Moduli Spaces of Pointed Rational Curves

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Lecture Notes

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1.1 A few generalities on moduli spaces

1.1.1 Moduli Problems

In order to pose a **moduli problem** we need the following ingredients:

1. A class of (geometric) objects $\mathcal{P}$ in some category $\mathcal{C}$.
2. A notion of **family** of such objects.
3. The notion of equivalence of families. Families over one point are precisely the objects of $\mathcal{P}$, and hence we are in particular assigning the notion of equivalence of objects.

Intuition. *A moduli space $\mathcal{M}_{\mathcal{P}}$ for the above problem consists of a space in the same category as the objects parameterized, such that:*

- *points are in bijective correspondence with equivalence classes of objects in $\mathcal{P}$.*
- *there is a natural bijection between families over a given base B and functions $B \to \mathcal{M}_{\mathcal{P}}$.*

We now define what a family is, in the familiar category of topological spaces. We leave it as an exercise to generalize the definition to any case we may be interested in.

Definition 1. *Let $\mathcal{P}$ define a class of objects in the category $\mathcal{C}$ of topological spaces. Then for any $B \in \mathcal{C}$, a* **family** *of $\mathcal{P}$-objects over B is a topological space X together with a surjective continuous function*

$$\pi : X \to B$$

such that the fiber at each point ($X_b := \pi^{-1}(b)$ for $b \in B$) is an object in $\mathcal{P}$.

Exercise 1. *Formulate the notion of a family in the language of categories. Note that you need to require your category $\mathcal{C}$ to have fiber products. Also, the role of the point is played by a terminal object in $\mathcal{C}$.*

If we have a space $\mathcal{M}_{\mathcal{P}}$ whose points are in bijection with objects in $\mathcal{P}$, then a family naturally defines a set function $\varphi_\pi : B \to \mathcal{M}_{\mathcal{P}}$:

$$\varphi_\pi(b) := [X_b] \tag{1.1}$$

The image of a point b is the point in the moduli space is the (equivalence class of the) fiber of that point. In order to call $\mathcal{M}_{\mathcal{P}}$ a moduli space we like the most (a **fine** moduli space) we make the following requirements:

1. That the set function φ_π is continuous (a morphism in $\mathcal{C}$).
2. That no two non-equivalent families give the same function.
3. That every continuous function from $f : B \to \mathcal{M}_{\mathcal{P}}$ arises as the function associated to a family.

When the three conditions above are verified, there a fruitful dictionary between the geometry of the moduli space and the geometry of families of objects. Becoming fluent with using this dictionary and translating questions back and forth is one of the main goals of this mini-course.

Intuition. *If this whole dictionary with families seems a bit weird, let me try to convince you that in fact it is quite natural. We would like the geometry of the moduli space to reflect the similarity of objects. For example, if our objects had (a complete set of) metric invariants, then we would like the moduli space to also have a metric, and objects with very close invariants correspond to points that are very near each other in the moduli space. Or, even better, that moving the invariants continuously would result in a continuous path in the moduli space.*

Of course the problem is making such statements general and precise. The notion of a family achieves precisely that: it tells us that over a base B objects are varying in an acceptable way if they fit together to form a larger object still in the right category.

Exercise 2. *Consider the moduli problem of isomorphism classes of unit length plane segments up to rigid motion in the plane. Note that if such a space exists it can have only one point. Construct two families of segments over a circle that are not equivalent to each other. But they both must give the constant function: hence 2 fails.*

1.1.2 Moduli Functors

We now reformulate the discussion in our previous section in categorical language. For the purposes of this mini-course this level of abstraction is hardly necessary, but it is useful to be able to connect to this type of language.

Definition 2. *A* **moduli problem** *(or moduli functor) for a class $\mathcal{P}$ of objects in a category $\mathcal{C}$ with fiber products is a contravariant functor:*

$$\mathcal{F}_{\mathcal{P}} : \mathcal{C} \to \mathcal{S}ets$$

defined by:

- *for an object $B \in \mathcal{C}$, $\mathcal{F}_{\mathcal{P}}(B)$ is the set of isomorphism classes of families of $\mathcal{P}$-objects.*
- *for a morphism $f : B' \to B$, $\mathcal{F}_{\mathcal{P}}(f)$ is the set map sending a family $X \to B$ to the pullback family $f^*(X) \to B'$.*

Definition 3. *An object $\mathcal{M}_{\mathcal{P}} \in \mathcal{C}$ represents the functor $\mathcal{F}_{\mathcal{P}}$ if the functor of points ($Hom_{\mathcal{C}}(-, \mathcal{M}_{\mathcal{P}})$) of $\mathcal{M}_{\mathcal{P}}$ is isomorphic (as a functor) to the moduli functor via a natural transformation $\mathbb{T}$. If such an object exsits, it is called a* **fine moduli space** *for the moduli problem.*

Exercise 3. *Meditate for a few minutes until you convince yourself that this is equivalent to the above discussion about families and the natural bijection with functions to the moduli space. In particular, note that evaluating the two functors on the terminal object in your category recovers the invoked bijection between points of the moduli space and (equivalence classes of) objects you want to parameterize.*

Intuition. *If you are funked out by this idea of understanding a space by thinking of its functor of points, think of it as a generalization of what we do when we talk about abstract manifolds. We give up the idea of understanding the manifold via a set of global coordinates, and rather focus on the local data. A differentiable atlas is the notion of understanding a collection of (injective) functions that cover the set of points of the manifold (the charts), plus the transition functions on double overlaps. Here instead we want to understand ALL possible functions into $\mathcal{M}_{\mathcal{P}}$, and all possible pullbacks. This is of course not a proof (the formal proof is given by Yoneda's lemma), but it should make it fairly plausible that the knowledge of the functor of points should be enough to recover the (scheme, manifold, etc) structure of the moduli space.*

1.1.3 Universal Families

When a moduli functor is representable by a fine moduli space, there is a very special object with a map to the moduli space, such that the fiber over each point $m \in \mathcal{M}_{\mathcal{P}}$ is precisely the (an) object (in the equivalence class) parameterized by the point m.

Definition 4. *Given a fine moduli space $\mathcal{M}_{\mathcal{P}}$, let $\mathbb{T}$ denote the natural transformation identifying the functor of points of $\mathcal{M}_{\mathcal{P}}$ with the moduli functor. The* **universal Family**

$$\pi : \mathcal{U}_{\mathcal{P}} \to \mathcal{M}_{\mathcal{P}}$$

is defined to be $\mathbb{T}(\mathbb{1}_{\mathcal{M}_{\mathcal{P}}})$*. The universal family has the property that for any* $m \in \mathcal{M}_{\mathcal{P}}$ *corresponding to an (equivalence class of) object(s)* $[X_m] \in \mathcal{P}$*,*

$$[\pi^{-1}(m)] = [X_m] \tag{1.2}$$

Exercise 4. *A nice consequence of the existence of a universal family is that any other family of* $\mathcal{P}$*-objects is obtained from the universal family by pullback. Show that this is just a formal consequence of the categorical definitions.*

Exercise 5. *Any scheme* X *is a fine moduli space for the functor "families of points of* X*", the universal family being* X *itself. This is all a big tautology, but make sure it makes perfect sense to you. It is a good exercise to unravel the definitions and keep them straight.*

1.2 Moduli of n Points on $\mathbb{P}^1$

We now introduce the moduli space $M_{0,n}$ of isomorphism classes of n ordered distinct marked points $p_i \in \mathbb{P}^1$. The subscript 0 denotes the genus of the curve $\mathbb{P}^1$.Since this is our first meaningful class of moduli spaces, we spell out all the ingredients.

We work in the category of algebraic varieties (or schemes if you prefer) over $\mathbb{C}$. The class of objects we consider is

$$\mathcal{P} = \{(p_1, \ldots, p_n) \mid p_i \in \mathbb{P}^1, \quad p_i \neq p_j \text{ for } i \neq j\}. \tag{1.3}$$

The equivalence relation on objects is

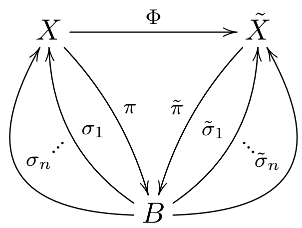
$$(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \text{ if there is } \Phi \in Aut(\mathbb{P}^1) \text{ s.t., for all } i, \Phi(p_i) = q_i. \tag{1.4}$$

A family of n distinct points on $\mathbb{P}^1$ over a base B consists of the following data:

- a (flat and proper) map $\pi : X \to B$, such that, for every point $b \in B$, $X_b = \pi^{-1}(b) \cong \mathbb{P}^1$.
- n disjoint sections $\sigma_i : B \to X$.

A function $s : B \to X$ is called a *section* of $\pi : X \to B$ if $\pi \circ s = \mathbb{1}_B$. This means that for every $b \in B$, $s(b) \in X_b$, i.e. s picks exactly one point in every fiber of the family. Two sections are called disjoint if their images are disjoint.

Two families $(X, B\pi, \sigma_1, \ldots, \sigma_n)$ and $(\tilde{X}, B, \tilde{\pi}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ are equivalent if there exists an isomorphism $\Phi : X \to \tilde{X}$ making the following diagram commute:



(1.5)

Finally we introduce our moduli functor.

Definition 5. *The moduli functor*

$$\mathcal{M}_{0,n} : Schemes \to Sets$$

assigns to any scheme B the set of equivalence classes of families of n distinct points on $\mathbb{P}^1$, and to any morphism $f : B' \to B$ the set map induced by the pullback of families.

Exercise 6. *Show that the definitions of family of n distinct points on $\mathbb{P}^1$ and of equivalence of families specialise to the definitions of objects and equivalence of objects when $B = pt$.*

Exercise 7. *For $n \leq 2$ show that the set $\mathcal{M}_{0,n}(pt.)$ consists of only one point. Show that for some base space B, there exist non-equivalent families. Conclude that a single point space does not represent the functor $\mathcal{M}_{0,n}$.*

For $n = 3$, since the automorphism group $Aut(\mathbb{P}^1) = \mathbb{P}GL_2(\mathbb{C})$ allows us to move any three points on $\mathbb{P}^1$ to the ordered triple $(0, 1, \infty)$ (see Exercise 11), the set $\mathcal{M}_{0,3}(pt.)$ consists of a single point. Given any family $(X, B, \pi, \sigma_1, \sigma_2, \sigma_3)$, consider the map $T : X \to B \times \mathbb{P}^1$ defined by:

$$T(x) = \Big(\pi(x), CR_{X_{\pi(x)}}(\sigma_1(\pi(x)), \sigma_2(\pi(x)), \sigma_3(\pi(x)), x)\Big),$$

where $CR_{X_{\pi(x)}}$ is the cross-ratio (as defined in Exercise 12) on the fiber containing the point x (we know such fiber is isomorphic to $\mathbb{P}^1$ and that the cross-ratio is independent of the choice of isomorphism one may choose to define it, hence T is well defined). T realizes an isomorphism of families between $(X, B, \pi, \sigma_1, \sigma_2, \sigma_3)$ and the family $(B \times \mathbb{P}^1, B, \pi_1, 0, 1, \infty)$, where π_1 is projection on the first factor and the sections are constant sections.

Exercise 8. *Show that the discussion in the previous paragraph proves that the functor $\mathcal{M}_{0,3}$ is represented by the algebraic variety $M_{0,3} = pt$. Show that a universal family is given by $U_3 = (\mathbb{P}^1, pt., \pi, 0, 1, \infty)$.*

Intuition. *The fact that any triple of points can be moved into $(0, 1, \infty)$ via an automorphism means that there is one equivalence class of triples, and so, if the functor is representable, it has to be represented by a one point* *space. The fact that there is exactly one automorphism moving a triple to the distinguished triple we chose to represent our equivalence class causes the fact that all families of triples can be trivialized. Equivalently, one can say that the only automorphism of $\mathbb{P}^1$ that fixes $(0,1,\infty)$ is the identity. As a general philosophy, when we parameterize objects which have no non-trivial automorphisms, we are not able to "glue" locally trivial families in non trivial ways. I.e., non-trivial automorphism of the objects we parameterize tend to give obstructions to the representability of the functor.*

Going one step up, the functor $\mathcal{M}_{0,4}$ is represented by the variety $M_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\}$: given a quadruple (p_1, p_2, p_3, p_4), we can always perform the unique automorphism of $\mathbb{P}^1$ sending (p_1, p_2, p_3) to $(0,1,\infty)$; the isomorphism class of the quadruple is then determined by the image of the fourth point.

A universal family U_4 is given by $(M_{0,4} \times \mathbb{P}^1, pt, \pi_1, 0, 1, \infty, \delta)$, where δ is the diagonal section $\delta(p) = p$. This family is represented in Figure 1.1.

