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THE CHOQUET INTEGRAL AS LEBESGUE INTEGRAL AND RELATED INEQUALITIES

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The integral inequalities known for the Lebesgue integral are discussed in the framework of the Choquet integral. While the Jensen inequality was known to be valid for the Choquet integral without any additional constraints, this is not more true for the Cauchy, Minkowski, Hölder and other inequalities. For a fixed monotone measure, constraints on the involved functions sufficient to guarantee the validity of the discussed inequalities are given. Moreover, the comonotonicity of the considered functions is shown to be a sufficient constraint ensuring the validity of all discussed inequalities for the Choquet integral, independently of the underlying monotone measure.

Keywords: Choquet integral, comonotone functions, integral inequalities, monotone measure, modularity

Classification: 28E10, 26D15

1. INTRODUCTION

The Lebesgue integral, supposing the σ-additivity of the underlying measure, is one of the most powerful tools in measure theory. As a typical example, recall probability theory, where the Lebesgue integral is just the expected value. The integral inequalities play an important role in applications of the Lebesgue integral. For example, the Minkowski inequality is essentially linked to L_p-norms. However, in several real situations, the additivity constraint of a measure does not allow to apply the Lebesgue integral, because of the non–additivity of involved measures. For example, in multicriteria problems, additivity means the non–interactivity of the involved criteria, which does not correspond to the majority of economical, sociological, etc. problems. A similar situation occurs in the case of search machines, e. g., in Google, where documents are listed based on their relevance to the given key words. To overcome the possible non–additivity defect problems, Choquet [3] has introduced an integral well defined for any monotone measure, and which – in the case when the underlying measure is σ–additive – coincides with the Lebesgue integral. The Choquet integral shares several properties with the Lebesgue integral, including some inequalities. For example, monotonicity and Jensen's inequality are common for both these integrals [7, 15]. However, there are some properties, including equalities and inequalities, which are valid for the Lebesgue integral, while in the case of the Choquet integral some additional constraints are needed. As a typical example in the case of equalities let us recall additivity, which is a genuine property of the Lebesgue integral, while for the Choquet integral it holds (for given functions f and g and any monotone measure m) only if the considered functions are comonotone [2, 10, 12, 13]. In the framework of inequalities, the subadditivity of the Choquet integral (for a given monotone measure m and any functions f and g) was shown to be equivalent to the submodularity of the involved monotone measure [2]. Similarly, the submodularity of the involved monotone measure is sufficient to guarantee the Hölder inequality [15]. However, the claim that the Hölder inequality holds for the Choquet integral in general [7] is not valid, see Example 4.3. The aim of this paper is the discussion of integral inequalities valid for the Lebesgue integral on any measure space (the Jensen, Cauchy, Hölder, Minkowski inequalities), in the framework of the Choquet integral. Having in mind the majority of applications, and because of the transparency reasons, we will deal with finite spaces and with non–negative functions only. Note that any simple function on an abstract space can be considered as a function on a finite space. Moreover, in the case of lower semi–continuous monotone measures, the Choquet integral on an abstract space can be introduced by means of simple functions [1]. Thus our results can easily be generalized to a general case, supposing the lower semi–continuity of the involved monotone measures. The paper is organized as follows. In the next section, the preliminary notions and definitions concerning monotone measures and the Choquet integral are given. In Section 3, the equality of the Lebesque and the Choquet integral for aggregated functions is studied. Based on the results from Section 3, in Section 4 several integral inequalities for the Choquet integral are proved. Finally, some concluding remarks are added.

2. THE CHOQUET INTEGRAL

Let $(X, \mathcal{A})$ be a measurable space (if X is finite, we put $\mathcal{A} = 2^X$ by convention). A set function $m\colon \mathcal{A} \to [0, \infty]$ is called a monotone measure whenever $m(\emptyset) = 0$, $m(X) > 0$ and for any $A, B \in \mathcal{A}$, $A \subseteq B$ it holds $m(A) \leq m(B)$. If $\lim_{n\to\infty} m(A_n) = m(A)$ whenever $(A_n) \in \mathcal{A}^{\mathbb{N}}$, $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \ldots$ and $A = \bigcup_{n\in\mathbb{N}} A_n$, the monotone measure m is called lower semi–continuous.

The Choquet integral was introduced in [3], see also [2, 10, 16].

