Quantum Theory.
A Mathematical Approach

Würzburg, March 18, 2015

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Contents

1. General introduction

Historically mathematics and physics were closely related subjects. All the famous mathematicians in the past were familiar with theoretical physics and made important contributions to it: in the first place Isaac Newton, his successors Joseph-Louis Lagrange, Pierre-Simon Laplace and William Rowan Hamilton. In the 19th and 20th century David Hilbert, John von Neumann, Henri Poincaré, Hermann Weyl, Elie Cartan, etc.. Since roughly the end of the second World War this is no longer the case. Reasons? Probably the influence of the Bourbaki movement, which revolutionized the formulation of mathematics, made it much more formal, 'abstract', more difficult to understand for physicists. And, of course, increasing specialization.

When I studied physics, mathematics students had to follow a few thorough courses in physics, in quantum mechanics, for example. Nowadays, certainly in the Netherlands, someone who studies mathematics won't in general learn anything about physics. As a consequence the present generation of mathematicians know little about modern physics, in particular very little about the two great theories that revolutionized 20th century physics, relativity and quantum theory.

Those who are nevertheless interested in these topics, find most physics books to be unaccessible, because of the loose, intuitive and sloppy mathematical language used.

Recently books have appeared that try to remedy this. Three to the best of my knowledge:

• Valter Moretti
Spectral Theory and Quantum Mechanics
Springer 2013,

• Brian Hall
Quantum Mechanics for Mathematicians
Springer 2014,

and finally my own book, published in December last year:

• Peter Bongaarts
Quantum Theory. A Mathematical Approach
Springer 2014.

See
`http://www.springer.com/physics/quantum`physics/book/978-3-319-09560-8

Not surprisingly these three books have a certain amount of overlap. Each has its strong points; they are complementary. This talk is based on my own book.

To understand the underlying mathematical structure of the great physical theories, in particular relativity and quantum theory, one needs to know such topics as functional analysis, Lie groups and algebra, differential geometry. That makes it easy for mathematicians to acquire a basic understanding of these theories. Physicist are not familiar with this kind of modern mathematics; physics textbooks do not use it. This makes getting a grip on the basic structure of quantum mechanics a long and relatively difficult procedure for physics students. Learning applications to specific explicit situations and applications is, of course, a different matter.

2. A bit of history

In the impressive building of classical physics, as it existed at the end of the 19th century, with as its main pillars Newton's classical mechanics and Maxwell's theory of electromagnetism, two small but embarrassing problems remained. One was the aether; this problem was solved by Einstein's special theory of relativity. The other was the problem of atomic spectra. Atoms can emit radiation. For example, NaCl gives in a flame yellow light, in fact with two slightly different sharp frequencies. This is a general phenomenon. All atoms have such systems of frequencies; all are different and characteristic for the type of atom. Most of these frequencies had been precisely measured by spectroscopists in the second half of the 19th century. The problem was that these spectra, and in particular their discreteness could not be understood by, and were in fact in total contradiction with classical physics.

The first step in the solution of this problem was taken by Max Planck. He found in 1900 that he could solve the longstanding problem of the distribution of frequencies in radiation in a cavity with non absorbing walls ('black-body radiation') by the ad hoc assumption that energy between the wall of the cavity and the radiation was exchanged in discrete energy packets.

• M. Planck
Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum
Verhandl. der Deutschen Physikal. Gesellschaft 17. 237-245 (1900)
Accessible at
`http://www.christoph.mettenheim.de/`
`planck-energieverteilung.pdf`

• M. Planck
On the Theory of the Energy Distribution Law of the Normal Spectrum. English translation:
`http://web.ihep.su/dbserv/compas/src/planck00b/eng.pdf`

Planck was abhorred by this idea of discreteness, even though he realized that it worked very well. He was too much a classical physicist.

Remark: There is a very nice book written by a historian of science about a German professor of theoretical physics, working towards the end of World War I, who worries about a lot and in particular about the new physics that he sees emerging, and which he does not like. This professor is fictitious, but every thing he does and says and all other things in the book reflect real persons and events, as is shown in the extensive notes at the end of the book.

• Russell McCormach
Night Thoughts of a Classical Physicist
Harvard University Press

The next step was taken by Niels Bohr, while working as a postdoctoral assistant with Ernest Rutherford in England. Rutherford had experimentally shown that an atom could be seen as a small planetary system: a nucleus encircled by electrons. To explain the discrete spectrum of the hydrogen atom, Bohr made in 1913 a brilliant ad hoc postulate, namely, that the electrons could only move in discrete orbits, occasionally jumping between different orbits, emitting or absorbing radiation in this process. Bohr was inspired by the work of Planck, but apart from that he had little theoretical background or justification for this idea, but he was able to derive from his assumption precise values for spectral lines.

• N. Bohr
Of the Constitution of Atoms and Molecules
Philos. Mag. 26, 1-24 (1913).

Accessible at:

`http://web.ihep.su/dbserv/compas/src/bohr13/eng.pdf`

Finally a complete theory explaining this was developed, within the span of a few years, basically between 1925 and 1927. It was the work of a few theoreticians, in the first place Werner Heisenberg, Max Born, Pasual Jordan, Erwin Schrödinger, then Wolfgang Pauli, Paul Dirac and many others. This is *quantum mechanics* as we know and use today. Its mathematical basis is functional analysis, in particular the theory of operators in Hilbert space, the understanding of which is due to John von Neumann, and is as such completely satisfactory.

At first it looked as if there were two different types of quantum mechanics. There was at first Heisenberg's *matrix mechanics*, in which the observables were noncommuting quantities, in fact infinite matrices. (A deep idea, on which I shall comment later). Then Schrödinger's *wave mechanics*. It was soon demonstrated that this was a matter of two different representations of the same mathematical theory.

Basic papers:

• W. Heisenberg
Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen
Z. Phys. 333, 879-893 (1925)

Accessible at

`http://www.chemie.unibas.ch/~steinhauser/documents/Heisenberg_1925_33_879-893.pdf`

An English translation can be found in the book "Sources of Quantum Mechanics"by B.L. van der Waerden. The complete book can be found at

`https://ia601208.us.archive.org/14/items/SourcesOfQuantumMechanics/VanDerWaerden-SourcesOfQuantumMechanics.pdf`

or

`https://archive.org/details/SourcesOfQuantumMechanics`

• M. Born, W. Heisenberg, P. Jordan
Zur Quantenmechanik
Z. Phys. 35, 557-615 (1926)
English translation. On Quantum Mechanics II.

