PARTITION BIJECTIONS, A SURVEY

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ABSTRACT. We present an extensive survey of bijective proofs of classical partitions identities. While most bijections are known, they are often presented in a different, sometimes unrecognizable way. Various extensions and generalizations are added in the form of exercises.

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1. Introduction

Constructive Partition Theory is a rich subject, with many classical and important results which influenced the development of Enumerative Combinatorics in the twentieth century. It is also a collection of various terminologies, notation and techniques, with a number of results rediscovered on many occasions, and some fundamental bijections remain in obscurity. This survey is an attempt to present the subject in a coherent way.

First, let us outline the framework of what we do in the survey. Our goal is to give direct combinatorial (bijective or involutive) proofs of partition identities, and occasionally some applications. To start, we translate the identities into equalities between the numbers of two types of partitions. In most cases, we represent these partitions graphically, by means of Young diagrams, and then use various combinatorial tools to transform one of the classes of partitions into the other. Although this approach appears to be simple, this is rather misleading as the resulting partition identities are often very powerful and at times difficult to prove by other means. As the reader will see, finding bijective proofs requires a great deal of ingenuity, but once found they are often not difficult to understand.

Historically, most partition identities were first proved analytically, and only much later combinatorially. The subject of this survey is so much intertwined with the subjects of partition identities, hypergeometric series, and q-series in general, that it is difficult to give an adequate historical presentation of one without the other. Nevertheless we cannot refrain from making a brief historical overview of the 120 years of effort of giving bijective proofs of partition identities. We emphasize the combinatorial part of the story and leave aside the meaning and history of partition identities (see section 10 for the references).

The Theory of Partitions as a subject started with Euler's celebrated treatise [66], where Chapter 16 introduced integer partitions as we know them. Back in 1748, Euler proved a variety of partition identities, most notably the Pentagonal Theorem and the "partitions of n into odd parts vs. partitions of n into distinct parts" theorem. In the next 250 years a great number of partition identities were proved, including those bearing the names of Gauss, Cauchy, Jacobi, Weierstrass, Sylvester, Heine, Lebesgue, Schur, MacMahon and Ramanujan.

As a research area, Partition Theory had trouble fitting in with other fields, perhaps due to its multidisciplinary nature. It originated as a part of the Analysis, but then quickly became a part of the Number Theory, when numerous applications has emerged. Older textbooks, such as [82], traditionally had at least one section devoted to Partition Theory. Later, Partition Theory was considered as a part of Combinatorial Analysis, a subject which evolved into modern day Combinatorics and Discrete Mathematics. Most recently, it seems, the subject gained the rights of its own.

The method of proving partition identities "constructively" was pioneered by Sylvester in [121]. To be fair, the monumental paper [121] with a playful title is a long survey of results of Sylvester himself, as well as his students and collaborators. The method was largely accepted after Franklin's involution was published [72]. Franklin was a student of Sylvester at Johns Hopkins University and wrote a few sections of [121]. Almost immediately an unexpected benefit of having combinatorial proofs was discovered by Cayley: He noted in a letter to Franklin (published in [53]) that Franklin's involution *"gives more,"* meaning it preserves a certain statistic on partitions, and thus proves a more general partition result.

With the two notable exceptions of Schur and MacMahon, few people worked in the field until the mid–1960s. Freeman Dyson [60] had to fight the following attitude at the time:

> *Professor Littlewood, when he makes use of an algebraic identity, always saves himself trouble of proving it; he maintains that an identity, if true, can be verified in a few lines by anybody obtuse enough to feel the need for verification.*

In about 1965 the "golden age" had begun. In a short span of less than 20 years many different people proved a large number of partition identities by combinatorial methods, giving an impression that one should expect "constructive" proofs of most if not all partition identities. This was the time when George Andrews arrived on the scene and played an important role in these developments. In his two fundamental papers [12, 13], which are somewhat forgotten now, he built a basis for both now standard techniques by which partition bijections are obtained. Thus in the late 1970s one had an impression that a unifying theory was in sight. Depending on one's point of view, the birth of the involution principle either confirmed or destroyed these hopes. The "golden age" was over.

In essence, Garsia and Milne showed that one can "mechanically" construct bijections out of existing bijections and involutions. Typically, these bijections turned out to be indirect and quite complicated. They introduced the involution principle, giving a long-awaited bijective proof of the Rogers–Ramanujan identities by combining the already-known Vahlen's involution, Sylvester's bijection and Schur's involution [76]. This approach was further extended in subsequent publications to give bijective proofs of many partition identities (see e.g. [48, 91, 109]).

Despite the clamorous claims of success of the "involution principle technology" (see [91]), from a traditional combinatorial of view this approach is unsatisfactory. First, the resulting bijective proofs are not simpler than the analytic proofs; in a sense they are not even new at all. The involution principle bijections are too complicated to follow and do not seem to produce new refinements of known partition identities. In fact, even the complexity of the Garsia–Milne bijection remains open. To quote Joichi and Stanton, *"The emphasis now should be placed on combinatorially important proofs rather than just a proof"* [91].

While it has been over twenty years since the Garsia–Milne paper [76], the state of "constructive partition theory" remains confused and the years uneventful. Few genuine new bijections have been discovered, as the importance of combinatorial proofs seem to have plunged once again in the anonymous "public opinion". We hope this survey will help to reverse this course.

Let us briefly summarize the state of art from a nontechnical point of view.[1] The way we see it, finding direct bijective or involutive proofs or most identities is an unfeasible task. Right now very few partition identities have such proofs and it seems there is little reason why the remaining identities should be so fortunate to have them, especially after resisting a combinatorial proof for so many years. Of course, just like bijections, partition identities are not created equal and one can make a plausible case that only the "important" partition identities should have a

[1]We suggest the diligent reader at this point first review "The language" section of the introduction (see below).

combinatorial proof. Few open problems we include in this survey present the first challenge to this thesis.

Right now the number of different combinatorial proofs (of "important" partition identities) remains rather small, many of them covered in this survey. The corresponding identities are often classical and their study is easy to justify. Unfortunately, many recent bijections give the same correspondences as the old ones, with the authors often aware of this. While new bijections are often easier to present than their older counterparts, there is a limit to this, and after a certain point little room is left for improvement. On the other hand, there is an appalling absence of new original ideas in bijection constructions. It is clear that large classes of partitions identities, such as Macdonald's eta-function identities in their full generality (see [101, 116]), require a new combinatorial approach.[2]

Another venue awaiting the exploration is our current lack of understanding where the "natural" bijections come from. It is conceivable that some partition results simply do not have direct combinatorial proofs. At the moment we are not aware of any negative results in this direction. Even finding a formal framework for such results is an important challenge. To be more specific, recall that after so many years of studies, the Rogers–Ramanujan's identities still lack a direct bijective proof despite having an essentially royal status in the world of Partition Theory. Should we assume that there is no such proof at least in the way we are trying to find it, or we are just awaiting for the right idea to come along and correct this oversight?

Let us conclude this brief excurse into the past, present and future of "Constructive Partition Theory" on an optimistic note. We believe there is a great deal of work yet to be done before we reach a better understanding of the combinatorial nature of partition identities. We hope this survey will provide a guidance and the ground floor for the future investigations. As D. J. Kleitman once said, *"Combinatorics will survive as long as mathematics does"* [94]. To paraphrase this, Partition Theory will survive as long as Combinatorics does, and we believe its future is as bright as it was imagined by Sylvester so many years ago...

Material Selection. As we mentioned before, in this survey we concentrate on bijections of what we view as the "important" partition identities. Of course, classifying partition identities into "important" and "unimportant" is not easy. This requires a good analytic background, work experience with partition identities, and an intangible "sense of beauty". Although we do not claim to possess either of these qualities, we hope the reader will agree with and appreciate our selection.

Upon going through some of the extensive partition literature, we were overwhelmed by the task. We can only agree with the sentiment expressed by George Andrews in [16] (in a much less general context), that *"the superficial sameness of these results leaves one daunted."* Later, on the same page Andrews continues, "[A] *compendium of Rogers-Ramanujan type identities leaves the impression that it is impossible to have any idea of what is really going on."* Thus we are conservative in our selection, hoping that combinatorial proofs of partition identities will help the reader to see beyond their "superficial sameness."

[2] A largely overlooked paper [98] makes the first step in this direction, but the proof stops shy of being bijective.

To summarize, we concentrate on a few key classical partition identities, and present a number of their extensions and generalizations. No attempt is made to cover the whole field or to be complete in references. A great deal of material is placed in the exercises which are interspersed with the main results. Because of space limitations, no hints or solutions are provided. We hope this survey will be useful to both beginners and experts in the field.

While we heavily borrow from the available literature, we often felt the need to significantly modify the original presentation for the sake of clarity and consistency. Some of the bijections are different enough from the original exposition that they probably constitute as new constructions. Since the line is virtually impossible to draw, we never claim authorship but always refer to the source.

