Subcomplete forcing and its forcing principles

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August 3, 2017
Conference in honor of Ronald Jensen's 80th birthday

Forcing principles

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- Forcing axioms
- Bounded forcing axioms
- Resurrection axioms

Focus: subcomplete forcing

Martin's Axiom

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- The Subcomplete Forcing Axiom, SCFA ($\Gamma =$the collection of all subcomplete forcings)

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- Magidor forcing (of length ω_1) is subcomplete. (F.)
- Every ω_2-distributive forcing is equivalent to a subcomplete forcing. (F.)

Forcing SCFA

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- Assuming the existence of a supercompact cardinal κ, one can iterate proper forcings with countable support, with iterands given by a Laver function for the supercompactness of κ, producing a model in which PFA $+ \kappa = \omega_2 = 2^\omega$ holds. (Baumgartner)

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- This can be modified to work for SCFA, by iterating subcomplete forcings. During the iteration, CH will be forced, and since no reals are added, the final model will satisfy SCFA $+ \kappa = \omega_2 +$ CH. (Jensen)

Lower bounds on the consistency strength of these forcing axioms can be proved by showing that these principles imply the failure of $\square$ principles.

$\square$-principles

Definition (Jensen)

Let κ be a cardinal. $\square_\kappa$ says that there is a $\square_\kappa$-sequence, that is, a sequence $\langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each C_α is club in α, $\mathrm{otp}(C_\alpha) \leqslant \kappa$ and for each β that is a limit point of C_α, $C_\beta = C_\alpha \cap \beta$.

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If λ is also a cardinal, then $\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence, i.e., a sequence $\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each $\mathcal{C}_\alpha$ has size at most λ, and each $C \in \mathcal{C}_\alpha$ is club in α, has order-type at most κ, and satisfies the coherency condition that if β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$.

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$\square_{\kappa,\kappa^+}$ holds trivially, so $\square^*_\kappa$ is the weakest nontrivial principle here.

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Thus, if we can show that a statement we're interested in implies the failure of $\square_\kappa$ for some singular cardinal κ, then $0^\#$ must exist (or else, by covering, the $\square_\kappa$-sequence from *L* would be a $\square_\kappa$ sequence in V).

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This kind of argument can be generalized to higher core models, mining more strength.

Stationary reflection

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Definition

Let λ be an uncountable regular cardinal, and let $S \subseteq \lambda$ be stationary. S reflects at an ordinal $\alpha < \kappa$ of uncountable cofinality iff $S \cap \alpha$ is stationary in α. It reflects iff it reflects at some such α.

Stationary reflection and $\square$

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Suppose $\square_\kappa$ holds. Then every stationary subset $S \subseteq \kappa^+$ has a stationary subset T that does not reflect.

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Now suppose that $\alpha < \kappa^+$ has uncountable cofinality and $T \cap \alpha$ is stationary. Then C'_α, the set of limit points of C_α, is club in α, and whenever $\beta \in C'_\alpha \cap T$, $C_\beta = C_\alpha \cap \beta$ has order type γ.

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This cannot be, since the $C_\alpha \cap \beta$s are longer and longer initial segments of C_α. $\square$

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Observation

Let κ be a cardinal. Then FP_{κ^+} implies the failure of $\square_\kappa$.

Otherwise, the set of ordinals below κ^+ of countable cofinality would have to have a stationary subset that does not reflect.

The failure of $\square$ under SCFA

Fact (Jensen)

If $\kappa > \omega_1$ is a regular cardinal and $A \subseteq \kappa$ is a stationary set consisting of ordinals of countable cofinality, then the forcing $\mathbb{P}_A$ to shoot a club of order type ω_1 through A is subcomplete.

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So, for regular $\kappa > \omega_1$, SCFA implies FP_κ, which implies that every stationary subset of S^κ_ω reflects, which implies that $\Box_{\bar{\kappa}}$ fails, if $\kappa = \bar{\kappa}^+$.

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Towards reaching the failure of weak square principles, stronger principles of stationary reflection will be useful.