The English translation is available at:

`http://fisica.ciens.ucv.ve/~svincenz/SQM333.pdf`

• E. Schrödinger
Quantization as a Problem of Proper Values. I,II
Collected papers on wave mechanics. AMS Chelsea 1982
English translation of the 1928 German edition.

For the contribution of von Neumann see his classical book:

• John von Neumann
Mathematische Grundlagen der Quantenmechanik
Springer 1996 (Original edition Springer 1932)
Mathematical Foundations of Quantum Mechanics
English translation. Princeton 1996.

The physics of atoms, their properties and structure, cannot be described with classical theories. Atoms consists of a (relatively) heavy nucleus, surrounded by a system of electrons. Quantum mechanics made *atomic physics* an important and fruitful new area of physics.

The next step was the study of the nucleus itself: *nuclear Physics*, which began in earnest just before the beginning of World War II. It lead to the construction of nuclear reactors and finally to the atomic bomb. For this the quantum mechanics of Heisenberg, Schrödinger, c.s. was no longer sufficient. A new version of quantum theory had to be developed: *quantum field theory.* The pioneer in this was Paul Dirac; it was later made into a broad physical theory by Richard Feynman, Julian Schwinger, Freeman Dyson, and independently, by Sin-Itiro Tomonaga, whose papers were initially written in Japanese. It was also very effective. By using quantum field theories theoreticians were able to predict certain experimental results with extreme precision, in some cases up to fifteen decimals.

However, in a mathematical sense, the theory was – and still is - very unsatisfactory. During calculations one finds at many places divergent integrals. These can be removed by a system of ad-hoc prescriptions, *renormalization.* Fifty years of hard work by mathematicians and mathematical physicists have been of no avail.

Two basic papers

• P.A.M. Dirac
The Quantum Theory of the Emission and Absorption of Radiation
Proc. Roy. Soc. A114, 243-265 (1927).

Accessible at

`http://hermes.ffn.ub.es/luisnavarro/nuevo_maletin/Dirac_QED_1927.pdf`

or

`http://wwwhome.lorentz.leidenuniv.nl/~boyarsky/media/Proc.R.Soc.Lond.-1927-Dirac-243-65.pdf`

• Julian Schwinger
The Theory of Quantized Fields. I. II. III. IV
Phys. Rev. 82, 914- (1951), 91, 713- (1953), 91, 728- (1953), 92, 1283- (1953)

Understanding the structure of nuclei led to the next step: *elementary particle physics* or *high energy physics*. This means breaking up the nucleus and investigate the properties of the new 'subatomic' particles that appear in collisions in high energy particle accelerators.

The role of quantum field theory, with its effectiveness but also with its mathematical problems, remains the same. Quantum field theory is the unique theoretical framework for elementary particle physics.

The final result so far is the so-called *Standard Model*, a phenomenological scheme in which all the known particles have a definite place, except the graviton, the supposed particle associated with some kind of quantum field for general relativity.

At the end of my talk I shall make a few remarks on the Standard Model, its problems, together with remarks concerning the general outlook for elementary particles and quantum theory.

This will do for the history of quantum theory.

3. Quantum theory. Introduction

In most physics textbooks the treatment of quantum theory follows a scheme which is in essence the scheme of the historical development of the theory, even though usually very few actual historical details are given. The history of quantum theory is in itself quite interesting. It shows how new theories come into being, with half understood heuristic ideas, with leaps and bounds, dead ends and false roads, which may be followed for some time.

Although I find the history of quantum theory – and the history of physics generally – of great interest, and although I believe that some knowledge should be a part of the general education of physicists, I shall nevertheless follow here in my exposition of quantum theory the opposite direction. This is in particular the best way to teach the subject to mathematicians.

This means that I shall formulate a number of precise mathematical 'axioms', together with rules for the physical interpretations of these axioms. There will be three levels of axiomatization.

1. That for elementary quantum mechanics.
2. That for quantum statistical mechanics.
3. That for quantum field theory.

For this third case I shall give the axioms for a special approach: algebraic quantum field theory. This is a very general axiomatization, in terms of abstract algebras, which has been used to attack the mathematical problems of quantum field theory.

4. Level 1. The axioms for elementary quantum mechanics

I shall state the axioms, as given – in a manner of speaking – by revelation from above. After that I shall illustrate them by explicit example of quantum systems. There will be five of these axioms.

Axiom I: *The state of a quantum system is represented by a unit vector ψ in a Hilbert space $\mathcal{H}$.*

A Hilbert space is a complex inner product space. The inner product of two vectors ψ and ϕ is denoted as (ψ, ϕ). It is conjugate linear in the first variable (physics convention). If the dimension is infinite, the usual case in quantum theory, it is separable, i.e. has a countable orthonormal basis, and is complete, i.e. each Cauchy sequence has a limit.

Axiom II: *An observable a of the system is represented by a self-adjoint operator A in $\mathcal{H}$.*

Intermezzo. Operators in Hilbert space

In finite dimensional linear algebra a hermitian operator can be represented by a hermitian matrix. In an infinite dimensional Hilbert space the notion of operator and more specifically that of hermitian operator is more complicated. In general operators in infinite dimensional Hilbert space may not be defined on all vectors of the Hilbert space, but only on a dense linear subspace $\mathcal{D}$. Such operators are called *unbounded operators*. An operator A in $\mathcal{H}$, defined on a linear subspace $\mathcal{D}$, is bounded iff there is a positive constant C such that

$$||A\psi|| \leq C||\psi||, \quad \forall \psi \in \mathcal{D}.$$

The infinum of all possible such numbers C is called the *norm* of A; it is denoted as $||A||$. If an operator is bounded it can be uniquely extended to all of $\mathcal{H}$. Most of the operators in quantum theory are in fact unbounded.

The infinite dimensional analogue of a hermitian operator is a hermitian symmetric, or, for short, *symmetric operator*. An operator A with domain of definition $\mathcal{D}$ is symmetric if and only if

$$(A\psi, \phi) = (\psi, A\psi), \quad \forall \psi, \phi \in \mathcal{D}.$$

For quantum theory symmetric operators are not good enough; a special property is needed. They should be *selfadjoint*. This property is fairly subtle; I shall not give the definition. Note that the general theory of unbounded selfadjoint operator is due to John von Neumann; it is one of the great achievements of 20th century mathematics. The most important property of selfadjoint operators, very relevant for selfadjoint operators is the *spectral theorem*. Further on more on this.

