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ON FUZZIFICATION OF THE NOTION OF QUANTALOID

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The paper considers a fuzzification of the notion of quantaloid of K. I. Rosenthal, which replaces enrichment in the category of $\bigvee$-semilattices with that in the category of modules over a given unital commutative quantale. The resulting structures are called quantale algebroids. We show that their constitute a monadic category and prove a representation theorem for them using the notion of nucleus adjusted for our needs. We also characterize the lattice of nuclei on a free quantale algebroid. At the end of the paper, we prove that the category of quantale algebroids has a monoidal structure given by tensor product.

Keywords: many-value topology, monadic category, nucleus, quantale, quantale algebra, quantale algebroid, quantale module, quantaloid, tensor product

Classification: 06F07, 03E72, 16G99, 18B99, 18A40

1. INTRODUCTION

The paper considers a fuzzification of the notion of *quantaloid* introduced by K. I. Rosenthal [35] as a category enriched in the symmetric, monoidal closed category **CSLat**$(\bigvee)$ of $\bigvee$-semilattices (the concept was also studied by A. Pitts [33] under the name of **SL**-*category*). It appeared soon that the notion has applications in different contexts of theoretical computer science. In particular, S. Abramsky and S. Vickers in their paper on quantales and process semantics [1] use quantaloids to introduce the notion of typing on processes. Moreover, R. Betti and S. Kasangian [4] indicate how categories enriched in a certain quantaloid provide an appropriate categorical framework for considering tree automata. Stimulated by the work of S. Kasangian and R. Rosebrugh [21], the ideas were further developed in, e. g., [37, 38, 39]. On the other hand, [36] introduces the notion of *Girard quantaloid* generalizing the concept of Girard quantale, which provides the partially ordered semantics for the linear logic of J. Girard [10]. Quite recently, U. Höhle [17] proposed quantaloids as a categorical basis for many-valued mathematics. Taking up the idea, we showed in [40] that the notion is a cornerstone for powerset operator foundations for categorically-algebraic fuzzy sets theories, with applications in algebra and topology.

It is important to notice that the concept of quantaloid has a solid historical background. It appeared as a generalization of the notion of *quantale* introduced by C. J. Mulvey [28] as an attempt to provide a possible setting for constructive foundations of quantum mechanics, as well as to study the spectra of non-commutative C^*-algebras, which are locales in the commutative case (the corresponding generalization in ring theory, from rings to *ringoids*, has already been studied in, e. g., [27]). The combination of "quantum logic" and "locale" gave rise to "quantale". Although the word itself was coined only in 1986, the study of quantale-like structures goes back to the late 1930's, when the pioneering papers of M. Ward and R. P. Dilworth [8, 56, 57, 58] appeared, where the authors propose to develop a systematic theory of lattices, over which an auxiliary operation of multiplication or residuation is defined. The investigation takes its roots in purely ring-theoretic considerations. In particular, the lattices of one-sided and two-sided ideals of a ring provided the first examples of such structures. Since then the ideal theory of rings has been successfully studied under the auspice of quantales. Nowadays, the notion can boast many areas of application, e. g., in the field of *non-commutative* topology [29, 30].

Of particular interest is the theory of *modules over a quantale* [23, 31, 32, 39], motivated by the successful developments of the theory of modules over a ring [3, 18]. The first lattice analogy of ring module appeared in the paper of A. Joyal and M. Tierney [20] in connection with analysis of descent theory, and although the authors work with commutative structures most of their results are valid for non-commutative case as well. Modules over a unital quantale form the central idea in the unified treatment of process semantics developed in [1]. Stimulated by the results, we considered in [43] some properties of the category Q-**Mod** of such structures. In particular, we showed that Q-**Mod** is a monadic construct and considered some of its intrinsic properties. The study was continued in [41], where we provided an extension of the standard procedure of completion of a partially ordered set through the collection of all its lower sets.

It appeared that the category Q-**Mod** has a natural generalization in the form of the category Q-**Alg** of algebras over a given unital commutative quantale Q, suggested by the concept of algebra over a unital commutative ring [3, 18]. In [44], we considered a representation theorem for quantale algebras, which generalized the result of K. I. Rosenthal stating that for every quantale Q, there exists a semigroup S and a quantic nucleus j on the powerset of S such that Q is isomorphic to the range of j. The study of nuclei on quantale algebras was continued in [42], where we used the technique of nucleus (adjusted for our specific context) to construct coproducts of objects in the category Q-**Alg**. Moreover, there exists a relation between nuclei and (*existential*) *quantifiers* on quantale algebras [47], the latter being motivated by monadic logic of P. Halmos [15]. Further generalizations of the topic and its applications to many-valued topology can be found in [40, 45, 46].

This paper considers a unification of the notions of quantale algebra and quantaloid in the structure called *quantale algebroid*, which essentially provides a fuzzification of the concept of quantaloid. It is a category **A** enriched in the symmetric, monoidal closed category Q-**Mod** of modules over a unital commutative quantale Q. We show that the category Q-**Abrds** of Q-algebroids is monadic and provide a representation theorem for its objects using the notion of *quantale algebroidal nucleus*. We also characterize the (complete) lattice of nuclei on a quantale algebroid free over a given category, through the lattice of congruences on this category. Moreover, we prove that the category Q-**Abrds** has a monoidal structure given by tensor product.

The motivation for the extension of quantaloids came from a particular problem in lattice-valued topology, i. e., the need of a common framework for many-valued topological setting of C. L. Chang [7], J. A. Goguen [12] and its stratified version of R. Lowen [25]. It appeared that the concept of quantale algebroid was precisely the missing point in the desired unifying approach [46]. On the other hand, the fruitfulness of the notion provided the need for a somewhat deeper investigation of some of its properties. It is the aim of this paper to answer the challenge, developing the theory in question in the direction suggested by its applications.

A word is due on the term "fuzzification" in the title of the paper. The underlying idea of the claim comes from the fact that the category **Qtlds** of quantaloids is the category of algebras for a monad, which extends the powerset monad in the category **Set** of sets [39, Theorem 3.1.2], whereas the category Q-**Abrds** is the category of algebras for a monad, which extends the lattice-valued powerset monad in **Set** (Theorem 3.2), the latter one in its turn providing a fuzzification (according to the *Principle of Fuzzification* of J. A. Goguen [11]) of the aforesaid powerset monad, replacing crisp sets with their many-valued counterparts. In one word, quantaloids are based in crisp sets, whereas quantale algebroids rely on lattice-valued ones.

A categorically-minded reader will notice immediately that many results of this manuscript follow from the already well-developed theory of *enriched categories* (see, e. g., the classical book of G. M. Kelly [22], the somewhat more compact account of F. Borceux [5, Chapter 6], or the specific developments of F. W. Lawvere [24] on generalized metric spaces, R. Street [48, 49, 50] on cosmoi, I. Stubbe [51, 52, 53, 54, 55] on categories enriched in a quantaloid). Indeed, every monoidal category $\mathcal{V}$ gives rise to the category $\mathcal{V}$-**CAT** of $\mathcal{V}$-*categories* and $\mathcal{V}$-*functors*. For example, the category **CSLat**$(\bigvee)$-**CAT** is isomorphic to the above-mentioned category **Qtlds**. Given a unital commutative quantale Q, both Q and **CSLat**$(\bigvee)$ are **CSLat**$(\bigvee)$-categories (the former one denoted by $\mathbf{Q}$) and, therefore, one can consider the **CSLat**$(\bigvee)$-functor category $[\mathbf{Q}^{op}, \mathbf{CSLat}(\bigvee)]$, which is canonically symmetric, monoidal closed and, moreover, is isomorphic to the already mentioned category Q-**Mod**. The category (Q-**Mod**)-**CAT** in its turn is isomorphic to the category Q-**Abrds**, which is the study topic of this paper. Applying the standard results of enriched category theory, it follows that Theorem 3.1 is a consequence of [5, Proposition 6.4.7], Theorem 7.1 is a particular instance of [5, Corollary 6.3.2] and Theorem 7.2 follows from [5, Proposition 6.4.6]. On the other hand, the just mentioned results of [5] and their respective analogues of [22, (1.17), (1.18), (2.39)] provide the general categorically-theoretic constructions for an arbitrary symmetric, monoidal closed category $\mathcal{V}$, never touching the concrete case of quantale modules of this paper. K. I. Rosenthal [35, 38, 39] has gone forward, explicitly considering the category **CSLat**$(\bigvee)$-**CAT** (quantaloids), whereas I. Stubbe [51] developed the idea in another direction, investigating the category **Cat**$(\mathbf{Q})$ of categories enriched in a quantaloid $\mathbf{Q}$ ($\mathbf{Q}$-*enriched categories*), with the aim to extend the theory of $\mathcal{V}$-categories to *bicategories* (the reader should notice that K.I. Rosenthal [39, Chapter 3] started a similar theory using somewhat different naming conventions). None of them, however, turned his attention to the rather natural case of the category (Q-**Mod**)-**CAT**. It is the main goal of this paper to provide the explicit development of the above-mentioned results in the latter case, paying much attention to how the standard crisp set-theoretic constructions get transformed into their lattice-valued analogues, providing a fuzzification (in the above-mentioned sense of J. A. Goguen [11]) of the respective procedures. In one word, it is not the result itself, but the fuzzy aspects of its underlying machinery that plays the most important role for us. Some further extensions of the proposed theory are mentioned in the last section Conclusion, whereas here we would like to underline two natural steps forward, i. e., the categories (Q-**Abrds**)-**CAT** (following K. I. Rosenthal) and **Cat**(**A**) for a quantale algebroid **A** (following I. Stubbe).

The necessary categorical background can be found in [2, 16, 26]. For algebraic notions we recommend [3, 23, 34, 39]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e. g., with that of an adjoint situation.

2. QUANTALE ALGEBROIDS

In this section we introduce the category Q-**Abrds** of algebroids over a given unital commutative quantale Q. Let us start by recalling the definition of quantale [34].

Definition 2.1. A *quantale* is a triple $(Q, \leqslant, \otimes)$ such that

1. $(Q, \leqslant)$ is a $\bigvee$-semilattice, i. e., a partially ordered set having arbitrary joins;
2. $(Q, \otimes)$ is a semigroup;
3. $q \otimes (\bigvee S) = \bigvee_{s\in S}(q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s\in S}(s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

A quantale Q is said to be *unital* provided that there exists an element $\imath \in Q$ such that $(Q, \otimes, \imath)$ is a monoid. Q is said to be *commutative* provided that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in Q$.

Every quantale, being a complete lattice, has the largest element $\top$ and the smallest element $\bot$. The following are typical examples of quantales.

Example 2.2. Every *frame*, i. e., a complete lattice L with $a \wedge (\bigvee S) = \bigvee_{s\in S}(a \wedge s)$ for every $a \in L$ and every $S \subseteq L$ [19], is a commutative unital quantale, where $\otimes = \wedge$ and $\imath = \top$. In particular, the chain $\mathbf{2} = \{\bot, \top\}$ is a quantale.

Example 2.3. Let $(A, \cdot)$ be a semigroup. The powerset $\mathcal{P}(A)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{s \cdot t \mid s \in S,\ t \in T\}$. If $(A, \cdot, \imath)$ is a monoid, then $\mathcal{P}(A)$ is unital, with the unit $\{\imath\}$. If $(A, \cdot)$ is commutative, then so is $\mathcal{P}(A)$.

Example 2.3 provides the free quantale over a given semigroup [34]. We will generalize the construction later on, while producing free quantale algebroids from categories. This will justify the title of the paper, since the new notion will employ lattice-valued powersets instead of crisp ones.

Example 2.4. Let X be a set and let $\mathcal{R}(X)$ denote the set of all binary relations on X. $\mathcal{R}(X)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{(x,y) \in X \times X \mid (x,z) \in T \text{ and } (z,y) \in S \text{ for some } z \in X\}$. $\mathcal{R}(X)$ is unital, with the diagonal relation $\triangle = \{(x,x) \mid x \in X\}$ being the unit.

It is shown in [6] that every unital quantale is isomorphic to a *relational quantale*, i. e., to a subset of $\mathcal{R}(X)$ containing $\triangle$ and closed under composition of relations, with $\bigvee$ being (in general) different from unions (see [14] for a more general result).

