SUBCOMPLETE FORCING, TREES AND GENERIC ABSOLUTENESS

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ABSTRACT. We investigate properties of trees of height ω_1 and their preservation under subcomplete forcing. We show that subcomplete forcing cannot add a new branch to an ω_1-tree. We introduce fragments of subcompleteness which are preserved by subcomplete forcing, and use these in order to show that certain strong forms of rigidity of Suslin trees are preserved by subcomplete forcings. Finally, we explore under what circumstances subcomplete forcing preserves Aronszajn trees of height and width ω_1. We show that this is the case if CH fails, and if CH holds, then this is the case iff the bounded subcomplete forcing axiom holds. Finally, we explore the relationships between bounded forcing axioms, preservation of Aronszajn trees of height and width ω_1 and generic absoluteness of Σ_1^1-statements over first order structures of size ω_1, also for other canonical classes of forcing.

1. INTRODUCTION

Much of the work in this paper is motivated by prior work of the first author in which it was observed that the countably closed maximality principle ($\mathsf{MP}_{<\omega_1\text{-closed}}(H_{\omega_2})$) implies countably closed-generic $\Sigma_2^1(H_{\omega_1})$-absoluteness, defined in Section 4, see [2]. The point here is that countably closed-generic $\Sigma_1^1(\omega_1)$-absoluteness is provable in ZFC. In [13], the maximality principle for subcomplete forcing was considered, and the question arose whether it has the same consequence. Dually to the situation with countably closed forcing, the underlying question is whether subcomplete generic $\Sigma_1^1(\omega_1)$-absoluteness is provable in ZFC. Subcomplete forcing was introduced by Jensen, who showed that it cannot add real numbers, yet may change cofinalities to be countable, and that it can be iterated with revised countable support. Moreover all countably closed forcing notions are subcomplete. What makes forcing principles for subcomplete forcing particularly intriguing is that they tend to be compatible with CH, while otherwise having consequences similar to those of other, more familiar forcing classes, such as proper, semi-proper, or stationary set preserving forcings, which imply the failure of CH. There is a close relationship between these generic absoluteness properties and the preservation of certain types of Aronszajn trees, and this led us to investigate properties of trees of height ω_1 and their preservation under subcomplete forcing. The main question we had in mind, stated in [13, Question 3.1.6], was whether subcomplete forcing can add a branch to an $(\omega_1, \leq \omega_1)$-Aronszajn tree, that is, a tree of height and width ω_1 that does not have a cofinal branch.

The work on properties of ω_1-trees and their preservation under subcomplete forcing, in particular on strong forms of rigidity, led us to consider weak forms of subcompleteness which themselves are preserved by subcomplete forcing. In Section 2, we recall the definition of subcompleteness, originally introduced by Jensen, investigate the relevant fragments of subcompleteness we call minimal subcompleteness and prove the preservation facts we need. In Section 3, we show that Suslin trees are preserved under minimally subcomplete forcing, and that such forcing cannot add new

branches to ω_1-trees. We then show that certain strong forms of rigidity of Suslin trees, introduced in [6], are preserved by subcomplete forcing. Finally, in Section 4, we establish the relationships between the preservation of wide Aronszajn trees, forms of generic absoluteness and the bounded subcomplete forcing axiom, of course building on Bagaria's work [1] on bounded forcing axioms and principles of generic absoluteness. The main results in this section are as follows. The first one is Theorem 4.21, which says:

Theorem. *Assuming* CH*, the following are equivalent.*

(1) *Subcomplete generic* $\mathbf{\Sigma}_1^1(\omega_1)$*-absoluteness.*
(2) BSCFA.
(3) *Subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees.*

Here, BSCFA is the bounded subcomplete forcing axiom. The corresponding equivalence holds for any other natural class of forcing notions (see Definition 4.9) that don't add reals. The second main result is Theorem 4.22, which settles our original question, whether subcomplete forcing can add a branch to an $(\omega_1, \leq\omega_1)$-Aronszajn tree:

Theorem. *Splitting in two cases, we have:*

(1) *If* CH *fails, then subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees.*
(2) *If* CH *holds, then subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees iff* BSCFA *holds.*

We have a complete analysis for other forcing classes as well. The following is Theorem 4.24.

Theorem. *Let* Γ *be the class of proper, semi-proper, stationary set preserving, ccc or subcomplete forcing notions. Consider the following properties.*

(1) BFA_Γ.
(2) Γ*-generic* $\mathbf{\Sigma}_1^1(\omega_1)$*-absoluteness.*
(3) *Forcings in* Γ *preserve* $(\omega_1, \leq\omega_1)$*-Aronszajn trees.*

Then $(1) \Longleftrightarrow (2) \Longrightarrow (3)$*, but* (3) *does not imply* $(1)/(2)$*.*

Of course, subcomplete forcing is the only one of these forcing classes whose bounded forcing axiom is consistent with CH, and it is in the presence of CH that the unusual situation arises that these conditions are equivalent for this class.

2. Fragments of subcompleteness and their preservation

We begin by recalling the concept of subcompleteness of a partial order, as introduced by Jensen (see [10]). If M and N are models of the same first order language, then we write $M \prec N$ to express that M is an elementary submodel of N, and we write $\sigma : M \prec N$ to say that σ is an elementary embedding from M to N. If X is a subset of the domain of N, then we write $X \prec N$ to express that the reduct $N|X$ of N to X is an elementary submodel of N.

Definition 2.1. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal γ such that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$, meaning that if $x \in N$, $f \in L_\gamma(N)$ and $f : x \longrightarrow N$, then $\mathrm{range}(f) \in N$.

Definition 2.2. For a poset $\mathbb{P}$, $\delta(\mathbb{P})$ is the minimal cardinality of a dense subset of $\mathbb{P}$.

Definition 2.3. Let $N = L_\tau^A = \langle L_\tau[A], \in, A \cap L_\tau[A] \rangle$ be a ZFC^- model where τ is a cardinal and A is a set, δ an ordinal and $X \cup \{\delta\} \subseteq N$. Then $C_\delta^N(X)$ is the smallest $Y \prec N$ such that $X \cup \delta \subseteq Y$.

Definition 2.4. A forcing $\mathbb{P}$ is *subcomplete* if there is a cardinal $\theta > \delta = \delta(\mathbb{P})$ which *verifies the subcompleteness of* $\mathbb{P}$, which means that $\mathbb{P} \in H_\theta$, and for any ZFC^- model $N = L_\tau^A$ with $\theta < \tau$ and $H_\theta \subseteq N$, any $\sigma : \overline{N} \prec N$ such that $\overline{N}$ is countable, transitive and full and such that $\mathbb{P}, \theta \in \mathrm{range}(\sigma)$, any $\overline{G} \subseteq \overline{\mathbb{P}}$ which is $\overline{\mathbb{P}}$-generic over $\overline{N}$, and any $s \in \mathrm{range}(\sigma)$, the following holds. Letting $\sigma(\overline{s}, \overline{\theta}, \overline{\mathbb{P}}) = s, \theta, \mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V with $p \in G$, there is in V$[G]$ a σ' such that

(1) $\sigma' : \overline{N} \prec N$,
(2) $\sigma'(\overline{s}, \overline{\theta}, \overline{\mathbb{P}}) = s, \theta, \mathbb{P}$,
(3) $(\sigma')``\overline{G} \subseteq G$,
(4) $C_\delta^N(\mathrm{range}(\sigma')) = C_\delta^N(\mathrm{range}(\sigma))$.

The three main properties of subcomplete forcings are that they don't add reals, preserve stationary subsets of ω_1, and that they can be iterated (with revised countable support). We now isolate key parts of what it means that a forcing is subcomplete, which are in a sense responsible for these preservation properties. The remaining parts are crucial for the iterability of the resulting class of forcings. We call the stripped down version of the definition of subcompleteness *minimal subcompleteness*.

Definition 2.5. If X is a set such that the restriction of $\in$ to X orders X extensionally, then let $\sigma_X : N_X \longrightarrow X$ be the inverse of the Mostowski collapse of X, where N_X is transitive.

Let N be a transitive model of ZFC^-, and let $\mathbb{P} \in X \prec N$, where X is countable. Then X *elevates to* $N^{\mathbb{P}}$ if the following holds: let $c \in N_X$, and let $\overline{\mathbb{P}} = \sigma_X^{-1}(\mathbb{P})$. Then, whenever $\overline{G}$ is generic over N_X for $\overline{\mathbb{P}}$, there is a condition $p \in \mathbb{P}$ such that if G is $\mathbb{P}$-generic over V and $p \in G$, then in V$[G]$, there is an elementary embedding $\sigma' : N_X \prec N$ such that $\sigma'``\overline{G} \subseteq G$ and $\sigma'(c) = \sigma_X(c)$.

A forcing notion $\mathbb{P}$ is *minimally subcomplete* if for all sets a, H, there is a transitive model of ZFC^- the form $N = L_\tau^A$ with $a \in N$ and $H \subseteq N$ such that

$$Z_{N,\mathbb{P},a} = \{\omega_1 \cap X \mid a \in X \text{ and } X \text{ elevates to } N^{\mathbb{P}}\}$$

contains a club subset of ω_1.

Let us first show that this is indeed a weakening of subcompleteness.

Observation 2.6. *If $\mathbb{P}$ is subcomplete, then $\mathbb{P}$ is minimally subcomplete.*

Proof. Let θ verify the subcompleteness of $\mathbb{P}$. Given sets a, H let $\tau > \theta$, $A \subseteq \tau$, and $H_\theta \in N = L_\tau[A]$ with $a \in N$ and $H \subseteq N$. Let $\mu = \tau^+$ and $\nu = \tau^{++}$. Let $N' = L_\mu[A]$. We claim that $Z_{N',\mathbb{P},a}$ contains a club. Let

$$Z = \{\omega_1 \cap Y \mid a \in Y \prec L_\nu[A], \ Y \text{ countable}\}$$

Then clearly, Z contains a club. Moreover, if $\omega_1 \cap Y \in Z$, where $Y \prec L_\nu[A]$ and Y is countable, then, letting $\overline{A} = (\sigma_Y)^{-1}(A)$, $\overline{\tau} = (\sigma_Y)^{-1}(\tau)$ and $\overline{\mu} = (\sigma_Y)^{-1}(\mu)$, it follows that N_Y is of the form $L_{\overline{\nu}}[\overline{A}]$, where $\overline{A} \subseteq \overline{\tau}$, so $\overline{A}$ is a bounded subset of $\overline{\mu}$, which is regular, and also the largest cardinal in N_Y. Letting $\overline{N} = L_{\overline{\mu}}[\overline{A}]$, it follows that $\overline{N}$ is full, as witnessed by N_Y (note that N_Y has the same bounded subsets of $\overline{\mu}$ as $\overline{N}$, and it is a model of ZFC^-). Let $\sigma = \sigma_Y \restriction \overline{N}$, $X = \mathrm{range}(\sigma) = Y \cap L_\mu[A]$. Then, since $\mathbb{P}$ is subcomplete, X elevates to $N'^{\mathbb{P}}$. Since $\omega_1 \cap X = \omega_1 \cap Y$, it follows that $Z_{N',\mathbb{P},a}$ contains a club, as claimed. $\square$

Next, let's check that minimal subcompleteness, while weaker than subcompleteness, is still strong enough to preserve the properties mentioned before.

Fact 2.7. *Let $\mathbb{P}$ be a minimally subcomplete forcing, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over* V.

(1) $\mathcal{P}(\omega)^V = \mathcal{P}(\omega)^{V[G]}$.

