λ-structures and s-structures: Translating the iteration strategies

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Abstract

Continuing the work [Fuc08], I show that the translation functions developed previously map iterable λ-structures to iterable s-structures and vice versa. To this end, I analyze how the translation functions interact with the formation of extender ultrapowers and normal iterations. This analysis makes it possible to translate iterations, and, in a last step, iteration strategies, thus arriving at the result.

1 Introduction

In this article, I continue the work begun in [Fuc08], and the first part is a prerequisite of the current paper. Both of these papers are based on my dissertation. In the first part, I introduced λ-structures and s-structures (for simplicity, I won't distinguish between the potential and "Pseudo" variants of these structures in this introduction). These are closely related to premice in the Friedman-Jensen and the Mitchell-Steel indexing convention, respectively. I developed functions which translate these structures in both directions. The aim of the current paper is to take the analysis of these structures further, turning to iterable λ-structures and s-structures. The main result is that the translation functions work for these structures as well, if an appropriate notion of iterability for λ-structures is chosen. The point is that when forming normal iterations, the model in the iteration tree to which an extender is applied depends on how the critical point of the extender fits into the sequence of the iteration indices of the previously used extenders. Since the indexing of extenders is different in s-structures and λ-structures, this means that the arising iteration trees may have a different structure as well. The solution to this problem is to introduce a notion of normal s-iteration of a λ-structure, which basically mimics the way normal iterations of s-structures are formed.

The paper is organized as follows: In section 2, I recall the main tools of the first part of the paper that will be needed, for the reader's convenience. Section 3 analyzes $\boldsymbol{\Sigma}_1^{(n)}$ definability in a λ-structure M and its counterpart s-structure $N = \mathtt{S}(M)$. This analysis is needed when comparing the outcome of forming fine structural extender-ultrapowers of these structures, which is done in section 4. The formation of such ultrapowers is the successor step in an iteration, which I analyze in the following section 5. There, I introduce the notion of a normal s-iteration of a λ-structure. A lot of things which are essential to get the theory going are verified in that section: That the s'-initial segment condition is preserved under normal s-iterations, and that there is a notion of s-coiteration such that the s-coiteration of normally s-iterable λ-structures terminates. I also show some basic results on the s'-initial segment condition: It is implied by the Z-initial segment condition used in the current Mitchell-Steel variant of premice, so that the notion of s-premice is not unduly restrictive, and it is preserved downwards to Σ_1-embeddable structures. After that, in section 6, I develop a method to "translate" a normal s-iteration $\mathcal{I}$ of a pλ-structure M to a normal iteration $\mathtt{S}(\mathcal{I})$, called the transliteration of $\mathcal{I}$, of the ps-structure $\mathtt{S}(M)$. The transliteration process works in the other direction as well, and it can be used to finally translate normal s-iteration strategies of pλ-structures to normal iteration strategies of ps-structures, and vice versa. So this shows that the translation functions translate normally s-iterable λ-structures to normally iterable s-structures, and vice versa, which is the main result of this paper. The last section collects some results that didn't fit in elsewhere: First, I show that normally iterable Mitchell-Steel-premice are normally iterable s-structures, then I analyze different notions of iterability and argue that transliterations of the arising iterations can be formed also, and finally I compare the procedure of passing to the squash of a type III structure, forming an ultrapower, and then inverting the squash, to the process of passing to its maximal continuation instead in both cases: They are equivalent.

In order to facilitate the orientation of the reader, I added a table of contents right after this introduction, and at the end of the paper, there is an index.

Contents

2 Preliminaries, and a quick review

In this section, I collect results proved in the first part of this paper, [Fuc08], which will be used here as well. The first part is a prerequisite to the current paper, and the latter cannot be understood without knowledge of the former, but let me briefly remind the reader what was done in the first part. I developed functions $\mathtt{S}$ mapping pPλ-structures to pPs-structures and Λ, which is the inverse of $\mathtt{S}$, preserving a considerable amount of fine structure; I'll be more explicit on this matter later. In order to summarize the main results on these functions, let me recall the following definitions. $\mathbf{pP\boldsymbol{\lambda}}$, $\mathbf{P\boldsymbol{\lambda}}$ and $\boldsymbol{\Lambda}$ are the classes of pPλ- Pλ and λ-structures, respectively. Analogously, $\mathbf{pP\boldsymbol{s}}$, $\mathbf{P\boldsymbol{s}}$ and $\mathfrak{S}$ are the classes of pPs-, Ps and s-structures, respectively. For the exact meaning of p and P in the definition of these structures, the reader is referred to the first part of this paper.

Theorem 2.1.

1. $\mathtt{S}$ *is a bijection between* $\mathbf{pP\boldsymbol{\lambda}}$ *and* $\mathbf{pP\boldsymbol{s}}$*, and* Λ *is the inverse of* $\mathtt{S}$*, hence a bijection between* $\mathbf{pP\boldsymbol{s}}$ *and* $\mathbf{pP\boldsymbol{\lambda}}$*.*
2. $\mathtt{S}{\restriction}\mathbf{P\boldsymbol{\lambda}}$ *is a bijection between* $\mathbf{P\boldsymbol{\lambda}}$ *and* $\mathbf{P\boldsymbol{s}}$*.* $\Lambda{\restriction}\ \mathbf{P\boldsymbol{s}}$ *is the inverse of* $\mathtt{S}{\restriction}\ \mathbf{P\boldsymbol{\lambda}}$*, hence a bijection between* $\mathbf{P\boldsymbol{s}}$ *and* $\mathbf{P\boldsymbol{\lambda}}$*.*
3. $\mathtt{S}{\restriction}\boldsymbol{\Lambda}$ *is a bijection between* $\boldsymbol{\Lambda}$ *and* $\mathfrak{S}$*.* $\Lambda{\restriction}\ \mathfrak{S}$ *is the inverse of* $\mathtt{S}{\restriction}\boldsymbol{\Lambda}$*, and hence a bijection between* $\mathfrak{S}$ *and* $\boldsymbol{\Lambda}$*.*

A λ-structure M is a premouse following the Friedman-Jensen indexing scheme, enhanced by an additional predicate, D_M, the use of which is that it allows us to define the function $\mathtt{S}$ restricted to initial segments of M in a simple way. The predicate is the following:

Definition 2.2. Let M be a weak j-ppm. Then let D_M be the set defined by:

$$\begin{aligned} D_M := \{\tau \in M \ \ | \ \ & (\mathrm{Lim}(\tau) \vee \tau = 0) \wedge \\ & \neg(\exists \nu \in M \quad E^M_{\omega\nu} \neq \emptyset \wedge s^+(\nu)^M < \tau \leq \nu)\}. \end{aligned}$$

For $\nu, \gamma \leq \mathrm{ht}(M)$, say that ν *hides* γ *in* M iff $M||\nu$ is active and $s^+(\nu)^M < \gamma \leq \nu$. So D_M consists of 0 and those limit ordinals of M that are not hidden by any $\nu < \mathrm{ht}(M)$.

The main reason why it is possible to work with these enhanced structures is that the function sending a weak j-ppm $\bar{M}$ (this is what is referred to as a pre-premouse in the Jensen approach) to $D_{\bar{M}}$ is what I refer to as an enhancement (an exact definition of this concept is given in the first part). The following lemma is the crucial fact on enhancements:

Lemma 2.3. *Let* $\langle A_M \mid M \text{ is a } j-ppm\rangle$ *be an enhancement. Fix a weak j-ppm* M *and let*

$$\pi : \langle M, A_M\rangle \longrightarrow^*_F \langle N, A\rangle \quad \textit{or} \quad \pi : \langle M, A_M\rangle \longrightarrow_F \langle N, A\rangle.$$

Let N *be transitive. Then* $A = A_N$*.*

Here are some basic facts on the specific enhancement $M \mapsto D_M$.

Lemma 2.4. *Let* M *be a pPλ-structure s.t.* $\mathrm{ht}(M)$ *is a limit ordinal. Then* D_M *is closed and unbounded in* On_M*.*

Corollary 2.5. *Let M be a pPλ-structure. Then*

$$\mathrm{ht}(N) = \begin{cases} \mathrm{otp}(D_M) & \textit{if } M \textit{ is passive,} \\ \cup\,\mathrm{otp}(D_M^*) & \textit{otherwise.} \end{cases}$$

Moreover, $h_M^1(\mathrm{ht}(N)) = |M|$.

If M is an active pPλ-structure, then $s^+(M)$ is the index its top-extender would have in the Mitchell-Steel indexing convention. The following lemma describes when $\mathtt{S}(M||\mu)$ is a segment of $\mathtt{S}(M)$, for $\mu \leq \mathrm{ht}(M)$.

Lemma 2.6. *Let M be a pPλ-structure. Let $\alpha < \mathrm{ht}(M)$. Then the following are equivalent:*

1. *There is no $\mu \leq \mathrm{ht}(M)$ such that $M||\mu$ is active and $s^+(M||\mu) \leq \alpha < \mu$.*
2. *$\mathtt{S}(M||\alpha)$ is a segment of N.*

In particular, this is true if $M||\alpha$ is active and $s^+(M||\alpha) \in D_M$.

The first item of the previous lemma can be easily expressed using the predicate D_M. It follows that the expressive power of a pPλ-structure is strong enough to describe its corresponding pPs-structure in a Σ_1 way, as follows:

Lemma 2.7. *There are Σ_1 formulae $\varphi_{\mathrm{V}}(x,y), \varphi_E(x,y), \varphi_F(x)$ such that for every pPλ-structure $M = \langle \mathrm{J}_\alpha^E, F, D_M\rangle$ with $\alpha > 1$, we have:*

(a) $|\widehat{\mathtt{S}(M)}| = \{z \mid M \models \varphi_{\mathrm{V}}[z, \alpha\dot{-}1]\}$.

(b) $E^{\widehat{\mathtt{S}(M)}} = \{z \mid M \models \varphi_E[x, \alpha\dot{-}1])\}$.

(c) $E_{\mathrm{top}}^{\widehat{\mathtt{S}(M)}} = \{z \mid M \models \varphi_F[z]\}$.

Here, let $\widehat{\mathtt{S}(M)} = \langle |\widehat{\mathtt{S}(M)}|, E^{\widehat{\mathtt{S}(M)}}, E_{\mathrm{top}}^{\widehat{\mathtt{S}(M)}}\rangle$.

Moreover, $\langle E^{\widehat{\mathtt{S}(M||\gamma)}} \mid \gamma < \mathrm{ht}(M)\rangle$ and $\langle |\widehat{\mathtt{S}(M||\gamma)}| : \gamma < \mathrm{ht}(M)\rangle$ are uniformly $\Sigma_1(M)$.

This is the key to the next tool, a way to translate Σ_1-formulae from a pPs-structure to its corresponding pPλ-structure.

Lemma 2.8. *There are functions $\hat{g}$ and g with the following property: If $M = \langle \mathrm{J}_\alpha^E, F, D\rangle$ $(\alpha > 1)$ is a pPλ-structure and φ is a Σ_1 formula, then $\hat{g}(\varphi)$ and $g(\varphi)$ are Σ_1 formulae such that for arbitrary $\vec{x}$, the following holds:*

(a) *If φ is a formula in the language of $\widehat{N}$, then $\hat{g}(\varphi)$ is a formula in the language of M, and*

$$\widehat{N} \models \varphi[\vec{x}] \iff M \models \hat{g}(\varphi)[\vec{x}, \alpha\dot{-}1].$$

(b) *If φ is a formula in the language of $\tilde{\mathcal{C}}_0(\widehat{N})$, then $\hat{g}(\varphi)$ is a formula in the language of $\tilde{\mathcal{C}}_0(M)$, and*

$$\tilde{\mathcal{C}}_0(\widehat{N}) \models \varphi[\vec{x}] \iff \tilde{\mathcal{C}}_0(M) \models \hat{g}(\varphi)[\vec{x}, \alpha\dot{-}1].$$

(c) *If M is a pλ-structure, and φ is a formula in the language of $\mathcal{C}_0(\widehat{N})$, then $\hat{g}(\varphi)$ is a formula in the language of $\mathcal{C}_0(M)$, and*

$$\mathcal{C}_0(\widehat{N}) \models \varphi[\vec{x}] \iff \mathcal{C}_0(M) \models \hat{g}(\varphi)[\vec{x}, \alpha\dot{-}1].$$

(d) *If* φ *is a formula in the language of* $\tilde{\mathcal{C}}_0(N)$*, then* $g(\varphi)$ *is a formula in the language of* $\tilde{\mathcal{C}}_0(M)$*, and*

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{x}] \iff \tilde{\mathcal{C}}_0(M) \models g(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

(e) *If* M *is a p*λ*-structure and* φ *is a formula in the language of* $\mathcal{C}_0(N)$*, then* $g(\varphi)$ *is a formula in the language of* $\mathcal{C}_0(M)$*, and*

$$\mathcal{C}_0(N) \models \varphi[\vec{x}] \iff \mathcal{C}_0(M) \models g(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

The main result for translating formulae in the other direction is this:

Corollary 2.9. *Let* M *be a pP*λ*-structure. Then there is a sequence* $F^N = \langle f^N_\mu \mid \mu \leq \mathrm{ht}(N)\rangle$ *of functions from* ω *to* ω *with the following properties (in the following, we write* f_μ *for* f^N_μ*):*

(a) $\Lambda(N||\mu) \models \varphi[\vec{\xi}] \iff N||\mu \models f_\mu(\varphi)[\vec{\xi}, \mu \dot{-} 1]$*, where* $\vec{\xi} < \omega\mu$*.*

(b) $f_\mu(\varphi)$ *is a* Σ_1*-formula, if* φ *is.*

(c) f_μ *is uniformly* $\Sigma_\omega(N||\mu)$*.*

(d) $F = \{\langle n, m, \gamma\rangle \mid n = f_\gamma(m) \wedge \gamma < \mathrm{ht}(N)\}$ *is uniformly* $\Sigma_1(N)$*.*

The following lemma describes the relationship between the fine structure of a pPλ-structure and its pendant s-structure.

Lemma 2.10. *Let* M *be a pP*λ*-structure. Then for* $n \geq 1$*:*

(a) $\omega\rho^n_M = \omega\rho^n_{\mathtt{S}(M)}$*,*

(b) $\mathbf{\Sigma}^{(n-1)}_1(M) \cap \mathcal{P}(H^n_M) = \mathbf{\Sigma}^{(n-1)}_1(\mathtt{S}(M)) \cap \mathcal{P}(H^n_{\mathtt{S}(M)})$*.*

It even follows that $\omega\rho^n_N$*,* $\mathbf{\Sigma}^{(n-1)}_1(N) \cap \mathcal{P}(H^n_N)$ *are the same for every* $N \in \{M,\ \mathtt{S}(M),\ \widehat{\mathtt{S}(M)},\ \tilde{\mathcal{C}}_0(M),\ \tilde{\mathcal{C}}_0(\mathtt{S}(M)),\ \tilde{\mathcal{C}}_0(\widehat{\mathtt{S}(M)}),\ \mathcal{C}_0(M),\ \mathcal{C}_0(\mathtt{S}(M)),\ \mathcal{C}_0(\widehat{\mathtt{S}(M)})\}$*.*

3 $\mathbf{\Sigma}^{(n)}_1$-definable sets in M and N

Lemma 3.1. *Let* M *be a pP*λ*-structure and* $N = \mathtt{S}(M)$*. Then there are* $q := \{\vec{\alpha}\} \in [\mathrm{ht}(N)]^{<\omega}$ *and functions* $f'_N : \omega \longrightarrow \omega$ *and* $\hat{f}'_N : \omega \longrightarrow \omega$*, so that the following holds:*

There is a fixed list $\vec{w}$ *of variables of the same length as* $\vec{\alpha}$*, so that for every Boolean combination* $\varphi(\vec{x})$ *of* Σ_1*-formulae in which the variables* $\vec{w}$ *don't occur,* $f'_N(\varphi)$ *is also a Boolean combination of* Σ_1*-formulae. The free variables of* $f'_N(\varphi)$ *are* $\{\vec{x}, \vec{w}\}$*, and we have for arbitrary* $\vec{a} \in |N|$*:*

$$M \models \varphi[(^{\vec{x}}/_{\vec{a}})] \iff N \models f'_N(\varphi)[(^{\vec{x}}/_{\vec{a}}), (^{\vec{w}}/_{\vec{\alpha}})].$$

Correspondingly, $\hat{f}'_N(\varphi)$ *is a Boolean combination of* Σ_1*-formulae, and we have for arbitrary* $\vec{a}$ *in* $|\widehat{N}|$*:*

$$M \models \varphi[(^{\vec{x}}/_{\vec{a}})] \iff \widehat{N} \models \hat{f}'_N(\varphi)[(^{\vec{x}}/_{\vec{a}}), (^{\vec{w}}/_{\vec{\alpha}})].$$

If $\mathrm{ht}(M)$ *and* $\mathrm{ht}(N)$ *are limit ordinals, then* $q = \emptyset$*.*

Proof. I concentrate on the definition of f'_N; the definition of $\hat{f}'_N$ is analogous. The new point here is that arbitrary members of $|N|$ are allowed, not only ordinals. I use the fact that the restriction to ordinals is not necessary for the translation in the opposite direction. Let $f : \mathrm{On}_N \longrightarrow |N|$ be a canonical $\Sigma_1(N)$-surjection (f is a partial function). Let $\varphi(\vec{x})$ be a Σ_1-formula (we define f'_N only for Σ_1-formulae, since it is obvious how to deal with Boolean combinations of such formulae). Note that f is $\Sigma_1(M)$ in $\mathrm{ht}(M)\dot{-}1$; this makes use of Lemma 2.8.

Let $\varphi^*(\vec{x}, y) = \varphi(f(\vec{x}))$ be the result of substituting $f(\vec{x})$ for $\vec{x}$, so

$$M \models \varphi^*[\vec{a}, \mathrm{ht}(M)\dot{-}1] \iff M \models \varphi[f(\vec{a})]$$

for $\vec{a} \in |M|$ (so each of these statements can only hold if $\vec{a} \in \mathrm{On}_N$). The map $\varphi \mapsto \varphi^*$ is uniform in the definition of f over M, and φ^* is Σ_1.

Using Lemma 2.5, let $q = \{\vec{\alpha}\} \in [\mathrm{ht}(N)]^{<\omega}$ be chosen so that $\mathrm{ht}(M)\dot{-}1 \in h^1_M(q)$. q can be chosen so that $\mathrm{ht}(N)\dot{-}1 \in q$. Let $\mathrm{ht}(M)\dot{-}1 = h^1_M(m, \vec{\alpha})$.

If $\mathrm{ht}(M)$ and $\mathrm{ht}(N)$ are limit ordinals, then $q = \emptyset$ is as desired. In this situation, the next substitution is obsolete.

Substituting $h^1_M(m, \vec{w})$ for y yields a Σ_1-formula $\tilde{\varphi}(\vec{x}', \vec{w})$ with

$$M \models \tilde{\varphi}[\vec{a}, \vec{\alpha}] \iff M \models \varphi[f(\vec{a})].$$

Using the function $f^N := f^N_{\mathrm{ht}(N)}$ from Corollary 2.9 yields:

$$\begin{aligned} M \models \varphi[\vec{a}] &\iff M \models \tilde{\varphi}[f^{-1}(\vec{a}), \vec{\alpha}] \\ &\iff N \models f^N(\tilde{\varphi})[f^{-1}(\vec{a}), \vec{\alpha}] \\ &\iff N \models \psi[\vec{a}, \vec{\alpha}], \end{aligned}$$

where $\psi(\vec{x}, \vec{w})$ is the following formula:

$$\exists \vec{x}' \quad \vec{x} = f(\vec{x}') \wedge f^N(\tilde{\varphi}).$$

The parameter $\mathrm{ht}(N)\dot{-}1$ does not need to be exhibited in $f^N(\tilde{\varphi})$, because it occurs in q already. Hence, setting $f'_N(\varphi) := \psi$ finishes the proof. $\square$

Lemma 3.2. *Let M be a pPλ-structure, $N = \mathtt{S}(M)$. Then there is a $q := \{\vec{\alpha}\} \in [\mathrm{ht}(N)]^{<\omega}$ and a fixed list $\vec{w}$ of variables such that for every $n < \omega$, there are functions $f^{(n)}_N$ and $\hat{f}^{(n)}_N$ with the following properties:*

For (Boolean combinations of) $\Sigma^{(n)}_1$-formulae $\varphi(\vec{z}^0, \ldots, \vec{z}^l)$ in which none of the variables $\vec{w}$ occur, $f^{(n)}_N(\varphi)$ and $\hat{f}^{(n)}_N(\varphi)$ are again (Boolean combinations of) $\Sigma^{(n)}_1$-formulae, so that for $\vec{a}^0 \in |N|, \vec{a}^1 \in H^1, \ldots, \vec{a}^l \in H^l$ we have: [1]

$$M \models \varphi[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l})] \iff N \models f^{(n)}_N(\varphi)[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l}), ({}^{\vec{w}^0}/_{\vec{\alpha}})],$$

and for $\vec{a}^0 \in |\widehat{N}|, \vec{a}^1 \in H^1, \ldots, \vec{a}^l \in H^l$ we have:

$$M \models \varphi[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l})] \iff \widehat{N} \models \hat{f}^{(n)}_N(\varphi)[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l}), ({}^{\vec{w}^0}/_{\vec{\alpha}})].$$

Proof. I concentrate of the functions $f^{(n)}_N$. I proceed by recursion on n.

For $n = 0$, I use the function f'_N and the parameter q from Lemma 3.1 and set:

$$f^{(0)}_N := f'_N,$$

[1] Here I write $H^m = H^m_M = H^m_N$.

in the sense that the types of the variables from φ are taken over in $f_N^{(n)}(\varphi)$; this is unproblematic, since for $m \geq 1$ $H_M^m = H_N^m$.

Now assume that $f_N^{(n)}$ has been defined. I derive how to define $f_N^{(n+1)}$. It suffices to give the definition of $f_N^{(n+1)}(\varphi)$ for $\Sigma_1^{(n+1)}$-formulae, since it is clear how to extend the definition to Boolean combinations.

So let ψ be a $\Sigma_1^{(n+1)}$-formula in which no variable from $\vec{w}$ occurs. Then ψ has the form

$$\exists \vec{z}^{n+1} \quad (Q_1 w_1^{n+1} \in v_1^{i_1} \cdots Q_m w_m^{n+1} \in v_m^{i_m} \quad \varphi),$$

where $i_1, \ldots, i_m \geq n+1$ and φ is a Boolean combination of $\Sigma_1^{(n)}$-formulae. Then I define:

$$f^{(n+1)}(\psi) := \exists \vec{z}^{n+1} \quad (Q_1 w_1^{n+1} \in v_1^{i_1} \cdots Q_m w_m^{n+1} \in v_m^{i_m} \quad f_N^{(n)}(\varphi)).$$

It follows from Lemma 2.10 that this definition works, or rather, it follows from the consequence of that lemma that $H_M^{n+1} = H_N^{n+1}$: Let $\vec{u}^0, \ldots, \vec{u}^{n+1}$ be the free variables of ψ. Then we have for $\vec{a}^0 \in |N|, \vec{a}^1 \in H^1, \ldots \vec{a}^{n+1} \in H^{n+1}$:

$$\begin{aligned}
& M \models \exists \vec{z}^{n+1} \quad (Q_1 w_1^{n+1} \in v_1^{i_1} \cdots Q_m w_m^{n+1} \in v_m^{i_m} \quad \varphi)[({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}})] \\
\iff \quad & \exists \vec{b}^{n+1} \in H^{n+1} Q_1 w_1^{n+1} \in v_1^{i_1} \cdots Q_m w_m^{n+1} \in v_m^{i_m} ({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}}) \\
& \quad M \models \varphi[({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}})] \\
\iff \quad & \exists \vec{b}^{n+1} \in H^{n+1} Q_1 w_1^{n+1} \in v_1^{i_1} \cdots Q_m w_m^{n+1} \in v_m^{i_m} ({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}}) \\
& \quad N \models f_N^{(n)}(\varphi)[({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}})] \\
\iff \quad & N \models f_N^{(n+1)}(\psi)[({}^{\vec{u}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{u}^{n+1}}/_{\vec{a}^{n+1}})].
\end{aligned}$$

This involves some abuse of notation, but it should be clear what's meant: In the second and third step, the first variable substitution has to be done by hand, so that if $v_j^{i_j} = u_k^{i_j}$, then $v_j^{i_j}$ has to be replaced with $\vec{a}_k^{i_j}$. □

One arrives at the following converse in the same way. Instead of the functions f'_N and $\hat{f}'_N$ from Lemma 3.1, now the functions g and $\hat{g}$ from Lemma 2.8 have to be used.

Lemma 3.3. *Let N be a pPs-structure, $M = \Lambda(N)$. For $n < \omega$, there are functions $g^{(n)}$ and $\hat{g}^{(n)}$, which map (Boolean combinations of) $\Sigma_1^{(n)}$-formulae to (Boolean combinations of) $\Sigma_1^{(n)}$-formulae, so that for all (Boolean combinations of) $\Sigma_1^{(n)}$-formulae $\varphi(\vec{z}^0, \ldots, \vec{z}^l)$ in which some fixed variable $\tilde{z}^0$ does not occur, and all elements $\vec{a}^0 \in |N|, \vec{a}^1 \in H^1, \ldots, \vec{a}^l \in H^l$, we have:*

$$\tilde{\mathcal{C}}_0(N) \models \varphi[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l})] \iff \tilde{\mathcal{C}}_0(M) \models g^{(n)}(\varphi)[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l}), ({}^{\tilde{z}^0}/_{\mathrm{ht}(M) \dot{-} 1})],$$

and analogously, for $\vec{a}^0 \in |\widehat{N}|, \vec{a}^1 \in H^1, \ldots, \vec{a}^n \in H^l$, we have:

$$\widehat{N} \models \varphi[({}^{\vec{z}^0}/_{\vec{a}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{a}^l})] \iff M \models \hat{g}^{(n)}(\varphi)[({}^{\vec{z}^0}/_{\vec{\alpha}^0}), \ldots, ({}^{\vec{z}^l}/_{\vec{\alpha}^l}), ({}^{\tilde{z}^0}/_{\mathrm{ht}(M) \dot{-} 1})].$$

Here is the lemma that one expected in this section:

Lemma 3.4. *Let M be a pPλ-structure and $N = \mathsf{S}(M)$. Then we have for $n < \omega$:*

$$\mathbf{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(|N|) = \mathbf{\Sigma}_1^{(n)}(N),$$

and

$$\mathbf{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(|\widehat{N}|) = \mathbf{\Sigma}_1^{(n)}(\widehat{N}).$$

Proof. Fix $n < \omega$.

The inclusions from right to left are obvious consequences of Lemma 3.3; for the first, one can apply the function $g^{(n)}$, for the second $\hat{g}^{(n)}$. These functions transform $\Sigma_1^{(n)}$-formulae from $\tilde{\mathcal{C}}_0(N)$ to $\tilde{\mathcal{C}}_0(M)$, but since in the same additional constants appear in the pseudo-Σ_0-codes of N and M, these can be treated like parameters. Since the result talks about definability in parameters, this is unproblematic. Also the parameter $\mathrm{ht}(N)\dot{-}1$ that may occur in the translated formula is harmless here.

For the opposite direction, a little argument is needed. I only show the first claim, the proof of the second one is analogous. So let $A \in \mathbf{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(|N|)$ in parameters $\vec{a}^0, \ldots, \vec{a}^l$. Since we have already seen that $|M| = h_M^1(\mathrm{ht}(N))$ (Lemma 2.5), there is a $p = \{\vec{\gamma}\} \in [\mathrm{ht}(N)]^{<\omega}$, so that A is $\mathbf{\Sigma}_1^{(n)}(M)$ in some parameters $\{\vec{\gamma}\}, \vec{a}^1, \ldots, \vec{a}^l$, as h_M^1 is a good $\Sigma_1^{(0)}$-function to H_M^0 and hence can be substituted for $\vec{a}^0$ in the formula defining A. Now let A be defined by

$$\vec{a} \in A \iff M \models \varphi[({}^{\vec{y}^i}/_{\vec{a}}), ({}^{\vec{x}^0}/_{\vec{\gamma}}), ({}^{\vec{x}^1}/_{\vec{a}^1}), \ldots, ({}^{\vec{x}^l}/_{\vec{a}^l})],$$

where φ is a $\Sigma_1^{(n)}$-formula. Let $f_N^{(n)}$ and $q = \{\vec{\alpha}\}$ be chosen as in Lemma 3.1. Then we have:

$$\vec{a} \in A \iff N \models f_N^{(n)}(\varphi)[({}^{\vec{y}^i}/_{\vec{a}}), ({}^{\vec{x}^0}/_{\vec{\gamma}}), ({}^{\vec{x}^1}/_{\vec{a}^1}), \ldots, ({}^{\vec{x}^l}/_{\vec{a}^l}), ({}^{\vec{w}}/_{\vec{\alpha}})],$$

and this shows that A is $\mathbf{\Sigma}_1^{(n)}(N)$, as claimed. □

Definition 3.5. Let M be a pPλ-structure, $N = \mathtt{S}(M)$. Let $\kappa \in |N|$, $\kappa < \lambda$, κ, λ p.r. closed. Set:

$$\begin{aligned} \bar{\Gamma}^*(M,\kappa) &:= \{f \in \Gamma^*(M,\kappa) \mid \mathrm{ran}(f) \subseteq |\widehat{N}|\}, \\ \bar{D}^*(M,\kappa,\lambda) &:= \{\langle \vec{\alpha}, f\rangle \in D^*(M,\kappa,\lambda) \mid f \in \bar{\Gamma}^*(M,\kappa)\}. \end{aligned}$$

Lemma 3.6. *Let M be a pPλ-structure, $N = \mathtt{S}(M)$. Let $\kappa \in |N|$, $\kappa < \lambda$, κ, λ p.r. closed, and let $\omega\rho^1 := \omega\rho_M^1 = \omega\rho_N^1 > \kappa$. Then*

$$\begin{aligned} \Gamma^*(\widehat{N},\kappa) &= \bar{\Gamma}^*(M,\kappa), \\ D^*(\widehat{N},\kappa,\lambda) &= \bar{D}^*(M,\kappa,\lambda). \end{aligned}$$

Remark: The corresponding is true of N as well, as is shown by the same proof. This is not needed here, though.

Proof. I will first need an observation which requires a new concept:

Let's call $\mathfrak{X} := \langle T, Z, \mathrm{arg}\rangle$ an *explicit rendering* of a good $\mathbf{\Sigma}_1^{(n)}(M)$-function if the following conditions are satisfied:

1. $T = \langle |T|, <_T\rangle$ is a finite tree on $\mathbb{Q}$ (that is, the nodes of T are rational numbers).

2. Z is a function with $\mathrm{dom}(Z) = |T|$, and if $s \in |T|$, then $Z(s)$ is a $\mathbf{\Sigma}_1^{(i)}(M)$-function to H_M^i, for some $i \leq n$.

3. For $s \in |T|$, $\mathrm{arg}(s)$ is an argument type of $Z(s)$. If s is not a leaf of T (that is, no maximal node), then let $\mathrm{succ}_T(s)$ be the set of immediate $<_T$-successors of s. In this case, the following is required: If $\mathrm{arg}_{\mathfrak{X}}(s) = \langle i_0, \ldots, i_{m-1}\rangle$, then $\mathrm{succ}_T(s) = \{j_0, \ldots, j_{m-1}\}$ (in increasing order) , and $Z(j_k)$ is a $\mathbf{\Sigma}_1^{(i_k)}(M)$-function to $H_M^{i_k}$.

4. For arbitrary leavesr p and q of T, $\mathrm{arg}(p) = \mathrm{arg}(q)$. The argument type common to all leaves is called the argument type of $\mathfrak{X}$, for which I write $\mathrm{arg}(\mathfrak{X})$.

For $s \in T$, define by $>_T$-recursion a function $\mathfrak{X}_s : H_M^{q_0} \times \cdots \times H_M^{q_{r-1}} \longrightarrow H_M^i$, where $\arg(\mathfrak{X}) = \langle q_0, \ldots, q_{r-1} \rangle$ and $Z(s)$ is a function to H_M^i:

If s is a leaf of T, then let $\mathfrak{X}_s = Z(s)$. Otherwise let $\mathrm{succ}_T(s) = \{j_0, \ldots, j_{m-1}\}$ (in increasing order). Set:

$$\mathfrak{X}_s(\vec{z}) := Z(s)(\mathfrak{X}_{j_0}(\vec{z}), \ldots, \mathfrak{X}_{j_{m-1}}(\vec{z})).$$

I shall say that $\mathfrak{X}$ is an explicit rendering of $\mathfrak{X}_\perp$, where $\perp$ is the root of T.

($*$) *Let f be a good $\mathbf{\Sigma}_1^{(n)}(M)$-function with $\mathrm{ran}(f) \subseteq |\widehat{N}|$. Then f has an explicit rendering $\mathfrak{X} = \langle T, Z, \arg \rangle$* [2]*, so that for every $s \in T$,*

$$\mathrm{ran}(Z(s)) \subseteq |\widehat{N}|.$$

Proof of ($*$). Assume the contrary. Let f be a counterexample, and set:

$$\mathcal{E} := \{\mathfrak{X} \mid \mathfrak{X} \text{ is an explicit rendering of } f\}.$$

Obviously, $\mathcal{E} \neq \emptyset$. For $\mathfrak{X} = \langle T, Z, \arg \rangle \in \mathcal{E}$, let

$$a(\mathfrak{X}) := \{s \in |T| \mid \mathrm{ran}(Z(s)) \not\subseteq |\widehat{N}|\}.$$

Let $N(\mathfrak{X})$ be the number of members of $a(\mathfrak{X})$, and choose a fixed $\mathfrak{X} \in \mathcal{E}$, so that $N(\mathfrak{X}) = \min N``\mathcal{E}$. By assumption, $N(\mathfrak{X}) > 0$. Let $s \in |T|$ be $<_T$-minimal with $\mathrm{ran}(Z(s)) \not\subseteq |\widehat{N}|$. Obviously, $Z(s)$ is a $\mathbf{\Sigma}_1^{(0)}(M)$-function to H_M^0, as $H_M^1 = H_{\widehat{N}}^1 \subseteq \widehat{N}$.

It follows that $s \neq \perp$: Otherwise, one could define $\mathfrak{X}' := \langle T, Z', \arg \rangle$ by setting, for $t \neq \perp$: $Z'(t) := Z(t)$, and $Z(\perp) := g$, where

$$g(\vec{w}) := \begin{cases} Z(\perp)(\vec{w}) & \text{if } Z(\perp)(\vec{w}) \in |\widehat{N}|, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $\mathrm{ran}(\mathfrak{X}_\perp) = \mathrm{ran}(f) \subseteq |\widehat{N}|$, it's obvious that $\mathfrak{X}'$ is also an explicit rendering of f. So $\mathfrak{X}' \in \mathcal{E}$. But $N(\mathfrak{X}') < N(\mathfrak{X})$, contradicting the choice of $\mathfrak{X}$.

So let $\bar{s}$ be the immediate $<_T$-predecessor of s. Let $\mathrm{succ}_T(\bar{s}) = \{j_0, \ldots, j_{m-1}\}$ (in increasing ordere), and $s = j_k$ ($k < m$). Let $g := Z(\bar{s})$ be a $\mathbf{\Sigma}_1^{(i)}(M)$-function to H_M^i, $\arg(\bar{s}) = \langle i_0, \ldots, i_{m-1} \rangle$. Then $i_k = 0$, as $Z(j_k) = Z(s)$ takes on values in $|M| \setminus |\widehat{N}|$, hence in $H_M^0 \setminus H_M^1$. Let $h = Z(s)$, and let $\arg(s) = \langle k_0, \ldots, k_{l-1} \rangle$. In the following, I want to construct a new explicit rendering of f in which the "bad function" h does not occur anymore, by substituting it in g. To this end, define a $\mathbf{\Sigma}_1^{(i)}(M)$-function $\tilde{g}$ to H_M^i with argument type $\langle i_0, \ldots, i_{k-1}, k_0, \ldots, k_{l-1}, i_{k+1}, \ldots, i_{m-1} \rangle$ as follows:

$$\begin{aligned} &\tilde{g}(v_0, \ldots, v_{k-1}, w_0, \ldots, w_{l-1}, v_{k+1}, \ldots, v_{m-1}) := \\ &\qquad g(v_0, \ldots, v_{k-1}, h(w_0, \ldots, w_{l-1}), v_{k+1}, \ldots, v_{m-1}). \end{aligned}$$

It follows from Lemma [Zem02, Lemma 1.8.1] that $\tilde{g}$ is a $\mathbf{\Sigma}_1^{(i)}(M)$-function to H_M^i. It's crucial here that h is a $\Sigma_1^{(0)}(M)$-function to H_M^0.

Let $\mathrm{succ}_T(s) = \{q_0, \ldots, q_{l-1}\}$ (in increasing order). Choose rational numbers $\{\tilde{q}_0, \ldots, \tilde{q}_{l-1}\} \subseteq \mathbb{Q} \setminus |T|$ with $j_{k-1} < \tilde{q}_0 < \ldots < \tilde{q}_{l-1} < j_{k+1}$ (where formally, $j_{k-1} = -\infty$, $j_{k+1} = +\infty$ if

[2] This means: $\mathfrak{X}_\perp = f$.

undefined). Set $K := |T| \setminus (\{s\} \cup \mathrm{succ}_T(s))$. Now define $\tilde{\mathfrak{X}} = \langle \tilde{T}, \tilde{Z}, \widetilde{\arg} \rangle$ as follows:

$$
\begin{array}{rcl}
|\tilde{T}| & := & K \cup \{\tilde{q}_0, \ldots, \tilde{q}_{l-1}\}, \\
p <_{\tilde{T}} q & \iff & ((p, q \in K \wedge p <_T q) \vee \\
 & & \vee(p \in \{\tilde{q}_0, \ldots, \tilde{q}_{l-1}\} \wedge \bar{s} \leq_T q) \vee \\
 & & \vee \displaystyle\bigvee_{k<l} (q = \tilde{q}_k \wedge p \leq_T q_k)) \qquad (\text{for } p, q \in |\tilde{T}|), \\
\tilde{Z}(q) & := & \begin{cases} Z(q) & \text{if } q \in K \setminus \{\bar{s}\}, \\ Z(q_k) & \text{if } q = \tilde{q}_k, \\ \tilde{g} & \text{if } q = \bar{s}\ . \end{cases} \\
\widetilde{\arg}(q) & := & \begin{cases} \arg(q) & \text{if } q \in K \setminus \{\bar{s}\}, \\ \arg(q_k) & \text{if } q = \tilde{q}_k, \\ \langle i_0, \ldots, i_{k-1}, k_0, \ldots, k_{l-1}, i_{k+1}, \ldots, i_{m-1} \rangle & \text{if } q = \bar{s}. \end{cases}
\end{array}
$$

Obviously then $\tilde{\mathfrak{X}} \in \mathcal{E}$ and $N(\tilde{\mathfrak{X}}) < N(\mathfrak{X})$, contradicting the choice of $\mathfrak{X}$. $\square_{(*)}$

Let's now turn to the main claim. It obviously suffices to prove that $\Gamma^*(\widehat{N}, \kappa) = \bar{\Gamma}^*(M, \kappa)$, since this immediately implies the second part of the claim.