Definition 2.5. Let Q_1 and Q_2 be quantales. A map $Q_1 \xrightarrow{f} Q_2$ is called a *quantale homomorphism* provided that f preserves $\otimes$ and $\bigvee$. A *unital quantale homomorphism* should additionally preserve the unit.

Definitions 2.1 and 2.5 give rise to the category **Quant** of quantales and quantale homomorphisms studied thoroughly in [23, 34]. On the next step, we recall the category Q-**Mod** of unital left modules over a given unital quantale Q [31, 39, 43]. Its definition is motivated by the classical category R-**Mod** of unital left modules over a unital ring R [3, 18].

Definition 2.6. Given a unital quantale Q, Q-**Mod** is the category, whose objects (*unital left Q-modules*) are pairs $(A, *)$, where A is a $\bigvee$-semilattice and $Q \times A \xrightarrow{*} A$ is a map (the *action* of Q on A) such that

1. $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$ and every $S \subseteq A$;
2. $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ for every $a \in A$ and every $S \subseteq Q$;
3. $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every q_1, $q_2 \in Q$ and every $a \in A$;
4. $1 * a = a$ for every $a \in A$;

and whose morphisms $(A, *) \xrightarrow{f} (B, *)$ (*unital left Q-module homomorphisms*) are $\bigvee$-preserving maps $A \xrightarrow{f} B$ with $f(q * a) = q * f(a)$ for every $a \in A$ and every $q \in Q$.

It should be noticed immediately that it is possible to define the category of modules over an arbitrary quantale by dropping off Item (4) of Definition 2.6. On the other hand, in [40], we showed that every category of modules over a non-unital quantale is equivalent to the category of unital modules over a unital extension of the quantale in question and, therefore, categories of non-unital modules (but not these modules themselves) as entities are redundant in mathematics.

For shortness sake, from now on, "Q-module" means "unital left Q-module". It is not difficult to see that the category **2-Mod** (recall the two-element quantale of Example 2.2) is isomorphic to the category $\mathbf{CSLat}(\bigvee)$ of $\bigvee$-semilattices and $\bigvee$-preserving maps. Notice as well that every unital quantale is a module over itself (with action given by multiplication).

On the next step, we define the category Q-**Alg** of algebras over a given unital commutative quantale Q. The definition is motivated by the category K-**Alg** of algebras over a commutative unital ring K [3, 18].

Definition 2.7. Given a unital commutative quantale Q, Q-**Alg** is the category, whose objects (Q-*algebras*) are triples $(A, *, \otimes)$ such that

1. $(A, *)$ is a Q-module;
2. $(A, \otimes)$ is a quantale;
3. $q * (a \otimes b) = (q * a) \otimes b = a \otimes (q * b)$ for every $a, b \in A$ and every $q \in Q$;

and whose morphisms $(A, *, \otimes) \xrightarrow{f} (B, *, \otimes)$ (Q-*algebra homomorphisms*) are those quantale homomorphisms $(A, \otimes) \xrightarrow{f} (B, \otimes)$, which are also Q-module homomorphisms.

It is not difficult to see that the category **2-Alg** is isomorphic to the category **Quant**. Notice as well that every unital commutative quantale is an algebra over itself (with action given by multiplication).

On the last step, we define the category of quantale algebroids, which will be the main object of our study. Recall from Introduction that the concept generalizes the notion of *quantaloid* of K. I. Rosenthal [39], defined as a category, whose hom-sets are $\bigvee$-semilattices, with composition in the category preserving $\bigvee$ in both variables. In the language of enriched category theory [22, 24, 49], this says that quantaloids are precisely the categories enriched in the above-mentioned symmetric, monoidal closed (or *autonomous* [39]) category **CSLat**$(\bigvee)$. Replacing **CSLat**$(\bigvee)$ with the category Q-**Mod** of Definition 2.6 (see [43, Proposition 8.15] for the proof of the result that the category in question is autonomous for a unital commutative quantale Q; see also [39, Theorem 5.3.1] for the stronger result on Q-**Mod** being a $*$-*autonomous category*), provides the following notion.

Definition 2.8. Given a unital commutative quantale Q, a Q-*algebroid* is a category **A** such that

1. for every **A**-objects A and B, the hom-set $\mathbf{A}(A, B)$ is a Q-module;
2. composition of morphisms in **A** preserves $\bigvee$ and the action of Q in both variables.

Notice that although formally one can drop both unitality and commutativity of the underlying quantale Q, the resulting category of Q-algebroids will be stripped of too many good properties. For example, removal of commutativity will make us to consider the category Q-**BMod** of Q-*bimodules*, the monoidal structure of which is not symmetric. Also the classical theory of algebras over a ring [3, 13, 18] uses both conditions, motivating our own setting to follow the suit.

Some important features of Q-algebroids are contained in the next lemma.

Lemma 2.9. Every Q-algebroid **A** has the following properties:

1. given **A**-morphisms f and g such that $f \circ g$ is defined, $q * (f \circ g) = (q * f) \circ g = f \circ (q * g)$ for every $q \in Q$;

2. given an **A**-object A, $\mathbf{A}(A, A)$ is a unital Q-algebra;
3. given $f \in \mathbf{A}(A, C)$ and $g \in \mathbf{A}(B, C)$, there exists $g \to_r f \in \mathbf{A}(A, B)$ such that for every $h \in \mathbf{A}(A, B)$, $g \circ h \leqslant f$ iff $h \leqslant g \to_r f$;
4. given $f \in \mathbf{A}(A, C)$ and $g \in \mathbf{A}(A, B)$, there exists $g \to_l f \in \mathbf{A}(B, C)$ such that for every $h \in \mathbf{A}(B, C)$, $h \circ g \leqslant f$ iff $h \leqslant g \to_l f$;
5. given $f \in \mathbf{A}(A, B)$ and $q \in Q$, there exists $q \twoheadrightarrow f \in \mathbf{A}(A, B)$ such that for every $h \in \mathbf{A}(A, B)$, $q * h \leqslant f$ iff $h \leqslant q \twoheadrightarrow f$.

The second item of Lemma 2.9 suggests the term "Q-algebroid". The following examples of quantale algebroids generalize (actually, fuzzify) the standard ones for quantaloids of, e. g., [35, 37, 39]. To avoid unnecessary repetitions in the examples, we assume that Q is a given unital commutative quantale.

Example 2.10. A quantale algebroid with one object is just a unital quantale algebra. In view of the remark, quantale algebroids can be thought of as quantale algebras "with many objects".

Example 2.11. The category Q-**Mod** is a Q-algebroid. In particular, $\mathbf{CSLat}(\bigvee)$ is a **2**-algebroid. On the other hand, the category Q-**Alg** is not even a quantaloid (the addition of quantale operation collapses everything).

In the next four examples, the required Q-module structure of the Q-algebroid in question is provided by the point-wise operations on hom-sets induced by Q.

Example 2.12. Q-**SetRel** is the category, whose objects are sets, and whose morphisms $X \xrightarrow{R} Y$ (*Q-relations*) are maps $X \times Y \xrightarrow{R} Q$. Given two morphisms $X \xrightarrow{R} Y$ and $Y \xrightarrow{S} Z$, their composition $X \xrightarrow{S \circ R} Z$ is defined by $S \circ R(x, z) = \bigvee_{y \in Y} S(y, z) \otimes R(x, y)$. Given a set X, the identity 1_X is defined by $1_X(x, y) = \imath$, if $x = y$; otherwise, $1_X(x, y) = \bot$.

Example 2.13. Q-**Ord** is the category, whose objects are *preordered sets* $(X, \leqslant)$ (the relation $\leqslant$ is reflexive and transitive), and whose morphisms $(X, \leqslant) \xrightarrow{R} (Y, \leqslant)$ (*Q-(order ideals)*) are Q-relations $X \xrightarrow{R} Y$ such that

1. if $x_1 \leqslant x_2$ in X, then $R(x_2, y) \leqslant R(x_1, y)$ for every $y \in Y$;
2. if $y_1 \leqslant y_2$ in Y, then $R(x, y_1) \leqslant R(x, y_2)$ for every $x \in X$.

Composition of morphisms is borrowed from the category Q-**SetRel**. Given a preordered set $(X, \leqslant)$, the identity $1_{(X,\leqslant)}$ is defined by $1_{(X,\leqslant)}(x, y) = \imath$, if $x \leqslant y$; otherwise, $1_{(X,\leqslant)}(x, y) = \bot$.

Example 2.14. Given a category $\mathbf{C}$, $\mathcal{P}_Q(\mathbf{C})$ is the category, whose objects are those of $\mathbf{C}$, and whose morphisms $A \xrightarrow{\alpha} B$ (*Q-subsets of* $\mathbf{C}(A, B)$) are maps $\mathbf{C}(A, B) \xrightarrow{\alpha} Q$. Given two morphisms $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$, their composition $A \xrightarrow{\beta \circ \alpha} C$ is defined by $\beta\circ\alpha(f) = \bigvee_{g\circ h=f} \beta(g)\otimes\alpha(h)$. Given a **C**-object C, the identity $1_C^{\mathcal{P}_Q(\mathbf{C})}$ is defined by $1_C^{\mathcal{P}_Q(\mathbf{C})}(f) = 1$, if $f = 1_C$; otherwise, $1_C^{\mathcal{P}_Q(\mathbf{C})}(f) = \bot$. $\mathcal{P}_Q(\mathbf{C})$ will be called the *free Q-algebroid on* **C**. The reason for the term is explained in the next section.

Example 2.15. Let **C** be a category and let A, B be **C**-objects. A *span from A to B* in **C** is a source $A \xleftarrow{f} \bullet \xrightarrow{g} B$. *Span*$(A,B)$ denotes the class of all spans from A to B. A *Q-crible* is a Q-subset of *Span*(A,B) such that if $(f,g) \in Span(A,B)$ and if $f\circ h$, $g\circ h$ are defined, then $\alpha(f,g) \leqslant \alpha(f\circ h, g\circ h)$.

Q-**Rel**(**C**) is the category, whose objects are those of **C**, and whose morphisms are Q-cribles $A \xrightarrow{\alpha} B$. Given two morphisms $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$, their composition $A \xrightarrow{\beta\circ\alpha} C$ is defined by $\beta\circ\alpha(f,h) = \bigvee\{\beta(g,h)\otimes\alpha(f,g) \mid (f,g) \in Span(A,B), (g,h) \in Span(B,C)\}$. For a **C**-object C, the identity $1_C^{Q\text{-}\mathbf{Rel}(\mathbf{C})}$ is given by $1_C^{Q\text{-}\mathbf{Rel}(\mathbf{C})}(f,g) = 1$, if $f = g$; otherwise, $1_C^{Q\text{-}\mathbf{Rel}(\mathbf{C})}(f,g) = \bot$.

The last example generalizes [39, Example (7) on p. 18], where the author constructs a quantaloid, which captures multiplication and residuation of ring ideals.

Example 2.16. Let $K = (K, +, \cdot, 0_K, 1_K)$ be a commutative unital ring. The set $\mathrm{Idl}(K)$ of ideals of K is a unital commutative quantale with the required operations defined as follows (notice that $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers):

1. given $A_i \in \mathrm{Idl}(K)$ for $i \in I$, $\bigvee_{i\in I} A_i = \{a_{i_1} + \ldots + a_{i_n} \mid a_{i_j} \in A_{i_j},\ n \in \mathbb{N}^+\}$;
2. given $A, B \in \mathrm{Idl}(K)$, $A \otimes B = \{a_1 \cdot b_1 + \ldots + a_n \cdot b_n \mid a_i \in A,\ b_i \in B,\ n \in \mathbb{N}^+\}$;
3. $1 = K$.