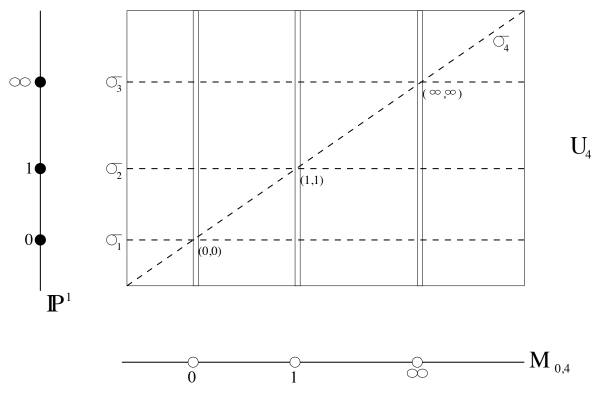


Figure 1.1: The universal family $U_4 \to M_{0,4}$.

Exercise 9. *By choosing to trivialize the first three sections, we have made a choice of a global coordinate system on $M_{0,4}$, or alternatively, we gave an injective map from $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to the set of configurations of four points on*

$\mathbb{P}^1$ *that meets each equivalence class exactly once. By trivializing a different choice of sections we make a different choice of coordinates, and correspondingly get a different universal family. Show explicitly (gotta do it once in your life) one change of coordinates for* $M_{0,4}$*, and write down the induced isomorphism of universal families.*

The general case is similar. Any n-tuple $\underline{p} = (p_1, \ldots, p_n)$ is equivalent to a n-tuple of the form $(0, 1, \infty, \Phi_{cr}(p_4), \ldots, \Phi_{cr}(p_n))$, where Φ_{cr} is the unique automorphism of $\mathbb{P}^1$ sending (p_1, p_2, p_3) to $(0, 1, \infty)$. This shows

$$M_{0,n} = \overbrace{M_{0,4} \times \ldots \times M_{0,4}}^{n-3 \text{ times}} \smallsetminus \{\text{all diagonals}\}.$$

If we define $U_n := M_{0,n} \times \mathbb{P}^1$, then the projection of U_n onto the first factor gives rise to a universal family

$$\begin{array}{c} U_n \\ \pi \downarrow\uparrow \sigma_i \\ M_{0,n} \end{array}$$

where the σ_i's are the universal sections:

$$\sigma_i(\underline{p}) = (\underline{p}, \Phi_{cr}(p_i)) \in U_n. \tag{1.6}$$

This family is tautological since the fibre over a moduli point, which is the class of a marked curve, is the marked curve itself.
With U_n as its universal family, the affine variety $M_{0,n}$ is a fine moduli space for isomorphism classes of n ordered distinct marked points on $\mathbb{P}^1$.

1.3 Compactification: first steps.

We now understand quite well the moduli space $M_{0,n}$. In particular, we notice it is not compact for $n \geq 4$. There are many reasons why compactness (or properness) is an extremely desirable property for moduli spaces. As an extremely practical reason, proper (and if possible projective) varieties are better suited for an intersection theory, which then allows for using our moduli spaces for enumerative geometric applications. Also, a compact moduli space encodes information on how objects can degenerate in families. A totally reasonable, if a bit naively phrased, question like

> *what happens when* $p_1 \to p_2$ *in* $M_{0,4}$*?*

would find an answer if we find a "good" compactification for $M_{0,4}$.

In general there are many ways to compactify a space. A "good" compactification $\overline{\mathcal{M}}$ of a moduli space $\mathcal{M}$ should have the following properties:

1. $\overline{\mathcal{M}}$ should be itself a moduli space, parametrizing some natural generalization of the objects of $\mathcal{M}$.

2. $\overline{\mathcal{M}}$ should not be a horribly singular space.

3. the boundary $\overline{\mathcal{M}} \smallsetminus \mathcal{M}$ should be a normal crossing divisor.

4. it should be possible to describe boundary strata combinatorially in terms of simpler objects. This point may appear mysterious, but it will be clarified soon enough.

We discuss the simple example of $M_{0,4}$, to provide intuition for the ideas and techniques used to compactify the moduli spaces of n-pointed rational curves for arbitrary n.

A natural first attempt is to just allow the points to come together, i.e. enlarge the collection of objects $\mathcal{P}$ that we are considering from $\mathbb{P}^1$ with 4 ordered distinct marked points to $\mathbb{P}^1$ with 4 ordered, not necessarily distinct, marked points. After all, as a set, $\mathcal{P} = (\mathbb{P}^1)^4$, which is pretty nice.

When we impose our equivalence relation, we notice two kinds of issues:

- We have equivalence classes corresponding to all points, three points, or two pairs of points having come together. Such equivalence classes behave like one or two-pointed curves; as we saw in Exercise 7 there exist nontrivial families giving rise to constant maps, which prevents representability of our functor.

- We have six equivalence classes corresponding to where exactly two points have come together. These points "look like" $M_{0,3}$'s, so they may be allright.

  But consider the families over $B = \mathbb{C}$ with coordinate t:

  $$C_t = (0, 1, \infty, t) \quad \text{and} \quad D_t = (0, t^{-1}, \infty, 1).$$

  For each $t \neq 0$, the automorphism of $\mathbb{P}^1$ $\phi_t(z) = \frac{z}{t}$ shows that $C_t \sim D_t$, thus corresponding to the same family in $M_{0,4}$. But for $t = 0$, C_0 has $p_1 = p_4$ whereas D_0 has $p_2 = p_3$. These configurations are not equivalent up to an automorphism of $\mathbb{P}^1$, hence are distinct points in our compactification of $M_{0,4}$. Thus, we have a family with two distinct limit points (the space is **nonseparated**).

Our failed attempt was not completely worthless though since it allowed us to understand that:

1. we should not allow too many points to coincide; in particular, we should always have at least three distinct marks on $\mathbb{P}^1$.

2. we want the condition $p_1 = p_4$ to coincide with $p_2 = p_3$, and likewise for the other two possible disjoint pairs.

On the one hand this is very promising: 3 is the number of points needed to compactify $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to $\mathbb{P}^1$. On the other hand, it is now mysterious what modular interpretation to give to this compactification.

The strategy is to achieve the modular interpretation for the compactification by compactifying simultaneoulsy the moduli space and the universal family. Let us then turn to the universal family, illustrated in Figure 1.1, and naively compactify the picture by filling in the three points on the base, and completing U_4 to $\mathbb{P}^1 \times \mathbb{P}^1$ where the sections are extended by continuity.

Intuition. *We notice a bothersome asymmetry in this picture: the point* p_4 *is the only one allowed to come together with all the other points: yet common sense, backed up by the explicit example just presented, suggests that there should be democracy among the four points. One way to restore democracy is to blow-up* $\mathbb{P}^1 \times \mathbb{P}^1$ *at the three points* $(0,0)$, $(1,1)$, (∞,∞). *This makes all the sections disjoint, and still preserve the smoothness and projectivity of our universal family.*

Exercise 10. *Refer back to Exercise 9, consider the isomorphism* ψ *of the two different universal families* U_4, U_4' *obtained when different choices of sections to be trivialized are made. Note that if the universal families are "closed" to* $\overline{U}_4$ *(resp.* $\overline{U}_4'$*)* $\cong \mathbb{P}^1 \times \mathbb{P}^1$, ψ *extends to a rational function that is indeterminate at the points* $(0,0)$, $(1,1)$, (∞,∞), *and contracts the fibers over* 0, 1, ∞. *Show that blowing up such three points in both* $\overline{U}_4$ *and* $\overline{U}_4'$ *resolves the indeterminacies of* ψ, *which now extends to an isomorphism*

$$\hat{\psi} : Bl(\overline{U}_4) \to Bl(\overline{U}_4').$$

Exercise 10 shows that the candidate for a universal family $Bl(\overline{U}_4)$ is well behaved with respect to the change of coordinates induced by different choices of three sections to trivialize. The fibres over the three exceptional points are $\mathbb{P}^1 \cup E_i$: nodal rational curves. These are the new objects that we have to allow in order to obtain a good compactification of $M_{0,4}$. This leads us to the notion of **stable rational pointed curve**, which we will expand upon in the next lecture.

1.4 Further exercises

Exercise 11 (Automorphisms of $\mathbb{P}^1$)**.** *Depending on your favorite point of view, we are parameterizing configurations of marked points on the complex projective line* $\mathbb{P}^1$ *(algebraic geometry) or on the Riemann sphere* $\hat{S}^2 = \mathbb{C} \cup \infty$ *(complex analysis). We are interested in the group of automorphisms of* $\mathbb{P}^1$, *that can be described as:*

$\mathbb{P}GL(2,\mathbb{C})$**:** *equivalence classes of* 2×2 *matrices with non-zero determinant, up to simultaneous non-zero scaling of all the coefficients, acting on points of* $\mathbb{P}^1$ *by matrix multiplication on the homogeneous coordinates.*

Moebius tranformations: *Meromorphic functions of the form:*

$$w = \frac{az + b}{cz + d},$$

with $ad - bc \neq 0$

If you are not familiar with these concepts, spend some time understanding each of them and their equivalence. Show that there is a unique automorphism of $\mathbb{P}^1$ *that maps any three distinct points of* $\mathbb{P}^1$ *to any other three distinct points.*

A slick way to do so is to show that for any (p_1, p_2, p_3)*, with* $p_i \in \mathbb{P}^1$ *and* $p_i \neq p_j$ *when* $i \neq j$*, there exists a unique automorpshism* $\Phi_{cr} \in Aut(\mathbb{P}^1)$ *such that*

$$\Phi_{cr}(p_1) = 0, \qquad \Phi_{cr}(p_2) = 1, \qquad \Phi_{cr}(p_3) = \infty. \tag{1.7}$$

Exercise 12 (Cross-ratio)**.** *Given* (p_1, p_2, p_3, p_4)*, 4 distinct points of* $\mathbb{P}^1$*, their* **cross-ratio** *is defined to be:*

$$CR(p_1, p_2, p_3, p_4) = \frac{(p_1 - p_4)(p_3 - p_2)}{(p_1 - p_2)(p_3 - p_4)} \tag{1.8}$$

- *With* Φ_{cr} *as in* (1.7)*, show:*

$$CR(p_1, p_2, p_3, p_4) = \Phi_c r(p_4).$$

- *Show that the cross ratio is a projective invariant: i.e., if* Φ *is any automorphism of* $\mathbb{P}^1$

$$CR(p_1, p_2, p_3, p_4) = CR(\Phi(p_1), \Phi(p_2), \Phi(p_3), \Phi(p_4)).$$

Exercise 13 (A new perspective on a pencil of conics)**.** *Consider the pencil of plane conics defined by the equation:*

$$\mathcal{S} := \{\lambda x(z - y) + \mu z(y - x) = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \tag{1.9}$$

1. *Recognize that the total space of this pencil is the blow-up of* $\mathbb{P}^2$ *at four points. For each* $i = 1, \ldots, 4$ *denote* E_i *the exceptional divisor corresponding to blowing up the* i*-th point.*

2. *Consider the commutative diagram:*

$$\begin{array}{ccc} U_4 = M_{0,4} \times \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^2 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} \\ M_{0,4} & \xrightarrow{\lambda \mapsto (\lambda:1)} & \mathbb{P}^1 \end{array}, \tag{1.10}$$

where F *is the function:*

$$F(\lambda, w) = (w(w - 1) : w(\lambda - 1) : \lambda(w - 1)). \tag{1.11}$$

(a) Show that the Zariski closure of $Im(F)$ in $\mathbb{P}^2 \times \mathbb{P}^1$ is $\mathcal{S}$.

(b) Show that for each $i = 1, \ldots, 4$, $F \circ \sigma_i$ gives a parameterization of the exceptional divisor E_i.

Thus we have realized the compactified universal family $\pi : \overline{U}_4 \to \overline{M}_{0,4}$ as the blow-up of $\mathbb{P}^2$ at 4 points, and identified the four tautological sections with the exceptional divisors.

Exercise 14. *Notice that $\overline{M}_{0,4}$ is equal to the universal family $U_{0,3}$, where the images of the three distinguished sections are the three points added to compactify $M_{0,4}$. Do you think this is a coincidence, or is there a reason for this?*

Lecture 2

Stable Pointed Curves: $\overline{M}_{0,n}$

At the end of our last lecture we learned that in order to compactify $M_{0,4}$ to a moduli space, we had to "invite" to the moduli problem some degenerations of the projective line: specifically, pairs of projective lines glued together at one point. In fact allowing rational curves with nodal singularities is a way to compactify $M_{0,n}$ for all n. We now make this precise.

Definition 6. *A* **stable rational n-pointed curve** *is a tuple $(C, p_1, \dots, p_n)$, such that:*

- *C is a connected curve of arithmetic genus 0, whose only singularities are nodes (i.e. locally analytically around a singular point $p \in Sing(C)$, C can be defined by the equation $xy = 0$.)*
- *$(p_1, \dots, p_n)$ are distinct points of $C \smallsetminus Sing(C)$.*
- *The only automorphism of C that preserves the marked points is $\mathbb{1}_C$.*

For more combinatorially minded people, this is equivalent to the notion of a stable tree of projective lines.