The substantial direction here is from right to left. So let $f \in \bar{\Gamma}^*(M, \kappa)$. Then f is a good $\mathbf{\Sigma}_1^{(n)}(M)$-function, where $\omega\rho_M^{n+1} > \kappa$; here, $-1 \leq n < \omega$. Using the [Zem02, S. 73] convention, I refer to functions that are members of $|M|$ as good $\mathbf{\Sigma}_1^{(-1)}(M)$-functions.

If $n \geq 0$, then by $(*)$, there is an explicit rendering $\mathfrak{X} = \langle T, Z, \arg \rangle$ of f as a good $\mathbf{\Sigma}_1^{(n)}(M)$-function, so that $\mathrm{ran}(Z(s)) \subseteq |\widehat{N}|$, for each $t \in |T|$. It follows that each $Z(s)$ can be restricted to $|\widehat{N}|$ without changing $\mathfrak{X}_\perp$ (if s is a leaf of T, then $\mathrm{dom}(Z(s)) \subseteq [\kappa]^m$, for some $m < \omega$ already). Denoting the resulting explicit rendering of f by $\mathfrak{X}' := \langle T, Z', \arg \rangle$, one sees that $Z'(s)$ is a subset of $|\widehat{N}|$, for every $s \in |T|$. By Lemma 3.4, this means that each such $Z'(s)$ is also a $\mathbf{\Sigma}_1^{(i)}(\widehat{N})$-function to $H^i_{\widehat{N}}$, and so, $\mathfrak{X}'$ is an explicit rendering of a good $\mathbf{\Sigma}_1^{(n)}(\widehat{N})$-function, namely the function $\mathfrak{X}'_\perp = f$.

On the other hand, if $f \in |M|$, then f is also a (good) $\mathbf{\Sigma}_1^{(0)}(M)$-function in the parameter f, hence also a (good) $\mathbf{\Sigma}_1^{(0)}(\widehat{N})$-function. By assumption, $\omega\rho^1_{\widehat{N}} = \omega\rho^1_M > \kappa$, hence in this case also, $f \in \Gamma^*(\widehat{N}, \kappa)$. $\square$

4 Ultrapowers

In this section, I analyze the formation of ultrapowers, the successor step in iterations.

4.1 Σ_0-extender ultrapowers of successor structures

In [Fuc09], I introduced the notion of a Σ_ω-ultrapower: The construction is analogous to the fine structural ultrapower, where the functions considered are all definable ones (using parameters). I proved the following theorem there:

Theorem 4.1. *Let $\vec{\dot{A}}, \vec{\dot{B}}$ be predicate symbols with interpretations $\vec{A}, \vec{B}$. Let $\bar{X}$ be a transitive set which is closed under functions rudimentary in $\vec{A}$, such that $\vec{A}, \vec{B} \subseteq \bar{X}$. Let $\bar{M} = \langle \bar{X}, \vec{A}, \vec{B} \rangle$ be* *definably well-ordered. Let* $X := \mathrm{rud}_{\vec{A}}(\bar{X})$ *and* $M = \langle X, \vec{A}, \vec{B}\rangle$*, and let* $\underset{\sim}{\Sigma}_\omega(\bar{M}) = X \cap \mathcal{P}(\bar{X})$*.*[3] *Let* F *be an extender on* $\bar{M}$ *and* M*.*[4] *Let*

$$\bar{\pi} : \bar{M} \longrightarrow_F^{\Sigma_\omega} \bar{M}',$$

where $\bar{M}' = \langle \bar{X}', \vec{A}', \vec{B}'\rangle$ *is transitive. Then the following is a correct definition of a function* π*:*

$$\pi(\mathtt{val}^{\bar{M}}[t](\vec{a})) := \mathtt{val}^{\bar{M}'}[t](\bar{\pi}(\vec{a})),$$

where $t \in \mathfrak{T}(\vec{A})$ *and* $\vec{a} \in \bar{X}$ *is an assignment of its free variables.*

Set $X' := \mathrm{rud}_{\vec{A}'}(\bar{X}')$ *and* $M' := \langle X', \vec{A}', \vec{B}'\rangle$*. Then*

$$\pi : M \longrightarrow_F M' \quad \textit{and} \quad \bar{\pi} \subseteq \pi.$$

4.2 Extender ultrapowers of M and $\widehat{N}$

Lemma 4.2. *Let* M *be a pP*λ*-structure,* $N = \mathtt{S}(M)$*. Let* F *be an extender on* M *and* N*. Let* $\omega\rho^1_M > \kappa := \mathrm{crit}(F)$*, and let*

$$\begin{aligned} \pi : M &\longrightarrow^*_F \quad M', \\ \hat{\pi} : \widehat{N} &\longrightarrow^*_F \quad N'. \end{aligned}$$

Then $N' = \widehat{\mathtt{S}(M')}$ *and* $\hat{\pi} \subseteq \pi$*.*

Proof. Let $\lambda = \mathrm{lh}(F)$.

Define a relation E on $D^*(\widehat{N}, \kappa, \lambda)$ by

$$\langle \vec{\alpha}, f\rangle E \langle \vec{\beta}, g\rangle \iff \prec\vec{\alpha}, \vec{\beta}\succ \in F(\{\prec\vec{\gamma}, \vec{\delta}\succ \mid f(\vec{\gamma}) \in g(\vec{\delta})\}).$$

Analogously, E' and I' on $D^*(M, \kappa, \lambda)$ are defined by

$$\begin{aligned} \langle \vec{\alpha}, f\rangle E' \langle \vec{\beta}, g\rangle &\iff \prec\vec{\alpha}, \vec{\beta}\succ \in F(\{\prec\vec{\gamma}, \vec{\delta}\succ \mid f(\vec{\gamma}) \in g(\vec{\delta})\}), \\ \langle \vec{\alpha}, f\rangle I' \langle \vec{\beta}, g\rangle &\iff \prec\vec{\alpha}, \vec{\beta}\succ \in F(\{\prec\vec{\gamma}, \vec{\delta}\succ \mid f(\vec{\gamma}) = g(\vec{\delta})\}). \end{aligned}$$

Let φ_{V} be the formula from Lemma 2.7, hence a Σ_1-formula defining uniformly over pPλ-structures $\tilde{M}$ the universe of $\widehat{\mathtt{S}(\tilde{M})}$. In the following, I suppress the additional parameter $\mathrm{ht}(\tilde{M}) \dot{-} 1$ occurring in that formula, since it is preserved by π.

(1) *Let* $\langle \vec{\alpha}, f\rangle \in D^*(M, \kappa, \lambda)$ *have the property that* $M' \models \varphi_{\mathrm{V}}[\pi(f)(\vec{\alpha})]$*. Then there is a* $\langle \vec{\alpha}, f'\rangle \in \bar{D}^*(M, \kappa, \lambda)$ *with* $\langle \vec{\alpha}, f\rangle I' \langle \vec{\alpha}, f'\rangle$*.*

Proof of (1). Letting $X := \{\prec\vec{\gamma}\succ \mid M \models \varphi_{\mathrm{V}}[f(\vec{\gamma})]\}$, we have

$$\prec\vec{\alpha}\succ \in F(X),$$

as follows from a Łoś theorem (see [Zem02, Lemma 3.1.11 (d)]; φ_{V} is $\Sigma^{(1)}_0$, $\omega\rho^1_M > \kappa$).

[3] If $\vec{B}$ is empty, then it is a general fact that $\underset{\sim}{\Sigma}_\omega(\bar{M}) = X \cap \mathcal{P}(\bar{X})$, see [Jen72, Cor. 1.7]. But otherwise, this need not be true, since X is the rudimentary closure of $\bar{X} \cup \{\bar{X}\}$ only under functions which are rudimentary in $\vec{A}$. So $\underset{\sim}{\Sigma}_\omega(\bar{M})$ will contain each B_i as an element, while this is not necessarily true of X.

[4] Note that, letting $\kappa = \mathrm{crit}(F)$, this implies that $\mathcal{P}(\kappa) \cap \bar{X} = \mathrm{dom}(F) = \mathcal{P}(\kappa) \cap X$.

Moreover, $X \in |M|$, since φ_{V} is Σ_1, and f is a good $\mathbf{\Sigma}_1^{(n)}(M)$-function. The substitution of f in φ_{V} yields another $\mathbf{\Sigma}_1^{(n)}(M)$ formula. Since $X \subseteq \kappa$ and $\omega\rho_M^{n+1} > \kappa$ it follows that $X \in |M|$.

Since $\prec\vec{\alpha}\succ \in F(X)$, X is not empty. So fix some $\vec{\xi} < \kappa$ with $f(\vec{\xi}) \in |\widehat{N}|$. Define a function $h : \kappa^{\mathrm{lh}(\vec{\alpha})} \longrightarrow \kappa$ by

$$h(\vec{\delta}) := \begin{cases} \langle\vec{\delta}\rangle & \text{if } \langle\vec{\delta}\rangle \in X \ , \\ \langle\vec{\xi}\rangle & \text{otherwise.} \end{cases}$$

Obviously, h is a good $\mathbf{\Sigma}_1^{(n)}(M)$-function; it is even a member of $|M|$. Now define $f' : \kappa^{\mathrm{lh}(\vec{\alpha})} \longrightarrow |\widehat{N}|$ by

$$f'(\vec{\gamma}) := f(h(\vec{\gamma})).$$

Ten $f' \in \bar{\Gamma}^*(M)$, and for $\prec\vec{\gamma}\succ \in X$, $f'(\vec{\gamma}) = f(\vec{\gamma})$. Let

$$X' := \{\prec\vec{\gamma}\succ \mid f(\vec{\gamma}) = f'(\vec{\gamma})\}.$$

Then $X \subseteq X'$. It follows that

$$\vec{\alpha} \in F(X) \subseteq F(X'),$$

and this means that $\langle\vec{\alpha}, f\rangle I' \langle\vec{\alpha}, f'\rangle$, as wished. $\square_{(1)}$

(2) *If* $\langle\vec{\alpha}, f\rangle \in D^*(\widehat{N}, \kappa, \lambda)$, *then* $\pi(f)(\vec{\alpha}) = \hat{\pi}(f)(\vec{\alpha})$.

Proof of (2). I show the claim by E-induction on $\langle\vec{\alpha}, f\rangle$. If it holds of all E-predecessors of $\langle\vec{\alpha}, f\rangle$, then

$$\begin{aligned} \pi(f)(\vec{\alpha}) &= \{\pi(g)(\vec{\beta}) \mid \langle\vec{\beta}, g\rangle E' \langle\vec{\alpha}, f\rangle\} \\ &\supseteq \{\pi(g)(\vec{\beta}) \mid \langle\vec{\beta}, g\rangle E \langle\vec{\alpha}, f\rangle\} \\ &= \{\hat{\pi}(g)(\vec{\beta}) \mid \langle\vec{\beta}, g\rangle E \langle\vec{\alpha}, f\rangle\} \\ &= \hat{\pi}(f)(\vec{\alpha}). \end{aligned}$$

So it remains to show the reverse inclusion. So let $\langle\vec{\beta}, g\rangle E' \langle\vec{\alpha}, f\rangle$. I have to show that $\pi(g)(\vec{\beta}) \in \hat{\pi}(f)(\vec{\alpha})$.

Letting $X := \{\prec\vec{\gamma}, \vec{\delta}\succ \mid g(\vec{\gamma}) \in f(\vec{\delta})\}$, we have

$$\prec\vec{\beta}, \vec{\alpha}\succ \in F(X).$$

Let $X' := \{\prec\vec{\gamma}, \vec{\delta}\succ \mid M \models \varphi_{\mathrm{V}}[g(\vec{\gamma})]\}$. Then $X \subseteq X'$, as $f \in \Gamma^*(\widehat{N}, \kappa)$. It follows easily that

$$\vec{\beta} \in F(\{\vec{\gamma} \mid M \models \varphi_{\mathrm{V}}[g(\vec{\gamma})]\}).$$

But this means, by a Łoś theorem, that $M' \models \varphi_{\mathrm{V}}[\pi(g)(\vec{\beta})]$. Now let, by (1), $\langle\beta, g'\rangle \in \bar{D}^*(M, \kappa, \lambda)$ be chosen in such a way that $\langle\vec{\beta}, g\rangle I' \langle\vec{\beta}, g'\rangle$. By Lemma 3.6, $\bar{\Gamma}^*(\kappa, M) = \Gamma^*(\widehat{N}, \kappa)$, and hence $\langle\vec{\beta}, g'\rangle \in D^*(\widehat{N}, \kappa, \lambda)$.

We have $\langle\vec{\beta}, g'\rangle E \langle\vec{\alpha}, f\rangle$, which means, by inductive hypothesis, and since $\langle\vec{\beta}, g\rangle I' \langle\vec{\beta}, g'\rangle$,

$$\pi(g)(\vec{\beta}) = \pi(g')(\vec{\beta}) = \hat{\pi}(g')(\vec{\beta}) \in \hat{\pi}(f)(\vec{\alpha}).$$

This is what I wanted to show. $\square_{(2)}$

(3) $\hat{\pi} \subseteq \pi$.

Proof of (3). For $a \in |\widehat{N}|$, $\hat{\pi}(a) = \hat{\pi}(\mathrm{const}_a)(0) = \pi(\mathrm{const}_a)(0) = \pi(a)$, by (2). $\square_{(3)}$

(4) $|N'| = |\widehat{\mathtt{S}(M')}|$.

Proof of (4). For the direction from left to right, let $a \in |N'|$. Then $a = \hat{\pi}(f)(\vec{\alpha}) = \pi(f)(\vec{\alpha})$, for some $\langle \alpha, f \rangle \in D^*(\widehat{N}, \kappa, \lambda)$. Hence $\langle \vec{\alpha}, f \rangle \in \bar{D}^*(M, \kappa, \lambda)$, and this means in particular that

$$\prec\vec{\alpha}\succ \in F(\{\prec\vec{\gamma}\succ \mid M \models \varphi_{\mathrm{V}}[f(\vec{\gamma})]\}),$$

since $\{\prec\vec{\gamma}\succ \mid M \models \varphi_{\mathrm{V}}[f(\vec{\gamma})]\} = \kappa$. By Łoś, it follows that

$$M' \models \varphi_{\mathrm{V}}[\pi(f)(\vec{\alpha})].$$

So, $a = \pi(f)(\vec{\alpha}) \in |\widehat{\mathtt{S}(M')}|$, by Lemma 2.7.

For the other direction, let $a \in |\widehat{\mathtt{S}(M')}|$. Let $a = \pi(f)(\prec\vec{\alpha}\succ)$ for some $\langle \vec{\alpha}, f \rangle \in D^*(M, \kappa, \lambda)$. Then

$$M' \models \varphi_{\mathrm{V}}[\pi(f)(\vec{\alpha})].$$

By (1), let $\langle \vec{\alpha}, f' \rangle \in \bar{D}^*(M, \kappa, \lambda) = D^*(\widehat{N}, \kappa, \lambda)$ have the property that $\langle \vec{\alpha}, f' \rangle I' \langle \vec{\alpha}, f \rangle$. It then follows by (2) that

$$a = \pi(f)(\vec{\alpha}) = \pi(f')(\vec{\alpha}) = \hat{\pi}(f')(\vec{\alpha}) \in |N'|.$$

$\square_{(4)}$

(5) $\dot{E}^{N'} = \dot{E}^{\widehat{\mathtt{S}(M')}}$ *and* $\dot{F}^{N'} = \dot{F}^{\widehat{\mathtt{S}(M')}}$.

Proof of (5). One can argue here as in the proof of (4), using the Σ_1-formulae φ_E and φ_F from Lemma 2.7. $\square_{(5),\ \text{Lemma}}$

Lemma 4.3. *Let M be an active pPλ-structure, $N = \mathtt{S}(M)$. Let F be an extender with critical point κ on M and N, and let*

$$\begin{aligned} \pi : M &\longrightarrow^*_F M', \\ \sigma : N &\longrightarrow^*_F N'. \end{aligned}$$

Let $\omega\rho^1_M > \kappa$. If $\pi(s(M)) = s(M')$, then $N' = \mathtt{S}(M')$ and $\sigma \subseteq \pi$.

Proof. One can argue like in the proof of Lemma 4.2, with the difficulty that one cannot use the Σ_1 formula φ_{V} here in order to define $|\mathtt{S}(M)|$ uniformly in M. But abstracting from how φ_{V} is formulated, it is obvious that $|\mathtt{S}(M)|$ is $\Sigma_1(M)$ in the parameter $s(M)$ (note that "$x < s^+(M)$" $\Sigma_1(M)$ in $s(M)$). As $\pi(s(M)) = s(M')$, one can deduce (as before) that $|N'| = |\mathtt{S}(M')|$.

In the same way, one show that $\dot{E}^{N'} = \dot{E}^{\mathtt{S}(M')}$, and that $\dot{F}^{N'} = \dot{F}^{\mathtt{S}(M')}$. The first is clear, as $\dot{E}^{\mathtt{S}(M)} = \dot{E}^{\widehat{\mathtt{S}(M)}} \restriction s^+(M)$; so with the help of the formula φ_E from Lemma 2.7, one can produce a Σ_1-formula defining $\dot{E}^{\mathtt{S}(M)}$ in the parameter $s(M)$, uniformly in M. It is easy to see that one can define $F^c = \dot{F}^{\mathtt{S}(M)}$ as well, going back to the way it was defined (see the first part of this paper, Def. 3.3). $\square$

The first part is not published yet, so this reference needs to be checked!

Lemma 4.4. *Let M be a pPλ-structure, $N = \mathtt{S}(M)$. Let F be an extender on M and N with critical point κ. Assume $(\kappa^+)^M$ exists. Let*

$$\begin{aligned} \pi : M &\longrightarrow^*_F M', \\ \sigma : \widehat{N} &\longrightarrow^*_F N'. \end{aligned}$$

Then $N' = \widehat{\mathtt{S}(M')}$ and $\sigma \subseteq \pi$.

Proof. If $\omega\rho^1_M > \kappa$, then Lemma 4.2 yields the claim. So let $\omega\rho^1_{\widehat{N}} = \omega\rho^1_M \leq \kappa$. Then π and σ are Σ_0-extender ultrapower embeddings.

Let $M = \langle \mathrm{J}^{E^M}_\mu, E^M_{\mathrm{top}}, D_M\rangle$, $N = \langle \mathrm{J}^{E^N}_\nu, E^N_{\mathrm{top}}\rangle$ and $\widehat{N} = \langle \mathrm{J}^{E^{\widehat{N}}}_{\hat{\nu}}, E^{\widehat{N}}_{\mathrm{top}}\rangle$. I distinguish two cases.

Case 1: μ is a successor ordinal.

Let $\mu = \bar{\mu}+1$. Then $\nu = \bar{\nu}+1$ is also a successor ordinal. Moreover, clearly, $N = \widehat{N}$. As $\tau := (\kappa^+)^M$ exists, F is an extender on $\bar{M} := M||\bar{\mu}$. τ is also a cardinal in N, since $|N| \subseteq |M|$. As $|M||\tau| = |\mathtt{S}(M||\tau)|$, and since $\mathtt{S}(M||\tau)$ is a segment of N, F is also an extender on $\bar{N} := N||\bar{\nu}$.

Moreover, the $*$-ultrapower of $\bar{M}$ by F exists, since one can define a canonical embedding $I\!D^*(\bar{M}, F) \longrightarrow \mathrm{Ult}(M, F)$ by $[\vec{\alpha}, f] \mapsto \pi(f)(\vec{\alpha})$. For the same reason, the $*$-ultrapower of $\bar{N}$ by F exists also. Let

$$\begin{array}{rlll} \bar{\pi}: & \bar{M} & \longrightarrow^*_F & \bar{M}' \\ \bar{\sigma}: & \bar{N} & \longrightarrow^*_F & \bar{N}'. \end{array}$$

Obviously, $\bar{N} = \mathtt{S}(\bar{M})$. Since $\mathcal{P}(\kappa) \cap |\bar{M}| = \mathcal{P}(\kappa) \cap |M|$, it follows that $\omega\rho^\omega_{\bar{M}} > \kappa$, hence $\omega\rho^\omega_{\bar{N}} = \omega\rho^\omega_{\hat{\bar{N}}} > \kappa$, too. So $\bar{\pi}$ and $\bar{\sigma}$ are Σ^*-preserving. As $R^*_{\bar{M}} \neq \emptyset \neq R^*_{\bar{N}}$, it even follows that $\bar{\pi}$ and $\bar{\sigma}$ are Σ_ω-preserving. If $\bar{M}$ and $\bar{N}$ are active, then as a consequence, $\bar{\pi}(s(\bar{M})) = s(\bar{M}')$.

Using Lemma 4.2, or Lemma 4.3 in case $\bar{M}$ and $\bar{N}$ are active, one gets:

$$\bar{N}' = \mathtt{S}(\bar{M}'), \text{ and } \bar{\sigma} \subseteq \bar{\pi}.$$

Let $\bar{X} = |\bar{M}|$, $X = |M|$, $\bar{X}' = |\bar{M}'|$, $\bar{Y} = |\bar{N}|$, $Y = |N|$ and $\bar{Y}' = |\bar{N}'|$. Then $X = \mathrm{rud}_{\dot{E}^{\bar{M}}, E^{\bar{M}}_{\mathrm{top}}}(\bar{X})$ and $Y = \mathrm{rud}_{\dot{E}^{\bar{N}}, E^{\bar{N}}_{\mathrm{top}}}(\bar{Y})$. Set:

$$\begin{array}{rcl} \tilde{M} & := & \langle X, \dot{E}^{\bar{M}}, E^{\bar{M}}_{\mathrm{top}}, D_{\bar{M}}\rangle, \\ \tilde{M}' & := & \langle |M'|, \dot{E}^{\bar{M}'}, E^{\bar{M}'}_{\mathrm{top}}, D_{\bar{M}'}\rangle, \\ \tilde{N} & := & \langle Y, \dot{E}^{\bar{N}}, E^{\bar{N}}_{\mathrm{top}}\rangle, \\ \tilde{N}' & := & \langle |N'|, \dot{E}^{\bar{N}'}, E^{\bar{N}'}_{\mathrm{top}}\rangle. \end{array}$$

Obviously,

$$\begin{array}{rlll} \pi: & \tilde{M} & \longrightarrow_F & \tilde{M}', \\ \sigma: & \tilde{N} & \longrightarrow_F & \tilde{N}'. \end{array}$$

Since $\bar{\pi}$ and $\bar{\sigma}$ are Σ_ω-preserving, Lemma 4.1 can be applied, showing that

$$\begin{array}{rcl} |M'| = |\tilde{M}'| & = & \mathrm{rud}_{\dot{E}^{\bar{M}'}, E^{\bar{M}'}_{\mathrm{top}}}(\bar{X}'), \\ |N'| = |\tilde{N}'| & = & \mathrm{rud}_{\dot{E}^{\bar{N}'}, E^{\bar{N}'}_{\mathrm{top}}}(\bar{Y}'), \\ \pi(\mathtt{val}^{\dot{E}^{\bar{M}}, E^{\bar{M}}_{\mathrm{top}}}[c](\vec{a}, \bar{X})) & = & \mathtt{val}^{\dot{E}^{\bar{M}'}, E^{\bar{M}'}_{\mathrm{top}}}[c](\bar{\pi}(\vec{a}), \bar{X}'), \\ \sigma(\mathtt{val}^{\dot{E}^{\bar{N}}, E^{\bar{N}}_{\mathrm{top}}}[c](\vec{b}, \bar{Y})) & = & \mathtt{val}^{\dot{E}^{\bar{N}'}, E^{\bar{N}'}_{\mathrm{top}}}[c](\bar{\sigma}(\vec{b}), \bar{Y}'), \end{array}$$

for $c \in \mathfrak{C}(\dot{E}, \dot{F})$ and $\vec{a} \in \bar{X}$, $\vec{b} \in \bar{Y}$. In particular, $\bar{\pi} \subseteq \pi$ and $\bar{\sigma} \subseteq \sigma$. Since $\mathtt{S}(\bar{M}') = \bar{N}'$, it follows that

$$|\mathtt{S}(M')| = |N'|.$$

One can conclude that $M' = \bar{M}' + 1$. The crucial point here is that

$$\dot{E}^{M'}(\alpha, b, x) \iff (\alpha < \bar{\mu}' \wedge \dot{E}^{\bar{M}'}(\alpha, b, x)) \vee (\alpha = \bar{\mu}' \wedge b \in \dot{F}^{\bar{M}'}(x)).$$

This follows from the fact that the corresponding is true of M and $\bar{M}$:

$$\begin{array}{rl} & \dot{E}^{M'}(\pi(f_1)(\vec{\alpha}^1), \pi(f_2)(\vec{\alpha}^2), \pi(f_3)(\vec{\alpha}^3)) \\ \iff & \prec\vec{\alpha}^1, \vec{\alpha}^2, \vec{\alpha}^3\succ \in F(\{\prec\vec{\gamma}^1, \vec{\gamma}^2, \vec{\gamma}^3\succ \mid \dot{E}^M(f_1(\vec{\gamma}^1), f_2(\vec{\gamma}^2), f_3(\vec{\gamma}^3))\}) \\ \iff & \prec\vec{\alpha}^1, \vec{\alpha}^2, \vec{\alpha}^3\succ \in F(\{\prec\vec{\gamma}^1, \vec{\gamma}^2, \vec{\gamma}^3\succ \mid \\ & \quad \mid \tilde{M} \models (f_1(\vec{\gamma}^1) < \bar{\mu} \wedge \dot{E}(f_1(\vec{\gamma}^1), f_2(\vec{\gamma}^2), f_3(\vec{\gamma}^3))) \vee \\ & \quad \vee(f_1(\vec{\gamma}^1) = \bar{\mu} \wedge f_2(\vec{\gamma}^2) \in \dot{F}(f_3(\vec{\gamma}^3)))\}) \\ \iff & (\pi(f_1)(\vec{\alpha}^1) < \pi(\bar{\mu}) \wedge \dot{E}^{\tilde{M}'}(\pi(f_1)(\vec{\alpha}^1), \pi(f_2)(\vec{\alpha}^2), \pi(f_3)(\vec{\alpha}^3))) \vee \\ & \quad \vee(\pi(f_1)(\vec{\alpha}^1) = \pi(\bar{\mu}) \wedge \pi(f_2)(\vec{\alpha}^2) \in \dot{F}^{\tilde{M}'}(\pi(f_3)(\vec{\alpha}^3))). \end{array}$$

Clearly, that $\dot{E}^{\tilde{M}'} = \dot{E}^{\bar{M}'}$ and $\dot{F}^{\tilde{M}'} = \dot{F}^{\bar{M}'}$, since $\dot{E}^{\tilde{M}} = \dot{E}^{\bar{M}}, \dot{F}^{\tilde{M}} = \dot{F}^{\bar{M}}$, and $\pi(\bar{M}) = \bar{M}'$.

This shows that $M' =$"$\bar{M}' + 1$". One shows analogously that $N' =$"$\bar{N}' + 1$". Hence

$$\mathtt{S}(M') = \mathtt{S}(\bar{M}' + 1) = \mathtt{S}(\bar{M}') + 1 = \bar{N}' + 1 = N'.$$

It remains to show that $\sigma \subseteq \pi$. One deduces from the preservation properties of π:

$$\begin{array}{rcl} \pi(|\bar{N}|, \dot{E}^{\bar{N}}, \dot{F}^{\bar{N}}) & = & \pi(|\mathtt{S}(\bar{M})|, \dot{E}^{\mathtt{S}(\bar{M})}, \dot{F}^{\mathtt{S}(\bar{M})}) \\ & = & |\mathtt{S}(\pi(\bar{M}))|, \dot{E}^{\mathtt{S}(\pi(\bar{M}))}, \dot{F}^{\mathtt{S}(\pi(\bar{M}))} \\ & = & |\mathtt{S}(\bar{M}')|, \dot{E}^{\mathtt{S}(\bar{M}')}, \dot{F}^{\mathtt{S}(\bar{M}')} \\ & = & |\bar{N}'|, \dot{E}^{\bar{N}'}, \dot{F}^{\bar{N}'}. \end{array}$$

Now let $x \in |N|$. Then there are $\vec{a} \in |\bar{N}|$ and some $c \in \mathfrak{C}(\dot{E}, \dot{F})$, such that $x = \mathtt{val}^{\dot{E}^{\bar{N}}, \dot{F}^{\bar{N}}}[c](\vec{a}, |\bar{N}|)$. Then

$$\begin{array}{rcl} \sigma(x) & = & \sigma(\mathtt{val}^{\dot{E}^{\bar{N}}, \dot{F}^{\bar{N}}}[c](\vec{a}, |\bar{N}|)) \\ & = & \mathtt{val}^{\dot{E}^{\bar{N}'}, \dot{F}^{\bar{N}'}}[c](\bar{\sigma}(\vec{a}), |\bar{N}'|) \\ & = & \mathtt{val}^{\pi(\dot{E}^{\bar{N}}), \pi(\dot{F}^{\bar{N}})}[c](\pi(\vec{a}), \pi(|\bar{N}|)) \\ & = & \pi(\mathtt{val}^{\dot{E}^{\bar{N}}, \dot{F}^{\bar{N}}}[c](\vec{a}, |\bar{N}|)) \\ & = & \pi(x). \end{array}$$

I used the fact here that the map $A, B, c, \vec{a} \mapsto \mathtt{val}^{A,B}[c](\vec{a})$ is Σ_1. This is what needed to be shown, so case 1 is dealt with.

Case 2: $\mathrm{ht}(M)$ is a limit ordinal.

I verify first that $\sigma \subseteq \pi$. For $\alpha < \widehat{\nu}$, set:

$$\begin{array}{rcl} \Gamma_\alpha & := & ({}^\kappa|\widehat{N}||\alpha|) \cap |N|, \\ \bar{\Gamma}_\alpha & := & ({}^\kappa|\widehat{N}||\alpha|) \cap |M|. \end{array}$$

Then $\Gamma_\alpha = \bar{\Gamma}_\alpha$:

It's clear that $\Gamma_\alpha \subseteq \bar{\Gamma}_\alpha$. I show the opposite inclusion. So let $f \in \bar{\Gamma}_\alpha$. We know that D_M is unbounded in On_M (Lemma 2.4). Hence $\beta + 1 < \mu$ can be chosen so that $f \in |M||\beta + 1|$ and additionally $\omega(\beta + 1) \in D_M$. Clearly, $f \in \mathbf{\Sigma}_1(M||\beta + 1)$, and by Lemma 3.4 it follows that $f \in \mathbf{\Sigma}_1(\mathtt{S}(M||\beta + 1))$. As $\omega(\beta + 1) \in D_M$, $\mathtt{S}(M||\beta + 1)$ is a segment of $\widehat{N}$. Let $\mathtt{S}(M||\beta) = \widehat{N}||\beta'$. Then $f \in \widehat{N}||\beta' + 1$, and in particular, $f \in |\widehat{N}|$. So $f \in \Gamma_\alpha$, as claimed. Familiar arguments now show that $\sigma{\restriction}|\widehat{N}||\alpha| = \pi{\restriction}|\widehat{N}||\alpha|$, and since this holds for every $\alpha < \mathrm{ht}(\widehat{N})$, this proves that $\sigma \subseteq \pi$.

In the following, I make use of the fact that σ is a Σ_0-extender ultrapower embedding, and is, in particular, cofinal:

$$\begin{aligned}
\text{“}\widehat{N} &= \bigcup_{\omega\alpha\in D_M} \mathtt{S}(M||\alpha)\text{”},\\
\text{“}N' &= \bigcup_{\omega\alpha\in \mathrm{On}_N} \sigma(\widehat{N}||\alpha)\text{”}.
\end{aligned}$$

So we get:

$$\text{“}N' = \bigcup_{\omega\alpha\in D_M} \sigma(\mathtt{S}(M||\alpha))\text{”}.$$

But since $\sigma \subseteq \pi$, and since $\pi : M \longrightarrow_{\Sigma_0} M'$ is cofinal, this means:

$$\begin{aligned}
N'^{\text{passive}} &= \text{“}\bigcup_{\omega\alpha\in D_M} \pi(\mathtt{S}(M||\alpha))\text{”}\\
&= \text{“}\bigcup_{\pi(\omega\alpha)\in D_{M'}} \mathtt{S}(M'||\pi(\alpha))\text{”}\\
&= \text{“}\bigcup_{\omega\beta\in D_{M'}} \mathtt{S}(M'||\beta)\text{”}\\
&= \widehat{\mathtt{S}(M')}^{\text{passive}}.
\end{aligned}$$

The above argument shows moreover that $\dot{E}^{N'} = \dot{E}^{\mathtt{S}(M')}$.

In case M is active, it is easily seen that $\dot{F}^{M'} = \dot{F}^{N'}$. This is because then $|M| = |\widehat{N}|$, hence $\pi = \sigma$, and $|M'| = |N'|$. It follows that for $a = \sigma(f)(\vec{\alpha}) \in |N'|$,

$$\begin{aligned}
\dot{F}^{N'}(a) &\iff \dot{F}^{N'}(\sigma(f)(\vec{\alpha}))\\
&\iff \prec\vec{\alpha}\succ \in F(\{\vec{\beta} \mid \dot{F}^{\widehat{N}}(f(\vec{\alpha}))\})\\
&\iff \prec\vec{\alpha}\succ \in F(\{\vec{\beta} \mid \dot{F}^{M}(f(\vec{\alpha}))\})\\
&\iff \dot{F}^{M'}(\pi(f)(\vec{\alpha}))\\
&\iff \dot{F}^{M'}(a).
\end{aligned}$$

This shows that $N' = \widehat{\mathtt{S}(M')}$, as wished. $\square$

Lemma 4.5. *Let M be a pPλ-structure, $N = \mathtt{S}(M)$. Let F be an extender on M and N with critical point κ. Assume $(\kappa^+)^M$ exists. Then* $\mathrm{Ult}^*(M, F)$ *exists iff* $\mathrm{Ult}^*(\widehat{N}, F)$ *exists, and is of the form $\widehat{N'}$, for a pPs-structure N'.*

Proof. If $\mathrm{Ult}^*(M, F)$ exists, then there is an embedding $k : I\!D^*(\widehat{N}, F) \longrightarrow I\!D^*(M, F)$, defined by

$$k([\vec{\alpha}, f]_{I\!D^*(\widehat{N},F)}) = [\vec{\alpha}, f]_{I\!D^*(M,F)}.$$

This works because, letting $\gamma := \mathrm{lh}(F)$, $D^*(\widehat{N}, \kappa, \gamma) \subseteq D^*(\kappa, M, \gamma)$, as follows from Lemma 3.6. Let's turn to the opposite direction.

Case 1: $\omega\rho^1_M = \omega\rho^1_{\widehat{N}} > \kappa$.

Then we're not dealing with Σ_0-ultrapowers. Let

$$\sigma : \widehat{N} \longrightarrow^*_F \widehat{N'},$$

and set $M' := \Lambda(N')$. This makes sense, as N' is a pPs-structure. I am going to construct an embedding

$$j : I\!D^*(M,F) \longrightarrow M'$$

which preserves the $\in$-relation, thus showing that $\mathrm{Ult}^*(M,F)$ is well-founded.

Consider the relation

$$R = \{\langle\langle m,q\rangle,x\rangle \mid x = h^1_M(m,q) \wedge q \in [\mathrm{ht}(\widehat{N})]^{<\omega}\}.$$

Obviously, R is uniformly $\Sigma_1(M)$ in $\mathrm{ht}(M)\dot{-}1$. Let h be a uniform uniformization of R which is $\mathbf{\Sigma}_1(M)$ in $\mathrm{ht}(M)\dot{-}1$, i.e., let

$$\begin{aligned}\forall x \in |M|((\exists m,q \in |M| \qquad & x = h^1_M(m,q) \wedge q \in [\mathrm{ht}(\widehat{N})]^{<\omega}) \\ & \longrightarrow x = h^1_M(h(x)) \wedge (h(x))^2_1 \in [\mathrm{ht}(\widehat{N})]^{<\omega}),\end{aligned}$$

and h is (uniformly) $\mathbf{\Sigma}_1(M)$ in $\mathrm{ht}(M)\dot{-}1$.

Then for $f \in \Gamma^*(M,\kappa)$, the function

$$\bar{f} := h \circ f$$

is also in $\Gamma^*(M,\kappa)$. Since moreover, $\mathrm{ran}(h) \subseteq |\widehat{N}|$, so that $\bar{f} \in \bar{\Gamma}^*(\kappa,M)$, it follows by Lemma 3.6 even that

$$\bar{f} \in \Gamma^*(\widehat{N},\kappa).$$

So the value $\sigma(\bar{f})$ can be made sense of as usual. This can be made use of in order to define:

$$j([\vec{\alpha},f]_{I\!D^*(M,F)}) := (h^1_{M'} \circ \sigma(\bar{f}))(\vec{\alpha}).$$

I prove the correctness of this definition. The same proof shows that j is Σ_1-preserving, which is more than needed in order to conclude the well-foundedness fact – it suffices to know:

$$[\vec{\alpha},f]_{I\!D^*(M,F)}E[\vec{\beta},g]_{I\!D^*(M,F)} \longrightarrow M' \models j([\vec{\alpha},f]_{I\!D^*(M,F)}) \in j([\vec{\beta},g]_{I\!D^*(M,F)}).$$

So let $\langle\vec{\alpha},f\rangle,\langle\vec{\beta},g\rangle \in \Gamma^*(M,\kappa)$, and let $[\vec{\alpha},f]_{I\!D^*(M,F)} = [\vec{\beta},g]_{I\!D^*(M,F)}$. I have to show that $h^1_{M'}(\sigma(\bar{f})(\vec{\alpha})) = h^1_{M'}(\sigma(\bar{g})(\vec{\beta}))$. Let $\chi(x,y)$ be uniformly Σ_1, so that

$$M \models \chi(x,y) \iff h^1_M(x) = h^1_M(y).$$

We have:

$$\begin{aligned} & [\vec{\alpha},f]_{I\!D^*(M,F)} = [\vec{\beta},g]_{I\!D^*(M,F)} \\ \iff & \prec\vec{\alpha},\vec{\beta}\succ \in F(\{\prec\vec{\gamma},\vec{\delta}\succ < \kappa \mid M \models f(\vec{\gamma}) = g(\vec{\delta})\}) \\ \iff & \prec\vec{\alpha},\vec{\beta}\succ \in F(\{\prec\vec{\gamma},\vec{\delta}\succ < \kappa \mid M \models \chi[\bar{f}(\vec{\gamma}),\bar{g}(\vec{\delta})]\}) \\ \iff & \prec\vec{\alpha},\vec{\beta}\succ \in F(\{\prec\vec{\gamma},\vec{\delta}\succ < \kappa \mid \widehat{N} \models \hat{f}_N(\chi)[\bar{f}(\vec{\gamma}),\bar{g}(\vec{\delta}),\mathrm{ht}(\widehat{N})\dot{-}1]\}) \\ \iff & \widehat{N'} \models \hat{f}_N(\chi)[\sigma(\bar{f})(\vec{\alpha}),\sigma(\bar{g})(\vec{\beta}),\mathrm{ht}(N')\dot{-}1] \\ \iff & \widehat{N'} \models \hat{f}_{N'}(\chi)[\sigma(\bar{f})(\vec{\alpha}),\sigma(\bar{g})(\vec{\beta}),\mathrm{ht}(N')\dot{-}1] \\ \iff & M' \models \chi[\sigma(\bar{f})(\vec{\alpha}),\sigma(\bar{g})(\vec{\beta})] \\ \iff & h^1_M(\sigma(\bar{f})(\alpha)) = h^1_M(\sigma(\bar{g})(\vec{\beta})),\end{aligned}$$

and that's what was to be shown. In the transition from $\widehat{N}$ to $\widehat{N'}$, I used the Łoś theorem, which is true for Σ_1-formulae, as $\omega\rho^1_{\widehat{N}} > \kappa$ (by [Zem02, Lemma 3.1.11 (d)], it even holds for

$\Sigma_0^{(1)}$-formulae). It follows from Corollary 2.9 that $\hat{f}_N = \hat{f}_{N'}$. As a side remark, it is easy to see that the map $\pi : M \longrightarrow M'$ defined by $\pi(x) = j([0, \mathrm{const}_x]_{\mathbb{D}^*(M,F)})$ is precisely the extender ultrapower embedding.