Theorem (Jensen)

Assume SCFA. *Let $\kappa > \omega_1$ be a regular cardinal. Then the Strong Friedman Principle* SFP_κ *holds at κ:*
Let $\langle A_i \mid i < \omega_1\rangle$ be a sequence of stationary subsets of S^κ_ω. Let $\langle D_i \mid i < \omega_1\rangle$ be a partition of ω_1 into stationary sets. Then there is a normal function $f : \omega_1 \longrightarrow \tau$ such that for every $i < \omega_1$, $f``D_i \subseteq A_i$.

Simultaneous stationary reflection

Definition (Cummings-Magidor)

Let μ be a cardinal, let λ be an uncountable regular cardinal, and let $S \subseteq \lambda$ be stationary. The simultaneous reflection principle $\mathrm{Refl}(\mu, S)$ holds iff for every sequence $\langle T_i \mid i < \mu \rangle$ of stationary subsets of S, there exists an $\alpha < \kappa$ of uncountable cofinality such that for all $i < \mu$, T_i reflects to α ("$\vec{T}$ reflects simultaneously at α").

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Observation

Let $\kappa > \omega_1$ be a regular cardinal. Then SFP_κ implies $\mathrm{Refl}(\omega_1, S^\kappa_\omega)$.

Simultaneous reflection and weak $\square$

Lemma (Cummings, Magidor)

If κ is singular and $\square_{\kappa,\mu}$ holds for some $\mu < \kappa$, then every stationary subset of κ^+ has a collection of $\mathrm{cf}(\kappa)$ many stationary subsets which do no reflect simultaneously at any point of uncountable cofinality.

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This lemma, together with our observations on SFP_{κ^+}, shows that if SCFA holds and κ is singular with $\mathrm{cf}(\kappa) \leqslant \omega_1$, then $\square_{\kappa,\mu}$ fails for every $\mu < \kappa$.

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This lemma shows that SCFA implies that for every uncountable cardinal κ and every $\mu < \mathrm{cf}(\kappa)$, $\square_{\kappa,\mu}$ fails.

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Fact

CH implies that $\square^*_{\omega_1}$ holds.

(Because CH implies the existence of a special ω_2-Aronszajn tree, and the existence of a special κ^+-Aronszajn tree is equivalent to $\square^*_\kappa$.)

The extent of weak $\square$ under SCFA

Theorem

Assume SCFA*, and let* λ *be an uncountable cardinal.*

1. *If* $\mathrm{cf}(\lambda) \leqslant \omega_1$*, then* $\square_{\lambda,\mu}$ *fails, for every* $\mu < \lambda$*.*
2. *If* $\mathrm{cf}(\lambda) \geqslant \omega_2$*, then* $\square_{\lambda,\mu}$ *fails for every* $\mu < \mathrm{cf}(\lambda)$*.*
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The situation is as with MM, except that:

- MM implies that CH fails, and that $\square^*_{\omega_1}$ fails, and
- MM implies that if $\mathrm{cf}(\lambda) = \omega$, then $\square^*_\lambda$ fails, while I don't know the status of $\square^*_\lambda$ under SCFA.

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It can be shown (using an argument of Cummings-Magidor) that the above results are optimal, i.e., from a supercompact cardinal, one can produce a model of SCFA in which, if $\mathrm{cf}(\lambda) = \omega_1$, then $\square^*_\lambda$ holds, etc.

Generalized stationary reflection

There is a principle of reflection of stationary subsets of $[H_\lambda]^\omega$, for regular $\lambda \geqslant \omega_2$, that follows from MM. If $\mathrm{cf}(\kappa) = \omega$, and the reflection principle holds for stationary subsets of $[H_{\kappa^+}]^\omega$, then $\square^*_\kappa$ fails.

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This principle follows from MA$^+(\sigma$-closed$)$, and hence also from SCFA$^+$.

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There is a principle of reflection of stationary subsets of $[H_\lambda]^\omega$, for regular $\lambda \geqslant \omega_2$, that follows from MM. If $\mathrm{cf}(\kappa) = \omega$, and the reflection principle holds for stationary subsets of $[H_{\kappa^+}]^\omega$, then $\square^*_\kappa$ fails.
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Todorčević's strong reflection principle is too much, since it implies that the nonstationary ideal on ω_1 is saturated, while SCFA is consistent with $\diamondsuit$.