Let $(R, *)$ be a K-algebra with the unit 1_R [3, 18] and let $\mathrm{Idl}_K(R)$ be the set of two-sided ideals A of R such that $k * a \in A$ for every $k \in K$ and every $a \in A$. Straightforward computations show that $\mathrm{Idl}_K(R)$ is a unital $\mathrm{Idl}(K)$-algebra with the action of $\mathrm{Idl}(K)$ given by $J * A = \{k_1 * a_1 + \ldots + k_n * a_n \mid k_i \in J,\ a_i \in A,\ n \in \mathbb{N}^+\}$ for every $J \in \mathrm{Idl}(K)$ and every $A \in \mathrm{Idl}_K(R)$, and the unit being the whole R. Similarly, one shows that the set $\mathrm{Sgr}_K(R)$ of subgroups of R closed under K-action in the above-mentioned sense is a unital $\mathrm{Idl}(K)$-algebra, with the unit the subgroup $\{k * 1_R \mid k \in K\}$. Moreover, the sets $\mathrm{LIdl}(R)$ (resp. $\mathrm{RIdl}(R)$) of left (resp. right) ideals of R closed under K-action are (in general non-unital) $\mathrm{Idl}(K)$-algebras.

$\mathbf{A}(R)$ is the category, whose objects are 0 and 1, and whose hom-sets are defined by $\mathbf{A}(R)(0,0) = \mathrm{Idl}_K(R)$, $\mathbf{A}(R)(0,1) = \mathrm{RIdl}_K(R)$, $\mathbf{A}(R)(1,0) = \mathrm{LIdl}_K(R)$, $\mathbf{A}(R)(1,1) = \mathrm{Sgr}_K(R)$. Composition of morphisms is given by ideal multiplication. The aforesaid remarks provide $\mathbf{A}(R)$ with the structure of $\mathrm{Idl}(K)$-algebroid.

Unfortunately, we are still unable to generalize the example suggested by the notion of *coverage* $\mathcal{C}$ on a category **C** [39, Example (6) on p. 18].

Definition 2.17. Let **A** and **B** be Q-algebroids. A *Q-algebroid homomorphism* is a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ such that on hom-sets it induces a Q-module homomorphism $\mathbf{A}(A, A') \to \mathbf{B}(F(A), F(A'))$.

Definitions 2.8 and 2.17 provide the (quasi)category Q-**Abrds** of Q-algebroids and Q-algebroid homomorphisms. Moreover, there exists the obvious forgetful functor $Q\text{-}\mathbf{Abrds} \xrightarrow{U} \mathbf{CAT}$ to the (quasi)category **CAT** of categories and functors. One can easily see that the (quasi)category **2-Abrds** is isomorphic to the (quasi)category **Qtlds** of quantaloids and quantaloid homomorphisms of [35, 39].

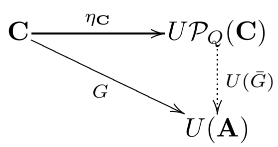
For convenience sake, from now on, we will not distinguish between (quasi)categories and categories. We also fix a unital commutative quantale Q and consider the category Q-**Abrds** of Q-algebroids.

3. THE CATEGORY OF QUANTALE ALGEBROIDS IS MONADIC

In this section we construct a monad on the category **CAT**, whose Eilenberg-Moore category [2] is isomorphic to Q-**Abrds**. To begin with, we provide an adjoint situation which will give us the desired monad.

Theorem 3.1. The forgetful functor $Q\text{-}\mathbf{Abrds} \xrightarrow{U} \mathbf{CAT}$ has a left adjoint.

Proof. It will be enough to show that every category $\mathbf{C}$ has an U-universal arrow, namely, a functor $\mathbf{C} \xrightarrow{\eta_{\mathbf{C}}} U\mathcal{P}_Q(\mathbf{C})$ such that every functor $\mathbf{C} \xrightarrow{G} U(\mathbf{A})$ extends to a unique Q-algebroid homomorphism $\mathcal{P}_Q(\mathbf{C}) \xrightarrow{\bar{G}} \mathbf{A}$, which makes the triangle



commute.

Let $\mathcal{P}_Q(\mathbf{C})$ be the category defined in Example 2.14. Notice that given a set X, every $x \in X$ gives rise to a Q-subset of X defined by $\alpha_x(y) = 1$, if $y = x$; otherwise, $\alpha_x(y) = \bot$. Define the desired functor $\mathbf{C} \xrightarrow{\eta_{\mathbf{C}}} U\mathcal{P}_Q(\mathbf{C})$ by $\eta_{\mathbf{C}}(C \xrightarrow{f} C') = C \xrightarrow{\alpha_f} C'$. Preservation of identities is clear. To show preservation of composition, notice that

$$(\eta_{\mathbf{C}}(g) \circ \eta_{\mathbf{C}}(f))(h) = \bigvee_{h' \circ h'' = h} (\eta_{\mathbf{C}}(g))(h') \otimes (\eta_{\mathbf{C}}(f))(h'') = \begin{cases} 1, & h = g \circ f \\ \bot, & \text{otherwise} \end{cases}$$
$$= (\eta_{\mathbf{C}}(g \circ f))(h).$$

For a given functor $\mathbf{C} \xrightarrow{G} U(\mathbf{A})$, define its required extension $\mathcal{P}_Q(\mathbf{C}) \xrightarrow{\bar{G}} \mathbf{A}$ by $\bar{G}(C \xrightarrow{\alpha} C') = G(C) \xrightarrow{\bigvee_{f \in \mathbf{C}(C,C')} \alpha(f) * G(f)} G(C')$. Straightforward computations show that $\bar{G}$ is the unique Q-algebroid homomorphism making the above-mentioned triangle commute. For example, to check that $\bar{G}$ preserves composition, notice that given $\mathcal{P}_Q(\mathbf{C})$-morphisms $C \xrightarrow{\alpha} C'$ and $C' \xrightarrow{\beta} C''$, $\bar{G}(\beta \circ \alpha) = \bigvee_{f \in \mathbf{C}(C,C'')} (\beta \circ \alpha)(f) * G(f) = \bigvee_{f \in \mathbf{C}(C,C'')} (\bigvee_{g \circ h = f} \beta(g) \otimes \alpha(h)) * G(f) = \bigvee_{f \in \mathbf{C}(C,C'')} \bigvee_{g \circ h = f} ((\beta(g) \otimes \alpha(h)) *$

$(G(g)\circ G(h)) = \bigvee_{f\in\mathbf{C}(C,C'')}\bigvee_{g\circ h=f}(\beta(g)*G(g))\circ(\alpha(h)*G(h)) = (\bigvee_{g\in\mathbf{C}(C',C'')}\beta(g)* G(g))\circ(\bigvee_{h\in\mathbf{C}(C,C')}\alpha(h)*G(h)) = \bar{G}(\beta)\circ\bar{G}(\alpha)$. □

The proof of Theorem 3.1 generalizes the construction of free quantaloid over a given category, suggested by K. I. Rosenthal [35, 39]. The crucial difference of our approach is the use of lattice-valued sets instead of the standard crisp ones (substitute the two-element quantale **2** with the quantale Q), and that justifies the claim in the title of this paper on fuzzification of the notion of quantaloid.

Another important remark is that Theorem 3.1 gives rise to the adjoint situation

$$(\eta,\varepsilon):\ \mathcal{P}_Q \longrightarrow\!\!\!\!\dashv U\ : Q\text{-}\mathbf{Abrds}\to\mathbf{CAT} \qquad (\mathcal{ADJ})$$

defined as follows

1. for a functor $\mathbf{C}\xrightarrow{H}\mathbf{D}$, $(\mathcal{P}_Q(H))(C\xrightarrow{\alpha}C') = H(C)\xrightarrow{\bigvee_{f\in\mathbf{C}(C,C')}\alpha(f)*\alpha_{H(f)}}H(C')$;
2. for a Q-algebroid $\mathbf{A}$, $\mathcal{P}_QU(\mathbf{A})\xrightarrow{\varepsilon_{\mathbf{A}}}\mathbf{A}$ is $\varepsilon_{\mathbf{A}}(A\xrightarrow{\alpha}A') = A\xrightarrow{\bigvee_{f\in\mathbf{A}(A,A')}\alpha(f)*f}A'$.

In a standard way [2], adjunction $(\mathcal{ADJ})$ gives rise to a monad $\mathtt{T}=(T,\eta,\mu)$ on **CAT** defined by $T=U\mathcal{P}_Q$ and $\mu=U\varepsilon\mathcal{P}_Q$. We are going to show that the Eilenberg-Moore category of the monad $\mathtt{T}$ is precisely the category Q-**Abrds**.

Start with some preliminary observations. In [43], we considered the *Q-powerset monad* on the category **Set** of sets and maps, based on the functor $\mathbf{Set}\xrightarrow{\mathcal{P}_Q}\mathbf{Set}$, $\mathcal{P}_Q(X\xrightarrow{f}Y) = Q^X\xrightarrow{f_Q^{\rightarrow}}Q^Y$, $(f_Q^{\rightarrow}(\alpha))(y) = \bigvee_{f(x)=y}\alpha(x)$, together with natural transformations $X\xrightarrow{\eta_X}\mathcal{P}_Q(X)$, $\eta_X(x)=\alpha_x$ and $\mathcal{P}_Q\mathcal{P}_Q(X)\xrightarrow{\mu_X}\mathcal{P}_Q(X)$, $(\mu_X(\hat{\alpha}))(x) = \bigvee_{\beta\in\mathcal{P}_Q(X)}\hat{\alpha}(\beta)*\beta(x)$. In particular, [43, Proposition 4.6] shows that the Eilenberg-Moore category of the monad is precisely the category Q-**Mod**. Given a set X, the algebra map $\mathcal{P}_Q(X)\xrightarrow{h}X$ yields the required module structure on X as follows (we make no distinction between a set and its characteristic function, since $\mathcal{P}(X)\cong\{\bot,1\}^X\subseteq Q^X$ for every Q with at least two elements): for every $x,y\in X$ let $x\leqslant y$ iff $h(\{x,y\})=y$; for every $S\subseteq X$ let $\bigvee S = h(S)$; for every $q\in Q$ and every $x\in X$ let $q*x = h(q*\{x\})$. With these preliminaries in hand, we can proceed to the main result of this section.

Theorem 3.2. The Eilenberg–Moore category of the monad $\mathtt{T}$ is precisely the category Q-**Abrds**.

P r o o f. Every algebra $(\mathbf{C},H)$ for the monad $\mathtt{T}$ comes equipped with the functor $\mathcal{P}_Q(\mathbf{C})\xrightarrow{H}\mathbf{C}$, which is the identity on objects, and on hom-sets it yields the maps $\mathcal{P}_Q(\mathbf{C}(A,B))\xrightarrow{H_{A,B}}\mathbf{C}(A,B)$ involving the above-mentioned Q-powerset monad. By the aforesaid discussion, $\mathbf{C}(A,B)$ is a $\bigvee$-semilattice, and functoriality of H implies that composition of morphisms in $\mathbf{C}$ preserves $\bigvee$ and the Q-action. For example, to show the former, notice that given $f\in\mathbf{C}(A,B)$ and $S\subseteq\mathbf{C}(B,C)$, $(\bigvee S)\circ f = H_{B,C}(S)\circ H_{A,B}(\{f\}) = H_{A,C}(S\circ\{f\}) = H_{A,C}(\{s\circ f\mid s\in S\}) = \bigvee\{s\circ f\mid s\in S\}$. □

Theorem 3.2 underlines the crucial difference between the categories **Qtlds** of quantaloids and Q-**Abrds** of Q-algebroids, the former being based on a generalization of the standard powerset monad $\mathcal{P}$ on the category **Set** [2, Example 20.2(3)] and the latter relying on the extension of the Q-powerset (essentially, lattice-valued) monad $\mathcal{P}_Q$. This was our main motivation in calling quantale algebroids a fuzzification of the notion of quantaloid.

Another remark should be made here. In [43, Proposition 4.12], we showed that the Kleisli category of the Q-powerset monad is precisely the category Q-**SetRel** of Example 2.12. It would be extremely useful for applications to generalize the result to our setting. Notice that the Kleisli category of the monad $\mathtt{T}$ has categories $\mathbf{C}$ as objects and functors $\mathbf{C} \xrightarrow{H} \mathcal{P}_Q(\mathbf{C}')$ as morphisms.

4. QUANTALE ALGEBROIDAL NUCLEI AND THEIR PROPERTIES

In this section we introduce the notion of *quantale algebroidal nucleus.* It generalizes the notions of quantaloidal nucleus [39, Definition 2.1.1] and quantale algebra nucleus [44, Definition 4.1], both taking their origin in the concept of *frame nucleus* [19].