Case 2: $\omega\rho^1_{\widehat{N}} = \omega\rho^1_M \le \kappa$.

So in this case, Σ_0-extender ultrapowers are formed. Let

$$\sigma : \widehat{N} \longrightarrow_F N'.$$

Let $\mathrm{ht}(N) = \nu$ and $\mathrm{ht}(M) = \mu$.

Case 2.1: $\nu = \bar{\nu} + 1$.

Then also $\mu = \bar{\mu} + 1$. Let $\bar{M} := M||\bar{\mu}$ and $\bar{N} := N||\bar{\nu}$. Then $\bar{\sigma} : \bar{N} \longrightarrow^*_F \bar{N}'$ exists, and $\omega\rho^\omega_{\bar{N}} > \kappa$, as can be shown using an argument of the proof of Lemma 4.4, in case 1. Arguing like in case 1 of the current proof, it can be shown that consequently, $\bar{\pi} : \bar{M} \longrightarrow^*_F \bar{M}'$ exists: The onlyl problematic case is that $\bar{N}$ is active. Then $\bar{N}$ is uniformly $\Sigma_1(\bar{M})$ in the parameter $s(\bar{M})$. If in the argument of case 1 one replaces $\widehat{N}$ with $\bar{N}$, $\widehat{N'}$ with $\bar{N}'$, M with $\bar{M}$, M' with $\bar{M}'$ and σ with $\bar{\sigma}$ everywhere, the result is a proof of the desired conclusion. Note that $\bar{\sigma}$ is Σ_ω-preserving, so that $\bar{\sigma}(s(\bar{N})) = s(\bar{N}')$. This is crucial, since now R is uniformly $\Sigma_1(\bar{M})$ in $s(\bar{M})$, and I need that the same Σ_1-definition in the parameter $\bar{\sigma}(s(\bar{M}))$ defines the right relation in $\bar{M}'$.

Now an application of Lemma 4.1 yields that $\mathrm{Ult}(M, F)$ exists.

Case 2.2: ν and μ are limit ordinals.

The let $\sigma : \widehat{N} \longrightarrow_F \widehat{N'}$, $M' = \Lambda(N')$. As in case 1, I am going to define an embedding from $\mathbb{D}^*(M, F)$ into M', verifying that $\mathbb{D}^*(M, F)$ is well-founded. Since we're in case 2, $\mathbb{D}^*(M, F) = \mathbb{D}^0(M, F)$. Wloglet M and N be passive; otherwise, $\Gamma^0(\kappa, M) = \Gamma^0(\kappa, \widehat{N})$ and the well-foundedness is trivial. So in the following, I don't need to distinguish between $\widehat{N}$ and N. I define $k : \mathbb{D}(M, F) \longrightarrow M'$ as follows.

Let $[\vec{\alpha}, f] \in \mathbb{D}(M, F)$. Then let $f = h^1_M(m, d)$, where $d \in [\mathrm{ht}(N)]^{<\omega}$. Set

$$k([\vec{\alpha}, f]) := h^1_{M'}(m, d)(\vec{\alpha}).$$

I'll show that this definition is correct. Again, the same proof will show that k is Σ_1-preserving.

So let $\langle\vec{\alpha}, f\rangle, \langle\vec{\beta}, g\rangle \in D^0(\kappa, \lambda, M)$, so that $[\vec{\alpha}, f] = [\vec{\beta}, g]$, where $\lambda = \mathrm{lh}(F)$. Let $f = h^1_M(m, d)$, $g = h^1_M(n, e)$. Let $\chi(u, v, w, x, y, z)$ be a Σ_1-formula that has the following property for every pPλ-structure P:

$$P \models \chi[a, b, c, d, e, f] \iff \text{“}h^1_P(a, b) \text{ and } h^1_P(c, d) \text{ are functions, and } h^1_P(a, b)(e) \cong h^1_P(c, d)(f).\text{”}$$

Set:

$$u := \{\prec\vec{\mu}, \vec{\nu}\succ < \kappa \mid f(\vec{\mu}) = g(\vec{\nu})\}.$$

$[\vec{\alpha}, f] = [\vec{\beta}, g]$ says precisely that $\prec\vec{\alpha}, \vec{\beta}\succ \in F(u) = \sigma(u)$ (F may be assumed to be *whole*).

Choose $\gamma < \mathrm{ht}(M)$ with $\omega\gamma \in D_M$, so that $f, g \in |M||\gamma|$ and $\mathtt{S}(M||\gamma)$ is a segment of N – for example, γ can be chosen to be a successor ordinal. Let $f = h^1_{M||\gamma}(\bar{m}, \bar{d})$, $g = h^1_{M||\gamma}(\bar{n}, \bar{e})$.

(1) $h^1_{\Lambda(\sigma(\mathtt{S}(M||\gamma)))}(\bar{m}, \sigma(\bar{d})) = h^1_{M'}(m, \sigma(d))$. *The corresponding is true of* $\bar{n}$, $\bar{e}$, n, e.

Proof of (1). Let $M||\gamma = h^1_M(q, z)$, $z \in [\mathrm{ht}(N)]^{<\omega}$, $q < \omega$. If ψ is a Σ_1-formula which expresses the desired property uniformly, then we have:

$$\begin{aligned}
& h^1_{h^1_M(q,z)}(\bar{m},\bar{d}) = h^1_M(m,d) \\
\iff\quad & M \models \psi[q,z,\bar{m},\bar{d},m,d] \\
\iff\quad & N \models f_N(\psi)[q,z,\bar{m},\bar{d},m,d] \\
\iff\quad & N' \models f_N(\psi)[q,\sigma(z),\bar{m},\sigma(\bar{d}),m,\sigma(d)] \\
\iff\quad & N' \models f_{N'}(\psi)[q,\sigma(z)\bar{m},\sigma(\bar{d}),m,\sigma(d)] \\
\iff\quad & M' \models \psi[q,\sigma(z),\bar{m},\sigma(\bar{d}),m,\sigma(d)] \\
\iff\quad & h^1_{h^1_{M'}(q,\sigma(z))}(\bar{m},\sigma(\bar{d})) = h^1_{M'}(m,\sigma(d)).
\end{aligned}$$

So it remains to show that $h^1_{M'}(q,\sigma(z)) = \Lambda(\sigma(\mathtt{S}(M||\gamma)))$. To this end, let $N||\bar{\gamma} = \mathtt{S}(M||\gamma)$. Then we have:

$$\begin{aligned}
& M \models N||\bar{\gamma} = \mathtt{S}(h^1_M(q,z)) \\
\iff\quad & M \models \psi'[N||\bar{\gamma},q,z] \\
\iff\quad & N \models f'_N(\psi')[N||\bar{\gamma},q,z] \\
\iff\quad & N' \models f'_{N'}(\psi')[\sigma(N||\bar{\gamma}),q,\sigma(z)] \\
\iff\quad & M' \models \sigma(N||\bar{\gamma}) = \mathtt{S}(h^1_{M'}(q,\sigma(z))) \\
\iff\quad & \sigma(\mathtt{S}(M||\gamma)) = \mathtt{S}(h^1_{M'}(q,\sigma(z))) \\
\iff\quad & \Lambda(\sigma(\mathtt{S}(M||\gamma))) = h^1_{M'}(q,\sigma(z)),
\end{aligned}$$

as claimed. Here, I used the functions f'_N and $f'_{N'}$ from Lemma 3.1. Their definitions are uniform, and in the case that the structures have limit height, as in the current case, no additional parameters are needed. The same proof shows the corresponding for $\bar{n}$, $\bar{e}$, n, e. $\Box_{(1)}$

(2) $\sigma(u) = \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid h^1_{M'}(m,\sigma(d))(\vec{\mu}) = h^1_{M'}(n,\sigma(e))(\vec{\nu})\}$.

Proof of (2).

$$\begin{aligned}
u &= \{\prec\vec{\mu},\vec{\nu}\succ < \kappa \mid M||\gamma \models \chi[\bar{m},\bar{d},\bar{n},\bar{e},\langle\vec{\mu}\rangle,\langle\vec{\nu}\rangle]\} \\
&= \{\prec\vec{\mu},\vec{\nu}\succ < \kappa \mid N||\bar{\gamma} \models f_{N||\bar{\gamma}}(\chi)[\bar{m},\bar{d},\bar{n},\bar{e},\langle\vec{\mu}\rangle,\langle\vec{\nu}\rangle,\bar{\gamma}\dot{-}1]\}.
\end{aligned}$$

So u is defined in N by a Σ_0-formula in the parameters $N||\bar{\gamma}$, $\bar{m}$, $\bar{d}$, $\bar{n}$, $\bar{e}$ and $\bar{\gamma}\dot{-}1$. As σ is Σ_1-preserving, it follows that

$$\begin{aligned}
\sigma(u) &= \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid \sigma(N||\bar{\gamma}) \models f_{N||\bar{\gamma}}(\chi)[\bar{m},\sigma(\bar{d}),\bar{n},\sigma(\bar{e}),\langle\vec{\mu}\rangle,\langle\vec{\nu}\rangle,\sigma(\bar{\gamma})\dot{-}1]\} \\
&= \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid \sigma(N||\bar{\gamma}) \models f_{\sigma(N||\bar{\gamma})}(\chi)[\bar{m},\sigma(\bar{d}),\bar{n},\sigma(\bar{e}),\langle\vec{\mu}\rangle,\langle\vec{\nu}\rangle,\sigma(\bar{\gamma})\dot{-}1]\} \\
&= \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid \Lambda(\sigma(N||\bar{\gamma})) \models \chi[\bar{m},\sigma(\bar{d}),\bar{n},\sigma(\bar{e}),\langle\vec{\mu}\rangle,\langle\vec{\nu}\rangle,\sigma(\bar{\gamma})\dot{-}1]\} \\
&= \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid h^1_{\Lambda(\sigma(N||\bar{\gamma}))}(\bar{m},\sigma(\bar{d}))(\vec{\mu}) = h^1_{\Lambda(\sigma(N||\bar{\gamma}))}(\bar{n},\sigma(\bar{e}))(\vec{\nu})\}.
\end{aligned}$$

By (1), this means precisely:

$$\sigma(u) = \{\prec\vec{\mu},\vec{\nu}\succ < \sigma(\kappa) \mid h^1_{M'}(m,\sigma(d))(\vec{\mu}) = h^1_{M'}(n,\sigma(e))(\vec{\nu})\},$$

as claimed. $\Box_{(2)}$

Clearly, $\prec\vec{\alpha},\vec{\beta}\succ \in F(u) = \sigma(u)$, and this means by (2) that

$$h^1_{M'}(m,\sigma(d))(\vec{\alpha}) = h^1_{M'}(n,\sigma(e))(\vec{\beta}),$$

which shows that the definition of k is correct. Obviously, the same proof shows that

$$[\vec{\alpha},f]E[\vec{\beta},g] \implies k([\vec{\alpha},f]) \in k([\vec{\beta},g]),$$

and hence the well-foundedness of $I\!\!D(M,F)$. $\Box$

5 Iterations

I now introduce in the next subsection a notion of normal iteration (called normal s-iteration) of a pPλ-structure, which mimics the way pPs-structures are usually iterated. In the following subsections, I develop the theory of this kind of iterations, and in the end I introduce the fitting notion of normal iteration of a pPs-structure, which is basically the same as the notion of maximal iteration used in the Mitchell-Steel setup. The presentation follows [Jen01].

5.1 Normal s-iterations

In the following definition, note that it doesn't matter for the formation of fine-structural ultrapowers of a (pP)λ-structure whether we take it to be its Σ_0-code or just the bare structure, since the Σ_0-code just has some additional parameters, and the functions with respect to which the ultrapowers are formed have to be boldface definable anyway, so the additional parameters won't make a difference.

Definition 5.1. Let $\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, D, \langle\nu_i \mid i \in D\rangle, \langle\eta_i \mid i < \theta\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ be an iteration (in the sense of [Jen97, §4, p. 3]) of pPλ-structures. Set: $s_i := s(\nu_i)^{M_i}$, $s_i^+ := s^+(\nu_i)^{M_i}$. Then $\mathcal{I}$ is a normal s-iteration with s-indices $\langle\langle s_i, s_i^+\rangle \mid i \in \theta\rangle$ iff the following hold:

(a) $\mathcal{I}$ is standard (in the sense of [Jen97, §4, p. 4]).

(b) $s_h^+ < \nu_i$ for $h, i \in D$ with $h < i$.

(c) $T(i+1) =$ the least $\xi \in D$ with $\kappa_i < s_\xi$, if $i \in D$; otherwise $T(i+1) = i$. [5]

(d) Let $i \in D$. Then there is no $\nu > \nu_i$ such that $E_\nu^{M_i} \neq \emptyset$ and $s^+(\nu)^{M_i} < s^+(\nu_i)^{M_i}$. I'll say that ν_i is *applicable* in M_i in order to express this.

Definition 5.2. A pPλ-structure M is *normally s-iterable* if it has a successful normal iteration strategy $\mathcal{S}$. This means: $\mathcal{S}$ is a partial function whose domain is contained in the class of normal s-iterations of M of limit length, so that if $\mathcal{I}$ is a normal s-iteration of M which lies in the domain of $\mathcal{S}$, $\mathcal{S}(\mathcal{I})$ is a cofinal branch through the iteration tree. I'll say that an iteration $\mathcal{I}$ of M is according to $\mathcal{S}$, if for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{I})$, $\mathcal{S}(\mathcal{I}|\lambda) = (<_T)``\{\lambda\}$ (here, $T = T^{\mathcal{I}}$ is the tree of the iteration $\mathcal{I}$). $\mathcal{S}$ is a successful normal iteration strategy for M if every normal iteration of M which is according to $\mathcal{S}$ can be continued according to $\mathcal{S}$. This means firstly that if $\mathcal{I}$ is such an iteration of limit length, then $b := \mathcal{S}(\mathcal{I})$ is defined and the direct limit of the structures on b is well-founded. Secondly, every normal iteration of M which is according to $\mathcal{S}$ and has successor length, has to be continuable in the sense that one can pick any extender index in the last model of the iteration which satisfies (b) of Definition 5.1, apply it to the model prescribed by that definition, and thus produce a well-founded model.

This notion can also be defined using an *iteration game*, as in [Ste00].

Fix a normal s-iteration $\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, D, \langle\nu_i \mid i \in D\rangle, \langle\eta_i \mid i < \theta\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ in the following.

Lemma 5.3. *Let $i = T(h+1)$ and $h+1 \leq_T j$. Then $\pi_{i,j}\restriction\kappa_h = \mathrm{id}\restriction\kappa_h$.*

Proof. Suppose the statement of the lemma fails to hold of i. Let j be the least counterexample.

Case 1: $j = h+1$.

[5] Henceforth, I denote the immediate $<_T$-predecessor of $i+1$ by $T(i+1)$.

In this case,

$$\pi_{i,j} : M_i||\eta_h \longrightarrow^*_{E^{M_h}_{\nu_h}} M_j,$$

and hence $\kappa_h = \mathrm{crit}(\pi_{i,j})$.

Case 2: j is the immediate $<_T$-successor of j' and $h+1 \leq_T j'$.

Then let $j = k+1$, hence $j' = T(k+1)$. As in case 1, it follows that $\kappa_k = \mathrm{crit}(\pi_{j',j})$. According to condition (c) of normality, $\kappa_k \geq s_i$, as $i < j' = T(k+1)$. Moreover, again by condition (c), $\kappa_h < s_i$. Hence $\kappa_h < s_i \leq \kappa_k = \mathrm{crit}(\pi_{j',j})$, that is, in particular,

$$\pi_{j',j}\restriction\kappa_h = \mathrm{id}\restriction\kappa_h.$$

By minimality of j, we have:

$$\pi_{i,j'}\restriction\kappa_h = \mathrm{id}\restriction\kappa_h.$$

It follows immediately that the corresponding also holds for $\pi_{i,j} = \pi_{j',j}\pi_{i,j'}$.

Case 3: j is a limit point of T.

This cannot happen, as follows immediately from the minimality of j and the basic properties of the direct limit.

So all cases are excluded, and hence the lemma is proven. □

Lemma 5.4. *Let $j \in D \cap i$. Then $\mathrm{J}^{E^{M_j}}_{s_j^+} = \mathrm{J}^{E^{M_i}}_{s_j^+}$, and we have: $s_j^+ = (s_j)^{+M_i}$. The proof shows moreover that for $i \in D$ we have: $\tau_i < \eta_i$.*

Proof. Assume the contrary. We may assume that the iteration is direct (meaning that $D = \cup\theta$ – see [Jen97, §4, p. 4]). Let i be minimal so that there is a $j < i$ such that the claim fails. Let j be least with this property.

Case 1: i is a successor ordinal.

Let $i = h+1$ and $\xi = T(i)$. Then $\xi \leq h < i$ and $j \leq h$. By minimality of i, it follows that

$$\mathrm{J}^{E^{M_\xi}}_{s_\xi^+} = \mathrm{J}^{E^{M_h}}_{s_\xi^+},$$

and s_ξ^+ is a cardinal in M_h. By choice of ξ, $\kappa_h < s_\xi$. It follows that $s_\xi^+ \geq (\kappa_h^+)^{\mathrm{J}^{E^{M_h}}_{\nu_h}} = \tau_h$. Hence $\mathrm{J}^{E^{M_\xi}}_{\tau_h} = \mathrm{J}^{E^{M_h}}_{\tau_h}$. In particular, $\eta_h \geq s_\xi^+ \geq \tau_h$.

(1) $\tau_h < \eta_h$.

Proof of (1). Assume the contrary. It follows that $\eta_h = s_\xi^+ = \tau_h$, and further that $\eta_h < \mathrm{ht}(M_\xi)$: Otherwise,

$$\mathrm{ht}(M_\xi) = \eta_h = s_\xi^+ \leq \nu_\xi \leq \mathrm{ht}(M_\xi),$$

hence $\eta_h = \nu_\xi = s_\xi^+ = \tau_h$, and it would follow that

$$\begin{aligned}(*)\ \lambda_\xi &= \text{the largest cardinal of } \mathrm{J}^{E^{M_\xi}}_{\nu_\xi}\\ &= \text{the largest cardinal of } \mathrm{J}^{E^{M_h}}_{\tau_h}\\ &= \kappa_h < s_\xi \leq \lambda_\xi,\end{aligned}$$

a contradiction.

The fact that $\eta_h < \mathrm{ht}(M_\xi)$ entails that $\omega\rho^\omega_{M_\xi||\eta_h} \leq \kappa_h < s_\xi$. Hence s_ξ^+ is not a cardinal in $M_\xi||(s_\xi^+ + 1) = M_\xi||(\eta_h + 1)$. Hence $s_\xi^+ = \nu_\xi$, since s_ξ^+ is a cardinal in $M_\xi||\nu_\xi$ when $s_\xi^+ < \nu_\xi$. Again, as a consequence, $\eta_h = \nu_\xi = s_\xi^+ = \tau_h$, from which the contradiction $(*)$ follows as above. $\square_{(1)}$

(2) $\mathrm{J}^{E^{M_i}}_{\nu_h} = \mathrm{J}^{E^{M_h}}_{\nu_h}$, *and* $\nu_h \in \mathrm{Card}^{M_i}$.

Proof of (2). It follows from (1) that

$$\mathrm{J}^{E^{M_i}}_{\nu_h} = \pi_{\xi,i}(\mathrm{J}^{E^{M_\xi}}_{\tau_h}) = \pi_{\xi,i}(\mathrm{J}^{E^{M_h}}_{\tau_h}) = \mathrm{J}^{E^{M_h}}_{\nu_h},$$

and since τ_h is a cardinal in $M_\xi||\eta_h$, $\nu_h = \pi_{\xi,i}(\tau_h)$ is a cardinal in M_i. $\square_{(2)}$

It follows from this that the statement of the lemma holds for h and i: As $\nu_h \geq s_h^+$, it follows that

$$\mathrm{J}^{E^{M_i}}_{s_h^+} = \mathrm{J}^{E^{M_h}}_{s_h^+}.$$

Moreover, s_h^+ is a cardinal in M_i: As $\tau_h < \eta_h$, either $s_h^+ = \nu_h = \pi_{\xi,i}(\tau_h) \in \mathrm{Card}^{M_i}$, hence the image of a successor cardinal in $M_\xi||\eta_h$, or $s_h^+ < \nu_h$, which implies that $s_h^+ \in \mathrm{Card}^{\mathrm{J}^{E^{M_h}}_{\nu_h}} = \mathrm{Card}^{\mathrm{J}^{E^{M_i}}_{\nu_h}} \subseteq \mathrm{Card}^{M_i}$, as $\nu_h \in \mathrm{Card}^{M_i}$. But then s_h^+ even is the successor cardinal of s_h in M_i, as this is true in $M_h||\nu_h$: If α were a cardinal in M_i between s_h and s_h^+, then there would be a surjection $f : s_h \longrightarrow \alpha$ with $f \in M_h||\nu_h \subseteq M_i$, a contradiction.

Hence the claim holds for h, i. It follows that $j < h$. As $h < i$, the claim is true of j, h, by minimality, that is,

$$s_j^+ \quad \text{is the successor cardinal of } s_j \text{ in } M_h \text{ and } \mathrm{J}^{E^{M_j}}_{s_j^+} = \mathrm{J}^{E^{M_h}}_{s_j^+}.$$

By (b) in the definition of normal s-iterations (Definition 5.1), $\nu_h > s_j^+$. Moreover, by (2), $\mathrm{J}^{E^{M_i}}_{\nu_h} = \mathrm{J}^{E^{M_h}}_{\nu_h}$, and ν_h is a cardinal in M_i. Hence

$$\mathrm{J}^{E^{M_i}}_{s_j^+} = \mathrm{J}^{E^{M_h}}_{s_j^+} = \mathrm{J}^{E^{M_j}}_{s_j^+},$$

and s_j^+ is the successor cardinal of s_j in $\mathrm{J}^{E^{M_h}}_{\nu_h} = \mathrm{J}^{E^{M_i}}_{\nu_h}$, hence in M_i. Hence this case is excluded.

Case 2: i is a limit ordinal.

Let $j < l <_T i$ be such that there are no truncation points in $(<_T \text{“}\{i\}) \setminus l$, meaning that if $l \leq h+1 <_T i$, then $\eta_h = \mathrm{ht}(M_{T(h+1)})$. That this is possible is a consequence of the definition of an iteration (see [Jen97, §4, p. 3]).

Pick h, h' so that $l <_T h+1 <_T h'+1$ are in immediate succession in T. According to condition (c) in the definition of normality it then follows that

$$s_j \leq \kappa_h < s_l \leq \kappa_{h'}.$$

By minimality of i,

$$s_j^+ = (s_j)^{+M_{h+1}} \text{ and } \mathrm{J}^{E^{M_j}}_{s_j^+} = \mathrm{J}^{E^{M_{h+1}}}_{s_j^+}.$$

By choice of l, $M_{h+1} = M_{h+1}||\eta_{h'}$ and hence $\kappa_{h'}$ is a limit cardinal in M_{h+1}. So $\kappa_{h'}$ is a limit cardinal in M_{h+1} greater than s_j, and $s_j^+ = (s_j)^{+M_{h+1}}$, hence $s_j^+ < \kappa_{h'}$. But since $\pi_{h+1,i}\restriction\kappa_{h'} = \mathrm{id}\restriction\kappa_{h'}$,

$$\mathrm{J}^{E^{M_i}}_{s_j^+} = \pi_{h+1,i}(\mathrm{J}^{E^{M_{h+1}}}_{s_j^+}) = \mathrm{J}^{E^{M_{h+1}}}_{s_j^+} = \mathrm{J}^{E^{M_j}}_{s_j^+},$$

and $s_j^+ = \pi_{h+1,i}(s_j^+) = \pi_{h+1,i}((s_j)^{+M_{h+1}}) = (s_j)^{+M_i}$. So this case is also eliminated, and the proof is complete. $\Box$

Lemma 5.5. *For $j \in D$ and $i \in D \cap j$,*

$$s_i^+ \leq s_j^+.$$

If $s_i^+ = s_{i+1}^+$, then $s_i^+ = \nu_i$, in particular, M_i is not modest. Moreover, in this case, $s_{i+1}^+ < \nu_{i+1} = \mathrm{ht}(M_{i+1})$ (and so $s_{i+1}^+ < s_{i+2}^+$, if $i+1, i+2 \in D$), and $\nu_i < \mathrm{ht}(M_{i+1})$.

Before beginning the proof of this lemma, let me recall a general fact that was proved in the first part of this paper:

Lemma 5.6. *Let M be an active, weak j-ppm. Then $|M| = h_M^1(s(M))$, in particular $\omega\rho_M^1 \leq s(M)$. Moreover, if $\mu < \nu \leq \mathrm{ht}(M)$, then $s^+(\mu)^M \neq s^+(\nu)^M$.*

Proof of Lemma 5.5. The iteration may be assumed to be direct (meaning that $D = \cup\theta$). Let $i < \theta$, $\xi = T(i+1)$ and $M^* = M_\xi||\eta_i$. Assume that $s_i^+ \geq s_{i+1}^+$.

(1) $\nu_{i+1} = \mathrm{ht}(M_{i+1})$.

Proof of (1). We have:

$$\omega\rho^1_{M_{i+1}||\nu_{i+1}} \leq s_{i+1} < s_{i+1}^+ \leq s_i^+ < \nu_{i+1}.$$

Hence $\nu_{i+1} = \mathrm{ht}(M_{i+1})$, since otherwise s_i^+ wouldn't be a cardinal in M_{i+1}, contradicting Lemma 5.4. $\Box_{(1)}$

Hence $E_{\mathrm{top}}^{M^*} \neq \emptyset \neq E_{\mathrm{top}}^{M_{i+1}}$.

(2) $s(M^*) < \tau_i$.

Proof of (2). Assume the contrary, so that $\tau_i \leq s(\eta_i)^{M_\xi} < \eta_i$. It follows that

$$s_{i+1}^+ > s(M_{i+1}) \geq \sup \pi_{\xi,i+1}``s(M^*) \geq \sup \pi_{\xi,i+1}``\tau_i = \nu_i \geq s_i^+.$$

For details, see the proof of Lemma 5.25. This chain of inequations contradicts the assumption that $s_{i+1}^+ \leq s_i^+$. $\Box_{(2)}$

(3) $s_\xi^+ \leq s^+(M^*)$.

Proof of (3). Otherwise, $s^+(M^*) < s_\xi^+$, hence $\eta_i \neq \nu_\xi$. Then it would have to be the case that $\eta_i < \nu_\xi$, for otherwise, $\eta_i > \nu_\xi$ would imply that

$$s^+(\eta_i)^{M_\xi} < s^+(\nu_\xi)^{M_\xi} \leq \nu_\xi < \eta_i,$$

contradicting condition (d) in the definition of normality; the application of the extender $E_{\nu_\xi}^{M_\xi}$ wouldn't have been allowed. So it follows that $s_\xi^+ > \eta_i$, for otherwise

$$\omega\rho^1_{M_\xi||\eta_i} \leq s(\eta_i)^{M_\xi} < s_\xi^+ \leq \eta_i < \nu_\xi,$$

so that s_ξ^+ wouldn't be a cardinal in $M_\xi||\nu_\xi$. But τ_i is a cardinal in $\mathrm{J}^{E^{M_i}}_{s_\xi^+} = \mathrm{J}^{E^{M_\xi}}_{s_\xi^+}$ (or $\tau_i = s_\xi^+$), while η_i is maximal with this property, so that $\eta_i \geq s_\xi^+ > \eta_i$, a contradiction. $\Box_{(3)}$

Thus far, we have seen:

(4) $s(\eta_i) < \tau_i \leq s_\xi^+ \leq s^+(\eta_i)$ *in* M_ξ.

(5) $\tau_i = s^+(\eta_i)^{M_\xi} = s_\xi^+$.

Proof of (5). We have: $\tau_i \in (s(\eta_i), (s(\eta_i)^+)^{M_\xi}]$ and τ_i is a cardinal in M^*. Hence $\tau_i = (s(\eta_i)^+)^{M^*} = s^+(\eta_i)^{M_\xi}$. Using (4), it follows that $\tau_i \le s_\xi^+ \le s^+(\eta_i)^{M_\xi} = \tau_i$. $\square_{(5)}$

(6) $\nu_\xi = \eta_i$.

Proof of (6). If it were the case that $\nu_\xi \neq \eta_i$, then it would follow that $s_\xi^+ \neq s^+(\eta_i)^{M_\xi}$, contradicting (5) – the map $\mu \mapsto (s^+(\mu))^{M_\xi}$ is injective, by Lemma 5.6 $\square_{(6)}$

Hence $s(\eta_i)^{M_\xi} = s(\nu_\xi)^{M_\xi} = s_\xi > \kappa_i$, so that

$$s(M_{i+1}) \ge \operatorname{lub} \pi_{\xi,i+1}``s_\xi > \pi_{\xi,i+1}(\kappa_i) = \lambda_i.$$

As a consequence,

$$s_{i+1}^+ = s^+(\nu_{i+1})^{M_{i+1}} \ge (\lambda_i^+)^{M_{i+1}} = \nu_i \ge s_i^+ \ge s_{i+1}^+;$$

in order to see that $(\lambda_i^+)^{M_{i+1}} = \nu_i$, note that $\mathrm{J}_{\nu_i}^{E^{M_i}} = \mathrm{J}_{\nu_i}^{E^{M_{i+1}}}$ and $\nu_i \in \mathrm{Card}_{M_{i+1}}$ – see the proof of Lemma 5.4, Case 1. So $s_i^+ = s_{i+1}^+$, and thus it is shown that $\langle s_i^+ \mid i \in D\rangle$ is non-decreasing. Moreover, $s_i^+ = \nu_i$, hence M_i is not modest, as claimed. Finally, by Lemma 5.4, $\tau_i < \eta_i$, and hence $\nu_i = \pi_{\xi,i+1}(\tau_i) \in M_{i+1}$. So we have shown:

$$s_{i+1}^+ = s_i^+ = \nu_i < \operatorname{ht}(M_{i+1}) = \nu_{i+1}.$$

$\square$

Lemma 5.7. *For $i < j < \theta$ with $i, j \in D$, $\lambda_j > s_i^+$.*

Proof. We may assume $\mathcal{I}$ is direct. By Lemma 5.4, s_i^+ is a successor cardinal in M_j, and by condition (b) in the definition of normality, $\nu_j > s_i^+$. It follows that s_i^+ is a successor cardinal in $\mathrm{J}_{\nu_j}^{E^{M_j}}$. Since moreover,

$$\mathrm{J}_{\nu_j}^{E^{M_j}} \models \quad \lambda_j \text{ is a limit cardinal and the largest cardinal,}$$

it follows that $s_i^+ < \lambda_j$. $\square$

Definition 5.8. For $i < j < \theta$, set:

$$\lambda_{i,j} \simeq \min\{\lambda_l \mid l \in [i,j) \cap D\}.$$

Remark: In order to reduce the notational complexity, I will use the following conventions in situations where several iterations occur, if possible: If $\mathcal{I}' = \langle\langle M_i' \mid i < \theta'\rangle, D', \langle \nu_i' \mid i \in D'\rangle, \langle \eta_i' \mid i < \theta'\rangle, T', \langle {\pi'}_{i,j} \mid i \le_{T'} j < \theta'\rangle\rangle$ is an iteration, I write:

$$\lambda_{i,j}' \simeq \min\{\lambda_l' \mid l \in [i,j) \cap D'\},$$

and similarly for $\bar{\mathcal{I}}$ and $\bar\lambda_{i,j}$, *etc.*

Corollary 5.9. *If $h < i < j < \theta$, $h \in D$, then $s_h^+ < \lambda_{i,j}$. Moreover, for $i < j < \theta$ with $i \in D$,*

$$\begin{aligned} s_i = \lambda_i \quad &\Longrightarrow \quad \lambda_{i,j} = \lambda_i, \\ s_i < \lambda_i \quad &\Longrightarrow \quad \lambda_{i,j} > s_i^+. \end{aligned}$$

So in each case, $\lambda_{i,j} \ge s_i$.

Proof. This follows from Lemma 5.7. □

Lemma 5.10. *For $i < j$ with $[i, j) \cap D \neq \emptyset$,*

$$\lambda_{i,j} \textit{ is a limit cardinal in } M_j, \textit{ and } \mathrm{J}^{E^{M_i}}_{\lambda_{i,j}} = \mathrm{J}^{E^{M_j}}_{\lambda_{i,j}}.$$

Proof. Assume the iteration is direct. Fix i. I proceed by induction on $j \in (i, \theta)$.

$\boxed{j = i + 1}$ Then $\lambda_{i,j} = \lambda_i < \nu_i$, and we have $\mathrm{J}^{E^{M_i}}_{\nu_i} = \mathrm{J}^{E^{M_{i+1}}}_{\nu_i}$, in particular, $\mathrm{J}^{E^{M_i}}_{\lambda_{i,j}} = \mathrm{J}^{E^{M_j}}_{\lambda_{i,j}}$. Hence λ_i is a limit cardinal in $\mathrm{J}^{E^{M_{i+1}}}_{\nu_i}$. But ν_i is a cardinal in M_{i+1}, since $\nu_i = \pi_{\xi,i+1}(\tau_i)$. Hence, by acceptability of M_{i+1}, λ_i is a limit cardinal also in the full structure M_{i+1}.

$\boxed{j \to j + 1}$ Firstly, $\mathrm{J}^{E^{M_i}}_{\lambda_{i,j+1}} = \mathrm{J}^{E^{M_{j+1}}}_{\lambda_{i,j+1}}$: By the inductive hypothesis,

$$\mathrm{J}^{E^{M_i}}_{\lambda_{i,j+1}} = \mathrm{J}^{E^{M_j}}_{\lambda_{i,j+1}},$$

since $\lambda_{i,j+1} \leq \lambda_{i,j}$. Moreover, it is easy to see that

$$\mathrm{J}^{E^{M_j}}_{\nu_j} = \mathrm{J}^{E^{M_{j+1}}}_{\nu_j}.$$

Since

$$\nu_j > \lambda_j \geq \lambda_{i,j+1}$$

it thus follows that

$$\mathrm{J}^{E^{M_i}}_{\lambda_{i,j+1}} = \mathrm{J}^{E^{M_j}}_{\lambda_{i,j+1}} = \mathrm{J}^{E^{M_j+1}}_{\lambda_{i,j+1}}.$$

It remains to be shown that $\lambda_{i,j+1}$ is a limit cardinal in M_{j+1}.

Case 1.: $\lambda_{i,j} = \lambda_{i,j+1}$.

In this case, $\lambda_j \geq \lambda_{i,j}$. By inductive assumption, $\lambda_{i,j}$ is a limit cardinal in M_j. Obviously, $\nu_j > \lambda_j \geq \lambda_{i,j}$. So we have:

$$\lambda_{i,j+1} = \lambda_{i,j} \text{ is a limit cardinal in } \mathrm{J}^{E^{M_j}}_{\nu_j} = \mathrm{J}^{E^{M_j+1}}_{\nu_j}.$$

As before, this implies that $\lambda_{i,j+1}$ is a limit cardinal in M_{j+1} as well.

Case 2.: $\lambda_{i,j} > \lambda_{i,j+1}$.

In this case, $\lambda_j = \lambda_{i,j+1} < \lambda_{i,j}$. Obviously, λ_j is a limit cardinal in $\mathrm{J}^{E^{M_j}}_{\nu_j} = \mathrm{J}^{E^{M_{j+1}}}_{\nu_j}$, which implies that λ_j is a limit cardinal in M_{j+1}. As $\lambda_j = \lambda_{i,j+1}$, this is all that's needed.

$\boxed{\mathrm{Lim}(j)}$ As the set of truncation points in the branch $<_T$ "$\{j\}$ is bounded in j, $i_0 + 1 <_T j$ can be chosen in such a way that the following conditions are satisfied:

(a) There are no truncations in $(<_T \text{“}j) \setminus i_0$.

(b) $i_0 \geq i$.

(c) $\lambda_{i,j} = \lambda_{i,i_0}$.

Now define a sequence $\langle i_n \mid n < \omega \rangle$ as follows: i_0 has been defined already. If i_n is defined, then let $i_{n+1} + 1$ be the immediate T-successor of $i_n + 1$ with $i_{n+1} + 1 <_T j$. So the sequence $\langle i_n + 1 \mid n < \omega \rangle$ enumerates the first ω members of the branch below j that are above $i_0 + 1$, $<_T$-increasingly. We have:

(∗) *There is an* $n \in \omega$*, such that* $\kappa_{i_{n+2}} \geq \lambda_{i_{n+1}}$.

Proof of (∗). Assume the contrary. So for each $n < \omega$,

$$\kappa_{i_{n+2}} < \lambda_{i_{n+1}}.$$

Since

$$\operatorname{crit}(\pi_{i_n+1,i_{n+1}+1}) = \kappa_{i_{n+1}} \text{ und } \pi_{i_n+1,i_{n+1}+1}(\kappa_{i_{n+1}}) = \lambda_{i_{n+1}},$$

it follows that

$$\begin{array}{rcl} \pi_{i_n+1,j}(\kappa_{i_{n+1}}) & = & \pi_{i_{n+1}+1,j}\pi_{i_n+1,i_{n+1}+1}(\kappa_{i_{n+1}}) \\ & = & \pi_{i_{n+1}+1,j}(\lambda_{i_{n+1}}) \\ & > & \pi_{i_{n+1}+1,j}(\kappa_{i_{n+2}}). \end{array}$$

This holds for all $n < \omega$, hence $\langle \pi_{i_n+1,j}(\kappa_{i_{n+1}}) \mid n < \omega \rangle$ is a decreasing $\in$-chain in M_j, which is well-founded, a contradiction. $\square_{(*)}$

Now pick n as in (∗). Since the sequence $\langle \kappa_{i_n} \mid 1 \leq n < \omega \rangle$ is strictly increasing (as $\kappa_{i_{n+1}} < s_{i_n+1} \leq \kappa_{i_{n+2}}$), it follows that

$$\operatorname{crit}(\pi_{i_{n+2}+1,j}) = \kappa_{i_{n+3}} > \kappa_{i_{n+2}} \geq \lambda_{i_{n+1}} \geq \lambda_{i,j}.$$

Set: $i' = i_{n+2} + 1$. Then by minimality of j,

$$\mathrm{J}^{E^{M_i}}_{\lambda_{i,i'}} = \mathrm{J}^{E^{M_{i'}}}_{\lambda_{i,i'}},$$

and $\lambda_{i,i'}$ is a limit cardinal in $M_{i'}$. As i_0 was chosen so that $\lambda_{i,j} = \lambda_{i,i_0}$, and as $i' > i_0$, it follows that $\lambda_{i,j} = \lambda_{i,i'}$. We get:

$$\mathrm{J}^{E^{M_j}}_{\lambda_{i,j}} = \pi_{i',j}(\mathrm{J}^{E^{M_{i'}}}_{\lambda_{i,i'}}) = \mathrm{J}^{E^{M_{i'}}}_{\lambda_{i,i'}} = \mathrm{J}^{E^{M_i}}_{\lambda_{i,i'}} = \mathrm{J}^{E^{M_i}}_{\lambda_{i,j}},$$

and $\lambda_{i,j} = \pi_{i',j}(\lambda_{i,i'})$ is a limit cardinal in M_j. $\square$

The proof of the following lemma illustrates how $\lambda_{i,j}$ takes over the role of λ_i in the usual Friedman-Jensen setting.