No references or attributions are given in the main body of the paper; we delay this at times extensive or controversial material until the last section. We tend to refer only to papers that were used directly in the writing or to the most recent papers containing further references which may be useful to the reader. At times, for the benefit of the reader, we also cite more recent references where the results have been rediscovered or presented in a different language. Normally, the bijections in such papers are equivalent or the same as those in this survey. Also, as we emphasize the bijections themselves, we tend to be less careful with combinatorial statistics on partitions. Given bijection descriptions, such statistics often give new or old partition identities, some of which are mentioned in the exercises.

The structure. Admittedly, we are heavily influenced by Andrews's and Stanley's celebrated monographs [24, 115]. In fact, one can view this paper as a supplement to either book, even if written in a rather different style. The order of the sections and the results within the sections reflect our notions of difficulty and importance. The exercises are placed directly after the relevant material. The placement of open problems is less obvious at times.

The theorems are denoted by ▼ and are rarely proved. Proofs are introduced by ► and end with ■. The examples and exercises we deem important for understanding the material are denoted by ($\Diamond$). We suggest that the reader unfamiliar with the subject should attempt to prove all the theorems and such exercises. Additional results and exercises are marked by ($\circ$), ($\circ\circ$) and ($\circ\circ\circ$); our choice reflects their difficulty on a log-scale: from simple to medium, from medium to hard, and from hard to very hard. We also include a number of questions and open problems, which are marked ($*$), ($**$) and ($***$), to indicate approximately their difficulty on the same scale. We should emphasize here that all identities mentioned in the open problems have already been proved; it is a combinatorial proof that is sought.

It was our intention to use pictures as much as possible, so a number of definitions and results are best understood upon examining the included figures. The formulas and theorems are not numbered; they are usually unique in a subsection so subsection numbers suffice. While the survey does not require any preliminary knowledge of the subject, it is written in a concise manner. The reader is assumed to be familiar with the generating functions and occasionally other standard combinatorial concepts for which we refer the reader to [115].

The language. We shall adopt the following conventions which we use throughout the paper. A *correspondence* between two set is a one-to-one function from one set into another. Typically, these two sets will be infinite sets of partitions, in which case the correspondence preserves the size of partitions. A *bijection* is a correspondence between two sets together with its lexical description. Naturally, the same correspondence can be described in many different ways. Thus we can say that two bijections *give the same correspondence*. We also say that two bijections are *identical* if their descriptions are essentially the same or sufficiently close to each other.

When describing a bijection, we refer to its construction as a *map* and then prove (or leave the proof to the reader) that it is well defined and one-to-one. Almost always, this is straightforward. We say that a bijection is *explicit* if its descriptions is sufficiently concise. Virtually all our bijections are explicit, except for those given by the involution principle. There is a way to formalize the notion of "explicit bijection" by treating it as an algorithm which may or may not be polynomial in the size of the input.

Informally, we refer say that a bijection is *natural* if we believe most people would agree with this characterization. For example, conjugation is a natural bijection between partitions of n into at most k parts and partitions of n into parts which are no larger than k.

We say that bijections $\varphi : \mathcal{A} \to \mathcal{B}$ and $\varphi' : \mathcal{A}' \to \mathcal{B}'$ are *equivalent* if there are natural bijections $\alpha : \mathcal{A} \to \mathcal{A}'$ and $\beta : \mathcal{B} \to \mathcal{B}'$ such that $\varphi \circ \beta = \alpha \circ \varphi'$. We say that a bijection or involution is *direct* if it uses no intermediate steps in its constructions. Of course, one uses common sense when deciding whether a particular bijection is direct or not; same with natural bijections.

We say that a proof is *combinatorial* if it is based on a sequence of bijections or involutions or double counting arguments. Similarly, a proof is *bijective* or *involutive* if it is based on a direct bijection or involution, respectively. Thus, for example, the proof of Rogers–Ramanujan's identities we present in 7.2 is combinatorial but neither bijective nor involutive.

The notation. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers. We use routinely both notations for partitions trusting this will not lead to confusion. Various sets of partitions of n are denoted by script capital letters $\mathcal{A}_n$, $\mathcal{B}_n$, etc. Occasionally some of these sets (such as partitions into distinct parts, $\mathcal{D}_n$) are preserved throughout the survey. Partitions are denoted by letters λ, μ, ν, and the bijections are denoted by different Greek letters: α, φ, ψ, etc. Parameters of partitions (the largest part, the number of parts, etc.) are denoted by Roman minuscules, such as $a(\lambda)$, $\ell(\mu)$, etc. Usually we write q-series with a parameter t instead of q. We do this for psychological reasons, to underscore their combinatorial, not analytical, context.

Acknowledgements.

I would like to thank Adriano Garsia for a permission to quote from his unpublished lecture notes [75]. Gilles Schaeffer made helpful comments on the structure of the paper in its early stages. David Jackson was very emphatic in his criticism of the first version of the introduction. Thanks to him, the introduction is nowhere near the brief preface it was originally envisioned. He also made a number of general comments on the project some of which we happily incorporated. Peter McNamara and Tim Chow proofread the paper and found numerous typos, mistakes and missing English articles.

I am extremely grateful to Richard Stanley for the encouragement and support of my Combinatorics endeavors. In fact, the idea of writing this survey came after

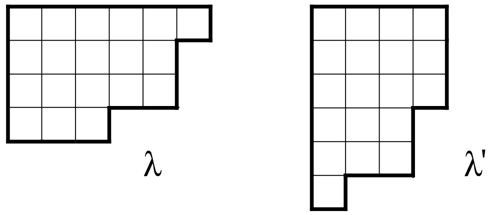

Figure 1. Young diagrams of partitions $\lambda = (6,5,5,3) = (35^2 6)$ and $\lambda' = (4,4,4,3,3,1) = (13^2 4^3)$.

our conversation. After the survey was written it was warmly welcomed by George Andrews, and I am very thankful for that. Many thanks to Robin Chapman, Freeman Dyson, Hershel Farkas, Mourad Ismail and Jiang Zeng for comments on the paper and help with the references.

The Harvard University archives were helpful in my search for the 19th century partition literature. My parents, Mark and Sofia Pak, were gracious enough not to throw away my handwritten research notes on partitions, which were left intact in my room for over 10 years. These notes formed a basis for this survey. Finally, the author was partially supported by NSA and NSF grants.

2. Basic results

2.1. **Partitions and Young diagrams.**

2.1.1. We define a *partition* λ to be an integer sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_l > 0$. We say that λ is a *partition of* n, denoted $\lambda \vdash n$ and $|\lambda| = n$, if $\sum_i \lambda_i = n$. We refer to the integers λ_i as the *parts of the partition.* Let $a(\lambda) = \lambda_1$ and $s(\lambda) = \lambda_\ell$ denote the largest and the smallest parts of the partition λ. The *number of parts* of λ we denote by $\ell(\lambda) = \ell$. Let $m_i = m_i(\lambda)$ be the number of parts of λ equal to i. We also use $\lambda = (1^{m_1} 2^{m_2} \dots)$ as an alternative notation for partitions. The *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of λ is defined by $\lambda'_i = |\{j : \lambda_j \ge i\}| = m_i + m_{i+1} + \dots$. Clearly, $|\ell(\lambda')| = a(\lambda)$.

For partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ define the *sum* $\lambda + \mu$ to be a partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$. Similarly, define the *union* $\lambda \cup \mu$ to be a partition with parts $\{\lambda_i, \mu_j\}$ (arranged in nonincreasing order). Observe that $(\lambda \cup \mu)' = \lambda' + \mu'$.

2.1.2. A *Young diagram* $[\lambda]$ of a partition $\lambda \vdash n$ is a collection of n 1×1 squares (i,j) on a square grid $\mathbb{Z}^2$, with $1 \le i \le \ell(\lambda)$, $1 \le j \le \lambda_i$. Pictorially, we adopt a convention (the so-called English convention), with the first coordinate i increasing downward, while the second coordinate j increases from left to right. The conjugate Young diagram $[\lambda']$ is then obtained by reflection of $[\lambda]$ through the line $i = j$ (see Figure 1). Young diagrams corresponding to the sum and the union of partitions are shown in Figure 2.

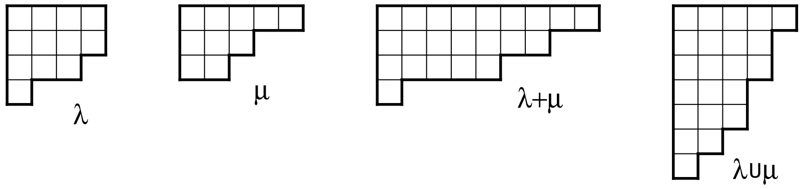

FIGURE 2. Young diagrams of partitions $\lambda = (4,4,3,1)$, $\mu = (5,3,2)$, $\lambda+\mu = (9,7,5,1)$, $\lambda\cup\mu = (5,4,4,3,3,2,1)$.