Definition 4.1. Let $\mathbf{A}$ be a Q-algebroid. A *quantale algebroidal nucleus* on $\mathbf{A}$ is a map (in the functorial sense, i. e., taking objects to objects and morphisms to morphisms) $\mathbf{A} \xrightarrow{J} \mathbf{A}$ such that for every $\mathbf{A}$-morphisms f, g and every $q \in Q$,

1. $J(A \xrightarrow{f} B) = A \xrightarrow{J(f)} B$;
2. $f \leqslant g$ implies $J(f) \leqslant J(g)$;
3. $f \leqslant J(f)$;
4. $J \circ J(f) \leqslant J(f)$;
5. $J(f) \circ J(g) \leqslant J(f \circ g)$ provided that $f \circ g$ is defined;
6. $q * J(f) \leqslant J(q * f)$.

For shortness sake from now on "nucleus" will mean "quantale algebroidal nucleus". It is important to notice that every nucleus is a lax functor [39], and that allows to make Definition 4.1 a bit shorter. For convenience of the reader as well as to streamline the subsequent proceedings, we decided to state all properties explicitly.

The next proposition shows some consequences of Definition 4.1 (recall that every map $X \xrightarrow{f} Y$ extends to the image operator $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$).

Lemma 4.2. Given a nucleus J on a Q-algebroid $\mathbf{A}$, for every $f \in \mathbf{A}(A, B)$, $g \in \mathbf{A}(B, C)$, $q \in Q$ and every $S \subseteq \mathbf{A}(A, B)$, the following hold:

1. $J \circ J(f) = J(f)$;
2. $J(\bigvee S) = J(\bigvee J^{\rightarrow}(S))$;
3. $J(g \circ f) = J(g \circ J(f)) = J(J(g) \circ f) = J(J(g) \circ J(f))$;
4. $J(q * f) = J(q * J(f))$.

P r o o f. The proof consists of straightforward computations. For (1) notice that $J(f) \leqslant J \circ J(f)$ and thus, $J \circ J(f) = J(f)$. For (3) notice that $J(g) \circ J(f) \leqslant J(g \circ f)$ and, therefore, $J(J(g) \circ J(f)) \leqslant J \circ J(g \circ f) = J(g \circ f)$. On the other hand, $f \leqslant J(f)$ and $g \leqslant J(g)$ yield $g \circ f \leqslant J(g) \circ J(f)$ and, therefore, $J(g \circ f) \leqslant J(J(g) \circ J(f))$. □

Corollary 4.3. Suppose J is a nucleus on a Q-algebroid $\mathbf{A}$. Let $\mathbf{A}_J$ be the subgraph of $\mathbf{A}$ with the same objects as $\mathbf{A}$, and for morphisms $f \in \mathbf{A}(A, B)$, $f \in \mathbf{A}_J(A, B)$ iff $J(f) = f$. Given $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathbf{A}_J$, define $g \circ_J f = J(g \circ f)$. For every $\mathbf{A}$-object A let $1_A^{\mathbf{A}_J} = J(1_A)$. Then $\mathbf{A}_J$ is a Q-algebroid with the following structure:

1. $\bigvee_J S = J(\bigvee S)$ for every $S \subseteq \mathbf{A}_J(A, B)$;
2. $q *_J f = J(q * f)$ for every $f \in \mathbf{A}_J(A, B)$ and every $q \in Q$.

Moreover, the restriction $\mathbf{A} \xrightarrow{J} \mathbf{A}_J$ is a Q-algebroid homomorphism.

P r o o f. Straightforward computations and Lemma 4.2 show that $\mathbf{A}_J$ with the above-mentioned structure is a Q-algebroid. As an example, we show the second part of Item (2) of Definition 2.8. Given $f \in \mathbf{A}_J(A, B)$, $g \in \mathbf{A}_J(B, C)$ and $q \in Q$, $q*_J(g\circ_J f) = J(q*J(g\circ f)) = J(q*(g\circ f)) = J((q*g)\circ f) = J(J(q*g)\circ f) = (q*_J g)\circ_J f$ and, similarly, $(q*_J g) \circ_J f = g \circ_J (q*_J f)$. The last statement of the corollary follows from the definition of $\mathbf{A}_J$. □

The next lemma provides the necessary and sufficient condition for a subgraph of a quantale algebroid to be the image of a nucleus (recall Lemma 2.9).

Lemma 4.4. Let $\mathbf{S}$ be a subgraph of a Q-algebroid $\mathbf{A}$ containing all objects of $\mathbf{A}$. $\mathbf{S}$ is of the form $\mathbf{A}_J$ for some nucleus on $\mathbf{A}$ iff the following conditions are fulfilled:

1. each hom-set $\mathbf{S}(A, B)$ is closed under $\bigwedge$;
2. if $f \in \mathbf{S}(A, C)$ and $g \in \mathbf{C}(B, C)$, then $g \rightarrow_r f \in \mathbf{S}(A, B)$;
3. if $f \in \mathbf{S}(A, C)$ and $g \in \mathbf{C}(A, B)$, then $g \rightarrow_l f \in \mathbf{S}(B, C)$;
4. if $f \in \mathbf{S}(A, B)$ and $q \in Q$, then $q \twoheadrightarrow f \in \mathbf{S}(A, B)$.

P r o o f. *Necessity*: Since $\mathbf{A}$ is a quantaloid, the first three items follow from [39, Proposition 2.1.2]. For the last item, notice that $q * J(q \twoheadrightarrow f) \leqslant J(q * (q \twoheadrightarrow f)) \leqslant J(f) = f$ and, therefore, $J(q \twoheadrightarrow f) \leqslant q \twoheadrightarrow f$. *Sufficiency*: Define $\mathbf{A} \xrightarrow{J} \mathbf{A}$ by $J(A \xrightarrow{f} B) = A \xrightarrow{\bigwedge\{g \in \mathbf{S}(A,B) \mid f \leqslant g\}} B$. By [39, Proposition 2.1.2], J is a quantaloidal nucleus on $\mathbf{A}$. To show that $q * J(f) \leqslant J(q * f)$, notice that given $g \in \mathbf{S}(A, B)$ such that $q * f \leqslant g$, $f \leqslant q \twoheadrightarrow g \in \mathbf{S}(A, B)$ and, therefore, $J(f) \leqslant q \twoheadrightarrow g$. It immediately follows that $q * J(f) \leqslant g$, implying $q * J(f) \leqslant J(q * f)$. □

The next lemma shows that the family of nuclei on a given quantale algebroid is closed under the formation of point-wise $\bigwedge$ and, therefore, is a $\bigwedge$-semilattice. The proof follows from the respective result for quantaloids and a bit of calculation to show the last item of Definition 4.1.

Lemma 4.5. Given a Q-algebroid $\mathbf{A}$ and a family $(J_s)_{s\in S}$ of nuclei on $\mathbf{A}$, $\mathbf{A} \xrightarrow{J} \mathbf{A}$ defined by $J(A \xrightarrow{f} B) = A \xrightarrow{\bigwedge_{s\in S} J_s(f)} B$ is a nucleus on $\mathbf{A}$.

5. A REPRESENTATION THEOREM FOR QUANTALE ALGEBROIDS

With the help of the notions and results of the previous sections, we are ready to introduce a representation theorem for quantale algebroids, generalizing [39, Theorem 3.2.1], which states that for every quantaloid $\mathbf{Q}$, there exists a category $\mathbf{C}$ and a quantaloidal nucleus J on the free quantaloid over $\mathbf{C}$ such that $\mathbf{Q}$ is isomorphic to the range of J. To begin with, we construct a nucleus on the free Q-algebroid $\mathcal{P}_Q U(\mathbf{A})$ (recall Theorem 3.1 and adjunction $(\mathcal{ADJ})$ afterwards). Notice that since nuclei are the identities on objects, it is enough to give their morphism action only.

Lemma 5.1. Given a Q-algebroid $\mathbf{A}$, there exists a nucleus J on $\mathcal{P}_Q U(\mathbf{A})$ defined by the formula $(J(\alpha))(f) = f \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha)$.

P r o o f. Straightforward computations provide the result. For example, to show that $J(\alpha)\circ J(\beta) \leqslant J(\alpha\circ\beta)$ for every $\mathcal{P}_Q U(\mathbf{A})$-morphisms $A \xrightarrow{\beta} B$ and $B \xrightarrow{\alpha} C$, notice that given $f \in \mathbf{A}(A,C)$, $(J(\alpha)\circ J(\beta))(f) * f = (\bigvee_{g\circ h=f}(J(\alpha))(g) \otimes (J(\beta))(h)) * f = \bigvee_{g\circ h=f}((J(\alpha))(g) \otimes (J(\beta))(h)) * (g\circ h) = \bigvee_{g\circ h=f}((J(\alpha))(g) * g) \circ ((J(\beta))(h) * h) \leqslant \bigvee_{g\circ h=f} \varepsilon_{\mathbf{A}}(\alpha)\circ\varepsilon_{\mathbf{A}}(\beta) \leqslant \varepsilon_{\mathbf{A}}(\alpha)\circ\varepsilon_{\mathbf{A}}(\beta) = \varepsilon_{\mathbf{A}}(\alpha\circ\beta)$ and, therefore, $(J(\alpha)\circ J(\beta))(f) \leqslant f \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha\circ\beta) = (J(\alpha\circ\beta))(f)$. □

In order to distinguish it in the subsequent developments, the nucleus obtained in Lemma 5.1 will be denoted by $\boldsymbol{J}$. The next result shows a useful property of $\boldsymbol{J}$.

Lemma 5.2. Given a Q-algebroid $\mathbf{A}$ and a $\mathcal{P}_Q U(\mathbf{A})$-morphism $A \xrightarrow{\alpha} B$, it follows that $\varepsilon_{\mathbf{A}} \circ \boldsymbol{J}(\alpha) = \varepsilon_{\mathbf{A}}(\alpha)$.

P r o o f. $\varepsilon_{\mathbf{A}} \circ \boldsymbol{J}(\alpha) = \bigvee_{f\in\mathbf{A}(A,B)}(f \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha)) * f$ and $\varepsilon_{\mathbf{A}}(\alpha) = 1 * \varepsilon_{\mathbf{A}}(\alpha) \leqslant (\varepsilon_{\mathbf{A}}(\alpha) \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha)) * \varepsilon_{\mathbf{A}}(\alpha) \leqslant \bigvee_{f\in\mathbf{A}(A,B)}(f \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha)) * f \leqslant \varepsilon_{\mathbf{A}}(\alpha)$ yield $\varepsilon_{\mathbf{A}} \circ \boldsymbol{J}(\alpha) = \varepsilon_{\mathbf{A}}(\alpha)$. □

Given a Q-algebroid $\mathbf{A}$, every $\mathbf{A}$-morphism $A \xrightarrow{f} B$ gives rise to a $\mathcal{P}_Q U(\mathbf{A})$-morphism $A \xrightarrow{\bar{\alpha}_f} B$ defined by $\bar{\alpha}_f(g) = g \twoheadrightarrow f$ for every $g \in \mathbf{A}(A,B)$. Notice that $\alpha_f \leqslant \bar{\alpha}_f$, providing the motivation for our notation.

Lemma 5.3. Given a Q-algebroid $\mathbf{A}$ and an $\mathbf{A}$-morphism $A \xrightarrow{f} B$,

1. $\varepsilon_{\mathbf{A}}(\bar{\alpha}_f) = f$;
2. $\bar{\alpha}_f$ is a $(\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}$-morphism.

P r o o f. For (1): Since $\bar{\alpha}_f(g) * g \leqslant f$ for every $g \in \mathbf{A}(A,B)$, $\varepsilon_{\mathbf{A}}(\bar{\alpha}_f) \leqslant f$. On the other hand, $1 \leqslant \bar{\alpha}_f(f)$ implies $f \leqslant \bar{\alpha}_f(f) * f \leqslant \varepsilon_{\mathbf{A}}(\bar{\alpha}_f)$. For (2): Given $g \in \mathbf{A}(A,B)$, $(\boldsymbol{J}(\bar{\alpha}_f))(g) = g \twoheadrightarrow \varepsilon_{\mathbf{A}}(\bar{\alpha}_f) = g \twoheadrightarrow f = \bar{\alpha}_f(g)$ by (1). □

Lemmas 5.2 and 5.3 give rise to the following representation theorem (the main result of this section).