Definition (Todorcevic, Jensen (?))

Let λ be a limit of limit ordinals. A sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ is coherent if for every limit $\alpha < \lambda$, $\mathcal{C}_\alpha \neq \emptyset$ and for every $C \in \mathcal{C}_\alpha$, C is club in α, and for every limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$.

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The principle $\square(\lambda, 1)$ is denoted $\square(\lambda)$.

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But it turns out that there is a route using stationary reflection.

The goal is to determine the extent of $\square(\kappa,\lambda)$ under SCFA.

Diagonal reflection

Definition (P. Larson)

The principle OSR$_{\omega_2}$ says that whenever $\langle T_\alpha \mid \alpha < \omega_2 \rangle$ is a sequence of stationary subsets of ω_2, each consisting of ordinals of countable cofinality, then there is a $\gamma < \omega_2$ with $\mathrm{cf}(\gamma) = \omega_1$ at which T_α reflects, for all $\alpha < \gamma$.

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Definition (F.)

Let λ be a regular cardinal, let $S \subseteq \lambda$ be stationary in λ, and let $\kappa < \lambda$. The diagonal reflection principle $\mathrm{DSR}({<}\kappa, S)$ says that whenever $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ is a sequence of stationary subsets of S, where $j_\alpha < \kappa$ for every $\alpha < \lambda$, then there is a $\gamma < \lambda$ of uncountable cofinality, and there is a club $F \subseteq \gamma$ such that for every $\alpha \in F$ and every $i < j_\alpha$, $S_{\alpha,i} \cap \gamma$ is stationary in γ. The version of the principle in which $j_\alpha \leqslant \kappa$ is denoted $\mathrm{DSR}(\kappa, S)$.

The point of diagonal stationary reflection in the present context are the following two theorems.

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Let λ be regular, $\kappa < \lambda$ a cardinal, and assume that $\mathrm{DSR}({<}\kappa, S)$ *holds, for some stationary $S \subseteq \lambda$. Then* $\square(\lambda, {<}\kappa)$ *fails.*

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Fortunately, diagonal reflection follows from SCFA.

Theorem (F.)

SCFA *implies that for every regular $\lambda > \omega_1$,* $\mathrm{DSR}(\omega_1, S^{\lambda}_{\omega})$ *holds.*

Effects of SCFA

Lemma (F.)

Assume SCFA.

1. *The principle* $\square(\omega_2, \omega)$ *fails, but it is consistent that* $\square(\omega_2, \omega_1)$ *holds.*
2. *If* $\lambda > \omega_2$ *is a regular cardinal, then* $\square(\lambda, \omega_1)$ *fails.*

Maximizing $\square$

Lemma (F.)

If the existence of a supercompact cardinal is consistent, then so is the existence of a supercompact cardinal κ such that for every regular cardinal $\lambda > \kappa$, the principle $\square(\lambda, \kappa)$ holds.

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The point is that in a model in which κ is supercompact and its supercompactness is indestructible by κ-directed closed forcing and GCH holds above κ, one can iterate to add a version of *indexed square* sequences of width κ at every $\lambda > \kappa$, using a forcing, due to Lambie-Hanson, that's κ-directed closed and λ-strategically closed.

The extent of $\square(\cdot,\cdot)$ under SCFA

Theorem (F.)

Assume the consistency of the existence of a supercompact cardinal. It is consistent that

1. $\mathsf{SCFA} + \mathsf{CH} + \diamondsuit$ *holds*
2. *for every regular* $\lambda > \omega_2$, $\square(\lambda, \omega_2)$ *holds.*

But in any model of $\mathsf{SCFA} + \mathsf{CH}$, *necessarily,* $\square(\lambda, \omega_1)$ *fails for all regular* $\lambda > \omega_2$, $\square(\omega_2, \omega)$ *fails, and* $\square(\omega_2, \omega_1)$ *holds.*

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Sketch: Starting in a model where κ is supercompact and $\square(\lambda, \kappa)$ holds, for every regular cardinal $\lambda > \kappa$, run the Baumgartner iteration. The resulting model will satisfy $\mathsf{SCFA} + \diamondsuit + \kappa = \omega_2$. The forcing is κ-c.c., so the $\square(\lambda, \kappa)$ sequences will survive and become $\square(\lambda, \omega_2)$ sequences. $\square(\omega_2, \omega_1)$ follows from CH. The claimed failure of $\square$ principles follows from the lemma from two slides earlier.