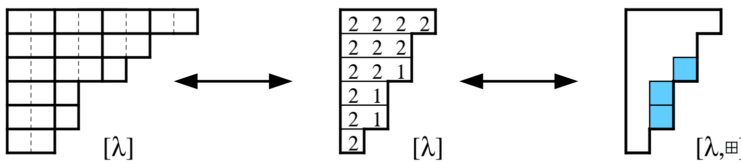

FIGURE 3. Young diagram $[8,6,5,4,3,3,2]$, the corresponding 2-modular diagram and MacMahon diagram.

2.1.3. A *MacMahon diagram* $\big[\lambda,\boxplus\big]$ is a Young diagram $[\lambda]$ and a subset of squares which we call "marked", such that a marked square can be only the rightmost square in a row, and no marked square can lie above an unmarked one. We refer to the partition λ as the *shape* of $\big[\lambda,\boxplus\big]$. Of course, there are many MacMahon diagrams of the same shape λ. By abuse of notation, we denote by $[\lambda]$ the diagram with no marked squares, and by $[\overline{\lambda}]$ the diagram with all rightmost squares marked. We denote by $\varkappa\big[\lambda,\boxplus\big]$ the number of marked squares in a MacMahon diagram $\big[\lambda,\boxplus\big]$. Clearly, $\varkappa[\overline{\lambda}] = \ell(\lambda)$.

MacMahon diagrams with marked squares only in the corners are called *standard MacMahon diagrams.* Observe that if $\big[\lambda,\boxplus\big]$ is a standard MacMahon diagram, then so is $\big[\lambda,\boxplus\big]'$. We define the sum $[\nu]+\big[\lambda,\boxplus\big]$ and the union $[\nu]\cup\big[\lambda,\boxplus\big]$ of a Young diagram and a standard MacMahon diagram in the obvious fashion.

2.1.4. *Modular diagrams.* A *2-modular diagram* $[\mu]_2$ to be a Young diagram with the integers 1 or 2 written in squares, such that 1 can appear only in the last square of a row, and no 2 can appear above 1. There exists a natural bijection between Young diagrams and 2-modular diagrams by collapsing two consecutive squares into one 2-square (see Figure 3). We denote by $[\lambda]_2$ the 2-modular diagram corresponding to partition λ. Finally, there is an obvious bijection between 2-modular diagrams and MacMahon diagrams as in Figure 3. We shall use this bijection later in the paper.

In general, a m-modular diagram $[\mu]_m$ is defined by having an integer m written in all squares of $[\mu]$ which are not the last of a row; any integer i such that $1\le i\le m$ can be written in the last square of a row.

2.1.5. ($\Diamond$) A partition λ is called *self-conjugate* if $\lambda = \lambda'$. Prove that the number of self-conjugate partitions is equal to the number of partitions into distinct odd parts.

2.1.6. ($\Diamond$) Prove bijectively the following summation formula:

$$\sum_{\lambda=(1^{m_1}2^{m_2}\ldots)\vdash n} \frac{1}{m_1!\,1^{m_1}\,m_2!\,2^{m_2}\,\cdots} \;=\; 1\,.$$

2.1.7. ($\circ$) Prove bijectively the following product formula:

$$\prod_{\lambda=(1^{m_1}2^{m_2}\ldots)\vdash n} m_1!\; m_2!\;\cdots \quad = \prod_{\lambda=(1^{m_1}2^{m_2}\ldots)\vdash n} 1^{m_1}\,2^{m_2}\,\cdots$$

2.2. **Generating functions.**

2.2.1. *Number of partitions.* Denote by $\mathcal{P}_n = \{\lambda \vdash n\}$ the set of partitions of n, and let $p(n) = |\mathcal{P}_n|$ be the number of partitions of n. Denote by $\mathcal{P} = \cup_n \mathcal{P}_n$ the set of all partitions. Let $\mathcal{P}_{n,k} = \{\lambda \vdash n : \ell(\lambda) \le k\}$ be the set of partitions of n with at most k parts, and let $p_k(n) = |\mathcal{P}_{n,k}|$. For convenience, let $p(0) = p_k(0) = 1$.

From the representation $\lambda = (1^{m_1}2^{m_2}\ldots)$ we immediately have:

$$P(t) := \sum_{n=0}^{\infty} p(n)t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}$$

Taking the conjugate partition, we obtain $p_k(n) = |\{\lambda \vdash n : a(\lambda) \le k\}|$. Therefore:

$$P_k(t) := \sum_{n=0}^{\infty} p_k(n)t^n = \prod_{i=1}^{k} \frac{1}{1-t^i}$$

Similarly, we obtain more general formulas:

$$P(t,s) := \sum_n \left(\sum_{\lambda \vdash n} s^{\ell(\lambda)} \right) t^n = \prod_{i=1}^{\infty} \frac{1}{1-st^i},$$

$$P_k(t,s) := \sum_n \left(\sum_{\lambda \vdash n,\, \ell(\lambda)\le k} s^{a(\lambda)} \right) t^n = \prod_{i=1}^{k} \frac{1}{1-st^i}.$$

2.2.2. ($*{*}*$) Prove combinatorially the following *Ramanujan's identity*:

$$\sum_{k=1}^{\infty} p(5k-1)\,t^k \;=\; 5\,\prod_{i=1}^{\infty} \frac{(1-t^{5i})^5}{(1-t^i)^6}\,.$$

2.2.3. *Euler's first row decomposition.* The following identity is due to Euler:

$$1+\sum_{n=1}^{\infty}\frac{s^n t^n}{(1-t)(1-t^2)\cdots(1-t^n)} \,=\, \prod_{i=1}^{\infty}\frac{1}{1-st^i}$$

Indeed, on the r.h.s. we have a generating function for all partitions:

$$\sum_{\lambda\in\mathcal{P}} s^{\ell(\lambda)}t^{|\lambda|} = \sum_n s^n \sum_{\lambda:\,\ell(\lambda)=n} t^{|\lambda|} = \sum_n s^n P_k(t),$$

which proves the result.

2.2.4. ($\Diamond$) Prove that the following sum is symmetric in a and b:

$$F(a,b;\,t) \,:=\, \sum_{n=1}^{\infty}\frac{a\,b^n\,t^n}{(1-at)(1-at^2)\cdots(1-at^n)}\,.$$

2.2.5. ($\Diamond$) Let $(k)_q = 1+q+\ldots+q^{k-1}$, $k!_q = (k)_q(k-1)_q\cdots(1)_q$. Define the Gaussian coefficients $\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q}$. Prove combinatorially:

$$\sum_{\lambda:\ a(\lambda)\le k,\,\ell(\lambda)\le\ell} q^{|\lambda|} \,=\, \binom{k+\ell}{k}_q$$

2.3. **Basic geometry of Young diagrams.**

2.3.1. *Durfee square.* The largest square $[\delta_r] = \{(i,j), 1\le i,j\le r\}$ which fits into a Young diagram $[\lambda]$ is called the *Durfee square* (see Figure 4). Observe that $[\lambda]\smallsetminus[\delta_r]$ is a disjoint union of two Young diagrams $[\mu]$ and $[\nu]$, such that $\mu,\nu'\in\mathcal{P}_{n,k}$. Define the map $\varphi:\mathcal{P}_n\to\bigsqcup_{r,k}\mathcal{P}_{n-k,r}\times\mathcal{P}_{k-r^2,r}$ by letting $\varphi(\lambda)=(\mu,\nu)$.

▼ *Map φ defined above is a bijection.*

This proves $P(t)=\sum_r t^{r^2}P_r(t)P_r(t)$ and implies *Euler's identity*:

$$\prod_{i=1}^{\infty}\frac{1}{1-t^i} = 1+\sum_{r=1}^{\infty}\frac{t^{r^2}}{(1-t)^2(1-t^2)^2\cdots(1-t^r)^2}\,,$$

More generally, we have $P(t,s)=\sum_r s^r t^{r^2}P_r(t)P_r(t,s)$, which implies *Cauchy's idenity*:

$$\prod_{i=1}^{\infty}\frac{1}{1-st^i} = 1+\sum_{r=1}^{\infty}\frac{s^r\,t^{r^2}}{(1-t)(1-t^2)\cdots(1-t^r)(1-st)(1-st^2)\cdots(1-st^r)}\,.$$

2.3.2. ($\circ$) Generalize Durfee squares to prove the following identity:

$$\sum_{a,b\ge 0}\frac{t^{(a^2-ab+b^2)}\,z^{(a-b)}}{(1-t)(1-t^2)\cdots(1-t^a)\,(1-t)(1-t^2)\cdots(1-t^b)} = \sum_{n=-\infty}^{\infty} t^{n^2}z^n\,\prod_{i=1}^{\infty}\frac{1}{1-t^i}\,.$$

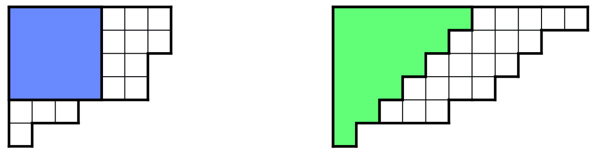

Figure 4. Young diagrams $\lambda = (7,7,6,6,4,1)$ with Durfee square $[\delta_4]$, and $\mu = (11,9,8,7,5,1) \in \mathcal{D}_{41}$ with Sylvester's triangle $[\rho_6]$.