(2) *If S is stationary in ω_1, then this remains true in* $V[G]$.

Proof. For (1)., the proof is exactly the same as with subcomplete forcing. Assume the contrary, and suppose toward a contradiction that there is name $\dot{r} \in V^{\mathbb{P}}$ for a subset of ω and a condition q forcing that $\dot{r}$ is new. Let $a = \langle q, \dot{r} \rangle$. By minimal subcompleteness, there is N with $H_\theta \subseteq N$ for some large enough θ, so that $Z_{N,\mathbb{P},a}$ contains a club. So there is $\alpha = \omega_1 \cap X \in Z_{N,\mathbb{P},a}$, where $X \prec N$ is countable. Then X elevates to $N^{\mathbb{P}}$. Let $\overline{G}$ be a generic filter for $\sigma_X^{-1}(\mathbb{P}) = \overline{\mathbb{P}}$ over N_X, such that $\sigma_X^{-1}(q) = \overline{q} \in \overline{G}$. So we have a condition $p \in \mathbb{P}$, where, letting G be $\mathbb{P}$-generic over V containing p, we have an elementary embedding $\sigma' : N_X \longrightarrow N$ such that $\sigma'``\overline{G} \subseteq G$. Thus $q \in G$ as well, so we have that $r = \dot{r}^G$ is new. But this is a contradiction as $r = \sigma'``\overline{r} = \sigma_X``\overline{r} = \overline{r} \in V$, where $\overline{r} = \sigma_X^{-1}(r)$.

For (2)., assume the contrary. Let $S \subseteq \omega_1$ stationary, and suppose towards a contradiction that there is a $\dot{C} \in V^{\mathbb{P}}$ such that for some $q \in \mathbb{P}$,

$$q \Vdash \text{``}\dot{C} \subseteq \check{\omega}_1 \text{ is club } \wedge \; \check{S} \cap \dot{C} = \emptyset.\text{''}$$

Let $a = \langle q, \dot{C}, S \rangle$, and, by minimal subcompleteness, let N be such that $Z_{N,\mathbb{P},a}$ contains a club, where $H_\theta \subseteq N$, for some θ which is sufficiently large to conclude that the fact displayed above holds in N. Let $\alpha \in S \cap Z_{N,\mathbb{P},a}$, and let X witness this. That is, $a \in X \prec N$, $\alpha = \omega_1 \cap X$, and X elevates to $N^{\mathbb{P}}$. Let $\overline{a} = \sigma_X^{-1}(a)$, $\overline{\mathbb{P}} = \sigma_X^{-1}(\mathbb{P})$, and let $c = \langle \overline{a}, \overline{\mathbb{P}} \rangle$.

Since $\sigma_X : N_X \prec N$ is elementary, $\overline{a}$ is of the form $\langle \overline{q}, \dot{\overline{C}}, \overline{S} \rangle$, where $\sigma_X(\overline{q}) = q$, $\sigma_X(\dot{\overline{C}}) = \dot{C}$, $\sigma_X(\overline{S}) = S$, and in N_X, $\overline{q}$ forces with respect to $\overline{\mathbb{P}}$ that $\dot{\overline{C}} \subseteq \check{\alpha}$ is club.

Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}$ with $\overline{q} \in \overline{G}$. Since X elevates to $N^{\mathbb{P}}$, there is a condition $p \in \mathbb{P}$ such that if we let G be $\mathbb{P}$-generic over V with $p \in G$, then there is in $V[G]$ an elementary embedding $\sigma' : \overline{N} \prec N$ with $\sigma'(c) = \sigma_X(c)$ and $\sigma'``\overline{G} \subseteq G$. So σ' lifts to an embedding $\sigma^* : N_X[\overline{G}] \prec N[G]$ in $V[G]$. Let $\overline{C} = \dot{\overline{C}}^{\overline{G}}$, $C = \dot{C}^G$. Since $q = \sigma'(\overline{q}) \in G$, we have that $C \subseteq \omega_1$ is club in $V[G]$ and $S \cap C = \emptyset$ in $N[G]$. However, $\alpha = \omega_1^{\overline{N}}$, so $\overline{q} \in \overline{G}$ implies that $\overline{C}$ is club in α. Since $\sigma^* \restriction \alpha = \mathrm{id}$, it follows that $C \cap \alpha = \overline{C} \cap \alpha$, and so, $\alpha < \omega_1^V$ is a limit point of C, so $\alpha \in C \cap S$, a contradiction. $\square$

It was shown in [13] that the subcompleteness of a forcing $\mathbb{P}$ is very fragile: it can be destroyed by countably closed forcing - note in this context that Jensen pointed out that every countably closed forcing is subcomplete. The same negative result remains true of minimal subcompleteness. In the following proposition, $\mathbb{N}$ denotes Namba forcing, which Jensen proved to be subcomplete, assuming CH (see [10] and [11]).

Proposition 2.8 ([13]). *Forcing with $\mathcal{C}oll(\omega_1, \omega_2) \times \mathbb{N}$ collapses ω_1.*

Thus, after forcing with $\mathcal{C}oll(\omega_1, \omega_2)$, the ground model version of Namba forcing collapses ω_1, hence adds a real, and is thus not even minimally subcomplete any longer (see Fact 2.7). However, the minimal fragment of subcompleteness survives countably distributive forcing of size at most ω_1, as we shall show presently.

Lemma 2.9. *Let $\mathbb{P}$ be subcomplete. Then after countably distributive forcing of size at most ω_1, $\mathbb{P}$ is minimally subcomplete.*

Proof. Let $\mathbb{Q}$ be countably distributive, and let $|\mathbb{Q}| \leq \omega_1$. Let $H \subseteq \mathbb{Q}$ be generic. Let's assume that the conditions in $\mathbb{Q}$ are countable ordinals, so that $H \subseteq \omega_1$. To show that $\mathbb{P}$ is minimally

subcomplete in $V[H]$, let $a = \dot{a}^H$, θ be given. In $V[H]$, we have to find a transitive $N \models \mathsf{ZFC}^-$ with $H_\theta \subseteq N$, such that the set $Z_{N,\mathbb{P},a}$ contains a club subset of ω_1.

In V, since $\mathbb{P}$ is subcomplete, we can pick a regular cardinal τ and an $A \subseteq \tau$ such that, letting $\mu = \tau^+$ and $\nu = \tau^{++}$, we have that $H_\theta \subseteq L_\tau[A]$, and whenever Y is countable, $\mathbb{P} \in Y$ and $Y \prec L_\nu[A]$, it follows that $X = Y \cap L_\mu[A]$ elevates to $L_\mu[A]^{\mathbb{P}}$, because in this situation, N_X is full, as in the proof of Observation 2.6.

In $V[H]$, let $A' = A \oplus H = \{\prec\alpha, \beta\succ \mid \alpha \in A \ \wedge \ \beta \in H\}$, where $\prec\alpha, \beta\succ$ is the Gödel code of $\langle \alpha, \beta \rangle$. Note that $L_\tau[A'] = L_\tau[A, H] = L_\tau[A][H]$. We claim that in $V[H]$, whenever $Y' \prec L_\nu[A']$ is countable, with $\mathbb{P}, A, A', \mathbb{Q}, H \in Y'$, it follows that $X' = Y' \cap L_\mu[A']$ elevates to $L_\mu[A']^{\mathbb{P}}$.

To see this, fix such Y' and X', and let $\overline{\tau}, \overline{\mu}, \overline{A}, \overline{A}', \overline{\mathbb{Q}}, \overline{H}$ be such that

$$\sigma_{Y'}(\overline{\tau}, \overline{\mu}, \overline{A}, \overline{A}', \overline{\mathbb{Q}}, \overline{H}) = \tau, \mu, A, A', \mathbb{Q}, H$$

Then $N_{Y'} = L_{\overline{\nu}}[\overline{A}']$, for some cardinal $\overline{\nu}$, and by elementarity of $\sigma_{Y'}$, it follows that in $N_{Y'}$ it is the case that $\overline{H}$ is $\overline{\mathbb{Q}}$-generic over $L_{\overline{\mu}}[\overline{A}]$. Let $\sigma = \sigma_{Y'} \upharpoonright L_{\overline{\mu}}[\overline{A}]$. So $\sigma = \sigma_X$, where $X = X' \cap L_\mu[A]$, and $\sigma : L_{\overline{\mu}}[\overline{A}] \prec L_\mu[A]$ is elementary.

In $V[H]$, let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $L_{\overline{\mu}}[\overline{A}']$, and let $c \in N_{X'}$ be given. There is then a $\dot{c} \in L_{\overline{\mu}}[\overline{A}]$ such that $c = \dot{c}^{\overline{H}}$. Let $Y = Y' \cap L_\nu[A]$, $X = X' \cap L_\mu[A]$. Since $\mathbb{Q}$ is countably distributive, it follows that $\overline{G}, X, Y \in V$, and $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\mu}}[\overline{A}]$. Since $X = Y \cap L_\mu[A]$, it follows that X elevates (in V) to $L_\tau[A]^{\mathbb{P}}$, as N_X is full. Hence, there is a condition $p \in \mathbb{P}$ that verifies this (with respect to $\dot{c}$). Thus, let G be $\mathbb{P}$-generic over $V[H]$ with $p \in G$. Then in $V[G]$, there is a $\sigma' : L_{\overline{\mu}}[\overline{A}] \prec L_\mu[A]$ with $\sigma' \text{``} \overline{G} \subseteq G$ and $\sigma'(\dot{c}) = \sigma_X(\dot{c})$. So σ' lifts to $\sigma^* : L_{\overline{\mu}}[\overline{A}][\overline{G}] \prec L_\mu[A][G]$.

Since $\overline{H} \in L_{\overline{\mu}}[\overline{A}']$ and $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $L_{\overline{\mu}}[\overline{A}']$, it follows by the product lemma that $\overline{H}$ is $\overline{\mathbb{Q}}$-generic over $L_{\overline{\mu}}[\overline{A}][\overline{G}]$, and since σ^* doesn't move $\overline{H}$, as $\overline{H} \subseteq \omega_1^{L_{\overline{\mu}}[\overline{A}]} = \omega_1^{L_{\overline{\mu}}[\overline{A}][\overline{G}]}$, it follows that σ^* lifts to

$$\sigma^{**} : L_{\overline{\mu}}[\overline{A}][\overline{G}][\overline{H}] \prec L_\mu[A][G][H]$$

Noting that

$$L_{\overline{\mu}}[\overline{A}][\overline{G}][\overline{H}] = L_{\overline{\mu}}[\overline{A}][\overline{H}][\overline{G}] = L_{\overline{\mu}}[\overline{A}'][\overline{G}]$$

and

$$L_\mu[A][G][H] = L_\mu[A][H][G] = L_\mu[A'][G]$$

we see that $\sigma^{**} \upharpoonright L_{\overline{\mu}}[\overline{A}']$ witnesses that X' elevates to $L_\mu[A']^{\mathbb{P}}$ in $V[H]$, since $\sigma^{**}(c) = \sigma^*(\dot{(c)}^{\overline{H}}) = \sigma'(\dot{c}^{\overline{H}}) = \sigma'(\dot{c})^H = \sigma_X(\dot{c})^H = \sigma_{X'}(c)$. This last equality holds because $\sigma_X = \sigma_{X'} \upharpoonright L_{\overline{\mu}}[\overline{A}]$, since X is transitive in X', in the sense that if $d \in X$ and $e \in d \cap X'$, then $e \in X$. It follows that $\sigma_{X'}$ is the lift of σ_X to $L_{\overline{\mu}}[\overline{A}']$, and hence that $\sigma_X(\dot{c})^H = \sigma_{X'}(c)$.