Definition 7. *A* ***stable n pointed tree of projective lines*** *is a tuple $(C, p_1, \dots, p_n)$, such that:*

- *C is connected and each irreducible component of C is isomorphic to $\mathbb{P}^1$. Irriducible components are called twigs.*
- *The points of intersection of the components are ordinary double points.*
- *There are no closed circuits in C, i.e., if any node is removed then the curve becomes disconnected.*
- *$p_1, \dots, p_n$ and the double points of C are all distinct and are called special points.*
- *Every twig has at least three special points.*

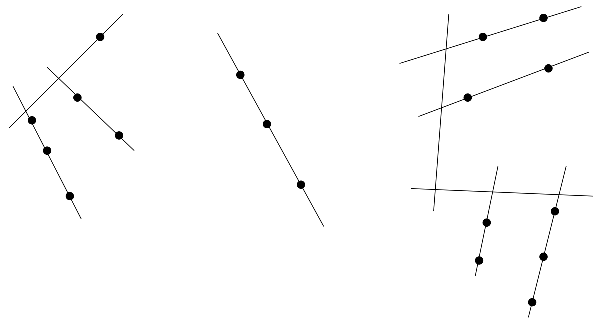


Figure 2.1: Stable marked trees of projective lines. The points should be labeled, but I got lazy...

We draw a marked tree as in Figure 2.1, where each line represents a twig.

Exercise 15. *Show that Definitions 6 and 7 are equivalent.*

Exercise 16. *Develop the natural definition of family of rational stable n-pointed curves over a base B and of isomorphism of families; describe the moduli functor $\overline{\mathcal{M}}_{0,n}$ for isomorphism classes of such families.*

Intuition. *Here's an intuitive (albeit imprecise) picture about the compactification to stable pointed curves. When points on a rational curve C want to coincide, we don't allow them as follows: at the moment of collision, we create a new bubble attached to the point of collision and transfer the points to the new bubble. At this point we are able to act via elements of $\mathbb{P}GL(2,\mathbb{C})$ on each component of the curve, but such elements must be compatible in the sense that they must preserve nodes. That is why nodes of C are considered just like marked points.*

With all these definitions in place, in Section 1.3 we proved the following statement.

Lemma 1. *$\overline{M}_{0,4} \cong \mathbb{P}^1$ represents the functor of isomorphism classes of families of n-pointed rational curves. The universal family $\overline{U}_4$ may be viewed as the second projection $Bl_{(0,0),(1,1),(\infty,\infty)}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1$, or as the blow up of a pencil of plane conics at its four base points.*

This result generalizes to arbitrary n.

Theorem 1 (Knudsen)**.** *There exists an irreducible, smooth, projective fine moduli space $\overline{M}_{0,n}$ for n-pointed rational stable curves, compactifying $M_{0,n}$. The universal family $\overline{U}_n$ is obtained from U_n via a finite sequence of blow-ups.*

We discuss the proof of this theorem and a few constructions of $\overline{M}_{0,n}$ in Exercise 38. For references in the literature, you may see [KV07] or [Knu83], [Knu12], [Kee92] or [Kap93b]. At this moment we turn our attention to exploring the geometry of this family of moduli spaces.

2.1 The boundary

We define the **boundary** to be the complement of $M_{0,n}$ in $\overline{M}_{0,n}$. It consists of all points parameterizing nodal stable curves. We begin by making some purely set-theoretical considerations.

The boundary of $\overline{M}_{0,4}$ consists of three points parameterizing the nodal pointed curves depicted in Figure 2.2.

The boundary of $\overline{M}_{0,5}$ is represented in Figure 2.3: it consists of 15 points parameterizing pointed curves with three irreducible components, and 10 copies of $M_{0,4}$ parameterizing rational pointed curves with two irreducible components.

In general the boundary of $\overline{M}_{0,n}$ is stratified by locally closed subsets parameterizing curves of a given topological type, together with a prescribed assignment of the marks to the irreducible components of the curve. A natural way to encode this data is via the dual graph to a pointed curve.

Definition 8. *Given a rational, stable n-pointed curve $(C, p_1, \ldots, p_n)$, its* **dual graph** *is defined to have:*

- *a vertex for each irreducible component of C;*
- *an edge for each node of C, joining the appropriate vertices;*
- *a labeled half edge for each mark, emanating from the appopriate vertex.*

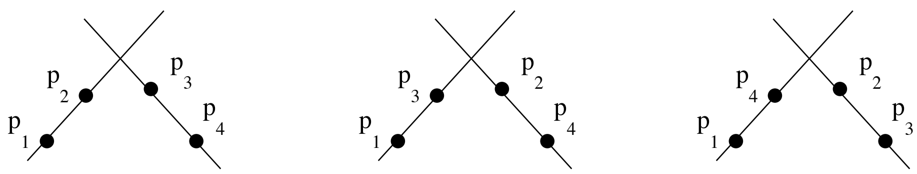


Figure 2.2: The boundary of $\overline{M}_{0,4}$ consists of three points.

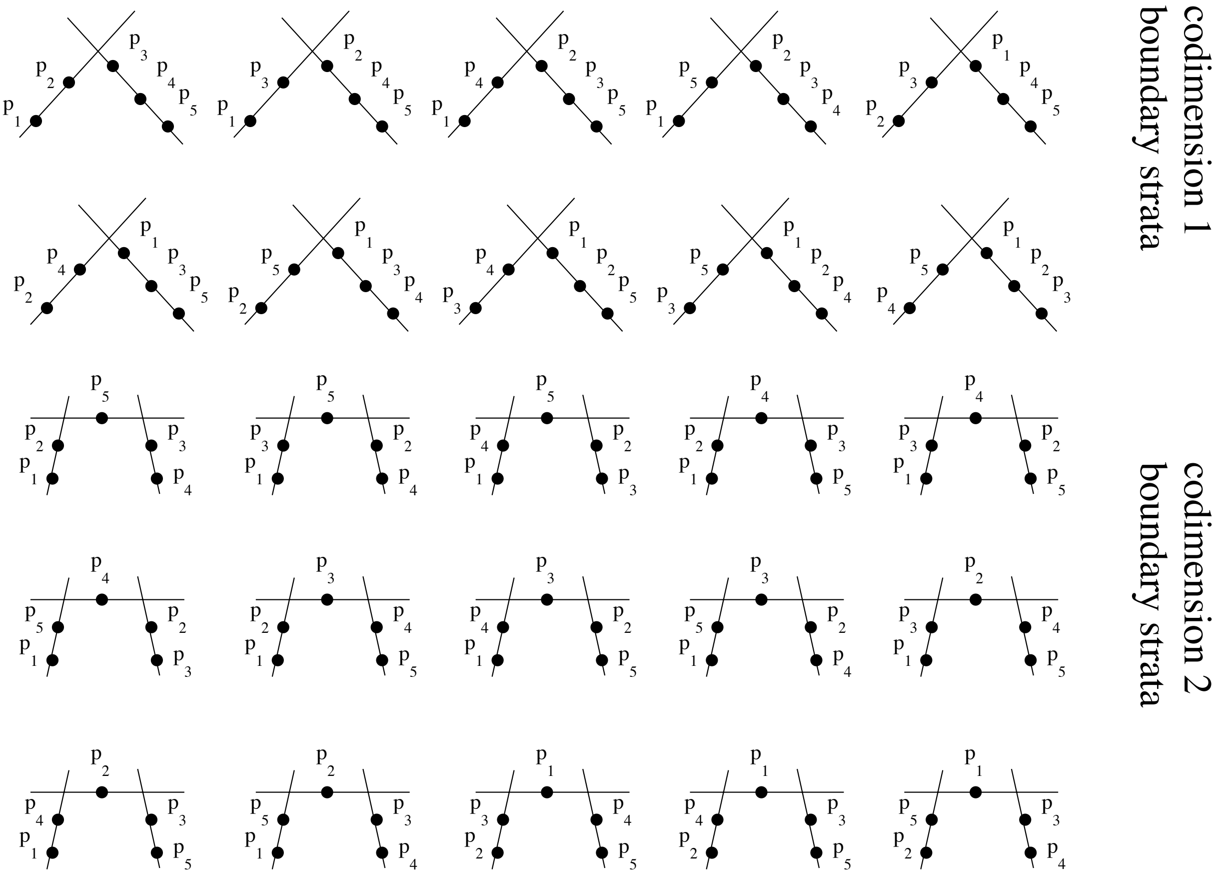


Figure 2.3: Boundary strata of $\overline{M}_{0,5}$

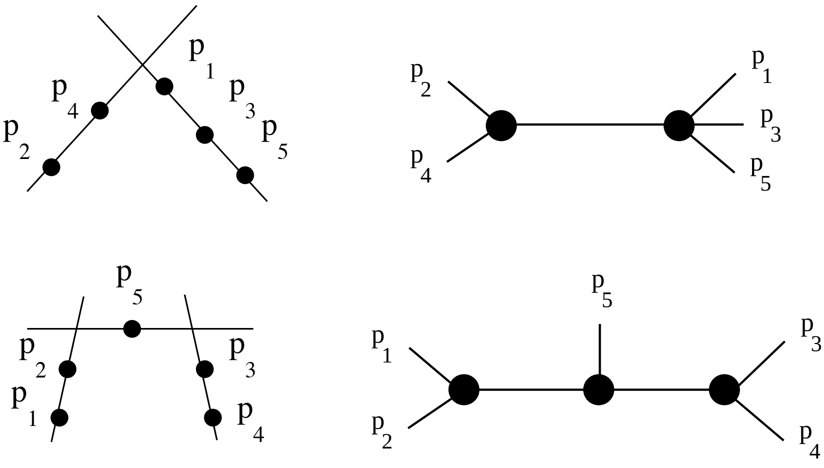


Figure 2.4: Two boundary strata of $\overline{M}_{0,5}$ on the left, and their dual graphs on the right.

Figure 2.4 gives an example of the dual graphs of some strata in $\overline{M}_{0,5}$. Now we can say that a boundary stratum S is identified by the dual graph Γ_S of the general curve that it parameterizes.

Exercise 17. *Show that the codimension of a boundary stratum S equals the number of nodes that any curve parameterized by S has, or, equivalently, the number of edges in its dual graph.*

The **closures** of the codimension 1 boundary strata of $\overline{M}_{0,n}$ are called the **irreducible boundary divisors**; they are in one-to-one correspondence with all ways of partitioning $[n] = A \cup A^c$ with the cardinality of both A and A^c strictly greater than 1. We denote $D(A) = D(A^c)$ the divisor corresponding to the partition A, A^c.

Exercise 18. *Describe the set theoretic intersection of two divisors $D(A) \cap D(A')$. In particular describe combinatorial conditions for A and A' that characterize when the intersection is empty.*

There is a natural partial order on the strata: $S_1 < S_2$ if the graph Γ_{S_2} can be obtained from Γ_{S_1} by contracting some edges. Geometrically, this means that S_1 is in the closure of S_2.

Exercise 19. *Describe explicitly the poset of boundary strata for $\overline{M}_{0,5}$. Show that each codimension-two stratum is in the closure of exactly two codimension-one strata.*

2.2 Natural morphisms

There is a wealth of natural morphisms among the various moduli spaces $\overline{M}_{0,n}$. To be as intuitive as possible, we now describe them as set functions, i.e. by describing how they act on the objects parameterized. There is a fair amount of technical work shoved under the rug that shows that all of these constructions work in families and are therefore honest morphisms of the moduli spaces.

Intuition. *Understanding these morphisms is a powerful tool in the study of the geometry of these moduli spaces: it shows that the boundary of $\overline{M}_{0,n}$ is "built" out of smaller moduli spaces of pointed rational curves, and it opens the way to an inductive study of any geometric question about $\overline{M}_{0,n}$ that can be "pushed to the boundary".*

2.2.1 Forgetful morphisms and universal families

The $(n+1)$-th forgetful morphism is the function that assignes to an $(n+1)$ pointed curve, the n pointed curve obtained by removing the last marked point:

$$\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n} \tag{2.1}$$

$$(C, p_1, \ldots, p_{n+1}) \mapsto (C, p_1, \ldots, p_n)$$

The function π_{n+1} is defined if p_{n+1} does not belong to a twig with only three special points, as the n-pointed curve obtained by forgetting p_{n+1} is still stable. If it does belong to such a twig, then the resulting tree is no longer stable. One obtains a stable curve again by contracting the unstable twigs, as illustrated in Figure 2.5.