Lemma 5.11. *Let* $\mathcal{I} = \langle \langle M_i \mid i < \theta \rangle, D, \langle \nu_i \mid i \in D \rangle, \langle \eta_i \mid i < \theta \rangle, T, \langle \pi_{i,j} \mid i \leq_T j < \theta \rangle \rangle$ *be a normal s-iteration of the pPλ-structure* M*, with s-indices* $\langle \langle s_i, s_i^+ \rangle \mid i \in D \rangle$*. Then for* $i \in D$*, the following hold:*

(a) If $\nu_i = \operatorname{ht}(M_i)$*, then* $\mathcal{P}(\tau_i) \cap \mathbf{\Sigma}_1(M_i) \subseteq \mathbf{\Sigma}_1(M_{T(i+1)}||\eta_i)$.

(b) If $\nu_i < \operatorname{ht}(M_i)$*, then* $E^{M_i}_{\nu_i}$ *is* Σ_1-*amenable (see [Jen97, §1, p. 12]) wrt.* $M_{T(i+1)}||\eta_i$.

Proof. Assume the contrary. Let $\mathcal{I}$ be a counterexample of minimal length. We may assume $\mathcal{I}$ is direct. Let $i \in D$ be such that (a) or (b) are not satisfied. Then $\theta = i + 2$, for otherwise $\mathcal{I}|(i+2)$ (this is the canonically defined initial segment of $\mathcal{I}$ of length $i + 2$) a shorter normal s-iteration which is a counterexample. For the same reason, (a) and (b) hold for all $j < i$.

(1) $T(i + 1) < i$.

Proof of (1). Otherwise, $T(i + 1) = i$ and hence (a) and (b) hold trivially. $\square_{(1)}$

Let $\nu = \nu_i$, $\kappa = \kappa_i$, $\delta = T(i + 1)$ and $\tau = \tau_i$.

(2) $\nu = \operatorname{ht}(M_i)$.

Proof of (2). Otherwise $\nu < \mathrm{ht}(M_i)$. Let $\alpha < \lambda_i = \mathrm{lh}(E_\nu^{M_i})$. By normality of $\mathcal{I}$, $\kappa < s_\delta \leq \lambda_{\delta,i}$; see Corollary 5.9. Moreover, $\lambda_{\delta,i}$ is a limit cardinal in M_i, by Lemma 5.10. Together with the acceptability of M_i, this yields: $(E_\nu^{M_i})_\alpha := \{x \subseteq \kappa \mid \alpha \in E_\nu^{M_i}(x)\} \in \mathrm{J}^{E^{M_i}}_{(\kappa^{++})^{M_i}} \subseteq \mathrm{J}^{E^{M_i}}_{\lambda_{\delta,i}} = \mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,i}}$.

But $\lambda_{\delta,i} \leq \eta_i$: Note that $\kappa < s_\delta < s_\delta^+ < \nu$, $s_\delta^+ \in \mathrm{Card}^{M_i}$. Hence, $(\kappa^+)^{M_i||\nu_i} = (\kappa^+)^{M_i||s_\delta^+} = (\kappa^+)^{M_i}$, so that $\mathcal{P}(\kappa) \cap M_i = \mathcal{P}(\kappa) \cap M_i||\nu$. Because $\mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,i}} = \mathrm{J}^{E^{M_i}}_{\lambda_{\delta,i}}$, it follows that $\eta_i \geq \lambda_{\delta,i}$, as claimed, since $\mathcal{P}(\kappa) \cap |M_i||\nu_i| = \mathcal{P}(\kappa) \cap |\mathrm{J}^{E^{M_i}}_{\lambda_{\delta,i}}| = \mathcal{P}(\kappa) \cap |\mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,i}}|$.

So $(E_\nu^{M_i})_\alpha \in M_\delta||\eta_i$. In particular, $(E_\nu^{M_i})_\alpha$ is $\mathbf{\Sigma}_1(M_\delta||\eta_i)$, hence i is not a counterexample, a contradiction. $\square_{(2)}$

(3) *i is not a limit ordinal.*

Proof of (3). Assume the contrary. As i is a counterexample, there is $A \in (\mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M_i)) \setminus \mathbf{\Sigma}_1(M_\delta||\eta_i)$. Let A be $\Sigma_1(M_i)$ in p. Pick $\alpha <_T i$, $\alpha > \delta$ such that $\kappa < \mathrm{crit}(\pi_{\alpha,i}) = \kappa_\beta$, where $\alpha = T(\beta+1)$. Moreover, fix α large enough that there are no truncations in $(<_T \text{“}\{i\}) \setminus \alpha$, and so that $p \in \mathrm{ran}(\pi_{\alpha,i})$. Let $\bar{p} := \pi_{\alpha,i}^{-1}(p)$.

Since the sequence of critical points along a branch of T is strictly increasing, it follows that $\mathcal{P}(\kappa_\beta) \cap M_\alpha = \mathcal{P}(\kappa_\beta) \cap M_i$, hence κ_β is a limit cardinal in M_i greater than κ, so $\kappa_\beta > \tau$. As $\pi_{\alpha,i}$ is at least Σ_1-preserving, and as $\mathrm{crit}(\pi_{\alpha,i}) > \tau$, it follows that A is $\Sigma_1(M_\alpha)$ in $\bar{p}$.

Define an s-iteration $\mathcal{I}' = \langle\langle M'_\beta \mid \beta < \theta'\rangle, D', \langle \nu'_\beta \mid \beta \in D'\rangle, \langle \eta'_\beta \mid \beta < \theta'\rangle, T', \langle {\pi'}_{\beta,\gamma} \mid \beta \leq_{T'} \gamma < \theta'\rangle\rangle$ of M as follows. Set: $\nu'_\alpha := \mathrm{ht}(M_\alpha)$. It follows from the Σ_1-preservation of $\pi_{\alpha,i}$ that $E^{M_\alpha}_{\nu'_\alpha} \neq \emptyset$, as $E^{M_i}_{\nu_i} \neq \emptyset$. Let $\kappa'_\alpha := \mathrm{crit}(E^{M_\alpha}_{\nu'_\alpha})$. $E^{M_\alpha}_{\nu'_\alpha}$ is an extender with critical point κ'_α, hence $E_\nu^{M_i}$ has critical point $\pi_{\alpha,i}(\kappa'_\alpha) = \kappa$, and hence $\kappa = \kappa'_\alpha$, as $\kappa < \mathrm{crit}(\pi_{\alpha,i})$. Moreover, $\pi_{\alpha,i}\restriction\mathcal{P}(\kappa) = \mathrm{id}\restriction(\mathcal{P}(\kappa) \cap M_\alpha)$. Set: $\nu'_j := \nu_j$ for $j < \alpha$, $T' := (T \cap (\alpha+1)^2) \cup \{\langle \xi, \alpha+1\rangle\}$, where ξ is the least $\mu \leq \alpha$ such that $\kappa_i = \kappa'_\alpha < s_\mu$. So $\xi = T'(\alpha+1) = T(i+1) = \delta$, $\tau = \tau_i = \tau'_\alpha$ and $\eta'_\alpha = \eta_i$, since $\kappa'_\alpha = \kappa$, $\tau < \kappa_\beta$ and $\delta < \alpha$. The s-indices s'_j and ${s'}_j^+$ are defined accordingly. Finally, $\theta' := \alpha + 2$. Then $M'_\xi||\eta'_\alpha = M_\delta||\eta_i$ is *-extendible by $E^{M_\alpha}_{\nu'_\alpha}$, which can be seen as follows:

$$\langle \mathrm{id}, \pi_{\alpha,i}\restriction\lambda'_\alpha\rangle : \langle M_\delta||\eta_i, E^{M_\alpha}_{\nu'_\alpha}\rangle \longrightarrow \langle M_\delta||\eta_i, E^{M_i}_{\nu_i}\rangle,$$

since for $X \in \mathcal{P}(\kappa) \cap M_\delta||\eta_i = \mathcal{P}(\kappa) \cap M_\alpha$ and $\alpha_1, \ldots, \alpha_n < \lambda'_\alpha$,

$$\prec\alpha_1, \ldots, \alpha_n\succ \in E^{M_\alpha}_{\nu'_\alpha}(X) \iff \prec\pi_{\alpha,i}(\alpha_1), \ldots, \pi_{\alpha,i}(\alpha_n)\succ \in E^{M_i}_{\nu_i}(\pi_{\alpha,i}(X)) = E^{M_i}_{\nu_i}(X),$$

and $\pi_{\alpha,i}\restriction\lambda'_i : \lambda'_\alpha \longrightarrow \lambda_i$. The desired extendibility follows from [Jen97, Kap.3, Lemma 1], as the identity map has all preservation properties one could wish for.

But $\mathcal{I}'$ is a normal s-iteration of M ($\nu'_\alpha \geq \nu_\alpha > s_j^+ = {s'}_j^+$ for $j < \alpha$) shorter than $\mathcal{I}$. Hence, α satisfies condition (a) in $\mathcal{I}'$. Since A is $\mathbf{\Sigma}_1(M_\alpha)$ it follows that $A \in \mathbf{\Sigma}_1(M'_{T(\alpha+1)}||\eta'_\alpha) = \mathbf{\Sigma}_1(M_\delta||\eta_i)$, contradicting the choice of A. $\square_{(3)}$

So let $i = h+1$, and set:$\xi := T(i)$, $M^* := M_\xi||\eta_h$ and $F := E^{M_h}_{\nu_h}$. So $\pi_{\xi,i} : M^* \longrightarrow^*_F M_i$.

(4) $\kappa < \kappa_h$. *Hence* $\pi_{\xi,i}\restriction(\tau^+)^{M^*} = \mathrm{id}$.

Proof of (4). By the preservation properties of $\pi_{\xi,i}$, it follows that $E^{M^*}_{\mathrm{top}} = E^{M_i}_{\mathrm{top}} \neq \emptyset$, as $E^{M_i}_{\mathrm{ht}(M_i)} \neq \emptyset$, by (2). Moreover, setting $\kappa' := \mathrm{crit}(E^{M^*}_{\mathrm{top}})$, it follows that $\kappa = \pi_{\xi,i}(\kappa')$.

It follows that $\kappa' < \kappa_h$: Otherwise, $\kappa = \pi_{\xi,i}(\kappa') \geq \pi_{\xi,i}(\kappa_h) = \lambda_h \geq s_h$. This would imply for $j < i$ that

$$\kappa \geq \lambda_h \geq \lambda_{j,h+1} \geq s_j,$$

by corollary 5.9. By definition of $\delta = T(i+1)$, this would entail that $T(i+1) = i$, contradicting (1).

The first part of the claim follows now, as $\kappa_h = \mathrm{crit}(\pi_{\xi,i})$:

$$\kappa = \pi_{\xi,i}(\kappa') = \kappa' < \kappa_h.$$

Turning to the second part, note that by weak amenability of F, $\mathcal{P}(\kappa_h) \cap M^* = \mathcal{P}(\kappa_h) \cap M_i$, and κ_h is a limit cardinal in M_i. As $\kappa < \kappa_h$ and $\tau = (\kappa^+)^{M_i}$, $\tau < \kappa_h$. But of course, κ_h is a limit cardinal in M^*, hence $(\tau^+)^{M^*} < \kappa_h = \mathrm{crit}(\pi_{\xi,i})$. This proves the claim. $\square_{(4)}$

(5) $\delta \leq \xi$.

Proof of (5). By (4) and normality of $\mathcal{I}$, it follows that $\kappa < \kappa_h < s_\xi$. But $\delta = T(i+1)$ is the least γ with $\kappa < s_\gamma$. $\square_{(5)}$

(6) *F is $\mathbf{\Sigma}_1-$amenable wrt. M^*.*

Proof of (6). This follows immediately from the minimality of $\mathrm{lh}(\mathcal{I})$. $\square_{(6)}$

(7) $\tau < \lambda_{\delta,i}$.

Proof of (7). We have $\kappa < s_\delta$. I'll use Corollary 5.9 in the following.
If $s_\delta = \lambda_\delta$, then $s_\delta = \lambda_\delta = \lambda_{\delta,i} > \tau$, as $\lambda_{\delta,i}$ is a limit cardinal in M_i.
If on the other hand, $s_\delta < \lambda_\delta$, then $\lambda_{\delta,i} > s_\delta^+ \geq \tau$, as s_δ^+ is a cardinal greater than κ in M_i. $\square_{(7)}$

(8) $\omega\rho^1_{M_i} \leq \tau$.

Proof of (8). Assume $\omega\rho^1_{M_i} > \tau$. Let $A \subseteq \tau$ be a $\mathbf{\Sigma}_1(M_i)-$set. By assumption, $A \in M_i$. By acceptability of M_i, $A \in \mathrm{J}^{E^{M_i}}_{(\tau^+)^{M_i}}$. But $(\tau^+)^{M_i} < \lambda_{\delta,i}$, as $\lambda_{\delta,i}$ is a limit cardinal in M_i, and because of (7). Since $\mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,i}} = \mathrm{J}^{E^{M_i}}_{\lambda_{\delta,i}}$, it follows that

$$A \in \mathrm{J}^{E^{M_i}}_{\lambda_{\delta,i}} = \mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,i}}.$$

Since A was arbitrary, (a) is satisfied at i. But vacuously, (b) is also satisfied at i, so that i was no counterexample after all. $\square_{(8)}$

(9) $\omega\rho^1_{M^*} \leq \tau$.

Proof of (9). By (6), $\pi_{\xi,i} : M^* \longrightarrow M_i$ is $\mathbf{\Sigma}^*-$preserving. By (4), $\pi_{\xi,i}\restriction(\tau^+)^{M^*} = \mathrm{id}$. The claim follows from (8), as $\pi_{\xi,i}``H^1_{M^*} \subseteq H^1_{M_i}$. $\square_{(9)}$

(10) $\mathcal{P}(\kappa_h) \cap \mathbf{\Sigma}_1(M_i) \subseteq \mathbf{\Sigma}_1(M^*)$.

Proof of (10). $\kappa_h = \mathrm{crit}(\pi_{\xi,i})$, hence (10) follows from [Jen97, §2, Corollary 6.5.]. $\square_{(10)}$

(11) $\delta < \xi$.

Proof of (11). Assume $\xi \leq \delta$. By (5), $\delta \leq \xi$, hence $\delta = \xi$. By (4), $\tau < \kappa_h$.
If $\eta_h = \eta_i$, (10) shows that property (a) is satisfied, a contradiction.
Otherwise, $\eta_h < \eta_i$, as $\tau < \kappa_h$. Then $M^* \in M_\delta||\eta_i$. But this implies:

$$\mathbf{\Sigma}_1(M_i) \cap \mathcal{P}(\tau) \subseteq \mathbf{\Sigma}_1(M^*) \subseteq M_\xi||\eta_i = M_\delta||\eta_i.$$

Hence the properties (a) and (b) are satisfied at i, a contradiction. $\square_{(11)}$

(12) *$M^* = M_\xi$ (hence $\eta_h = \mathrm{ht}(M_\xi)$).*

Proof of (12). Assume the contrary, so $\eta_h < \mathrm{ht}(M_\xi)$. By (8), $\omega\rho^1_{M_i} \leq \tau$. Moreover, $\pi_{\xi,i} : M^* \longrightarrow_{\Sigma^*} M_i$, and by (4), $\kappa_h > \tau$, hence $\omega\rho^1_{M^*} = \omega\rho^1_{M_i} \leq \tau$. Now let $A \subseteq \tau$ be $\mathbf{\Sigma}_1(M_i)$ and a counterexample for (a) at i. By (10), A is $\mathbf{\Sigma}_1(M_\xi||\eta_h)$. But as $M_\xi||\eta_h \in M_\xi$, it follows that $A \in \mathcal{P}(\tau) \cap M_\xi$. By (7), $\tau < \lambda_{\delta,i} \leq \lambda_{\delta,\xi}$, and as $\lambda_{\delta,\xi}$ is a cardinal in M_ξ, it follows by acceptability that

$$A \in \mathcal{P}(\tau) \cap M_\xi \subseteq \mathrm{J}^{E^{M_\xi}}_{\lambda_{\delta,\xi}} = \mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,\xi}}.$$

Note that $\eta_i \geq \lambda_{\delta,\xi}$. This is because

$$\mathcal{P}(\kappa) \cap \mathrm{J}^{E^{M_i}}_{\nu_i} = \mathcal{P}(\kappa) \cap \mathrm{J}^{E^{M_\delta}}_{s^+_\delta} = \mathcal{P}(\kappa) \cap \mathrm{J}^{E^{M_\delta}}_{\nu_\delta}$$

by acceptability; for the first identity, note that $\mathrm{J}^{E^{M_\delta}}_{s^+_\delta} = \mathrm{J}^{E^{M_i}}_{s^+_\delta}$ and s^+_δ is a cardinal in M_i. For the second one, it suffices to remark that s^+_δ is a cardinal in $\mathrm{J}^{E^{M_\delta}}_{\nu_\delta}$, if $s^+_\delta < \nu_\delta$. As $\nu_\delta > \lambda_{\delta,\xi}$, this implies that $\eta_i \geq \lambda_{\delta,\xi}$.

Hence,

$$A \in \mathrm{J}^{E^{M_\delta}}_{\lambda_{\delta,\xi}} \subseteq M_\delta||\eta_i,$$

which contradicts the choice of A. $\Box_{(12)}$

As was shown in (4), $\kappa = \kappa' := \mathrm{crit}(E^{M^*}_{\eta_h})$ and $\tau = (\kappa^+)^{M_i} = (\kappa'^+)^{M^*}$. As in the proof of (3) it can be shown that $M_\delta||\eta_i$ is *-extendible by $E^{M^*}_{\eta_h}$, since

$$\langle \mathrm{id}\restriction M_\delta||\eta_i, \pi_{\xi,i}\rangle : \langle M_\delta||\eta_i, E^{M^*}_{\eta_h}\rangle \longrightarrow \langle M_\delta||\eta_i, E^{M_i}_{\nu_i}\rangle,$$

and $M_\delta||\eta_i$ is *-extendible by $E^{M_i}_{\nu_i}$ by assumption. Now, in analogy to the proof of (3), one can define an normal s-iteration $\mathcal{I}' := \langle\langle M'_\alpha \mid \alpha < \theta'\rangle, D', \langle \nu'_\alpha \mid \alpha \in D'\rangle, \langle \eta'_\alpha \mid \alpha < \theta'\rangle, T', \langle {\pi'}_{\alpha,\beta} \mid \alpha \leq_{T'} \beta < \theta'\rangle\rangle$ of M with length $\theta' := \xi + 2$, by setting: $\nu'_\xi := \eta_h = \mathrm{ht}(M_\xi)$ and $\nu'_j := \nu_j$ for $j < \xi$. So $\kappa'_\xi = \kappa$, $\tau'_\xi = \tau$, hence $T'(\xi+1) = T(i+1)$ and $\eta'_\xi = \eta_i$, as $\delta < \xi$. By (4), (10) and (12),

$$\mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M_i) \subseteq \mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M_\xi).$$

As $\xi = T(i)$, $\xi < i$, and hence, $\mathcal{I}'$ is a normal s-iteration of M shorter than $\mathcal{I}$. It follows by minimality of $\mathrm{lh}(\mathcal{I})$ that

$$\mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M_\xi) = \mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M'_\xi) \subseteq \mathbf{\Sigma}_1(M'_{T'(\xi+1)}||\eta'_\xi) = \mathbf{\Sigma}_1(M_\delta||\eta_i).$$

Taken together, this shows:

$$\mathcal{P}(\tau) \cap \mathbf{\Sigma}_1(M_i) \subseteq \mathbf{\Sigma}_1(M_\delta||\eta_i).$$

So i does satisfy the conditions (a) and (b) after all. $\Box$

5.2 Initial Segment Conditions

Since I use an initial segment condition for pλ-structures that differs from what's called "s-ISC" in [Jen01], it needs to be checked that the new condition has all the properties needed. Thus, it should have the usual properties an initial segment condition should have: Being preserved under iterations (in the present case under normal s-iterations), and guaranteeing that the coiteration process of two coiterable structures terminates. I will prove the first part in the present section. Another property that's crucial in the context of the present paper is that the s'-ISC shouldn't be too restrictive. More precisely, it should be a consequence of the Z-ISC. [6] So what I will

[6] I refer to the initial segment condition of [Ste00] as Z-ISC, since the notion of a type-Z-extender is used in its formulation.

show is that hereditarily continuable Mitchell-Steel-premice satisfy the s'-ISC. It is obvious that normally iterable Mitchell-Steel-premice are hereditarily continuable, so that this assumption can be dropped for Mitchell-Steel-mice.

First, let me recall the definition of the s'-ISC and the s'-MISC, on which it builds.

Definition 5.12. Let M be an active extender structure. M satisfies the minimal *s'-initial segment condition* (s'-MISC), iff, letting $F := E^M_{\mathrm{top}}$, for every cutpoint[7] $\xi \in [\tau(F), s(F))$ of F, $(\xi^+)^M \neq (\xi^+)^{[M]_\xi}$.

Definition 5.13. Let M be a potential Pseudo-λ- or s-structure. The s'-initial segment condition (s'-ISC) for M says that for every $\alpha \leq \mathrm{ht}(M)$ with $F = E^M_\alpha \neq \emptyset$ and each cutpoint $\xi \in [\tau(F), s(F))$ of F,

(a) If $[M||\alpha]_\xi$ satisfies the s'-MISC, then $[M||\alpha]_\xi \in \widehat{M||\alpha}$.

(b) If $[M||\alpha]_\xi$ satisfies the s'-MISC and $\xi' \in [\tau(F), \xi)$ is such that $[M||\alpha]_{\xi'}$ satisfies the s'-MISC, then $[M||\alpha]_{\xi'} \in [M||\alpha]_\xi$.

The following is a folkloristic fact (which I made use of in the first part of this paper already). For the reader's convenience, I include a proof here.

Lemma 5.14. *Let $N = \langle \mathrm{J}^E_\nu, F\rangle$ be an active pPλ- or pPs-structure. Let $s = s(F)$, $\tau = \tau(F)$ and $\tau \leq \xi < \zeta \leq s$. Then*

$$\mathrm{crit}(\sigma^N_{\xi,\zeta}) \simeq \min((\zeta \cap \mathrm{gen}_F) \setminus \xi).$$

In particular, if $[\xi, \zeta)$ contains no generators, then $\sigma_{\xi,\zeta} = \mathrm{id}\restriction|[N]_\xi|$.

Proof. If $(\zeta \cap \mathrm{gen}_F) \setminus \xi = \emptyset$, then obviously, $[N]_\xi = [N]_\zeta$ and $\sigma_{\xi,\zeta} = \mathrm{id}\restriction|[N]_\xi|$, as claimed. So let $\xi' = \min(\zeta \cap \mathrm{gen}_F) \setminus \xi$ exist. In the following, I write π_α, $\sigma_{\alpha,\beta}$ for π^N_α, $\sigma^N_{\alpha,\beta}$, respectively.

(1) $\xi' \subseteq \mathrm{ran}(\sigma_{\xi,s})$.

Proof of (1). Let $\gamma < \xi'$. Then there exist $\vec{\alpha} \in \mathrm{gen}_F \cap \xi'$ such that $\gamma = \pi_s(f)(\vec{\alpha})$ for some $f \in {}^{\kappa^n}\kappa \cap \mathrm{J}^{E^N}_\tau$. But then $\vec{\alpha} < \xi$, as $\xi' = \min(\zeta \cap \mathrm{gen}_F) \setminus \xi$. Hence,

$$\gamma = \sigma_{\xi,s}(\pi_\xi(f)(\vec{\alpha})),$$

so that $\gamma \in \mathrm{ran}(\sigma_{\xi,\zeta})$. $\square_{(1)}$

(2) $\sigma_{\xi,\zeta}\restriction\xi' = \mathrm{id}\restriction\xi'$.

Proof of (2). By (1), $\sigma_{\xi,s}\restriction\xi' = \mathrm{id}\restriction\xi'$, as $\sigma_{\xi,s}$ is order preserving. The claim follows, since $\sigma_{\xi,s} = \sigma_{\zeta,s}\sigma_{\xi,\zeta}$. $\square_{(2)}$

(3) $\xi' \notin \mathrm{ran}(\sigma_{\xi,s})$.

Proof of (3). Otherwise, there is a function $f \in {}^{\kappa^n}\kappa \cap \mathrm{J}^{E^N}_\tau$ and ordinals $\vec{\alpha} < \xi$ with $\xi' = \sigma_{\xi,s}(\pi_\xi(f)(\vec{\alpha})) = \pi_s(f)(\vec{\alpha})$, which is impossible, as ξ' is a generator of F. $\square_{(3)}$

(4) $\sigma_{\zeta,s}\restriction\zeta = \mathrm{id}\restriction\zeta$.

[7] As a reminder, a cutpoint here is an s-cutpoint. So in the present context, ξ is a cutpoint of F if $\xi = s(F|\xi)$. And $s(F|\xi) = \mathrm{lub}(\tau(F|\xi) \cup \mathrm{gen}_{F|\xi})$. For more details, the reader is referred to the first part.

Proof of (4). For $\gamma < \zeta$, $\sigma_{\zeta,s}(\gamma) = \sigma_{\zeta,s}(\pi_\zeta(\mathrm{id})(\gamma)) = \pi_s(\mathrm{id})(\gamma) = \gamma$. $\square_{(4)}$

It follows immediately by (1) and (3) that

(5) $\xi' = \mathrm{crit}(\sigma_{\xi,s})$.

(6) $\xi' = \mathrm{crit}(\sigma_{\xi,\zeta})$.

Proof of (6). Otherwise (2) implies $\xi' = \sigma_{\xi,\zeta}(\xi')$. So since $\xi' < \zeta$, (4) implies:

$$\sigma_{\xi,s}(\xi') = \sigma_{\zeta,s}(\sigma_{\xi,\zeta}(\xi')) = \sigma_{\zeta,s}(\xi') = \xi',$$

which contradicts (5). $\square_{(6),Lemma}$

Corollary 5.15. *Let $N = \langle \mathrm{J}^E_\nu, F\rangle$ be an active pPλ- or pPs-structure. Let $s = s(F)$, $\tau = \tau(F)$ and $\tau \leq \xi < \zeta \leq s$. Suppose*

$$\xi' = \min((\zeta \cap \mathrm{gen}_F) \setminus \xi)$$

exists. Then

$$E^{[N]_\xi}\restriction\xi' = E^{[N]_\zeta}\restriction\xi'.$$

5.2.1 The Z-initial segment condition

The following definition is from [Ste00] or [SSZ02], and is used for the formulation of a variant of the initial segment condition.

Definition 5.16. Let $M = \langle \bar{M}, F\rangle$ be a continuable extender-structure. Then F of type Z iff $s(F) = \lambda + 1$, for some limit ordinal λ, so that λ is a cutpoint of F with the property that $(\lambda^+)^M = (\lambda^+)^{[M]_\lambda}$.

Remark 5.17. The definition of this concept given in [Ste00, p. 9] uses a different formulation which is equivalent to the present one for continuable structures. There, it is demanded that $(\lambda^+)^M = (\lambda^+)^{\mathrm{Ult}(M,F|\lambda)}$. If M is continuable, then $(\lambda^+)^{[M]_\lambda} = (\lambda^+)^{\mathrm{Ult}(M,F|\lambda)}$, since, letting $\tau = \tau(F)$ and $\pi : M \longrightarrow_{F|\lambda} \mathrm{Ult}(M, F|\lambda)$, $\pi(\mathrm{J}^{E^M}_\tau)$ is a segment of $\mathrm{Ult}(M, F|\lambda)$ the height of which is a cardinal in $\mathrm{Ult}(M, F|\lambda)$ which is greater than λ (as $\lambda < s(F) \leq \pi(\mathrm{crit}(F)) < \pi(\tau)$). Since $|\pi(\mathrm{J}^{E^M}_\tau)| = |[M]_\lambda|$, the equivalence of the two definitions follows by acceptability.

Lemma 5.18. *Let M be an active pPs-structure with top extender F. Let ξ be a cutpoint of F. If $[M]_\xi$ satisfies the s'-MISC, then $F\restriction\xi$ is not of type Z.*

Proof. $F\restriction\xi$ is an extender on $[M]_\xi$. Assume $F\restriction\xi$ if of type Z. Then $\xi = \bar{\xi} + 1$, where $\bar{\xi}$ is a cutpoint of F. But then, $\bar{\xi}$ is also a cutpoint of $E^{[M]_\xi}_{\mathrm{top}}$, so that, as $[M]_\xi$ satisfies the s'-MISC, it follows that $(\bar{\xi}^+)^{[M]_\xi} \neq (\bar{\xi}^+)^{[M]_{\bar{\xi}}}$, because $[[M]_\xi]_{\bar{\xi}} = [M]_{\bar{\xi}}$. This contradicts the assumption that $F|\xi$ is of type Z. $\square$

Definition 5.19. A pPs-structure N satisfies the *Z-initial segment condition* (Z-ISC) iff for every $\alpha \leq \mathrm{ht}(N)$ such that $F := E^N_\alpha$ is an extender, the following holds:

For each cutpoint $\eta < s(F)$ such that $F\restriction\eta$ is not of type Z, one of the following is true:

(A) There is a $\gamma < \alpha$ such that $E_\gamma = (F\restriction\eta)^*$,[8] or

(B) $E^N_\eta \neq \emptyset$, and there is a $\gamma < \alpha$ such that $E^{\widehat{N||\eta}}_\gamma = (F\restriction\eta)^*$.

[8]Here I use the following terminology: $F^* = (\widehat{F^{\mathfrak{f}}}|s^+(F))^{\mathfrak{h}}$.

Again, the formulation differs from the original slightly (see [Ste00, Definition 2.4]), but is equivalent for continuable structures.

Lemma 5.20. *Let M be a pPs-structure that satisfies the Z-ISC. Then M satisfies the s'-ISC, too.*

Proof. Let $\nu \leq \mathrm{ht}(M)$, $N := M||\nu$ active. Let $F = E^N_{\mathrm{top}}$, $\tau = \tau(F)$ and $s = s(F)$. Finally, let $\xi < s$ be a cutpoint of F, such that $[N]_\xi$ satisfies the s'-MISC. I have to show two things:

(a) $[N]_\xi \in \widehat{N}$.

Proof of (a). By Lemma 5.18, $F{\upharpoonright}\xi$ is not of type Z. So by the Z-ISC entweder, one of (A) and (B) is true:

(A) $\exists \eta < \nu \quad E^N_\eta = (F{\upharpoonright}\xi)^*$.

In this case, $[N]_\xi = \widehat{N||\eta} \in \widehat{N}$.

(B) $E^N_\xi \neq \emptyset$*, and there is a* γ *such that* $(F{\upharpoonright}\xi)^* = E^{\widehat{N||\xi}}_\gamma$.

Then $\mathrm{J}^{E^N}_\tau$, $E^N_\nu|\xi \in \widehat{N||\xi} \in \widehat{N}$, since $\widehat{N}^{\mathrm{passive}}$ is a ZFC^--model. For the same reason, it follows that $[N]_\xi \in \widehat{N}$. $\square_{(a)}$

(b) *Let $\zeta < \xi$ be such that $[N]_\zeta$ satisfies the s'-MISC. Then $[N]_\zeta \in [N]_\xi$.*

Proof of (b). First, set
$$\xi' = \min(\mathrm{gen}_F \setminus \xi),$$
which makes sense, as $\xi < s$. By Lemma 5.14, $\xi' = \mathrm{crit}(\sigma_{\xi,s})$. It may be assumed that ζ is a cutpoint of F, since, letting $\bar{\zeta} = s(F|\zeta)$, $\bar{\zeta}$ is a cutpoint of F, and $[N]_\zeta = [N]_{\bar{\zeta}}$, $\bar{\zeta} \leq \zeta < \xi$ – so one could work with $\bar{\zeta}$ instead of ζ. Again, it follows by Lemma 5.18 that $F|\zeta$ is not of type Z. So the Z-ISC may be exploited, providing the following case distinction.

(A) $\exists \eta < \nu \quad E^N_\eta = (F{\upharpoonright}\zeta)^*$.

As $\tau(F) = \tau(F{\upharpoonright}\zeta)$,
$$[N]_\zeta = \widehat{N||\eta},$$
and
$$\eta = (s(E^N_\eta)^+)^{\widehat{N||\eta}} = (\zeta^+)^{[N]_\zeta}.$$
Since $[N]_\xi$ satisfies the s'-MISC and ζ is a cutpoint of $E^{[N]_\xi}_{\mathrm{top}}$, it follows that
$$(\zeta^+)^{[N]_\zeta} < (\zeta^+)^{[N]_\xi}.$$
As $\zeta < \xi \leq \xi'$ and ξ', being the critical point of $\sigma_{\xi,s}$, is a cardinal in $[N]_\xi$, this implies that
$$\eta = (\zeta^+)^{[N]_\zeta} < (\zeta^+)^{[N]_\xi} \leq \xi'.$$
By the coherency of pPs-structures, together with Corollary 5.15, this yields, in particular,
$$E^N{\upharpoonright}(\eta+1) = E^{[N]_s}{\upharpoonright}(\eta+1) = E^{[N]_\xi}{\upharpoonright}(\eta+1).$$
This results in:
$$[N]_\zeta = \widehat{[N]_\xi||\eta} \in [N]_\xi,$$
as ${[N]_\xi}^{\mathrm{passive}}$ is a ZFC^--model.

(B) $E^N_\zeta \neq \emptyset$*, and there is a* γ*, such that* $(F|\zeta)^* = E_\gamma^{\widehat{N||\zeta}}$.

We know that $E^N\restriction\xi = E^{[N]_\xi}\restriction\xi$, hence $N||\zeta = ([N]_\xi)||\zeta$. So

$$\widehat{N||\zeta} \in [N]_\xi,$$

as $[N]_\xi{}^{\text{passive}}$ is a ZFC^--model. Hence,

$$(F|\zeta)^* = E_\gamma^{\widehat{N||\zeta}} \in [N]_\xi.$$

Since

$$N||\tau = [N]_\xi||\tau \in [N]_\xi,$$

this shows that $[N]_\zeta \in [N]_\xi$ as well, as claimed. $\Box_{(b),\text{Lemma}}$

5.3 Preservation of the s'-ISC

In this section, I am aiming at showing that the s'-ISC is preserved by s-iterations, that is, that normal s-iterates of pλ-structures are pλ-structures. Recall the following definition:

Definition 5.21. Let M be an active extender structure. Set:

$$\begin{aligned} C_M = \{\xi \mid \tau \leq \xi < s(M),\ &\xi \text{ is a cutpoint of } E^M_{\text{top}} \\ &\text{and } [M]_\xi \text{ satisfies the } s'\text{-MISC }\}. \end{aligned}$$

I gave a proof of the following lemma in [Fuc08, Lemma 8.25].

Lemma 5.22. *Let* M *be an active* pλ*-structure. Let* $\tau(M) \leq \xi < s(M)$ *be a cutpoint with the property that* $\xi \notin C_M$*. Then* $\xi = \bar{\xi}+1$ *for a cutpoint* $\bar{\xi}$ *of* $F = E^M_{\text{top}}$*. (So* $\bar{\xi}$ *is a limit of generators of* F*). Moreover,* $(\bar{\xi}^+)^{[M]_{\bar{\xi}}} = (\bar{\xi}^+)^{[M]_\xi}$ *- the proof shows that* $\bar{\xi}$ *is the only cutpoint less than* ξ *with this property.*

This lemma has some useful consequences.

Lemma 5.23. *Let* M *be a* pλ*-structure. Then* M *satisfies* s'*-MISC.*

Proof. Assuming the contrary, let M be a counterexample of minimal height. Then M is active. Let $F = E^M_{\text{top}}$. By choice of M, the statement of the lemma is true of all proper initial segments of M. So let ξ be a cutpoint of F such that $(\xi^+)^{[M]_\xi} = (\xi^+)^M$. Then $[M]_\xi \notin M$, as otherwise, in M there would be a surjection from ξ onto $|[M]_\xi|$, which would imply that $(\xi^+)^M > (\xi^+)^{[M]_\xi}$. But since $[M]_\xi \notin M$, it cannot be that $[M]_\xi$ satisfies the s'-MISC. Since ξ is a cutpoint, this means by Lemma 5.22, that $\xi = \bar{\xi}+1$, where $\bar{\xi}$ is a generator of F, which is a limit of generators. As $[M]_{\bar{\xi}}$ satisfies the s'-MISC, by the same Lemma, and hence is a member of M, it can be concluded, again by the same lemma:

$$(\xi^+)^M = (\bar{\xi}^+)^M > (\bar{\xi}^+)^{[M]_{\bar{\xi}}} = (\bar{\xi}^+)^{[M]_\xi} = (\xi^+)^{[M]_\xi},$$

contradicting the choice of ξ. $\Box$

Lemma 5.24. *Let* M *be a* pλ*-structure. Then* C_M *is closed in* $s(M)$*.*

Proof. This follows immediately from Lemma 5.22. Let ξ be a limit of C_M. Then ξ is obviously a limit of generators, and hence a cutpoint. So the lemma can be applied to show that $[M]_\xi$ satisfies the s'-MISC, for otherwise ξ would have to be a successor ordinal. So $\xi \in C_M$. $\Box$

Lemma 5.25. *Let M be an active pPλ-structure. Let $\sigma : M \longrightarrow_G M'$, with* $\mathrm{crit}(G) < s(M)$ *(to emphasize: this is a Σ_0-ultrapower). Then* $s(M') = \mathrm{lub}\,\sigma``s(M)$.

Proof. Let $F = E^M_{\mathrm{top}}$, $\kappa = \mathrm{crit}(F)$, $\tau = \tau(F)$, $s = s(M)$, $F' = E^{M'}_{\mathrm{top}}$, $\kappa' = \mathrm{crit}(F')$, $\tau' = \tau(F')$, $s' = s(M')$ and $\tilde{\kappa} = \mathrm{crit}(G)$. Since being a generator of F is $\Pi_1(M)$, and since σ is Σ_1-preserving, it follows that

$$\mathrm{lub}\,\mathrm{gen}_{F'} \geq \mathrm{lub}\,\sigma``\mathrm{gen}_F = \mathrm{lub}\,\sigma``\,\mathrm{lub}\,\mathrm{gen}_F.$$

In case $s(M) = \tau$, it obviously follows that $\tau' = \sigma(\tau) \geq \mathrm{lub}\,\sigma``\tau$. Thus, so far it has been shown that $s(M') \geq \mathrm{lub}\,\sigma``s(M)$.

Letting $\pi = \pi^M_s$ and $\pi' = \pi^{M'}_{s'}$, we have:

$(*)$ $\sigma \circ \pi = \pi' \circ \sigma$.