Thus, $H_\theta^{V[G]} \subseteq L_\mu[A']$ and $Z_{L_\mu[A'],\mathbb{P},a}$ contains a club, because in $V[H]$, $\{Y' \cap \omega_1 \mid Y' \prec L_\nu[A']\}$ contains a club. $\qquad\square$

Remark 2.10. Slight variations of the proof of the previous lemma show the following.

(1) Minimal subcompleteness is preserved by countably distributive proper (that is, strongly proper) forcing of size ω_1. In a sense, the modified proof is somewhat easier than the original one.

(2) The following slightly strengthened version of minimal subcompleteness of a forcing $\mathbb{P}$, which is still weaker than subcompleteness, is preserved by countably distributive forcing of size ω_1: for any set a, there is a τ and an $A \subseteq \tau$, such that, letting $\mu = \tau^+$ and $\nu = \tau^{++}$, we have that $a, \mathbb{P} \in L_\mu[A]$, and for every countable $Y \prec L_\nu[A]$, $Y \cap L_\mu[A]$ elevates to $L_\mu[A]^{\mathbb{P}}$.

To formulate a corollary to the previous lemma, recall that, given a cardinal μ, Jensen introduced a version of subcompleteness called *subcompleteness above μ,* which requires the elevated embedding to coincide with the originally given embedding up to the preimage of μ. There is a natural version of minimal subcompleteness above μ, which we make precise presently.

Definition 2.11. Let μ be a cardinal, N a transitive model of ZFC^-, $\mathbb{P}$ a forcing notion and $X \prec N$ countable with $\mu, \mathbb{P} \in X$. Then X *elevates to $N^{\mathbb{P}}$ above μ* if for every $\overline{G}$ which is generic over N_X for $\sigma_X^{-1}(\mathbb{P})$, and for every $c \in N_X$, there is a condition $p \in \mathbb{P}$ such that whenever G is generic over V for $\mathbb{P}$, then in V[G], there is an elementary $\sigma' : N_X \prec N$ with $(\sigma')``\overline{G} \subseteq G$, $\sigma'(c) = \sigma_X(c)$ and $\sigma' \restriction \overline{\mu} = \sigma_X \restriction \overline{\mu}$, where $\overline{\mu} = \sigma_X^{-1}(\mu)$.

The proof of Lemma 2.9 then shows the following.

Corollary 2.12. *Let $\mathbb{P}$ be subcomplete above μ. Then after countably distributive forcing of size at most μ, $\mathbb{P}$ is minimally subcomplete above μ.*

3. Minimal subcompleteness and the preservation of properties of ω_1-trees

Countably closed forcing does not add cofinal branches through ω_1-trees, so it is natural to wonder whether other subcomplete forcing cannot do this either. Indeed, we see below that this is true of minimally subcomplete forcing as well. The proof for subcomplete forcing is given in [13]. Let's begin by establishing some terminology on trees.

Definition 3.1. A *tree* is a partial order $T = \langle |T|, <_T \rangle$ in which the predecessors of any member of $|T$ are well-ordered by $<_T$ and there is a unique minimal element called the *root*.

- The members of $|T|$ are called the *nodes* of T, and we will tend to conflate the tree T with its underlying set $|T|$.
- The *height of a node $t \in T$* is the order type of the set of its predecessors under the restriction of the tree order. We write T_α for the αth level of T, the set of nodes having height α. The *height of a tree T*, height(T), is the strict supremum of the heights of its nodes.
- We write $T \restriction \alpha$ for the subtree of T of nodes having height less than α. An ω_1-*tree* is a normal tree of height ω_1 where all levels are countable. A tree of height ω_1 is *normal* if every node has (at least) two immediate successors, nodes on limit levels are uniquely determined by their sets of predecessors, and every node has successors on all higher levels up to ω_1.
- We write T_t to denote the subtree of T consisting of the nodes $s \in T$ with $s \geq_T t$. For nodes $t \in T$, by succ$_T(t)$ we mean the set of successors $s \geq_T t$ in the tree.
- A *branch b* in T is a maximal linearly ordered subset of T, and the length of the branch is its order type. For α less than the length of b, we write $b(\alpha)$ for the node in b that has height α. We write $[T]$ for the set of *cofinal branches* of T, that is, those branches containing nodes on every nonempty level of T. If $t \in T$ is a node, then we write b_t to mean the "branch" below t: $b_t = \{s \in T \mid s <_T t\}$.
- An ω_1-tree is an *Aronszajn* tree if it has no cofinal branches. Two nodes t and s in T are *compatible*, written $s \parallel t$, if there is $r \in T$ such that $r \geq_T t$ and $r \geq_T s$. This is the same as demanding that either $s <_T t$, $s >_T t$, or $s = t$, or, in other words, that s and t are *comparable*. Otherwise, they are incompatible, written $s \perp t$. An *antichain* in a tree is a set of pairwise incompatible elements. A *Suslin tree* is an ω_1-tree with no uncountable antichain. When forcing with a tree, we reverse the order, so that stronger conditions are higher up in the tree. Consequently, Suslin trees are *ccc* as notions of forcing. A *Kurepa tree* is an ω_1-tree with at least ω_2-many cofinal branches.

Lemma 3.2. *Let T be an ω_1-tree. If $\mathbb{P}$ is minimally subcomplete and G is $\mathbb{P}$-generic then $[T] = [T]^{V[G]}$.*

Proof. Assume not. Let $\dot{b}$ be a name for a new cofinal branch through $T \subseteq H_{\omega_1}$; let $p \in \mathbb{P}$ be a condition forcing that $\dot{b}$ is a new cofinal branch through $\check{T}$. Let N be a transitive model of ZFC^- with $\mathbb{P}, \dot{b}, p \in N$ and $H_\theta \subseteq N$, such that $Z = Z_{N,\mathbb{P},\langle \dot{b}, p \rangle}$ contains a club, where θ is large enough to ensure that inside N, p forces that $\dot{b}$ is a new cofinal branch through $\check{T}$. Let $\alpha \in Z$, and let X witness this. Let $\alpha = \omega_1 \cap X$, $\overline{N} = N_X$ and $\sigma = \sigma_X$. As usual, let $\overline{p}, \overline{\mathbb{P}}, \dot{\overline{b}} = \sigma^{-1}(p, \mathbb{P}, \dot{b})$.

By elementarity, we have that $\overline{p}$ forces $\dot{\overline{b}}$ to be a new cofinal branch over $\overline{N}$. As we construct a generic $\overline{G}$ for $\overline{\mathbb{P}}$ over $\overline{N}$, we will use the countability of $\overline{N}$ to diagonalize against all "branches" as seen on level α of the tree T in N, thereby obtaining a contradiction.

Toward this end, enumerate the dense sets $\langle \overline{D}_n \mid n < \omega \rangle$ of $\overline{\mathbb{P}}$ that belong to $\overline{N}$. Also denote the sequence of downward closures of nodes on level α of T, the "branches" through $T \upharpoonright \alpha$ that extend to have nodes of higher height in T, as $\langle b_n \mid n < \omega \rangle$. Now define a sequence of conditions of the form $\overline{p}_n$ for $n < \omega$ that decide values of $\dot{\overline{b}}$ in $\overline{T}$ differently from b_n. Ensure along the way that for all n,

- $\overline{p}_{n+1} \in \overline{D}_n$ and
- $\overline{p}_{n+1} \leq \overline{p}_n$.

The construction (in V) may go as follows:

Let $\overline{p}_0 := \overline{p} \in \overline{N}$. For each $n < \omega$, note that there must be two conditions $\overline{p}_{n+1}^0 \perp \overline{p}_{n+1}^1$, both extending $\overline{p}_n$, that decide the value of the branch $\dot{\overline{b}}$ to differ on some value. This always has to be possible since these conditions always extend $\overline{p}$, that forces $\dot{\overline{b}}$ to be new. Say $\overline{p}_{n+1}^1 \Vdash \check{x}_n \in \dot{\overline{b}}$ and $\overline{p}_{n+1}^0 \Vdash \check{x}_n \notin \dot{\overline{b}}$. Let $\overline{p}_{n+1}$ be a condition in $\overline{D}_n$ extending $\overline{p}_{n+1}^1$ if $x_n \notin b_n$, or a condition in $\overline{D}_n$ extending $\overline{p}_{n+1}^0$ otherwise.

Let $\overline{G}$ be the generic filter generated by the $\langle \overline{p}_n \mid n < \omega \rangle$, let $\dot{\overline{b}}^{\overline{G}} = \overline{b}$. Since $\mathbb{P}$ is minimally subcomplete, there is a condition $q \in \mathbb{P}$ such that whenever G is $\mathbb{P}$-generic with $q \in G$, by minimal subcompleteness we have $\sigma' \in V[G]$ such that:

- $\sigma' : \overline{N} \prec N$
- $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{p}, \dot{\overline{b}}) = \theta, \mathbb{P}, T, p, \dot{b}$
- $\sigma' \text{``} \overline{G} \subseteq G$.

So below q there is a lift $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$ extending σ' with $\sigma^*(\overline{b}) = \sigma'(\dot{\overline{b}})^G = \dot{b}^G = b$, and $\sigma^*(\overline{T}) = \sigma'(\overline{T})^G = T$. The point is that since $\overline{p} \in \overline{G}$, we have $N[G] \models p \in G$, so b is a branch through T.

Furthermore, α is the critical point of the embedding σ^*. So below α the tree T, and thus the branch b, is fixed. In particular, in $N[G]$, $b \upharpoonright \alpha = \overline{b}$. However, $\overline{b}$ was constructed so as to not be equal to any of the b_ns, so it cannot be extended to become a branch through T, since it can't have a node on the αth level. This is a contradiction. $\qquad \square$

So in particular, minimally subcomplete forcing preserves Aronszajn trees. The following theorem shows that after forcing with a minimally subcomplete forcing, not only are there no new cofinal branches added to a Suslin tree T, but no uncountable antichains either. The proof is exactly the same as is given by Jensen [9, Chapter 3 p. 10].

Lemma 3.3. *Minimally subcomplete forcing preserves Suslin trees.*

Proof. Let T be a Suslin tree. Let $\mathbb{P}$ be minimally subcomplete. Suppose toward a contradiction that $p \in \mathbb{P}$ forces that $\dot{A}$ is a maximal antichain of size ω_1. Let N be a transitive model of ZFC^- with $p, \mathbb{P}, \dot{A} \in N$, and with $H_\theta \subseteq N$, where θ is large enough that N is sufficiently correct about what p forces with respect to $\mathbb{P}$, such that $Z_{N,\mathbb{P},\langle p,\mathbb{P},\dot{A}\rangle} \neq \emptyset$. Let $X \prec N$ with $p, \mathbb{P}, \dot{A} \in X$ such that X elevates to $N^{\mathbb{P}}$. Let $\sigma = \sigma_X$, $\overline{N} = N_X$.

Letting $\alpha = \omega_1^{\overline{N}}$, we have that $\overline{T} = T \restriction \alpha$ as usual. Let M be a countable, transitive ZFC^- model with both $\overline{N}, T \restriction (\alpha + 1) \in M$. Let $\overline{G} \subseteq \overline{\mathbb{P}}$ be generic over M with $\overline{p} \in \overline{G}$. So $\overline{G}$ is also generic over $\overline{N}$.