Definition 2.1. Let $(X, \mathcal{A})$ be a measurable space, $m\colon \mathcal{A} \to [0, \infty]$ a monotone measure and $f\colon X \to [0, \infty]$ an $\mathcal{A}$-measurable function. The Choquet integral of f with respect to m is given by

$$(C)\int_X f\,\mathrm{d}m = \int_0^\infty m(f > t)\,\mathrm{d}t, \tag{1}$$

where the integral on the right-hand side is the (improper) Riemann integral.

Note that (1) can be rewritten as

$$(C)\int_X f\,\mathrm{d}m = \int_0^\infty m(f \geqslant t)\,\mathrm{d}t.$$

Moreover, if m is σ–additive, the Choquet and Lebesgue integrals coincide,

$$(C)\int_X f\,\mathrm{d}m = \int_X f\,\mathrm{d}m.$$

Evidently, the Choquet integral satisfies the following property:
(P) for any two monotone measures m_1, $m_2\colon \mathcal{A}\to[0,\infty]$ coinciding on $\mathcal{F}=\{\{f\geqslant t\}\}_{t=0}^{\infty}$ (or on $\{\{f>t\}\}_{t=0}^{\infty}$), the Choquet integrals with respect to them also coincide, i. e.,

$$(C)\int_X f\,\mathrm{d}m_1 = (C)\int_X f\,\mathrm{d}m_2.$$

Compare with the integral pair equivalence of (m_1, f) and (m_2, f) introduced in [5].

The next result can be straightforwardly derived from [1], compare also Theorem 4.4 in [14]. However, for the transparency of the paper, we add here an alternative proof.

Proposition 2.2. Let $(X,\mathcal{A})$ be a measurable space, $m\colon \mathcal{A}\to[0,\infty]$ a monotone measure and $f\colon X\to[0,\infty]$ an $\mathcal{A}$-measurable function. If X is a finite set or $f\colon X\to[0,\infty]$ is a simple function, i. e., *Ran* f is finite, then there exists a σ–additive measure $\mu_f\colon \sigma(\mathcal{F})\to[0,\infty]$ such that $\mu|_{\mathcal{F}} = m|_{\mathcal{F}}$, where $\sigma(\mathcal{F})$ is the smallest σ–algebra containing $\mathcal{F}=\{\{f\geqslant t\}\}_{t=0}^{\infty}$, such that

$$(C)\int_X f\,\mathrm{d}m = (C)\int_X f\,\mathrm{d}\mu_f = \int_X f\,\mathrm{d}\mu_f. \tag{2}$$

P r o o f. Suppose that X is finite (or $f\colon X\to[0,\infty]$ is simple, i. e., *Ran* f is finite). Therefore the set system $\mathcal{F}$ is a finite chain. Let $\sigma(\mathcal{F})$ be the smallest σ–algebra containing $\mathcal{F}$. Let *Ran* $f=\{a_1,\dots,a_k\}$, $0\leqslant a_1<\cdots<a_k$. Then

$$(C)\int_X f\,\mathrm{d}m = \sum_{i=1}^{k} a_i\left(m\left(f>a_{i-1}\right)-m\left(f>a_i\right)\right)$$

with convention $\{f>a_0\}=X$. Note that then $\sigma(\mathcal{F})$ is an atomic algebra with atoms

$$A_i=\{f>a_{i-1}\}\setminus\{f>a_i\}, \quad i=1,\dots,k$$

and the corresponding σ–additive measure $\mu_f\colon\sigma(\mathcal{F})\to[0,\infty]$ is determined by the values

$$\mu_f(A_i)=m\left(f>a_{i-1}\right)-m\left(f>a_i\right).$$

Therefore there exists a σ–additive measure $\mu_f\colon\sigma(\mathcal{F})\to[0,\infty]$ such that $\mu_f|_{\mathcal{F}} = m|_{\mathcal{F}}$, and by the property (P) it holds (2). □

Remark 2.3. In the general case, when $(X,\mathcal{A})$ is a measurable space, $m\colon\mathcal{A}\to[0,\infty]$ a monotone lower continuous measure and $f\colon X\to[0,\infty]$ an $\mathcal{A}$-measurable function then the set system $\mathcal{F}$ is a chain and thus closed under finite intersections and finite unions. Therefore, if m is lower semi–continuous, there is a σ–additive measure $\mu_f\colon\sigma(\mathcal{F})\to[0,\infty]$ such that $\mu_f|_{\mathcal{F}} = m|_{\mathcal{F}}$, where $\sigma(\mathcal{F})$ is the smallest σ–algebra containing $\mathcal{F}$ and by the property (P) it holds (2).