Proof of $(*)$. Firstly, it's obvious that the domains of the functions on the left and on the right are equal, namely $|\mathrm{J}^{E^M}_\tau|$. For $X \in \mathcal{P}(\kappa) \cap M$, it follows firstly that

$$\sigma(\pi(X)) = \sigma(F(X)) = F'(\sigma(X)) = \pi'(\sigma(X)).$$

But each member of $\mathrm{J}^{E^M}_\tau$ can be coded in a Σ_1 uniform way by a subset X of κ that belongs to M. This implies the claim. $\square_{(*)}$

In order to see that $s(M') \leq \mathrm{lub}\,\sigma``s(M)$, assume the contrary. Let $\xi \in \mathrm{gen}_{F'}$ be such that $\xi \geq \mathrm{lub}\,\sigma``s(M)$. Pick $f \in ({}^{\tilde{\kappa}^n}\kappa) \cap M$ and $\vec{\alpha} < \mathrm{lh}(G)$ so that

$$\xi = \sigma(f)(\vec{\alpha}).$$

As $f \in M$, by coherency of M, there are a function $g \in ({}^{\kappa^m}\mathrm{J}^{E^M}_\tau) \cap \mathrm{J}^{E^M}_\tau$ and ordinals $\vec{\beta} < s$ so that

$$f = \pi(g)(\vec{\beta}).$$

But this implies:

$$\begin{aligned} \xi &= \sigma(f)(\vec{\alpha}) \\ &= \sigma(\pi(g)(\vec{\beta}))(\vec{\alpha}) \\ &= \sigma(\pi(g))(\sigma(\vec{\beta}))(\vec{\alpha}) \\ &= \sigma(\pi(\tilde{g}))(\sigma(\vec{\beta}), \vec{\alpha}). \end{aligned}$$

Here, $\tilde{g} \in {}^{m+n}\kappa \cap M$ is defined by:

$$\tilde{g}(\vec{\gamma}, \vec{\delta}) := \begin{cases} g(\vec{\gamma})(\vec{\delta}) & \text{if } g(\vec{\gamma}) \text{ is a function with } \vec{\delta} \in \mathrm{dom}(g(\vec{\gamma})), \\ 0 & \text{otherwise.} \end{cases}$$

So by $(*)$,

$$\xi = \pi'(\sigma(\tilde{g}))(\sigma(\vec{\beta}), \vec{\alpha}).$$

But since $\vec{\beta} < s$, $\sigma(\vec{\beta}) < \mathrm{lub}\,\sigma``s \leq \xi$, and as $\tilde{\kappa} < s$, it follows that $\vec{\alpha} < \mathrm{lh}(G) \leq \sigma(\tilde{\kappa}) < \mathrm{lub}\,\sigma``s \leq \xi$. Hence $\sigma(\vec{\beta}), \vec{\alpha} < \xi$ and $\sigma(\tilde{g}) \in {}^{\kappa'^{m+n}}\mathrm{J}^{E^{M'}}_{\tau'} \cap \mathrm{J}^{E^{M'}}_{\tau'}$. Thus, ξ is not a generator of F' after all, a contradiction. $\square$

Lemma 5.26. *Let $\sigma : M \longrightarrow_{\Sigma_2} M'$, where M is a pPλ-structure. Then* $\mathrm{lub}\,\sigma``s(M) \leq s(M') \leq \sigma(s(M))$.

Proof. See [Jen01, §1, Lemma 3.6] □

For the reader's convenience, let me recall the different types of extender-structures:

Definition 5.27. Let M be an active extender-structure. Then M is of...

$$\begin{array}{rll} \ldots \text{ type I} & \text{iff} & s(M) = \tau(M), \\ \ldots \text{ type II} & \text{iff} & s(M) = \xi + 1 \text{ for some ordinal } \xi, \\ \ldots \text{ type III} & \text{iff} & \tau(M) < s(M) \text{ is a limit ordinal.} \end{array}$$

Remark 5.28. If M is a pλ-structure of type II, the then there is a maximal $\eta \in C_M$, by Lemma 5.24.

If in this situation, a generator ξ does not belong to C_M, then ξ is an isolated generator of E^M_{top}, for otherwise ξ would be a limit of generators, hence a cutpoint, so that by Lemma 5.22 $[M]_\xi$ would have to satisfy the s'-MISC.

In the following, I am going to treat the different types of structures separately. In essence, I'll carry out the corresponding case of the inductive proof showing that the s'-ISC is preserved under normal s-iterations. I'll show even more, namely that the $*$-ultrapower of an active pλ-structure M by an extender with critical point less than $s(M)$ yields a pλ-structure of the same type.

5.3.1 Type I

Lemma 5.29. *Let M be a pλ-structure and $\pi : M \longrightarrow^*_G M'$, where M is of type I and* $\text{crit}(G) <$ $s(M)$*. Then M' is also a pλ-structure of type I.*[9]

Proof. Obviously, $\tau(M') = \pi(\tau(M))$. So it follows that

$$\pi(s(M)) = s(M') = \tau(M'),$$

because *either* $\text{crit}(G) \geq \omega\rho^1_M$, in which case π is a Σ_0-extender ultrapower, and by Lemma 5.25, $s(M') = \text{lub}\,\pi``s(M) \leq \pi(s(M)) = \pi(\tau(M)) = \tau(M') \leq s(M')$, *or* $\text{crit}(G) < \omega\rho^1_M$, and then π is Σ_2-preserving (see [Zem02, Lemma 3.1.11(c)]), so that Lemma 5.26 can be applied. This shows that $s(M') \leq \pi(s(M)) = \pi(\tau(M)) = \tau(M')$. But by definition, also $s(M') \geq \tau(M')$, which proves the claim.

As in the following lemmas, it will suffice to show that those parts of the s'-ISC which refer to the top extender are satisfied in M'. The rest will obviously be satisfied, as π is either cofinal or Σ_2-preserving, depending on the location of the critical point of G. So since M' is of type I in the present case, there is nothing to show, since $[\tau(M'), s(M')) = \emptyset$ – see Definition 5.13. □

5.3.2 Type II

I would like to remind the reader of the following definition, from [Fuc08]:

Definition 5.30. Let M be a pPλ-structure of type II. Then

$$q_M := F|\max C_M.$$

Lemma 5.31. *Let M be a pλ-structure of type II, $\pi : M \longrightarrow^*_G M'$ and* $\text{crit}(G) < s(M)$*. Then M' is also a pλ-structure of type II, $\pi(s(M)) = s(M')$ and $\pi(q_M) = q_{M'}$.*

[9]Here, as in the following lemmas, I assume the extender ultrapower is well-founded.

Proof. Let $M = \langle \mathrm{J}_\nu^E, F\rangle$ and $M' = \langle \mathrm{J}_{\nu'}^{E'}, F'\rangle$. Let $s(M) = \xi + 1$, noting that $\xi > \tau := \tau(F)$. Finally, set $\kappa = \mathrm{crit}(F)$, $\kappa' = \mathrm{crit}(F')$ and $\tau' = \tau(F')$.

Claim: $\pi(s(M)) = s(M') = \pi(\xi) + 1$.

Proof of claim. There are two cases:

Case 1: $\mathrm{crit}(G) \geq \omega\rho_M^1$.

In this case, $\pi : M \longrightarrow_G M'$, and by Lemma 5.25

$$s(M') = \mathrm{lub}\, \pi``s(M) = \pi(\xi) + 1 = \pi(\xi + 1) = \pi(s(M)),$$

as claimed.

Case 2: $\mathrm{crit}(G) < \omega\rho_M^1$.

Then $\pi : M \longrightarrow_{\Sigma_2} M'$, and Lemma 5.26 yields:

$$\pi(\xi) + 1 = \mathrm{lub}\, \pi``s(M) \leq s(M') \leq \pi(s(M)) = \pi(\xi) + 1,$$

where $\pi(\xi) + 1 = \pi(\xi + 1) = \pi(s(M))$. $\square_{Claim}$

The rest of the proof proceeds by cases:

Case 1: $\xi = \max C_M$.

Then ξ is a limit of generators, as ξ is a cutpoint. Moreover, since M satisfies the s'-ISC, it follows that $[M]_\xi \in M$, and hence also that $F|\xi \in M$. The statement "$x = F|\xi$" is $\Pi_1(M)$ in ξ:

$$\begin{aligned} x = F|\xi \iff \quad M \models \quad & (x \text{ is a function with domain contained in } \mathcal{P}(\kappa) \wedge \\ & \forall X \forall Y (Y = F(X) \to x(X) = Y \cap \xi)) \end{aligned}$$

Let $x = F|\xi$. Then $\pi(x)$ satisfies this statement in M', where F and ξ have to be replaced by F' and $\pi(\xi) = \xi'$, respectively. Hence, $\pi(F|\xi) = F'|\xi'$ and $\pi([M]_\xi) = [M']_{\xi'}$. It only remains to show that $\xi' = \max C_M$.

As $\pi\restriction[M]_\xi : [M]_\xi \longrightarrow_{\Sigma_\omega} [M']_{\xi'}$, the property of ξ of being a limit of generators of $E_{\mathrm{top}}^{[M]_\xi}$ is preserved, because this can be expressed in $[M]_\xi$. Hence, ξ' is a limit of generators of $E_{\mathrm{top}}^{[M']_{\xi'}}$, and hence a limit of generators of F'. So ξ' is a *cutpoint* of F'. It remains to show that $[M]_{\xi'}$ satisfies the s'-MISC. But also the statement expressing that $[M]_\xi$ satisfies the s'-MISC is $\Sigma_\omega([M]_\xi)$, and hence is carried over to $[M']_{\xi'}$.

This proves the lemma in case 1.

Case 2: $\xi \notin C_M$.

Then ξ is not a limit of generators of F, by Remark 5.28. Set:

$$\bar{\xi} = \sup(\mathrm{gen}_F \cap \xi) \text{ and } \bar{\xi}' = \pi(\bar{\xi}).$$

Hence $\bar{\xi} < \xi$.

Case 2.1: $\bar{\xi} \notin \mathrm{gen}_F$.

Then $\bar{\xi}$ is a limit of generators, and by Lemma 5.22, $\bar{\xi} \in C_M$. Hence $[M]_{\bar{\xi}} \in M$. It follows that $\bar{\xi}'$ is a limit of generators of F', as $\bar{\xi}$ is a limit of generators of $E_{\mathrm{top}}^{[M]_{\bar{\xi}}}$ and $\pi\restriction[M]_{\bar{\xi}} \longrightarrow_{\Sigma_\omega} [M']_{\bar{\xi}'}$. Hence $\bar{\xi}'$ is a cutpoint of F', and $[M']_{\bar{\xi}'}$ satisfies the s'-MISC, again by elementarity. So $\bar{\xi}' \in C_{M'}$. It suffices to show that $\bar{\xi}' = \max C_{M'}$, and for this, in turn, it suffices to see that $[\bar{\xi}', \xi') \cap \mathrm{gen}_{F'} = \emptyset$, because then there is in M' no cutpoint greater than $\bar{\xi}'$ that's less than $s(M')$.

We know that $[\bar{\xi}, \xi) \cap \mathrm{gen}_F = \emptyset$. Hence $[M]_{\bar{\xi}} = [M]_\xi \in M$. So $F_\xi := F|\xi \in M$. Again, F_ξ is characterized by a $\Pi_1(M)$-statement in ξ, which is preserved by π. Hence $F'_{\xi'} \in M'$ and we have $\pi([M]_\xi) = [M']_{\xi'}$. But this implies:

$$[M']_{\bar{\xi}'} = \pi([M]_{\bar{\xi}}) = \pi([M]_\xi) = [M']_{\xi'},$$

so that $[\bar{\xi}', \xi') \cap \mathrm{gen}_{F'} = \emptyset$, as claimed. The other parts of the s'-ISC for M' are easily verified.

Case 2.2: $\bar{\xi} \in \mathrm{gen}_F$.

Case 2.2.1: $\bar{\xi} + 1 \in C_M$, or $\bar{\xi} = \kappa$ (and $\tau \in C_M$).

One can argue similarly as in case 2.1. It follows that $[M']_{\bar{\xi}'+1} \in M'$, that $\bar{\xi}' + 1$ (or τ') is a cutpoint of F', and that $[M']_{\bar{\xi}'+1}$ satisfies the s'-MISC, hence that $\bar{\xi}' + 1$ (or τ') belongs to $C_{M'}$. Finally, one can argue that $(\bar{\xi}', \xi') \cap \mathrm{gen}_{F'} = \emptyset$, like before.

Case 2.2.2: $\bar{\xi} + 1 \notin C_M$, and $\bar{\xi} \neq \kappa$.

As $\bar{\xi} \in \mathrm{gen}_F$ and $\bar{\xi} > \tau$, $\bar{\xi}+1$ is a cutpoint of F, and by Lemma 5.22, $\bar{\xi}$ is a limit of generators of F. Hence $\bar{\xi} = \max C_M$, again by the same lemma. As before, it follows that

(+) $[M]_{\bar{\xi}} \in M, \pi(q_M) = F'|\bar{\xi}'$ *and* $\pi([M]_{\bar{\xi}}) = [M']_{\bar{\xi}'} \in M'$.

Moreover, $[M']_{\bar{\xi}'}$ satisfies the s'-MISC, hence $\bar{\xi}' \in C_{M'}$.

(1) $(\bar{\xi}^+)^{[M]_{\bar{\xi}}} = \xi$.

Proof of (1). By Lemma 5.22 it's clear that

$$(\bar{\xi}^+)^{[M]_{\bar{\xi}}} = (\bar{\xi}^+)^{[M]_{\bar{\xi}+1}}.$$

Moreover, ξ is a cardinal in $[M]_{\bar{\xi}+1}$, as $\xi = \mathrm{crit}(\sigma^M_{\bar{\xi}+1})$. Hence, obviously,

$$(\bar{\xi}^+)^{[M]_{\bar{\xi}}} = (\bar{\xi}^+)^{[M]_{\bar{\xi}+1}} \leq \xi.$$

Assume that $\zeta := (\bar{\xi}^+)^{[M]_{\bar{\xi}}} < \xi$.

Then $\zeta = \sigma^M_{\bar{\xi}+1}(\zeta) = (\bar{\xi}^+)^M$, which yields the contradiction

$$\zeta = (\bar{\xi}^+)^M > \mathrm{ht}([M]_{\bar{\xi}}) \geq (\bar{\xi}^+)^{[M]_{\bar{\xi}}} = \zeta,$$

since there is in M a surjection from $\bar{\xi}$ onto $|[M]_{\bar{\xi}}|$, as $[M]_{\bar{\xi}} \in M$. $\square_{(1)}$

(2) $(\bar{\xi}'^+)^{[M']_{\bar{\xi}'}} = \xi'$.

Proof of (2). This follows immediately from (1) and the preservation properties of π. $\square_{(2)}$

(3) $(\bar{\xi}', \xi') \cap \mathrm{gen}_{F'} = \emptyset$.

Proof of (3). Suppose there was a $\gamma \in (\bar{\xi}', \xi') \cap \mathrm{gen}_{F'}$. Then

$$\xi' = (\bar{\xi}'^+)^{[M']_{\bar{\xi}'}} \leq (\bar{\xi}'^+)^{[M']_\gamma} \leq \gamma < \xi',$$

as γ, being the critical point of $\sigma^{M'}_\gamma$, is a cardinal in $[M']_\gamma$. $\square_{(3)}$

(4) $(\bar{\xi}'^+)^{[M']_{\bar{\xi}'}} = (\bar{\xi}'^+)^{[M']_{\bar{\xi}'+1}}$.

Proof of (4). It follows by (2) that

$$\xi' = (\bar{\xi}'^+)^{[M']_{\bar{\xi}'}} \leq (\bar{\xi}'^+)^{[M']_{\bar{\xi}'+1}} \leq (\bar{\xi}'^+)^{[M']_{\xi'}} \leq \xi',$$

hence all these are equal. I used here that ξ', being the critical point of $\sigma^{M'}_{\xi', s(M')}$, is a cardinal in $[M']_{\xi'}$. $\square_{(4)}$

As $\bar{\xi}'$ is a limit of generators of F', a by now familiar argument shows that $\bar{\xi}'$ is a cutpoint of F', and (4) shows that $[M']_{\bar{\xi}'+1}$ does not satisfy the s'-MISC. Hence $\bar{\xi}' + 1 \notin C_{M'}$. Using (3), it follows that $\bar{\xi}' = \max C_{M'}$; we have already seen that $\bar{\xi}' \in C_{M'}$. So (+) shows the claim also in this case.

Again, the other parts of the s'-ISC are easily checked. $\square$

5.3.3 Type III

Lemma 5.32. *Let M be an active pλ-structure of type III. Then $\omega\rho_M^1 = s(M)$.*

Proof. It is obvious that $\omega\rho_M^1 \le s(M)$, since there is a $\Sigma_1(M)$-surjection from $s(M)$ onto $|M|$. Suppose $\omega\rho_M^1 < s(M)$. Then let A be a $\Sigma_1(M)$ set, using a parameter p, such that $A \cap \omega\rho_M^1 \notin M$. Set $F := E_{\mathrm{top}}^M$.

Then for $\delta \in \mathrm{gen}_F$ and $\delta < \delta' \le s$, $\mathrm{crit}(\sigma_{\delta,\delta'}^M) = \delta$, by Lemma 5.14. Moreover,

$(*)$ $|M| = \bigcup_{\delta \in \mathrm{gen}_F} \mathrm{ran}(\sigma_{\delta,s}^M)$.

Proof of $(*)$. By definition of s,

$$\pi_s : \mathrm{J}_\tau^{E^M} \longrightarrow_{F|s} \mathrm{J}_\nu^{E^M},$$

where $\tau = \tau(M)$ and $\nu = \mathrm{ht}(M)$. Let $x \in \mathrm{J}_\nu^{E^M}$ and $\kappa = \mathrm{crit}(F)$. Then there are $n, \vec{\alpha} \in s^n$ and a function $f : \kappa^n \to \mathrm{J}_\tau^{E^M}$ with $f \in \mathrm{J}_\tau^{E^M}$, so that

$$x = \pi_s^M(f)(\vec{\alpha}).$$

Let $\max(\vec{\alpha}) < \delta \in \mathrm{gen}_F$ (such a δ exists, as gen_F has no maximal element). Then

$$\sigma_{\delta,s}^M(\pi_\delta^M(f)(\vec{\alpha})) = \pi_s^M(f)(\vec{\alpha}) = x \in \mathrm{ran}(\sigma_{\delta,s}^M).$$

$\square_{(*)}$

Now let μ be a cutpoint of F with the following properties:

- $p \in \mathrm{ran}(\sigma_{\mu,s}^M)$.
- $\mu \ge \omega\rho_M^1$.
- $[M]_\mu$ satisfies die s'-MISC.

Finding such a μ is no problem, using Lemma 5.22. Then $\sigma_{\mu,s}^M : [M]_\mu \longrightarrow_{\Sigma_0} M$ is cofinal, hence Σ_1-preserving.

Let $\bar{A}$ be $\boldsymbol{\Sigma}_1([M]_\mu)$ in $\bar{p}$ by the same definition as A is $\boldsymbol{\Sigma}_1(M)$ in p, where $\sigma_{\mu,s}^M(\bar{p}) = p$. As $\omega\rho_M^1 \le \mu \le \mathrm{crit}(\sigma_{\mu,s}^M)$, and since $\sigma_{\mu,s}^M$ is a Σ_1-preserving embedding, it follows that

$$\bar{A} \cap \omega\rho_M^1 = A \cap \omega\rho_M^1.$$

But $[M]_\mu$ satisfies the s'-MISC and M is a pλ-structure, so $[M]_\mu \in M$. Hence everything definable in $[M]_\mu$ belongs to M, in particular the set $\bar{A} \cap \omega\rho_M^1 = A \cap \omega\rho_M^1$, a contradiction. $\square$

Lemma 5.33. *Let M be an active pλ-structure of type III and G an extender on M with*

$$\omega\rho_M^2 \le \tilde{\kappa} := \mathrm{crit}(G) < \omega\rho_M^1.$$

Let $\pi : M \longrightarrow_G^* M'$. Then

$$\begin{aligned} \pi\restriction H_M^1 : M^{1,\emptyset} &\longrightarrow_G M'^{1,\emptyset} \text{ and} \\ s(M') = \omega\rho_{M'}^1 &= \sup \pi``s(M). \end{aligned}$$

Proof. Set
$$\tilde{M} := M^{1,\emptyset}.$$
Let $F = E^M_{\text{top}}$, $s = s(M)$, $\kappa = \kappa(M)$, $\tau = \tau(M)$, $\nu = \text{ht}(M)$.

Remark. $p_{M,1} = \emptyset$, since using the top extender of M a surjection from $s = \omega\rho^1_M$ onto J^E_ν can be defined in M without using parameters.

Let
$$\bar{\pi} : \tilde{M} \longrightarrow_G \tilde{M}^* = \langle \mathrm{J}^{\tilde{E}^*}_{s^*}, B\rangle.$$

(0) $\bar{\pi} = \pi{\restriction}|\tilde{M}|$.

Proof of (0). The proof proceeds in three steps:

(0.1) *If* $f \in \Gamma^*(M, \tilde{\kappa})$ *and* $\text{ran}(f) \subseteq x \in |\tilde{M}|$ *for some* x *(that is,* $\text{ran}(f)$ *is bounded in* $|\tilde{M}|$*), then* $f \in |\tilde{M}|$.

Proof of (0.1). Firstly, it's clear that if $f \in \Gamma^*(M, \tilde{\kappa}) \cap |M|$ and $\text{ran}(f) \subseteq x \in |\tilde{M}|$, then $f \in |M|$ – this follows from the acceptability of M. Now let $f \in \Gamma^*(M, \tilde{\kappa})$, $\text{ran}(f) \subseteq x \in |\tilde{M}|$. By the above, it suffices to show that $f \in |M|$. But if $f \notin |M|$, then f is a good $\mathbf{\Sigma}^{(n)}_1(M)$-function, where $\omega\rho^{n+1} > \tilde{\kappa}$. As $\tilde{\kappa} \in [\omega\rho^2_M, \omega\rho^1_M)$, this means that $n = 0$, hence f is a Σ_1-function. But as $\text{ran}(f) \subseteq x \in |\tilde{M}|$, there is some $\alpha < \text{ht}(|\tilde{M}|) = \rho^1_M$ such that $f \subseteq |M||\alpha|$. This implies that $f \in |M|$, by definition of ρ^1_M. $\square_{(0.1)}$

Set $\tilde{\lambda} := \text{lh}(G)$, and define for $\langle \vec{\alpha}, f\rangle, \langle \vec{\beta}, g\rangle \in D^*(M, \tilde{\kappa}, \tilde{\lambda})$:
$$\begin{aligned}
\langle \vec{\alpha}, f\rangle E' \langle \vec{\beta}, g\rangle &\iff \prec\vec{\alpha}, \vec{\beta}\succ \in G(\{\prec\gamma, \delta\succ < \tilde{\kappa} \mid f(\vec{\gamma}) \in g(\vec{\delta})\}),\\
\langle \vec{\alpha}, f\rangle I' \langle \vec{\beta}, g\rangle &\iff \prec\vec{\alpha}, \vec{\beta}\succ \in G(\{\prec\gamma, \delta\succ < \tilde{\kappa} \mid f(\vec{\gamma}) = g(\vec{\delta})\}).
\end{aligned}$$

Denote the restrictions of E', I' to $D(\tilde{M}, \tilde{\kappa}, \tilde{\lambda})$ by E, I, respectively.

(0.2) *Let* $\langle \beta, g\rangle \in D^*(M, \tilde{\kappa}, \tilde{\lambda})$, $\langle \vec{\alpha}, f\rangle \in D(\tilde{M}, \tilde{\kappa}, \tilde{\lambda})$, *and let* $\langle \vec{\beta}, g\rangle E' \langle \vec{\alpha}, f\rangle$. *Then there is a* $\bar{g}$*, so that* $\langle \vec{\beta}, \bar{g}\rangle \in D^*(M, \tilde{\kappa}, \tilde{\lambda})$, $\text{ran}(\bar{g})$ *is bounded in* $\tilde{M}$*, and* $\langle \vec{\beta}, \bar{g}\rangle I' \langle \vec{\alpha}, f\rangle$. *By (2),* $\bar{g} \in |\tilde{M}|$.

Proof of (0.2). Note that $f \in |\tilde{M}|$. Let $\alpha < \text{ht}(\tilde{M})$ be so that $f \in x := |\tilde{M}||\alpha|$. Define $h : |M| \longrightarrow |M|$ by:
$$h(a) := \begin{cases} a & \text{if } a \in x, \\ \emptyset & \text{otherwise.} \end{cases}$$
Then $\bar{g} := h \circ g$ is a good $\mathbf{\Sigma}^{(0)}_1(M)$-function with the desired properties:

Since $\{\prec\vec{\gamma}\succ \mid g(\vec{\gamma}) \in x\} \subseteq \{\prec\vec{\gamma}\succ \mid g(\vec{\gamma}) = \bar{g}(\vec{\gamma})\}$, it follows that
$$\prec\vec{\beta}\succ \in G(\{\prec\vec{\gamma}\succ \mid g(\vec{\gamma}) \in x\}) \subseteq G(\{\prec\vec{\gamma}\succ \mid g(\vec{\gamma}) = \bar{g}(\vec{\gamma})\}).$$
Hence $\langle \vec{\beta}, \bar{g}\rangle I' \langle \vec{\beta}, g\rangle$, and $\bar{g}$ is as wished. $\square_{(0.2)}$

(0.3) *For* $\langle \vec{\alpha}, f\rangle \in D(\tilde{M}, \tilde{\kappa}, \tilde{\lambda})$, $\pi(f)(\vec{\alpha}) = \bar{\pi}(f)(\vec{\alpha})$.

Proof of (0.3). The proof proceeds by E-induction on $\langle \vec{\alpha}, f\rangle$. Suppose the claim holds for all E-predecessors of $\langle \vec{\alpha}, f\rangle$. We have:
$$\pi(f)(\vec{\alpha}) = \{\pi(g)(\vec{\beta}) \mid \langle \vec{\beta}, g\rangle E' \langle \vec{\alpha}, f\rangle\}.$$

But for $\langle\vec{\beta}, g\rangle E'\langle\vec{\alpha}, f\rangle$, by (0.2), there is a $\bar{g}$ so that $\langle\vec{\beta}, \bar{g}\rangle I'\langle\vec{\beta}, g\rangle E'\langle\vec{\alpha}, f\rangle$, and $\langle\vec{\beta}, \bar{g}\rangle \in D(\tilde{M}, \tilde{\kappa}, \tilde{\lambda})$. Hence

$$\begin{aligned} \pi(f)(\vec{\alpha}) &= \{\pi(g)(\vec{\beta}) \mid \langle\vec{\beta}, g\rangle E\langle\vec{\alpha}, f\rangle\} \\ &= \{\bar{\pi}(g)(\vec{\beta}) \mid \langle\vec{\beta}, g\rangle E\langle\vec{\alpha}, f\rangle\} \\ &= \bar{\pi}(f)(\vec{\alpha}). \end{aligned}$$

This was to be shown. $\square_{(0.3)}$

The claim follows from (0.3): $\pi(x) = \pi(\mathrm{const}_x)(0) = \bar{\pi}(\mathrm{const}_x)(0) = \bar{\pi}(x)$ for $x \in |\tilde{M}|$. $\square_{(0)}$

Set:

$$\tilde{F} := \langle (F|\mu) \mid \mu < s\rangle.$$

(1) *The relation $\tilde{F}$ is rudimentary in s, κ, $A_M^{1,\emptyset}$.*

Proof of (1). The statement „$x = \tilde{F}(\mu)$“ says that

$$\mu < s \wedge x = (F|\mu),$$

and that statement is $\Pi_1(M)$:

$$\begin{aligned} x = (F|\mu) \quad \Longleftrightarrow \quad & M \models \text{“}x \text{ is a function”} \wedge \mathrm{dom}(x) = \mathcal{P}(\kappa) \\ & \wedge \forall w \in \mathrm{dom}(x) \forall y \quad (\dot{F}(y, w) \longrightarrow y \cap \mu = x(w)). \end{aligned}$$

By Lemma 5.22, $(F|\mu) \in M$ for $\mu < s$. It follows that

$$\langle x, \mu\rangle \in \tilde{F} \iff M \models \varphi[\kappa, s, \mu, x]$$

for some Π_1-formula φ. If i is the Gödel number of $\neg\varphi$, then, consequently:

$$\langle x, \mu\rangle \in \tilde{F} \iff \neg A_M^{1,\emptyset}(i, \langle\kappa, s, \mu, x\rangle),$$

from which one sees, that $\tilde{F}$ is rudimentary in s, κ and $A_M^{1,\emptyset}$. $\square_{(1)}$

By Lemma 5.22, it follows that for each $\xi < s$, we have:

$$[M]_\xi \in \tilde{M}.$$

Let F^* be the function, that's rudimentary in $B, \bar{\pi}(\kappa)$, s^* by the same definition, by which $\tilde{F}$ is rudimentary in A_M^1, κ, s. For $\bar{\pi}(\tau) \leq \gamma < s^*$, denote the maximal continuation of $\mathrm{J}_{\bar{\pi}(\tau)}^{\tilde{E}^*}$ according to $F^*(\gamma)$ by M_γ^*. Using (0) and (1), it is the easy to see that

$$\pi([M]_\xi) = M^*_{\bar{\pi}(\xi)}.$$

For $\gamma \leq \delta \leq s^*$, let $\sigma^*_{\gamma,\delta}$ be the canonical embedding

$$\sigma^*_{\gamma,\delta} : M^*_\gamma \longrightarrow M^*_\delta.$$

We have:

$$\langle M, \langle \sigma^M_{\mu,s} \mid \tau \leq \mu < s\rangle\rangle = \mathrm{dir\ lim}(\langle [M]_\mu \mid \tau \leq \mu < s\rangle, \langle \sigma^M_{\mu,\delta} \mid \tau \leq \mu \leq \delta < s\rangle),$$

because by claim $(*)$ of the proof of Lemma 5.32, $\mathrm{J}_\nu^E = \bigcup_{\tau \leq \mu < s} \mathrm{ran}(\sigma^M_{\mu,s})$, and for $\tau \leq \mu \leq \delta < s$, obviously, $\sigma^M_{\delta,s}\sigma_{\mu,\delta} = \sigma^M_{\mu,s}$. Now let

$$\langle M^*, \langle \sigma^*_{\bar{\pi}(\mu)} \mid \tau \leq \mu < s\rangle\rangle = \mathrm{dir\ lim}(\langle M^*_{\bar{\pi}(\mu)} \mid \tau \leq \mu < s\rangle, \langle \sigma^*_{\bar{\pi}(\mu),\bar{\pi}(\delta)} \mid \tau \leq \mu \leq \delta < s\rangle),$$

where wfc (M^*) is transitive.

(2) *M^* is well-founded.*

Proof of (2). Define an embedding

$$j : M^* \longrightarrow M'$$

by

$$j(\sigma^*_{\bar{\pi}(\mu)}(\bar{\pi}(f)(\vec{a})) := \sigma^{M'}_{\pi(\mu),s(M')}(\vec{a}).$$

The correctness of this definition is implicit in the following proof that j is Σ_0-preserving: Let φ be a Σ_0-formula, and assume that

$$M^* \models \varphi[\vec{a}].$$

Let $\mu < s$ be large enough that $\vec{a} \in \mathrm{ran}(\sigma^*_{\bar{\pi}(\mu)})$. Let $\vec{\bar{a}} \in M^*_{\bar{\pi}(\mu)}$ be such that $\sigma^*_{\bar{\pi}(\mu)}(\vec{\bar{a}}) = \vec{a}$. Since $\sigma^*_{\gamma,\delta}$ is Σ_1-preserving whenever $\bar{\pi}(\tau) \le \gamma \le \delta < s^*$, it follows that $\sigma^*_{\bar{\pi}(\mu)}$ is also Σ_1-preserving, hence we have

$$M^*_{\bar{\pi}(\mu)} \models \varphi[\vec{\bar{a}}].$$

Now we know:

$$M^*_{\bar{\pi}(\mu)} = \pi([M]_\mu) = [M']_{\pi(\mu)}.$$

Hence we have:

$$\begin{aligned} M^*_{\bar{\pi}(\mu)} \models \varphi[\vec{\bar{a}}] \quad &\iff \quad [M']_{\pi(\mu)} \models \varphi[\vec{\bar{a}}] \\ &\iff \quad M' \models \varphi[\sigma^{M'}_{\pi(\mu),s(M')}(\vec{\bar{a}})] \\ &\iff \quad M' \models \varphi[j(\vec{a})]. \end{aligned}$$

Of course, the well-foundedness of M' implies that of M^*. $\Box_{(2)}$

Define a map $\pi' : M \longrightarrow M^*$ by

$$\pi'(\sigma^M_{\mu,s}(x)) := \sigma^*_{\bar{\pi}(\mu)}(\bar{\pi}(x)).$$

(3) $\pi' \restriction s = \bar{\pi} \restriction s$.

Proof of (3).

(3.1) $\sigma^M_{\gamma,s} \restriction \gamma = \mathrm{id} \restriction \gamma$ *for* $\tau \le \gamma < s$.

Proof of (3.1.). By Lemma 5.14, $\mathrm{crit}(\sigma^M_{\gamma,s}) = \min(\mathrm{gen}_F \setminus \gamma) \ge \gamma$. $\Box_{(3.1)}$

(3.2) $\sigma^*_\gamma \restriction \gamma = \mathrm{id} \restriction \gamma$, *for* $\bar{\pi}(\tau) \le \gamma < s^*$.

Proof of (3.2). Clearly,

$$\sigma^*_{\gamma,\delta} \restriction \gamma = \mathrm{id} \restriction \gamma \text{ fr } \bar{\pi}(\tau) \le \gamma \le \delta < s^*.$$

This implies the claim: Assume the contrary. Let α be minimal such that

(-) *There is a $\gamma < s$ such that $\alpha < \bar{\pi}(\gamma)$ and $\sigma^*_{\bar{\pi}(\gamma)}(\alpha) > \alpha$.*

Choose such a γ. Pick γ' with $\gamma \le \gamma' < s$, so that $\alpha \in \mathrm{ran}(\sigma^*_{\bar{\pi}(\gamma')})$. Let $\bar{\alpha} = (\sigma^*_{\bar{\pi}(\gamma')})^{-1}(\alpha)$. Then we have:

$$\sigma^*_{\bar{\pi}(\gamma)}(\alpha) = \sigma^*_{\bar{\pi}(\gamma')}(\alpha),$$

since $\sigma^*_{\bar{\pi}(\gamma')}(\alpha) = \sigma^*_{\bar{\pi}(\gamma')}(\sigma^*_{\bar{\pi}(\gamma),\bar{\pi}(\gamma')}(\alpha)) = \sigma^*_{\bar{\pi}(\gamma)}(\alpha)$. I used here that $\alpha < \bar{\pi}(\gamma) \le \mathrm{crit}(\sigma^*_{\bar{\pi}(\gamma),\bar{\pi}(\gamma')})$.

Hence

$$\sigma^*_{\bar{\pi}(\gamma')}(\alpha) = \sigma^*_{\bar{\pi}(\gamma)}(\alpha) > \alpha = \sigma^*_{\bar{\pi}(\gamma')}(\bar{\alpha}),$$

which implies that $\bar{\alpha} < \alpha$. Moreover,

$$\bar{\alpha} < \bar{\pi}(\gamma'),$$

since $\bar{\alpha} < \alpha < \bar{\pi}(\gamma) \leq \bar{\pi}(\gamma')$. Hence, (-) is also satisfied by $\bar{\alpha}$, as witnessed by γ', as $\bar{\alpha} < \bar{\pi}(\gamma')$ and $\sigma^*_{\bar{\pi}(\gamma')}(\bar{\alpha}) = \alpha > \bar{\alpha}$. This contradicts the choice of α. $\square_{(3.2)}$

Now let's turn to the claim itself:

Let $\alpha < s$. As s is a limit, $\alpha + 1 < s$. Using (3.1) and (3.2), it follows that

$$\pi'(\alpha) = \pi'(\sigma^M_{\alpha+1,s}(\alpha)) = \sigma^*_{\bar{\pi}(\alpha+1)}(\bar{\pi}(\alpha)) = \bar{\pi}(\alpha).$$

$\square_{(3)}$

(4) $\pi' : M \longrightarrow_{\Sigma_2} M^*$.

Proof of (4). It's obvious that π' is Σ_1-preserving. So it suffices to show that Σ_2-formulae are preserved downwards. Let $\varphi(x, y)$ be a Σ_0-formula, and assume

$$M^* \models \exists x \forall y \quad \varphi(x, y)$$

(I suppress parameters in the range of π'). Let $a \in M^*$ be such that

$$M^* \models (\forall y \quad \varphi(x, y))[a].$$

Let $\tau \leq \mu < s$ and $\bar{a} \in \bar{\pi}([M]_\mu)$ be such that $a = \sigma^*_{\bar{\pi}(\mu)}(\bar{a})$. As $\sigma^*_{\bar{\pi}(\mu)}$ is Σ_1-preserving, it follows that

$$\bar{\pi}([M]_\mu) = M^*_{\bar{\pi}(\mu)} \models (\forall y \quad \varphi(x, y))[\bar{a}],$$

hence

$$M^*_{\bar{\pi}(\mu)} \models \exists x \forall y \quad \varphi(x, y),$$

so that

$$\tilde{M}^* \models (\exists x \forall y \quad \varphi(x, y))_{M^*_{\bar{\pi}(\mu)}}.$$

This is $\Sigma_0(\tilde{M}^*)$ in $M^*_{\bar{\pi}(\mu)}$, hence

$$\tilde{M} \models (\exists x \forall y \quad \varphi(x, y))_{[M]_\mu},$$

which implies that

$$[M]_\mu \models \exists x \forall y \quad \varphi(x, y).$$

Now let $\bar{b} \in [M]_\mu$ be such that

$$[M]_\mu \models (\forall y \quad \varphi(x, y))[\bar{b}].$$

As $\sigma^M_{\mu,s} : M_\mu \longrightarrow_{\Sigma_1} M$, it follows that

$$M \models (\forall y \quad \varphi(x, y))[\sigma_{\mu,s}(\bar{b})],$$

in particular,

$$M \models \exists x \forall y \quad \varphi(x, y).$$

$\square_{(4)}$

In particular, M^* is a J-model, as this is expressible by a Q-statement which is true in M. Let

$$M^* = \langle \mathrm{J}^{E^*}_{\nu^*}, E^*_{\nu^*} \rangle$$

and $\kappa^* = \pi'(\kappa)$, $\tau^* = \pi'(\tau)$, and $\lambda^* = \pi'(\lambda)$.

(5) $s^* = \omega\rho^1_{M^*}$.

Proof of (5). Two directions have to be verified.

$\boxed{s^* \geq \omega\rho^1_{M^*}}$ This follows, as $M^* = h_{M^*}(s^*)$: Let $x \in M^*$. Then let $\mu \in [\tau, s)$ and $\bar{x} \in M^*_{\bar{\pi}(\mu)}$ be such that $x = \sigma^*_{\bar{\pi}(\mu)}(\bar{x})$. Obviously, $M^*_{\bar{\pi}(\mu)} = h_{M^*_{\bar{\pi}(\mu)}}(\bar{\pi}(\mu))$, since $M^*_{\bar{\pi}(\mu)}$ is the maximal continuation of $\mathrm{J}^{\tilde{E}^*}_{\bar{\pi}(\tau)}$ according to $\tilde{F}^*(\bar{\pi}(\mu))$. So let $\xi < \bar{\pi}(\mu)$ and $m < \omega$ be such that $\bar{x} = h_{M^*_{\bar{\pi}(\mu)}}(\xi, m)$.