We can now work below a condition in $G \subseteq \mathbb{P}$ generic to obtain a $\sigma' \in V[G]$ such that:

- $\sigma' : \overline{N} \prec N$
- $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{p}, \overline{\dot{A}}) = \theta, \mathbb{P}, T, p, \dot{A}$
- $\sigma'``\overline{G} \subseteq G$.

As usual we have a lift $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$. Letting $\overline{A} = \overline{\dot{A}}^{\overline{G}}$ and $\dot{A}^G = A$ we have that $\sigma^*(\overline{A}) = A$. Let $\langle b_t \mid t \in T_\alpha \rangle$ be the collection of partial branches below the nodes of level α of the tree T.

Every node in T above level α has to have a predecessor in level α. For each $t \in T_\alpha$, $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}[b_t]$ since b_t is $\overline{T}$-generic over $\overline{N}$ - as cofinal branches through Suslin trees are generic. By the product lemma, each b_t is $\overline{T}$-generic over $\overline{N}[\overline{G}]$. Since $\overline{A}$ is maximal, $b_t \cap \overline{A} \neq \emptyset$. Thus $\overline{A}$ is sealed in $\overline{T} = T \restriction \alpha$, meaning it has no elements above level α. But since $\overline{A} \subseteq A$ and A is maximal, this means that A is countable, so T remains Suslin as desired.   □

The following rigidity properties were introduced in [6], where it was shown, among other things, that Suslin trees exhibiting these properties can be constructed, assuming the $\diamondsuit$ principle holds.

Definition 3.4. A normal ω_1-tree T has the *unique branch property* (is UBP) so long as

$$\mathbb{1} \Vdash_T \text{``}\check{T} \text{ has exactly one new cofinal branch.''}$$

That is, after forcing with the tree, T has exactly one cofinal branch that was not in the ground model. We say that T has the *n-fold* UBP so long as forcing with T^n adds exactly n branches.

A Suslin tree is *Suslin off the generic branch* so long as after forcing with T to add a generic branch b, for any node t not in b, the tree T_t remains Suslin. Let n be a natural number. A Suslin tree T is *n-fold Suslin off the generic branch* so long as after forcing with with the tree n times, or forcing with T^n that adds n branches $b_1, \ldots, b_n$, T_p remains Suslin for any p not on any b_i.

Combining the results from Sections 2 and 3, we can conclude that these strong rigidity properties of Suslin trees are preserved by subcomplete forcing.

Theorem 3.5. *The following properties of an ω_1-tree T are preserved by subcomplete forcing:*

(1) T is Aronszajn
(2) T is not Kurepa
(3) T is Suslin
(4) T is Suslin and UBP
(5) T is Suslin off the generic branch
(6) T is n-fold Suslin off the generic branch (for $n \geq 2$)
(7) T is $(n-1)$-fold Suslin off the generic branch and n-fold UBP (for $n \geq 2$)

Proof. Items (1). and (2). are immediate corollaries of Lemma 3.2. Item (3). is Lemma 3.3. In fact, these properties are even preserved by minimally subcomplete forcing.

For the remaining proofs, let $\mathbb{P}$ be a subcomplete forcing, and let G be generic for $\mathbb{P}$ over V.

Proof of (4). First we show upward absoluteness. Let T be a Suslin tree with the UBP. We have already seen that T is still Suslin in V$[G]$. To see that it is still UBP, let b be T-generic over V$[G]$. In $V[b]$, b is the unique cofinal branch through T, and we have that $\mathbb{P}$ is still minimally subcomplete by Lemma 2.12. Since minimally subcomplete forcing doesn't add branches to ω_1-trees by Lemma 3.2, b is still the unique cofinal branch of T in $V[b][G] = V[G][b]$. So T still has the UBP in $V[G]$.

For downward absoluteness, suppose T has the UBP in $V[G]$ but does not have the UBP in V. Let $p \in G$ force that T has the UBP. Let b be a generic branch for T over V such that in $V[b]$ the tree T has at least two branches. Let G' be $\mathbb{P}$-generic over $V[b]$ with $p \in G'$. Then T has at least two branches in $V[b][G'] = V[G'][b]$, so T is not UBP in $V[G']$, contradicting that $p \in G'$.

Proof of (5). For upward absoluteness, let T be Suslin off the generic branch, and let b be a generic branch for T over V$[G]$. T is still Suslin in $V[G]$ by (3). In $V[b]$, we have that $\mathbb{P}$ is still minimally subcomplete by Lemma 2.9. We have that for any node t not in b, the tree T_t remains Suslin in $V[b][G] = V[G][b]$ since after minimally subcomplete forcing T_t remains Suslin by Lemma 3.3. So T remains Suslin off the generic branch after forcing with $\mathbb{P}$.

For downward absoluteness, suppose T is Suslin off the generic branch in $V[G]$ but not in V. Let $p \in G$ force that T is Suslin off the generic branch. Let b be a V-generic branch through T such that in $V[b]$ the tree T is not Suslin off the generic branch. So there is $t \in T$ off of b such that T_t is not Suslin in V$[b]$. Let G' be $\mathbb{P}$-generic over $V[b]$ with $p \in G'$. Then, in $V[b][G'] = V[G'][b]$, T_t is not Suslin, so that T is not Suslin off the generic branch in V$[G']$, contradicting that $p \in G'$.

Proof of (6). For upward absoluteness, let T be n-fold Suslin off the generic branch. Let $b_1 \times b_2 \times \ldots \times b_n$ be T^n-generic over V$[G]$. Since T is n-fold Suslin off the generic branch, it follows that T^n is countably distributive, so again we know that $\mathbb{P}$ is still minimally subcomplete in V$[b_1, b_2, \ldots, b_n]$ by Lemma 2.9. In $V[b_1, \ldots, b_n][G] = V[G][b_1, \ldots, b_n]$, for any node t not in one of the generic branches $b_1, \ldots, b_n$, we have that T_t is Suslin by Lemma 3.3. So T remains n-fold Suslin off the generic branch after forcing with $\mathbb{P}$. Downward absoluteness works as in (5).

Proof of (7). For upward absoluteness, suppose that T is $(n-1)$-fold Suslin off the generic branch and n-fold UBP for some $n \geq 1$. Let $b_1 \times b_2 \times \ldots \times b_n$ be T^n-generic over V$[G]$. Since T is $(n-1)$-fold Suslin off the generic branch, it follows that T^n is countably distributive, so again we know that $\mathbb{P}$ is still minimally subcomplete in V$[b_1, b_2, \ldots, b_n]$ by Lemma 2.9. Again we have $V[b_1, \ldots, b_n][G] = V[G][b_1, \ldots, b_n]$ where $b_1, \ldots, b_n$ are the unique cofinal branches through T, since $\mathbb{P}$ does not add branches to T^n over $V[b_1, \ldots, b_n]$. So T has the n-fold UBP in $V[G]$. We have already seen in (6). that T stays $(n-1)$-fold Suslin off the generic branch in V$[G]$.

Downward absoluteness is again the same as in (4). $\square$

4. Generic absoluteness and the preservation of wide Aronszajn trees

In Lemma 3.2, we showed that subcomplete forcing cannot add a new branch to an ω_1-tree, and in particular, that it preserves Aronszajn trees. In the present section, we will explore slightly stronger preservation properties. Let us introduce versions of κ-trees in which the requirement that the levels have size less than κ is relaxed.

Definition 4.1. Let κ and λ be cardinals. We shall say that T is a $(\kappa, \leq\lambda)$-*tree* if T is a tree of height κ with levels of size less than or equal to λ. We shall refer to the size restriction on the levels in the tree in the second coordinate as the tree's width, so that a $(\kappa, \leq\lambda)$-tree has width $\leq \lambda$.

An *Aronszajn $(\kappa, \leq\lambda)$-tree* is a $(\kappa, \leq\lambda)$-tree with no cofinal branch.

It is easy to see that in general, countably closed forcing can't add a (cofinal) branch to any $(\omega_1, \leq\kappa)$-Aronszajn tree, for any κ. The guiding question for the work in the present section, as stated in [13], is as follows.

Question 4.2. Can subcomplete forcing add cofinal branches to an $(\omega_1, \leq\omega_1)$-Aronszajn tree?

In the remainder of the present section, we will answer this question fully. Let's first make the simple observation that even countably closed forcing may add branches to a tree of height ω_1 and width $\leq 2^\omega$ (but such a tree can never be Aronszajn, by our earlier remarks).

Observation 4.3. *Subcomplete (or even countably closed) forcing may add a cofinal branch to an $(\omega_1, \leq 2^\omega)$-tree.*

Proof. The point here is that the poset $\mathcal{A}dd(\omega_1, 1)$ is subcomplete since it is countably closed, but it may be viewed as a tree of height ω_1 that has levels of size up to 2^ω. Of course this tree is not Aronszajn - it is Kurepa. Every cofinal branch through the tree corresponds to a subset of ω_1, of which there are already more than ω_1-many of in the ground model. $\square$

If we allow the size of the levels of the height ω_1 tree to be large, then we can obtain a slightly more complicated example of an Aronszajn tree to which subcomplete forcing can add a cofinal branch, using the forcing denoted by Jensen as $\mathbb{P}_A$. In the following definition, we write $\mathrm{cof}(\omega)$ for the class of ordinals with countable cofinality.

Definition 4.4. Let $\kappa > \omega_1$ be regular, and let $A \subseteq \kappa \cap \mathrm{cof}(\omega)$. We write $\mathbb{P}_A$ to denote the forcing designed to shoot a cofinal, normal sequence of order type ω_1 through A. The conditions of $\mathbb{P}_A$ consist of normal functions of the form $p : \nu + 1 \to A$, where $\nu < \omega_1$, and extension is defined in the usual way, by $p \leq q$ if and only if $q \subseteq p$.

Jensen showed that $\mathbb{P}_A$ is subcomplete, see [10]. If $(\kappa \cap \mathrm{cof}(\omega)) \setminus A$ is stationary in $\kappa \cap \mathrm{cof}(\omega)$, then $\mathbb{P}_A$ is not countably closed. A $\mathbb{P}_A$-generic filter G gives rise to the function $\cup G : \omega_1 \to A$ which is normal and cofinal in κ. The forcing $\mathbb{P}_A$ is used to show that the subcomplete forcing axiom SCFA implies *Friedman's Principle*, which states that for every regular cardinal $\kappa > \omega_1$ and every stationary set $A \subseteq \kappa \cap \mathrm{cof}(\omega)$, there is a normal function $f : \omega_1 \to A$, that is, A contains a closed set of order type ω_1.

Proposition 4.5. *Suppose that Friedman's Principle fails for ω_2. Then subcomplete forcing may add a cofinal branch to an $(\omega_1, \leq\omega_2 \cdot 2^\omega)$-Aronszajn tree.*

Proof. Let $A \subseteq \mathrm{cof}(\omega) \cap \omega_2$ witness the failure of Friedman's Principle. Consider the forcing poset $\mathbb{P}_A$ as a tree. It has size $\omega_1 \cdot \omega_2^\omega = \omega_2^\omega$, since it consists of functions with domain in ω_1, and each condition is from a countable ordinal to ω_2.

Considering $\mathbb{P}_A$ as a tree, it has height ω_1. Each level has size less than or equal to $\omega_2^\omega = \omega_2 \cdot 2^\omega$. Moreover, since Friedman's Principle fails for ω_2, the tree $\mathbb{P}_A$ has no cofinal branches and is thus Aronszajn, yet forcing with the tree will add a cofinal branch. $\square$

However, we may slightly tweak the proof of Lemma 3.2 to see that subcomplete forcing does preserve $(\omega_1, <2^\omega)$-Aronszajn trees. Note that this shows that the failure of CH implies a negative answer to Question 4.2.