Example: Our standard example of a quantum system is that of a nonrelativistic particle of mass m, in ordinary 3-dimensional space, with space coordinates $\mathbf{x} = (x_1, x_2, x_3)$, moving in a central potential $V(\mathbf{x})$. The Hilbert space $\mathcal{H}$ is the space of square integrable functions $\psi(\mathbf{x})$. The inner product of two such 'wave functions' ψ and ϕ is

$$(\psi, \phi) = \int_{-\infty}^{+\infty} \overline{\psi(\mathbf{x})}\phi(\mathbf{x}))\, d\mathbf{x}.$$

There are two sets of three observables from which all other observables can be constructed. One has the three components of position, represented by the operators Q_j, for $j = 1, 2, 3$, acting in $\mathcal{H}$ as multiplication operators,

$$(Q_j\psi)(x_1, x_2, x_3) = x_j\psi(x_1, x_2, x_3),$$

and three (linear) momentum operators P_j, for $j = 1, 2, 3$, acting as differential operators,

$$(P_j\psi)(x_1, x_2, x_3) = \frac{\hbar}{i}\frac{\partial}{\partial x_j}\psi(x_1, x_2, x_3).$$

Why these particular operators have been chosen for these particular observables is a story in which I cannot enter here, except stating that, using general principles, it is, essentially, the only possibility.

Commutation rules between operators that represent observables, expressions of the form $[A, B] = AB - BA$, are of great importance in quantum theory. For the above example the basic commutation rules are the relations between the operators for position and momentum.

$$[P_j, P_k] = [Q_j, Q_k] = 0, \quad [P_j, Q_k] = \frac{\hbar}{i}\delta_{j,k}, \quad j, k = 1, 2, 3,$$

the *canonical commutation relations.*

All other observables of this system can be constructed from these P_j and Q_k. The most important one is the *energy*:

$$H = \frac{P^2}{2m} + V(\mathbf{x}).$$

This operator is usually called the *Hamiltonian* because it generates the time development of the system, as will be discussed further on.

This definition illustrates the general manner in which quantum observables are obtained from suggestions from the classical descriptions of the system. This is however in general not without problems, because the road from a polynomial in classical p_j and q_k to a corresponding polynomial in quantum P_j and Q_k is not always unique. Obviously $pq = qp$ but $PQ \neq QP$. This means that quantization of a classical system is not a unique procedure. If there are different possibilities experiments should decide which is the correct one. This problem does not occur in the simple case of a particle in a potential.

Two points are worth noting here:

1. The appearance of Planck's constant in a formula means that one is dealing with quantum theory. There is a relation between quantum and classical systems. By taking the limit $\hbar \to 0$ (classical limit) one obtains the corresponding classical system. Note however that $\hbar$ is a constant of nature. It has a dimension (of an action), so it numerical value depends on the system of units that one is using. Taking the classical limit means therefore taking a limit of the system of units that is being used.

2. The noncommuting of operators that represent observables is very typical for and important in quantum theory. *It means that the corresponding observables cannot be measured simultaneously with arbitrary precision*. I shall come back to this later.

Axiom III: *Physical interpretation of Axioms I and II.*

Axioms I and II, as I have presented them, do not tell us anything about the physical meaning of quantum theory. To explain this meaning we need the *spectral theorem for selfadjoint operators.*

The spectral theorem for selfadjoint operators

In finite dimensional linear algebra a hermitian matrix can be diagonalized. On the diagonal there will be real numbers $\alpha_1, \alpha_2, \alpha_3, \ldots$, the eigenvalues of the linear operator A associated with the matrix, belonging to an orthonormal system of eigenvectors $\psi_1, \psi_2, \psi_3, \ldots$.

In an infinite dimensional Hilbert space one has something in the same spirit, but much more sophisticated. In the first place there may be *continuous spectrum*, which is not associated with eigenvectors. In stead one has in general a so-called *spectral resolution* for a given A, a system $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ of nondecreasing, mutually commuting projection operators such that, in operator language,

$$\lim_{\alpha\to-\infty} E_\alpha = 0, \qquad \lim_{\alpha\to+\infty} E_\alpha = 1.$$

and

$$A = \int_{-\infty}^{+\infty} \alpha \, dE_\alpha,$$

or in terms of vector-valued integrals

$$A\psi = \int_{-\infty}^{+\infty} \alpha \, d(E_\alpha \psi), \qquad \forall \psi \in \mathcal{D},$$

or in terms of numerical integrals

$$(\psi_1, A\psi_2) = \int_{-\infty}^{+\infty} d(\psi_1, E_\alpha \psi_2), \qquad \forall \Psi_1, \psi_2 \in \mathcal{D}.$$

This is the general form of the *spectral theorem*. (The various integrals are Riemann-Stieltjes integrals). *It is important to note that this theorem does not hold for symmetric operators; one definitely does need selfadjointness*.

In some cases one still has (discrete) eigenvalues and eigenvectors. For the sake of simplicity I shall not discuss the general case, but only the discrete case. Most physics textbooks do this, in addition they write integrals instead of infinite sums in the case of continuous spectrum.

The probabilistic nature of quantum theory

The most important and striking feature of quantum theory is that most statements are probabilistic.

In classical physics, such as classical statistical mechanics, probability means that one has insufficient knowledge of the underlying state of the system. In quantum theory there is *no* underlying situation; the probabilistic nature is fundamental. This is a philosophical problem; in the submicroscopic world there is no strict causality. This has been discussed since the beginning of quantum mechanics, with Einstein and Bohr as its principal antagonists. Vivid discussions continue. There are different theoretical positions, with interesting experiments being done, the results of which may decide which theory is correct.

Axiom III (continued)

Let me now state Axiom III in a physically more explicit manner. I do it only for the discrete case.

Consider an observable a represented by the selfadjoint operator A, with real eigenvalues $\alpha_1 < \alpha_2 < \ldots$. There is an orthonormal system of eigenvectors $\{\psi_j\}_{j=1,2,\ldots}$ of A. Consider a state represented by a unit vector ψ.

1. *The probability of finding in a measurement the value α for the observable a in the interval $[\lambda_1, \lambda_2]$ is $\sum |(\psi, E_j\psi)|^2$, for j running*

through the values for which $\lambda_1 \leq \alpha_j \leq \lambda_2$, and with E_j the orthogonal projection operator on ψ_j.