Intuition. *Generically, the data of an $(n+1)$ pointed curve can be thought as of the choice of a point - namely p_{n+1} - on an n pointed curve. Intuitively this means that one should be able to identify the universal family of $\overline{M}_{0,n}$ with $\overline{M}_{0,n+1}$. This is indeed the case, and it is at the core of Knudsen's construction of the moduli space in [Knu83, Knu12].*

The contraction morphism

$$\mathbf{c} : \overline{M}_{0,n+1} \to \overline{U}_{0,n} \tag{2.2}$$

assigns to $(C, p_1, \ldots, p_{n+1})$:

- $p_{n+1} \in (C, p_1, \ldots, p_n)$ if $(C, p_1, \ldots, p_n)$ is stable;
- $p_k \in \pi_{n+1}(C, p_1, \ldots, p_{n+1})$, if p_{n+1} is on a twig with three special points, one of which is p_k;

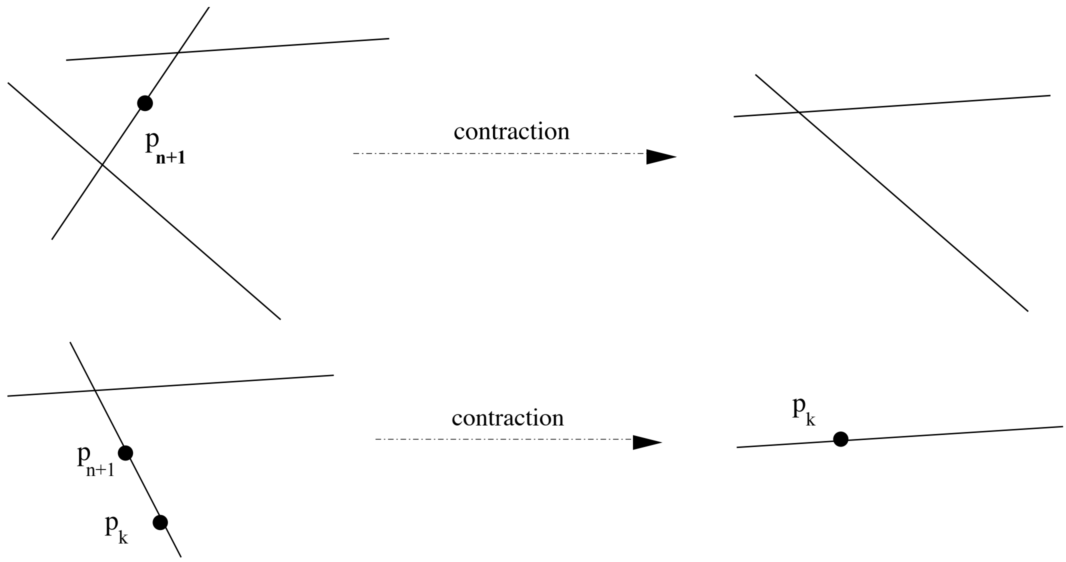


Figure 2.5: Contracting twigs that become unstable after forgetting the p_{n+1}. In the second case the mark p_k is placed where the node used to be.

- $n \in \pi_{n+1}(C, p_1, \ldots, p_{n+1})$, if p_{n+1} is on a twig with three special points, the other two being nodes; n is the node which is formed after contracting the twig that contained p_{n+1}.

The stabilization morphism

$$\mathbf{s} : \overline{U}_{0,n} \to \overline{M}_{0,n+1} \tag{2.3}$$

is defined by:

- $\mathbf{s}[p \in (C, p_1, \ldots, p_n)] = (C, p_1, \ldots, p_n, p)$ if p is a smooth point of C and it is not a mark;

- $\mathbf{s}[p_k \in (C, p_1, \ldots, p_n)]$ is the curve defined by attaching a twig at the position of the i-th mark, and putting p_k and p_{n+1} on this twig, as in Figure 2.6;

- if n is a node of C, then $\mathbf{s}[n_k \in (C, p_1, \ldots, p_n)]$ is the curve defined by inserting a twig between the two shadows of the node n and placing p_{n+1} on this twig, as in Figure 2.7.

Theorem 2 ([Knu83, Knu12]). *The maps* $\mathbf{c}, \mathbf{s}$ *are well defined morphisms, they are inverses of each other, and they make the following diagram com-*

Figure 2.6: The image of a mark under the stabilization morphism

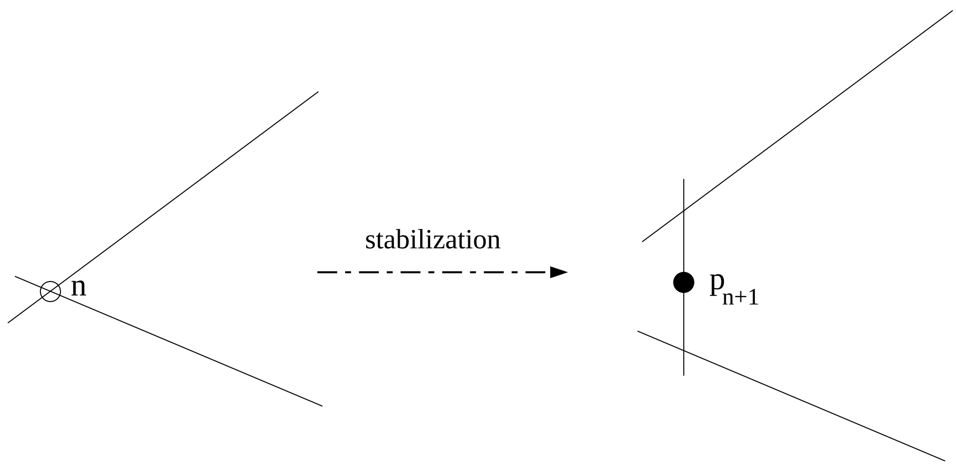
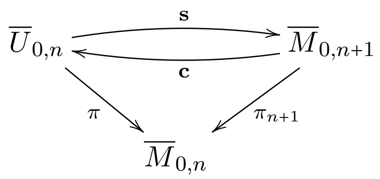


Figure 2.7: The image of a node under the stabilization morphism

mute:



This identifies $(\overline{M}_{0,n+1}, \pi_{n+1})$ *with the universal family of* $\overline{M}_{0,n}$*.*

Exercise 20. *Describe the image of* $\mathbf{s} \circ \sigma_i$ *in* $\overline{M}_{0,n+1}$*.*

Remark. There is nothing special about forgetting the last mark. We have forgetful morphisms π_i that forget any mark we want (and therefore different ways of viewing $\overline{M}_{0,n+1}$ as a universal family over $\overline{M}_{0,n}$). Also, we may compose forgetful morphisms to forget more than one point at a time.

2.3 Gluing morphisms

There are also natural gluing morphisms; for $[n] = I \cup I^c$ a partition of the set of indices, with $|I|$ and $|I^c| \geq 2$,

$$gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$$

takes a pair of pointed curves and identifies the points marked $\star$ and $\bullet$. The resulting nodal curve is clearly stable. The map gl_I is an isomorphism onto its image, which is the boundary divisor corresponding to the partition $[n] = I \cup I^c$. This shows that the closure of the codimension one boundary strata are the irreducible components of the boundary.

One can realize the closure of any boundary stratum as the isomorphic image of an appropriate gluing morphism. This statement makes precise the earlier claim that the boundary of $\overline{M}_{0,n}$ is "built out" of products of smaller spaces of rational pointed stable curves.

2.4 Intersection Theory on $\overline{M}_{0,n}$

2.4.1 A quick and dirty introduction to intersection theory

The main character here is the Chow ring, $A^*(X)$, of a smooth algebraic variety X. The ring $A^*(X)$ is, in some loose sense, the algebraic counterpart of the cohomology ring $H^*(X)$, and it makes precise in the algebraic category the intuitive concepts of oriented intersection in topology.

Elements of the group $A^n(X)$ are formal finite sums of codimension n closed subvarieties (cycles), modulo an equivalence relation called rational equivalence. $A^*(X) = \oplus_0^{dim X} A^n(X)$ is a graded ring with product given by intersection.

Intersection is independent of the choice of representatives for the equivalence classes.

In topology, if we are interested in the cup product of two cohomology classes **a** and **b**, we can choose representatives a and b that are transverse to each other. We can assume this since transversality is a generic condition: if a and b are not transverse then we can perturb them ever so slightly and make them transverse while not changing their classes. This being the case, then $a \cap b$ represents the cup product class $\mathbf{a} \cup \mathbf{b}$.

In algebraic geometry, even though this idea must remain the backbone of our intuition, things are a bit trickier. Exceptional divisors in blow-ups are examples of cycles that are rigid, in the sense that their representative is unique, and hence "unwigglable". Still, we can define an algebraic product that reduces to the "geometric" one when transversality can be achieved (see [Ful98]).

Example: the Chow Ring of Projective Space.

$$A^*(\mathbb{P}^n) = \frac{\mathbb{C}[\mathbf{H}]}{(\mathbf{H}^{n+1})},$$

where $\mathbf{H} \in A^1(\mathbb{P}^n)$ is the class of a hyperplane H.

2.4.2 Chern Classes of Bundles

For every vector bundle there is a natural section $s_0 : B \to \mathcal{E}$ defined by

$$s_0(b) = (b, 0) \in \{b\} \times \mathbb{C}^n.$$

It is called the zero section, and it gives an embedding of B into $\mathcal{E}$.

A natural question to ask is if there exists another section $s : B \to \mathcal{E}$ which is disjoint form the zero section, i.e. $s(b) \neq s_0(b)$ for all $b \in B$. The **Euler class** of this vector bundle ($\mathbf{e}(\mathcal{E}) \in A^n(B)$) is defined to be the class of the self-intersection of the zero section: it measures obstructions for the above question to be answered affirmatively. This means that $\mathbf{e}(\mathcal{E}) = 0$ if and only if a never vanishing section exists. It easily follows from the Poincaré-Hopf theorem that for a manifold M, the following formula holds:

$$\mathbf{e}(TM) \cap [M] = \chi(M).$$

That is, the degree of the Euler class of the tangent bundle is the Euler characteristic.

The Euler class of a vector bundle is the first and most important example of a whole family of "special" cohomology classes associated to a bundle, called the **Chern classes** of $\mathcal{E}$. The k-th Chern class of $\mathcal{E}$, denoted $\mathbf{c}_k(\mathcal{E})$, lives in $A^k(B)$. In the literature you can find a wealth of definitions for

Chern classes, some more geometric, dealing with obstructions to finding a certain number of linearly independent sections of the bundle, some purely algebraic. Such formal definitions, as important as they are (because they assure us that we are talking about something that actually exists!), are not particularly illuminating. In concrete terms, what you really need to know is that Chern classes are cohomology classes associated to a vector bundle that satisfy a series of properties, which we now recall.

Let $\mathcal{E}$ be a vector bundle of rank n:

identity: by definition, $\mathbf{c}_0(\mathcal{E}) = 1$.

normalization: the n-th Chern class of $\mathcal{E}$ is the Euler class:

$$\mathbf{c}_n(\mathcal{E}) = \mathbf{e}(\mathcal{E}).$$

vanishing: for all $k > n$, $\mathbf{c}_k(\mathcal{E}) = 0$.

pull-back: Chern classes commute with pull-backs:

$$f^*\mathbf{c}_k(\mathcal{E}) = \mathbf{c}_k(f^*\mathcal{E}).$$

tensor products: if L_1 and L_2 are line bundles,

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Whitney formula: for every extension of bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

the k-th Chern class of $\mathcal{E}$ can be computed in terms of the Chern classes of $\mathcal{E}'$ and $\mathcal{E}''$, by the following formula:

$$\mathbf{c}_k(\mathcal{E}) = \sum_{i+j=k} \mathbf{c}_i(\mathcal{E}')\mathbf{c}_j(\mathcal{E}'').$$

Using the above properties it is immediate to see:

1. all the Chern classes of a trivial bundle vanish (except the 0-th);
2. for a line bundle L, $\mathbf{c}_1(L^*) = -\mathbf{c}_1(L)$.

To show how to use these properties to work with Chern classes, we calculate the first Chern class of the tautological line bundle over $\mathbb{P}^1$. The tautological line bundle is

$$\begin{array}{c} \mathcal{S} \\ \pi \downarrow \\ \mathbb{P}^1, \end{array}$$

where $\mathcal{S} = \{(p,l) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid p \in l\}$. It is called tautological because the fiber over a point in $\mathbb{P}^1$ is the line that point represents.

Our tautological family fits into the short exact sequence of vector bundles over $\mathbb{P}^1$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{S} & \to & \mathbb{C}^2 \times \mathbb{P}^1 & \to & \mathcal{Q} & \to & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & \mathbb{P}^1 & & & & \end{array}$$

where $\mathcal{Q}$ is the bundle whose fibre over a line $l \in \mathbb{P}^1$ is the quotient vector space $\mathbb{C}^2/l$. Notice that $\mathcal{Q}$ is also a line bundle. From the above sequence, we have that

$$0 = \mathbf{c}_1(\mathbb{C}^2 \times \mathbb{P}^1) = \mathbf{c}_1(\mathcal{S}) + \mathbf{c}_1(\mathcal{Q}). \tag{2.4}$$

Since $\mathbb{P}^1$ is topologically a sphere, which has Euler characteristic 2, then

$$2 = \mathbf{c}_1(T\mathbb{P}^1) = \mathbf{c}_1(\mathcal{S}^*) + \mathbf{c}_1(\mathcal{Q}) = -\mathbf{c}_1(\mathcal{S}) + \mathbf{c}_1(\mathcal{Q}). \tag{2.5}$$

The second equality in 2.5 holds because $T\mathbb{P}^1$ is the line bundle $Hom(\mathcal{S},\mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$. It now follows from (2.4) and (2.5) that $\mathbf{c}_1(\mathcal{S}) = -1$.