Theorem 5.4. (Representation theorem) Given a Q-algebroid $\mathbf{A}$, there exists a Q-**Abrds**-isomorphism $\mathbf{A} \xrightarrow{\rho} (\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}$ defined by $\rho(f) = \bar{\alpha}_f$.

P r o o f. Straightforward computations show that ρ is a Q-algebroid homomorphism. For example, for preservation of $\bigvee$, notice that given $S \subseteq \mathbf{A}(A, B)$ and $f \in \mathbf{A}(A, B)$, $(\bigvee_{\boldsymbol{J}} \rho^{\rightarrow}(S))(f) = (\boldsymbol{J}(\bigvee \rho^{\rightarrow}(S)))(f) = f \twoheadrightarrow \varepsilon_{\mathbf{A}}(\bigvee \rho^{\rightarrow}(S)) = f \twoheadrightarrow \bigvee(\varepsilon_{\mathbf{A}} \circ \rho)^{\rightarrow}(S) = f \twoheadrightarrow \bigvee S = (\rho(\bigvee S))(f)$.

Let $(\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}} \xrightarrow{\sigma} \mathbf{A}$ be the restriction of $\varepsilon_{\mathbf{A}}$ to $(\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}$. Easy computations show that σ is a quantale algebroid homomorphism. For example, to show that σ preserves composition, notice that given $(\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}$-morphisms α and β with $\alpha \circ_{\boldsymbol{J}} \beta$ defined, $\sigma(\alpha \circ_{\boldsymbol{J}} \beta) = \sigma \circ \boldsymbol{J}(\alpha \circ \beta) = \varepsilon_{\mathbf{A}} \circ \boldsymbol{J}(\alpha \circ \beta) = \varepsilon_{\mathbf{A}}(\alpha \circ \beta) = \varepsilon_{\mathbf{A}}(\alpha) \circ \varepsilon_{\mathbf{A}}(\beta) = \sigma(\alpha) \circ \sigma(\beta)$ by Lemma 5.2.

It is not difficult to check that $\sigma \circ \rho = 1_{\mathbf{A}}$ and $\rho \circ \sigma = 1_{(\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}}$, e. g., to show the last equality notice that given $\alpha \in (\mathcal{P}_Q U(\mathbf{A}))_{\boldsymbol{J}}(A, B)$ and $g \in \mathbf{A}(A, B)$, $\alpha(g) = (\boldsymbol{J}(\alpha))(g) = g \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha) = (\rho \circ \sigma(\alpha))(g)$. □

It follows that every quantale algebroid arises via a quantale algebroidal nucleus on a free quantale algebroid. In case of $Q = \mathbf{2}$, Theorem 5.4 turns into the representation theorem for quantaloids of K. I. Rosenthal [39, Theorem 3.2.1], since $\mathcal{P}_Q U(\mathbf{A}) \cong \mathcal{P}U(\mathbf{A})$ (using the notation of [39]). Given an $\mathbf{A}$-morphism $A \xrightarrow{f} B$, $\bar{\alpha}_f$ is the *lower set* $\downarrow f = \{g \in \mathbf{A}(A, B) \mid g \leqslant f\}$; given a $\mathcal{P}_Q U(\mathbf{A})$-morphism S, $\varepsilon_{\mathbf{A}}(S) = \bigvee S$. In particular, the nucleus $\boldsymbol{J}$ on $\mathcal{P}_Q U(\mathbf{A})$ is defined by $\boldsymbol{J}(S) = \downarrow(\bigvee S)$.

6. NUCLEI ON FREE QUANTALE ALGEBROIDS

In this section we characterize the lattice of nuclei on a free quantale algebroid $\mathcal{P}_Q(\mathbf{C})$ through the lattice of congruences on the category $\mathbf{C}$. Since every Q-algebroid is obtainable from a particular nucleus on $\mathcal{P}_Q(\mathbf{C})$ (Theorem 5.4), such a characterization is very important. Start by recalling the definition of *categorical congruence* [16].

Definition 6.1. Given a category $\mathbf{C}$, an equivalence relation $\sim$ on the class of morphisms of $\mathbf{C}$ is called a *congruence* on $\mathbf{C}$ provided that

1. every equivalence class under $\sim$ is contained in the hom-set $\mathbf{C}(A, B)$ for some $\mathbf{C}$-objects A, B;
2. whenever $f \sim f'$ and $g \sim g'$, it follows that $g \circ f \sim g' \circ f'$ provided that the compositions are meaningful.

The next two lemmas show a way of going from congruences to nuclei and back (recall our notations from Theorem 3.1).

Lemma 6.2. Let $\sim$ be a congruence on a category $\mathbf{C}$. Define $\mathcal{P}_Q(\mathbf{C}) \xrightarrow{J_\sim} \mathcal{P}_Q(\mathbf{C})$ by $(J_\sim(\alpha))(g) = \bigvee_{g \sim f} \alpha(f)$. Then $J_\sim$ is a nucleus on $\mathcal{P}_Q(\mathbf{C})$.

P r o o f. The proof consists of straightforward computations. For example, to show idempotency of J, notice that given $\alpha \in \mathcal{P}_Q(\mathbf{C})(A, B)$ and $g \in \mathbf{C}(A, B)$, $(J_\sim \circ J_\sim(\alpha))(g) = \bigvee_{g \sim f}(J_\sim(\alpha))(f) = \bigvee_{g \sim f} \bigvee_{f \sim h} \alpha(h) = \bigvee_{g \sim k} \alpha(k) = (J_\sim(\alpha))(g)$. □

Lemma 6.3. Given a category $\mathbf{C}$ and a nucleus J on $\mathcal{P}_Q(\mathbf{C})$, define a relation $\sim_J$ on $\mathbf{C}$-morphisms by $f \sim_J g$ iff $J(\alpha_f) = J(\alpha_g)$. Then $\sim_J$ is a congruence on $\mathbf{C}$.

Proof. By the definition, $\sim_J$ is an equivalence relation on $\mathbf{C}$-morphisms such that every equivalence class is contained in some $\mathbf{C}(A, B)$. To show that it is compatible with $\circ$, notice that given $f, f' \in \mathbf{C}(A, B)$ and $g, g' \in \mathbf{C}(B, C)$ such that $f \sim_J f'$ and $g \sim_J g'$, $\alpha_{g\circ f} = \alpha_g \circ \alpha_f \leqslant J(\alpha_g) \circ J(\alpha_f) = J(\alpha_{g'}) \circ J(\alpha_{f'}) \leqslant J(\alpha_{g'} \circ \alpha_{f'}) = J(\alpha_{g'\circ f'})$ and, therefore, $J(\alpha_{g\circ f}) \leqslant J(\alpha_{g'\circ f'})$. The converse inequality follows similarly. □

Let $\mathrm{N}(\mathcal{P}_Q(\mathbf{C}))$ denote the complete lattice of nuclei on $\mathcal{P}_Q(\mathbf{C})$ (recall Lemma 4.5) and let $\mathrm{Con}(\mathbf{C})$ denote the complete lattice of congruences on $\mathbf{C}$. Lemmas 6.2 and 6.3 give two maps: $\mathrm{Con}(\mathbf{C}) \xrightarrow{F} \mathrm{N}(\mathcal{P}_Q(\mathbf{C}))$, $F(\sim) = J_\sim$ and $\mathrm{N}(\mathcal{P}_Q(\mathbf{C})) \xrightarrow{G} \mathrm{Con}(\mathbf{C})$, $G(J) = \sim_J$. In the following, we show that the pair (G, F) provides a *Galois connection* [9, Chapter 0-3] between the partially ordered sets $\mathrm{N}(\mathcal{P}_Q(\mathbf{C}))$ and $\mathrm{Con}(\mathbf{C})$.

Lemma 6.4. Given $\sim \in \mathrm{Con}(\mathbf{C})$ and $J \in \mathrm{N}(\mathcal{P}_Q(\mathbf{C}))$, $F(\sim) \leqslant J$ iff $\sim \leqslant G(J)$. Moreover, $G \circ F = 1_{\mathrm{Con}(\mathbf{C})}$. If Q has at least two elements, then $F \circ G \neq 1_{\mathrm{N}(\mathcal{P}_Q(\mathbf{C}))}$.

Proof. Suppose $F(\sim) \leqslant J$. Given $g, f \in \mathbf{C}(A, B)$ such that $g \sim f$, $\alpha_f(f) = 1 = \alpha_g(g) \leqslant \bigvee_{f\sim f'} \alpha_g(f') = ((F(\sim))(\alpha_g))(f) \leqslant J(\alpha_g)(f)$ and, therefore, $\alpha_f \leqslant J(\alpha_g)$ that implies $J(\alpha_f) \leqslant J(\alpha_g)$. Similarly, $J(\alpha_g) \leqslant J(\alpha_f)$.

Suppose $\sim \leqslant G(J)$. For a $\mathcal{P}_Q(\mathbf{C})$-morphism α, $((F(\sim))(\alpha))(g) = \bigvee_{g\sim f} \alpha(f) \leqslant \bigvee_{gG(J)f} \alpha(f) = \bigvee_{J(\alpha_g)=J(\alpha_f)} \alpha(f)$. Since $J(\alpha_g) = J(\alpha_f)$ implies $\alpha(f) = (\alpha(f) * \alpha_g)(g) \leqslant (\alpha(f) * J(\alpha_g))(g) = (\alpha(f) * J(\alpha_f))(g) \leqslant (J(\alpha(f) * \alpha_f))(g) \leqslant (J(\alpha))(g)$, then $((F(\sim))(\alpha))(g) \leqslant (J(\alpha))(g)$ and thus, $((F(\sim))(\alpha) \leqslant J(\alpha)$.

To show that $G\circ F = 1_{\mathrm{Con}(\mathbf{C})}$, notice that given $\sim \in \mathrm{Con}(\mathbf{C})$ and $f, g \in \mathbf{C}(A, B)$, it follows that $g(G\circ F(\sim))f$ iff $(F(\sim))(\alpha_g) = (F(\sim))(\alpha_f)$ iff $\bigvee_{h\sim k} \alpha_g(k) = \bigvee_{h\sim k} \alpha_f(k)$ for every $h \in \mathbf{C}(A, B)$ iff ($h \sim g$ iff $h \sim f$, for every $h \in \mathbf{C}(A, B)$) iff $g \sim f$.

For the last statement, consider the nucleus $\boldsymbol{J}$ of Lemma 5.1. Given some $\alpha \in \mathcal{P}_Q U(\mathbf{A})(A, B)$ and $g \in \mathbf{A}(A, B)$, $((F \circ G(\boldsymbol{J}))(\alpha))(g) = \bigvee_{gG(\boldsymbol{J})f} \alpha(f) = \bigvee_{\boldsymbol{J}(\alpha_g)=\boldsymbol{J}(\alpha_f)} \alpha(f)$. Since $\boldsymbol{J}(\alpha_g) = \boldsymbol{J}(\alpha_f)$ yields $1 \leqslant g \twoheadrightarrow g = g \twoheadrightarrow \bigvee_{h\in\mathbf{C}(A,B)} \alpha_g(h) * h = g \twoheadrightarrow \varepsilon_{\mathbf{A}}(\alpha_g) = (\boldsymbol{J}(\alpha_g))(g) = (\boldsymbol{J}(\alpha_f))(g) = g \twoheadrightarrow f$, it follows that $g \leqslant f$ and, similarly, $f \leqslant g$. Thus $((F \circ G(\boldsymbol{J}))(\alpha))(g) = \alpha(g)$ and, therefore, $(F \circ G(\boldsymbol{J}))(\alpha) = \alpha$, i.e., $F \circ G(\boldsymbol{J}) = 1_{\mathcal{P}_Q U(\mathbf{A})}$. If Q has at least two elements, then for every $\mathbf{A}$-objects A and B, $\underline{\bot} \in \mathcal{P}_Q U(\mathbf{A})(A, B)$ ($\underline{\bot}$ is the constant map with value $\bot$) yields $(\boldsymbol{J}(\underline{\bot}))(\bot) = \bot \twoheadrightarrow \bot = \top \neq \bot = \underline{\bot}(\bot)$, i.e., $\boldsymbol{J}$ is not the identity on $\mathcal{P}_Q U(\mathbf{A})$. □

Given a category $\mathbf{C}$ and a congruence $\sim$ on $\mathbf{C}$ let $\mathbf{C}/\sim$ stand for the quotient category of $\mathbf{C}$ under $\sim$. The respective equivalence class of a $\mathbf{C}$-morphism f is denoted by $[f]_\sim$.