Effects of PFA

PFA does not imply $\mathrm{Refl}(\omega_1, S^\lambda_\omega)$, since PFA is compatible with $\square_{\kappa,\omega_2}$, for every $\kappa \geqslant \omega_2$; compare with the effects of simultaneous stationary reflection on the failure of weak squares by Cummings-Magidor. In particular, it does not imply $\mathrm{DSR}(\omega_1, S^\lambda_\omega)$. So the argument using PFA necessarily has to be different. But it turns out that the original Todorčević argument for $\square(\lambda)$ generalizes.

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Assume PFA. *Then the principle* $\square(\lambda, \omega_1)$ *fails for every regular* $\lambda > \omega_1$.

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Lemma

Assume PFA. *Then the principle* $\square(\lambda, \omega_1)$ *fails for every regular* $\lambda > \omega_1$*.*

Maximizing works exactly as before.

The extent of $\square(\cdot,\cdot)$ under PFA or MM

Theorem (F.)

Assume the consistency of the existence of a supercompact cardinal. Then it is consistent that

1. MM *or* PFA *holds*
2. *for every regular* $\lambda \geqslant \omega_2$, $\square(\lambda, \omega_2)$ *holds.*

In a model of (1), necessarily, $\square(\lambda, \omega_1)$ *fails, for every* $\lambda \geqslant \omega_2$.

A limitation

One might hope that diagonal stationary reflection can be used to settle the question about $\square^*_{\aleph_\omega}$ under SCFA. This is not so.

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Theorem (F.)

Assuming the consistency of infinitely many supercompact cardinals, it is consistent that for every nonzero $n < \omega$, $\mathrm{DSR}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$ holds, and moreover, $\square^*_{\aleph_\omega}$ holds.

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There is a model constructed by Foreman-Cummings-Magidor in which $\square^*_{\aleph_\omega}$ holds and also $\mathrm{Refl}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$ holds. One can check that that model actually satisfies $\mathrm{DSR}(\aleph_n, S^{\aleph_{\omega+1}}_{<\aleph_n}, \aleph_n)$, for all $n < \omega$.

Bounded forcing axioms

Definition (Goldstern-Shelah)

Let Γ be a class of forcings, and λ be a cardinal. Then BFA$(\Gamma, \leqslant\lambda)$ is the statement that if $\mathbb{P}$ is a forcing in Γ, $\mathbb{B}$ is its complete Boolean algebra, and $\mathcal{A}$ is a collection of at most ω_1 many maximal antichains in $\mathbb{B}$, each of which has size at most λ, then there is a filter in $\mathbb{B}$ that meets each antichain in $\mathcal{A}$. If Γ is the class of proper, semi-proper, stationary set preserving or subcomplete forcings, I write BPFA, BSPFA, BMM, BSCFA (respectively) for BFA$(\Gamma, \leqslant\omega_1)$. In general, for a cardinal λ, BPFA$(\leqslant\lambda)$, BSPFA$(\leqslant\lambda)$, BMM$(\leqslant\lambda)$, BSCFA$(\leqslant\lambda)$, then have the obvious meaning.

Definition (Goldstern-Shelah)

A regular cardinal κ is *reflecting* if for every $a \in H_\kappa$ and every formula $\varphi(x)$, the following holds: if there is a regular cardinal $\theta \geqslant \kappa$ such that $H_\theta \models \varphi(a)$, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(a)$.

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Theorem (Goldstern-Shelah)

BPFA *is equiconsistent with the existence of a reflecting cardinal.*

There is a proof of one direction of this equiconsistency result (showing that ω_2^V is reflecting in L), due to Todorčević, the idea of which generalizes from proper forcing to subcomplete forcing. The other direction generalizes very easily, given the iterability of subcomplete forcing.