Due to [1] we also have the next result.

Proposition 2.4. Let $(X,\mathcal{A})$ be a measurable space and $m\colon \mathcal{A}\to[0,\infty]$ a lower semi–continuous monotone measure. Then for each $\mathcal{A}$-measurable function $f\colon X\to[0,\infty]$ it holds

$$(C)\int_X f\,\mathrm{d}m = \sup\left\{(C)\int_X s\,\mathrm{d}m \,|\, s \text{ is a simple function, } s\leqslant f\right\}.$$

Finally, we recall a result related to submodular monotone measures, see Proposition 10.3 in [2]. Note that $m\colon\mathcal{A}\to[0,\infty]$ is submodular whenever for all $A,\,B\in\mathcal{A}$ it holds

$$m(A\cap B)+m(A\cup B)\leqslant m(A)+m(B). \tag{3}$$

If the inequality in (3) is replaced by equality, the measure is called modular.

Proposition 2.5. Let $(X,\mathcal{A})$ be a measurable space and $m\colon\mathcal{A}\to[0,\infty]$ a submodular monotone measure. Then the set

$$M=\{\mu\,|\,\mu\colon\mathcal{A}\to[0,\infty],\ \mu\leqslant m,\ \mu \text{ is additive}\}$$

is non–empty and for each $\mathcal{A}$-measurable function $f\colon X\to[0,\infty]$ it holds

$$(C)\int_X f\,\mathrm{d}m=\sup\left\{(C)\int_X f\,\mathrm{d}\mu\,|\,\mu\in M\right\}.$$

Moreover, for each $A\in\mathcal{A}$, $m(A)=\sup\{\mu(A)\,|\,\mu\in M\}$.

Clearly, if X is finite (or f simple) then $(C)\int_X f\,\mathrm{d}\mu=\int_X f\,\mathrm{d}\mu$ is the Lebesgue integral for each $\mu\in M$.

3. THE CHOQUET INTEGRAL = THE LEBESGUE INTEGRAL?

As already observed, see Proposition 2.2 and Remark 2.3, if a monotone measure m is lower semi-continuous, then for any function f there is a σ-additive measure μ_f such that $(C)\int_X f\,\mathrm{d}m=\int_X f\,\mathrm{d}\mu_f$. Evidently, for any set system $\mathcal{S}\subseteq\mathcal{A}$ and any monotone measure m and σ-additive measure μ (defined on $\sigma(\mathcal{S})$), if $f\colon X\to[0,\infty]$ is $\mathcal{S}$–measurable, i. e., if $\mathcal{F}\subseteq\mathcal{S}$, and $m|_{\mathcal{F}}=\mu|_{\mathcal{F}}$, then the Choquet integral $(C)\int_X f\,\mathrm{d}m$ and the Lebesgue integral $\int_X f\,\mathrm{d}\mu$ coincide.

Lemma 3.1. Let X be a finite space and $f,g:X\to[0,\infty]$ two functions. Denote by

$$\mathcal{T}_{f,g}=\left\{\bigcup_{i=1}^{p}A_i\cap B_i\ |\ p\in\mathbb{N},\,A_i\in\mathcal{F},\,B_i\in\mathcal{G}\right\}$$

the smallest set system in 2^X containing both $\mathcal{F}=\{\{f\geqslant t\}\}_{t=0}^{\infty}$ and $\mathcal{G}=\{\{g\geqslant t\}\}_{t=0}^{\infty}$, which is closed under unions and intersections. Then evidently both f and g are script $\mathcal{T}_{f,g}$ - measurable, and moreover for any mapping $T:[0,\infty]^2\to[0,\infty]$ non-decreasing in both coordinates also the function $T(f,g):X\to[0,\infty]$ given by $T(f,g)(x)=T(f(x),g(x))$ is $\mathcal{T}_{f,g}$-measurable.