Lemma 6.5. Given a congruence $\sim$ on a category $\mathbf{C}$, the Q-algebroids $\mathcal{P}_Q(\mathbf{C}/\sim)$ and $(\mathcal{P}_Q(\mathbf{C}))_{J_\sim}$ are isomorphic.

Proof. Define $\mathcal{P}_Q(\mathbf{C}/\sim) \xrightarrow{H} (\mathcal{P}_Q(\mathbf{C}))_{J_\sim}$ by $H(A \xrightarrow{\alpha} B) = A \xrightarrow{H(\alpha)} B$, where $(H(\alpha))(f) = \alpha([f]_\sim)$. We prove that H is a Q-algebroid isomorphism. To show that

$J_{\sim}(H(\alpha)) = H(\alpha)$, notice that for $g \in \mathbf{C}(A,B)$, $(J_{\sim}(H(\alpha)))(g) = \bigvee_{g\sim f}(H(\alpha))(f) = \bigvee_{g\sim f}\alpha([f]_{\sim}) = \alpha([g]_{\sim}) = (H(\alpha))(g)$. Preservation of identities by H is easy. For preservation of composition, use the fact that given $\alpha \in \mathcal{P}_Q(\mathbf{C}/\sim)(A,B)$ and $\beta \in \mathcal{P}_Q(\mathbf{C}/\sim)(B,C)$, $(H(\beta\circ\alpha))(g) = (\beta\circ\alpha)([g]_{\sim}) = \bigvee_{[f]_{\sim}\circ[h]_{\sim}=[g]_{\sim}}\beta([f]_{\sim})\otimes\alpha([h]_{\sim}) = \bigvee_{[f\circ h]_{\sim}=[g]_{\sim}}\beta([f]_{\sim})\otimes\alpha([h]_{\sim}) = \bigvee_{f\circ h\sim g}\beta([f]_{\sim})\otimes\alpha([h]_{\sim}) = \bigvee_{g\sim k}\bigvee_{f\circ h=k}\beta([f]_{\sim})\otimes\alpha([h]_{\sim}) = \bigvee_{g\sim k}\bigvee_{f\circ h=k}(H(\beta))(f)\otimes(H(\alpha))(h) = \bigvee_{g\sim k}(H(\beta)\circ H(\alpha))(k) = (J_{\sim}(H(\beta)\circ H(\alpha)))(g) = (H(\beta)\circ_{J_{\sim}} H(\alpha))(g)$. Similar computations show that H preserves $\bigvee$ as well as the action of Q.

For injectivity of H on morphisms, notice that given $\alpha, \beta \in \mathcal{P}_Q(\mathbf{C}/\sim)(A,B)$ such that $H(\alpha) = H(\beta)$, $\alpha([f]_{\sim}) = (H(\alpha))(f) = (H(\beta))(f) = \beta([f]_{\sim})$. For surjectivity, notice that given $\alpha \in (\mathcal{P}_Q(\mathbf{C}))_{J_{\sim}}(A,B)$, one can define $\beta \in \mathcal{P}_Q(\mathbf{C}/\sim)(A,B)$ by $\beta([f]_{\sim}) = \alpha(f)$. Since $J_{\sim}(\alpha) = \alpha$, the definition of β is correct. Moreover, $(H(\beta))(f) = \beta([f]_{\sim}) = \alpha(f)$ yields $H(\beta) = \alpha$. □

It is important to notice that Lemma 6.5 generalizes [39, Proposition 3.2.3], which is stated for quantaloids. Our result provides a fuzzification of the achievement, bringing it into the realm of lattice-valued mathematics. The reader should also be aware of the fact that [35, 39] characterize nuclei on free quantaloids in terms of the so-called *cover systems*. Unfortunately, we are still unable to provide a many-valued analogue of the notion suitable for quantale algebroids.

7. TENSOR PRODUCT OF QUANTALE ALGEBROIDS

By Theorem 3.1, we know that the functor $\mathbf{CAT} \xrightarrow{\mathcal{P}_Q} Q\text{-}\mathbf{Abrds}$ preserves colimits and, in particular, coproducts. On the other hand, in [39, Proposition 2.2.1], K. I. Rosenthal constructed the *tensor product* $\mathbf{Q}_1 \boxtimes \mathbf{Q}_2$ of two quantaloids $\mathbf{Q}_1$, $\mathbf{Q}_2$, showing [39, Example on p. 24] that the new operation is compatible with the free quantaloid functor $\mathbf{CAT} \xrightarrow{\mathcal{P}_{\mathbf{2}}} \mathbf{Qtlds}$, in the sense that $\mathcal{P}_{\mathbf{2}}(\mathbf{C}_1 \times \mathbf{C}_2) \cong \mathcal{P}_{\mathbf{2}}(\mathbf{C}_1) \boxtimes \mathcal{P}_{\mathbf{2}}(\mathbf{C}_2)$. The result was motivated by the respective achievement of A. Joyal and M. Tierney [20] stated for two $\bigvee$-semilattices A, B and the standard powerset operator $\mathcal{P}$. It is the aim of this section to extend the property to quantale algebroids, thereby providing a fuzzification of the machinery of K. I. Rosenthal.

Theorem 7.1. The category Q-**Abrds** has (binary) tensor products.

P r o o f. Given two Q-algebroids $\mathbf{A}$ and $\mathbf{B}$, define the category $\mathbf{A}\boxtimes\mathbf{B}$ (the object-part of the desired tensor product) as follows.

The class of objects of the category in question is precisely the family $\{(A,B)\mid A \in \mathbf{A}, B \in \mathbf{B}\}$, whose elements (A,B) are formally denoted by $A\boxtimes B$. Given two objects $A\boxtimes B$ and $A'\boxtimes B'$, the hom-set $\mathbf{A}\boxtimes\mathbf{B}(A\boxtimes B, A'\boxtimes B')$ is the tensor product $\mathbf{A}(A,A') \boxdot \mathbf{B}(B,B')$ of the Q-modules $\mathbf{A}(A,A')$ and $\mathbf{B}(B,B')$, constructed in a straightforward way in, e. g., [31, 43]. Given objects $A\boxtimes B$, $A'\boxtimes B'$ and $A''\boxtimes B''$, the composition law $\circ$ is defined by commutativity of the diagram

$$\begin{array}{ccc}
(\mathbf{A}(A',A'')\boxdot\mathbf{B}(B',B''))\times(\mathbf{A}(A,A')\boxdot\mathbf{B}(B,B')) & \overset{\circ}{\dashrightarrow} & \\
\downarrow{\scriptstyle -\boxdot-} & & \\
(\mathbf{A}(A',A'')\boxdot\mathbf{B}(B',B''))\boxdot(\mathbf{A}(A,A')\boxdot\mathbf{B}(B,B')) & & \mathbf{A}(A,A'')\boxdot\mathbf{B}(B,B''), \\
\downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \bar{\circ}_{\mathbf{A}}\boxdot\bar{\circ}_{\mathbf{B}}} & \\
(\mathbf{A}(A',A'')\boxdot\mathbf{A}(A,A'))\boxdot(\mathbf{B}(B',B'')\boxdot\mathbf{B}(B,B')) & &
\end{array}$$

where $\bar{\circ}_{\mathbf{C}}$ for $\mathbf{C}\in\{\mathbf{A},\mathbf{B}\}$ is the unique morphism given by commutativity of the triangle

$$\begin{array}{ccc}
\mathbf{C}(C',C'')\times\mathbf{C}(C,C') & \xrightarrow{-\boxdot-} & \mathbf{C}(C',C'')\boxdot\mathbf{C}(C,C') \\
 & \searrow{\scriptstyle \circ_{\mathbf{C}}} & \vdots\ {\scriptstyle \bar{\circ}_{\mathbf{C}}} \\
 & & \mathbf{C}(C,C'').
\end{array}$$

Explicitly, the composition law is defined by $(f'\boxdot g')\circ(f\boxdot g)=(f'\circ f)\boxdot(g'\circ g)$. The identity morphism $1_{A\boxtimes B}$ on a given object $A\boxtimes B$ is provided by $1_A\boxdot 1_B$.

Straightforward computations show that $\mathbf{A}\boxtimes\mathbf{B}$ is a Q-algebroid. For example, to check that the composition law is associative, notice that given $\mathbf{A}\boxtimes\mathbf{B}$-morphisms $A\boxtimes B\xrightarrow{f\boxdot g}A'\boxtimes B'\xrightarrow{f'\boxdot g'}A''\boxtimes B''\xrightarrow{f''\boxdot g''}A'''\boxtimes B'''$, $(f''\boxdot g'')\circ((f'\boxdot g')\circ(f\boxdot g))=(f''\boxdot g'')\circ((f'\circ f)\boxdot(g'\circ g))=(f''\circ(f'\circ f))\boxdot(g''\circ(g'\circ g))=((f''\circ f')\circ f)\boxdot((g''\circ g')\circ g)=((f''\circ f')\boxdot(g''\circ g'))\circ(f\boxdot g)=((f''\boxdot g'')\circ(f'\boxdot g'))\circ(f\boxdot g)$. Moreover, preservation of $\bigvee$ and the action of Q by $\circ$ (in both components) follows directly from its definition through the above-mentioned rectangle, since each of the three maps involved in the construction has the feature in question.

To verify the universal property of tensor product, we define its morphism-part, namely, the functor $\mathbf{A}\times\mathbf{B}\xrightarrow{-\boxtimes-}\mathbf{A}\boxtimes\mathbf{B}$ by the formula $-\boxtimes-((A,B)\xrightarrow{(f,g)}(A',B'))=A\boxtimes B\xrightarrow{f\boxdot g}A'\boxtimes B'$. To show that the functor preserves composition, notice that given $\mathbf{A}\times\mathbf{B}$-morphisms $(A,B)\xrightarrow{(f,g)}(A',B')\xrightarrow{(f',g')}(A'',B'')$, $-\boxtimes-((f',g')\circ(f,g))=-\boxtimes-(f'\circ f,g'\circ g)=(f'\circ f)\boxdot(g'\circ g)=(f'\boxdot g')\circ(f\boxdot g)=-\boxtimes-(f',g')\circ-\boxtimes-(f,g)$. Moreover, the functor $-\boxtimes-$ is a Q-*algebroid bimorphism*, i.e., a Q-algebroid homomorphism in each component, or, restated differently,

1. for every $\mathbf{A}$-morphism $A\xrightarrow{f}A'$ and every $\mathbf{B}$-morphism $B\xrightarrow{g_i}B'$ together with $q_i\in Q$ for $i\in I$, $-\boxtimes-(\bigvee_{i\in I}(f,q_i*g_i))=-\boxtimes-((f,\bigvee_{i\in I}q_i*g_i))=f\boxdot(\bigvee_{i\in I}q_i*g_i)=\bigvee_{i\in I}f\boxdot(q_i*g_i)=\bigvee_{i\in I}q_i*(f\boxdot g_i)=\bigvee_{i\in I}q_i*((-\boxtimes-)(f,g_i))$;

2. for every $\mathbf{B}$-morphism $B\xrightarrow{g}B'$ and every $\mathbf{A}$-morphism $A\xrightarrow{f_i}A'$ together with $q_i\in Q$ for $i\in I$, $-\boxtimes-(\bigvee_{i\in I}(q_i*f_i,g))=\bigvee_{i\in I}q_i*((-\boxtimes-)(f_i,g))$.

