $\Phi(K)$ is relatively compact subset of $L^2(Q)$.

In addition to this for upper semi continuity of Φ, let $\zeta_\varepsilon \in K$, $\zeta_\varepsilon \to \zeta$ in $L^2(Q)$ and $v_\varepsilon \to v$ in $L^2(Q)$, where $v_\varepsilon \in \Phi(\zeta_\varepsilon)$ and let u_ε be the corresponding controls. Hence by (42) and (49), it follows that on a subsequence

$$\begin{aligned} u_\varepsilon &\to u \quad \text{weakly in } L^2(Q) \\ v_\varepsilon &\to v \quad \text{weakly in } L^2(0,T;H_0^1(I)\cap H^2(I))\cap H^1(0,T;L^2(I)), \\ &\qquad\;\; \text{strongly in } C(0,T;L^2(I)). \end{aligned}$$

Moreover, $(\widehat{g}_\varepsilon(s,t,\zeta_\varepsilon)v_\varepsilon - \widehat{g}(s,t,\zeta)v) \to 0$ weakly in $L^2(Q)$ and hence there is $v \in K$ such that $v \in \Phi(v)$. Consequently by replacing (u,v) by $(u_\varepsilon, v_\varepsilon)$ in (48) and passing to the limit we get the optimal pair (u,v) satisfying $v(s,T) \equiv 0$ as claimed. This concludes the proof. □

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