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Proof. We may assume that $0^\#$ does not exist, as otherwise, every Silver indiscernible is reflecting in L. Let $\kappa = \omega_2$, fix $a \in L_\kappa = (H_\kappa)^L$, a formula $\varphi(x)$, a singular cardinal $\gamma > \kappa$, and let $\theta = \gamma^+ = (\gamma^+)^L$, by covering. Assume that $L_\theta \models \varphi(a)$. It suffices to show that there is an L-cardinal $\bar{\theta} < \kappa$ such that $L_{\bar{\theta}} \models \varphi(a)$.

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Let $\langle C_\xi \mid \xi$ is a singular ordinal in $L\rangle$ be the canonical global $\square$ sequence for L. It is Σ_1-definable in L and has the properties that for every L-singular ordinal ξ, the order type of C_ξ is less than ξ, and if ζ is a limit point of C_ξ, then ζ is singular in L and $C_\zeta = C_\xi \cap \zeta$.

Let $B = \{\xi < \theta \mid \kappa < \xi < \theta \text{ and } \mathrm{cf}(\xi) = \omega\}$. By covering, every $\xi \in B$ is singular in L. So C_ξ is defined for every $\xi \in B$, and since the function $\xi \mapsto \mathrm{otp}(C_\xi)$ is regressive, there is a stationary subset A of B on which this function is constant.

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This is a Σ_1 statement about the parameters ω_1 and a. So by BSCFA, the same statement is true in V. Let $\bar{\theta}, \bar{C}$ witness this. Since $\omega_1, a \in H_{\omega_2}$, such witnesses for a Σ_1 formula can be found in H_{ω_2}, so we may take $\bar{\theta} < \omega_2 = \kappa$.

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The point is now that $\bar{\theta}$ must be regular in L. The reason is that if $\bar{\theta}$ were singular in L, then $C_{\bar{\theta}}$ would be defined. Note that $\mathrm{cf}(\bar{\theta}) = \omega_1$. So, letting $C'_{\bar{\theta}}$ be the set of limit points of $C_{\bar{\theta}}$, $C'_{\bar{\theta}} \cap \bar{C}$ is club in $\bar{\theta}$. Now take $\xi < \zeta$, both in $C'_{\bar{\theta}} \cap \bar{C}$. Then, since $\xi, \zeta \in \bar{C}$, C_ξ and C_ζ have the same order type, but since both are limit points of $C_{\bar{\theta}}$, $C_\xi = C_{\bar{\theta}} \cap \xi$, which is a proper initial segment of $C_\zeta = C_{\bar{\theta}} \cap \zeta$.

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So $\bar{\theta}$ is a regular cardinal in L, $\bar{\theta} < \omega_2$, and $H^L_{\bar{\theta}} = L_{\bar{\theta}} \models \varphi(a)$, showing that ω_2 is reflecting in L. □

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Definition (Miyamoto)

Let κ be a regular cardinal, α an ordinal, and $\lambda = \kappa^{+\alpha}$. Then κ is H_λ-reflecting, or I will say $+\alpha$-reflecting, iff for every $a \in H_\lambda$ and any formula $\varphi(x)$, the following holds: if there is a cardinal θ such that $H_\theta \models \varphi(a)$, then the set of $N \prec H_\lambda$ such that

1. N has size less than κ,
2. $a \in N$,
3. if $\pi_N : N \longrightarrow H$ is the Mostowski-collapse of N, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(\pi_N(a))$

is stationary in $\mathcal{P}_\kappa(H_\lambda)$.

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Miyamoto's proof generalizes the original Goldstern-Shelah argument for BPFA, but the idea of Todorčević's argument generalizes to the subcomplete context.

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Proof.

BSCFA($\leqslant\omega_3$) implies SFP_{ω_2} and SFP_{ω_3}, which implies the failure of $\square(\omega_2)$ and $\square(\omega_3)$, and also $2^\omega \leqslant \omega_2$. This constellation implies that the axiom of determinacy holds in $L(\mathbb{R})$, by Schimmerling and Steel. $\square$

The weak hierarchy

So I'm looking for strengthenings of $\mathsf{BFA}_\Gamma(\leqslant\omega_2)$ that are weaker than $\mathsf{BFA}_\Gamma(\leqslant\omega_3)$, in consistency strength.