2.3.3. ($\circ\circ$) Generalize Durfee squares to prove the *Rogers-Fine identity*:

$$1 + \sum_{n=1}^{\infty} \frac{(1+at)(1+at^3)\cdots(1+at^{2n-1})\, z^n t^{2n}}{(1-bt^2)(1-bt^4)\cdots(1-bt^{2n})}$$

$$= \sum_{r=0}^{\infty} \frac{(1+azt^{4r+3})b^r z^r t^{2r(r+1)}}{(1-zt^{2(r+1)})} \prod_{i=1}^{r} \frac{(1+at^{2i-1})(1+ab^{-1}zt^{2i+1})}{(1-bt^{2i})(1-zt^{2i})}\,.$$

Hint: The l.h.s. is a generating function for partitions λ into parts such that odd parts are not repeated. Now consider a maximal $2r \times (r+1)$ rectangle which fits into $[\lambda]$, interpret the remaining pieces of $[\lambda]$ accordingly, and sum over all $r \geq 0$.

2.3.4. *Sylvester's triangle.* Let $\mathcal{D}_n$ be the set of partitions $\lambda \vdash n$ into *distinct parts*: $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$, and let $\mathcal{D} = \cup_n \mathcal{D}_n$. Clearly,

$$D(t,s) := 1 + \sum_{\lambda\in\mathcal{D}} s^{\ell(\lambda)} t^{|\lambda|} \; = \; \prod_{i=1}^{\infty}(1+st^i)\,.$$

Consider a diagram $[\rho_k] = \{(i,j) : i+j \leq k+1\}$, with $k = \ell(\lambda)$ (see Figure 4). We shall refer to $[\rho_k]$ as *Sylvester's triangle*. Observe that the horizontal parts of the diagram $[\lambda] \smallsetminus [\rho_k]$ for a partition. This gives $D(t,s) = \sum_r s^{\ell(\rho_r)} t^{|\rho_r|} P_r(t)$, and implies another *Euler identity*:

$$\prod_{i=1}^{\infty}(1+st^i) = 1 + \sum_{r=1}^{\infty} \frac{s^r\, t^{\frac{r(r+1)}{2}}}{(1-t)(1-t^2)\cdots(1-t^r)}\,.$$

2.3.5. *Frobenius coordinates.* Let $\mathcal{D}'_n$ be the set of partitions $\lambda \vdash n$ into *nonnegative distinct parts*: $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 0$, and let $\mathcal{D}' = \cup_n \mathcal{D}'_n$. When $\lambda_l = 0$, we say that λ contains the *empty part*, and let $\ell(\lambda) = l$. Note that we distinguish here partitions $\lambda \in \mathcal{D}'$ with and without the empty part. Clearly,

$$D'(t,s) := 1 + \sum_{\lambda\in\mathcal{D}'} s^{\ell(\lambda)} t^{|\lambda|} \; = \; \prod_{i=0}^{\infty}(1+st^i)\,.$$

When drawing a Young diagram of $\lambda \in \mathcal{D}'$, we add an interval ont the bottom for the auxiliary empty part.

Let $\mathcal{D}_{n,k} = \{\lambda \in \mathcal{D}_n : \ell(\lambda) = k\}$, $\mathcal{D}'_{n,k} = \{\lambda \in \mathcal{D}'_n : \ell(\lambda) = k\}$. Observe that $|\mathcal{D}'_{n,k}| = |\mathcal{D}_{n,k}| + |\mathcal{D}_{n,k-1}|$. Let us show that $|\mathcal{P}_n| = \sum_k \sum_m |\mathcal{D}_{m,k}| \cdot |\mathcal{D}'_{n-m,k}|$. Indeed, start with $\lambda \vdash n$ and let k be the side of the Durfee square $\delta_k \subset [\lambda]$. Now split the Young diagram $[\lambda]$ into two parts: one on or above the diagonal $i - j = 0$, and one below the diagonal. Now read the rows of the first part and the columns of the second diagram (see Figure 5). The resulting pair of partitions

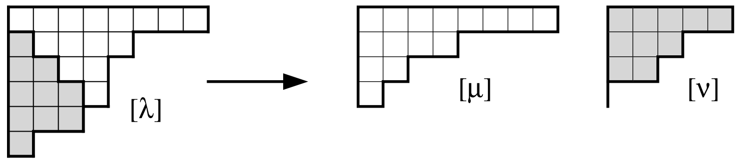

FIGURE 5. Partition $\lambda = (8,5,4,4,3,1) \vdash 25$, and its Frobenius coordinates (μ,ν), where $\mu = (8,4,2,1) \in \mathcal{D}_{15,4}$, $\nu = (5,3,2,0) \in \mathcal{D}_{10,4}$.

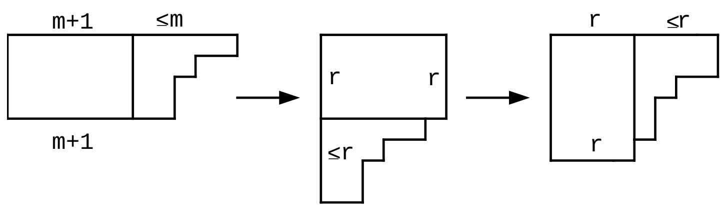

FIGURE 6. Bijection φ.

are the *Frobenius coordinates* (μ,ν) of λ, where $\mu \in \mathcal{D}_{m,k}$, $\nu \in \mathcal{D}'_{n-m,k}$. Let $\varphi : \mathcal{P}_n \to \bigsqcup_{m,k} \mathcal{D}_{m,k} \times \mathcal{D}'_{n-m,k}$ defined by $\varphi(\lambda) := (\mu,\nu)$. Then the above formula follows from the following result:

▼ *Map φ defined above is a bijection.*

2.3.6. Consider the following simple *Ramanujan's identity*:

$$\sum_{m=0}^{\infty} \frac{t^m}{(1-t^{m+1})\cdots(1-t^{2m})} = \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(1-t^{m+1})\cdots(1-t^{2m+1})}$$

Let $\mathcal{A}_n$ be the set of all partitions $\lambda \vdash n$ with unique smallest part $s(\mu)$, and $a(\lambda) \le 2s(\lambda)$. Let $\mathcal{B}_n$ be the set of all partitions $\mu \vdash n$ with odd $a(\mu)$, and $a(\mu) < 2s(\mu)$. The above identity is equivalent to $|\mathcal{A}_n| = |\mathcal{B}_n|$, for all $n > 0$.

We define a bijection $\varphi : \mathcal{B}_n \to \mathcal{A}_n$ as follows. Start with a partition $\mu \in \mathcal{B}_n$ with $a(\mu) = 2m+1$. Split the Young diagram $[\mu]$ into a $(m+1) \times \ell(\mu)$ rectangle and the remaining part $[\nu]$. Move $[\nu]$ so it attaches below the rectangle. Let $[\lambda]$ be the conjugate of the resulting Young diagram. Define $\varphi(\mu) = \lambda$ (see Figure 6).

▼ *The map $\varphi : \mathcal{B}_n \to \mathcal{A}_n$ defined above is a bijection.*

2.3.7. *Vahlen's involution.* Consider the following trivial identity:

$$\prod_{i=1}^{k}(1-t^i) \prod_{r=1}^{k} \frac{1}{1-t^r} = 1$$

Observe that the coefficient of t^n on the l.h.s. is equal to $\sum_{(\lambda,\mu)}(-1)^{\ell(\mu)}$, where the summation is over all $\lambda \in \mathcal{P}$, $\mu \in \mathcal{D}$, $|\lambda| + |\mu| = n$, and $a(\lambda), a(\mu) \le k$. Define an involution on $\mathcal{P} \times \mathcal{D}$ by moving the smallest part $s(\lambda)$ from λ to μ, if $s(\lambda) < s(\mu)$,

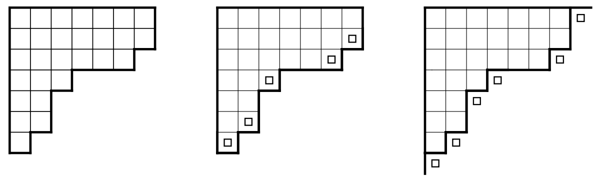

FIGURE 7. Young diagram $[\lambda]$, $\lambda = (7^2, 6, 3, 2^2, 1)$, with $\gamma(\lambda) = 5$ corners and $\gamma(\lambda) + 1 = 6$ outside corners.

or by moving part $s(\mu)$ from μ to λ, if $s(\lambda) \geq s(\mu)$. This map is called *Vahlen's involution*. By the construction, it has no fixed points for all $n \geq 1$.