This statement is only true when the spectrum of A is nondegenerate, i.e. when there is – up to a phase factor – only one unit eigenvector for each eigenvalue. The eigenspaces on which the E_j project are then one-dimensional. For the degenerate case one denotes the eigenvectors as as ψ_j^s, with $j = 1, 2, \ldots$ and $s = 1, 2, \ldots$. The eigenprojections E_j are then higher dimensional. It represents a simple generalization in which I shall not go.

2. *The average value (expectation or expectation value) for such a measurement is*

$$\overline{a}_\psi = (\psi, A\psi).$$

Note that is not hard to derive 2 from 1.

From these formulas it is clear that a quantum system can only be found with values for an observable in the spectrum of the corresponding operator, a very nonclassical fact.

Commensurable and noncommensurable observables

Operators that commute represent *commensurable observables*. For a system of such observables a state determines a *simultaneous or joint probability distribution*, just as one has has in classical physics. (This is a slight extension of Axiom III).

Examples of such systems are the three components of position or of momenta. Noncommuting operators represent *incommensurable observables*. An example is the pair (P_j, Q_j). For a system of such incommensurable operators there is no joint probability distribution. They cannot be measured at the same time with arbitrary precision.

There is, for example, an inequality which restricts the precision of simultaneous measurement of position of momenta. It is one of the basic formula of quantum mechanics:

$$\Delta p \, \Delta x \geq \frac{\hbar}{2},$$

the *Heisenberg uncertainty relation*. It means that if a particle has a sharply determined position its momentum is ill defined, and vice versa. A typical quantum consequence is that particles have *no well-defined trajectories*. An even more typical consequence is that in a system of two identical particles one cannot distinguish the separate particles. It does not make sense to speak of particle 1 and particle 2. This leads to *Pauli's Exclusion Principle*, one of the corner stones of atomic physics. I shall not discuss this.

Example (Harmonic oscillator) :

I consider a simplified version of my earlier example, now in 1-dimensional space: the 1-dimensional harmonic oscillator.

The *classical harmonic oscillator* in its simplest form is a point particle, moving along a straight line and attracted to the origin by a force proportional with its distance to the origin. Its total energy consists of potential and kinetic energy, is constant, and equal to

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2,$$

with ω its frequency. This energy can have all possible nonnegative values. The precise motion of such a classical oscillator can be easily derived (first year classical mechanics).

The *quantum harmonic oscillator*. Its Hilbert space is $L^2(\mathbb{R})$; the two basic operators are P (momentum) and Q (position); the Hamiltonian (energy) operator is, not surprisingly,

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2.$$

The eigenfunction and eigenvalues of H can be found (standard second year quantum mechanics course). The eigenvalues are definitely nonclassical, i.e.

$$E_n = n\,\hbar\omega + \frac{1}{2}\hbar\omega,$$

which means that only a series of discrete values of the energy are possible, and moreover that the oscillator has a nonzero lowest energy. (Zero point energy).

Example (Hydrogen atom). The obvious 3-dimensional example is the *hydrogen atom*, in which an electron moves around the nucleus attracted by a Coulomb potential. Classically this just Keppler's model of planetary motion: a planet moving in the gravitational field of the sun. The main feature of this model are the closed orbits (ellipses) of a planet. All such orbits are possible; they form a continuum. (Calculating this is second year's classical mechanics).

The quantized version is the hydrogen atom with the electron moving in the electrostatic Coulomb field of the nucleus. The energy operator is obtained again by substituting in the the classical expression the variables p_j, q_k by the operators P_j, Q_k, which gives

$$H = \frac{P^2}{2m} + V(\mathbf{x}),$$

with the Coulomb potential

$$V(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{\mathbf{r}}.$$

It should be noted that, together with the harmonic oscillator, this is the only quantum model which can be solved explicitly, i.e. of which the energy eigenvalues and eigenfunctions can be found in closed form. (Second year quantum mechanics).

There are discrete eigenvalues

$$E_n = -\frac{me^4}{8\epsilon_0^2 h^2 n^2} \quad n = 1, 2, \ldots,$$

with $h = 2\pi\hbar$. These give in fact Bohr's electron orbits, calculated by him in a rather heuristic manner. Apart from the case $n = 1$, they are all degenerate, i.e. with more than one eigenfunction for each eigenvalue. Corresponding with these are eigenfunctions, expressions in terms of well-known classical orthogonal polynomials. They describe the *bound states* of the atom. The one with the lowest energy is called the *ground state*. The other states converge to $E = 0$. Above this value there is a continuum spectrum, going to $+\infty$; this corresponds with the *scattering states* in which the electron is no longer bound to the nucleus. (All this a bit more advanced than the theory of the harmonic oscillator, but still second year quantum mechanics material).

Basic papers

• E. Schrödinger
Quantisation as a Problem of Proper Values (Part I)
English translation of the German original
Ann. Phys. 79 (1926).
Accessible at:

http://einstein.drexel.edu/~bob/Quantum_Papers/Schr_1.pdf

• E. Schrödinger
Quantisierung als Eigenwertproblem (Zweite Mitteilung)
Ann. Phys. 79, 489-525 (1926).

Accessible at:

http://dieumsnh.qfb.umich.mx/archivoshistoricosmq/ModernaHist/Schrodinger1926c.pdf

These two papers are highlights in 20th century theoretical physics.

The next two axioms are less surprising. Quantum mechanics is described in terms of Hilbert space and its operators. The natural automorphisms of Hilbert space are unitary operators.

Axiom IV : *Time development. This is described by a one-parameter group* $\{U(t)\}_{t\in\mathbb{R}}$ *of unitary operators in* $\mathcal{H}$ *(Stone's theorem), i.e. with*

$$U(t_2 + t_1) = U(t_2)U(t_1), \quad \forall t_1, t_2 \in \mathbb{R}.$$

This group is continuous in t*, i.e. the vectors* $U(t)\psi$ *are continuous in* t*, in the strong topology, determined by the norm of the vectors. As a consequence* $U(t)$ *can be written as an exponential*

$$U(t) = e^{-\frac{i}{\hbar}tH},$$

with the generator H *the generator of the group, the Hamiltonian, usually the energy operator.*

Example (continued) : For a 3-dimensional particle in a potential the time development can be described by a partial differential equation

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\right)\psi(\mathbf{x},t),$$

with Δ the Laplace operator. This the *Schrödinger equation.*

Axiom V : *Symmetries in quantum mechanics are described by systems* $\{U(g)\}_{g\in\mathcal{G}}$ *of unitary operators in* $\mathcal{H}$*, with* $\mathcal{G}$ *usually a 'continuous' or Lie group. The* $U(g)$ *should commute with all the time development operators* $U(t)$ *.*

Lie groups have systems of generators which form a *Lie algebra*. A representative A of such a Lie algebra in $\mathcal{H}$ is *constant of the motion*, meaning that for every state ψ the results of measuring the observable associated with A is independent of time. For symmetry under spatial translation one has the three components of momentum as constants of the motion, for rotational symmetry the three components of angular momentum.