2.4.3 The Chow ring of $\overline{M}_{0,n}$

In [Kee92], Keel constructs $\overline{M}_{0,n}$ by a sequence of blow-ups along smooth, disjoint, codimension 2 subvarieties starting from $\overline{M}_{0,n-1} \times \mathbb{P}^1$. This allows him to give a presentation of the Chow ring.

Theorem 3 ([Kee92])**.**

$$A^\star(\overline{M}_{0,n}) = \mathbb{Z}[\{D(A)\}_{A\subset[n],2\le|A|\le n-2}]/I_R,$$

where $\{D(A)\}$ denotes the set of boundary divisors and the ideal of relations I_R is generated by:

symmetry $D(A) = D(A^c)$;

disjointness $D(A)D(B) = 0$ *if their set theoretic intersection is empty (in Exercise 18 you were asked for the combinatorial condition characterizing this);*

WDVV $f^*(0) = f^*(\infty)$ *where f is any forgetful morphism $f : \overline{M}_{0,n} \to \overline{M}_{0,4}$, and $0,\infty \in \overline{M}_{0,4}$ are two boundary points.*

Exercise 21. *Write down explicitly the combinatorial condition corresponding to the WDVV relations.*

Intuition. *We should interpret Keel's theorem as a statement that things just went as well as possible. Boundary divisors generate the Chow ring, and the relations are only the inevitable ones coming from set theoretic disjointness, plus relations arising from the fact that any two points in $\mathbb{P}^1 = \overline{M}_{0,4}$ are equivalent.*

On the one hand, Keel's result is a complete and conclusive result about the intersection theory of $\overline{M}_{0,n}$, especially if one is interested mainly in its algebraic structure. If one is more directly interested in the geometry of intersections, then such a presentation is still a bit cryptic. It is not immediate that intersections of degree greater than the dimension of $\overline{M}_{0,n}$ vanish, for example. Also, it takes some thought to deal with self intersections of divisors. In the next few exercises we explore these issues.

Exercise 22. *Show that for any collection of $n-2$ distinct divisors in $\overline{M}_{0,n}$, their product vanishes.*

Exercise 23. *Show that all codimension two strata in $\overline{M}_{0,5}$ are equivalent. However, show that for each such stratum there is a unique expression as product of divisors which is true at the level of cycles.*

Exercise 24. *Compute the self intersection $D(\{1,2\})^2 \in A^2(\overline{M}_{0,5})$ as follows: replace one copy of $D(\{1,2\})$ using a WDVV relation, and reduce the problem to a sequence of transverse intersections.*

The last few exercises hopefully are convincing that one can indeed extract any geometric information from Keel's presentation; however that the process is somewhat laborious. This motivates us to introduce new Chow classes, which help us getting a more direct handle on geometric intersections of strata in $\overline{M}_{0,n}$.

2.5 Psi classes

For $i = 1,\dots,n$, we define the class $\psi_i \in A^1(\overline{M}_{0,n})$. We give a couple pseudo-definitions, which are good to develop intuition, before we give a formal one.

Take 1. Let $\mathbb{L}_i \to \overline{M}_{0,n}$ be a line bundle whose fiber over each point $(C, p_1,\dots,p_n)$ is canonically identified with $T^*_{p_i}(C)$. The line bundle $\mathbb{L}_i$ is called the i-th **cotangent** (or **tautological**) line bundle. Then

$$\psi_i := c_1(\mathbb{L}_i). \tag{2.6}$$

Take 2. Let $(X, B, \pi, \sigma_1,\dots,\sigma_n)$ be a family of rational stable n-pointed curves, and assume X is smooth. Denote f_π the map to $\overline{M}_{0,n}$ corresponding to such a family. Then

$$f^*_\pi(\psi_i) := -c_1(\sigma_i^* N_{\sigma_i/X}). \tag{2.7}$$

Intuition. *These two pseudo-definitions can be made precise, and they both reflect how one exploits the dictionary between geometry of a moduli space and geometry of the objects parameterized. We now provide an bona fide definition of psi classes.*

Take 3. Define the **relative dualizing sheaf** $\omega_\pi \to \overline{U}_{0,n}$ to be

$$\omega_\pi := \Omega^1(\overline{U}_{0,n})/\pi^*\Omega^1(\overline{M}_{0,n}). \tag{2.8}$$

If we restrict ω_π to a fiber of π correspoding to a curve C, we obtain the dualizing sheaf ω. This is a sheaf on C whose sections are meromorphic differential forms on the normalization of C such that they only have poles and at most of order one at the shadows of nodes of C, and the residues at the two shadows of a node must match. The following exercise is the local computation that is at the base of proving this fact.

Exercise 25. *Consider the neighborhood of a node, i.e. the family $\mathbb{C}^3 \to (\mathbb{C}, t)$ given by the equation:*

$$xy = t.$$

Show that the sections of the sheaf $\Omega^1_{\mathbb{C}^3}/\pi^*\Omega^1_{\mathbb{C}}$ indeed restrict to the central fiber to meromorphic forms with at most one poles at the node and matching residues there.

Definition 9. *The class $\psi_i \in A^1(\overline{M}_{0,n})$ is defined to be:*

$$\psi_i := c_1(\sigma_i^*\omega_\pi)$$

As a warm-up, consider a divisor of the form $D(\{i,j\}) \subset \overline{M}_{0,n}$, and let us describe the class of its self-intersection. Recall that we can think of $\overline{M}_{0,n}$ together with the morphism forgetting the j-th point as a universal family for $\overline{M}_{0,n-1}$. Then the divisor $D(\{i,j\})$ is the image of the section σ_i. Using the second pseudo-definition, we obtain:

$$D(\{i,j\})^2 = \sigma_{i*}\left(e\left(\sigma_i^*(N_{\sigma_i/\overline{M}_{0,n}})\right)\right) = -\sigma_{i*}(\psi_i).$$

Exercise 26. *Show that on $\overline{M}_{0,3}$ all psi classes vanish (trivially) and on $\overline{M}_{0,4}$, for any i we have $\psi_i = [pt.]$.*

2.5.1 Smoothing nodes and the normal bundle to a boundary divisor

Consider the gluing morphism:

$$gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$$

whose image is the divisor $D(I)$. We describe (pull-back of) the normal bundle $N_{D(I)/\overline{M}_{0,n}}$ in terms of tautological bundles on the factors.

Intuition. *The game is to relate the geometry of the moduli space with the geometry of the objects parameterized. To describe the normal direction to the divisor $D(I)$ we consider an infinitesimal family of curves, where the central fiber belongs to $D(I)$ and the generic fiber doesn't - which means that the generic fiber is a smooth $\mathbb{P}^1$. We note that "moving out" of the divisor corresponds to smoothing the node of the central fiber.*

Let C_t denote a one parameter family of rational stable n-pointed curves, such that $C_0 \in D(I)$ and $C_t \in M_{0,n}$ for $t \neq 0$. We can interpret t as a coordinate in a direction normal to $D(I)$ at C_0. Since we are interested in infinitesimal (first order) information, we can look at the local analytic equation for C_t around the node of the central fiber, and notice that it has the form $xy = t$. We observe that x and y can be identified with sections of the tangent spaces of the two axes in $\mathbb{C}^2$. These are the tangent spaces at the shadows of the node in the normalization of C_0. Finally we observe that once we give this interpretation, the relation provided by the local equation $xy = t$ gives a relation among sections of line bundles on $D(I)$. This is an impressionistic argument for the following statement.

Lemma 2. *Consider the gluing morphism* $gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$. *Then :*

$$gl_I^*(N_{D(I)/\overline{M}_{0,n}}) \cong \mathbb{L}_\star^\vee \boxtimes \mathbb{L}_\bullet^\vee. \tag{2.9}$$

Lemma 2 allows us to describe the self-intersection of a boundary divisor in terms of psi classes.

Lemma 3. *Consider the gluing morphism* $gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$ *and let* π_1 *and* π_2 *denote the two projections from* $\overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet}$ *onto the factors. Then:*

$$D(I)^2 = gl_{I*}(-\pi_1^*(\psi_\star) - \pi_2^*(\psi_\bullet)). \tag{2.10}$$

Exercise 27. *Recompute the self intersection* $D(\{1,2\})^2 \in A^2(\overline{M}_{0,5})$ *using Lemma 3.*

Exercise 28. *Describe a combinatorial algorithm that determines the intersection of any two boundary strata in* $\overline{M}_{0,n}$ *in terms of boundary strata and psi classes.*

2.5.2 Properties of psi classes

In Section 2.5.1 we showed that we can understand non-transversal intersection of boundary strata in terms of psi classes. This motivates us to seek for a better understanding of these classes. Because of Keel's presentation, we know that psi classes must be equivalent to a linear combination of boundary divisors, but is there a nice description for such a linear combination? Also, what if we want to deal with a product of psi classes? In an extreme case, suppose we are given a monomial in psi classes of degree $n-3$, which is just a multiple of the class of a point - is there a "quick" way to determine this multiple? In this section we answer all these questions in the positive.

The first property of psi classes we explore is how they restrict to boundary divisors (or strata in general). Informally, a boundary stratum is a product of moduli spaces; the restriction of ψ_i to the stratum equals the class ψ_i pulled back from the factor containing the $i-th$ mark. More formally:

Lemma 4. *Consider the gluing morphism* $gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$. *Assume that* $i \in I$ *and denote by* $\pi_1 : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,I\cup\star}$ *the first projection. Then:*

$$gl^*(\psi_i) = \pi_1^*(\psi_i).$$

Intuition. *This lemma is obvious if one accepts our first pseudo-definition of psi classes, as then the class* ψ_i *only knows what happens in an infinitesimal neighborhood of the point* p_i. *We leave it to the interested reader to convert this idea into an actual proof.*

Exercise 29. *Prove that*

$$\psi_i \cdot D(\{i,j\}) = 0.$$

The next result is a comparison between a psi class on $\overline{M}_{0,n+1}$ and the pull-back of the corresponding psi class on $\overline{M}_{0,n}$. This result is essential in giving an inductive description of psi classes.

Lemma 5. *Consider the forgetful morphism* $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$. *Then, for every* $i = 1, \ldots, n$,

$$\psi_i = \pi_{n+1}^*(\psi_i) + D(\{i, n+1\}).$$

In the spirit of this section, instead of a rigorous proof we provide a couple of quasi-proofs that have the advantage of illustrating what is going on.

Using the first "definition" of ψ_i we observe that the fibers of $\mathbb{L}_i$ and $\pi_{n+1}^*(\mathbb{L}_i)$ are canonically identified except along the divisor $D(\{i, n+1\})$. Hence it must be that

$$\mathbb{L}_i \cong \pi_{n+1}^*(\mathbb{L}_i) \otimes \mathcal{O}(kD(\{i, n+1\})),$$

for some integer k. Taking the first Chern class:

$$\psi_i = \pi_{n+1}^*(\psi_i) + kD(\{i, n+1\}). \tag{2.11}$$

We determine that $k = 1$ in the following exercise.

Exercise 30. *Determine the value of* k *in two different ways:*

1. *Multiply relation* (2.11) *by* $D(\{i, n+1\})$;
2. *Multiply relation* (2.11) *by the boundary stratum consisting of a chain of projective lines, with the points* $1, i$ *and* $n+1$ *on the first twig of the chain, and exactly three special points on each other twig.*

Using the second "definition" of ψ_i, we consider a one parameter family of n pointed curves $(X, B, \pi, \sigma_1, \dots, \sigma_n)$. We now consider an arbitrary section σ_{n+1}, not necessarily disjoint from the other sections (but let us assume that the intersections $\sigma_{n+1} \cap \sigma_i$ for all i are transversal). The family of rational stable $(n+1)$ pointed curves $(\hat{X}, B, \hat{\pi}, \hat{\sigma}_1, \dots, \hat{\sigma}_n, \hat{\sigma}_{n+1})$ is obtained by blowing up the intersection points of σ_{n+1} with the other sections, and taking the proper transforms of the sections. Denote $f_{\hat{\pi}} : B \to \overline{M}_{0,n+1}$ the corresponding map to the moduli space.

Identifying B with $\sigma_i(B)$ we have that

$$\sigma_i(B)^2 = -f_{\hat{\pi}}^*(\pi_{n+1}^*(\psi_i)).$$

Identifying B with $\hat{\sigma}_i(B)$,

$$\hat{\sigma}_i(B)^2 = -f_{\hat{\pi}}^*(\psi_i).$$

Finally, we use the fact that

$$\sigma = \hat{\sigma} + E_{i,n+1},$$

where $E_{i,n+1}$ is the exceptional divisor correponding to copies of $\mathbb{P}^1$ mapping to each of the intersection points of σ_i with σ_{n+1}. It follows that for any one parameter family $f_{\hat{\pi}}$,

$$f_{\hat{\pi}}^*(\psi_i) = f_{\hat{\pi}}^*(\pi_{n+1}^*(\psi_i) + D(\{i, n+1\})). \tag{2.12}$$

Lemma 5 allows a combinatorial description of cycles representing the psi classes.