Theorem 4.6. *Subcomplete forcing cannot add (cofinal) branches to $(\omega_1, <2^\omega)$-trees.*

Proof. Assume not. Let $\dot{b}$ be a name for a new branch through $T \subseteq H_{\omega_1}$ an $(\omega_1, < 2^\omega)$-tree; let p be a condition forcing that $\dot{b}$ is a new cofinal branch through $\check{T}$. Let θ verify the subcompleteness of $\mathbb{P}$ and let's place ourselves in the standard setup:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where X is countable and $\overline{N}$ is full.
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{p}, \dot{\overline{b}}) = \theta, \mathbb{P}, T, p, \dot{b}$.

Let $\alpha = \omega_1^{\overline{N}}$, the critical point of the embedding σ. By elementarity, we have that $\overline{p}$ forces $\dot{\overline{b}}$ to be a new branch over $\overline{N}$. We will construct continuum-many generics $\overline{G}_r$ for $\overline{\mathbb{P}}$ over $\overline{N}$, indexed by reals, each of which will correspond to continuum-many different values of the generic branch on level α of the tree T in N, to obtain a contradiction.

Toward this end, enumerate the dense sets $\langle D_n \mid n < \omega \rangle$ of $\overline{\mathbb{P}}$ that belong to $\overline{N}$, so that $\overline{p} \in D_0$.

We would like to construct binary trees of conditions in $\overline{\mathbb{P}}$ and branches in $\overline{T}$: $P = \langle \overline{p}_x \mid x \in 2^{<\omega} \rangle$, $B = \langle \overline{b}_x \mid x \in 2^{<\omega} \rangle$ such that, letting $|x|$ be the length of x, we have the following:

- $\overline{p}_x \in D_{|x|}$
- $x \subseteq y \implies \overline{p}_y \leq \overline{p}_x \leq \overline{p}$
- for some $\beta > |x|$, $p_x \Vdash \dot{\overline{b}} \restriction \check{\beta} = \overline{b}_x$
- $\overline{b}_{x^\frown \langle 0 \rangle} \neq \overline{b}_{x^\frown \langle 1 \rangle}$.

To do this, let $\overline{p}_0 := \overline{p} \in \overline{N}$. As we noted in the proof of Lemma 3.2, there must be two conditions $\overline{q}_1^0 \perp \overline{q}_1^1$, both extending $\overline{p}_0$, that decide the value of the branch $\dot{\overline{b}}$ differently. This always has to be true since these conditions extend $\overline{p}$, that forces that there is a new cofinal branch $\dot{\overline{b}}$. In particular, by our reasoning there is $\beta < \overline{\omega_1}$, and two conditions $\overline{q}_1^0 \leq \overline{p}_0$ and $\overline{q}_1^1 \leq \overline{p}_0$ such that

$$\overline{q}_1^0 \Vdash \dot{\overline{b}}(\beta) = t^0 \quad \text{and} \quad \overline{q}_1^1 \Vdash \dot{\overline{b}}(\beta) = t^1,$$

where $t^0 \neq t^1$. Let $\overline{b}_{\langle 0 \rangle}$ be the branch in $\overline{T}$ below t^0 and $\overline{b}_{\langle 1 \rangle}$ be the branch in $\overline{T}$ below t^1. It must be that $\overline{b}_{\langle 0 \rangle} \neq \overline{b}_{\langle 1 \rangle}$. Moreover we may extend $\overline{q}_1^0$ and $\overline{q}_1^1$ to conditions $\overline{p}_{\langle 0 \rangle}, \overline{p}_{\langle 1 \rangle} \in \overline{D}_1$.

Continuing in such a fashion, we may recursively continue to define $\overline{p}_x$ for $x \in 2^{<\omega}$.

In particular, suppose $\overline{p}_x$ and b_x are defined for x of length n. By our reasoning above there is $\beta > |x|$, and two conditions above each $\overline{p}_x$, such that $\overline{q}_x^0 \leq \overline{p}_x$ and $\overline{q}_x^1 \leq \overline{p}_x$ so that

$$\overline{q}_x^0 \Vdash \dot{\overline{b}}(\beta) = t^0 \quad \text{and} \quad \overline{q}_x^1 \Vdash \dot{\overline{b}}(\beta) = t^1,$$

where $t^0 \neq t^1$. Let $\overline{b}_{x^\frown \langle 0 \rangle}$ be the branch in $\overline{T}$ below t^0 and $\overline{b}_{x^\frown \langle 1 \rangle}$ be the branch in $\overline{T}$ below t^1. Extending each of these incompatible conditions so as to land in the $(n+1)$th dense set, we find $\overline{p}_{x^\frown \langle 0 \rangle} \leq \overline{q}_x^0$ and $\overline{p}_{x^\frown \langle 1 \rangle} \leq \overline{q}_x^1$ such that $\overline{p}_{x^\frown \langle 0 \rangle}, \overline{p}_{x^\frown \langle 1 \rangle} \in D_{n+1}$. Then, as desired, for each x of length n we have that $\overline{p}_{x^\frown \langle 0 \rangle}, \overline{p}_{x^\frown \langle 1 \rangle} \in D_{n+1}$. We've also designed it so that $\overline{p}_{x^\frown \langle 1 \rangle} \leq \overline{p}_x \leq \overline{p}$ and $\overline{p}_{x^\frown \langle 0 \rangle} \leq \overline{p}_x \leq \overline{p}$. Since we know that there is some m' such that $\overline{p}_x \Vdash \dot{\overline{b}} \restriction m' = \overline{b}_x$ we know that $p_{x^\frown \langle 1 \rangle}$ and $p_{x^\frown \langle 0 \rangle}$ force the same thing since they both extend $\overline{p}_x$. Thus our construction gives us that $\overline{p}_{x^\frown \langle 0 \rangle} \Vdash \dot{\overline{b}} \restriction m = \overline{b}_{x^\frown \langle 0 \rangle}$ and $\overline{p}_{x^\frown \langle 1 \rangle} \Vdash \dot{\overline{b}} \restriction m' = \overline{b}_x$.

So we have our binary trees P and B as desired. Any chain of conditions in the binary tree P will generate a generic filter $\overline{G}_r$; every real $r : \omega \to 2$ codes a path in the binary tree of conditions generating the generic. This is because our conditions were chosen to meet all of the dense sets in our list. Moreover, each generic filter $\overline{G}_r$ corresponds to a branch $\overline{b}_r$, where for each initial segment

t of r satisfies that for some $\beta > |x|$, $\bar{b}_r \restriction m = \bar{b}_x$. Because of how we chose P, this gives us that

$$\overline{N}[\overline{G}] \models \bar{\dot{b}}^{\overline{G}} = \bar{b}_r.$$

For each r let $\bar{\dot{b}}^{\overline{G}_r} = \bar{b}_r$. Since $\mathbb{P}$ is subcomplete, for each r there is a condition $q_r \in \mathbb{P}$ such that whenever G is $\mathbb{P}$-generic with $q_r \in G$, by subcompleteness we have $\sigma_r \in V[G]$ such that:

- $\sigma_r : \overline{N} \prec N$
- $\sigma_r(\bar{\theta}, \overline{\mathbb{P}}, \overline{T}, \bar{p}, \bar{\dot{b}}) = \theta, \mathbb{P}, T, p, \dot{b}$
- $\sigma_r \text{``} \overline{G}_r \subseteq G$.

So below each q_r there is a lift $\sigma_r^* : \overline{N}[\overline{G}_r] \prec N[G]$ extending σ_r with $\sigma_r^*(\bar{b}_r) = \sigma_r(\bar{\dot{b}})^G = \dot{b}^G = b_r$, and $\sigma_r^*(\overline{T}) = \sigma_r(\overline{T})^G = T$.

This means that we may force over N with $\mathbb{P}$ to obtain continuum many cofinal branches through the tree, $\langle b_r \mid r \in 2^\omega \rangle$. Each of these branches must, of course, have a node on level α. Since each cofinal branch $\bar{b}_r$ is unique, and since α is the critical point, this means that there are 2^ω-many nodes on level α of the tree T in N, a contradiction.   □

Note that Proposition 4.5 shows that if CH fails, then the previous theorem is optimal, that is, it cannot be extended to $(\omega_1, \leq 2^\omega)$-trees.

Next, we will look at the preservation of wide Aronszajn trees from a different angle, and show that under CH, dropping the restriction to trees with countable levels amounts to making a statement about a certain form of generic absoluteness which we introduce in the following.

Definition 4.7. Let n be a natural number, let $\mathbb{P}$ be a notion of forcing, and let κ be a cardinal. Then $\mathbb{P}$-*generic* $\boldsymbol{\Sigma}_n^1(\kappa)$-*absoluteness* is the statement that for any model $M = \langle M, \vec{A} \rangle$ of size κ for a countable first order language and every Σ_n^1-sentence φ over the language of M, the following holds:

$$(M \models \varphi)^V \iff 1_{\mathbb{P}} \Vdash_{\mathbb{P}} (M \models \varphi)$$

Note that we don't distinguish between the first and second order satisfaction symbol.

For a forcing class Γ, Γ-generic $\boldsymbol{\Sigma}_n^1(\kappa)$-absoluteness is the statement that $\mathbb{P}$-generic $\boldsymbol{\Sigma}_n^1(\kappa)$-absoluteness holds for every $\mathbb{P} \in \Gamma$. The classes of interest to us are the classes of ccc, proper, semi-proper, stationary set preserving or subcomplete forcings.

We will mostly be interested in $\boldsymbol{\Sigma}_1^1(\kappa)$-absoluteness. Note that by upward absoluteness, $\boldsymbol{\Sigma}_1^1(\kappa)$-absoluteness for a forcing notion $\mathbb{P}$ can be equivalently expressed by saying that for M as in the above definition and a Σ_1^1-formula φ, if $M \models \varphi$ holds in every forcing extension by $\mathbb{P}$, then $M \models \varphi$ holds in V. It is a ZFC fact that countably-closed $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness holds, and more generally, $<\kappa$-closed $\boldsymbol{\Sigma}_1^1(\kappa)$-absoluteness holds, for regular κ, see [2]. Much is known about the case $\kappa = \omega$. For example, by Shoenfield absoluteness, $\mathbb{P}$-generic $\boldsymbol{\Sigma}_2^1(\omega)$-generic absoluteness holds for any forcing notion $\mathbb{P}$. Here, we will mostly be interested in the case $\kappa = \omega_1$. The following lemma gives an equivalent characterization under CH for forcing notions which do not add reals, in terms of preserving wide Aronszajn trees.

Lemma 4.8. *Assume* CH. *Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent.*

(1) *Whenever T is an $(\omega_1, \leq \omega_1)$-Aronszajn tree, then it is not the case that $1_{\mathbb{P}}$ forces that $\check{T}$ has a cofinal branch, and it is not the case that $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "there is a new real".*

(2) *$\mathbb{P}$-generic $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness holds.*

Proof. The direction $(2) \implies (1)$ is clear: if $\mathbf{\Sigma}_1^1(\omega_1)$-statements are absolute for $\mathbb{P}$, then it cannot be that $1_\mathbb{P}$ forces that a real is added, because otherwise, by CH, one could use a predicate $A \subseteq \omega_1$ which lists all reals, and the Σ_1^1-statement "there is a subset a of ω which is not listed in A" would hold in any forcing extension by $\mathbb{P}$, but not in V. Similarly, let T be an $(\omega_1, \leq\omega_1)$-Aronszajn tree. It cannot be that $1_\mathbb{P}$ forces that $\check{T}$ has a cofinal branch, because otherwise the Σ_1^1 statement "$\check{T}$ has a cofinal branch" would be true in $V^\mathbb{P}$ but not in V - the nodes of T can be assumed to be countable ordinals, and the tree ordering can be used as a binary predicate to express this.