5. Level 2. The axioms for quantum statistical mechanics

When studying a system of a very large number of particles, e.g. a container filled with a gas, it is impossible to describe the motion of the separate particles. It also would be of no interest. Interesting are global properties such as pressure or temperature. Such properties can be studied by calculating averages, by using probability theory. For this one uses *ensembles*, the physical term for probability distribution on the *phase space* of a system, i.e. the space of position and momentum variables of all the particles, a space of very high dimensions, of course, but that is in itself no problem.

For the quantum description of such a system on needs a more general system of axioms. In this case this starts with the observables, not with the states.

Axiom I$'$: *The observables of the system are the selfadjoint operators in an ambient Hilbert space $\mathcal{H}$.*

Axiom II$'$: *The states are density operators in $\mathcal{H}$, i.e. selfadjoint trace class operators with trace* 1.

A *trace class operator* D in a Hilbert space is a bounded operator such that for an orthonormal basis $\{\psi_j\}_{j=1,2,\ldots}$ one has

$$\sum_{j=1,2,\ldots}^{\infty} ((D^*D)^{1/2}\psi_j, \psi_j) < \infty.$$

If this is true for one orthonormal basis it is true for all. The expression $\sum_{j=1,2,\ldots}^{\infty}(D\psi_j, \psi_j)$ is then called the *trace* of D. A trace class operator has always discrete spectrum; all the eigenvalues not equal to 0 have finite multiplicity. If one has a fixed orthonormal basis in mind the matrix representing D is called the *density matrix*.

Axiom III$'$: *Physical interpretation of Axioms I$'$ and II$'$.*

For the statistical description of system of many particles, both classical and quantum mechanically, such system has to be enclosed in a finite box. This implies that in the quantum case one has purely discrete spectrum.

For the sake of simplicity I restrict myself to the situation in which the state is stationary, i.e. with D constant in time, which is, in fact, usually the case in quantum en classical statistical mechanics. This means that D and the Hamiltonian H have a common orthonormal basis $\{\psi_j\}_j$ of eigenvectors. Note that the energy eigenvalues are very narrowly spaced. We also assume – again for simplicity's sake – that there is no degeneracy.

With all these assumptions we have

1. *The probability of finding in a measurement the value α for the observable a in the interval $[\lambda_1, \lambda_2]$ is $|(\psi_j, d_j E_j \psi_j)|^2$, for j running through the values for which $\lambda_1 \leq \alpha_j \leq \lambda_2$, with E_j the orthogonal projection operator on ψ_j, and with the d_j the (nonnegative) eigenvalues of D.*

The numbers d_j are weights connected with the ψ_j, contributing to the total probability of the state.

2. *The average value (expectation or expectation value) for such a measurement is* Trace (DA).

The next two axioms are much the same as in Level 1, except that the groups go unitary operators now act on the density operator.

Axiom IV$'$: *The time development. The one-parameter group $\{U(t)\}_{t\in\mathbb{R}}$ of unitary operators acts on the density operator D as*

$$D(t) = U(t)D(0)U(t)^{-1} = e^{-\frac{i}{\hbar}tH}D(0)e^{\frac{i}{\hbar}tH}, \quad \forall t \in \mathbb{R}.$$

Most of the density operators in quantum statistical mechanics, like the corresponding classical ensembles, are time-invariant.

Axiom V$'$: *Symmetries. Groups $\{U(g)\}_{g\in\mathcal{G}}$ of unitary operators act on D as*

$$D \mapsto U(g)DU(g)^{-1}, \;\; \forall g \in \mathcal{G},$$

with $\mathcal{G}$ a group, usually a Lie group. The $U(g)$ should again commute with the time development operators.

Note that if the density operator is just a one-dimensional projection operator on a unit vector ψ, we are back in the situation of Level 1, with ψ as state vector. This shows that Level 2 is a generalization of Level 1, to which it may return in special cases.

6. Level 3. The axioms for quantum field theory

One may consider four successive approaches to quantum field theory:

1. Quantum field theory according to Feynman, Dyson c.s.: Physically very successful. Elementary particle physics is based on it. Mathematically not understood. No axiom system.

2. Wightman axiomatic quantum field theory. Has a system of axioms.

3. Constructive quantum field theory of Jaffe and Glimm. No axioms.

4. Algebraic quantum field theory. Has a system of axioms.

Approaches 2-4 try or have tried to find a mathematical basis for 1. I shall discuss all four briefly and state the axioms for 4.

- **1.** The main reason quantum field theory is mathematically much more difficult than quantum mechanics is that it describes – or tries to describe – systems with an infinite number of degrees of freedom. Dirac was one the first to discuss the quantization of fields (1927).

• P.A.M Dirac
The Quantum Theory of the Emission and Absorption of Radiation
Proc. Roy. Soc. A 114, 243-265 (1927)

Accessible at:

`http://hermes.ffn.ub.es/luisnavarro/`
`nuevo_maletin/Dirac_QED_1927.pdf`

or

`http://rspa.royalsocietypublishing.org/`
`content/114/767/243.full.pdf`

It was soon discovered that this new theory was plagued with serious difficulties. At most points in the calculations so-called ‘infinities’, i.e. divergent integrals, appeared. A practical solution was found in the late 1940s by several theoreticians, Richard Feynman, Julian

Schwinger, Freeman Dyson, and independently in Japan, Sin-Itiro Tomonaga. It was called *renomalization*, a set of prescriptions in which the infinities are bundled together and then removed. The method is very technical, without any mathematical basis, but it results in numbers that agree in an astonishing degree with experimental results. So in a practical sense quantum field theory is a great success; it remains to this day the main theoretical basis for elementary particle physics. But as a mathematically rigorous theory is just does not exist.

Here are two critical quotes:

Dirac, in 1975: "Most physicists are very satisfied with the situation. They say: 'Quantum electrodynamics is a good theory and we do not have to worry about it any more.' I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!"