As $\sigma^*_{\bar{\pi}(\mu)} : M^*_{\bar{\pi}(\mu)} \longrightarrow_{\Sigma_1} M^*$ and $\sigma^*_{\bar{\pi}(\mu)} \restriction \bar{\pi}(\mu) = \mathrm{id} \restriction \bar{\pi}(\mu)$, it follows that

$$x = \sigma^*_{\bar{\pi}(\mu)}(\bar{x}) = \sigma^*_{\bar{\pi}(\mu)}(h_{M^*_{\bar{\pi}(\mu)}}(\xi, m)) = h_{M^*}(\xi, m).$$

$\boxed{s^* \leq \omega\rho^1_{M^*}}$ For this direction, one can argue as in the proof of Lemma 5.32: Assume the contrary, so that $s^* > \omega\rho^1_{M^*}$. Let A be $\Sigma_1(M^*)$ in p so that $A \cap \omega\rho^1_{M^*} \notin M^*$. Let $\mu \geq \tau$, $\mu < s$ be such that $\omega\rho^1_{M^*} \leq \bar{\pi}(\mu)$ and $p \in \mathrm{ran}(\sigma^*_{\bar{\pi}(\mu)})$. Let $\bar{p} = (\sigma^*_{\bar{\pi}(\mu)})^{-1}(p)$. Now let $\bar{A}$ be $\Sigma_1(M^*_{\bar{\pi}(\mu)})$ in $\bar{p}$ by the same Σ_1-definition as A.

It has to be checked that $M^*_{\bar{\pi}(\mu)} \in M^*$, for then it can be concluded that $A \cap \bar{\pi}(\mu) = \bar{A} \cap \bar{\pi}(\mu) \in M^*$, a contradiction.

In order to see this, let $\nu \in (\mu, s)$ be chosen so that $[M]_\mu \in [M]_\nu$ – this is easy, using part (b) of the s'-ISC. Let $a = \pi([M]_\mu) = M^*_{\bar{\pi}(\mu)}$. Then in $M^*_{\bar{\pi}(\nu)}$ the Σ_1-statement "$a = [M^*_{\bar{\pi}(\nu)}]_{\bar{\pi}(\mu)}$", which I denote by $\varphi[a, \bar{\pi}(\mu)]$, is true. By (3.2), $\mathrm{crit}(\sigma^*_{\bar{\pi}(\nu)}) \geq \bar{\pi}(\nu)$, hence by the Σ_1-preservation property of $\sigma^*_{\bar{\pi}(\nu)}$, it follows that $\sigma^*_{\bar{\pi}(\nu)}(a) = [M^*]_{\bar{\pi}(\mu)}$. But since $F^{M^*} | \bar{\pi}(\mu) = F^{M^*_{\bar{\pi}(\nu)}} | \bar{\pi}(\mu)$ and $\mathrm{J}^{E^{M^*}}_{\bar{\pi}(\tau)} = \mathrm{J}^{E^{M^*_{\bar{\pi}(\nu)}}}_{\bar{\pi}(\tau)}$, this means that $M^*_{\bar{\pi}(\mu)} = \sigma^*_{\bar{\pi}(\nu)}(a) \in M^*$. $\Box_{(5)}$

(6) $s^* = s(\nu^*)^{M^*}$.

Proof of (6). First, it is obvious that $s^* \leq s(\nu^*)^{M^*}$, since $\pi' : M \longrightarrow_{\Sigma_2} M^*$ and $\pi' \restriction s = \bar{\pi} \restriction s$, hence $s^* = \mathrm{lub}\, \bar{\pi}``s = \mathrm{lub}(\pi')``s \leq s(\nu^*)^{M^*}$, by Lemma 5.26.

In order to see the converse, define

$$i : \mathrm{J}^{E^*}_{\tau^*} \longrightarrow_{E^*_{\nu^*}} \mathrm{J}^{E^*}_{\nu^*}.$$

As M^* is a direct limit, it follows that

$$|\mathrm{J}^{E^*}_{\nu^*}| = \bigcup_{\tau \leq \mu < s} \mathrm{ran}(\sigma^*_{\bar{\pi}(\mu)}).$$

As before, we get:

$$\begin{aligned} \langle M^*, \langle \sigma^{M^*}_{\bar{\pi}(\mu), s^*} \mid \tau \leq \mu < s \rangle \rangle &= \mathrm{dir\ lim}(\langle [M^*]_{\bar{\pi}(\mu)} \mid \tau \leq \mu < s \rangle, \\ &\qquad \langle \sigma^{M^*}_{\bar{\pi}(\mu), \bar{\pi}(\delta)} \mid \tau \leq \mu \leq \delta < s \rangle) \\ &= \langle M^*, \langle \sigma^*_{\bar{\pi}(\mu)} \mid \tau \leq \mu < s \rangle \rangle. \end{aligned}$$

So we have

$$\begin{aligned} |\mathrm{J}^{E^*}_{\nu^*}| &= \bigcup_{\tau \leq \mu < s} \mathrm{ran}(\sigma^{M^*}_{\bar{\pi}(\mu), s^*}) \\ &= \bigcup_{\tau \leq \mu < s} \{\pi^{M^*}_{s^*}(f)(\vec{\alpha}) \mid f \in \mathrm{J}^{E^*}_{\tau^*} \wedge \vec{\alpha} < \bar{\pi}(\mu)\} \\ &= \{\pi^{M^*}_{s^*}(f)(\vec{\alpha}) \mid f \in \mathrm{J}^{E^*}_{\tau^*} \wedge \vec{\alpha} < s^*\}, \end{aligned}$$

since $s^* = \sup \bar{\pi}``s$. This shows that $s(M^*) \leq s^*$, as wished. $\Box_{(6)}$

(7) $\pi = \pi'$ *and* $M^* = M'$.

Proof of (7). It suffices to show:

$$\pi' : M \longrightarrow^*_G M^*.$$

In order to see this, several points need verification.

- M^* *is transitive.*

- $\pi' : M \longrightarrow_{\Sigma_0^{(n)}} M'$ *if* $\omega\rho^n_M > \tilde{\kappa}$.

Proof. In the current constellation, $\omega\rho^2_M \le \tilde{\kappa} < \omega\rho^1_M$, so it only needs to be checked that π' is Σ_0- and $\Sigma_0^{(1)}$-preserving. By (4), π' is even Σ_2-preserving, which of course implies the former, but also $\Sigma_0^{(1)}$-preservation is a consequence. By [Zem02, Lemma 1.7.1], it suffices firstly to show that $(\pi')``\Gamma^1_M \subseteq \Gamma^1_{M^*}$, which is trivial here. Secondly, the following has to be shown:

If $p \in \Gamma^1_M$ and $p' = \pi'(p)$, then

$$\pi' \restriction H^1_M : M^{1,p} \longrightarrow_{\Sigma_0} (M^*)^{1,p'}.$$

To this end, let p, p' be as in the claim. Because of the known preservation properties of π' it suffices to show: $\langle i, x\rangle \in A^{1,p}_M \iff \langle i, \pi'(x)\rangle \in A^{1,p'}_{M^*}$. But this is obvious. $\Box$

- $\mathrm{crit}(\pi') = \tilde{\kappa}$.

Proof. This is obvious, as $\mathrm{crit}(\pi') = \mathrm{crit}(\bar{\pi}) = \tilde{\kappa}$. $\Box$

- $G = \langle \tilde{\lambda} \cap \pi'(X) \mid X \in \mathcal{P}(\tilde{\kappa}) \cap M\rangle$.

Proof. This is also obvious. Since $s = \omega\rho^1_M$ is a cardinal in M, and since $\tilde{\kappa} < s$, it follows by acceptability of M that $\mathcal{P}(\tilde{\kappa}) \cap M = \mathcal{P}(\tilde{\kappa}) \cap \tilde{M}$. Since $\bar{\pi} : \tilde{M} \longrightarrow_G \tilde{M}^*$, it follows that $G = \langle \tilde{\lambda} \cap \bar{\pi}(X) \mid X \in \mathcal{P}(\tilde{\kappa}) \cap \tilde{M}\rangle$. The fact that $\pi' \restriction H^1_M = \bar{\pi} \restriction H^1_M$ now implies the claim. $\Box$

- $M^* = \{\pi'(f)(\vec{\alpha}) \mid \vec{\alpha} < \tilde{\lambda} \wedge f \in \Gamma^*(M, \tilde{\kappa})\}$.

Proof. Since $\omega\rho^2_M \le \tilde{\kappa} < s = \omega\rho^1_M$, $\Gamma^*(M, \tilde{\kappa})$ consists of functions f which are members of M or have a $\mathbf{\Sigma}_1^{(0)}$-definition over M.

So let $x \in M^*$. As $M^* = h^1_{M^*}(s^*)$, there are $i < \omega$ and $\xi < s^*$ so that

$$x = h^1_{M^*}(\langle i, \xi\rangle).$$

But since $\bar{\pi} : \tilde{M} \longrightarrow_G \tilde{M}^*$ there are a function $f \in \tilde{M}$ and ordinals $\vec{\alpha} < \tilde{\lambda}$, so that $\xi = \bar{\pi}(f)(\vec{\alpha})$. Define $g : (\mathrm{On} \cap M^*) \to M^*$ by $g(\beta) = h^1_{M'}(\langle i, \beta\rangle)$. Then g is a $\Sigma_1(M^*)$-function. Let $\bar{g}$ be the $\Sigma_1(M)$-function defined over M by the same formula. Set: $h := \bar{g} \circ f$. Then h is also a $\Sigma_1(M)$-function, and we have:

$$\pi'(h)(\vec{\alpha}) = (g \circ \pi'(f))(\vec{\alpha}) = g(\pi'(f)(\vec{\alpha})) = g(\xi) = h^1_{M^*}(\langle i, \xi\rangle) = x.$$

I used here that $\pi' \restriction \mathrm{J}^E_s = \bar{\pi} \restriction \mathrm{J}^E_s$. $\Box$

That's all that was needed for (7), and hence for the lemma, so that the proof is complete. $\Box_{(7),\mathrm{Lemma}}$

For the proof that the s'-ISC is preserved in the case of type III-structures, I need the following general observation.

Lemma 5.34. *Let $M = \langle \mathrm{J}_\alpha^A, \vec{B}\rangle$ be an acceptable J-structure. Let F be an extender on M with critical point κ. Let $\pi : M \longrightarrow_F^* M'$. If $\kappa < \omega\rho_M^{n+1}$, then $\omega\rho_{M'}^n = \pi(\omega\rho_M^n)$.*

Proof. Set $\rho = \omega\rho_M^n$.

$\boxed{\omega\rho_{M'}^n \leq \pi(\rho)}$ The statement $\forall\xi^n \quad \xi^n < \rho$ is $\Pi_1^{(n)}(M)$ in ρ, and since π is $\Sigma_0^{(n+1)}$-preserving ($\Sigma_1^{(n)}$ would suffice), that statement holds in M' of $\pi(\rho)$. I used [Jen97, §2, Lemma 3 (b)] here.

$\boxed{\omega\rho_{M'}^n \geq \pi(\rho)}$ Assume the contrary. Let $\rho' := \omega\rho_{M'}^n < \pi(\rho)$. In M', the following $\Pi_1^{(n)}$-statement holds:

$$\forall\xi^n \quad \xi^n \neq \rho'.$$

Let $\rho' = \pi(f)(\vec{\gamma})$ for some $f \in \Gamma^*(M,\kappa)$ and $\vec{\gamma} < \mathrm{lh}(F)$. Moreover, f can be chosen so that $\mathrm{ran}(f) \subseteq \rho$. It follows that

$$M' \models \forall\xi^n \quad \xi^n \neq \pi(f)(\vec{\gamma}).$$

By [Jen97, Lemma 3 (d)], one can apply a kind of Łóz theorem to that statement, as $\omega\rho_M^{n+1} > \kappa$, which yields that this is equivalent to

$$\vec{\gamma} \in F(\underbrace{\{\vec{\zeta} < \kappa \mid M \models \forall\xi^n \quad \xi^n \neq f(\vec{\zeta})\}}_{Z}).$$

In particular, $Z \neq \emptyset$. So let $\vec{\zeta} \in Z$. Then we have:

$$M \models \forall\xi^n \quad \xi^n \neq f(\vec{\zeta}),$$

but by choice of f, $f(\vec{\zeta}) < \rho = \omega\rho_M^n$ - a contradiction. $\square$

Lemma 5.35. *Let M be an active $p\lambda$-structure of type III, G an extender on M with $\kappa = \mathrm{crit}(G) < s(M)$ and*

$$\pi : M \longrightarrow_G^* M'.$$

Then M' is also a $p\lambda$-structure of type III (assuming the well-foudedness of M').

Proof. Due to the preservation properties of π, it suffices to prove only those aspects of the s'-ISC that relate to the top extender of M'. Thus, it suffices to show that $[M']_\zeta \in M'$ for arbitrarily large $\zeta < s(M')$.

As M is of type III, Lemma 5.32 says that $\omega\rho_M^1 = s(M)$. So $\kappa < \omega\rho_M^1$.

Case 1: $\omega\rho_M^2 \leq \kappa < \omega\rho_M^1$.

Using Lemma 5.22, it is easy to see that $[M]_\zeta \in M$ for arbitrarily large $\zeta < s(M)$. It follows that for such ζ, $[M']_{\pi(\zeta)} = \pi([M]_\zeta) \in M'$. But by Lemma 5.33, $s(M') = \sup \pi``s(M)$, so we're done.

Case 2: $\kappa < \omega\rho_M^2$.

Then $\pi : M \longrightarrow_{\Sigma_2^{(1)}} M'$ by [Jen97, Lemma 3, (b)]. Using Lemma 5.34, it follows that $s' := \omega\rho_{M'}^1 = \pi(\omega\rho_M^1) = \pi(s)$. Moreover, Lemma 5.26 says that $s(M') \leq \pi(s) = s'$, as π is also Σ_2-preserving. Letting $F := E_{\mathrm{top}}^M$, the following statement, call it Ψ, holds in M:

$$\forall\xi^1\exists x^1 \quad x^1 = F|\xi^1;$$

that the existential quantification may be bounded in this way is justified as in the proof of Lemma 5.33. Here, "$x^1 = F|\xi^1$" is expressed by

$$\text{“}x^1 \text{ is a function”} \wedge (\mathrm{dom}(x^1) \subseteq \bar{\kappa}) \wedge \forall a^0\forall b^0 \quad (b^0 = F(a^0) \longrightarrow x^1(a^0) = b^0 \cap \xi^1),$$

a $\Pi_1^{(0)}$-statement (I denote the critical point of F by $\bar{\kappa}$ here). So Ψ is $\Pi_2^{(1)}$ in $\bar{\kappa}$. Hence, the same statement holds in M', so that

$$M' \models \forall \xi < s' \quad F'|\xi \in M'.$$

So $s' = s(M')$, since it follows that $s(M') \geq s'$ (otherwise $M' \in M'$). So M' is of type III and satisfies the s'-ISC. $\square$

5.3.4 Putting things together

Now I am ready to prove that the s'-ISC of pλ-structures is preserved under normal s-iterations.

Theorem 5.36. *Any normal s-iterate of a pλ-structure is a pλ-structure.*

Proof. It suffices to show that the s'-ISC is preserved. So let M be a pλ-structure, and let $\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, D, \langle\nu_i \mid i \in D\rangle, \langle\eta_i \mid i \in D\rangle, \langle\kappa_i \mid i \in D\rangle, \langle\tau_i \mid i \in D\rangle, \langle\lambda_i \mid i \in D\rangle, \langle s_i \mid i \in D\rangle, \langle s_i^+ \mid i \in D\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ be a normal s-iteration of M (i.e., with $M = M_0$). I want to show by induction on $i < \theta$ that M_i satisfies the s'-ISC. If M_i is passive, then this is trivial. What was proved in the previous three subsections can be used to prove the successor step of the induction. So assume M_j satisfies the s'-ISC, for every $j \leq i$. I want to show that M_{i+1} does, too. Again, it may be assumed that M_{i+1} is active. Let $\xi = T(i+1)$ and $M^* = M_\xi||\eta_i$, so that

$$\pi_{\xi,i+1} : M^* \longrightarrow^*_{E_{\nu_i}^{M_i}} M_{i+1}.$$

By the way s-iterations are constructed, $\kappa_i < s_\xi$, and in the current case, M^* is active. Since $|M^*||s_\xi^+| = |M_i||s^+\xi|$, it follows that $\eta_i \geq s_\xi^+$.
Claim: $\kappa_i < s(M^*)$.
Proof of Claim. This is clear if $\eta_i = \nu_\xi$, for then, $s(M^*) = s_\xi$. Consider the case that $\eta_i < \nu_\xi$. It must be that $s(M^*) \geq s_\xi^+$, for otherwise we would have: $\omega\rho^\omega_{M_\xi||\eta_i} \leq s(M^*) < s_\xi^+ \leq \eta_i < \nu_\xi$, so that s_ξ^+ wouldn't be a cardinal in $M_\xi||\nu_\xi$. So clearly, $\kappa_i < s_\xi < s_\xi^+ \leq s(M^*)$, as desired. Finally, consider the case that $\eta_i > \nu_\xi$, and assume (towards a contradiction) that $s(M^*) \leq \kappa_i$. The definition of η_i implies that τ_i is a cardinal in M^*, so since $s(M^*) \leq \kappa_i$, it follows that $s^+(M^*) \leq \tau_i$. Since $\kappa_i < s_\xi$ and s_ξ^+ is a cardinal in $M_i||\nu_i$, it follows that $\tau_i \leq s_\xi^+$. So $s^+(M_\xi||\eta_i) \leq s_\xi^+ = s^+(M_\xi||\nu_\xi)$, which even implies that $s^+(M_\xi||\eta_i) < s^+(M_\xi||\nu_\xi)$. But that would mean that ν_ξ wasn't applicable in M_ξ, since $\eta_i > \nu_\xi$, a contradiction. $\square_{Claim}$

Since M_ξ satisfies the s'-ISC, and hence, M^* obviously does, too, it follows by Lemmas 5.29,5.31 and 5.35 (depending on the type of M^*) that this is also true of M_{i+1}, as desired. These lemmas are applicable because of the claim.

So suppose now that i is a limit ordinal and for all $j < i$, M_j satisfies the s'-ISC. Again, assume that M_i is active. Let $\xi <_T i$ be such that $[\xi, i)_T$ contains no truncations. I treat the different types separately again:

Case 1: M_ξ is of type I.

Then it is easy to show by induction on $\zeta \in [\xi, i]_T$ that M_ζ is of type I, using Lemma 5.29 in the induction step. In the limit step, all one needs to know is that the property of being a generator is expressible in a Π_1 way. In particular, M_i is of type I, and hence it satisfies the s'-ISC by fiat.

Case 2: M_ξ is of type II.

It is straightforward to show by induction on $\zeta \in [\xi, i]_T$ that M_ζ is of type II and that $\pi_{\xi,\zeta}(s(M_\xi)) = s(M_\zeta)$, this time using Lemma 5.31 in the successor step. So M_i is of type II and $\pi_{\xi,i}(s(M_\xi)) = s(M_i)$. Now the rest of the proof of Lemma 5.31 (starting at case 1) goes through.

It didn't matter for the rest of that argument that π was an ultrapower embedding by a short extender.

Case 3: M_ξ is of type III.

Again, it can easily be shown by induction on $\zeta \in [\xi, i]_T$ that M_ζ is of type III. Knowing this, it suffices to prove that $[M_i]_\gamma \in M_i$, for arbitrarily large $\gamma < s(M_i)$. So pick some generator $\gamma < s(M_i)$. Pick $\zeta <_T i$ such that $\gamma \in \operatorname{ran}(\pi_{\zeta,i})$. Then $\pi^{-1}_{\zeta,i}(\gamma)$ is a generator of $E^{M_\zeta}_{\mathrm{top}}$, so inductively, there is a generator γ' of $E^{M_\zeta}_{\mathrm{top}}$ which is larger than $\pi^{-1}_{\zeta,i}(\gamma)$ and is such that $[M_\zeta]_{\gamma'} \in M_\zeta$. Then $\pi_{\zeta,i}(\gamma') > \gamma$, $\pi_{\zeta,i}(\gamma')$ is a generator of $E^{M_i}_{\mathrm{top}}$, and $\pi_{\zeta,i}([M_\zeta]_{\gamma'}) = [M_i]_{\pi_{\zeta,i}(\gamma')} \in M_i$, as wished. $\Box$

5.3.5 Downwards preservation of the s'-ISC, and some results on ps-structures

The results in this section were presented in [Fuc08] already, since they were needed there. I repeat them here, because they fit in the present context much better.

Lemma 5.37. *Let M be a λ-structure of type II and $\sigma : \bar{M} \longrightarrow_{\Sigma_1} M$ an embedding with $q_M, s(M) \in \operatorname{ran}(\sigma)$. Then $\bar{M}$ is also a λ-structure of type II, $\sigma(s(\bar{M})) = s(M)$ and $\sigma(q_{\bar{M}}) = q_M$. The corresponding statement is true, when M is a pλ-structure of type II.*

Lemma 5.38. *There is a Π_1-formula $\psi(x, y)$ such that for every active pPs-structure N and every ordinal ξ, the following is true: If $E^{\widehat{N}}_{\mathrm{top}}|\xi \in |\widehat{N}|$, then $\langle a, f\rangle$ is the $\prec_N$-minimal[10] element of $\Gamma(N, \kappa(N))$ with $a \in [s(N)]^{<\omega}$ and $\pi^N_{s(N)}(f)(a) = E^{\widehat{N}}_{\mathrm{top}}|\xi$, if and only if*

$$\tilde{\mathcal{C}}_0(N) \models \psi[\langle a, f\rangle, \xi].$$

As a consequence, the following analog of Lemma 5.37 for ps-structures holds:

Lemma 5.39. *Let N be a ps-structure of type II and $\sigma : \bar{N} \longrightarrow_{\Sigma_1} N$ an embedding with $\dot{q}^{\mathcal{C}_0(N)}, \dot{s}^{\mathcal{C}_0(N)} \in \operatorname{ran}(\sigma)$. Then $\bar{N}$ is also a ps-structure of type II, $\sigma(\dot{s}^{\mathcal{C}_0(\bar{N})}) = \dot{s}^{\mathcal{C}_0(N)}$ and $\sigma(\dot{q}^{\mathcal{C}_0(\bar{N})}) = \dot{q}^{\mathcal{C}_0(N)}$.*

5.4 s-coiterations

Definition 5.40. Let $M = \langle J^E_\nu, E_{\omega\nu}\rangle$ be a pPλ-structure. Then $\tilde{E}^M := \langle \tilde{E}_\mu \mid \mu \le \nu\rangle = \langle \tilde{E}^M_\mu \mid \mu \le \nu\rangle$ is defined by

$$\begin{array}{rcll} \tilde{E}_{s^+(\mu)^M} & := & E_\mu, & \text{if } E_\mu \neq \emptyset, \\ \tilde{E}_\mu & := & \emptyset, & \text{if there is no } \xi \text{ such that } E_\xi \neq \emptyset \text{ and } \mu = s^+(\xi)^M. \end{array}$$

Note that this definition is correct, as $\nu \neq \nu' \implies s^{+M}(\nu) \neq s^{+M}(\nu')$ (Lemma 5.6).

Definition 5.41. Let M and M' be pPλ-structures. Set:

$$s(M, M') := \min\{\mu \mid \tilde{E}^M_\mu \neq \tilde{E}^{M'}_\mu \wedge \mu \le \operatorname{ht}(M) \cap \operatorname{ht}(M')\}.$$

So $s(M, M')$ is the least s-Index, at which M and M' differ. The basic idea is that in constructing the s-coiteration of M and M', these differences are eliminated in the order determined by s.

[10] For the definition of $\prec_N$, see [Fuc08].

Definition 5.42. Let M^0 and M^1 be pPλ-structures. An s-coiteration of M^0 and M^1 with s-indices $\langle s_i^+ \mid i \in D^0\rangle$ is a pair $\mathcal{I} = \langle \mathcal{I}^0, \mathcal{I}^1\rangle$ of iterations of M^0 and M^1,

$$\mathcal{I}^h = \langle\langle M_i^h \mid i < \theta^h\rangle, D^h, \langle \nu_i^h \mid i \in D^h\rangle, \langle \eta_i^h \mid i < \theta^h\rangle, T^h, \langle {\pi^h}_{i,j} \mid i \leq_{T^h} j < \theta^h\rangle\rangle \quad (h < 2)$$

with s-indices $\langle s_j^+ \mid j \in D^h\rangle$, where $\theta^0 = \theta^1 = \theta$, so that the following conditions are met:

(a) $\mathcal{I}^h$ is standard.

(b) If M_i^0 and M_i^1 are incompatible, meaning that neither of these structure is a segment of the other, then $s_i^+ = s(M_i^0, M_i^1)$, otherwise s_i^+ is undefined. If s_i^+ is undefined, then $\theta = i+1$, $i \notin D^h$, and the coiteration terminates at i.

(c) If $h \in 2$ has the property that $\tilde{E}^{M_i^h}_{s_i^+} \neq \emptyset$ (and there is at least one such h unless the coiteration terminates at i), then let $\bar{\nu}_i^h$ be such that $s^+(\bar{\nu}_i^h)^{M_i^h} = s_i^+$. Moreover, let $\bar{\kappa}_i^h = \mathrm{crit}(\tilde{E}^{M_i^h}_{s_i^+})$ and $\bar{\tau}_i^h = (\bar{\kappa}_i^{h+})^{M_i^h || \bar{\nu}_i^h}$. If $\tilde{E}^{M_i^h}_{s_i^+} = \emptyset$, then these are undefined. Let $\xi_i^h =$ be the least $\xi \in D^h$ with $\bar{\kappa}_i^h < s(\nu_\xi^h)^{M_\xi^h}$, if $\bar{\nu}_i^h$ is defined, $\xi_i^h = i$ otherwise.

If $\bar{\nu}_i^h$ is undefined, then set $\bar{M}_{i+1}^h := M_i^h$.

Otherwise let $\bar{\eta}_i^h \leq \mathrm{ht}(M_{\xi^h}^h)$ be maximal so that we have:

$$(\bar{\kappa}_i^{h+})^{M_{\xi^h}^h || \bar{\eta}_i^h} = \bar{\tau}_i^h.$$

Now let $\bar{M}_{i+1}^h$ and $\bar{\pi}_i^h$ be defined by:

$$\bar{\pi}_i^h : M_{\xi_i^h}^h || \bar{\eta}_i^h \longrightarrow^*_{\tilde{E}^{M_i^h}_{s_i^+}} \bar{M}_{i+1}^h.$$

If $\bar{M}_{i+1}^0$ is a segment of M_i^1, then we have:

$$M_{i+1}^1 = M_i^1,\ i \notin D^1$$

and

$$i \in D^0,\ M_{i+1}^0 = \bar{M}_{i+1}^0,\ T^0(i+1) = \xi_i^0,\ \nu_i^0 = \bar{\nu}_i^0,\ \eta_i^0 = \bar{\eta}_i^0, s_i^+ = s^{+M_i^0}(\nu_i^0),$$

and $\pi^0_{\xi_i^0, i+1} = \bar{\pi}_i^0$.

If this is not the case, but vice versa, $\bar{M}_{i+1}^1$ is a segment of M_i^0, then

$$M_{i+1}^0 = M_i^0,\ i \notin D^0$$

and

$$i \in D^1,\ M_{i+1}^1 = \bar{M}_{i+1}^1,\ T^1(i+1) = \xi_i^1,\ \nu_i^1 = \bar{\nu}_i^1,\ \eta_i^1 = \bar{\eta}_i^1, s_i^+ = s^{+M_i^1}(\nu_i^1),$$

and $\pi^1_{\xi_i^1, i+1} = \bar{\pi}_i^1$.

Remark: I refer to these situations as *exceptions*. In case of an exception, the coiteration terminates in the next step, as M_{i+1}^0 and M_{i+1}^1 are compatible.

Finally, if no such situation occurs, then

$$M_{i+1}^h = \bar{M}_{i+1}^h,\ T^h(i+1) \simeq \xi_i^h,\ \nu_i^h \simeq \bar{\nu}_i^h,\ \eta_i^h \simeq \bar{\eta}_i^h.$$

Moreover, $i \in D^h$ iff ν_i^h is defined, and $s_i^+ = s^+(\nu_i^h)$ for some (and then each) $h < 2$ with $i \in D^h$. [11]

[11]In the above definition, "$\simeq$" is to be understood in the sense of Kleene. So the value on the left hand side is defined iff the one on the right hand side is, and if so, they are equal.

The necessity for this rather technical definition results from complications that may occur when pPλ-structures M^0 and M^1 are coiterated, and at least one of them is not modest.

5.4.1 Existence and normality of the s-coiteration

Lemma 5.43. *Let M^0, M^1 be normally s-iterable pPλ-structures. Then there is a coiteration $\mathcal{I}$ of M^0 and M^1. Both sides of this coiteration are normal s-iterations.*

Proof. I show that if $\mathcal{I} = \langle \mathcal{I}^0, \mathcal{I}^1 \rangle$ is an s-coiteration of M^0 and M^1, then $\mathcal{I}^0$ and $\mathcal{I}^1$ are normal s-iterations. The existence then follows from normal s-iterability. I have to show conditions (b) and (d) from Definition 5.1 are satisfied.

(1) *Sei $h < 2$, $i < \mathrm{lh}(\mathcal{I}^h)$. There is no $\tilde{\nu} > \nu_i^h$ such that $s^+(\tilde{\nu})^{M_i^h} < s_i^+$. So condition (d) of normality is satisfied.*

Proof of (1). Assume the contrary. Let $h = 0$. Then

$$E := E_{\tilde{\nu}}^{M_i^0} = \tilde{E}^{M_i^0}_{s^+(\tilde{\nu})^{M_i^0}} = \tilde{E}^{M_i^1}_{s^+(\tilde{\nu})^{M_i^0}} = E_{\tilde{\nu}}^{M_i^1};$$

the last identity is valid since an extender determines its λ-index, because it determines the power set of its critical point. Let $\kappa = \mathrm{crit}(E)$ and $\tau = (\kappa^+)^{\mathrm{J}_{\tilde{\nu}}^{E^{M_i^0}}} = (\kappa^+)^{\mathrm{J}_{\tilde{\nu}}^{E^{M_i^1}}}$ (again an extender determines its critical point). Let

$$\begin{aligned} \pi^0 : \mathrm{J}_\tau^{E^{M_i^0}} &\longrightarrow_E \mathrm{J}_{\tilde{\nu}}^{E^{M_i^0}}, \\ \pi^1 : \mathrm{J}_\tau^{E^{M_i^1}} &\longrightarrow_E \mathrm{J}_{\tilde{\nu}}^{E^{M_i^1}}. \end{aligned}$$

(1.1) $\mathrm{J}_\tau^{E^{M_i^0}} = \mathrm{J}_\tau^{E^{M_i^1}}$.

Proof of (1.1). I have to show: $E^{M_i^0} \restriction \tau = E^{M_i^1} \restriction \tau$. To this end, let $\mu < \tau$ be such that $E_\mu^{M_i^0} \neq \emptyset$. Then

$$s^+(\mu)^{M_i^0} \leq \mu < \tau \leq s^+(\tilde{\nu})^{M_i^0} < s_i^+.$$

Since $\tilde{E}^{M_i^0} \restriction s_i^+ = \tilde{E}^{M_i^1} \restriction s_i^+$, it follows that

$$E_\mu^{M_i^0} = \tilde{E}^{M_i^0}_{s^+(\mu)^{M_i^0}} = \tilde{E}^{M_i^1}_{s^+(\mu)^{M_i^0}} = E_\mu^{M_i^1}.$$

The opposite direction is proved analogously. $\square_{(1.1)}$

It follows immediately that

(1.2) $\mathrm{J}_{\tilde{\nu}}^{E^{M_i^0}} = \mathrm{J}_{\tilde{\nu}}^{E^{M_i^1}}$.

(1.3) $s^+(\nu_i^0)^{M_i^0} = s^+(\nu_i^0)^{M_i^1}$.

Proof of (1.3). By (1.2), $E_{\nu_i^0}^{M_i^0} = E_{\nu_i^0}^{M_i^1}$, from which it follows that

$$s(\nu_i^0)^{M_i^0} = s(\nu_i^0)^{M_i^1},$$

and

$$s^+(\nu_i^0)^{M_i^0} = (s(\nu_i^0)^{M_i^0})^{+\mathrm{J}_{\nu_i^0}^{E^{M_i^0}}} = (s(\nu_i^0)^{M_i^1})^{+\mathrm{J}_{\nu_i^0}^{E^{M_i^1}}} = s^+(\nu_i^0)^{M_i^1}.$$

$\square_{(1.3)}$ Hence we get:

$$\tilde{E}^{M_i^0}_{s_i^+} = \tilde{E}^{M_i^0}_{s^+(\nu_i^0)^{M_i^0}} = E^{M_i^0}_{\nu_i^0} = E^{M_i^1}_{\nu_i^0} = \tilde{E}^{M_i^1}_{s^+(\nu_i^0)^{M_i^1}} = \tilde{E}^{M_i^1}_{s_i^+},$$

contradicting the definition of $s_i^+ = s(M_i^0, M_i^1)$. $\square_{(1)}$

I now turn to condition (b) in the definition of normality.

(2) *Let $i \in D^h$, $j \in D^h \cap i$. Then $s_j^+ = s^+(\nu_j^h)^{M_j^h} < \nu_i$.*

Proof of (2). Induction on i. Suppose the claim holds for all $i' < i$. If $i = \min(D^h \setminus (l+1))$ and $l \in D^h$, then I show that $\nu_i > s_l^+$. Wlog, let $i = l+1$. As $\mathcal{I}^h|(l+1)$ is a normal s-iteration, it follows for $\alpha \in D^h \cap l$ that $s_\alpha^{+h} \le s_l^{+h} < \nu_i$. The following observations make this clear:

(2.1) $E^{M_i^k} \restriction (\nu_l^h + 1) = E^{M_l^h} \restriction \nu_l^h$ *for $l \in D^k$.*

Proof of (2.1). This follows from the coherency of $\mathrm{J}^{E^{M_l^h}}_{\nu_l^h}$. $\square_{(2.1)}$

(2.2) $\mathrm{J}^{E^{M_l^0}}_{s_l^+} = \mathrm{J}^{E^{M_l^1}}_{s_l^+}$.

Proof of (2.2). Like the proof of (1.1). $\square_{(2.2)}$

It follows from (2.1) and (2.2) that

$$E^{M_i^0} \restriction (s_l^+ + 1) = E^{M_l^0} \restriction s_l^+ = E^{M_l^1} \restriction s_l^+ = E^{M_i^1} \restriction (s_l^+ + 1),$$

and that means that $\nu_{i+1}^h > s_i^+$.

This proves the claim in case $i = \min(D^h \setminus (l+1))$.

If i is a limit point of T^h, then let $b = <_{T^h}$ "$\{i\}$ and $j \in b \cap D^h$. I have to show that $s_j^+ < \nu_i^h$. There are only finitely many truncations in b, so by methods established earlier on, one can find a $j' \in b \setminus (j+1)$ so that, setting $\kappa := \mathrm{crit}(\pi^h_{j',i})$ and $\tilde{\kappa} := \mathrm{crit}(\pi^{1-h}_{j',i})$, it follows that $\kappa, \tilde{\kappa} > s_j^+$. Then $\mathrm{J}^{E^{M_i^h}}_{s_j^+ + 1} = \mathrm{J}^{E^{M_{j'}^h}}_{s_j^+ + 1} = \mathrm{J}^{E^{M_{j'}^{1-h}}}_{s_j^+ + 1} = \mathrm{J}^{E^{M_i^{1-h}}}_{s_j^+ + 1}$, and this implies that $\nu_i^h > s_j^+$, as wished. $\square$

5.4.2 Coherency of the s-coiteration

Lemma 5.44. *Let M^0 and M^1 be $pP\lambda$-structures with an s-coiteration $\mathcal{I} = \langle \mathcal{I}^0, \mathcal{I}^1 \rangle$, $\mathcal{I}^h = \langle \langle M_i^h \mid i < \theta^h \rangle, D^h, \langle \nu_i^h \mid i \in D^h \rangle, \langle \eta_i^h \mid i < \theta^h \rangle, T^h, \langle \pi^h{}_{i,j} \mid i \le_{T^h} j < \theta^h \rangle \rangle$ being an iteration of M^h. Then for $i \le j < \theta = \theta^0 = \theta^1$,*

$$s_i^+ \le s_j^+.$$

If $s_i^+ = s_{i+1}^+$, then we have: There is an $h < 2$ so that i, $i+1 \in D^h$ and $s_i^+ = \nu_i^h$; in particular, M_i^h is not modest. Moreover, in this case, $s_{i+1}^+ < \nu_{i+1}^h = \mathrm{ht}(M_{i+1}^h)$.

Proof. Assume the contrary. So let $s_{i+1}^+ \le s_i^+$.

Case 1: There is an $h \in 2$ so that i, $i+1 \in D^h$.

Both sides of the s-coiteration are normal. So it follows from Lemma 5.5 that $s_{i+1}^+ = s_i^+ = \nu_i^h$, and that $s_{i+1}^+ < \nu_{i+1}^h = \mathrm{ht}(M_{i+1}^h)$, as desired.

Case 2: Case 1 fails.

Then there is an $h \in 2$, so that $i \in D^h \setminus D^{1-h}$ and $i+1 \in D^{1-h} \setminus D^h$. Wlog, let $h = 0$. Then

$$M_i^1 = M_{i+1}^1,\ \tilde{E}^{M_i^1}_{s_i^+} = \tilde{E}^{M_{i+1}^1}_{s_i^+} = \emptyset;$$

i is not an exception, as $i+1 \in D^1$. Then $s^+_{i+1} < s^+_i$, as it follows from $i+1 \in D^1$ and $i \notin D^1$ that

$$\tilde{E}^{M^1_{i+1}}_{s^+_{i+1}} \neq \emptyset = \tilde{E}^{M^1_i}_{s^+_i} = \tilde{E}^{M^1_{i+1}}_{s^+_i}.$$

Hence:

$$\tilde{E}^{M^1_{i+1}}_{s^+_{i+1}} = \tilde{E}^{M^1_i}_{s^+_{i+1}} = \tilde{E}^{M^0_i}_{s^+_{i+1}},$$

as $\tilde{E}^{M^1_i} \restriction s^+_i = \tilde{E}^{M^0_i} \restriction s^+_i$. But it follows immediately from $\tilde{E}^{M^1_i}_{s^+_{i+1}} = \tilde{E}^{M^0_i}_{s^+_{i+1}}$ that

$$F := E^{M^1_i}_{\nu^1_{i+1}} = E^{M^0_i}_{\nu^1_{i+1}}.$$

Let $\kappa := \mathrm{crit}(F)$. Then

$$\tau := (\kappa^+)^{\mathrm{J}^{E^{M^0_i}}_{\nu^1_{i+1}}} = (\kappa^+)^{\mathrm{J}^{E^{M^1_i}}_{\nu^1_{i+1}}} \leq s^+_{i+1} < s^+_i,$$

hence $\mathrm{J}^{E^{M^0_i}}_\tau = \mathrm{J}^{E^{M^1_i}}_\tau$, as $\mathrm{J}^{E^{M^0_i}}_{s^+_i} = \mathrm{J}^{E^{M^1_i}}_{s^+_i}$. Let

$$\bar{\pi} : \mathrm{J}^{E^{M^0_i}}_\tau \longrightarrow_F \mathrm{J}^{E^{M^0_i}}_{\nu^1_{i+1}}.$$

Then

$$\bar{\pi} : \mathrm{J}^{E^{M^1_i}}_\tau \longrightarrow_F \mathrm{J}^{E^{M^1_i}}_{\nu^1_{i+1}},$$

hence $M^0_i||\nu^1_{i+1} = M^1_i||\nu^1_{i+1}$. So $\nu^1_{i+1} < \nu^0_i$, since otherwise it would follow that $M^0_i||\nu^0_i = M^1_i||\nu^0_i$. But then

$$M^0_{i+1}||\nu^1_{i+1} = M^0_i||\nu^1_{i+1} = M^1_i||\nu^1_{i+1} = M^1_{i+1}||\nu^1_{i+1}$$

would be a contradiction. So it can only occur in case 1 that $s^+_{i+1} \leq s^+_i$. □

Corollary 5.45. *Let M^0 and M^1 be pPλ-structures. Let $\mathcal{I} = \langle \mathcal{I}^0, \mathcal{I}^1 \rangle$ be an s-coiteration of M^0 and M^1, where $\mathcal{I}^h = \langle \langle M^h_i \mid i < \theta^h \rangle, D^h, \langle \nu^h_i \mid i \in D^h \rangle, \langle \eta^h_i \mid i < \theta^h \rangle, T^h, \langle {\pi^h}_{i,j} \mid i \leq_{T^h} j < \theta^h \rangle \rangle$, for $h \in 2$. If $i+2 < \theta$, then*

$$s^+_i = s^+_{i+1} \implies s^+_{i+1} < s^+_{i+2}.$$

Proof. Assume the contrary. An exception could only occur at $i+2$. But this would only have an effect on M^h_{i+3}, so that the possibility of an exception can be ignored here. So let

$$s^+_i = s^+_{i+1} = s^+_{i+2}.$$

Then by the previous lemma, there is an $h < 2$ so that

$$i, i+1 \in D^h.$$

Wlog, let $h = 0$. By Lemma 5.5 then $i+2 \notin D^0$, as $\mathcal{I}^0$ is a normal s-iteration and $s^+_{i+1} < \nu^0_{i+1}$ by Lemma 5.44, so that if $i+2$ were a member of D^0, it would have to be the case that $s^+_{i+1} < s^+_{i+2}$.