Exercise 31. *Recover that on $\overline{M}_{0,4}$, $\psi_i = [pt.]$ by using Lemma 5. Describe cycles representing ψ_i in $\overline{M}_{0,5}$.*

Once you have warmed up, you can then prove the following general statement.

Lemma 6. *For any choice of i, j, k distinct, we have the following equation in $A^1(\overline{M}_{0,n})$:*

$$\psi_i = \sum_{i \in I,\ j,k \notin I} D(I).$$

Show that any two such expressions for ψ_i are WDVV equivalent.

Exercise 32. *Prove that for any $k \geq 1$.*

$$\psi_i^k = \pi_{n+1}^*(\psi_i)^{k-1}(\pi_{n+1}^*(\psi_i) + D(\{i, n+1\})).$$

Now we investigate how a monomial in psi classes pushes forward with respect to a forgetful morphism.

Lemma 7 (String Equation)**.** *Consider the forgetful morphism* $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$*. Then*

$$\pi_{n+1*}\left(\prod_{i=1}^{n} \psi_i^{k_i}\right) = \sum_{j|k_j\neq 0} \psi_j^{k_j-1} \prod_{i\neq j} \psi_i^{k_i}.$$

Exercise 33. *Prove Lemma 7, by rewriting each of the* $\psi_i^{k_i}$ *as in Exercise 32, and observing which terms in the expansion of such produce do not vanish immediately, or after the push-forward.*

Lemma 7 allows us to explicity evaluate all top degree monomials in psi classes.

Exercise 34. *Let* $\sum k_i = n-3$*. Then*

$$\int_{\overline{M}_{0,n}} \prod_{i=1}^{n+1} \psi_i^{k_i} = \binom{n-3}{k_1, \ldots, k_{n+1}},$$

where the integral sign denotes push-forward to the class of a point.

Exercise 35. *Consider the forgetful morphism* $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$*. What is* $\pi_{n+1\star}(\psi_{n+1})$*?*

2.6 Further exercises

Exercise 36. *Consider the two models of* $\overline{M}_{0,5}$ *we have constructed:*

$$Bl_{(0,0),(1,1),(\infty,\infty)}(\mathbb{P}^1 \times \mathbb{P}^1)$$

and

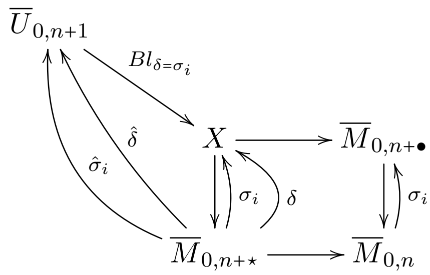
$$Bl_{(0:0:1),(0:1:0),(0:0:1),(1:1:1)}(\mathbb{P}^2).$$

In each of the two cases identify all the boundary strata.

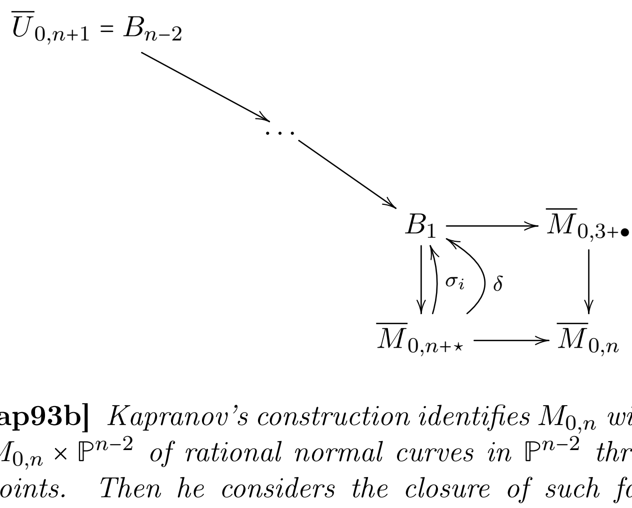
Exercise 37. *The* **boundary complex** *is the cone complex naturally associated to the poset structure of the boundary strata in* $\overline{M}_{0,n}$*. For each codimension* i *stratum you have a copy of* $(\mathbb{R}^{\geq 0})^i$*, and the poset structure naturally indicates how to identify cones with faces of other cones. Make this statement precise, and describe the boundary complex of* $\overline{M}_{0,5}$*. If you are familiar with Peterson graph, recognize that it is the cone over the Peterson graph.*

Exercise 38 (Constructions of $\overline{M}_{0,n}$)**.** *This exercise is not so much an exercise, as it is a quick and dirty sketch of some constructions of* $\overline{M}_{0,n}$*, in hope that it may help you read the relative literature if you decide to dig deeper into it. Try to follow these constructions in the cases* $n = 4, 5$ *and get a feel of why they work in general.*

Knudsen [Knu83] *Knudsen's construction is inductive, and it relies essentially on the identification of* $\overline{U}_{0,n}$ *with* $\overline{M}_{0,n+1}$ *given by the contraction and stabilization morphisms. Suppose we have realized* $\overline{M}_{0,n}$ *as a fine moduli space, which means in particular that we have constructed the universal family* $\overline{U}_{0,n}$*. We then take* $\overline{M}_{0,n+1}$ *to be equal to* $\overline{U}_{0,n}$*, and we must construct a universal family over it. First consider the fiber product of* $\overline{M}_{0,n+1}$ *with itself over* $\overline{M}_{0,n}$*. This gives a family of* n *pointed curves* X*. We can add one more section to this family by considering the diagonal section* $\star = \bullet$*. Then we stabilize this family, to obtain a family of rational stable* $(n+1)$*-pointed curves over* $\overline{M}_{0,n+1}$*. This is the universal family.*



Keel [Kee92] *Keel's construction is in a way similar to Knudsen, in that it starts from a fiber product of* $\overline{M}_{0,n+1}$ *over* $\overline{M}_{0,n}$ *to get a family of* n*-pointed curves over* $\overline{M}_{0,n+1}$*. His fiber product is with* $\overline{M}_{0,4}$ *so that now the* n *sections intersect wildly. Next he adds the* $n+1$*-th section just like Knudsen. Finally he resolves in steps the intersections of the sections, by iteratively blowing up loci where the largest number of sections intersect. By doing so he is insuring that he is always blowing up along smooth codimension two centers.*



Kapranov [Kap93b] *Kapranov's construction identifies* $M_{0,n}$ *with the family* $\mathcal{R} \subset M_{0,n} \times \mathbb{P}^{n-2}$ *of rational normal curves in* $\mathbb{P}^{n-2}$ *through* $n-1$ *general points. Then he considers the closure of such family* $\overline{\mathcal{R}} \subset \overline{M}_{0,n} \times \mathbb{P}^{n-2}$ *to be the universal family over* $\overline{M}_{0,n}$ *and, hence,* $\overline{M}_{0,n+1}$*.*

This construction constructs $\overline{M}_{0,n+1}$ as a sequence of blowups along smooth centers: first one blows up the $n-1$ points in general position, then the proper transforms of the lines joining each pair of points, then the proper transforms of the planes containing any three such points, etc...

Exercise 39. *Consider the forgetful morphism $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$. We define:*

$$\pi_{n+1\star}(\psi_{n+1}^{i+1}) := \kappa_i.$$

Kappa classes are important in the theory of moduli spaces of higher genus curves, as they give us interesting classes that exists on the moduli space of unpointed curves, and that are not entirely supported on the boundary. In the genus zero theory everything is boundary, and you need at least three points. For this reason I am not aware of these classes playing a prominent role in this theory. However a perfectly sensible question, whose answer I couldn't find with a quick literature search, is whether there is a good combinatorial representation of kappa classes in terms of boundary strata (along the lines of the description of psi classes in Lemma **??***).*

Lecture 3

Weighted Pointed Curves

3.1 Stability and Hasset spaces

Intuition. *In our last lecture we introduced the notion of stability as a way to compactify the moduli space $M_{0,n}$. We allowed curves to acquire nodal singularities, requested that there are no non-trivial automorphisms preserving the marks, and still mantained that we want all points to be distinct and on the smooth locus of the curve. We now interpret the notion of stability as a numerical condition on a divisor on the curve. This then allows us to naturally generalize the notion of stability and obtain alternative compactifications of $M_{0,n}$.*

Lemma 8. *A rational n-pointed curve $(C, p_1, \ldots, p_n)$ is stable if and only if the twisted dualizing sheaf $\omega_C(p_1 + \ldots + p_n)$ is ample.*

Recall that the dualizing sheaf of a nodal rational curve C restricts to an irreducible component X of C to the line bundle associated to the divisor $K_X + \sum n_j + \sum p_i$, where the n_j's are the nodes of C belonging to X, and the second sum runs over the marks of C that are on X. Since on $\mathbb{P}^1$ the degree of the canonical divisor is -2 and a divisor is ample if and only if it has positive degree, it is evident that ampleness is met if and only if X has at least three special points.

We can now tweak the condition of stability by assigning "weights" to the marked points, and requiring the weighted twisted canonical to be an ample divisor.

Definition 10. *A* **weight data** $\mathcal{A} = (a_1, \ldots, a_n)$ *consists of an n-tuple of positive rational numbers such that $\sum a_i > 2$. A rational n-pointed curve $(C, p_1, \ldots, p_n)$ is called* $\mathcal{A}$**-stable** *if the following three conditions are met:*

- *the twisted dualizing sheaf $\omega_C(a_1p_1 + \ldots + a_np_n)$ is ample;*
- *the marked points are all on the smooth locus of C;*

- *if a subset of points coincide, then the sum of their weights is less than or equal to* 1.

This corresponds to the fact that on every irreducible component of C the number of nodes plus the sum of the weights of the marks lying on the component is strictly greater than 2.

Intuition. *Going along with our physical metaphor, marked points now carry a certain amount of "mass" (or energy if you prefer). When points collide, such mass adds. If the total mass surpasses the critical value of* 2*, then a new twig bubbles off and carries the colliding points. Note that the second and third conditions are in a sense forced on us by the first one. When a point collides with a node, no matter how light the point is, the node must be split by a new component. Such component has now two nodes, which together with the weight of the mark surpass critical mass and make the twig stable. On the other hand, the third condition allows for limits of families where light points want to collide.*

Exercise 40. *With this definition of weighted-stability on objects, formulate the natural notion of weighted stability of families of rational pointed curves. Then you can formulate the notion of the moduli functor for isomorphism classes of $\mathcal{A}$-stable families of rational n-pointed curves.*

That such functor is representable is a result of Hassett.

Theorem 4 (Hassett, [Has03])**.** *There exists a connected, smooth and projective variety $\overline{M}_{0,\mathcal{A}}$ representing the functor of isomorphism classes of $\mathcal{A}$-stable families of rational n-pointed curves. We denote by $\overline{U}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{A}}$ the universal family.*

We note that when the weight data consists of all weights equal to 1, then we recover the usual notion of stability (also called Deligne-Mumford (DM) stability).

Exercise 41. *An interesting variant of this moduli problem is obtained by allowing marked points to have weight* 0*. A weight zero mark is allowed to collide with a node (why?). Prove that if $\mathcal{A} = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$ and $\mathcal{A}^+ = (a_{k+1}, \ldots, a_n)$, where all the $a_i > 0$, then*

$$\overline{M}_{0,\mathcal{A}} = \underbrace{\overline{U}_{0,\mathcal{A}^+} \times_{\overline{M}_{0,\mathcal{A}}} \cdots \times_{\overline{M}_{0,\mathcal{A}}} \overline{U}_{0,\mathcal{A}^+}}_{k \quad times} . \tag{3.1}$$

Show that these moduli spaces are singular by analyzing the local equation of a node and checking what happens when fiber products are taken.

Moduli spaces of weighted stable curves admit natural forgetful morphisms; as in the DM stable case, after forgetting some of the marks, twigs that become unstable must be contracted. There are now also morphisms among spaces with the same number of points, but with different weights.

Definition 11. *Given two weight data $\mathcal{A}, \mathcal{B}$, we say that $\mathcal{B} \leq \mathcal{A}$ if for every i, $b_i \leq a_i$. Then there exists a regular* **reduction** *morphism:*

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{B}} \tag{3.2}$$

s.t. $r(C, p_1, \ldots, p_n)$ is obtained by contracting twigs that become unstable when the weights of the points is "lowered" from a_i to b_i.

Lemma 9. *If $\mathcal{C} \leq \mathcal{B} \leq \mathcal{A}$, then*

$$\rho_{\mathcal{C},\mathcal{A}} = \rho_{\mathcal{C},\mathcal{A}} \circ \rho_{\mathcal{B},\mathcal{A}}.$$

The polytope (with some faces removed) P obtained by intersecting the hypercube $(0,1]^n$ with the half-space $\sum a_i > 2$ parameterizes weight data for n-pointed curves. The moduli functor remains constant in chambers determined as the complement of the hyperplanes $\sum_{i \in I} a_i = 1$, for I any subset of the set of indices. The moduli spaces are all birational, as they all contain $M_{0,n}$ as a dense open set.