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Fact (Claverie-Schindler)

$\mathsf{BFA}(\{\mathbb{Q}\}, \leqslant\kappa)$ *is equivalent to the following statement: if* $M = \langle |M|, \in, \langle R_i \mid i < \omega_1\rangle\rangle$ *is a transitive model for the language of set theory with* ω_1 *many predicate symbols* $\langle \dot{R}_i \mid i < \omega_1\rangle$*, of size* κ*, and* $\varphi(x)$ *is a* Σ_1*-formula, such that* $\Vdash_{\mathbb{Q}} \varphi(\check{M})$*, then there is in* V *a transitive* $\bar{M} = \langle |\bar{M}|, \in, \langle \bar{R}_i \mid i < \omega_1\rangle\rangle$ *and an elementary embedding* $j : \bar{M} \prec M$ *such that* $\varphi(\bar{M})$ *holds.*

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Inspired this characterization, Bagaria, Gitman and Schindler introduced the weak proper forcing axiom, wPFA. By keeping track of the size of the model in question, one arrives at a hierarchy of these weak forcing axioms.

The weak hierarchy

Definition

Let Γ be a class of forcings, and let κ be an uncountable cardinal. The weak κ-bounded forcing axiom for Γ, $\mathrm{wBFA}(\Gamma, \leqslant\kappa)$, says that whenever $M = \langle |M|, \in, \ldots, R_i, \ldots \rangle_{i<\omega_1}$ is a transitive model of size κ for a language $\mathcal{L}$ with ω_1 many predicates $\langle \dot{R}_i \mid i < \omega_1 \rangle$ and the binary relation symbol $\dot{\in}$, and if $\varphi(x)$ is a Σ_1-formula and $\mathbb{P}$ is a forcing in Γ that forces that $\varphi(\check{M})$ holds, then there is (in V) a transitive model $\bar{M} = \langle |\bar{M}|, \in, \langle \bar{R}_i \mid i < \omega_1 \rangle \rangle$ for $\mathcal{L}$ such that $\varphi(\bar{M})$ holds (in V), and such that in $\mathrm{V}^{\mathrm{Col}(\omega, |\bar{M}|)}$, there is an elementary embedding $j : \bar{M} \prec M$.

If Γ is the class of subcomplete forcings, then $\mathrm{wBSCFA}(\leqslant\kappa)$ is $\mathrm{wBFA}(\Gamma, \leqslant\kappa)$. Similarly, we abbreviate these axioms for the class of proper forcings by $\mathrm{wBPFA}(\leqslant\kappa)$.

$\mathrm{wBFA}(\Gamma, <\kappa)$ says that $\mathrm{wBFA}(\Gamma, \leqslant \bar{\kappa})$ holds for every $\bar{\kappa} < \kappa$, and $\mathrm{wBSCFA}(<\kappa)$, $\mathrm{wBPFA}(<\kappa)$ have the obvious meaning.

The large cardinal for wPFA turns out to be:

Definition (Schindler)

A regular cardinal κ is remarkable if for every regular $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\bar{\lambda}})}$, there is an elementary embedding $j : H_{\bar{\lambda}}^{\mathrm{V}} \prec H_{\lambda}^{\mathrm{V}}$ with $j(\mathrm{crit}(j)) = \kappa$.

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Definition (F.)

Let κ be an inaccessible cardinal and let $\lambda \geqslant \kappa$ be a cardinal. κ is remarkably $\leqslant\lambda$-reflecting if the following holds: for any $X \subseteq H_\lambda$ and any formula $\varphi(x)$, if there is a regular cardinal $\theta > \lambda$ such that $\langle H_\theta, \in\rangle \models \varphi(X)$, then there are cardinals $\bar\kappa \leqslant \bar\lambda < \bar\theta < \kappa$ such that $\bar\theta$ is regular, and there is a set $\bar X \subseteq H_{\bar\lambda}$ such that $\langle H_{\bar\theta}, \in\rangle \models \varphi(\bar X)$, and a generic embedding $j : \langle H_{\bar\lambda}, \in, \bar X, \bar\kappa\rangle \prec \langle H_\lambda, \in, X, \kappa\rangle$ (meaning that j exists in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\bar\lambda})}$) such that $j\restriction\bar\kappa = \mathrm{id}$.