Now comes the crucial universal property of the pair $(-\boxtimes-,\mathbf{A}\boxtimes\mathbf{B})$, i.e., given a Q-algebroid bimorphism $\mathbf{A}\times\mathbf{B}\xrightarrow{F}\mathbf{C}$, we have to verify that there exists a unique

Q-algebroid homomorphism $\mathbf{A} \boxtimes \mathbf{B} \xrightarrow{\bar{F}} \mathbf{C}$, making the following triangle commute:

$$\begin{array}{ccc} \mathbf{A} \times \mathbf{B} & \xrightarrow{-\boxtimes-} & \mathbf{A} \boxtimes \mathbf{B} \\ & {\scriptstyle F} \searrow & \vdots\ {\scriptstyle \bar{F}} \\ & & \mathbf{C}. \end{array} \tag{$\mathcal{D}1$}$$

To show the existence, define the action of $\bar{F}$ on objects by $\bar{F}(A \boxtimes B) = F(A, B)$. For the action on morphisms, notice that given $\mathbf{A} \times \mathbf{B}$-objects (A, B) and (A', B'), $\mathbf{A}(A, A') \times \mathbf{B}(B, B') \xrightarrow{F_{ABA'B'}} \mathbf{C}(F(A, B), F(A', B'))$ is a Q-module bimorphism (Q-module homomorphism in each component) and, therefore, there exists a unique Q-module morphism $\mathbf{A}(A, A') \boxdot \mathbf{B}(B, B') \xrightarrow{\bar{F}_{ABA'B'}} \mathbf{C}(F(A, B), F(A', B'))$, making the following triangle commute:

$$\begin{array}{ccc} \mathbf{A}(A, A') \times \mathbf{B}(B, B') & \xrightarrow{-\boxdot-} & \mathbf{A}(A, A') \boxdot \mathbf{B}(B, B') \\ & {\scriptstyle F_{ABA'B'}} \searrow & \vdots\ {\scriptstyle \bar{F}_{ABA'B'}} \\ & & \mathbf{C}(F(A, B), F(A', B')). \end{array}$$

In view of the remark, one can define $\bar{F}(\hbar) = \bar{F}_{ABA'B'}(\hbar)$ for every $\mathbf{A} \boxtimes \mathbf{B}$-morphism $A \boxtimes B \xrightarrow{\hbar} A' \boxtimes B'$. It is easy to see that $\bar{F}$ is a Q-algebroid homomorphism. For example, to show that $\bar{F}$ preserves composition, notice that given $\mathbf{A} \boxtimes \mathbf{B}$-morphisms $A \boxtimes B \xrightarrow{f \boxdot g} A' \boxtimes B' \xrightarrow{f' \boxdot g'} A'' \boxtimes B''$, $\bar{F}((f' \boxdot g') \circ (f \boxdot g)) = \bar{F}((f' \circ f) \boxdot (g' \circ g)) = \bar{F}_{ABA''B''}((f' \circ f) \boxdot (g' \circ g)) = \bar{F}_{ABA''B''} \circ (- \boxdot -)(f' \circ f, g' \circ g) = F_{ABA''B''}((f', g') \circ (f, g)) = F_{A'B'A''B''}(f', g') \circ F_{ABA'B'}(f, g) = \bar{F}(f' \boxdot g') \circ \bar{F}(f \boxdot g)$. Moreover, straightforward computations show that the aforesaid Diagram ($\mathcal{D}1$) commutes.

Uniqueness of the functor $\bar{F}$ is the consequence of the fact that given a Q-algebroid homomorphism $\mathbf{A} \boxtimes \mathbf{B} \xrightarrow{G} \mathbf{C}$ such that $G \circ - \boxtimes - = F$, it follows that $G(A \boxtimes B) = F(A, B)$ and, moreover, for every $\mathbf{A} \times \mathbf{B}$-objects (A, B) and (A', B'), the triangle

$$\begin{array}{ccc} \mathbf{A}(A, A') \times \mathbf{B}(B, B') & \xrightarrow{-\boxdot-} & \mathbf{A}(A, A') \boxdot \mathbf{B}(B, B') \\ & {\scriptstyle F_{ABA'B'}} \searrow & \vdots\ {\scriptstyle G_{ABA'B'}} \\ & & \mathbf{C}(F(A, B), F(A', B')) \end{array}$$

commutes, that yields $G_{ABA'B'} = \bar{F}_{ABA'B'}$. □

All preliminaries in their places, we can state the main result of this section.

Theorem 7.2. Given Q-algebroids $\mathbf{A}$ and $\mathbf{B}$, $\mathcal{P}_Q(\mathbf{A} \times \mathbf{B}) \cong \mathcal{P}_Q(\mathbf{A}) \boxtimes \mathcal{P}_Q(\mathbf{B})$.

P r o o f. In view of Theorem 3.1, there exists a unique Q-algebroid homomorphism $\mathcal{P}_Q(\mathbf{A}\times\mathbf{B}) \xrightarrow{F} \mathcal{P}_Q(\mathbf{A}) \boxtimes \mathcal{P}_Q(\mathbf{B})$, making the following diagram commute:

$$\begin{array}{ccc} \mathbf{A}\times\mathbf{B} & \xrightarrow{\eta_{\mathbf{A}\times\mathbf{B}}} & U\mathcal{P}_Q(\mathbf{A}\times\mathbf{B}) \\ {\scriptstyle \eta_{\mathbf{A}}\times\eta_{\mathbf{B}}}\downarrow & & \vdots\,{\scriptstyle UF} \\ U\mathcal{P}_Q(\mathbf{A})\times U\mathcal{P}_Q(\mathbf{B}) & \xrightarrow[-\boxtimes-]{} & U(\mathcal{P}_Q(\mathbf{A})\boxtimes\mathcal{P}_Q(\mathbf{B})). \end{array}$$

On the other hand, it is possible to define $\mathcal{P}_Q(\mathbf{A})\times\mathcal{P}_Q(\mathbf{B}) \xrightarrow{G} \mathcal{P}_Q(\mathbf{A}\times\mathbf{B})$ by $G((A,B) \xrightarrow{(\alpha,\beta)} (A',B')) = (A,B) \xrightarrow{\alpha\circledast\beta} (A',B')$ with the latter map being just $\mathbf{A}(A,A')\times\mathbf{B}(B,B') \xrightarrow{\alpha\circledast\beta} Q = \mathbf{A}(A,A')\times\mathbf{B}(B,B') \xrightarrow{\alpha\times\beta} Q\times Q \xrightarrow{\otimes} Q$. To show that G is a functor, we have to verify preservation of both composition and identities. For the former property, notice that given $\mathcal{P}_Q(\mathbf{A})\times\mathcal{P}_Q(\mathbf{B})$-morphisms $(A,B) \xrightarrow{(\alpha,\beta)} (A',B') \xrightarrow{(\alpha',\beta')} (A'',B'')$, $(G((\alpha',\beta')\circ(\alpha,\beta)))(f'',g'') = (G(\alpha'\circ\alpha,\beta'\circ\beta))(f'',g'') = ((\alpha'\circ\alpha)\circledast(\beta'\circ\beta))(f'',g'') = (\alpha'\circ\alpha)(f'')\otimes(\beta'\circ\beta)(g'') = (\bigvee_{f'\circ f=f''}(\alpha'(f')\otimes\alpha(f)))\otimes(\bigvee_{g'\circ g=g''}(\beta'(g')\otimes\beta(g))) = \bigvee_{f'\circ f=f''}\bigvee_{g'\circ g=g''}(\alpha'(f')\otimes\alpha(f)\otimes\beta'(g')\otimes\beta(g)) = \bigvee_{f'\circ f=f''}\bigvee_{g'\circ g=g''}(\alpha'(f')\otimes\beta'(g')\otimes\alpha(f)\otimes\beta(g)) = \bigvee_{(f',g')\circ(f,g)=(f'',g'')}(\alpha'(f')\otimes\beta'(g')\otimes\alpha(f)\otimes\beta(g)) = \bigvee_{(f',g')\circ(f,g)=(f'',g'')}(G(\alpha',\beta'))(f',g')\otimes(G(\alpha,\beta))(f,g) = (G(\alpha',\beta')\circ G(\alpha,\beta))(f'',g'')$. For the latter property, use the fact that given a $\mathcal{P}_Q(\mathbf{A})\times\mathcal{P}_Q(\mathbf{B})$-identity $(A,B) \xrightarrow{(1_A^{\mathcal{P}_Q(\mathbf{A})},\,1_B^{\mathcal{P}_Q(\mathbf{B})})} (A,B)$,

$$\begin{aligned}(G(1_A^{\mathcal{P}_Q(\mathbf{A})},\,1_B^{\mathcal{P}_Q(\mathbf{B})}))(f,g) = 1_A^{\mathcal{P}_Q(\mathbf{A})}(f)\otimes 1_B^{\mathcal{P}_Q(\mathbf{B})}(g) &= \begin{cases} 1_Q, & (f,g)=(1_A,1_B) \\ \bot, & \text{otherwise} \end{cases} \\ &= 1_{(A,B)}^{\mathcal{P}_Q(\mathbf{A}\times\mathbf{B})}(f,g).\end{aligned}$$

To show that G is a Q-algebroid bimorphism, use the fact that given a $\mathcal{P}_Q(\mathbf{A})$-morphism $A \xrightarrow{\alpha} A'$ and a $\mathcal{P}_Q(\mathbf{B})$-morphism $B \xrightarrow{\beta_i} B'$ together with $q_i\in Q$ for $i\in I$, $(G(\alpha,\bigvee_{i\in I} q_i * \beta_i))(f,g) = \alpha(f)\otimes(\bigvee_{i\in I} q_i*\beta_i)(g) = \bigvee_{i\in I} q_i*(\alpha(f)\otimes\beta_i(g)) = \bigvee_{i\in I} q_i*((G(\alpha,\beta_i))(f,g)) = (\bigvee_{i\in I} q_i * G(\alpha,\beta_i))(f,g)$ and, similarly, for the other component in question.

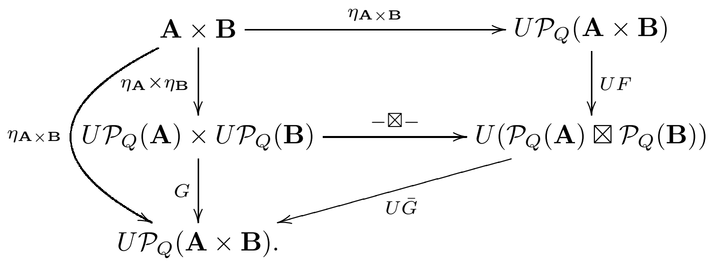
The just proved properties imply the existence of a unique Q-algebroid homomorphism $\mathcal{P}_Q(\mathbf{A})\boxtimes\mathcal{P}_Q(\mathbf{B}) \xrightarrow{\bar{G}} \mathcal{P}_Q(\mathbf{A}\times\mathbf{B})$, making the following triangle commute:

$$\begin{array}{ccc} \mathcal{P}_Q(\mathbf{A})\times\mathcal{P}_Q(\mathbf{B}) & \xrightarrow{-\boxtimes-} & \mathcal{P}_Q(\mathbf{A})\boxtimes\mathcal{P}_Q(\mathbf{B}) \\ & {\scriptstyle G}\searrow & \vdots\,{\scriptstyle \bar{G}} \\ & & \mathcal{P}_Q(\mathbf{A}\times\mathbf{B}). \end{array}$$

To verify the equality $\bar{G} \circ F = 1_{\mathcal{P}_Q(\mathbf{A}\times\mathbf{B})}$, notice that given an $\mathbf{A} \times \mathbf{B}$-morphism $(A, B) \xrightarrow{(f,g)} (A', B')$,

$$((G \circ (\eta_{\mathbf{A}} \times \eta_{\mathbf{B}}))(f, g))(f', g') = (G(\alpha_f, \beta_g))(f', g') = \alpha_f(f') \otimes \beta_g(g') = \begin{cases} 1_Q, & (f', g') = (f, g) \\ \bot, & \text{otherwise} \end{cases} = (\eta_{\mathbf{A}\times\mathbf{B}}(f, g))(f', g')$$

for every $\mathbf{A} \times \mathbf{B}$-morphism $(A, B) \xrightarrow{(f',g')} (A', B')$, and that gives commutativity of the diagram

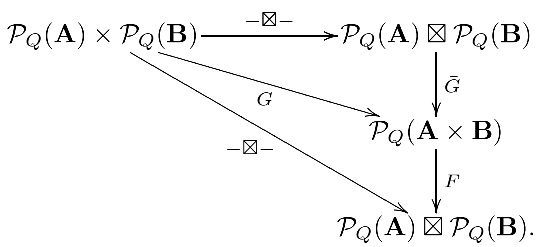


The desired identity now follows from the properties of the universal arrow $\eta_{\mathbf{A}\times\mathbf{B}}$ described in Theorem 3.1.