Exercise 42. *Completely classify the moduli problems of 4-pointed weighted stable curves. Note that for all weight data, the moduli space $\overline{M}_{0,\mathcal{A}} \cong \mathbb{P}^1$, but the universal family changes in the various chambers.*

Exercise 43. *In the case of five pointed curves, what moduli spaces $\overline{M}_{0,\mathcal{A}}$ are actual contractions of $\overline{M}_{0,5}$?*

3.2 Relation to Kapranov's Construction

In [Kap93b], $\overline{M}_{0,n}$ is obtained from a sequence of blow-ups of $\mathbb{P}^{n-3}$. In the first step, $n-1$ points in general position, which can be taken to be the torus fixed points plus the identity of the torus, are blown up; then all (proper transforms of) lines spanned by pairs of these points are blown up; then (proper transforms of) all planes spanned by triples of such points, and so on until codimension two linear subspaces spanned by $n-4$ of the points are blown up.

The space $\mathbb{P}^{n-3}$ can be interpreted as a space of weighted stable curves and the contraction map $\pi : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ as a reduction morphism. Consider the weight data $\mathcal{A}_0 = \left(\frac{1}{n-2}, \ldots, \frac{1}{n-2}, 1\right)$, where any $n-2$ of the first $n-1$ marks are allowed to come together; the moduli space $\overline{M}_{0,\mathcal{A}_0} \cong \mathbb{P}^{n-3}$ and $\pi = \rho_{\mathcal{A}_0,(1^n)}$.

Exercise 44. *Make friends with the above claim, by identifying each of the torus orbits of $\mathbb{P}^{n-3}$ with a locus of curves in $\overline{M}_{0,\mathcal{A}_0}$ where a certain subset of the marks have come together.*

Now consider the weight data $\mathcal{A}_1 = \left(\frac{1}{n-3}, \ldots, \frac{1}{n-3}, 1\right)$; now any $n-3$ of the first $n-1$ marks are allowed to come together. The moduli space $\overline{M}_{0,\mathcal{A}_1}$ is the blow up of $\mathbb{P}^{n-3}$ at $n-1$ points, i.e. the first step in Kapranov's sequence of blow-ups.

In general, the weight data $\mathcal{A}_k = \left(\frac{1}{n-2-k}, \ldots, \frac{1}{n-2-k}, 1\right)$ yields a moduli space $\overline{M}_{0,\mathcal{A}_k}$ which is the blow-up along the (proper transforms of) the dimension $k-1$ linear subspaces spanned by subsets of k of the $n-1$ distinguished points. We have thus factored Kapranov's construction as:

$$\overline{M}_{0,n} = \overline{M}_{0,\mathcal{A}_{n-4}} \to \ldots \to \overline{M}_{0,\mathcal{A}_k} \to \ldots \to \overline{M}_{0,\mathcal{A}_0} \cong \mathbb{P}^{n-3}. \tag{3.3}$$

It is interesting to note that one can imagine varying the weight data continuously from $\mathcal{A}_0$ to $\mathcal{A}_{n-4}$. This traces a segment in the parameter space of moduli spaces of weighted stable rational n-pointed curves, and the factorization above can be thought of as a wall-crossing phenomenon.

3.3 Psi classes on Hasset spaces

Moduli spaces of weighted stable rational curves also have psi classes, which are defined in the same way as in $\overline{M}_{0,n}$. As the moduli problem changes from one chamber to the next, so do the psi classes. It is important to notice that what really matters is how the universal family changes. We illustrate this by looking at the example of 4 pointed weighted stable curves. In this case we know that all moduli spaces are isomorphic to $\mathbb{P}^1$. Recall that for $\overline{M}_{0,4}$, $\psi_i = [pt.]$ for all i.

Weight data $\mathcal{A} = (1, 1, 1/2, 1/2)$ In this case the third and fourth points are allowed to coincide. The universal family $\pi_1 : \overline{U}_{0,\mathcal{A}} \cong Bl_{(0,0),(1,1)}\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, with the sections s_1, s_2, s_3 being the (proper transforms of the) constant sections $0, 1, \infty$ and s_4 equal the diagonal section. We note immediately that the local picture around the first two sections is identical to the case of $\overline{M}_{0,4}$. Therefore $\psi_1 = \psi_2 = [pt.]$. The third section is now the class of a horizontal fiber; therefore $s_3^2 = 0$, which implies $\psi_3 = 0$. We can now argue that $\psi_4 = 0$ as well by symmetry.

Exercise 45. *Compute $\psi_4 = 0$ directly, by computing the self intersection of the fourth section.*

Weight data $\mathcal{A} = (2/3, 2/3, 2/3, 1/3)$ In this case the fourth point is allowed to coincide with any one of the first three. The universal family $\pi_1 : \overline{U}_{0,\mathcal{A}} \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, with the sections s_1, s_2, s_3 being the constant sections $0, 1, \infty$ and s_4 equal the diagonal section. The first three sections are in the class of a horizontal fiber; therefore $\psi_1 = \psi_2 = \psi_3 = 0$. The fourth section has class $(1, 1)$. It follows that $s_4^2 = 2[pt.]$, which in turn implies $\psi_4 = -2[pt.]$.

Exercise 46. *Compute the psi classes for the above weight data by using as a model for the universal family a blow-up of* $\mathbb{P}^2$*, along the lines of Kapranov's construction.*

The following two lemmas allow us to relate ψ classes on weighted stable curve spaces on different spaces which are related by forgetful or reduction morphisms.

Lemma 10. *Consider the weight data* $\mathcal{A} = (a_1, \dots, a_{n+1})$ *and* $\mathcal{A}' = (a_1, \dots, a_n)$ *and the forgetful morphism* $\pi_{n+1} : \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{A}'}$*. For* $i = 1, \dots, n$*, we have:*

$$\psi_i = \begin{cases} \pi_{n+1}^*(\psi_i) & \text{if } a_i + a_{n+1} \leq 1 \\ \pi_{n+1}^*(\psi_i) + D(\{i, n+1\}) & \text{if } a_i + a_{n+1} > 1. \end{cases} \tag{3.4}$$

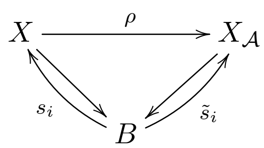
The proof of this lemma boils down to the observation that when $a_i + a_{n+1} \leq 1$, the image of s_i in $\overline{U}_{0,\mathcal{A}}$ is pulled back from $\overline{U}_{0,\mathcal{A}'}$, essentially because the $n+1$-th section is allowed to intersect s_i. When $a_i + a_{n+1} > 1$, then the situation is identical to the DM stable case.

Lemma 11. *Consider the reduction morphism* $\rho_{\mathcal{A},(1^n)} : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$*. For* $i = 1, \dots, n$*, we have:*

$$\psi_i = \rho_{\mathcal{A},(1^n)}^*(\psi_i) + \sum_{\substack{I \ni i, \\ \sum_{j \in I} a_j \leq 1}} D(I). \tag{3.5}$$

Intuition. *Informally,* ψ_i *is corrected by all boundary divisors where the* i*-th mark sits on a twig that gets contracted by* $\rho_{\mathcal{A},(1^n)}$*.*

The proof consists in noting that for every diagram of one parameter families of curves



X is obtained by blowing up $X_{\mathcal{A}}$ at all points of intersections of sections that are allowed to coincide for the weight data $\mathcal{A}$. To compute ψ_i we are interested only in points that are in the image of the section $\tilde{s}_i$. We have the equation

$$s_i = \rho^*(\tilde{s}_i) - \sum E(I), \tag{3.6}$$

where $E(I)$ denotes the exceptional $\mathbb{P}^1$'s coming from intersection of sections in subsets I that contain i and are allowed to come together. The proof is concluded by observing that (the pushforward of the) self intersection of a section gives (minus) the corresponding psi class, and by relating the self intersection of the section s_i and of the section $\tilde{s}_i$ via (3.6).

Given that we can relate psi classes on spaces of weighted stable curves to psi classes on $\overline{M}_{0,n}$ plus boundary corrections, and that we know everything about intersections of psi classes and boundary strata in $\overline{M}_{0,n}$, we could argue that we know everything about intersection of psi classes on on spaces of weighted stable curves. While this may be a true statement for a geometer, it certainly it is not satisfactory for a combinatorialist. A systematic exploration of the combinatorial structure of psi class intersections in moduli spaces of weighted stable curves has not yet been carried out. Here is an example of one family of ψ-intersections which yields a nice formula.

Lemma 12 (Nate Zbacnik, 2015)**.** *Consider the weight data* $\mathcal{A} = (1/2, 1/2, \dots, 1/2)$*. Then if* $n \geq 5$ *we have:*

$$\int_{\overline{M}_{0,\mathcal{A}}} \hat{\psi}_1^{n-3} = 2 - n.$$

We put a hat over the psi class on the weighted space and denote ψ_i the psi class on $\overline{M}_{0,n}$. Then we have:

$$\int_{\overline{M}_{0,\mathcal{A}}} \hat{\psi}_1^{n-3} = \int_{\overline{M}_{0,n}} \left(\psi_1 - \sum_{j=2}^{n} D(\{1,j\})\right)^{n-3} =$$

$$\int_{\overline{M}_{0,n}} \left(\psi_1^{n-3} + \sum_{j=2}^{n} (-1)^{(n-3)+(n-4)} \psi_\star^{n-4} D(\{1,j\})\right) = 1 - \sum_{j=2}^{n} 1 = 2 - n$$

Here, the first equality uses the formula in Lemma 11; in the second equality we use the fact that all divisors $D(\{1,j\})$ do not intersect each other and do not intersect ψ_i.Then we iterate the self-intersection formula $D(\{1,j\})^2 = -\psi_\star D(\{1,j\})$, where $\psi_\star$ denotes the psi class at the node pulled back from the $(n-1)$-pointed factor. Finally we evaluate the intersections of psi classes and boundary divisors in $\overline{M}_{0,n}$.

3.4 Losev-Manin spaces

Among spaces of weighted stable curves, the following family was studied by Losev and Manin in [LM00].

Definition 12. *The* **Losev-Manin** *space* $\overline{L}_n$ *is the moduli space of rational weighted stable corresponding to the weight data* $\mathcal{A} = (1, 1, \underbrace{\epsilon, \dots, \epsilon}_{n \ \ times})$.

We already know that $\overline{L}_n$ is a smooth projective variety of dimension $n-1$, which represents a moduli functor for families of weighted stable curves. When Losev and Manin introduced this space, they gave a more explicit geometric description of such a moduli functor.

- A LM**-chain** of projective lines of length k consists of $T = T_1 \cup T_2 \cup \dots \cup T_k$, where each of the T_i is a parameterized $\mathbb{P}^1$ and for $i = 2, \dots, n$, the point $\infty_{i-1} \in T_{i-1}$ is glued to the point $0_i \in T_i$. The group of automorpshisms $Aut(T)$ is the k-dimensional torus $\mathbb{G}_m^k$, where the i-th factor acts naturally on T_i.

- A LM**-configuration of** n **marks** on an LM-chain consist of length k of an n-tuple $(p_1,\ldots,p_n)$, where each marked point belongs to $T \smallsetminus \bigcup_{i=1}^k \{0_i,\infty_i\}$. We are not requiring the marked points to be distinct.
- Two LM-configurations $(p_1,\ldots,p_n)$ and $(q_1,\ldots,q_n)$ on T are **isomorphic** if there exists $\phi \in Aut(T)$ such that $(q_1,\ldots,q_n) = (\phi(p_1),\ldots,\phi(p_n))$.
- An LM-configuration is **stable** $(p_1,\ldots,p_n)$ if the only automorphism of T preserving the marked points is the identity. This is equivalent to requiring that each component of T hosts at least one marked point.

With these definitions in place, we leave it to the reader to formulate the natural notion of family of stable configurations of n marks on LM chains, and of isomorphism of families.

Theorem 5 ([LM00]). *The variety $\overline{L}_n$ represents the moduli functor for isomorphism classes of families of stable configurations of n marks on LM chains (of arbitrary length).*

The stability condition ensures that $\overline{L}_n$ parameterizes configurations of marks on chains of length at most n.

Exercise 47. *Make friends with this functor being the same as the functor for weighted stable curves corresponding to weight data* $\mathcal{A} = (1,1,\underbrace{\epsilon,\ldots,\epsilon}_{n \ \ times})$.

The boundary of $\overline{L}_n$ has irreducible components indexed by partitions of of set of marks $S_1 \cup \ldots \cup S_k = [n] = \{1,\ldots,n\}$, together with an ordering of the subset. This corresponds to having the marks in S_i on the component T_i of a chain of lenght k. A partition of $[n]$ into k subsets corresponds to a boundary stratum of codimension $k-1$; the closure of such a

Exercise 48. *Develop the appropriate combinatorial conditions on ordered partitions that describe when (the closure of) two boundary strata do not intersect, when they intersect, and when one stratum is in the closure of another.*

Exercise 49. *Study $\overline{L}_3$ as a blow-down of $\overline{M}_{0,5}$. Notice that $\overline{L}_3$ is a toric variety, and the modular boundary coincides with the toric boundary. Identify the torus orbits with boundary strata of $\overline{L}_3$.*

All Losev-Manin spaces are toric varieties. $\overline{L}_n$ is obtained from $\mathbb{P}^{n-1}$ by blowing up all torus fixed point, then the proper transforms of torus invariant lines, then the proper transform of the closures of two dimensional torus orbits and so on...this is the first step in the refined factorization of Kapranov's construction in Exercise 51.