2.4. **Number of distinct parts.**

2.4.1. Denote by $\gamma(\lambda)$ the number of distinct parts of a partition λ. The following result is, perhaps, a bit surprising at first glance:

$$\sum_{\lambda \in \mathcal{P}} \gamma(\lambda)\, t^{|\lambda|} \, = \, \frac{t}{1-t} \prod_{i=1}^{\infty} \frac{1}{1-t^i} \, .$$

▶ In a Young diagram $[\lambda]$, a *corner* is a square $(i, j) \in [\lambda]$, such that $(i, j+1), (i+1, j) \notin [\lambda]$. Similarly, an *outside corner* is a square $(i, j) \notin [\lambda]$, such that $(i, j-1) \in [\lambda]$, if $j > 1$, and $(i-1, j) \in [\lambda]$, if $i > 1$. Observe that $\gamma(\lambda)$ is equal to the number of corners of λ. Similarly, $\gamma(\lambda) + 1$ is the number of outside corners of λ (see Figure 7).

Denote by $[\widetilde{\lambda}] = [\lambda] - (i, j)$, a Young diagram obtained by removal of a square (i, j) from $[\lambda]$. Obviously, if $\lambda \vdash n$, then $\widetilde{\lambda} \vdash n - 1$. We have:

$$\sum_{\lambda \vdash n} \gamma(\lambda) = \sum_{[\widetilde{\lambda}] \subset [\lambda]} 1 = \sum_{\widetilde{\lambda} \vdash n-1} \bigl(\gamma(\widetilde{\lambda}) + 1\bigr) = p(n-1) + \sum_{\widetilde{\lambda} \vdash n-1} \gamma(\widetilde{\lambda}) = 1 + p(1) + p(2) + \ldots + p(n-1).$$

This immediately implies the result. ■

2.4.2. ($\circ\circ$) In a Young diagram $[\lambda]$, the *boundary* $\partial[\lambda]$ is a collection of squares $(i, j) \in [\lambda]$ such that $(i+1, j+1) \notin [\lambda]$. Similarly, the *outside boundary* $\overline{\partial}[\lambda]$ is a collection of squares $(i, j) \notin [\lambda]$ such that either $(i, j-1)$, or $(i-1, j)$m or $(i-1, j-1) \in [\lambda]$. Define a *rim hook* R in $[\lambda]$ to be a rookwise connected sequence of squares $R \subset \partial[\lambda]$, such that $[\lambda] - R$ is also a Young diagram of a partition. Similarly, an *outside hook* R' (outside of $[\lambda]$) is a rookwise connected sequence of squares $R' \subset \overline{\partial}[\lambda]$, such that $[\lambda] \cup R$ is a Young diagram of a partition. The *height* and *length* of a hook are the dimensions of the smallest rectangle which contains the hook.

Prove that for every λ, the number of rim hooks of height h and length ℓ in λ is one less than the number of hooks of height h and length ℓ outside of λ (see Figure 8). When $k = \ell = 1$ this was a crucial observation in the proof above. Compute a generating function $\sum_\lambda \eta(\lambda, k, \ell) t^{|\lambda|}$ for the number $\eta(\lambda, k, \ell)$ of hooks of height k and length ℓ in λ.

2.5. **Dyson's rank.**

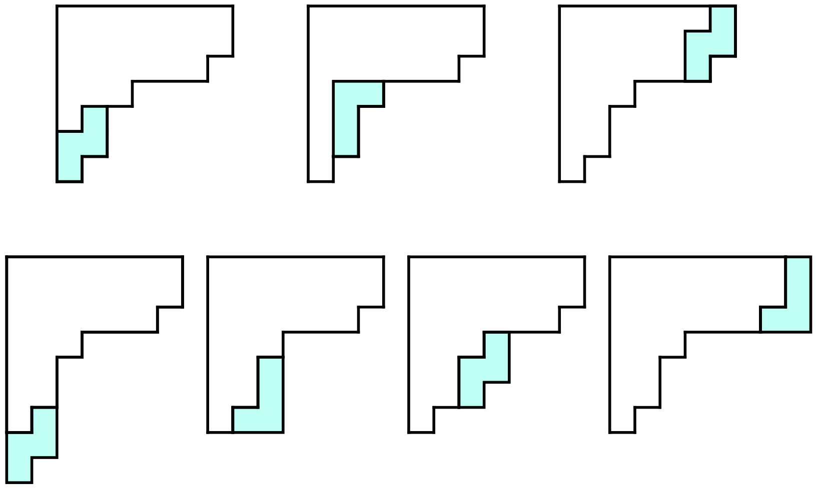

FIGURE 8. Young diagram $[7^2 632^2 1]$. Three rim hooks and four outside hooks of height 3 and length 2.

2.5.1. *Fine–Dyson symmetry relations.* Define the *rank* of a partition λ to be $r(\lambda) = a(\lambda) - \ell(\lambda)$. Denote by $p(n,r)$ the number of partitions $\lambda \vdash n$ of rank r. Denote by $\mathcal{H}_{n,r}$ and $\mathcal{G}_{n,r}$ the set of partitions of n with rank at most r and at least r, respectively. Let $h(n,r) = |\mathcal{H}_{n,r}|$. Clearly, $p(n,r) = h(n,r) - h(n,r-1)$. By conjugation, $|\mathcal{G}_{n,r}| = |\mathcal{H}_{n,-r}| = h(n,-r)$. Since $\mathcal{P}_n = \mathcal{H}_{n,r} \cup \mathcal{G}_{n,r+1}$, we also have *complementarity relations*:

$$h(n,r) \, + \, h(n,-1-r) \, = \, p(n)\,.$$

The following relations are called the *Fine-Dyson symmetry relations*:

$$h(n,1+r) \, = \, h(n+r,1-r).$$

We shall prove this formula by an explicit bijection $\psi_r : \mathcal{H}_{n,r+1} \to \mathcal{G}_{n+r,r-1}$. Start with a partition $\lambda \in \mathcal{H}_{n,r+1}$. Remove the leftmost column in $[\lambda]$, with $\ell = \ell(\lambda)$ squares. Add the top row with $(\ell + r)$ squares. Let $[\mu]$ be the resulting Young diagram (see Figure 9.) We call the map $\psi_r : \lambda \to \mu$ *Dyson's map.*

▼ *Dyson's map* $\psi_r : \mathcal{H}_{n,r+1} \to \mathcal{G}_{n+r,r-1}$ *is a bijection.*

▶ By assumption on λ, we have $r(\lambda) = a(\lambda) - \ell \le r+1$, so $\ell + r \ge a(\lambda) - 1$, and the top row $a(\mu)$ is the largest indeed. Clearly, $|\mu| = |\lambda| - \ell + (\ell + r) = n + r$. Also, $r(\mu) = a(\mu) - \ell(\mu) = \ell(\lambda) + r - (\lambda'_2 + 1) \ge r - 1$. Therefore, $\mu = \psi_r(\lambda) \in \mathcal{G}_{n+r,r-1}$. ■

2.5.2. (○) Deduce from Fine-Dyson symmetry the following *Fine's relations*:
1) $p(n) - p(n-1) = p(n+r+1,r)$, for $r+3 \ge n \ge 1$,
2) $p(n+1,0) - p(n,0) + 2\,p(n-1,3) = p(n+1) - p(n)$, for $n \ge 1$,
3) $p(n,r+1) - p(n-1,r) = p(n-r-3,r+4) - p(n-r-2,r+3)$, for $n \ge r+3$.

2.5.3. (◊) Interpret the l.h.s. in Fine's relation 1) (see above) as the number of partitions $\lambda \vdash n$, with $s(\lambda) \ge 2$. Now prove 1) bijectively.

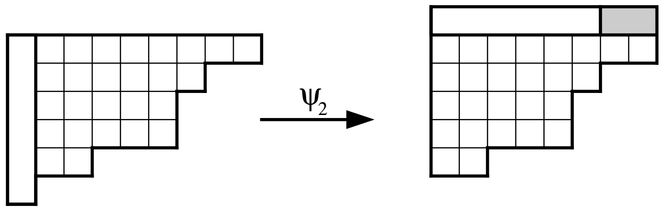

FIGURE 9. Dyson's map $\psi_r : \lambda \to \mu$, where $\lambda = (9,7,6,6,3,1) \in \mathcal{H}_{32,r+1}$, $\mu = (8,8,6,5,5,2) \in \mathcal{G}_{32+r,r-1}$, and $r = 2$.

2.5.4. *Generating function.* ($\lozenge$) Let $G_r(t) = \sum_{n=1}^{\infty} h(n,-r)\, t^r$ be the generating function for $|\mathcal{G}_{n,r}|$. Use the complementarity relations and the Fine-Dyson symmetry relations to obtain the following two identities:

$$1 + G_r(t) + G_{1-r}(t) = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)}\,, \quad G_r(t) = t^{r+1}\big(1 + G_{-2-r}(t)\big)\,.$$

Deduce from these identities that

$$G_r(t) \,=\, t^{r+1} \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} \,-\, t^{r+1}\, G_{r+3}(t)\,.$$

Iterate the above equation to conclude:

$$G_r(t) \,=\, \sum_{m=1}^{\infty} (-1)^{m-1} t^{\frac{m(3m-1)}{2}+rm} \prod_{i=1}^{\infty} \frac{1}{(1-t^i)}\,.$$

2.5.5. ($\circ$) Use Dyson's map 2.5.1 to give a bijective proof of the generating function above.