P r o o f. Recall that $T(f,g)=h$ is $\mathcal{H}$-measurable, where $\mathcal{H}=\{\{T(f,g)\geq t\}\}_{t=0}^{\max h}$. Let $Ran\, f=\{a_1,\dots,a_k\}$, with $0\leqslant a_1<\dots<a_k$, $Ran\, g=\{b_1,\dots,b_n\}$, with $0\leqslant b_1<\dots<b_n$ and $Ran\, h=\{c_1,\dots,c_r\}$, with $0\leqslant c_1<\dots<c_r$. Then $\mathcal{H}=\{\{h\geqslant c_i\}\}_{i=1}^{r}$ and there is a partition $\{C_1,\dots,C_r\}$ of $\{a_1,\dots,a_k\}\times\{b_1,\dots,b_n\}$ such that for each $i\in\{1,\dots,r\}$,

$$C_i=\left\{\left(a_{j_1}^{(i)},b_{j_1}^{(i)}\right),\dots,\left(a_{j_{k_i}}^{(i)},b_{j_{k_i}}^{(i)}\right)\right\}$$

and

$$T\left(a_{j_{k_1}}^{(i)},b_{j_{k_1}}^{(i)}\right)=\dots=T\left(a_{j_{k_i}}^{(i)},b_{j_{k_i}}^{(i)}\right)=c_i.$$

However, then

$$\{h\geqslant c_i\}=\bigcup_{q=1}^{k_i}\left(\left\{f\geqslant a_{j_q}^{(i)}\right\}\cap\left\{g\geqslant b_{j_q}^{(i)}\right\}\right)\in\mathcal{T}_{f,g}.$$

□

Observe that supposing the continuity of T, a similar result can be shown for an arbitrary space, modifying the finite unions in $\mathcal{T}_{f,g}$ by countable unions. The proof of this statement is related to the proof of Theorem 4.4 in [2] concerning the measurability of the sum $f+g$.

Theorem 3.2. Let X be a finite space and $m\colon 2^X\to[0,\infty]$ a monotone measure. Let functions $f,\ g\colon X\to[0,\infty]$ be given, such that using the notation from Lemma 3.1, m is modular on $\mathcal{T}_{f,g}$. Then there is an additive measure $\mu\colon 2^X\to[0,\infty]$ such that for any mapping $T\colon[0,\infty]^2\to[0,\infty]$ non–decreasing in both coordinates, it holds

$$(C)\int_X T(f,g)\,\mathrm{d}m=\int_X T(f,g)\,\mathrm{d}\mu. \tag{4}$$

P r o o f. By Lemma 3.1, it is enough to show that there is an additive measure μ on X such that $\mu|_{\mathcal{T}}=m|_{\mathcal{T}}$. Under the notation of Lemma 3.1, denote $F_i=\{f\geqslant a_i\}$, $i=1,\dots,k$, $G_j=\{g\geqslant b_j\}$, $j=1,\dots,n$. Then $X=F_1\supsetneqq\dots\supsetneqq F_k\neq\emptyset$, $X=G_1\supsetneqq\dots\supsetneqq G_n\neq\emptyset$, and thus $\{D_1,\dots,D_k\}$ where $D_i=F_i\setminus F_{i+1}$ (with convention $F_{k+1}=\emptyset$) and $\{E_1,\dots,E_n\}$ where $E_j=G_j\setminus G_{j+1}$ (with convention $G_{n+1}=\emptyset$) are the partitions of X. It is easy to check that $\{D_i\cap E_j\mid i\in\{1,\dots,k\},\ j\in\{1,\dots,n\}\}$ is the atomic partition of $\sigma(\mathcal{T}_{f,g})$, the smallest σ–algebra containing $\mathcal{T}_{f,g}$. Note that because of the finitness of X it is enough to consider algebras. Now, it is enough to define a set function μ on non–empty atoms $D_i\cap E_j$ by

$$\mu(D_i\cap E_j)=m(F_i\cap G_j)-m(F_i\cap G_{j+1})-m(F_{i+1}\cap G_j)+m(F_{i+1}\cap G_{j+1}),$$

and to extend it to $\sigma(\mathcal{T}_{f,g})$, supposing the additivity of μ. Evidently, from the modularity of m, it follows that $\mu|_{\mathcal{T}_{f,g}}=m|_{\mathcal{T}_{f,g}}$. If the atoms of $\sigma(\mathcal{T}_{f,g})$ are not singletons only, i.e., $2^X\neq\sigma(\mathcal{T})$, there are several possible extensions of μ to 2^X preserving the additivity of μ. □

As already mentioned in Introduction, the Choquet integral is not additive, in general.