Let's prove $(1) \implies (2)$. Since we are assuming CH, we have that for any $(\omega_1, \leq 2^\omega)$-tree T, it is not the case that $1_\mathbb{P}$ forces that $\check{T}$ has a cofinal branch. Upward absoluteness between V and $V^\mathbb{P}$ clearly holds for Σ_1^1-statements. To show downward absoluteness, let $\vec{A}$ be a finite list of finitary predicates on ω_1, $\vec{A} \in V$.

Let $\psi(\vec{A})$ be the following statement:

$$\exists X \; (\omega_1, \vec{A}, X) \models \varphi$$

where φ is a first order sentence in the language of set theory with predicate symbols for $\vec{A}$ and X. Assume that $\psi(\vec{A})$ is true in $V^\mathbb{P}$. Let $\dot{X}$ be a $\mathbb{P}$-name such that $1_\mathbb{P}$ forces that $\dot{X}$ is a witness that $\psi(\vec{A})$ holds.

In V, let T be the tree consisting of nodes of the form (α, x) such that $x \subseteq \alpha$, $\alpha < \omega_1$ and $(\alpha, \vec{A} \restriction \alpha, x) \models \varphi(a)$, where $\vec{A} \restriction \alpha$ is the list whose elements are of the form $A_i \cap \alpha^{m_i}$, m_i being the arity of A_i. The tree ordering $\leq$ is defined by setting

$$(\alpha, x) \leq (\beta, y) \iff (\alpha, \vec{A} \restriction \alpha, x) \prec (\beta, \vec{A} \restriction \beta, y).$$

Notice that T has cardinality ω_1 in V, by CH.

Now, if G is an arbitrary filter $\mathbb{P}$-generic over V, then by a standard Löwenheim-Skolem style argument, applied in $V[G]$, the set

$$C = \left\{ \alpha < \omega_1 \mid (\alpha, \vec{A} \restriction \alpha, \dot{X}^G \cap \alpha) \prec (\omega_1, \vec{A}, \dot{X}^G) \right\}$$

is club in ω_1. Thus the set $\left\{ (\alpha, \dot{X}^G \cap \alpha) \mid \alpha \in C \right\}$ defines a cofinal branch through T in $V[G]$, since for all countable α, we have $\dot{X}^G \cap \alpha \in V$ as $\mathbb{P}$ doesn't add reals.

Since this works for any G, and since we assumed that for any $(\omega_1, \leq\omega_1)$-Aronszajn tree, it is not the case that $1_\mathbb{P}$ forces that it has a cofinal branch, it follows that T is not an $(\omega_1, \leq\omega_1)$-Aronszajn tree. Hence, T has a cofinal branch in V, call it b. Let

$$X' = \bigcup \{x \mid \exists \alpha < \omega_1 \; (\alpha, x) \in b\}$$

Since $(\omega_1, \vec{A}, X')$ is the union of an elementary chain of models satisfying φ, this model must also satisfy φ in V, and thus $\psi(\vec{A})$ holds in V as witnessed by X'. $\qquad\square$

The formulation of condition 1. in the previous lemma seems a little cumbersome, and there is a clearer variant of the lemma, using a slightly modified version of $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness, which we define presently.

Definition 4.9. Let $\mathbb{P}$ be a poset and κ a cardinal. Then *strong $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness* is the principle asserting that for any model $M = \langle M, \vec{A} \rangle$ of size κ for a countable first order language and any Σ_1^1-sentence φ over that language, whenever $G \subseteq \mathbb{P}$ is generic over V, then $M \models \varphi$ iff $(M \models \varphi)^{V[G]}$. Similarly, if Γ is a forcing class, then *strong Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness* says that strong $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds for every $\mathbb{P} \in \Gamma$.

If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing, then we say that $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if they produce the same forcing extensions, that is, for every $\mathbb{P}$-generic G, there is a $\mathbb{Q}$-generic H such that $V[G] = V[H]$ and vice versa.

Let us also introduce the notation $\mathbb{P}_{\leq p}$ for the restriction of the ordering of $\mathbb{P}$ to the set of conditions $q \leq p$. Call a forcing class Γ *natural* if whenever $\mathbb{P} \in \Gamma$ and $p \in \mathbb{P}$, then $\mathbb{P}_{\leq p}$ is forcing equivalent to some $\mathbb{Q} \in \Gamma$.

In other words, using Σ_1^1-upward absoluteness, for a model M as above and a Σ_1^1 sentence over the language of M, *strong* $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness says that for any $\mathbb{P}$-generic G, if $(M \models \varphi)^{V[G]}$ holds, then $M \models \varphi$ holds. Regular $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness, on the other hand, says that if for every $\mathbb{P}$-generic G, $(M \models \varphi)^{V[G]}$ holds, then $M \models \varphi$ holds. To clarify the difference, let's consider the class Γ of all forcing notions $\mathbb{P}$ such that $\mathbb{P}$ is ccc and $\mathbb{P}$ has an atom. Then Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$ always holds, because for $\mathbb{P} \in \Gamma$ and M, φ as before, if $(M \models \varphi)^{V[G]}$ holds for every $\mathbb{P}$-generic G, then it holds for some G that contains an atom, in which case $V[G] = V$, and thus, $M \models \varphi$. On the other hand, strong Γ-generic $\mathbf{\Sigma}_1^1(\omega_1)$-absoluteness implies that every Aronszajn tree is special, because for an Aronszajn tree T, we can consider the lottery sum of a ccc forcing notion specializing T and a trivial forcing, consisting of one atom. That forcing notion is in Γ. Let G be generic for the nontrivial part of the forcing. If we let M be an elementary submodel of H_{ω_1} of size ω_1, equipped with T as a predicate, then the existence of a function specializing T can be expressed as a Σ_1^1 sentence over M, and it holds in $V[G]$, hence in V, which means that T is special in V.

It is easy to see that the notion of forcing equivalence introduced in the previous definition is first order expressible. Clearly, if $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent, then (strong) $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness is equivalent to (strong) $\mathbb{Q}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness. The following is essentially a reformulation of [7, Corollary 3.11].

Fact 4.10. *The class of subcomplete forcing notions is natural.*

Proof. If $\mathbb{P}$ is subcomplete and $p \in \mathbb{P}$, then by [7, Corollary 3.11], $\mathbb{P}_{\leq p}$ is $\delta(\mathbb{P})$-subcomplete (in the sense of [5]). This means that $\mathbb{P}_{\leq p}$ is essentially subcomplete in the sense of [5, Definition 2.2], and [5, Observation 2.4] then implies that $\mathbb{P}_{\leq p}$ is forcing equivalent to a subcomplete forcing. $\square$

Let us make a simple observation relating strong absoluteness to the previously introduced version of absoluteness.

Observation 4.11. *Let κ be a cardinal, let $\mathbb{P}$ be a forcing notion, and let Γ be a forcing class.*

(1) *Strong $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness is equivalent to saying that $\{\mathbb{P}_{\leq p} \mid p \in \mathbb{P}\}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds.*

(2) *If Γ is natural, then strong Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness.*

(3) *If Γ is either the class of all c.c.c., proper, semi-proper, countably closed, stationary set preserving or subcomplete forcing notions, then Γ is natural, and hence strong Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness.*

Proof. For (1), assume that strong $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds, let M be a κ-sized model of a countable first order language, let φ be a Σ_1^1-sentence of in that language, and let $p \in \mathbb{P}$ be a condition. Assume that $1_{\mathbb{P}_{\leq p}}$ forces (with respect to $\mathbb{P}_{\leq p}$) that $M \models \varphi$. If $G \ni p$ is $\mathbb{P}$-generic, then in $V[G]$, it is the case that $M \models \varphi$. Thus, by strong $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness, it is true in V that $M \models \varphi$. This shows that $\{\mathbb{P}_{\leq p} \mid p \in \mathbb{P}\}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds.

For the converse, assume that $\{\mathbb{P}_{\leq p} \mid p \in \mathbb{P}\}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds, let M and φ be as before, let G be $\mathbb{P}$-generic over V, and assume that in V$[G]$, it is the case that $M \models \varphi$. There is then a condition $p \in G$ which forces that $M \models \varphi$. But then it follows that $1_{\mathbb{P}_{\leq p}}$ forces that $M \models \varphi$. Hence, by $\mathbb{P}_{\leq p}$-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness, it follows that $M \models \varphi$ holds in V.

Now (2) and (3) follow immediately from (1), using Fact 4.10 and the remark preceding this fact. $\qquad\square$

All of this could be done for Γ-generic $\mathbf{\Sigma}_n^1$-absoluteness as well, of course, but we will not need this generality here. We obtain the following version of the previous lemma.

Lemma 4.12. *Assume* CH. *Let* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent.*

(1) $\mathbb{P}$ *preserves* $(\omega_1, \leq\!\omega_1)$-*Aronszajn trees and does not add reals.*
(2) Strong $\mathbb{P}$-*generic* $\mathbf{\Sigma}_1^1(\omega_1)$-*absoluteness holds.*

There is a natural version of the second order absoluteness properties introduced where one talks about a certain canonical structure M, defined by a formula to be re-interpreted in V$[G]$. For example, let's define $\mathbb{P}$-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness to mean

$$(\langle H_{\omega_1}, \vec{A}\rangle \models \varphi)^V \iff (\langle H_{\omega_1}, \vec{A}\rangle \models \varphi)^{V^{\mathbb{P}}}$$

for any finite set of finitary predicates $\vec{A}$ and any Σ_1^1-sentence φ, where H_{ω_1} is *re-interpreted* in $V^{\mathbb{P}}$ on the right hand side, rather than working with the same model on both sides. To be clear, the statement on the right hand side of the displayed equivalence means that $1_{\mathbb{P}}$ forces that $\langle H_{\omega_1}, \vec{A}\rangle$, in the sense of the forcing extension, satisfies φ. Further, Γ-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness means that this holds for every $\mathbb{P} \in \Gamma$.

It turns out that Γ-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness, where $\kappa = 2^\omega$, if Γ is a natural forcing class. Here, and in the following, we will indicate second order quantification by upper case variables. To see the claimed equivalence, first suppose Γ-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness holds. It follows that $H_{\omega_1} = H_{\omega_1}^{V[G]}$ whenever G is generic for some $\mathbb{P} \in \Gamma$. Clearly, $H_{\omega_1} \subseteq H_{\omega_1}^{V^{\mathbb{P}}}$, for any $\mathbb{P}$. But if we had $H_{\omega_1}^V \subsetneq M^{V[G]}$, then we could take $A = H_{\omega_1}^V$, and in V$[G]$ it would be true that $\langle H_{\omega_1}^{V[G]}, A\rangle \models \exists x \quad \neg\dot{A}(x)$, but clearly this is not true in V. So it follows that Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness also holds, because $H_{\omega_1}^V$ has size κ and doesn't change by forcing in Γ.