2.5.6. ($*{*}*$) Prove combinatorially Dyson's combinatorial interpretation of Ramanujan's congruence:

$$\sum_{r \equiv i \bmod 5} p(5k-1, r) \,=\, \frac{1}{5}\, p(5k-1)\,, \text{ for all integers } i,\, k > 0\,.$$

2.5.7. ($\circ\circ$) Let $\mathcal{R} = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$ be a set of triples of partitions (λ,μ,ν), such that $\lambda \in \mathcal{D}$. Let $|(\lambda,\mu,\nu)| = |\lambda| + |\mu| + |\nu|$, $vr(\lambda,\mu,\nu) = \ell(\mu) - \ell(\nu)$, and $\mathcal{R}_{n,r} = \{(\lambda,\mu,\nu) \in \mathcal{R} : |(\lambda,\mu,\nu)| = n,\ \varrho(\lambda,\mu,\nu) = r\}$. Consider *Garvan's weighted sum*:

$$M(n,r) \,=\, \sum_{(\lambda,\mu,\nu) \in \mathcal{R}_{n,r}} (-1)^{\ell(\lambda)}$$

Check that $M(n,r) = M(n,-r)$. Use Vahlen's involution 2.3.7 to prove that $\sum_r M(n,r) = p(n)$. Prove combinatorially the analogue of the Fine-Dyson symmetry relations in this case.

2.5.8. ($\circ$) Prove combinatorially:

$$\sum_{r=-\infty}^{\infty} r^2\, M(n,r) \,=\, 2\,n\,p(n)\,.$$

2.5.9. ($\circ\circ$) Define the *crank* of a partition λ as follows: $cr(\lambda) = -\ell(\lambda)$ if $t := \lambda_1 - \lambda_2 = 0$, and $cr(\lambda) = t - \lambda_{t+1}$ if $t > 0$. Let $N(n,r)$ be the number of partitions $\lambda \vdash n$ with crank $cr(\lambda) = r$. Prove combinatorially that $M(n,r) = N(n,r)$.

2.6. q-binomial theorem.

2.6.1. The following classical identity is usually called the *q-binomial theorem*:

$$\sum_{k=1}^{\infty} \frac{(1+a)(1+at)\cdots(1+at^{k-1})\,z^k t^k}{(1-t)(1-t^2)\dots(1-t^k)} \;=\; \prod_{i=1}^{\infty} \frac{1+azt^i}{1-zt^i}$$

Let us show that both sides are equal to the generating function

$$M(a,t,z) \,=\, \sum_{[\lambda,\boxplus]} a^{\varkappa[\lambda,\boxplus]}\, z^{\ell(\lambda)} t^{|\lambda|},$$

where the sum is over all standard MacMahon diagrams with n squares. In other words, we present two bijections between sets of partitions, one for each side of the identity, and standard MacMahon diagrams.

For the r.h.s. this is straightforward. Start with partitions $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{D}$, corresponding to the denominator and numerator, respectively. Let $[\overline{\mu}]$ be the corresponding standard MacMahon diagram with a marked square in each row. Now consider a standard MacMahon diagram $[\nu,\boxplus] = [\lambda] \cup [\overline{\mu}]$, which gives the desired interpretation of the r.h.s. Set $\psi(\lambda,\mu) = [\nu,\boxplus]$ (see Figure 10).

For the l.h.s, start with a pair of partitions $\upsilon \in \mathcal{P}$, $\omega \in \mathcal{D}'$, with $a(\upsilon) \le k$, $a(\omega) \le k-1$. Attach to $[\omega]$ a row of length k, the term corresponding to t^k, and denote by $[\pi]$ the resulting Young diagram. Now consider a standard MacMahon diagram $[\pi,\boxplus]$ of shape π, with a marked square in each corner, except perhaps for the square $(1,k)$. Mark the latter only if ω contains part (0). Now let $[\tau,\boxplus] = [\pi,\boxplus] \cup [\upsilon]$, and define $\varphi(\upsilon,\omega) = [\tau,\boxplus]'$.

▼ *The maps φ, ψ are bijections.*

Now check that $\ell(\nu) = \ell(\lambda) + \ell(\mu)$, $\varkappa[\nu,\boxplus] = \ell(\mu)$, and $|\nu| = |\lambda| + |\mu|$. Similarly, $\ell(\tau) = a(\pi) = k$, $\varkappa[\tau,\boxplus] = \ell(\omega)$, and $|\tau| = |\upsilon| + |\omega| + k$. Thus we obtain the q-binomial theorem.

2.6.2. ($\circ$) Deduce identities 2.2.3 and 2.2.5 from the q-binomial theorem.

2.6.3. ($*$) Prove combinatorially the following extension of the q-binomial theorem:

$$(1+a)\sum_{r=m}^{n} \frac{(1+abt)(1+abt^2)\cdots(1+abt^{r-1})\,bt^r}{(1-bt)(1-bt^2)\cdots(1-bt^r)} \;=\; \prod_{i=1}^{n} \frac{(1+abt^i)}{(1-bt^i)} - \prod_{j=1}^{m-1} \frac{(1+abt^j)}{(1-bt^j)}\,.$$

2.7. Heine transformation.

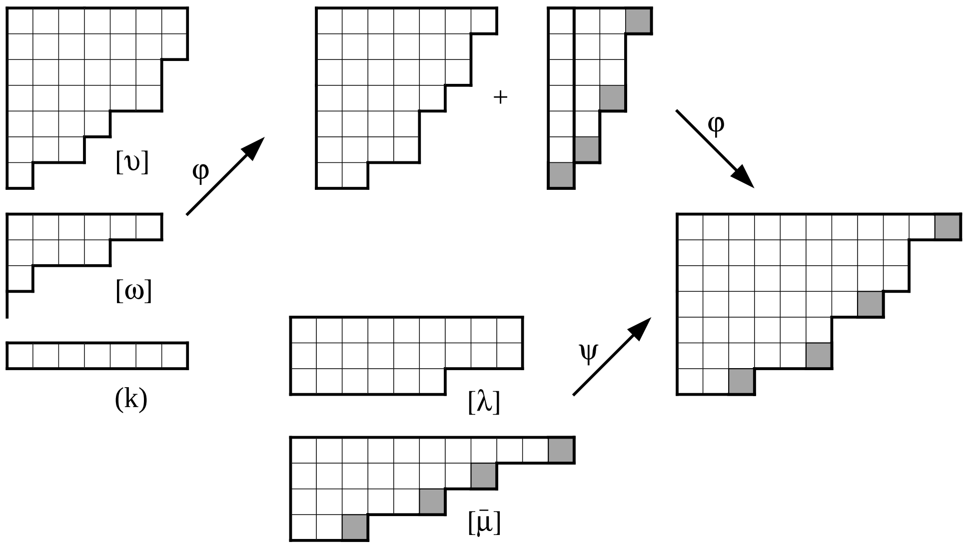

FIGURE 10. Example of bijections $\psi : (\lambda, \mu) \to \big[\nu, \boxplus\big]$ and $\varphi : (\upsilon, \omega) \to \big[\tau, \boxplus\big]$. Here $\lambda = (7, 7, 6, 6, 4, 3, 1)$, $\omega = (6, 4, 1, 0)$, $\lambda = (9, 9, 6)$, $\mu = (11, 8, 6, 3)$, and $\big[\nu, \boxplus\big] = \big[\tau, \boxplus\big]$.

2.7.1. The classical *Heine transformation* can be written as follows:

$$\sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{(1-at^i)(1-bt^i)}{(1-t^{i+1})(1-ct^i)}\, z^k = \prod_{r=0}^{\infty} \frac{(1-azt^r)(1-bt^r)}{(1-ct^{r+1})(1-zt^r)} \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{(1-cb^{-1}t^i)(1-zt^i)}{(1-t^{i+1})(1-azt^i)}$$

This is equivalent to $F(a, z, b, c;\, t) \,=\, F(c, b, z, a;\, t)$, where

$$F(a, z, b, c;\, t) \,=\, \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{(1+at^i)}{(1-t^{i+1})} \prod_{m=1}^{\infty} \frac{(1+cbt^{k+m})}{(1-bt^{k+m})}\, z^k t^k\,.$$

The proof idea is based on a symmetric combinatorial interpretation of the coefficients in $F(a, z, b, c;\, t)$.