But then, $i+2 \in D^1$. But since $s^+_{i+1} = s^+_{i+2}$, again by the previous lemma, there has to be an $h' \in 2$ so that $i+1, i+2 \in D^{h'}$. As $i+2 \notin D^0$, it follows that $h' = 1$, hence $i+1, i+2 \in D^1$.

It then even follows that $i \in D^1$, for otherwise, $M^1_i = M^1_{i+1}$, and since $i+1 \in D^1$, it would follow that

$$\emptyset \neq \tilde{E}^{M^1_{i+1}}_{s^+_{i+1}} = \tilde{E}^{M^1_i}_{s^+_{i+1}} = \tilde{E}^{M^1_i}_{s^+_i},$$

hence $i \in D^1$ after all, contradicting the assumption that $i \notin D^1$.

But $\mathcal{I}^1$ is a normal s-iteration, so the constellation

$$i, i+1, i+2 \in D^1 \text{ and } s_i^+ = s_{i+1}^+ = s_{i+2}^+$$

contradicts lemma 5.5. $\square$

5.4.3 s-coiterations terminate

Theorem 5.46. *Let M^0 and M^1 be pλ-structures with s-coiteration $\mathcal{I} = \langle \tilde{\mathcal{I}}^0, \tilde{\mathcal{I}}^1 \rangle$, where all structures occurring in $\tilde{\mathcal{I}}^0$ and $\tilde{\mathcal{I}}^1$ are pλ-structures. Then*

$$\mathrm{lh}(\mathcal{I}) \leq \max(\overline{\overline{M^0}}^+, \overline{\overline{M^1}}^+).$$

Proof. Assume the contrary. Let $\theta = \max(\overline{\overline{M^0}}^+, \overline{\overline{M^1}}^+) < \mathrm{lh}(\mathcal{I})$. Set:

$$\mathcal{I}^h := \tilde{\mathcal{I}}^h | \theta + 1 \quad \text{for } h \in 2.$$

Wlog, let D^h be unbounded in θ, for each $h < 2$: If, e.g., D^1 is bounded in θ, then D^0 has to be unbounded in θ, hence $\langle s_i^+ \mid i \in D^0 \rangle$ is unbounded in θ. Let $j = \sup D^1$. Then $\mathrm{ht}(M_j^1) < \theta$, hence there is an $i \in D^0 \setminus j$ with $s_i^+ > \mathrm{ht}(M_j^1) = \mathrm{ht}(M_i^1)$, so that $s(M_i^0, M_i^1) = s_i^+ > \mathrm{ht}(M_i^1)$, contradicting the definition of $s(M_i^0, M_i^1)$.

Let $\tau > \theta^+$ be such that τ is regular and

$$\langle \mathcal{I}^0, \mathcal{I}^1 \rangle \in H_\tau.$$

It is a straightforward matter to construct $\sigma : \bar{H} \prec H_\tau$ so that the following conditions are satisfied:

(a) $\bar{H}$ is transitive and $\overline{\overline{(\bar{H})}} < \theta$.

(b) $\theta, \langle \mathcal{I}^0, \mathcal{I}^1 \rangle \in \mathrm{ran}(\sigma)$. Let $\sigma(\bar{\theta}, \bar{\mathcal{I}}^0, \bar{\mathcal{I}}^1) = \theta, \mathcal{I}^0, \mathcal{I}^1$.

(c) $\sigma \restriction \bar{\theta} = \mathrm{id} \restriction \bar{\theta}$.

Let $\bar{\mathcal{I}}^h = \langle \langle \bar{M}_i^h \mid i \leq \bar{\theta} \rangle, \bar{D}^h, \langle \bar{\nu}_i^h \mid i \in \bar{D}^h \rangle, \langle \bar{\eta}_i^h \mid i \leq \bar{\theta}^h \rangle, \bar{T}^h, \langle \bar{\pi}_{i,j}^h \mid i \leq_{\bar{T}^h} j \leq \bar{\theta} \rangle \rangle$. Then obviously, for $i, j < \bar{\theta}$:

$$\begin{array}{rcl} \bar{M}_i^h = & M_i^h & = \sigma(\bar{M}_i^h), \\ \bar{\pi}_{i,j}^h = & \pi_{i,j}^h & = \sigma(\bar{\pi}_{i,j}^h), \\ \bar{T}^h \cap \bar{\theta}^2 & = & T^h \cap \bar{\theta}^2, \\ <_{\bar{T}^h} \text{“}\{\bar{\theta}\} & = & \bar{\theta} \cap (<_{T^h} \text{“}\{\theta\}). \end{array}$$

Hence $\bar{\theta}$ is a limit point of $<_{T^h}$ “$\{\theta\}$, so that

$$\bar{\theta} T^h \theta,$$

as follows from the properties of iteration trees; see [MS94, Def. 5.0.1] or [Jen97, §4, S.2, Def. and the following remark].

(1) $\langle \bar{M}_{\bar{\theta}}^h, \bar{\pi}_{i,\bar{\theta}}^h \rangle = \text{dir lim}_{i \leq_{\bar{T}} j <_{\bar{T}} \bar{\theta}} (\langle \bar{M}_i^h \rangle, \langle \bar{\pi}_{i,j}^h \rangle) = \text{dir lim}_{i \leq_T j <_T \bar{\theta}} (\langle M_i^h \rangle, \langle \pi_{i,j}^h \rangle) = \langle M_{\bar{\theta}}^h, \pi_{i,\bar{\theta}}^h \rangle$.

Hence

(2) $M^h_{\bar\theta} = \bar{M}^h_{\bar\theta}$, $\pi^h_{i,\bar\theta} = \bar\pi^h_{i,\bar\theta}$ *(for $i \le_T \bar\theta$).*

(3) $\sigma \restriction \mathrm{dom}(\pi^h_{\bar\theta,\theta}) = \pi^h_{\bar\theta,\theta}$.

Proof of (3). Let $x \in \mathrm{dom}(\pi^h_{\bar\theta,\theta}) \subseteq M^h_{\bar\theta}$. Let $i <_{T^h} \bar\theta$ and $x' \in M^h_i$, be so that $x = \pi^h_{i,\bar\theta}(x')$. Then we have:

$$\begin{array}{rcccl} \sigma(x) & = & \sigma(\pi^h_{i,\bar\theta}(x')) & = & \sigma(\bar\pi^h_{i,\bar\theta}(x')) \\ & = & \sigma(\bar\pi^h_{i,\bar\theta})(\sigma(x')) & = & \pi^h_{i,\theta}(x') \\ & = & \pi^h_{\bar\theta,\theta}(\pi^h_{i,\bar\theta}(x')) & = & \pi^h_{\bar\theta,\theta}(x). \end{array}$$

$\Box_{(3)}$

Now, for $h < 2$ choose the least ξ_h satisfying

$$\bar\theta <_{T^h} \xi_h + 1 <_{T^h} \theta \text{and} \xi_h \in D^h.$$

It's obvious that

(4) $\kappa^h_{\xi_h} = \mathrm{crit}(\pi^h_{\bar\theta,\xi_h+1}) = \mathrm{crit}(\pi^h_{\bar\theta,\theta}) = \mathrm{crit}(\sigma) = \bar\theta$.

Set: $\kappa := \kappa^h_{\xi_h} (= \bar\theta)$.

(5) *For* $X \in \mathrm{dom}(E^{M^h_{\xi_h}}_{\nu^h_{\xi_h}})$, $\quad E^{M^h_{\xi_h}}_{\nu^h_{\xi_h}}(X) \cap s^h_{\xi_h} = \sigma(X) \cap s^h_{\xi_h}$.

Proof of (5). The point is that $\mathrm{crit}(\pi^h_{\xi_h+1,\theta}) \ge s^h_{\xi_h}$ (since if $\xi_h+1 = T^h(\mu+1)$, then $\mathrm{crit}(\pi^h_{\xi_h+1,\theta}) = \mathrm{crit}(\pi^h_{\xi_h+1,\mu+1}) = \kappa^h_\mu < s^h_{\xi_h+1}$, and $\xi_h + 1$ is minimal with this property). Using this together with (3) for $\alpha < s^h_{\xi_h}$ and the fact that $X \in \mathrm{dom}(E^{M^h_{\xi_h}}_{\nu^h_{\xi_h}})$ one can argue as follows:

$$\begin{array}{rcl} \alpha \in E^{M^h_{\xi_h}}_{\nu^h_{\xi_h}}(X) & \iff & \alpha \in \pi^h_{\bar\theta,\xi_h+1}(X) \\ & \iff & \pi^h_{\xi_h+1,\theta}(\alpha) \in \pi^h_{\bar\theta,\theta}(X) \\ & \iff & \alpha \in \sigma(X). \end{array}$$

$\Box_{(5)}$

So letting $\gamma = s^0_{\xi_0} \cap s^1_{\xi_1}$, we get:

$$E^{M^0_{\xi_0}}_{\nu^0_{\xi_0}} | \gamma = E^{M^1_{\xi_1}}_{\nu^1_{\xi_1}} | \gamma.$$

Now let, wlog,

$$\xi_0 \le \xi_1.$$

Then $s^+_{\xi_0} \le s^+_{\xi_1}$. Moreover,

(6) $s^+_{\xi_0} \in \mathrm{Card}^{M^1_{\xi_1} || \nu^1_{\xi_1}}$.

Proof of (6). This is trivial if $s^+_{\xi_0} = s^+_{\xi_1}$. If $s^+_{\xi_0} < s^+_{\xi_1}$, then

$$s^+_{\xi_0} \in \mathrm{Card}^{J^{E^{M^0_{\xi_1}}}_{s^+_{\xi_1}}} = \mathrm{Card}^{J^{E^{M^1_{\xi_1}}}_{s^+_{\xi_1}}} \subseteq \mathrm{Card}^{J^{E^{M^1_{\xi_1}}}_{\nu^1_{\xi_1}}};$$

the inclusion follows from the fact that $s^+_{\xi_1} \in \mathrm{Card}^{J^{E^{M^1_{\xi_1}}}_{\nu^1_{\xi_1}}}$. $\Box_{(6)}$

(7) $\tau := \tau^0_{\xi_0} = \tau^1_{\xi_1}$.

Proof of (7). It follows from (6) that

$$\tau^0_{\xi_0} = (\kappa^+)^{\mathrm{J}^{E^{M^0_{\xi_0}}}_{\nu^0_{\xi_0}}} = (\kappa^+)^{\mathrm{J}^{E^{M^0_{\xi_0}}}_{s^+_{\xi_0}}} = (\kappa^+)^{\mathrm{J}^{E^{M^0_{\xi_1}}}_{s^+_{\xi_0}}} = (\kappa^+)^{\mathrm{J}^{E^{M^1_{\xi_1}}}_{s^+_{\xi_0}}} = (\kappa^+)^{\mathrm{J}^{E^{M^1_{\xi_1}}}_{\nu^1_{\xi_1}}} = \tau^1_{\xi_1}.$$

$\square_{(7)}$

Now let $h \in 2$ be such that

$$s^h_{\xi_h} \leq s^{1-h}_{\xi_{1-h}}.$$

Case 1: $s^h_{\xi_h} < s^{1-h}_{\xi_{1-h}}$.
Then let

$$\bar{\pi} : \mathrm{J}^{E^{M^{1-h}_{\xi_{1-h}}}}_{\tau} \xrightarrow[E^{M^{1-h}_{\xi_{1-h}}}_{\nu^{1-h}_{\xi_{1-h}}}|s^h_{\xi_h}]{} \mathrm{J}^{E'}_{\nu'}.$$

As $E^{M^{1-h}_{\xi_{1-h}}}_{\nu^{1-h}_{\xi_{1-h}}}|s^h_{\xi_h} = E^{M^h_{\xi_h}}_{\nu^h_{\xi_h}}|s^h_{\xi_h}$, it follows that

$$[M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}]_{s^h_{\xi_h}} = \langle \mathrm{J}^{E'}_{\nu'}, \bar{\pi}\restriction\mathcal{P}(\kappa)\rangle = M^h_{\xi_h}||\nu^h_{\xi_h};$$

note that $\mathrm{J}^{E^{M^0_{\xi_0}}}_{\tau} = \mathrm{J}^{E^{M^1_{\xi_1}}}_{\tau}$.

Since $s^h_{\xi_h}$ is a cutpoint of $E^{M^{1-h}_{\xi_{1-h}}}_{\nu^{1-h}_{\xi_{1-h}}}$ which is less than $s^{1-h}_{\xi_{1-h}}$, the s'-MISC for $M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}$ implies:

($*$) *$s^+_{\xi_h}$ is not a cardinal in $M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}$;*

$M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}$ satisfies the s'-ISC, hence also the s'-MISC (see Lemma 5.23).

For the same reason, $M^h_{\xi_h}||\nu^h_{\xi_h}$ satisfies the s'-MISC, hence $M^h_{\xi_h}||\nu^h_{\xi_h} = [M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}]_{s^h_{\xi_h}} \in M^{1-h}_{\xi_{1-h}}||\nu^{1-h}_{\xi_{1-h}}$. In particular,

$$\nu^h_{\xi_h} < \nu^{1-h}_{\xi_{1-h}}.$$

It follows from (6) and ($*$) that $h = 1$, so $\omega\rho^1_{M^1_{\xi_1}||\nu^1_{\xi_1}} \leq s^1_{\xi_1} < s^0_{\xi_0} < s^+_{\xi_0} \leq s^+_{\xi_1} \leq \nu^1_{\xi_1} < \nu^0_{\xi_0}$. So $s^+_{\xi_0}$ is not a cardinal in $M^0_{\xi_0}||\nu^0_{\xi_0}$, a contradiction. N.B.: $M^1_{\xi_1}||\nu^1_{\xi_1} = [M^0_{\xi_0}||\nu^0_{\xi_0}]_{s^0_{\xi_0}} \in M^0_{\xi_0}||\nu^1_{\xi_1}$.

So case 1 cannot occur.

Case 2: $s^h_{\xi_h} = s^{1-h}_{\xi_{1-h}} = \gamma$.
Set:

$$F := E^{M^0_{\xi_0}}_{\nu^0_{\xi_0}}|\gamma = E^{M^1_{\xi_1}}_{\nu^1_{\xi_1}}|\gamma.$$

Then

$$F' := E^{M^0_{\xi_0}}_{\nu^0_{\xi_0}} = \widehat{F} = E^{M^1_{\xi_1}}_{\nu^1_{\xi_1}}.$$

Since an extender determines its λ-index, it follows that

$$\nu := \nu^0_{\xi_0} = \nu^1_{\xi_1},$$

and using coherency of $M^0_{\xi_0}||\nu$ and $M^1_{\xi_1}||\nu$, it even follows that

$$M^0_{\xi_0}||\nu = M^1_{\xi_1}||\nu;$$

again I use that $\mathrm{J}_{\tau^0_{\xi_0}}^{E^{M^0_{\xi_0}}} = \mathrm{J}_{\tau^1_{\xi_1}}^{E^{M^1_{\xi_1}}}$. Hence, $\xi_0 \neq \xi_1$. So it must be the case that

$$\xi_0 < \xi_1.$$

Moreover, we know that

$$s^+_{\xi_0} = s^+_{\xi_1}.$$

It follows from Lemma 5.44 and Corollary 5.45 that

$$\xi_1 = \xi_0 + 1.$$

Moreover,

$$\nu^0_{\xi_0} = \nu = \nu^1_{\xi_1} > s^+_{\xi_0}$$

by normality. Hence,

$$\xi_1 = \xi_0 + 1 \notin D^0$$

by Lemma 5.5, since otherwise $s^+_{\xi_1} > s^+_{\xi_0}$. Again by Corollary 5.45 it follows that

$$\xi_0, \xi_1 \in D^1 \text{ und } \nu^1_{\xi_0} = s^+_{\xi_0}.$$

Also,

$$\nu^1_{\xi_1} = \mathrm{ht}(M^1_{\xi_1}),$$

by Lemma 5.5. So we're in the situation

$$M^1_{\xi_0+1} = M^1_{\xi_1}||\nu = M^0_{\xi_0}||\nu,$$

which contradicts the definition of s-coiterations: At stage ξ_0 of the coiteration, by Definition 5.42(c), $M^0_{\xi_0}$ was not allowed to be moved, and $M^1_{\xi_0+1}$ would have had to be a segment of $M^0_{\xi_0+1} = M^0_{\xi_0}$ (so that termination had occurred at stage $\xi_0 + 1$). This contradiction finishes the proof. $\square$

5.5 Normal iterations of pPs-structures

Definition 5.47. $\mathcal{I} = \langle\langle N_i \mid i < \theta\rangle, D, \langle\eta_i \mid i \in D\rangle, \langle\kappa_i \mid i \in D\rangle, \langle\tau_i \mid i \in D\rangle, \langle\lambda_i \mid i \in D\rangle, \langle s_i \mid i \in D\rangle, \langle s^+_i \mid i \in D\rangle, T, \langle\sigma_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ is a normal iteration of the pPs-structure N if:

(a) T is an iteration tree (in the sense of [Jen97]).

(b) For $i < \theta$, N_i is a pPs-structure, and $N_0 = N$.

(c) If $i \notin D$, then $i <_T i+1$, $N_{i+1} = N_i$ and $\sigma_{i,i+1} = \mathrm{id}\restriction|N_i|$.

(d) If $i \in D$, then we have:

  (1) $i + 1 < \theta$.

  (2) $E^{N_i}_{s^+_i} \neq \emptyset$, and $\kappa_i = \mathrm{crit}(E^{N_i}_{s^+_i})$, $s_i = s(E^{N_i}_{s^+_i})$, $\lambda_i = \lambda(E^{N_i}_{s^+_i})$, $\tau_i = \tau_i(E^{N_i}_{s^+_i})$.

  (3) $\xi := T(i+1) = \min\{\delta \mid \delta = i \vee (\delta \in D \wedge \kappa_i < s_\delta)\}$.

  (4) $\eta_i = \max\{\eta \leq \mathrm{ht}(N_\xi) \mid (\kappa^+_i)^{N_\xi||\eta} = (\kappa^+_i)^{N_i||s^+_i}\}$.

  (5) $\sigma_{\xi,i+1} : \widehat{N_\xi||\eta_i} \longrightarrow^*_{E^{N_i}_{s^+_i}} \widehat{N_{i+1}}$.

(6) If $j \in D \cap i$, then $s_j^+ < \mathrm{ht}(\widehat{N_i||s_i^+})$.

(e) If $\lambda < \theta$ is a limit ordinal, then

$$\langle N_\lambda, \langle \sigma_{i,\lambda} \mid i <_T \lambda \rangle \rangle = \mathrm{dir\ lim}(\langle N_i \mid i <_T \lambda \rangle, \langle \sigma_{i,j} \mid i <_T j <_T \lambda \rangle).$$

(f) For $i <_T j <_T k$, $\sigma_{i,k} = \sigma_{j,k}\sigma_{i,j}$.

(g) For $j < \theta$ $\{i \mid i+1 \leq_T j \wedge \eta_i < \mathrm{ht}(N_{T(i+1)})\}$ is finite. As before, I use $T(j)$ to denote the immediate $<_T$-predecessor of j, if it exists. Also, I refer to $i+1$ as a *truncation point* if $\eta_i < \mathrm{ht}(N_{T(i+1)})$. So for each $j < \theta$, there are only finitely many truncation points below j.

Definition 5.48. Complementing the the notion of normal s-iterability of pPλ-structures in Definition 5.2, I define now when a pPs-structure N is normally iterable. This shall mean that there is a successful normal iteration strategy for N, which is formulated precisely as before. It should be pointed out, though, that the continuation of an iteration has to consist of pPs-structures. In particular, the additional structure must be hereditarily continuable.

This finishes the treatment of normal iterations of pPs-structures. The reader may wonder why I don't go on proving things like that the s'-ISC of ps-structures is preserved under these iterations, et cetera. The reason is that I am going to show in the next section that the translation functions can even be used to translate iterations in such a very nice way that in order to see that the s'-ISC of a ps-structure N is preserved under a normal iteration, one can argue as follows: Let N' be a normal iterate of N. Let $M = \Lambda(N)$, $M' = \Lambda(N')$. M then satisfies the s'-ISC for pλ-structures, and M' is a normal s-iterate of M. So since s-iterations preserve the s'-ISC of pλ-structures, M' satisfies the s'-ISC. This is again preserved by the translation function, so that $N' = \mathtt{S}(M')$ satisfies the s'-ISC as well. It is also easy to check that the comparison process works, the main point being that Λ-images of initial segments of a pPs-structure are initial segments of the Λ-image of the whole structure. So in principle, one could just coiterate pPs-structures by coiterating their translates and retranslating the outcomes.

6 Transliterations

I put the results obtained up to now together in this section in order to translate iterations, forming what I call transliterations. Before doing this, let me recall a general observation which was shown in the first part of the paper:

Lemma 6.1. *Let M and N be acceptable J-models with:*

(i) $\mathbf{\Sigma}_1(M) \cap \mathcal{P}(H^1_M) = \mathbf{\Sigma}_1(N) \cap \mathcal{P}(H^1_N)$.

(ii) for $\alpha \in \mathrm{Card}^M \cap \mathrm{Card}^N$ $H^M_\alpha = H^N_\alpha$.

Then for every $n \geq 1$*:*

(a) $\omega\rho^n_M = \omega\rho^n_N$,

(b) $\mathbf{\Sigma}^{(n-1)}_1(M) \cap \mathcal{P}(H^n_M) = \mathbf{\Sigma}^{(n-1)}_1(N) \cap \mathcal{P}(H^n_N)$.

I shall also need the following observation on normal s-iterations:

Lemma 6.2. *Let $\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, D, \langle\nu_i \mid i \in D\rangle, \langle\eta_i \mid i < \theta\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ be a normal s-iteration with s-indices $\langle\langle s_i, s_i^+\rangle \mid i \in \theta\rangle$. Let $i \in D$ and $j = T(i+1)$. Then there is no $\mu > \eta_i$ such that $\mu \leq \mathrm{ht}(M_j)$ and $s^+(M_j||\mu) \leq \eta_i$.*

Proof. Assume the contrary. Pick a counterexample $\mu \leq \mathrm{ht}(M_j)$ so that $s^+(M_j||\mu) \leq \eta_i < \mu$. It is then obvious that $i+1$ is a truncation point.

(1) $s^+(M_j||\mu) \leq \kappa_i$.

Proof of (1). Assume $\kappa_i < s^+(M_j||\mu)$. As in $M_j||\eta_i + 1$, a new subset of κ_i appears, η_i is collapsed there to κ_i. But $\kappa_i < s^+(M_j||\mu) \leq \eta_i$, hence $s^+(M_j||\mu)$ is not a cardinal in $M_j||\eta_i + 1$, hence it's not a cardinal in $M_j||\mu$ either, as $\eta_i < \mu$, a contradiction. $\square_{(1)}$

Hence we have

(2) $s^+(M_j||\mu) \leq \kappa_i < s_j < \nu_j$.

Proof of (2). By definition of $T(i+1)$ in s-iterations, $\kappa_i < s_j$ $\square_{(2)}$

(3) $\mu < \nu_j$.

Proof of (3). Assume the contrary. Then $\nu_j < \mu$, as $s^+(M_j||\mu) \neq s^+(M_j||\nu_j)$: $s^+(M_j||\mu) \leq \kappa_i < s_j < s^+(M_j||\nu_j)$, so in particular, $\mu \neq \nu_j$. So it would follow that $s^+(M_j||\mu) < \nu_j < \mu$, meaning that ν_j was not applicable in $\mathcal{I}$. $\square_{(3)}$

(4) $\eta_i \geq s_j^+ = s^+(M_j||\nu_j)$.

Proof of (4). This follows from Lemma 5.4. As s_j^+ is a cardinal greater than κ_i in M_i, and since $s_j^+ \leq s_i^+$, it follows that

$$\mathcal{P}(\kappa_i) \cap |M_i||\nu_i| = \mathcal{P}(\kappa_i) \cap |M_i||s_j^+| = \mathcal{P}(\kappa_i) \cap |M_j||s_j^+|.$$

This implies the claim, since by normality, η_i is maximal with this property. $\square_{(4)}$

So this is the situation:

$$s^+(M_j||\mu) \leq \kappa_i < s_j < s_j^+ \leq \eta_i < \mu < \nu_j.$$

But $[s^+(M_j||\mu), \mu] \cap \mathrm{Card}^{M_j||\nu_j} = \emptyset$, as $\nu_j > \mu$. On the other hand, s_j^+ is a cardinal in $M_j||\nu_j$, and s_j^+ is in that interval, a contradiction. $\square$

This will be used in the proof of the next lemma, which introduces the notion of a transliteration.

Lemma 6.3. *Let M be a pPλ-structure, and $\mathcal{I} = \langle\langle M_i \mid i < \theta\rangle, D, \langle\nu_i \mid i \in D\rangle, \langle\eta_i \mid i < \theta\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ a normal s-iteration of M with s-indices $\langle\langle s_i, s_i^+\rangle \mid i \in D\rangle$. Let $N = \mathtt{S}(M)$. For $i < \theta$, set*

$$N_i = \mathtt{S}(M_i).$$

Moreover, for $i \in D$, define η'_i, as follows:

Let $\xi = T(i+1)$, $M_i^* = M_\xi||\eta_i$ and $N_i^* = \mathtt{S}(M_i^*)$. Set:

$$\eta'_i := \mathrm{ht}(N_i^*).$$

Then there are uniquely determined maps $\langle\sigma_{i,j} \mid i <_T j < \theta\rangle$, so that

$$\begin{aligned}\mathcal{I}' \quad := \quad & \langle\langle N_i \mid i < \theta\rangle, D, \langle\eta'_i \mid i \in D\rangle, \langle\kappa_i \mid i \in D\rangle, \langle\tau_i \mid i \in D\rangle, \langle\lambda_i \mid i \in \theta\rangle, \\ & \langle s_i \mid i \in D\rangle, \langle s_i^+ \mid i \in D\rangle, T, \langle\sigma_{i,j} \mid i \leq_T j < \theta\rangle\rangle\end{aligned}$$

is a normal iteration of N. I call this iteration the transliteration *of $\mathcal{I}$, and denote it by $\mathtt{S}(\mathcal{I})$.*

Proof. The uniqueness of the maps is obvious, so it suffices to prove their existence. Wlog, let $\mathcal{I}$ be direct. I prove by induction on $1 \le \gamma \le \theta$:

(∗) $\mathtt{S}(\mathcal{I}|\gamma)$ *exists, and for* $\alpha < \beta \le \gamma$, *the iteration* $\mathtt{S}(\mathcal{I}|\alpha)$ *is an initial segment of* $\mathtt{S}(\mathcal{I}|\beta)$. *In the notation of the lemma, we have for* $i \le_T j < \gamma$: $\sigma_{i,j} \subseteq \pi_{i,j}$.

So let $\gamma \ge 1$, and assume (∗) to hold for all $\bar{\gamma} < \gamma$.
Case 1: $\gamma = 1$.
Trivial.
Case 2: $\gamma > 1$, and γ is a successor ordinal.
Then let $\gamma = \bar{\gamma} + 1$ and

$$\begin{aligned}\mathtt{S}(\mathcal{I}|\bar{\gamma}) \quad = \quad & \langle\langle N_i \mid i < \bar{\gamma}\rangle, D \cap (\cup\bar{\gamma}), \langle\eta'_i \mid i \in D \cap (\cup\bar{\gamma})\rangle, \langle\kappa_i \mid i \in D \cap (\cup\bar{\gamma})\rangle, \\ & \langle\tau_i \mid i \in D \cap (\cup\bar{\gamma})\rangle, \langle\lambda_i \mid i \in \bar{\gamma}\rangle, \langle s_i \mid i \in D \cap (\cup\bar{\gamma})\rangle, \langle s_i^+ \mid i \in D \cap (\cup\bar{\gamma})\rangle, \\ & T \cap \bar{\gamma}^2, \langle\sigma_{i,j} \mid i \le_T j < \bar{\gamma}\rangle\rangle.\end{aligned}$$

Case 2.1: $\bar{\gamma}$ is a successor ordinal.
Set $\bar{\gamma} = \zeta + 1$. I shall prove:

(1) $E^{N_\zeta}_{s^+_\zeta} = ((E^{M_\zeta}_{\nu_\zeta}|s^+_\zeta)^{\mathtt{h}})^c \neq \emptyset$.

(2) $\xi := T(\zeta + 1) = \min\{\delta \mid \delta = \zeta \vee (\kappa_\zeta < s_\delta)\}$.

(3) $\eta'_\zeta = \max\{\eta \le \mathrm{ht}(N_\xi) \mid (\kappa^+_\zeta)^{N_\xi||\eta} = (\kappa^+_\zeta)^{N_\zeta||s^+_\zeta}\}$.

(4) Letting $\sigma_{\xi,\zeta+1} : \widehat{N_\xi||\eta'_\zeta} \longrightarrow^*_{E^{N_\zeta}_{s^+_\zeta}} N'$, $\sigma_{\xi,\zeta+1} \subseteq \pi_{\xi,\zeta+1}$ and $N' = \widehat{N_{\zeta+1}}$.

(5) For $j \in D \cap i$, $s^+_j < \mathrm{ht}(\widehat{N_i||s^+_i})$.

Proof of (1). By definition of normality, ν_ζ is applicable in M_ζ (Def. 5.1). By Lemma 2.6, $\mathtt{S}(M||\nu_\zeta)$ is a segment of $N_\zeta = \mathtt{S}(M_\zeta)$. This implies the claim immediately.
Proof of (2). This is trivial, as $\mathcal{I}$ is a normal s-iteration.
Proof of (3). As $\mathcal{I}$ is a normal s-iteration, it follows that

$$\eta_\zeta = \min\{\eta \le \mathrm{ht}(M_\xi) \mid (\kappa^+_\zeta)^{M_\xi||\eta} = (\kappa^+_\zeta)^{M_\zeta||\nu_\zeta} = \tau_\zeta\}.$$

It was mentioned in the proof of (1) already that $\mathtt{S}(M||\nu_\zeta)$ is a segment of N_ζ. So

$$\mathtt{S}(M_\zeta||\nu_\zeta) = N_\zeta||s^+_\zeta.$$

By Lemma 6.2, also $\mathtt{S}(M_\xi||\eta_\zeta)$ is a segment of N_ξ, hence

$$\mathtt{S}(M_\xi||\eta_\zeta) = N_\xi||\eta'_\zeta.$$

As $|N_\xi||\eta'_\zeta| \subseteq |M_\xi||\eta_\zeta|$, τ_ζ is a cardinal in $N_\xi||\eta'_\zeta$.[12] As τ_ζ is a cardinal in $M_\xi||\eta_\zeta$, it follows that

$$|M_\xi||\tau_\zeta| = |N_\xi||\tau_\zeta|.$$

[12] At this point, one has to use a different argument to prove the converse of this Lemma.

Hence

$$(\kappa_\zeta, \tau_\zeta) \cap \mathrm{Card}_{N_\xi||\eta'_\zeta} = (\kappa_\zeta, \tau_\zeta) \cap \mathrm{Card}_{N_\xi||\tau_\zeta} = (\kappa_\zeta, \tau_\zeta) \cap \mathrm{Card}_{M_\xi||\tau_\zeta} = \emptyset.$$

This shows that $(\kappa_\zeta^+)^{N_\xi||\eta'_\zeta} = \tau_\zeta$.

If $\eta'_\zeta < \mathrm{ht}(N_\xi)$, it remains to show that η'_ζ is maximal with the above property. But then it is also the case that $\eta_\zeta < \mathrm{ht}(M_\xi)$, and that means that $\omega\rho^\omega_{M_\xi||\eta_\zeta} \le \kappa_\zeta$. Applying Lemma 6.1 yields:

$$\omega\rho^\omega_{N_\xi||\eta'_\zeta} = \omega\rho^\omega_{M_\xi||\eta_\zeta} \le \kappa_\zeta.$$

This shows the maximality.

Proof of (4). Define N' and $\sigma_{\xi,\zeta+1}$ by:

$$\sigma_{\xi,\zeta+1} : \widehat{N_\xi||\eta'_\zeta} \longrightarrow^*_{E^{N_\zeta}_{s^+_\zeta}} N'.$$

This extender ultrapower is well-founded, by Lemma 4.5.

By Lemma 4.4, $N' = \mathtt{S}(\widehat{M_{\zeta+1}}) = \widehat{N_{\zeta+1}}$, and $\sigma_{\xi,\zeta+1} \subseteq \pi_{\xi,\zeta+1}$, as desired.

Proof of (5). For $j \in D \cap i$ $s^+_j < \nu_i = \mathrm{ht}(\widehat{N_i||s^+_i})$, as $\mathcal{I}$ is a normal s-iteration. Define for $i <_T \xi$:

$$\sigma_{i,\zeta+1} := \sigma_{\xi,\zeta+1} \circ \sigma_{i,\xi}.$$

Then (1)-(5) implies that

$$\begin{aligned}\mathtt{S}(\mathcal{I}|\gamma) \quad = \quad & \langle\langle N_i \mid i < \gamma\rangle, D \cap (\cup\gamma), \langle \eta'_i \mid i \in D \cap (\cup\gamma)\rangle, \langle \kappa_i \mid i \in D \cap (\cup\gamma)\rangle,\\ & \langle \tau_i \mid i \in D \cap (\cup\gamma)\rangle, \langle \lambda_i \mid i \in \gamma\rangle, \langle s_i \mid i \in D \cap (\cup\gamma)\rangle, \langle s^+_i \mid i \in D \cap (\cup\gamma)\rangle,\\ & T \cap \gamma^2, \langle \sigma_{i,j} \mid i \le_T j < \gamma\rangle\rangle\end{aligned}$$

is as wished.

Case 2.2: $\bar\gamma$ is a limit ordinal.

Let

$$\langle \tilde N, \langle \sigma_{i,\bar\gamma} \mid i <_T \bar\gamma\rangle\rangle := \mathrm{dir\ lim}(\langle \widehat{N_i} \mid i <_T \bar\gamma\rangle, \langle \sigma_{i,j} \mid i <_T j <_T \bar\gamma\rangle),$$

where for notational ease, I'll assume that no truncations occur in that tree. If this direct limit is well-founded, I'll identify it with its transitive isomorph, as usual.

Now define a model $\tilde M = \langle |\tilde M|, \dot E^{\tilde M}, \dot F^{\tilde M}\rangle$ by

$$\begin{aligned}|\tilde M| \quad &:= \quad \{\pi_{i,\bar\gamma}(a) \mid i <_T \bar\gamma \wedge a \in |\widehat{N_i}|\},\\ \dot E^{\tilde M}(\pi_{i,\bar\gamma}(a)) \quad &\iff \quad a \in |\widehat{N_i}| \wedge \dot E^{\widehat{N_i}}(a),\\ \dot F^{\tilde M}(\pi_{i,\bar\gamma}(a)) \quad &\iff \quad a \in |\widehat{N_i}| \wedge \dot F^{\widehat{N_i}}(a).\end{aligned}$$

$|\tilde M|$ *is transitive:* Assume the contrary. Then let $a \in |\tilde M|$ be such that $a \not\subseteq |\tilde M|$. Choose $b \in a \setminus |\tilde M|$. Let $i' <_T \bar\gamma$ and $a' \in |\widehat{N_{i'}}|$ be such that $\pi_{i',\bar\gamma}(a') = a$. Now let $i \ge_T i'$, $i <_T \bar\gamma$ be such that there is a $\bar b \in |M_i|$ with the property that $b = \pi_{i,\bar\gamma}(\bar b)$. Set: $\bar a = \pi_{i',i}(a')$. Then $\bar a \in |\widehat{N_i}|$, since $a' \in |\widehat{N_{i'}}|$, and $\pi_{i',i}\upharpoonright|\widehat{N_{i'}}| = \sigma_{i',i} : \widehat{N_{i'}} \longrightarrow \widehat{N_i}$. Moreover, $\pi_{i,\bar\gamma}(\bar a) = \pi_{i,\bar\gamma}(\pi_{i',i}(a')) = \pi_{i',\bar\gamma}(a') = a$. So thus far, we know: $a = \pi_{i,\bar\gamma}(\bar a)$, $b = \pi_{i,\bar\gamma}(\bar b)$, $\bar a \in |\widehat{N_i}|$ and $\bar b \in \bar a$. But $|\widehat{N_i}|$ is transitive, hence $\bar b \in |\widehat{N_i}|$, and this implies by definition of $|\tilde M|$, that $b = \pi_{i,\bar\gamma}(\bar b) \in |\tilde M|$. This is a contradiction.

I'll now show that

$$\langle \tilde{M}, \langle \pi_{i,\bar{\gamma}} \restriction |\widehat{N_i}| \mid i <_T \bar{\gamma} \rangle \rangle = \langle \tilde{N}, \langle \sigma_{i,\bar{\gamma}} \mid i <_T \bar{\gamma} \rangle \rangle,$$

making use of the fact that

$$\langle \tilde{N}, \langle \sigma_{i,\bar{\gamma}} \mid i <_T \bar{\gamma} \rangle \rangle = \mathrm{dir\ lim}(\langle \widehat{N_i} \mid i <_T \bar{\gamma} \rangle, \langle \sigma_{i,j} \mid i <_T j <_T \bar{\gamma} \rangle).$$

Obviously, for $i \leq_T j <_T \bar{\gamma}$, $\pi_{j,\bar{\gamma}} \sigma_{i,j} = \pi_{i,\bar{\gamma}} \restriction \widehat{N_i}$. Moreover, $|\tilde{M}|$ is transitive, and by definition, $|\tilde{M}| = \bigcup_{i <_t \bar{\gamma}} \mathrm{ran}(\pi_{i,\bar{\gamma}} \restriction |\widehat{N_i}|)$. This is all we need.