Corollary 3.3. Let X be a finite space and $m\colon 2^X \to [0,\infty]$ a monotone measure. Let functions f, $g\colon X \to [0,\infty]$ be given, such that using the notation from Lemma 3.1, m is modular on $\mathcal{T}_{f,g}$. Then

$$(C)\int_X (f+g)\,\mathrm{d}m = (C)\int_X f\,\mathrm{d}m \ + (C)\int_X g\,\mathrm{d}m.$$

P r o o f. Since the requirements of Theorem 3.2 are satisfied, then obviously the additivity of the Choquet integral follows from the additivity of the Lebesgue integral. Indeed, it is enough to put $T_1(u,v) = u+v$, $T_2(u,v) = u$ and $T_3(u,v) = v$. Then

$$\begin{aligned}(C)\int_X (f+g)\,\mathrm{d}m &= \int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu \ + \int_X g\,\mathrm{d}\mu \\ &= (C)\int_X f\,\mathrm{d}m \ + (C)\int_X g\,\mathrm{d}m.\end{aligned}$$

□

Obviously, if f and g are comonotone, then $\mathcal{T}_{f,g} = \mathcal{F} \cup \mathcal{G}$ is a chain, and thus any monotone measure on X is modular on $\mathcal{T}_{f,g}$, which gives an alternative proof of the comonotone additivity of the Choquet integral. Of course, it holds in general, without any restriction of the cardinality of the space X, see [2, 10, 16].

4. INTEGRAL INEQUALITIES FOR THE CHOQUET INTEGRAL

Based on Theorem 3.2, several integral inequalities valid for the Lebesgue integral can also be proved for the Choquet integral. Observe again that if functions f, $g\colon X \to [0,\infty]$ are comonotone, the modularity of m over $\mathcal{T}_{f,g}$ is guaranteed for any monotone measure m on X. Recall that functions f, $g\colon X \to [0,\infty]$ are said to be comonotone whenever for all x, $y \in X$ it holds

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0.$$

Note that in this section we always suppose the finiteness of X.

Corollary 4.1. Let $\varphi : [0,\infty] \to [0,\infty]$ be a non-decreasing convex function. Then the Jensen inequality holds for the Choquet integral.

P r o o f. It is enough to consider $f = g$ and $T_1(u,v) = u$, $T_2(u,v) = \varphi(u)$, where $\varphi\colon [0,\infty] \to [0,\infty]$ is a convex non–decreasing function. Supposing $m(X) = 1$, we obtain

$$\varphi\left((C)\int_X f\,\mathrm{d}m\right) = \varphi\left(\int_X f\,\mathrm{d}\mu\right) \leqslant \int_X \varphi(f)\,\mathrm{d}\mu = (C)\int_X \varphi(f)\,\mathrm{d}m.$$

□

Note that we cannot omit the non-decreasingness of φ in the above Corollary. Take, for example, $X = \{x_1, x_2\}, f : X \to [0,\infty]$ given by $f(x_1) = 1, f(x_2) = 0.5, \mu : 2^X \to [0,1]$ given by $\mu(\varnothing) = 0, \mu(\{1\}) = \mu(\{2\}) = 1/4, \mu(X) = 1$, and consider a convex function $\varphi : [0,\infty] \to [0,\infty]$ given by $\varphi(x) = \max(0, 1-x)$. Then $(C)\int f\,dm = 5/8$, and $\varphi(5/8) = 3/8 > (C)\int \varphi(f)\,dm = 1/8$, violating the Jensen inequality. For the discussion of Jensen's inequality for the Choquet integral on an abstract space we recommend [15].

Note that the Jensen inequality for the Choquet integral also holds in a general case, see [15].

Corollary 4.2. Under the requirements of Theorem 3.2, the Cauchy, Hölder and Minkowski inequalities hold for the Choquet integral.

Proof. Recall that the Hölder inequality for the Lebesgue integral is of the form

$$\int_X fg\,\mathrm{d}\mu \leqslant \left(\int_X f^p\,\mathrm{d}\mu\right)^{1/p} \cdot \left(\int_X g^q\,\mathrm{d}\mu\right)^{1/q} \tag{5}$$

where p and $q \in]1,\infty[$, $\frac{1}{p} + \frac{1}{q} = 1$. Observe that if $p = q = 2$, then (5) turns to the Cauchy inequality. Now, it is enough to put $T_1(u,v) = uv$, $T_2(u,v) = u^p$, $T_3(u,v) = v^q$, and then

$$\begin{aligned} (C)\int_X fg\,\mathrm{d}m &= \int_X fg\,\mathrm{d}\mu \\ &\leqslant \left(\int_X f^p\,\mathrm{d}\mu\right)^{1/p} \cdot \left(\int_X g^q\,\mathrm{d}\mu\right)^{1/q} \\ &= \left((C)\int_X f^p\,\mathrm{d}m\right)^{1/p} \cdot \left((C)\int_X g^q\,\mathrm{d}m\right)^{1/q}. \end{aligned} \tag{6}$$