2.7.2. Using the bijections 2.6.1, let us first give a combinatorial interpretation to the coefficients in the two products inside the series F. We obtain:

$$t^k \prod_{i=0}^{k-1} \frac{(1+at^i)}{(1-t^{i+1})} \,=\, \sum_{[\lambda,\boxplus]\,:\, a(\lambda)=k} a^{\varkappa[\lambda,\boxplus]}\, t^{|\lambda|},$$

where the summation is over all standard MacMahon diagrams $\big[\lambda, \boxplus\big]$ with k squares in the first row. Indeed, use the bijection φ' defined as ϕ in 2.6.1, but without conjugation in the last step. Similarly, for the second product we have:

$$\prod_{m=1}^{\infty} \frac{(1+cbt^{k+m})}{(1-bt^{k+m})} \,=\, \sum_{[\mu,\boxplus]\,:\, s(\mu)=k+1} b^{\ell(\mu)} c^{\varkappa[\mu,\boxplus]}\, t^{|\mu|},$$

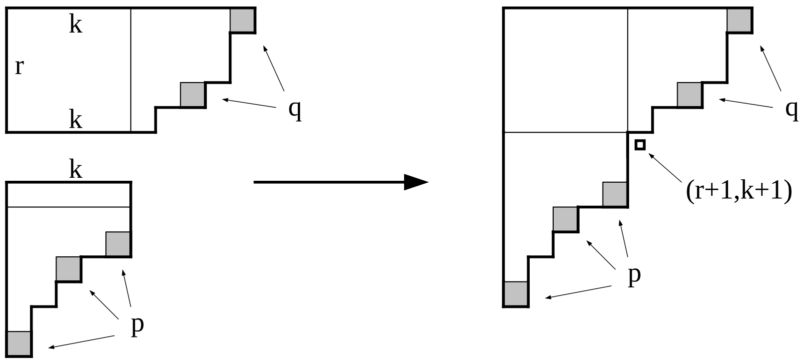

FIGURE 11. Bijection in the proof of Heine transformation.

where the summation is over all standard MacMahon diagrams $[\mu, \boxplus]$ which are either empty or have at least $k+1$ squares in the last row. Indeed, use the bijection ψ, defined in 2.6.1. Therefore, we have:

$$
\begin{aligned}
F(a, z, b, c;\, t) \,:=\, & \sum_{(p,l,r,q,n)} f(p, l, r, q;\, n)\, a^p\, z^l\, b^r\, c^q\, t^n \\
= & \sum_{k=0}^{\infty} \sum_{[\lambda,\boxplus]\,:\, a(\lambda)=k} \ \sum_{[\mu,\boxplus]\,:\, k<s(\mu)} a^{\varkappa[\lambda,\boxplus]}\, z^k\, b^{\ell(\mu)}\, c^{\varkappa[\mu,\boxplus]}\, t^{|\lambda|+|\mu|}\,,
\end{aligned}
$$

Attaching $[\mu, \boxplus]$ right above $[\lambda, \boxplus]$, we get a diagram $[\nu, \boxplus]$. From the equation above, we see that $f(p, k, r, q;\, n)$ is equal to the number of standard MacMahon diagrams $[\nu, \boxplus]$ with n squares, with an outside corner in $(r{+}1, k{+}1)$, with p marked squares in the rows that are $\le k$, with q marked squares in the rows that are $> k$ (see Figure 11). Conjugating $[\mu, \boxplus]$, we deduce $f(p, k, r, q;\, n) = f(q, r, k, p;\, n)$, which completes the proof of the Heine transformation.

2.7.3. ($\Diamond$) Convert the above proof into a direct bijection between quadruples of partitions, representing coefficients in $F(a, z, b, c;\, t)$ and $F(c, b, z, a;\, t)$.

2.7.4. ($\circ$) Deduce the following *Heine identity*:

$$
1 + \sum_{k=1}^{\infty} (abc)^k \prod_{i=0}^{k-1} \frac{(1 + b^{-1}q^i)(1 + c^{-1}q^i)}{(1 - aq^i)(1 - q^{i+1})} \;=\; \prod_{i=0}^{\infty} \frac{(1 + abq^i)(1 + acq^i)}{(1 - aq^i)(1 - abcq^i)}\,.
$$

2.7.5. ($\circ$) Deduce identities 2.3.1 and 2.3.4 from the Heine identity.

2.7.6. ($**$) Prove combinatorially *Ramanujan's ${}_1\psi_1$-summation*:

$$
\begin{aligned}
& \sum_{k=-\infty}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-b)(1-bq)\cdots(1-bq^{n-1})}\; z^k \\
= & \prod_{n=0}^{\infty} \frac{(1 - azq^n)(1 - a^{-1}z^{-1}q^{n+1})(1 - q^{n+1})(1 - a^{-1}bq^n)}{(1 - zq^n)(1 - a^{-1}bq^n)(1 - bq^n)(1 - a^{-1}q^{n+1})}\,.
\end{aligned}
$$

3. Euler's Theorem

3.1. **Partitions into distinct parts vs. partitions into odd parts.** Recall that $\mathcal{D}_n$ denotes the set of partitions into distinct parts. Denote by $\mathcal{O}_n$ the set of partitions of n into odd parts.

▼ Euler's Theorem. $|\mathcal{O}_n| = |\mathcal{D}_n|$.

The proof is straightforward:

$$1+\sum_{n=1}^{\infty} |\mathcal{O}_n|\, t^n \,=\, \prod_{r=1}^{\infty} \frac{1}{(1-t^{2r-1})} \,=\, \prod_{r=1}^{\infty} \frac{(1-t^r)(1+t^r)}{(1-t^{2r})(1-t^{2r-1})}$$
$$= \prod_{i=1}^{\infty}(1+t^i) \,=\, 1+\sum_{n=1}^{\infty} |\mathcal{D}_n|\, t^n\,.$$

In this section we present three bijective proofs of Euler's Theorem and a number of extensions. Further generalizations including Andrews' Theorem 8.1.1 will be presented in Section 8.

3.2. **Glaisher's bijection.**

3.2.1. *Glaisher's bijection* $\varphi : \mathcal{O}_n \to \mathcal{D}_n$ is defined as follows. Let $\lambda = (1^{m_1}3^{m_3}\ldots) \in \mathcal{O}_n$ be a partition with odd parts. For every odd i, let $\varphi(\lambda)$ contain part $i\cdot 2^r$, if and only if the integer m_i written in binary has 1 at the r-th position.

In the other direction, let $\phi : \mathcal{D}_n \to \mathcal{O}_n$ be defined by an iterative procedure. Start with $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{D}_n$. Substitute every even part (λ_i) with two parts $(\lambda_i/2)$. Repeat until the resulting partition μ has no even parts, and set $\phi(\lambda) := \mu$.

▼ *Maps $\varphi : \mathcal{O}_n \to \mathcal{D}_n$ and $\phi : \mathcal{D}_n \to \mathcal{O}_n$ are well defined bijections, inverse to each other: $\phi = \varphi^{-1}$.*

3.2.2. ($\Diamond$) Let $\mathcal{B}_n \subset \mathcal{D}_n$ be the set of all partitions $\lambda \vdash n$ into distinct parts, such that $\lambda_i \equiv 0, 1$ or $2 \bmod 4$. Let $\mathcal{Q}_n \subset \mathcal{O}_n$ be the set of all partitions $\mu \vdash n$ into odd parts, such that all parts $i \equiv 3 \bmod 4$ appear an even number of times. Finally, let $\mathcal{A}_n$ be the set of all partitions $\nu \vdash n$, such that $\nu_i \equiv 1, 5$ or $6 \bmod 8$. Check that $\mathcal{Q}_n = \phi(\mathcal{B}_n)$. Conclude that $|\mathcal{B}_n| = |\mathcal{A}_n|$.

3.2.3. ($\Diamond$) *Glaisher's Theorem.* For any $d \geq 2$, prove that the number of partitions $\lambda \vdash n$ with no part divisible by d is equal to the number of partitions $\mu \vdash n$ with no part repeated $\geq d$ times.

3.2.4. ($\circ$) Let $p_e(n)$ and $p_o(n)$ be the number of partitions of n into an even and an odd number of even parts, respectively. Prove combinatorially that $p_e(n)-p_o(n)$ is equal to the number of partitions of n into distinct odd parts.

3.2.5. ($\circ$) *Vector partitions.* Fix an integer $k \geq 1$. Consider nonnegative integer vectors $\mathbf{c} = (c_1, \ldots, c_k)$. Define a *vector partition* of $\mathbf{c}$ to be a presentation of $\mathbf{c}$ as a sum of nonnegative integer vectors, regardless of the order. A vector is called *odd* if it has at least one odd component. Then the number of vector partitions of $\mathbf{c}$ into odd vectors is equal to the number of vector partitions into *distinct* (i.e. unequal) vectors. Extend Glaisher's bijection to prove this result.

3.3. **Franklin's Extension.**

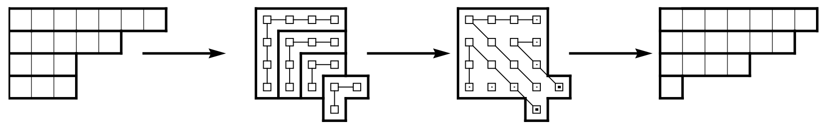

FIGURE 12. Sylvester's bijection $\psi : (7,5,3,3) \to (7,6,4,1)$.