This description tells us that $\overline{L}_n$ is the toric variety associated to the **permutohedron** (`https://en.wikipedia.org/wiki/Permutohedron`). In fact, after identifying the boundary of $\overline{L}_n$ with ordered partitions of the set $[n]$, this statement is more or less tautological, given that the permutohedron may be defined as the polytope associated to the poset of ordered partitions of $[n]$, which coincides with the boundary poset of $\overline{L}_n$.

There is a natural combinatorial description for the fan of $\overline{L}_n$. We have $N = (\mathbb{Z}^n/\langle(1,1,\ldots,1)\rangle) \otimes \mathbb{R}$, and we let $e_1,\ldots,e_n$ denote the standard lattice generators for $\mathbb{Z}^n$. For any two part ordered partition $S_1 \cup S_2$, consider the ray ρ_{S_1} spanned by the vector $\sum_{i\in S_1}[e_i]$. For any k-part partition $S_1 \cup \ldots \cup S_k$ consider the $k-1$ dimensional cone spanned by the rays $\rho_{S_1}, \rho_{S_1\cup S_2}, \ldots, \rho_{S_1\cup\ldots\cup S_{k-1}}$.

Exercise 50. *Check that indeed this construction yields the fan for $\overline{L}_n$.*

3.5 Further exercises

Exercise 51. *There is a more refined factorization of Kapranov's construction of $\overline{M}_{0,n}$, which singles out one point, let us say, p_n, to play a special role. We start with $\mathbb{P}^{n-3}$ The sequence of blow-ups is as follows.*

- *Blow up the points $p_1,\ldots,p_{n-1}$, then (the proper transforms of) all lines spanned by any two of these points, then (the proper transforms of) planes spanned by any three such points, and so on... (and from now on we omit saying "proper transforms of")*
- *Blow up the point p_n, then all lines spanned by p_n and another point $p_i \neq p_{n-1}$, then all planes spanned by p_n and two points p_i, p_j both different from p_{n-1} and so on...*
- *Blow up the line spanned by p_n and p_{n-1}, then the planes spanned by those two points plus a $p_i \neq p_{n-2}$, and so on...*

You continue with this pattern until at the last step you blow up the codimension two linear subspace spanned by $p_n, p_{n-1}, \ldots, p_3$.

1. *prove that this sequence of blow-ups is equivalent to the one described in Section 3.2.*
2. *determine a possible weight data for the weighted moduli spaces at the end of each group of blow ups as collected in the bulleted list above.*
3. *interpret this factorization in terms of pointed curves: which groups of sections are you "separating" at each step?*

4. *observe that after the first group of blow-ups, the resulting intermediate space is* $\overline{L}_{n-2}$.

Exercise 52 (Psi Classes)**.** *Consider* $\overline{L}_n$*, denote the two marks with weight one* p_0 *and* p_∞*, and the* n *"light" marks* $p_1, \ldots, p_n$.

1. *Prove that for* $i = 1, \ldots, n$, $\psi_i = 0$.
2. *Compute*
$$\int_{\overline{L}_n} \psi_0^k \psi_\infty^{n-1-k}.$$

Lecture 4

Tropical $M_{0,n}$

4.1 Dirty introduction to tropical geometry

Intuition. *Tropical geometry is a wild degeneration that turns classical algebro-geometric objects (e.g. curves), into piecewise-linear, combinatorial objects (e.g. graphs). One of the scopes of tropical geometry is to understand when combinatorial invariants of the tropical objects contain information about classical invariants of the original objects. When a correspondence theorem is established, one gains a powerful tool towards the computation of classical invariants.*

We denote by $K = \mathbb{C}\{\{t\}\}$ the field of Puiseaux series; its elements have form $x(t) = t^{\frac{p}{q}}a(t^{1/q})$, where q is a positive integer, p an arbitrary integer and a is an invertible power series. Informally, Puiseaux series are Laurent series in some (formal) q-th root of the variable. This is not such an extravagant choice of field: it is the algebraic closure of the quotient field of the ring of power series. There is a natural valuation $val : K \to \mathbb{Q}$ defined by:

$$val\left(t^{\frac{p}{q}}a(t^{1/q})\right) = \frac{p}{q}.$$

Let $Y \subset (\mathbb{C}^*)^n$ be a closed subvariety of the n-dimensional algebraic torus, defined by an ideal $I \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then we look at the K^*-valued points of the subvariety $V(I)$, and obtain a closed subvariety $Y_K \subset (K^*)^n$.

Definition 13. *The* **tropicalization** $Trop(Y)$ *of* Y *is the closure (with respect to the euclidean topology) in* $\mathbb{R}^n$ *of the image of the map* $Trop : Y_K \to \mathbb{Q}^n$ *defined by*

$$Trop(x_1(t), \dots, x_n(t)) := (val(x_1(t)), \dots, val(x_n(t))).$$

The set $Trop(Y)$ has the structure of a polyhedral fan in $\mathbb{R}^n$.

Intuition. *If you are not familiar with this notion of tropicalization, let me attempt to provide some inutition. The main idea is that we are going from thinking of a geometric object as of something "static", to thinking of it as something "dynamic": we introduce a time variable* t*, and we seek solutions of system of equations given by the ideal* I *that are "motions", i.e. functions of the time variable* t*. We are in particular interested in studying these motions as* $t \to 0$*. As it is the case when studying limits, we are not concerned with what happens at* $t = 0$*: this is why we work in a torus rather than affine space. We want to allow motions that run to zero or infinity in some coordinate directions. In fact, these are the most exciting solutions, because in this case we can look at the order of magnitude with which these coordinates vanish or diverge. The map* $Trop$ *in fact does just that: for any motion in* Y *it tells us the order of magnitude that each of the coordinates has when* t *approaches* 0*.*

If you are familiar with the definition of tropicalization in terms of min-plus semiring, the connection is quite simple. Given an equation $f = 0$*, a necessary condition for the existence of a "motion" solution of the form* $(x_1(t), \dots x_n(t))$ *is that the lowest order of* $f(x_1(t), \dots x_n(t))$ *vanishes. This amounts precisely to be on the corner locus of the piecewise linear function* $Trop(f)$ *obtained by interpreting* f *in the min-plus semiring. Then the equivalence of the definitions consists of the fact that Puiseaux series give you enough flexibility that once you line up the lowest order term, you are guaranteed to be able to actually find solutions.*

Finally, motions which tropicalize to a point with integral coordinates have the asymptotic behavior (as $t \to 0$*) of* 1*-parameter subgroups of the torus. So a one parameter subgroup* ρ *belongs to* $Trop(Y)$ *if there is a motion on* Y *with that asymptotic behavior. Recall that for a toric variety* X *a one parameter subgroup* ρ *belongs to a cone in the fan of* X *when the limit as* $t \to 0$ *of* ρ *is a point in* X *(and in particular, it is the distinguished point in the torus orbit corresponding to the cone). With this in mind, the following lemmas feel very natural.*

Lemma 13 (Tevelev, [Tev07])**.** *Identify* $\mathbb{R}^n$ *with* $N_{\mathbb{R}}$*, the vector space spanned by the lattice of one-parameter subgroups of the* n*-dimensional torus* T*. Let* σ *be a cone in* $N_{\mathbb{R}}$*, and* X_σ *the corresponding affine toric variety. Assume* X_σ *is smooth. Then* $Trop(Y)$ *intersects* σ *if and only if the closure of* Y *in* X_σ *intersects the closed orbit of* X_σ*.*

Lemma 14 (Tevelev, [Tev07])**.** *Let* $\Sigma \subset N_{\mathbb{R}}$ *be a fan, and* X_Σ *the corresponding toric variety. Then* $\overline{Y} \subset X_\Sigma$ *is proper if and only if* $Trop(Y)$ *is contained in the support of* Σ*.*

Exercise 53. *Consider* $Y_1 = V(\langle x + y + 1 \rangle)$ *and* $Y_2 = V(\langle x + y \rangle)$*. Describe* $Trop(Y_1)$ *and* $Trop(Y_2)$*. Which is the minimal toric variety in which* Y_i *compactifies?*

Definition 14. *Let Y be a closed subvariety of the torus T and X_Σ a toric variety whose dense torus is T. Then $\overline{Y} \subset X_\Sigma$ is a* **schön tropical compactification** *if $\overline{Y}$ is proper, and the multiplication map $T \times \overline{Y} \to \overline{Y}$ is smooth and faithfully flat.*

4.2 Tropical $M_{0,n}$

The tropicalization of a punctured curve embedded in an n-dimensional torus is a graph embedded in $\mathbb{R}^n$, with certain additional combinatorial decorations/conditions. However, in our study of rational pointed curves we did not pay particular attention to whether curves were embedded or not. Similarly, one can make an definition of an abstract tropical curve, study the combinatorial objects that parameterize abstract tropical curves, and then wonder whether such combinatorial moduli spaces are related to the classical ones by tropicalization.

Definition 15. *A* **tropical rational stable n-pointed curve** *is a metric tree with n marked ends, such that all (interior) vertices are at least trivalent. Metric means that all compact edges of the tree are assigned a length in $\mathbb{R}^{\geq 0} \cup \infty$. The marked ends are considered to be unbounded (hence length ∞).*

The moduli space $M_{0,n}^{trop}$ is defined to be the parameter space for tropical rational stable n-pointed curves. It naturally has the structure of a cone complex, where cones are indexed by topological types of marked trees, and inclusions of cones as faces are given by contraction of edges.

Exercise 54. *Show that $M_{0,n}^{trop}$ can be in fact identified with the boundary complex of $\overline{M}_{0,n}$.*

Exercise 55. *Show that $M_{0,4}^{trop}$ can be identified with a tropical line.*

The moduli spaces $\overline{M}_{0,n}$ are in fact schön tropical compactifications: there is an embedding of $M_{0,n}$ into a torus, and a toric variety X_Σ (whose fan is $M_{0,n}^{trop}$) such that the closure of $M_{0,n}$ in X_Σ is $\overline{M}_{0,n}$. This result first appears in [Tev07], who obtains it by "stitching together" results of [SS04, Kap93a], and is then spelled out in greater detail in [GM10].

We start by observing that $M_{0,n}$ can be viewed as the quotient $Gr(2,n)/T^{n-1}$, where the torus should be thought as the quotient of the n-dimensional torus by the diagonal one-parameter subgroup. The action is defined in the natural way: the i-th coordinate of the torus T^n scales the i-th column of the matrix representing a point in the Grassmannian.

Kapranov in [Kap93a] shows that $\overline{M}_{0,n}$ is then obtained as the Chow or Hilbert quotient $Gr(2,n)//^{*}T^{n-1}$, i.e. the closure of $Gr(2,n)/T^{n-1}$ in an appropriate Chow variety or Hilbert scheme.

The Grassmannian can be embedded in $\mathbb{P}^{\binom{n}{2}}$ via the Plücker embedding, and the action of the torus T^{n-1} can be naturally extended to $\mathbb{P}^{\binom{n}{2}}$ in such a way that it preserves this embedding. Speyer and Sturmfels [SS04] show that the tropicalization $Trop(A)$ of the affine cone of the Plücker embedding is a fan of dimension $2n-3$ in $\mathbb{R}^{\binom{n}{2}}$ with lineality space L of dimension n.

The tropicalization of $M_{0,n} = Gr(2,n)/T^{n-1}$ is then shown to be the quotient $Trop(A)/L$, a balanced fan in $\mathbb{R}^{\binom{n}{2-n}}$. Finally it is easy to see that $M_{0,n}^{trop}$ is naturally identified with $Trop(M_{0,n})$. The fan $Trop(M_{0,n})$ defines a toric variety that by Lemma 14 contains a compactification of $M_{0,n}$. It takes a little extra work to show that this compactification is indeed $\overline{M}_{0,n}$, but essentially this boils down to the fact that $M_{0,n}^{trop}$ contains the data of the toroidal boundary of $\overline{M}_{0,n}$.

Exercise 56. *Try to fill in the details of the argument above in the case* $n = 4$.

4.3 Tropical Weighted stable curves

We conclude this quick incursion into tropical geometry by quickly mentioning the works [Uli15, CHMR16], where the authors asked the question of whether Hassett spaces of weighted stable curves also are tropical compactifications. The answer is somewhat surprising in that it is quite restrictive. The only Hassett spaces which are tropical compactifications are spaces of rational pointed weighted stable curves with only *heavy/light points*, i.e. where the weight data is of the form $(1,\dots,1,\epsilon,\dots,\epsilon)$. These spaces can be embedded in a toric variety whose fan may be identified with the moduli space of tropical weighted stable curves (which is to be thought of as the space of metrizations on dual graphs of weighted stable curves).

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