To show that $F \circ \bar{G} = 1_{\mathcal{P}_Q(\mathbf{A})\boxtimes\mathcal{P}_Q(\mathbf{B})}$, notice that given a $\mathcal{P}_Q(\mathbf{A}) \times \mathcal{P}_Q(\mathbf{B})$-morphism $(A, B) \xrightarrow{(\alpha,\beta)} (A', B')$,

$$\begin{aligned} F \circ \bar{G}(\alpha, \beta) &= \bigvee_{(f,g)\in\mathbf{A}\times\mathbf{B}((A,B),(A',B'))} (G(\alpha, \beta))(f, g) * (\alpha_f \boxdot \beta_g) = \\ &\bigvee_{(f,g)\in\mathbf{A}\times\mathbf{B}((A,B),(A',B'))} (\alpha(f) \otimes \beta(g)) * (\alpha_f \boxdot \beta_g) = \\ &\bigvee_{(f,g)\in\mathbf{A}\times\mathbf{B}((A,B),(A',B'))} (\alpha(f) * \alpha_f) \boxdot (\beta(g) * \beta_g) = \\ &\bigvee_{f\in\mathbf{A}(A,A')} \bigvee_{g\in\mathbf{B}(B,B')} (\alpha(f) * \alpha_f) \boxdot (\beta(g) * \beta_g) = \\ (\bigvee_{f\in\mathbf{A}(A,A')} \alpha(f) * \alpha_f) \boxdot (&\bigvee_{g\in\mathbf{B}(B,B')} \beta(g) * \beta_g) = \alpha \boxdot \beta = - \boxtimes -(\alpha, \beta), \end{aligned}$$

and that provides commutativity of the diagram:



The needed identity now follows from the properties of the universal arrow $-\boxtimes-$ shown in Theorem 7.1. □

Theorem 7.2 gives the promised fuzzification of the result of K. I. Rosenthal. Some additional properties of tensor product of quantaloids can be found in [51, 54, 55]. It will be the topic of our further research to fuzzify the already obtained theory.

8. CONCLUSION

Motivated by a particular problem of many-valued topology, in the paper we introduced the notion of *quantale algebroid* as a fuzzification of the notion of quantaloid and generalized some properties of quantaloids to the new setting. Among other results, we characterized quantale algebroids in terms of nuclei (adjusted for our needs) and provided a link between the latter notion and congruences on categories. Moreover, we showed that the category of quantale algebroids has a natural monoidal structure in the form of tensor product. The main advantage of the developments is the shift of the classical theory of quantaloids into the lattice-valued world.

By our opinion, it would be useful to fuzzify other properties of quantaloids as well, in order to bring the new theory to its completion. In particular, [35, 39] consider the notion of *nondeterministic functor* [35] or *relational presheaf* [39], which is a functor $\mathbf{C}^{op} \xrightarrow{F} \mathbf{SetRel}$ from the dual of some category $\mathbf{C}$ to the category $\mathbf{SetRel}$ of sets and relations. It appears that the concept has applications in different contexts (see [39] for a throughout discussion of the topic). Moreover, there exists a nice connection between relational presheaves and categories enriched in free quantaloids [39, Proposition 3.3.1]. By analogy with the above-mentioned concept, it is possible to define a *Q-relational presheaf* as a functor $\mathbf{C}^{op} \xrightarrow{F} Q\text{-}\mathbf{SetRel}$. It will be the topic of our further research to develop the theory of such structures, which will result in a lattice-valued modification of the theory of sheaves.

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REFERENCES

[1] S. Abramsky and S. Vickers: Quantales, observational logic and process semantics. Math. Struct. Comput. Sci. *3* (1993), 161–227.

[2] J. Adámek, H. Herrlich, and G. E. Strecker: Abstract and Concrete Categories: The Joy of Cats. Dover Publications, Mineola, New York 2009.

[3] F. W. Anderson and K. R. Fuller: Rings and Categories of Modules. Second edition. Springer-Verlag, 1992.

[4] R. Betti and S. Kasangian: Tree automata and enriched category theory. Rend. Ist. Mat. Univ. Trieste *17* (1985), 71–78.

[5] F. Borceux: Handbook of Categorical Algebra. Volume 2: Categories and Structures. Cambridge University Press, 1994.

[6] C. Brown and D. Gurr: A representation theorem for quantales. J. Pure Appl. Algebra *85* (1993), 1, 27–42.

[7] C. L. Chang: Fuzzy topological spaces. J. Math. Anal. Appl. *24* (1968), 182–190.

[8] R. P. Dilworth: Non-commutative residuated lattices. Trans. Amer. math. Soc. *46* (1939), 426–444.

[9] G. Gierz, K. Hofmann, et al.: Continuous Lattices and Domains. Cambridge University Press, 2003.

[10] J. Girard: Linear logic. Theor. Comput. Sci. *50* (1987), 1–102.

[11] J. A. Goguen: L-fuzzy sets. J. Math. Anal. Appl. *18* (1967), 145–174.

[12] J. A. Goguen: The fuzzy Tychonoff theorem. J. Math. Anal. Appl. *43* (1973), 734–742.

[13] P. A. Grillet: Abstract Algebra. Second edition. Springer-Verlag, 2007.

[14] R. Gylys: Involutive and relational quantaloids. Lith. Math. J. *39* (1999), 4, 376–388.

[15] P. Halmos: Algebraic Logic. Chelsea Publishing Company, 1962.

[16] H. Herrlich and G. E. Strecker: Category Theory. Third edition. Heldermann Verlag, 2007.

[17] U. Höhle: Quantaloids as categorical basis for many valued mathematics. In: Abstracts of the 31st Linz Seminar on Fuzzy Set Theory (P. Cintula, E. P. Klement, L. N. Stout, eds.), Johannes Kepler Universität, Linz 2010, pp. 91–92.

[18] T. Hungerford: Algebra. Springer-Verlag, 2003.

[19] P. T. Johnstone: Stone Spaces. Cambridge University Press, 1982.

[20] A. Joyal and M. Tierney: An extension of the Galois theory of Grothendieck. Mem. Am. Math. Soc. *309* (1984), 1–71.

[21] S. Kasangian and R. Rosebrugh: Decomposition of automata and enriched category theory. Cah. Topologie Géom. Différ. Catég. *27* (1986), 4, 137–143.

[22] G. M. Kelly: Basic concepts of enriched category theory. Repr. Theory Appl. Categ. *10* (2005), 1–136.

[23] D. Kruml and J. Paseka: Algebraic and categorical aAspects of quantales. In: Handbook of Algebra (M. Hazewinkel, ed.), 5, Elsevier, 2008, pp. 323–362.

[24] F. W. Lawvere: Metric spaces, generalized logic and closed categories. Repr. Theory Appl. Categ. *1* (2002), 1–37.

[25] R. Lowen: Fuzzy topological spaces and fuzzy compactness. J. Math. Anal. Appl. *56* (1976), 621–633.

[26] S. Mac Lane: Categories for the Working Mathematician. Second edition. Springer-Verlag, 1998.

[27] B. Mitchell: Rings with several objects. Adv. Math. *8* (1972), 1–161.

[28] C. Mulvey: & Rend. Circ. Mat. Palermo *II* (1986), 12, 99–104.

[29] C. J. Mulvey and J. W. Pelletier: On the quantisation of points. J. Pure Appl. Algebra *159* (2001), 231–295.

[30] C. J. Mulvey and J. W. Pelletier: On the quantisation of spaces. J. Pure Appl. Algebra *175* (2002), 1-3, 289–325.

[31] J. Paseka: Quantale Modules. Habilitation Thesis, Department of Mathematics, Faculty of Science, Masaryk University, Brno 1999.

[32] J. Paseka: A note on nuclei of quantale modules. Cah. Topologie Géom. Différ. Catégoriques *43* (2002), 1, 19–34.

[33] A. M. Pitts: Applications of sup-lattice enriched category theory to sheaf theory. Proc. Lond. Math. Soc. III. *57* (1988), 3, 433–480.

[34] K. I. Rosenthal: Quantales and Their Applications. Addison Wesley Longman, 1990.

[35] K. I. Rosenthal: Free quantaloids. J. Pure Appl. Algebra *72* (1991), 1, 67–82.

[36] K. I. Rosenthal: Girard quantaloids. Math. Struct. Comput. Sci. *2* (1992), 1, 93–108.

[37] K. I. Rosenthal: Quantaloidal nuclei, the syntactic congruence and tree automata. J. Pure Appl. Algebra *77* (1992), 2, 189–205.

[38] K. I. Rosenthal: Quantaloids, enriched categories and automata theory. Appl. Categ. Struct. *3* (1995), 3, 279–301.

[39] K. I. Rosenthal: The Theory of Quantaloids. Addison Wesley Longman, 1996.

[40] S. Solovjovs: Powerset operator foundations for categorically-algebraic fuzzy sets theories. In: Abstracts of the 31st Linz Seminar on Fuzzy Set Theory (P. Cintula, E. P. Klement, L. N. Stout, ed.), Johannes Kepler Universität, Linz 2010, pp. 143–151.

[41] S. Solovyov: Completion of partially ordered sets. Discuss. Math., Gen. Algebra Appl. *27* (2007), 59–67.

[42] S. Solovyov: On coproducts of quantale algebras. Math. Stud. (Tartu) *3* (2008), 115–126.

[43] S. Solovyov: On the category Q-**Mod**. Algebra Univers. *58* (2008), 35–58.

[44] S. Solovyov: A representation theorem for quantale algebras. Contr. Gen. Alg. *18* (2008), 189–198.

[45] S. Solovyov: Sobriety and spatiality in varieties of algebras. Fuzzy Sets Syst. *159* (2008), 19, 2567–2585.

[46] S. Solovyov: From quantale algebroids to topological spaces: fixed- and variable-basis approaches. Fuzzy Sets Syst. *161* (2010), 9, 1270–1287.

[47] S. Solovyov: On monadic quantale algebras: basic properties and representation theorems. Discuss. Math., Gen. Algebra Appl. *30* (2010), 1, 91–118.

[48] R. Street: Elementary cosmoi I. Lect. Notes Math. *420* (1974), 134–180.

[49] R. Street: Cauchy characterization of enriched categories. Repr. Theory Appl. Categ. *4* (2004), 1–16.

[50] R. Street: Enriched categories and cohomology. Repr. Theory Appl. Categ. *14* (2005), 1–18.

[51] I. Stubbe: Categorical structures enriched in a quantaloid: categories, distributors and functors. Theory Appl. Categ. *14* (2005), 1–45.

[52] I. Stubbe: Categorical structures enriched in a quantaloid: Orders and ideals over a base quantaloid. Appl. Categ. Struct. *13* (2005), 3, 235–255.

[53] I. Stubbe: Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories. Cah. Topol. Géom. Différ. Catég. *46* (2005), 2, 99–121.

[54] I. Stubbe: Categorical structures enriched in a quantaloid: tensored and cotensored categories. Theory Appl. Categ. *16* (2006), 283–306.

[55] I. Stubbe: $\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices. Theory Appl. Categ. *19* (2007), 4, 50–60.

[56] M. Ward: Residuation in structures over which a multiplication is defined. Duke math. J. *3* (1937), 627–636.

[57] M. Ward: Structure residuation. Ann. Math. *39* (1938), 558–568.

[58] M. Ward and R. P. Dilworth: Residuated lattices. Trans. Am. Math. Soc. *45* (1939), 335–354.

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