Similarly, one can prove the Minkowski inequality, i.e., that for any $p \in [1,\infty[$ it holds

$$\left((C)\int_X (f+g)^p\,\mathrm{d}m\right)^{1/p} \leqslant \left(\int_X f^p\,\mathrm{d}m\right)^{1/p} + \left(\int_X g^p\,\mathrm{d}m\right)^{1/p}. \tag{7}$$

□

Note that the above inequalities do not hold in general, if the modularity of m on $\mathcal{T}_{f,g}$ is violated.

Example 4.3. Let $m = m_*$ be the weakest normed monotone measure m on X, i.e, $m_*(X) = 1$ and if $A \neq X$, then $m_*(A) = 0$. It holds $(C)\int_X f\,\mathrm{d}m_* = \min\{f(x) \mid x \in X\}$ ($= \min f$, for short) and then the Hölder inequality (6) turns into

$$\min(fg) \leq (\min f) \cdot (\min g), \tag{8}$$

while the Minkowski inequality (7) turns into

$$\min(f+g) \leq (\min f) + (\min g). \tag{9}$$

Obviously, the inequalities opposite to (8) and (9) are always satisfied, and thus inequalities (8) and (9) can be changed into equalities

$$\min(fg) = (\min f) \cdot (\min g), \tag{10}$$

and

$$\min(f+g) = (\min f) + (\min g), \tag{11}$$

respectively. For $\emptyset \neq A,\ B \subseteq X$, let $f = 2 - 1_A$, $g = 2 - 1_B$. Then $\min f = \min g = 1$. However, if $A \cap B = \emptyset$, then $\min(fg) = 2$ and $\min(f+g) = 3$, i. e., both equalities (10) and (11) are violated. Note that in this case

$$\mathcal{F} = \{\emptyset, A^c, X\}, \quad \mathcal{G} = \{\emptyset, B^c, X\} \quad \text{and} \quad \mathcal{T}_{f,g} = \{\emptyset, (A \cup B)^c, A^c, B^c, X\}.$$

However, then

$$m_*(A^c) = m_*(B^c) = m_*(A^c \cap B^c) = 0,$$

while

$$m_*(A^c \cup B^c) = m_*(X) = 1,$$

i. e. m_* is not modular on $\mathcal{T}_{f,g}$.

Remark 4.4.

(i) All discussed inequalities hold for any monotone measure m on X whenever f and g are comonotone. This is a consequence of already mentioned fact that the comonotonicity of f and g ensures the validity of Theorem 3.2 for any monotone measure m on X.

(ii) Proposition 2.5 allows to show all considered inequalities to be valid whenever the considered monotone measure m is submodular, compare also [15].

(iii) If $(X, \mathcal{A})$ is an abstract measurable space (not necessarily finite) and m is a monotone measure on $(X, \mathcal{A})$ which is lower semi–continuous, one can apply Proposition 2.4 to show the validity of all discussed inequalities. Here the modularity of m on $\mathcal{T}_{f,g}$ - the smallest set system in $\mathcal{A}$ containing $\mathcal{F}$ and $\mathcal{G}$, and closed under unions and intersections – is again considered.

5. CONCLUDING REMARKS

We have introduced a framework in which the Choquet integral with respect to a monotone measure m coincides with the Lebesgue integral with respect to a σ–additive measure μ. Consequently, several integral inequalities known for the Lebesgue integral are also valid for the Choquet integral in this framework. The comonotonicity of the involved functions was shown to be sufficient to ensure the validity of all discussed integral inequalities for the Choquet integral with respect to any monotone measure. Note that similar attempts in the framework of the Sugeno integral and also some other kinds of integrals have recently been done in several works, e. g., in [4, 6, 8, 9, 11]. Though the nature of the Sugeno integral is rather different from the nature of the Lebesgue and Choquet integrals – the Sugeno integral is related to the lattice operations $\vee$ and $\wedge$, while the Lebesgue and Choquet integrals are based on the arithmetic operations $+$ and $\cdot$ on $\mathbb{R}$ – also there the comonotonicity of the involved functions was shown to be essential for proving some integral inequalities.

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