κ is remarkably $<\lambda$-reflecting iff it is remarkably $\leqslant\bar\lambda$-reflecting, for every cardinal $\bar\lambda < \lambda$ with $\kappa \leqslant \bar\lambda$.

Equiconsistencies for the weak hierarchy

Theorem (F.)

Let λ be a cardinal.

1. *If $\lambda \geqslant \omega_2$ and* wBSCFA$(\leqslant\lambda)$ *holds, then ω_2 is remarkably $\leqslant\lambda$-reflecting in L.*
2. *If $\lambda \geqslant \omega_2$ and* wBSCFA$(<\lambda)$ *holds, then ω_2 is remarkably $<\lambda$-reflecting in L.*
3. *If κ is remarkably $\leqslant\lambda$-reflecting, where $\kappa \leqslant \lambda$, then* wBSCFA$(\leqslant\lambda)$ *holds in a κ-c.c. subcomplete forcing extension.*
4. *If κ is remarkably $<\lambda$-reflecting, where $\lambda > \kappa$, then* wBSCFA$(<\lambda)$ *holds in a κ-c.c. subcomplete forcing extension.*

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The resurrection axiom for Γ, introduced by Hamkins and Johnstone, strengthens this by saying that for every $\mathbb{P} \in \Gamma$, there is a $\dot{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$ and

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These principles can be strengthened and generalized to H_κ, with $\kappa > \omega_2$, by using elementary embeddings rather than elementary substructures.

Definition (after Hamkins, Johnstone, Tsaprounis)

Let $\kappa \geqslant \omega_2$ be a cardinal, and let Γ be a class of forcings. The resurrection axiom for Γ at H_κ, $\mathrm{RA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P} \in \Gamma$, then there is a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, λ is a cardinal and there is an elementary embedding

$$j : \langle H_\kappa^{\mathrm{V}}, \in\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in\rangle$$

The principle $\underset{\sim}{\mathrm{RA}}_\Gamma(H_\kappa)$ says that for every $A \subseteq H_\kappa$ and every G as above, there is a $\mathbb{Q}$ as above such that for every H as above, in $\mathrm{V}[G][H]$, there are a B and a j such that

$$j : \langle H_\kappa^{\mathrm{V}}, \in, A\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in, B\rangle,$$

and such that if κ is regular, then λ is regular in $\mathrm{V}[G][H]$.

Equiconsistencies at ω_2

Definition (Hamkins-Johnstone)

An inaccessible cardinal κ is uplifting if there are arbitrarily large inaccessible cardinals λ such that $\langle \mathrm{V}_\kappa, \in \rangle \prec \langle \mathrm{V}_\lambda, \in \rangle$. It is strongly uplifting if for every $A \subseteq \kappa$, there are arbitrarily large inaccessible λ such that for some $B \subseteq \lambda$, $\langle \mathrm{V}_\kappa, \in, A \rangle \prec \langle \mathrm{V}_\lambda, \in, B \rangle$.

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Theorem (Hamkins-Johnstone)

For Γ the class of proper forcing notions, $\mathsf{RA}_\Gamma(H_{\omega_2})$ is equiconsistent with an uplifting cardinal, and $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$ is equiconsistent with a strongly uplifting cardinal.

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Theorem (Minden)

The same is true for the class of subcomplete forcings, and for the class of countably closed forcings.

At H_{ω_3}, a leap in consistency strength occurs again.
Hence, it is natural to consider the hierarchy of the “virtual” resurrection axioms, where the elementary embeddings are added by some further forcing.

Virtual resurrection

Let $\kappa \geqslant \omega_2$ be a cardinal, and let Γ be a class of forcings.