To see the converse, assume that Γ-generic $\mathbf{\Sigma}_1^1(\kappa)$-absoluteness holds. We claim that it follows that $H_{\omega_1} = H_{\omega_1}^{V[G]}$ whenever G is generic for some $\mathbb{P} \in \Gamma$. Suppose otherwise. Then $H_{\omega_1} \subsetneq H_{\omega_1}^{V[G]}$ for some G generic for some $\mathbb{P} \in \Gamma$, which means that V$[G]$ has a new real. But then, in V$[G]$, the second order formula $\exists X \subseteq \omega \quad \forall x \quad x \neq X$, holds in the structure $\langle H_{\omega_1}^V, \in\rangle$, while this is not true in V. Thus, it follows that Γ-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness holds.

Remark 4.13. In the Lemmas 4.8 and 4.12, $\mathbf{\Sigma}_1^1(\omega_1)$-absoluteness can be replaced with $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness, since under CH, H_{ω_1} has size ω_1. On the other hand, if CH fails, then $\mathcal{A}dd(\omega_1, 1)$-generic $\mathbf{\Sigma}_1^1(H_{\omega_1})$-absoluteness fails, even though $\mathcal{A}dd(\omega_1, 1)$, being countably closed, preserves Aronszajn trees of any width. This is because $\mathcal{A}dd(\omega_1, 1)$ forces CH, and this can be expressed as a $\mathbf{\Sigma}_1^1(H_{\omega_1})$ statement true in $V^{\mathcal{A}dd(\omega_1,1)}$ but false in V.

We will now explore a fruitful connection between subcomplete generic $\mathbf{\Sigma}_1^1(\omega_1)$-absoluteness and the bounded subcomplete forcing axiom.

The bounded forcing axiom was originally introduced in [8] in the context of proper forcing. The bounded forcing axiom for a poset $\mathbb{P}$ says that if $\mathbb{B}$ is the complete Boolean algebra of $\mathbb{P}$, then for

any collection of up to ω_1 many maximal antichains in $\mathbb{B}$, each having size at most ω_1, there is a filter in $\mathbb{B}$ that meets each antichain. The bounded forcing axiom for a class Γ of forcings says that each $\mathbb{P} \in \Gamma$ satisfies the bounded forcing axiom for $\mathbb{P}$. We write BSCFA for the bounded forcing axiom for the class of all subcomplete forcings. The following is a version of a characterization of the bounded forcing axiom, due to Bagaria, tailored to the present context.

Theorem 4.14 ([1, Theorem 5]). *Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:*

(1) *The bounded forcing axiom holds for $\mathbb{P}$.*
(2) *$\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds, meaning: if $\varphi(\vec{x})$ is a Σ_1-formula in the language of set theory and $\vec{a} \in H_{\omega_2}$, then $\varphi(\vec{a})$ iff $\Vdash_{\mathbb{P}} \varphi(\vec{a})$.*

A very useful way of reformulating this theorem is as follows.

Theorem 4.15. *Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:*

(1) *The strong bounded forcing axiom holds for $\mathbb{P}$, meaning that the bounded forcing axiom holds for $\{\mathbb{P}_{\leq p} \mid p \in \mathbb{P}\}$.*
(2) *Strong $\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds: if G is $\mathbb{P}$-generic over V, then*

$$\langle H_{\omega_2}, \in \rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in \rangle^{V[G]}.$$

Of course, in the previous theorem, H_{ω_2} is reinterpreted in $V[G]$ on the right hand side of (2).. Thus, (2). of Theorem 4.15 can be taken as a characterization of the bounded forcing axiom for a natural forcing class Γ.

We will show next that property (2). is equivalent to $\mathbb{P}$-generic (or strong $\mathbb{P}$-generic in the case of Theorem 4.15) $\mathbf{\Sigma}_1^1(\omega_1)$-absoluteness. For this, we will need an observation that is probably a folklore fact, but since it is important in the present context, we will provide a proof. We will work with the following natural way of coding elements of H_{ω_2}.

Definition 4.16. A *code* is a pair $\langle R, \alpha \rangle$, where $R \subset \omega_1 \times \omega_1$, $\alpha < \omega_1$ and $\langle \omega_1, R \rangle$ is extensional and well-founded.

If $\langle R, \alpha \rangle$ is a code, then let U_R, σ_R be the unique objects (given by Mostowski's isomorphism theorem) such that U_R is transitive and $\sigma_R : \langle U_R, \in \restriction U_R \rangle \longrightarrow \langle \omega_1, R \rangle$ is an isomorphism. The *set coded by* $\langle R, \alpha \rangle$ is

$$c_{R,\alpha} = \sigma_R^{-1}(\alpha).$$

Clearly, every member of H_{ω_2} has a code, and only members of H_{ω_2} have codes. Using codes, Σ_1 statements over $\langle H_{\omega_2}, \in \rangle$ can be translated into Σ_1^1 statements over $\langle H_{\omega_1}, \in \rangle$.

Observation 4.17. *Let $\varphi(v_0, \ldots, v_{n-1})$ be a Σ_1-formula. Then there is a Σ_1^1-formula φ^c with free variables $X_0, x_0 \ldots, X_{n-1}, x_{n-1}$ (upper case variables being second order and lower case ones being first order) such that the following holds. Let $a_0, \ldots, a_{n-1} \in H_{\omega_2}$, and let $\langle R_0, \alpha_0 \rangle, \ldots, \langle R_{n-1}, \alpha_{n-1} \rangle$ be codes, such that a_i is coded by $\langle R_i, \alpha_i \rangle$, for $i < n$. Then*

$$\langle H_{\omega_2}, \in \rangle \models \varphi(a_0, \ldots, a_{n-1}) \iff \langle L_{\omega_1}, \in \rangle \models \varphi^c(R_0, \alpha_0, \ldots, R_{n-1}, \alpha_{n-1}).$$

Proof. The construction of φ^c proceeds by induction on φ. We will assume that φ is presented in such a way that the only subformulas of φ that are negated are atomic. Any formula can be written in this form.

If φ is of the form $v_0 = v_1$, then $\varphi^c(X_0, x_0, X_1, x_1)$ is defined in such a way that it expresses: there is an injective function $F : \omega_1 \longrightarrow \omega_1$ with $F(x_0) = x_1$, such that whenever $\beta_0 X_0 \beta_1 \ldots X_0 \beta_m X_0 x_0$, then $F(\beta_0) X_1 F(\beta_1) \ldots X_1 F(\beta_m) R_1 x_1$ and vice versa. Expressing the existence of such a function

requires a second order existential quantification. Hence, the resulting formula $\varphi^c(X_0, x_0, X_1, x_1)$ can be written as a Σ_1^1 formula.

If φ is of the form $v_0 \in v_1$, then $\varphi^c(X_0, x_0, X_1, x_1)$ is defined to express: there is a $\beta < \omega_1$ such that $\beta X_1 x_1$, and such that the sentence of the form $(v_0 = v_1)^c$ holds of X_0, x_0, X_1, β (reducing to the previous case). The second order existential quantification occurring in $(v_0 = v_1)^c$ can be pushed in front of the first order quantification ("there exists a $\beta < \omega_1$"), in this case simply because both are existential quantifications.

If φ is of the form $\neg(v_0 = v_1)$, then $\varphi^c(X_0, x_0, X_1, x_1)$ is defined to express: there are U_0, U_1, F such that U_0 is closed under X_0-predecessors, U_1 is closed under X_1-predecessors and $F : \langle U_0, X_0 \cap U_0^2 \rangle \longrightarrow \langle U_1, X_1 \cap U_1^2 \rangle$ is a maximal isomorphism, meaning that F cannot be expanded beyond U_0, and it is not the case that $x_0 \in U_0$, $x_1 \in U_1$ and $F(x_0) = x_1$.

If φ is of the form $\neg(v_0 \in v_1)$, then this can be expressed equivalently by $\forall v \in v_1 \neg(v_0 = v)$. We already know how to translate $\neg(v_0 = v)$, and we can then use the definition in the case of bounded quantification below.

The inductive steps corresponding to the logical connectives $\wedge$ and $\vee$ can be dealt with in the obvious way, setting $(\varphi \wedge \psi)^c = \varphi^c \wedge \psi^c$ and $(\varphi \vee \psi)^c = \varphi^c \vee \psi^c$.

Let's look at the case that φ is of the form $\forall u \in w \;\; \psi(u, w, v_0, \ldots, v_{n-1})$. Define the formula $\varphi^c(Y, y, X_0, x_0, \ldots, X_{n-1}, x_{n-1})$ to express: for all $\beta Y y$, $\psi(u, w, v_0, \ldots, v_{n-1})^c$ is true of $Y, \beta, Y, y, X_0, x_0, \ldots, X_{n-1}, x_{n-1}$. The resulting formula has a universal first order quantification over a Σ_1^1 formula. Since ω_1-sequences of subsets of ω_1 can be coded by single subsets of ω_1, the second order quantification can be pulled out in front of the first order quantifier, resulting in a Σ_1^1 formula.

The case of existential bounded quantification is easier, so we omit it here.

Thus, we have described how to translate Σ_0-formulas. The remaining case is that φ is of the form $\exists u \;\; \psi(u, v_0, \ldots, v_{n-1})$, where $\psi(u, v_0, \ldots, v_{n-1})$ is a Σ_0-formula. In this case, the translated formula $\varphi^c(X_0, x_0, \ldots, X_{n-1}, x_{n-1})$ expresses that there are an S (this is second order) and an α such that $\langle S, \alpha \rangle$ is a code and such that $\psi^c(S, \alpha, X_0, x_0 \ldots, X_{n-1}, x_{n-1})$ holds. Expressing that $\langle S, \alpha \rangle$ is a code amounts to saying that it is extensional, which is first order expressible, and that it is well-founded. In order to do this, we use an additional existential second order quantification, saying that there is an $F \subseteq \omega_1 \times \omega_1 \times \omega_1$ such that, if we set $f_\xi = \{\langle \gamma, \delta \rangle \mid \langle \xi, \gamma, \delta \rangle \in F\}$, then $f_\xi : \langle \xi, S \cap (\xi \times \xi) \rangle \longrightarrow \langle \omega_1, < \rangle$ is an order preserving function, for every $\xi < \omega_1$. This can be expressed in a first order way, using the predicates S and F inside L_{ω_1}, and it follows that S is well-founded, because any decreasing ω-sequence in S would be bounded by some $\xi < \omega_1$, contradicting that f_ξ is order preserving. And if S is well-founded, then so is every initial segment $\langle \xi, S \cap (\xi \times \xi) \rangle$, hence there is an f_ξ as described. $\qquad \square$

Note that the proof of the previous observation contained a concrete translation procedure $\varphi \mapsto \varphi^c$ which ZFC-provably has the properties described, that is, the same translation procedure works in any ZFC-model. Note also that we could have used any other model of (a fairly weak fragment of) ZFC$^-$ that contains ω_1 in place of L_{ω_1}. We will use this uniformity of the translation procedure in the following proof.

Observation 4.18. *Let $\mathbb{P}$ be a notion of forcing that preserves ω_1. Then the following are equivalent:*

(1) *$\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds.*
(2) *$\mathbb{P}$-generic $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness holds.*

Proof. The implication $(1) \implies (2)$ is easy to see, because $\mathcal{P}(\omega_1) \subseteq H_{\omega_2}$, so a second order existential quantification over ω_1 can be expressed as a first order existential quantification over the elements of H_{ω_2} which are subsets of ω_1.