Feynman, one of the creators of renormalized quantum field theory, in 1985: "The shell game that we play ... is technically called 'renormalization'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate."

One of the fundamental papers:

• F.J. Dyson
Divergence of perturbation theory in quantum electrodynamics
Phys. Rev. 85, 631632 (1952).

Introductory texts:

• B. Delamotte
A hint of renormalization
Am. J. Phys. 72, 170-184 (2004)

Accessible at
`http://arxiv.org/pdf/hep-th/0212049v3.pdf`

and

• John Baez
Renormalization Made Easy
Web page, December 2009
`http://math.ucr.edu/home/baez/renormalization.html`

- **2.** Starting in the late 1960s several mathematical physicists started a search for a rigorous basis for quantum field theory. The first was Arthur Wightman. He proposed to characterize a quantum field theory by its *vacuum expectation values of the product of field operators*. A beautiful scheme in which a few interesting general theorems could be proved, but so far it has been in impossible to fit a nontrivial theory in this scheme.

• Arthur Wightman
Quantum field theory in terms of vacuum expectation values
Phys. Rev. 101, 860 (1956).

A good book on this approach

• R.F. Streater, A.S. Wightman
PCT, Spin and Statistics and All That
Princeton University Press 2000.

- **3.** The next step was *constructive quantum field theory*, mainly by James Glimm and Arthur Jaffe, an approach in which one tried to construct interacting quantum field models by working in lower spacetime dimension and using position and momentum cut-offs, which at the end of the process might be removed by taking limits. In this spirit one obtained a few well-defined models in 2 or 3 dimensional spacetime, but not much more.

An overview of this work by the two main protagonists:

• J. Glimm, A. Jaffe
Quantum Physics. A Functional Integral Point of View
Springer 1987.

For another overview:

• Stephen J. Summers
A Perspective on Constructive Quantum Field Theory. 2012
`http://arxiv.org/pdf/1203.3991v1.pdf`

\- **4.** Finally a fourth step: *algebraic quantum field theory.* This approach takes the algebra of observables as its starting point. A Hilbert space $\mathcal{H}$ appears later. I give here one of the possible slightly different set of axioms.

Axioms for algebraic quantum theory

Axiom I$''$: *There is an abstract* $*$*-algebra* $\mathcal{A}$ *of pre-observables.*

$\mathcal{A}$ is complex, associative, with unit element, and – typical for the quantum situation – noncommutative. The $*$-operation or conjugation is a conjugate-linear map, $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$, with the property $(ab)^* = b^*a^*$. A selfadjoint element is of course characterized by $a^* = a$. Note that the selfadjoint elements do not form a subalgebra of $\mathcal{A}$.

Axiom II$''$: *The state of the system is described by a positive, normalized, linear functional* ω *on* $\mathcal{A}$.

Positivity of ω means $\omega(a^*a) \geq 0$, for all a in $\mathcal{A}$; ω normalized means $\omega(1_{\mathcal{A}}) = 1$.

Axiom III$''$: *The interpretation needs the so-called* GNS construction*, depending on* ω*, which leads back to the interpretation according to Levels 1 and 2.*

Intermezzo: **The GNS construction**.

The GNS construction (Gelfand-Naimark-Segal) is an example of something that mathematicians like: to take an object, let it act on itself and then as a result get a new object with interesting properties.

Start from the algebra $\mathcal{A}$ and an arbitrary state ω. Take a copy of $\mathcal{A}$, use it as a vector space and call it $\mathcal{H}_0$. Elements of this space, in fact elements of $\mathcal{A}$ will, in their new role, be denoted as $\underline{a}$. Define a sesquilinear for on $\mathcal{H}_0$

$$(\underline{a}, \underline{b}) = \omega(a^*b), \quad \forall a, b \in \mathcal{A}.$$

The form $(\cdot, \cdot)$ is not yet a true inner product, because there are still null-vectors. They can be squeezed out by a simple quotient procedure. Let me call the resulting quotient space $\mathcal{H}_1$. The vectors in $\mathcal{H}_1$ are equivalence classes in $\mathcal{H}_0$, and can be denoted as $[\underline{a}]$. $\mathcal{H}_1$ is not yet a Hilbert space, because it is not complete. It can be completed by the standard completion prescription in terms of Cauchy sequences. The result is the ω-dependent Hilbert space $\mathcal{H}_\omega$, with a representation π_ω of the algebra $\mathcal{A}$ by bounded operators $\pi_\omega(a)$ in the representation space $\mathcal{H}_\omega$, according to

$$\pi_\omega(a)[\underline{b}] = [\underline{ab}], \quad \forall a, b \in \mathcal{A}.$$

One needs of course to check that this is indeed well-defined, in term of equivalence classes.

There is a special unit vector in $\mathcal{H}_\omega$, namely $[\underline{1_\mathcal{A}}]$; it is usually denoted as Ω_0. It is cyclic, meaning that the set of all vectors $\pi_\omega(a)\Omega_0$ is dense in $\mathcal{H}_\omega$. This state is in most systems the ground state.

With this we are back in the situation of Level 1, and in an adapted version of Level 2.

Axiom IV$''$: *The time development of the system is – not surprisingly – given by a one-parameter group $\{\phi(t)\}_{t\in\mathbb{R}}$ of $*$-automorphisms of the algebra $\mathcal{A}$.*

We consider states together with time development automorphisms that leave these states invariant, i.e. with

$$\omega(\phi(t)a) = \omega(a), \quad \forall a \in \mathcal{A},\ t \in \mathbb{R}.$$

As a consequence of this the automorphisms $\phi(t)$ are *unitarily implementable* in $\mathcal{H}_\omega$, which means there are unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ such that

$$\pi_\omega(\phi(t)a) = U(t)\pi_\omega(a)\, U(t)^{-1}, \quad \forall a \in \mathcal{A},\, t \in \mathbb{R}.$$

Another consequence is that the ground state $\Omega_0 = \underline{1_\mathcal{A}}$ is time-invariant.

At this point the algebra $\mathcal{A}$ has not yet a topology. However, if the group of automorphisms is in an appropriate sense continuous in t, then the group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ has a a selfadjoint generator, the Hamiltonian for this particular situation.