Definition

The virtual resurrection axiom for Γ at H_κ, $\mathrm{vRA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P} \in \Gamma$, then there is a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, there is some further forcing $\mathbb{R} \in \mathrm{V}[G][H]$ such that if I is generic for $\mathbb{R}$ over $\mathrm{V}[G][H]$, then in $\mathrm{V}[G][H][I]$, there is an elementary embedding

$$j : \langle H_\kappa^{\mathrm{V}}, \in \rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in \rangle$$

I will call such an embedding *virtual*.

Definition

The boldface virtual resurrection axiom for Γ at H_κ, $\underset{\sim}{\text{vRA}}_\Gamma(H_\kappa)$, says that for every $A \subseteq \kappa$ and every G as before, there is a $\mathbb{Q}$ as before such that for every H as before, there are a $B \in \mathrm{V}[G][H]$ and an $\mathbb{R}$ as before such that for every I as before, there is a j in $\mathrm{V}[G][H][I]$ such that

$$j : \langle H_\kappa^{\mathrm{V}}, \in, A \rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in, B \rangle$$

and such that, if κ is regular in V, then λ is regular in $\mathrm{V}[G][H]$. Finally, the virtual unbounded resurrection axiom vUR_Γ says that $\text{vRA}_\Gamma(H_\kappa)$ holds for every cardinal $\kappa \geqslant \omega_2$.

Virtual super extendibility

Definition

Let κ be an inaccessible cardinal and α an ordinal. Then κ is virtually super α-extendible if there are arbitrarily large inaccessible cardinals γ such that for some β, there is an elementary embedding j in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\kappa+\alpha})}$ such that

$$j : \langle H^{\mathrm{V}}_{\kappa+\alpha}, \in, \kappa \rangle \prec \langle H^{\mathrm{V}}_{\gamma+\beta}, \in, \gamma \rangle$$

where $j \restriction \kappa = \mathrm{id}$ (equivalently, $j \restriction H_\kappa = \mathrm{id}$). Here, κ and γ are used as predicates in these structures, and it follows that $j(\kappa) = \gamma$ if $\alpha > 0$.

Definition

κ is strongly virtually super α-extendible if for every $A \subseteq \kappa^{+\alpha}$, there are arbitrarily large inaccessible cardinals γ such that for some β and some $B \subseteq H_{\gamma+\beta}$ (in V), there is an elementary embedding j in $\mathrm{V}^{\mathrm{Col}(\omega,\theta)}$, for some large enough θ, such that

$$j : \langle H^{\mathrm{V}}_{\kappa+\alpha}, \in, A, \kappa\rangle \prec \langle H^{\mathrm{V}}_{\gamma+\beta}, \in, B, \gamma\rangle$$

with $j \upharpoonright \kappa = \mathrm{id}$, and such that, if $\kappa^{+\alpha}$ is regular, then $\gamma^{+\beta}$ is regular.

κ is virtually super $<\alpha$-extendible if it is virtually super $\bar{\alpha}$-extendible for every $\bar{\alpha} < \alpha$.

Theorem (F.)

Let Γ be the class of semiproper, proper, countably closed or subcomplete forcings.

1. *If κ is virtually super $<\theta$-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\text{vRA}_\Gamma(H_{\omega_{2+\bar{\theta}}})$ holds, for every $\bar{\theta} < \theta$.*
2. *If κ is strongly virtually super $<\theta$-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\underset{\sim}{\text{vRA}}_\Gamma(H_{\omega_{2+\bar{\theta}}})$ holds, for every $\bar{\theta} < \theta$.*
3. *If κ is virtually extendible, then vUR_Γ holds in a κ-c.c. forcing extension by a forcing in Γ.*
4. *If $\text{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds, then ω_2 is virtually super θ-extendible in L.*
5. *If $\underset{\sim}{\text{vRA}}_\Gamma(H_{\omega_{2+\theta}})$ holds, where $\mathrm{cf}(\omega_{2+\theta}) > \omega$, then ω_2 is strongly virtually super θ-extendible in L.*
6. *The consistency strength of vUR_Γ is a virtually extendible cardinal.*

Thank you, Ronald!