For the direction $(2) \implies (1)$, let $\vec{a} = a_0, \ldots, a_{n-1}$ be a list of parameters in H_{ω_2}, $\varphi(\vec{x})$ a Σ_1-formula, and suppose that

$$\langle H_{\omega_2}^{\mathrm{V}[G]}, \in \rangle \models \varphi(\vec{a})$$

whenever G is $\mathbb{P}$-generic over V. In V, let $\langle R_0, \alpha_0 \rangle, \ldots, \langle R_{n-1}, \alpha_{n-1} \rangle$ be codes for $a_0, \ldots, a_{n-1}$, respectively, and let φ^c be the Σ_1^1 translation of φ given by Observation 4.17. Since the same codes work in V$[G]$, the translation procedure is uniform, we can conclude that

$$(L_{\omega_1} \models \varphi^c(R_0, \alpha_0, \ldots, R_{n-1}, \alpha_{n-1}))^{\mathrm{V}[G]}.$$

Clearly, φ^c can be replaced by a Σ_1^1-sentence $\tilde{\varphi}^c$ in the language with predicate/constant symbols $\dot{C}_0, \dot{c}_0, \ldots, \dot{C}_{n-1}, \dot{c}_{n-1}$ for the codes, so that we get

$$(\langle L_{\omega_1}, \in, R_0, \alpha_0 \ldots, R_{n-1}, \alpha_{n-1} \rangle \models \tilde{\varphi}^c)^{\mathrm{V}[G]}$$

Since this model is in V, and it has size ω_1 there, noting that this holds for every $\mathbb{P}$-generic G, it follows from $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness that

$$\langle L_{\omega_1}, \in, R_0, \alpha_0, \ldots, R_{n-1}, \alpha_{n-1} \rangle \models \tilde{\varphi}^c$$

holds in V, that is,

$$\langle L_{\omega_1}, \in \rangle \models \varphi^c(R_0, \alpha_0, \ldots, R_{n-1}, \alpha_{n-1})$$

holds in V, which means that, undoing the translation, which is uniform, we get that

$$\langle H_{\omega_2}, \in \rangle \models \varphi(\vec{a})$$

as desired. $\square$

The same proof shows the equivalence of the strong forms of these generic absoluteness conditions. Note that either condition (1) or (2) of the following observation implies that $\mathbb{P}$ preserves ω_1.

Observation 4.19. *Let $\mathbb{P}$ be a notion of forcing. Then the following are equivalent:*

(1) *Whenever G is generic for $\mathbb{P}$ over V, we have that*

$$\langle H_{\omega_2}, \in \rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in \rangle^{\mathrm{V}[G]}.$$

(2) *Strong $\mathbb{P}$-generic $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness holds.*

Note that either 1. or 2. in the previous observation imply that $\mathbb{P}$ preserves ω_1.

In general, we have the following simple observation.

Observation 4.20. *Let Γ be a natural class of forcing notions, and consider the following statements.*

(1) BFA_Γ.

(2) *Γ-generic $\boldsymbol{\Sigma}_1^1(\omega_1)$-absoluteness.*

(3) *Forcings in Γ preserve $(\omega_1, \leq_{\omega_1})$-Aronszajn trees.*

Then $(1) \iff (2) \implies (3)$.

Proof. $(1) \iff (2)$ follows from Observations 4.11.(2), 4.19 and Theorem 4.15. The implication $(2) \implies (3)$ follows because if T were an $(\omega_1, \leq_{\omega_1})$-Aronszajn tree that acquires a cofinal branch in V$[G]$, where G is $\mathbb{P}$-generic for some $\mathbb{P} \in \Gamma$, then the existence of such a branch would be a $\Sigma_1^1(T)$ statement true in V$[G]$ but false in V, contradicting (2). $\square$

So, writing BSCFA for the bounded subcomplete forcing axiom, we get the following equivalences, using the fact that the class of subcomplete forcing notions is natural (see Observation 4.11), as well as Theorem 4.15, Lemma 4.12 and Observation 4.18.

Theorem 4.21. *Assuming* CH*, the following are equivalent.*

(1) BSCFA*.*
(2) Subcomplete generic $\Sigma^1_1(\omega_1)$-absoluteness.
(3) Subcomplete forcing preserves $(\omega_1, \leq\omega_1)$-Aronszajn trees.

Actually, (1) and (2) are equivalent, regardless of whether CH *holds or not, and (1)/(2) always implies (3), but for the converse, we need* CH*, since the failure of* CH *implies (3) (by Theorem 4.6), but not (1)/(2).*

Obviously, this theorem generalizes to any natural class Γ of forcing notions that don't add reals.

Let us make some remarks on the consistency strength of BSCFA and its relationship to CH. It was shown in [3] that the consistency strength of BSCFA is a reflecting cardinal. Moreover, looking at the construction there, one sees that the consistency strengths of BSCFA and of BSCFA + CH are the same. Namely, if BSCFA holds, then $\kappa = \omega_2$ is reflecting in L, and one can perform a subcomplete (in the sense of L) forcing over L to reach a model $L[g]$ where $\omega_2 = \kappa$. Since the forcing is subcomplete in L, it does not add reals, and hence preserves CH. In a sense, in the context of BSCFA, it is natural to assume CH, since it holds in the "natural" models, and since CH is implied by natural strengthenings of BSCFA, such as the resurrection axiom or the maximality principle for subcomplete forcing, see [13], [4].

However, BSCFA does not imply CH, and in fact, the first author, in joint work with Corey Switzer, observed that the consistency strength of BSCFA + ¬CH is equal to that of BSCFA, that is, the existence of a reflecting cardinal. Thus, assuming ¬CH, condition 3. in the previous theorem holds, while the consistency strength of the equivalent conditions 1. and 2. is a reflecting cardinal, showing that the implication cannot be reversed.

This puts us in a position to answer Question 4.2, asking whether subcomplete forcing may add a cofinal branch to an $(\omega_1, \leq\omega_1)$-Aronszajn tree, completely. Recall Theorem 4.6, which gives us part 1. of the following theorem.

Theorem 4.22. *Splitting in two cases, we have:*

(1) If CH *fails, then subcomplete forcing preserves $(\omega_1, \leq\omega_1)$-Aronszajn trees.*
(2) If CH *holds, then subcomplete forcing preserves $(\omega_1, \leq\omega_1)$-Aronszajn trees iff* BSCFA *holds.*

It is now interesting to explore the relationships between bounded forcing axioms, the forms of generic Σ^1_1-absoluteness introduced above, and the property of $(\omega_1, \leq\omega_1)$-Aronszajn tree preservation for other canonical classes of forcing. Let's first observe some limitations on Γ-generic $\Sigma^1_1(\kappa)$-absoluteness.

Observation 4.23. *(1) If* CH *fails, then $\mathcal{A}dd(\omega_1, 1)$-generic $\Sigma^1_1(\omega_2)$-absoluteness fails.*
(2) $\mathrm{Col}(\omega_1, \omega_2)$-generic $\Sigma^1_1(\omega_2)$-absoluteness fails.
(3) If $\mathbb{P}$ is a forcing that adds a real, then $\mathbb{P}$-generic $\Sigma^1_1(2^\omega)$-absoluteness fails.

Proof. For (1), if CH fails, we can take M to be an elementary submodel of H_{ω_1} of size ω_2, containing ω_2 many distinct reals. If G is generic for $\mathcal{A}dd(\omega_1, 1)$ over V, then in V$[G]$, M satisfies the Σ^1_1 statement that there is a function F with domain ω_1 that lists all the reals of M, but it is false in V.

For (2), we can take M to be the structure ω_2, equipped with an ordinal pairing function and a constant symbol for ω_1. Then if G is $\mathrm{Col}(\omega_1, \omega_2)$-generic over V, in $\mathrm{V}[G]$, the structure M satisfies the Σ_1^1 sentence expressing that there is a surjection F from ω_1 onto the universe of M. This is not true in V.

For (3), if $\mathbb{P}$ adds a real, then the Σ_1^1-formula "there is an $X \subseteq \omega$ such that for all x, $x \neq X$" holds, from the point of view of $\mathrm{V}^{\mathbb{P}}$, in the structure $\langle H_{\omega_1}^{\mathrm{V}}, \in \rangle$, but not from the point of view of V. $\qquad\square$

So, by (2) of the above observation, if Γ is the class of proper, semi-proper or stationary set preserving, subcomplete, or countably closed forcings, then Γ-generic $\Sigma_1^1(\omega_2)$-absoluteness fails.

Theorem 4.24. *Let Γ be the class of proper, semi-proper, stationary set preserving, ccc or subcomplete forcing notions. Consider the following properties.*

(1) BFA_Γ.
(2) Γ-generic $\Sigma_1^1(\omega_1)$-absoluteness.
(3) Forcings in Γ preserve $(\omega_1, \leq\omega_1)$-Aronszajn trees.

Then (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3), but (3) does not imply (1)/(2).

Proof. By Observation 4.20, we know that (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) holds. Let's show that (3) does not imply (2).

In the case of subcomplete forcing, we have already seen that (3) follows from $\neg\mathsf{CH}$, while (1)/(2) have consistency strength a reflecting cardinal.

For the case of ccc forcing, recall that is known that CH is consistent with the statement that every Aronszajn tree is special, see [12]. But if every Aronszajn tree is special, then ccc forcing cannot add a cofinal branch to any $(\omega_1, \leq\kappa)$-Aronszajn tree T, no matter how wide it is: assume $\mathbb{P}$ were a c.c.c. forcing that did. Let $\dot{b}$ be a $\mathbb{P}$-name for a cofinal branch through T, and let $p \in \mathbb{P}$ force this. Let X be the set of members x of T such that some $q \leq p$ forces that $\check{x} \in \dot{b}$. Then X is closed under T-predecessors, because if $x \leq y \in X$ and $q \leq p$ forces that $\check{y} \in \dot{b}$, then q also forces that $\check{x} \in \dot{b}$. Also, the set X has nodes at arbitrarily large heights less than ω_1, since p forces that $\dot{b}$ is a cofinal branch. Moreover, for any $\alpha < \omega_1$, X has at most countably many nodes at level α of T, because for any such node, there is a condition below p that forces that that node is in $\dot{b}$, and these conditions have to be pairwise incompatible, so that the claim follows from the fact that $\mathbb{P}$ is c.c.c. This shows that the restriction $\overline{T}$ of T to X is an ω_1-tree, hence an Aronszajn tree, and hence special. Now we have a contradiction, since $\mathbb{P}_{\leq p}$ adds a branch to $\overline{T}$. This is impossible, since $\mathbb{P}$ preserves ω_1. But now, in any model of CH in which every Aronszajn tree is special, (3) is satisfied, while (1) and (2) fail, since (1)/(2) imply the failure of CH.

To cover the remaining cases, we will show that if Γ is a natural forcing class containing all proper forcing notions, then the assertion that forcing notions in Γ preserve $(\omega_1, \leq\omega_1)$-Aronszajn trees does not imply BFA_Γ. To see this, recall that it follows from MA_{ω_1} that every $(\omega_1, \leq\omega_1)$-Aronszajn tree is special, and hence that every such tree is preserved by every ω_1-preserving forcing. But the consistency strength of MA_{ω_1} is the same as that of ZFC, while the consistency strength of BFA_Γ is at least a reflecting cardinal. $\qquad\square$

Recall that under CH, the versions of the three conditions listed in the previous theorem for subcomplete forcing are equivalent. The proof showed that this is not the case for c.c.c. forcing (for (3) is consistent with CH, in this case, while (1) is not). Of course, subcomplete forcing is the only class considered here whose bounded forcing axiom is consistent with CH, and it is in the context of CH that we have this unusual equivalence between the three conditions.

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