Axiom V$''$: *Symmetries are described by $*$-automorphisms, usually groups $\{U(g)\}_{g\in G}$ of such automorphisms.*

For a state ω invariant under the $\{\phi(g)\}_{g\in G}$ such symmetries can be implemented by unitary operators $\{U(g)\}_{g\in\mathcal{G}}$, meaning that

$$\pi_\omega(\phi(g)a) = U(g)\pi_\omega(a)\, U(g)^{-1}, \quad \forall a \in \mathcal{A},\, g \in \mathcal{G}.$$

The idea of an algebraic formulation of quantum mechanics goes back to Irving Segal (1947 paper):

• I.E. Segal
Postulates for general quantum mechanics
Ann. of Math. 48, 930-948 (1947).

For a long time this paper remained fairly unknown. Some twenty years later Rudolf Haag and Daniel Kastler took up its ideas and applied it to quantum field theory, then, as now, mathematically speaking, a very problematic topic. This resulted in a fundamental paper (1964):

• Rudolf Haag, Daniel Kastler
An algebraic approach to quantum field theory
J. Math. Phys. 5, 848-861 (1964).

Important further work in this direction was done by Huzihiro Araki. He published (in 1999) an overview of general algebraic quantum field theory in his book:

• Huzihiro Araki
Mathematical Theory of Quantum Fields
Oxford University Press 1999.

A good recent review is

• Hans Halvorson, Michael Mueger
Algebraic Quantum Field Theory
`http://arxiv.org/pdf/math-ph/0602036v1.pdf`

The main application of algebraic quantum theory is quantum field theory. This is a *relativistic* theory. This is the first time that I mention relativity theory in this lecture. Therefore a brief intermezzo.

Intermezzo:*Relativity*. Relativity theory is a new description of spacetime due to Albert Einstein (paper in 1905).

According to special relativity spacetime is not just the combination of 3-dimensional space and 1-dimensional time; it is a truly 4-dimensional linear space. It is called *Minkowski space* and has a indefinite bilinear form, the *Minkowski metric*. The symmetry group of special relativity is the (in homogenous) Lorentz group, a pseudo-orthogonal matrix group which leaves the Minkowski inner product invariant.

Note that relativity here means *special relativity*. There is also a generalization, *general relativity* (1912 paper by Einstein), which gives a description of gravity in terms of the curvature of space time. I do not need this here, except for a brief remark at the end of my talk.

Strangely enough the explicit idea of a four dimensional spacetime is not due to Einstein but to the mathematician Hermann Minkowski, a number theorist, who gave a talk on this in 1908 and subsequently published it in a paper in 1909. Here is a quote:
"*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality*".

Basic historical literature on relativity .

• A. Einstein
Zur Elektrodynamik bewegter Körper
Ann. Phys. 891-921 (1905).
Accessible at
`https://web.archive.org/web/20050220050316/`
`http://www.pro-physik.de/Phy/pdfs/ger_890_921.pdf`
English translations:
"On the Electrodynamics of Moving Bodies"
at:
`http://www.fourmilab.ch/etexts/einstein/specrel/www/`

A collection of the principal early papers on special relativity:

• A. Einstein, H.A. Lorentz, H. Weyl, H. Minkowski
The Principle of Relativity
A collection of original papers on the special and general theory of relativity
Translated from the 1913 German edition, Dover 1952

The application of algebraic Level 3 scheme to quantum field theory is as follows. Consider the collection of all open sets $\mathcal{O}$ in $\mathbb{R}^4$ (spacetime). Inclusion gives this collection a natural partial order. Suppose one has a $*$-algebra $\mathcal{A}_{\mathcal{O}}$ for every $\mathcal{O}$, generated by all field operators with support in $\mathcal{O}$. These algebras will be C^*-algebras, a very important class of abstract normed $*$-algebras.

Brief Characterization of C^*-algebras

A C^*-algebra $\mathcal{A}$ is a complete normed $*$-algebra, so a **$*$-Banach algebra**, with the the additional property

$$||a^*a|| = ||a||^2, \quad \forall a \in \mathcal{A}.$$

Because of this innocent looking property C^*-algebras are among the most important and widely studied objects in functional analysis.

For each pair $(\mathcal{O}_1, \mathcal{O}_2)$, such that $\mathcal{O}_1 \subset \mathcal{O}_2$, there will be an injective $*$-homomorphism $\mathcal{O}_1 \rightarrow \mathcal{O}_2$. By means of a projective limit construction the algebras $\mathcal{A}_{\mathcal{O}}$ together give a single large C^*-algebra $\mathcal{A}_{\mathrm{q.l.}}$, the algebra of *quasi-local algebras*, in my terminology the algebra of *pre-observables*. The difficult task now is to find a Lorentz invariant state ω on this $\mathcal{A}_{\mathrm{q.l.}}$. So far such states are known only for systems of free fields, i.e. fields that describe noninteracting particles. This is a bit disappointing, of course.

However, the algebraic scheme for Level 3 quantum theory, as formulated here, is appealing as an elegant way of describing general physical systems, both classical and quantum. A brief sketch of this idea is given in the next section.

7. Algebraic dynamical systems

Algebras first appeared in quantum mechanics, in the spirit of the general formalism that we have in mind, in the pioneering work of Irving Segal (1918-1998), in his 1947 paper, mentioned earlier.

The realization that algebraic descriptions have a much wider scope than algebraic quantum mechanics and field theory is more recent. This more general point of view is probably due to Alain Connes, the French mathematician and Field medalist.

Connes introduced *noncommutative geometry*, in first instance a way of making geometry a part of algebra. This goes back to the classical Gelfand-Naimark theorem which states that there is a one-to-one correspondence between compact topological spaces and commutative C^*-algebras. There is, however, an interesting and far reaching generalization. A noncommutative C^*-algebra can be seen as an algebraic description of a fictitious 'noncommutative manifold', with properties that generalizes the properties of an ordinary manifold. Connes studied in this way in particular 'noncommutative Riemannian manifolds'.

Connes wrote a fascinating book about this:

• Alain Connes
Noncommutative Geometry
Academic Press 1994
It can be freely downloaded from his official website:
`http://www.alainconnes.org/en/`

It was extensively reviewed in 1996 by Irving Segal in the Bulletin of the American Mathematical Society. The review is well worth reading. It can be found at:

`http://www.ams.org/bull/1996-33-04/`
`S0273-0979-96-00687-8/S0273-0979-96-00687-8.pdf`

Here is a brief quote:

"It is more in the nature of a long discourse or letter to friends."

Connes was not pleased by this remark.

Alain Connes is one of the few present-day mathematicians with a deep interest in and great knowledge of physics. Another is Michael Atyah.