So $\sigma_{i,\bar{\gamma}} \subseteq \pi_{i,\bar{\gamma}}$, for $i <_T \bar{\gamma}$. Moreover, it's easy to see that $\tilde{M} = \widehat{\mathtt{S}(M_{\bar{\gamma}})}$:

$$\begin{array}{rcl} \pi_{i,\bar{\gamma}}(a) \in |\widehat{\mathtt{S}(M_{\bar{\gamma}})}| & \iff & M_{\bar{\gamma}} \models \varphi_{\mathrm{V}}[\pi_{i,\bar{\gamma}}(a), \pi_{i,\bar{\gamma}}(\mathrm{ht}(M_{\bar{\gamma}}) \dot{-} 1)] \\ & \iff & M_i \models \varphi_{\mathrm{V}}[a, \mathrm{ht}(M_i) \dot{-} 1] \\ & \iff & a \in |\widehat{N_i}|, \end{array}$$

for $i <_T \bar{\gamma}$ and $a \in |M_i|$. Here, I used the formula φ_{V} from Lemma 2.7 again.

Hence:

$$\begin{array}{rcl} |\widehat{\mathtt{S}(M_{\bar{\gamma}})}| & = & \{\pi_{i,\bar{\gamma}}(a) \mid i <_T \bar{\gamma} \wedge a \in |\widehat{N_i}|\} \\ & = & |\tilde{M}|. \end{array}$$

Analogously, one sees that $\dot{E}^{\tilde{M}} = \dot{E}^{\widehat{\mathtt{S}(M_{\bar{\gamma}})}}$ and $\dot{F}^{\tilde{M}} = \dot{F}^{\widehat{\mathtt{S}(M_{\bar{\gamma}})}}$. Hence,

$$\tilde{N} = \widehat{\mathtt{S}(M_{\bar{\gamma}})} = \widehat{N_{\bar{\gamma}}}.$$

So it's obvious that $\mathtt{S}(\mathcal{I}|\gamma)$ exists, as demanded.

Case 3: γ is a limit ordinal.

In this case, one can just set:

$$\mathtt{S}(\mathcal{I}|\gamma) := \text{“} \bigcup_{1 \leq \alpha < \gamma} \mathtt{S}(\mathcal{I}|\alpha) \text{”},$$

in the obvious sense. □

Here is the transliteration for the opposite direction:

Lemma 6.4. *Let N be a pPs-structure, and $\mathcal{I} = \langle \langle N_i \mid i < \theta \rangle, D, \langle \eta_i \mid i \in D \rangle, \langle \kappa_i \mid i \in D \rangle, \langle \tau_i \mid i \in D \rangle, \langle \lambda_i \mid i \in D \rangle, \langle s_i \mid i \in D \rangle, \langle s_i^+ \mid i \in D \rangle, T, \langle \sigma_{i,j} \mid i \leq_T j < \theta \rangle \rangle$ a normal iteration of N. Let $M = \Lambda(N)$. For $i < \theta$, let*

$$M_i = \Lambda(N_i).$$

For $i \in D$ let $\xi = T(i+1)$, and $N_i^* = N_\xi || \eta_i$. Let $M_i^* = \Lambda(N_i^*)$. Set:

$$\eta_i' := \mathrm{ht}(M_i^*).$$

Then there are uniquely determined maps $\langle \pi_{i,j} \mid i <_T j < \theta \rangle$, so that

$$\mathcal{I}' := \langle \langle M_i \mid i < \theta \rangle, D, \langle \eta_i' \mid i \in D \rangle, \langle \kappa_i \mid i \in D \rangle, \langle \tau_i \mid i \in D \rangle, \langle \lambda_i \mid i \in \theta \rangle, T, \langle \sigma_{i,j} \mid i \leq_T j < \theta \rangle \rangle$$

is a normal s-iteration of M with s-indices $\langle \langle s_i, s_i^+ \rangle \mid i \in D \rangle$. I denote this iteration by $\Lambda(\mathcal{I})$, and call it the transliteration *of $\mathcal{I}$.*

Proof. The proof of the opposite direction goes through, up to some minor modifications. Using the same numbering as in that proof, the first change in the argumentation occurs in case 2.1, in the proof of claim (3). It has to be shown that

(3) $\eta'_\zeta = \max\{\eta \leq \mathrm{ht}(M_\xi) \mid (\kappa_\zeta^+)^{M_\xi||\eta} = (\kappa_\zeta^+)^{M_\zeta||\nu_\zeta}\}$.

Proof of (3). I first show:

(a) $(\kappa_\zeta^+)^{M_\xi||\eta'_\zeta} = \tau_\zeta$.

Suppose not. By Lemma 5.4,

$$\mathrm{J}_{s_\xi^+}^{E^{M_\xi}} = \mathrm{J}_{s_\xi^+}^{E^{M_\zeta}},$$

and as $\kappa_\zeta < s_\xi < s_\xi^+$, and since s_ξ^+ is a cardinal in M_ζ, it follows that

$$(\kappa_\zeta^+)^{M_\xi||s_\xi^+} = \tau_\zeta.$$

As $|\mathtt{S}(M_\xi||s_\xi^+)| = |N_\xi||s_\xi^+|$, it follows that $(\kappa_\zeta^+)^{N_\xi||s_\xi^+} = \tau_\zeta$, and hence $\eta_\zeta \geq s_\xi^+$. This implies that

$$\eta'_\zeta \geq \eta_\zeta \geq s_\xi^+.$$

By assumption, $\eta'_\zeta > s_\xi^+$, since $(\kappa_\zeta^+)^{M_\xi||s_\xi^+} = \tau_\zeta$. Hence, $\alpha \in (s_\xi^+, \eta'_\zeta)$ can be chosen to be minimal with the property that $(\kappa_\zeta^+)^{M_\xi||\alpha} > \tau_\zeta$. As s_ξ^+ is a cardinal in $M_\xi||\nu_\xi$, $(\kappa_\zeta^+)^{M_\xi||s_\xi^+} = (\kappa_\zeta^+)^{M_\xi||\nu_\xi}$. This means that

$$\alpha > \nu_\xi.$$

Moreover, it is obvious that α is a successor ordinal. Let $\alpha = \bar{\alpha} + 1$. Then

$$\omega\rho^\omega_{M_\xi||\bar{\alpha}} \leq \kappa_\zeta.$$

($*$) *$\mathtt{S}(M_\xi||\bar{\alpha})$ is a segment of N_ξ.*

Proof of ($*$). By Lemma 2.6, it suffices to show that there is no $\mu \leq \mathrm{ht}(M)$ with $M||\mu$ active and $s^+(M_\xi||\mu) \leq \omega\bar{\alpha} < \mu$. Assuming this is not the case let μ have this property. Since $\omega\rho^\omega_{M_\xi||\bar{\alpha}} \leq \kappa_\zeta$,

$$\mathrm{Card}_{M_\xi||\mu} \cap (\kappa_\zeta, \omega\bar{\alpha}] = \emptyset.$$

As $s^+(M_\xi||\mu) \leq \omega\bar{\alpha}$ it even follows that $s^+(M_\xi||\mu) \leq \kappa_\zeta < s_\xi^+$. Moreover, it was already shown that $\alpha > \nu_\xi$. Hence, $\mu > \alpha > \nu_\xi$. Altogether,

$$s^+(M_\xi||\mu) < s^+(M_\xi||\nu_\xi) \leq \nu_\xi < \mu,$$

which contradicts the applicability of ν_ξ in M_ξ. $\square_{(*)}$

So let $N_\xi||\alpha' = \mathtt{S}(M_\xi||\bar{\alpha})$. Then

$$\omega\rho^\omega_{N_\xi||\alpha'} = \omega\rho^\omega_{M_\xi||\bar{\alpha}} \leq \kappa_\zeta.$$

But obviously, $\tau_\zeta < \alpha' < \eta_\zeta$, which contradicts the fact that $(\kappa_\zeta^+)^{N_\xi||\eta_\zeta} = \tau_\zeta$. $\square_{(a)}$

(b) *η'_ζ is maximal with the property in (a).*

Proof of (b). If $\eta_\zeta = \mathrm{ht}(N_\xi)$, then $\eta'_\zeta = \mathrm{ht}(M_\xi)$, and there is nothing to prove. Otherwise, $\omega\rho^\omega_{M_\xi||\eta'_\zeta} = \omega\rho^\omega_{N_\xi||\eta_\zeta} \leq \kappa_\zeta$, since $M_\xi||\eta'_\zeta = \Lambda(N_\xi||\eta_\zeta)$. Hence $(\kappa^+_\zeta)^{M_\xi||\eta'_\zeta+1} \geq \eta'_\zeta \geq \eta_\zeta > \tau_\zeta$. So no $\gamma > \eta'_\zeta$ can possibly have the property in (a). $\square_{(b),(3)}$

One thing that didn't need a proof in the proof of the other direction is that ν_ζ is applicable in M_ζ. But that's easy to see: As s^+_ζ indexes an extender in $N_\zeta = \mathtt{S}(M_\zeta)$, $\mathtt{S}(M_\zeta||\nu_\zeta) = N_\zeta||s^+_\zeta$, which is a segment of N_ζ. Now Lemma 2.6 yields that ν_ζ is applicable.

The second change needed occurs here:

Case 2.2: $\bar\gamma$ is a limit ordinal.

Assume $\Lambda(\mathcal{I}|\bar\gamma)$ has been defined already, and let $\sigma_{i,j} \subseteq \pi_{i,j}$ for all $i \leq_T j < \bar\gamma$.

Set $M' := \Lambda(N_{\bar\gamma})$. Define for $i <_T \bar\gamma$ (again assuming no truncations occur in the branch between i and $\bar\gamma$) a map $\pi_{i,\bar\gamma} : M_i \longrightarrow M_{\bar\gamma}$ by:

$$\pi_{i,\bar\gamma}(h^1_{M_i}(j,q)) := h^1_{M'}(j,\sigma_{i,\bar\gamma}(q)),$$

where $j < \omega$ and $q \in [\mathrm{ht}(N_i)]^{<\omega}$. I again make use of the fact that $|M_i| = h^1_{M_i}(\mathrm{ht}(\mathtt{S}(M_i)))$. The correctness follows by the usual argument: It can be expressed uniformly by a Σ_1-formula that $h^1_{M_i}(j,q) = h^1_{M_i}(k,r)$, which is transported by f_{N_i} to N_i. The transformed formula then holds, modulated by $\sigma_{i,\bar\gamma}$, in $N_{\bar\gamma}$. So since $f_{N_i} = f_{N_{\bar\gamma}}$, the original formula holds in M', which by the uniformity means that $h^1_{M'}(j,\sigma_{i,\bar\gamma}(q)) = h^1_{M'}(k,\sigma_{i,\bar\gamma}(r))$.

It is easy to see that the so-defined functions verify the existence of $\Lambda(\mathcal{I}|\gamma)$ (setting $M_{\bar\gamma} := M'$, of course). For example, to show that every $a \in |M'|$ is in the range of $\pi_{i,\bar\gamma}$, let $a = h^1_{M'}(j,q')$, for some $j < \omega$, $q \in [\mathrm{ht}(N_{\bar\gamma})]^{<\omega}$. Then there exist $i <_T \bar\gamma$ and $q \in [\mathrm{ht}(N_i)]^{<\omega}$ so that $q' = \sigma_{i,\bar\gamma}(q)$. Let $\bar a := h^1_{M_i}(j,q)$. Then $a = \pi_{i,\bar\gamma}(\bar a)$.

This concludes the treatment of case 2.2.

The other parts of the proof work as before. $\square$

6.1 Translating strategies

Having developed the method of transliterations, the following lemma is straightforward.

Lemma 6.5. *Let M be a normally iterable pPλ-structure. Let $\mathcal{S}$ be a normal iteration strategy for M. Then there is a normal iteration strategy $\mathtt{S}(\mathcal{S})$ for $\mathtt{S}(M)$.*

The corresponding statement holds true of normally iterable pPs-structures as well.

Proof. I define

$$\mathtt{S}(\mathcal{S})(\mathcal{I}) \cong \mathcal{S}(\Lambda(\mathcal{I})).$$

It follows by induction on the length of iterations $\mathcal{I}$ of $\mathtt{S}(M)$ of limit length that for every iteration $\mathcal{I}$ of N which is according to $\mathtt{S}(\mathcal{S})$, $\Lambda(\mathcal{I})$ is according to $\mathcal{S}$, and that $\mathcal{S}(\Lambda(\mathcal{I}))$ is defined. The converse is shown in the same way. $\square$

The following is the main theorem of this paper:

Theorem 6.6. *The restriction of $\mathtt{S}$ to the class of normally s-iterable λ-structures is a bijection between this class and the class of normally iterable s-structures. The restriction of Λ to the latter class is the inverse of this bijection.*

The corresponding statement is true for the class of normally iterable pPλ, pλ and Pλ-structures.

Proof. This follows from Lemma 6.5, together with Theorem 2.1. $\square$

7 Further results

7.1 Iterable Mitchell-Steel premice

At this point, I am returning to an issue I raised in the introduction to the first part of this paper. From the very beginning, I tried to keep the definitions of the structures used as liberal as possible. It was necessary, though to demand continuability of the pPs-structures, in order to insure that they will have a counterpart with λ-indexing. I am going to prove presently that normally iterable Mitchell-Steel-premice, as introduced in [Ste00] are normally iterable s-structures. This shows that continuability is not restrictive in the realm of Mitchell-Steel mice. I am not going to use the notion of k-iterations here, but instead use the $*$-fine structure theory to form $*$-iterations, like elsewhere in the present paper. The Mitchell-Steel premice are precisely the weak s-structures defined in the following definition.

Definition 7.1. N ia a *weak ps-structure* iff N has all the properties of a ps-structure, with the exception of hereditary continuability, that is the continuability of all active segments, including the whole structure, if active, and N is modest (meaning that for every $\alpha \leq \mathrm{ht}(N)$ that indexes an extender, $s(N||\alpha) < \lambda(N||\alpha)$).

Analogously, N is a weak s-structure iff N is modest and has all properties of an s-structure except hereditary continuability.

Lemma 7.2. *Let N be a weak (p)s-structure which is normally iterable. Then it is a normally iterable (p)s-structure.*

Proof. I will assume familiarity with iterations of Mitchell-Steel-premice, as described in [Ste00] or [MS94]. Instead of k-extender ultrapowers, I will use $*$-ultrapowers, though.

Obviously, N it suffices to show that N is normally iterable, as a (p)s-structure (this implies immediately that N is hereditarily continuable, and Lemma 5.20 yields that N satisfies the s'-ISC). In order to see this, it has to be shown that there is a successful normal iteration strategy for N in that sense. The point is that if a normal iteration of N is continued, the new structures have to be hereditarily continuable. So the proof is complete if the following can be shown:

If $\mathcal{S}$ is a successful normal iteration strategy for N as a weak (p)s-structure (i.e., as a Mitchell-Steel premouse), then all the models in any normal iteration of N which is according to $\mathcal{S}$ are hereditariliy continuable weak (p)s-structures (and hence (p)s-structures). This shows then that $\mathcal{S}$ is also a successful normal iteration strategy for N, viewed as a (p)s-structure.

Assume the contrary. Let $\mathcal{I} = \langle\langle N_i \mid i < \theta\rangle, D, \langle\eta_i \mid i \in D\rangle, \langle\kappa_i \mid i \in D\rangle, \langle\tau_i \mid i \in D\rangle, \langle s_i \mid i \in D\rangle, \langle s_i^+ \mid i \in D\rangle, T, \langle\pi_{i,j} \mid i \leq_T j < \theta\rangle\rangle$ be a normal iteration of N according to $\mathcal{S}$ which is a counterexample to the claim of minimal length θ. Obviously then θ is a successor ordinal, since otherwise, every N_i is hereditarily continuable ($i < \theta$): N_i appears in $\mathcal{I}|(i+1)$, a normal iteration of N which has length less than θ and is according to $\mathcal{S}$.

So let $\theta = \gamma + 1$. Then the previous argument shows that N_γ is not hereditarily continuable, while N_i is, whenever $i < \gamma$. So let α be an extender index of N_γ witnessing this, so that $N_\gamma||\alpha$ is not continuable. I show:

($*$) *There is a $j \in D$ with $s_j^+ \geq \alpha$.*

Proof of ($*$). Assuming the contrary, it would be possible to continue the iteration by letting $s_\gamma^+ := \alpha$, $\kappa_\gamma := \mathrm{crit}(E_\alpha^{N_\gamma})$, so that at stage γ, the extender $E_\alpha^{N_\gamma}$ is used. This would be possible since by assumption, $\alpha > s_j^+$, for every $j \in D$. [13] So let η_γ, $\xi := T'(\gamma+1)$ be defined like they have to be defined in order to produce a normal iteration $\mathcal{I}'$ with iteration tree T' that continues $\mathcal{I}$ as described. Since $\mathcal{I}$ was formed according to $\mathcal{S}$, this is possible, and the result is that

$$\sigma_{\xi,\gamma+1} : \mathcal{C}_0(N_\xi||\eta_\gamma)^{\mathrm{sq}} \longrightarrow^*_{E_\alpha^{N_\gamma}} \mathcal{C}_0(N_{\gamma+1})^{\mathrm{sq}}$$

exists. By definition of η_γ, and due to the coherency of normal iterations of Mitchell-Steel-structures it follows that $\tau_\gamma = (\kappa_\xi^+)^{N_\xi||\eta_\gamma} \leq s_\xi^+$, and $\mathrm{J}_{s_\xi^+}^{E^{N_\xi}} = \mathrm{J}_{s_\xi^+}^{E^{N_\gamma}}$, so that $s_\xi^+ \leq \eta_\gamma$, and hence $|\mathrm{J}_{\tau_\gamma}^{E^{N_\gamma}}| \subseteq |N_\xi||\eta_\gamma|$. So there is an embedding $k : I\!\!D(\mathrm{J}_{\tau_\gamma}^{E^{N_\gamma}}, E_{s_\gamma^+}^{N_\gamma}) \longrightarrow N_{\gamma+1}$ defined by

$$k([\vec{\alpha}, f]) := \sigma_{\xi,\gamma+1}(f)(\vec{\alpha}).$$

But this shows that $I\!\!D(\mathrm{J}_{\tau_\gamma}^{E^{N_\gamma}}, E_{s_\gamma^+}^{N_\gamma})$ is well-founded, so that $N_\gamma||\alpha$ is continuable, contradicting the assumption. $\square_{(*)}$

Now let $j \in D$ be least with $s_j^+ \geq \alpha$. By the strong coherency properties of normal iterations of Mitchell-Steel premice, it follows that

$$\mathrm{J}_{s_j^+}^{E^{N_j}} = \mathrm{J}_{s_j^+}^{E^{N_\gamma}}, \text{ and } E_{s_j^+}^{N_\gamma} = \emptyset.$$

As α indexes an extender in N_γ, it follows that $s_j^+ > \alpha$. So $N_j||\alpha = N_\gamma||\alpha$. But N_j is hereditarily continuable, as $j < \gamma$, and hence $N_j||\alpha$ is continuable, so also $N_\gamma||\alpha$, a contradiction. This proves the claim, and hence the lemma. $\square$

7.2 Other notions of iterability

There are diverse notions of iterability. For example, one can restrict the length of the iterations that can be formed. This yields the notion of θ-iterability of [Ste96, Def. 2.9]). The methods of the previous section show that that the translation functions $\mathtt{S}$ and Λ provide a correspondence between (ω_1+1)-iterable structures, for example.

It is also true that an (ω_1+1)-iterable weak (p)s-structure is an (ω_1+1)-iterable (p)s-structure: Let $\mathfrak{S}$ be an ω_1+1-iteration strategy for the weak (p)s-structure N. It follows from the proof of Lemma 7.2 that all models occurring in an iteration of length less than ω_1 are hereditarily continuable. The only new point is that the directed limit model along the cofinal branch through an iteration of length ω_1 determined by the strategy is also hereditarily continuable. It is clear that the iteration of length ω_1 can be translated to an iteration of $\Lambda(N)$ which has the same length, and that the limit along the main branch of the translated iteration is well-founded, just because ω_1 has uncountable cofinality. A decreasing epsilon-chain in the limit model would already yield such a chain in a previous model. The limit model of the translated iteration will be the Λ-image of the limit model of the original iteration (see case 2.2 of the proof of Lemma 6.3), and the existence of this image shows in particular that the pre-image is hereditarily continuable.

Another frequently useful variant is the one that's just referred to as iterability in [Jen97, §4, S. 26]. It postulates the existence of a *good iteration strategy*. In essence, a good iteration is just a composition of normal iterations, so that the base model of each component iteration is a segment of the target model of the previous component iteration. As usual, direct limits are

[13] Note that in [MS94] and [Ste00], it is demanded that the sequence of extender indices in a normal iteration is strictly increasing. This works, since the structures considered in these works only contain extenders whose s-index is less than their λ-index. If this assumption is dropped, then one has to deal with normal iterations in which the s-indices are not strictly increasing, otherwise one cannot show that coiterations terminate. But the structure at hand is a Mitchell-Steel-pm, so that the sequence of extender indices is strictly increasing.

formed at limit stages. This notion of iteration is used to prove the Dodd-Jensen-Lemma. I'll call the corresponding notion of iterability *good iterability.*

One can formulate an iterability notion for (p)s-structures in precisely the same way. In order to arrive at the corresponding notion for (p)λ-structures, an additional requirement is needed, though: If at the beginning of one of the component iteration, some model M_i is truncated, say to η, then there may be no $\nu \le \mathrm{ht}(M_i)$ such that $s^+(\nu)^{M_i} \le \eta \le \nu$ (this is relevant because of Lemma 2.6). The methods developed so far show that the translation functions constitute a precise correspondence between this notion of s-iterability for λ-structures and good iterability for s-structures. The corresponding results obviously hold true for pPs-, pPλ and the corresponding weak pPs-structures.

7.3 On the *squashed* (Pseudo)-Σ_0-codes

Recall the types of Σ_0-codes introduced in the first part of this paper, first for (pP)s-structures:

Definition 7.3. Let $\mathcal{L}$ be the language of set theory with additional symbols $\dot{E}$, $\dot{F}$, $\dot{\kappa}$ and $\dot{s}$. Let $N = \langle \mathrm{J}^E_\alpha, F\rangle$ be a pPs-structure. Then its *Pseudo-*Σ_0*-code,* $\tilde{\mathcal{C}}_0(N)$, is an $\mathcal{L}$-structure, which is defined as follows:

1. If N is passive, then $\tilde{\mathcal{C}}_0(N)$ has the universe $|\mathrm{J}^E_\alpha|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \dot{s}^{\tilde{\mathcal{C}}_0(N)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E\restriction\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = \emptyset$.
2. If N is active of type I or II, then $\tilde{\mathcal{C}}_0(N)$ has the universe $|\mathrm{J}^E_\alpha|$ again, but in that case, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = F$.
3. If N is active of type III, then the universe of $\tilde{\mathcal{C}}_0(N)$ is $|\widehat{N}|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E^{\widehat{N}}\restriction\mathrm{ht}(\widehat{N})$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = E^{\widehat{N}}_{\mathrm{top}}$.

In addition, the *squashed-Pseudo-*Σ_0*-code*, $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ of N, is defined as follows: If N is passive or active of type I or II, then $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}} = \tilde{\mathcal{C}}_0(N)$. If, on the other hand, N is active of type III, then let $s = s(F)$. The universe of $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ is then $|\mathrm{J}^E_s|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = E\restriction s$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = F^{\mathfrak{h}}\restriction s = \{\langle\alpha, X\rangle \mid \alpha \in (F^{\mathfrak{f}}(X)) \cap s\}$.

Analogously, the code $\tilde{\mathcal{C}}_0(\widehat{N})$ is defined as follows.

1. If N is passive, then $\tilde{\mathcal{C}}_0(\widehat{N}) = \tilde{\mathcal{C}}_0(N)$.
2. If N is active of type I or II, then $\tilde{\mathcal{C}}_0(\widehat{N})$ has universe $|\widehat{N}|$, and I set: $\dot{\kappa}^{\tilde{\mathcal{C}}_0(\widehat{N})} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(\widehat{N})} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(\widehat{N})} = E^{\widehat{N}}\restriction\mathrm{ht}(\widehat{N})$ and $\dot{F}^{\tilde{\mathcal{C}}_0(\widehat{N})} = E^{\widehat{N}_{\mathrm{top}}}$.
3. If N is active of type III, then $\tilde{\mathcal{C}}_0(\widehat{N}) = \tilde{\mathcal{C}}_0(N)$.

The corresponding codes for (pP)λ-structures were defined as follows:

Definition 7.4. Let $\tilde{\mathcal{L}}$ be the language of set theory with additional symbols $\dot{D}$, $\dot{E}$, $\dot{F}$, $\dot{\kappa}$ and $\dot{s}$. Let $M = \langle \mathrm{J}^E_\alpha, F, D_M\rangle$ be a pPλ-structure. Then its *Pseudo-*Σ_0*-code,* $\tilde{\mathcal{C}}_0(M)$ is the $\tilde{\mathcal{L}}$-structure defined as follows. The universe of $\tilde{\mathcal{C}}_0(M)$ is $|\tilde{\mathcal{C}}_0(M)| = |\mathrm{J}^E_\alpha|$ and $\dot{D}^{\tilde{\mathcal{C}}_0(M)} = D_M$, and

1. If M is passive, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \dot{s}^{\tilde{\mathcal{C}}_0(M)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = \emptyset$.
2. If M is active of type I or II, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(M)} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = F$.

3. If M is active of type III, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(M)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = F$.

I am going to deal with the squashed codes in this section for the first (and last) time, showing that $(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$ and $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ are "fine structurally equivalent" in case N is a ps-structure of type III. I'll also show that the notion of normal iteration of ps-structures is equivalent to the one used in [MS94], if the structures are hereditarily continuable and $*$-ultrapowers are used.

Lemma 7.5. *Let N be an active ps-structure of type III. Let $s = s(E^N_{\mathrm{top}})$. Then $s = \omega\rho^1_{\tilde{\mathcal{C}}_0(N)}$, and $p_{\tilde{\mathcal{C}}_0(N),1} = \langle\emptyset\rangle$.*

Proof. This is shown like Lemma 5.32; it's obvious that one can define a Σ_1-surjection from s onto $|\widehat{N}|$ in $\widehat{N}$, using the top extender predicate. $\square$

Lemma 7.6. *Let N be an active ps-structure of type III. Then the structures $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ and $\mathcal{C}_0(N)^{\mathrm{sq}}$ are amenable.*

Proof. Let $s = s(E^N_{\mathrm{top}})$. Since $|\widehat{N}| = |\Lambda(N)|$, it follows from Lemma 5.22 that for each $\alpha < s$, $(E^{\widehat{N}}_{\mathrm{top}}|\alpha) \in |\widehat{N}|$. It has to be shown that for such α,

$$|\mathrm{J}^{E^N}_\alpha| \cap \dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} \in |\mathrm{J}^{E^N}_s|,$$

where it may be assumed that $\alpha \geq \tau(N)$. Let $\tilde{F} = \dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}$. Then $\tilde{F} = \{\langle\gamma, X\rangle \mid s > \gamma \in E^{\widehat{N}}_{\mathrm{top}}(X)\}$. Hence,

$$\bar{F} := |\mathrm{J}^{E^N}_\alpha| \cap \tilde{F} = \{\langle\gamma, X\rangle \mid \widehat{N} \models \gamma < \omega\alpha \wedge \exists Y \quad (\dot{F}(Y, X) \wedge \gamma \in Y)\}.$$

So this is a $\mathbf{\Sigma}_1(\widehat{N})$-subset of $|\mathrm{J}^{E^{\widehat{N}}}_\alpha|$, and $\alpha < s = \omega\rho^1_{\widehat{N}}$ by Lemma 7.5. Hence, $\bar{F} \in |\widehat{N}|$, and thus by acceptability of $\widehat{N}$, it follows that $\bar{F} \in |\widehat{N}||(\alpha^+)^{\widehat{N}}|$. But $(\alpha^+)^{\widehat{N}} \leq \omega\rho^1_{\widehat{N}} = s$, hence $\bar{F} \in |\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}|$, as claimed. $\square$

Lemma 7.7. *Let N be an active ps-structure of type III. Then $Z := |\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}| = |\mathcal{C}_0(N)^{\mathrm{sq}}| = |\tilde{\mathcal{C}}_0(N)^{1,\emptyset}|$, and the following assertions hold:*

(a) for $q \in Z$, a set A is $\Sigma_1(\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}})$ in q, iff A is $\Sigma_1(\tilde{\mathcal{C}}_0(N)^{1,\emptyset})$ in q.

(b) for $q \in Z$, a set A is $\Sigma_1(\mathcal{C}_0(N)^{\mathrm{sq}})$ in q, iff A is $\Sigma_1(\mathcal{C}_0(N)^{1,\emptyset})$ in q.

Proof. Note that $\tilde{\mathcal{C}}_0(N)$ is essentially the same as $\widehat{N}$, as N is of type III – the only additional constant that's not interpreted as 0 in $\tilde{\mathcal{C}}_0(N)$ is $\dot{\kappa}$, and that is easily definable in $\widehat{N}$ as well.

It suffices obviously to prove the claims concerning $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, since $\mathcal{C}_0(N)^{\mathrm{sq}}$ differs from $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ only by the additionally available constant $\dot{q}^{\mathcal{C}_0(N)^{\mathrm{sq}}} = \emptyset$, which is irrelevant.

It follows immediately from Lemma 7.5 that $|\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}| = |\mathcal{C}_0(N)^{\mathrm{sq}}| = |\tilde{\mathcal{C}}_0(N)^{1,\emptyset}|$. Moreover, we know that $s := s(E^N_{\mathrm{top}}) = \omega\rho^1_{\tilde{\mathcal{C}}_0(N)}$. Set: $\bar{N} := (\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$.

Two directions have to be shown. I'll deal with the easier one first, the direction from left to right. Essentially, the proof reduces to expressing the predicate $\tilde{F} := \dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}$ over $\bar{N}$ by a Σ_0-formula. We have for $\alpha < s$:

$$\begin{aligned} \tilde{F}(\alpha, X) \quad &\iff \quad \tilde{\mathcal{C}}_0(N) \models \underbrace{\exists Y \quad (\dot{F}(Y, X) \wedge \alpha \in Y)}_{\varphi_i[\langle\alpha, X\rangle]} \\ &\iff \quad \bar{N} \models \dot{A}(i, \langle\alpha, X\rangle). \end{aligned}$$

So if a set A is defined in $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ by a Σ_1-formula $\varphi(x,y)$ in the parameter q, then one just has to replace every occurrence of $\dot{F}(v,w)$ in the formula by "$\dot{A}(i,\langle v,w\rangle)$", in order to arrive at a formula which defines the same set in $\bar{N}$.

A little more work is needed for the converse. I'll translate the formula in two steps.

Step 1: Let $e : s \longrightarrow |\tilde{N}|$ be the monotone enumeration of $|\tilde{N}|$ according to the canonical well-ordering of $\tilde{N}$. Note that $\mathrm{J}_s^{E^N}$ is a ZF^--model, and hence the order type of this well-ordering is s. Moreover, e is uniformly Σ_1. Let $\bar{A} = \{\langle i,\xi\rangle \mid \xi < s \wedge A^{1,\emptyset}_{\tilde{\mathcal{C}}_0(N)}(i,e(\xi))\}$. Set:

$$\tilde{N} := \langle \mathrm{J}_s^{E^{\widehat{N}}}, \bar{A}\rangle.$$

I'll show that every set that is Σ_1-definable in $\bar{N}$ using parameters is also Σ_1-definable in $\tilde{N}$, using the same parameters, and vice versa. Let $\varphi(\vec{x})$ be a Σ_1-formula. I'll first define another Σ_1-formula $\varphi^*(\vec{x})$ such that $\bar{N} \models \varphi(\vec{b})$ iff $\tilde{N} \models \varphi^*(\vec{b})$. Here is the deduction of the definition of φ^*, as well as the proof that φ^* behaves as desired:

$$\begin{aligned}
& \bar{N} \models \varphi(\vec{b}) \\
\iff\quad & \exists\gamma < s \quad (\vec{b} \in \bar{N}|\gamma \wedge \bar{N}|\gamma \models \varphi(e(\vec{\xi})) \\
\iff\quad & \tilde{N} \models \big(\exists e'\exists u\exists a\exists\gamma \quad (\text{“}\vec{b} \in u = \mathrm{J}^E_\gamma\text{”} \wedge \\
& \qquad\qquad \text{“}(e' \subseteq e) \wedge (u \subseteq \mathrm{ran}(e'))\text{”} \wedge \\
& \qquad\qquad \text{“}a = u \cap A^{1,\emptyset}_{\tilde{\mathcal{C}}_0(N)}\text{”} \wedge \\
& \qquad\qquad \varphi_{\langle u,\in,E\cap u,a\cap u\rangle}(\vec{b}))\big) \\
\iff\quad & \tilde{N} \models \varphi^*(\vec{b}).
\end{aligned}$$

Here, "$(e' \subseteq e) \wedge (u \subseteq \mathrm{ran}(e'))$" expresses that e' is a function that satisfies the uniform Σ_1-definition of e on its domain, and is large enough that $u \subseteq \mathrm{ran}(e')$. "$a = u \cap A^{1,\emptyset}_{\tilde{\mathcal{C}}_0(N)}$" expresses that $a = \{\langle i,x\rangle \mid i,x \in u \wedge \bar{A}(i,e'^{-1}(x))\}$, which is easy to express explicitly in $\tilde{N}$, where $\bar{A}$ is available as a predicate.

Step 2: Now I want to show that every set that's Σ_1-definable in $\tilde{N}$ using a parameter, is also Σ_1-definable in $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, using the same parameter. Again, the main problem is expressing the predicate $\bar{A}$ over $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$:

$$\begin{aligned}
\bar{A}(i,\xi) \quad\iff\quad & \xi < s \wedge \tilde{\mathcal{C}}_0(N) \models \varphi_i(e(\xi),\emptyset) \\
\iff\quad & \tilde{\mathcal{C}}_0(N)^{\mathrm{sq}} \models \xi \in \mathrm{On} \wedge \\
& \qquad \wedge \dot{F}(\xi,\{\zeta < \kappa(N) \mid N||\tau(N) \models \varphi_i[e^{N||\tau(N)}(\xi),\emptyset]\}).
\end{aligned}$$

This is obviously equivalent to a Σ_0-formula in the parameter $\mathrm{J}^{E^N}_{\tau(N)+1}$, for $e^{N||\tau(N)}$ is an element of this structure, and it is definable over $N\tau(N)$. By replacing each occurrence of $\dot{A}(i,a)$ with the above formula in a Σ_1-formula which defines some set B in a parameter q over $\tilde{N}$, one produces a Σ_1-formula φ' defining the same set over $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ in q and the parameter $\mathrm{J}^{E^N}_{\tau(N)+1}$. So the latter parameter has to be eliminated. Obviously, it suffices to check that $\{\tau(N)\}$ is lightface Σ_1-definable in $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$.

Let $\pi : \mathrm{J}^{E^N}_\tau \longrightarrow_{E^N_{\mathrm{top}}} \widehat{N}^{\mathrm{passive}}$. Let $f : \kappa \longrightarrow \kappa$ be defined by: $f(\alpha) = (\alpha^+)^{\mathrm{J}^{E^N}_\tau} = (\alpha^+)^{\mathrm{J}^{E^N}_{\kappa(N)}}$ (this last identity makes use of the acceptability of N; $\kappa(N)$ is a cardinal in N). Then we have: $\tau(N) = \pi(f)(\kappa) = \pi(\mathrm{id})(\tau)$, as is easily seen. Hence, by a Łoś theorem, $\tau(N)$ is the unique ξ with:

$$\dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}(\prec\kappa,\xi\succ,\{\prec\alpha,\beta\succ < \kappa(N) \mid (\alpha^+)^{\mathrm{J}^{E^N}_{\kappa(N)}} = \beta\}).$$

As $\kappa(N)$ is available in $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ as a constant (which isn't crucial here, as $\kappa(N)$ can be defined to be the unique ζ with $\dot{F}^{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}(\zeta,\zeta)$), this can obviously be expressed by a Σ_1-formula over $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ without parameters. It was such a description of $\tau(N)$ that we were looking for. $\square$

By induction on n, this implies:

Lemma 7.8. *Let N be an active ps-structure of type III, $n<\omega$. Then $\omega\rho^n_{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = \omega\rho^{n+1}_{\tilde{\mathcal{C}}_0(N)} = \omega\rho^n_{(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}}$, $Z := |\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}| = |(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}|$, and for $q\in Z$ we have:*

(a) A is $\Sigma_1^{(n)}(\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}})$ in q, iff A is $\Sigma_1^{(n)}(\tilde{\mathcal{C}}_0(N)^{1,\emptyset})$ in q.

(b) A is $\Sigma_1(\mathcal{C}_0(N)^{\mathrm{sq}})$ in q, iff A is $\Sigma_1(\mathcal{C}_0(N)^{1,\emptyset})$ in q.

In particular, $p_{\tilde{\mathcal{C}}_0(N)^{1,\emptyset}} = p_{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}$, $q^n_{\tilde{\mathcal{C}}_0(N)^{1,\emptyset}} = q^n_{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}}$, and $\tilde{\mathcal{C}}_0(N)^{1,\emptyset}$ is sound iff $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ is sound, etc. In short: $\tilde{\mathcal{C}}_0(N)^{1,\emptyset}$ and $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ are fine structurally equivalent.

Proof. It was shown in Lemma 7.7 already that $Z := |\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}| = |(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}|$. It also follows from that lemma that $\omega\rho^1_{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = \omega\rho^1_{(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}}$, and in particular that $H^1_{\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}} = H^1_{(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}}$. Moreover, it shows that the assumptions of Lemma 6.1 are satisfied by $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ and $(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$. It follows that all projecta of these structures coincide.

Moreover, Lemma 7.7 yields equivalents to Lemma 3.1, where no additional parameters are needed, and M is replaced with $(\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$, $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, and N is replaced with $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, $\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$, respectively. Correspondingly, one gets analogs of the succeeding Lemma 3.2, where again, M is replaced with $\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$, $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, and N is replaced with $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, $\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$, respectively, and no additional parameters are needed. This obviously shows (a) and (b).

It follows from (a) that $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ and $\tilde{\mathcal{C}}_0(N))^{1,\emptyset}$ are fine structurally equivalent. The analogous statement is obviously also true of $\mathcal{C}_0(N)^{\mathrm{sq}}$ and $\tilde{\mathcal{C}}_0(N)^{1,\emptyset}$. $\square$

This gives:

Lemma 7.9. *Let N be a ps-structure of type III and F an extender on N with critical point less than $s(N)$. Let $\bar{\pi} : \tilde{\mathcal{C}}_0(N)^{\mathrm{sq}} \longrightarrow^*_F \bar{N}$. Let $\pi : \widehat{N} \longrightarrow^*_F \widehat{N'}$ (assuming N' is well-founded). Then $\bar{N} = \tilde{\mathcal{C}}_0(N')^{\mathrm{sq}}$.*

Proof. It was already shown that $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ is fine structurally equivalent to $\widehat{N}^{1,\emptyset}$ (save the additional constant $\kappa(N)$ that's available in $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$ – but this has no influence on the class of functions used when forming the fine structural extender ultrapower). Moreover, $\emptyset \in R^1_{\widehat{N}}$. But it is well-known that in this case, forming the $*$-extender ultrapower of a structure is the same as forming the $*$-extender ultrapower of the reduct and then lifting the embedding the the full structure, using the upwards extension of embeddings lemma (see [Zem02, p. 13]). $\square$

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