The idea that both commutative and noncommutative algebraic schemes can be used to give general characterizations of physical systems will not be surprising to most mathematical physicists, but discussions of this point of view in the literature are rare. To my knowledge there are two fairly recent books:

• L.D. Fadeev, O.A. Yakubowskiǐ
Lectures on Quantum Mechanics for Mathematics Students
American Mathematical Society 2009.

• F. Strocchi
An Introduction to the Mathematical Structure of Quantum Mechanics
World Scientific 2005.

The first book is an elegant introduction. It is however brief and contains few details. The approach of the second book is much too restricted. In my own book on quantum theory algebraic descriptions of physical systems was one of the main themes.

Here is a list of the elements of what I call an *algebraic dynamical system.*

An algebraic dynamical system consists of

1. a $*$-algebra $\mathcal{A}$ of observables,
2. a positive normalized linear expectation functional ω on $\mathcal{A}$,
3. rules for the physical interpretation of 1 and 2,
4. a 1-parameter group $\{\phi(t)\}_{t\in\mathbb{R}}$ of time evolution $*$-automorphisms of $\mathcal{A}$,
5. Groups of symmetry $*$-automorphisms $\{\phi(g)\}_{g\in\mathcal{G}}$ of $\mathcal{A}$.

If $\mathcal{A}$ is commutative we have a classical system; If $\mathcal{A}$ is noncommutative a quantum system. In the noncommutative case Planck's constant $\hbar$ will appear in the formulas. Letting $\hbar$ go to 0, i.e. taking the limit of the system of units, such that the numerical value of $\hbar$ goes to zero, gives the *classical limit* of the quantum system.

However, not all classical limits of quantum systems are physically meaningful. A Maxwell quantum or photon quantum field, in elementary particle, has as classical limit the classical Maxwell field. A meson quantum field, also in elementary particle physics, has a classical limit which is mathematically well-defined but physically meaningless. It may however play a role as an auxiliary object, used for constructing the quantum field.

There is an even more problematic situation, in which the classical limit is not only an object without physical meaning, but in which the limit of the noncommuting algebra of quantum obersevables is not commutative, but what I call 'almost commutative', i.e. commutative up to minus signs. An example is the electron-positron quantum field. Such a 'pseudo-classical' system can still be used for constructing the quantum system. It also leads to an interesting recently developed field of mathematics, with notions such as super algebras, super manifolds, etc. I cannot discuss this here.

Elementary quantum mechanics (Level 1) is nonrelativistic. This is obvious for the Schrödinger equation; it is first order in time and second order in space. Time and space should be treated on equal footing. The same holds for quantum statistical mechanics (Level 2). The distinction between space and time depends on the coordinate system. Therefore time development has no intrinsic meaning. It is part of the general symmetry under the action of the inhomogeneous

Lorentz group. Relativistic theories have to be *Lorentz covariant*, their equations *Lorentz invariant*. For this reason I employ for relativistic physical systems, such as relativistic quantum field theory, the term *algebraic covariance system*, instead of *algebraic dynamical system*.

What I have said here about general algebraic dynamical systems is a sketch of what I consider to be an interesting idea. Many important technical details need to be worked out. One obvious question is what types of algebras $\mathcal{A}$ should be used. C^*algebras? They are the foremost sort of 'abstract' algebras in functional analysis, of which must is known. They cannot be used universally, as is suggested in the book of Strocchi, because they have in general not enough projection operators, the main tool for applications in quantum physics. Von Neumann algebras, the most important type of operator algebras, do better in this respect, in fact all projections of a selfadjoint operator in such an algebra belong to this algebra. (Note that an algebraic description of classical statistical mechanics, or more general of classical probability theory uses commutative von Neumann algebras). The natural commutative algebras describing classical mechanics are the algebras of smooth functions on phase space, obviously not C^*- or von Neumann algebras, but some kind of generalizations of Fréchet algebras.

I should finally remark that the notion of a general algebraic system or an algebraic covariance system has no great practical value when doing explicit calculations in some explicit model. It should however appeal to theoretical and mathematical physicists for who like structural elegance and who appreciate possibilities of unification of theories.

8. Concluding remarks

I have discussed non-relativistic quantum mechanics and quantum statistical mechanics as theories that are both mathematically and physically completely satisfactory (Level 1 and 2). The situation is less good for relativistic quantum mechanics and field theory (Level 3). The physical side is all right; a proper mathematical basis has yet not been found. Level 4, quantization of general relativity, is still very much work in progress. A few remarks on this in this section may be useful.

General relativity, as developed by Albert Einstein is a classical field theory (1912), which gave a new description of gravity, in terms of a 4-dimensional spacetime manifold, with a curvature tensor as its main feature. Its predictions have been verified with great precision by various experiments (redshift of light, perihelion precession of the orbit of the planet Mercury, deflection of light by the sun).

An acceptable quantum version of the gravitational field has not been found. The main problem is that the standard renormalization procedure of Feynman c.s. does not work. The theory is *nonrenormalizable*.

Various attempts have been made to solve this problem, by bypassing the renormalization procedure. One approach is called loop gravity. The most important and most visible contender in this field is *string theory*, in which the basic elements are no longer point particles in 4-dimensional spacetime, but tiny strings, open or closed, in much higher dimension. This approach has been studied by many people, making it clear that there would be no possibilities for experimental verification of its predictions even in a distant future. Moreover it needs *supersymmetry*, a phenomenon, predicted forty years ago, no trace of which has until now been experimentally observed. This together means that string theory is not a physical theory. It is *science fiction*.

So far the relevance of string theory for physics is zero. Its usefulness lies in its contributions to mathematics. This is due to work, and sometimes just heuristically suggestions by people like Nathan Seiberg, Juan Maldacena and in particular Ed Witten, Fields Medalist in mathematics in 1990. Examples of topics that greatly profited from this are, for example, Calabi-Yau manifolds and vertex operator algebras.

String theory is an example where ideas from physics have been important for pure mathematics. Another example is the theory of operator algebras, C^*-algebras and von Neumann algebras. There is a difference: physics articles by people as Haag, Kastler are perfectly readable for mathematicians, this contrary to most articles in string theory.

See for an introduction to string theory

• T. Hübsch
A Hitchhikers Guide to Superstring Jump Gates and Other Worlds
`https://web.archive.org/web/20101207045114/http://homepage.mac.com/thubsch/HSProc.pdf`

and for a very critical evaluation

• Peter Woit
String Theory: An Evaluation
`http://arxiv.org/pdf/physics/0102051v1.pdf`