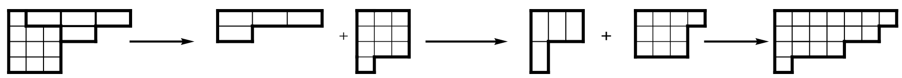

FIGURE 13. The second version of Sylvester's bijection ζ.

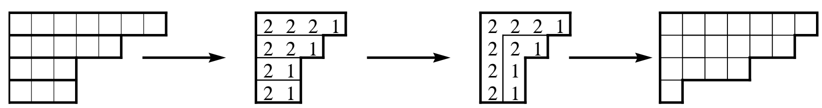

FIGURE 14. The third version of Sylvester's bijection η.

3.3.1. For a partition λ, denote by $\gamma_{\mathcal{O}}(\lambda)$ the number of even part sizes, and by $\gamma_{\mathcal{D}}(\lambda)$ the number of repeated part sizes. *Franklin's extension* of Euler's Theorem states that the number of partitions $\lambda \vdash n$ with $\gamma_{\mathcal{O}}(\lambda) = k$ is equal to the number of partitions $\mu \vdash n$ with $\gamma_{\mathcal{D}}(\mu) = k$. When $k = 0$ we obtain Euler's Theorem.

As before, let $\mathcal{P}_n$ be the set of all partitions of n. Define the following extension $\overline{\varphi} : \mathcal{P}_n \to \mathcal{P}_n$ of Glaisher's bijection. Start with $\lambda \in \mathcal{P}_n$. Suppose $\lambda = \pi \cup \nu$, where π is a partition into even parts and ν is a partition into odd parts. Divide each part of π into two, and denote this partition by $\pi/2$. Now let $\overline{\varphi}(\lambda) = \varphi(\nu) \cup \pi/2 \cup \pi/2$. Clearly, $\overline{\varphi} : \mathcal{O}_n \to \mathcal{D}_n$.

▼ *The map* $\overline{\varphi} : \mathcal{P}_n \to \mathcal{P}_n$ *defined above is a bijection. Moreover, if* $\overline{\varphi}(\lambda) = \mu$*, then* $\gamma_{\mathcal{O}}(\lambda) = \gamma_{\mathcal{D}}(\mu)$*.*

3.3.2. ($\Diamond$) Find a similar extension of Glaisher's Theorem 3.2.3.

3.4. **Sylvester's bijection.**

3.4.1. *Sylvester's bijection.* The following is a different bijective proof of Euler's Theorem 3.1. In fact, we will present three different bijections giving the same correspondence.

Sylvester's bijection $\psi : \mathcal{O}_n \to \mathcal{D}_n$ is best described by a picture. We arrange all odd parts symmetrically, folding them as hooks, and then read them diagonally, as shown on Figure 12.

For the second bijection $\zeta : \mathcal{O}_n \to \mathcal{D}_n$, we divide the diagram $[\lambda]$ into two parts, along the line $j = 1 + 2i$. Read the parts above and below as diagrams of partitions α and β', respectively. Now let $\zeta(\lambda) = (2 \cdot (\alpha/2)')' + \beta$, see Figure 13.

To exhibit the third bijection, define $\eta : \mathcal{O}_n \to \mathcal{D}_n$ as follows. Draw a 2-modular diagram $[\lambda]_2$ corresponding $\lambda \in \mathcal{O}_n$. Draw successive hooks $H_1, H_2, \ldots,$ as in

Figure 14. Let μ_1 be the number of squares in H_1, let μ_2 be the number of 2-s in H_1, let μ_3 be the number of squares in H_2, let μ_4 be the number of 2-s in H_2, etc. Now let $\eta(\lambda) = \mu$.

> ▼ *The maps $\psi, \zeta, \eta : \mathcal{O}_n \to \mathcal{D}_n$ are bijections giving identical correspondence: $\psi = \zeta = \eta$.*

3.4.2. ($\Diamond$) *Fine's Theorem.* Prove that the number of partitions $\mu \in \mathcal{D}_n$ with $a(\mu) = k$ is equal to the number of partitions $\lambda \in \mathcal{O}_n$ with $a(\lambda) + 2\ell(\lambda) = 2k + 1$.

3.4.3. ($\Diamond$) Let $\mathcal{O}_n^1$ and $\mathcal{O}_n^3$ be the sets of partitions λ of n into odd parts, such that the largest part $a(\lambda) = \lambda_1 \equiv 1$ and $3 \bmod 4$, respectively. Let $\mathcal{D}_n^0$ and $\mathcal{D}_n^1$ be the sets of partitions λ of n into distinct parts, such that the largest part $a(\lambda) = \lambda_1$ is even and odd, respectively. Apply Fine's Theorem to show that $\psi : \mathcal{O}_n^1 \to \mathcal{D}_n^0$, $\mathcal{O}_n^3 \to \mathcal{D}_n^1$, when n is even, and $\psi : \mathcal{O}_n^1 \to \mathcal{D}_n^1$, $\mathcal{O}_n^3 \to \mathcal{D}_n^0$, when n is odd. Use 5.2.2 to compute $|\mathcal{O}_n^1| - |\mathcal{O}_n^3|$.

3.4.4. ($\circ$) *Sylvester's Theorem.* Let $\gamma(\lambda)$ be the number of distinct parts in $\lambda \in \mathcal{P}$. For every $\mu \in \mathcal{D}$, let $\zeta(\mu)$ be the number of *contiguous sequences* of parts in μ, i.e. the number sequences of consecutive integers in a partition $(\mu_1, \mu_2, \dots)$. Prove that the number of partitions $\lambda \in \mathcal{O}_n$ with $\gamma(\lambda) = k$ is equal to the number of partitions $\mu \in \mathcal{D}_n$ with $\zeta(\lambda) = k$, for all $n \ge k \ge 1$.

3.4.5. ($\circ$) Denote by $|\lambda|_a$ the alternating sum of parts of a partition λ: $|\lambda|_a = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \dots$ Prove that the number of partitions $\lambda \vdash n$ into k odd parts is equal to the number of partitions $\mu \vdash n$ into distinct parts with $|\mu|_a = k$.

3.4.6. ($\circ$) Let $\lambda \in \mathcal{P}$ have *type* (c, m) if the parts appear, alternately, starting with the largest part, c times, $(m - c)$ times, c times, $(m - c)$ times, etc. Let $1 \le c < m$. Generalize Sylvester's bijection to prove that the number of partitions of n with parts $\equiv c \bmod m$ is equal to the number of partitions of n of type (c, m). When $c = 1$ and $m = 2$, this is Euler's Theorem. Extend Fine's and Sylvester's theorems to partitions of type (c, m).

3.5. **Iterated Dyson's map.** Let $\mathcal{D}_{n,r}$ be the set of partitions $\mu \in \mathcal{D}_n$ with rank $r(\mu) = r$. Let $\mathcal{U}_{n,2k+1}$ be the set of partitions $\lambda \in \mathcal{O}_n$, such that the largest part $a(\lambda) = 2k + 1$. The following identity is a refinement of Euler's Theorem:

$$|\mathcal{U}_{n,2r+1}| = |\mathcal{D}_{n,2r+1}| + |\mathcal{D}_{n,2r}|.$$

Recall the construction of Dyson's map $\psi_r : \mathcal{H}_{n,r+1} \to \mathcal{G}_{n+r,r-1}$ defined in 2.5.1. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. Consider a sequence of partitions ν^1, ν^2, $\dots$, ν^ℓ, such that $\nu^\ell = (\lambda_\ell)$, and ν^i is obtained by applying Dyson's map ψ_{λ_i} to ν^{i+1}. Now let $\mu = \nu^1$. We shall call new map $\xi : \lambda \to \mu$ the *iterated Dyson's map*. See Figure 15 for an example.

> ▼ *The map $\xi : \mathcal{O}_n \to \mathcal{D}_n$ is a bijection. Moreover, $\xi\big(\mathcal{U}_{n,2r+1}\big) = \mathcal{D}_{n,2r+1} \cup \mathcal{D}_{n,2r}$.*

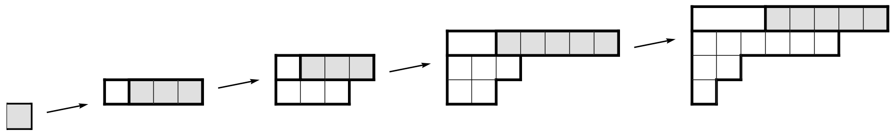

FIGURE 15. The iterated Dyson's map $\zeta : \lambda \to \mu$, where $\lambda = (5,5,3,3,1) \in \mathcal{U}_{17,5}$ and $\mu = (8,6,2,1) \in \mathcal{D}_{17,4}$.

4. Partition Theorems of Lebesgue, Göllnitz and Schur

4.1. **Lebesgue identity.** The following result is called *Lebesgue identity*:

$$\sum_{r=1}^{\infty} t^{\binom{r+1}{2}} \frac{(1+zt)(1+zt^2)\cdots(1+zt^r)}{(1-t)(1-t^2)\cdots(1-t^r)} = \prod_{i=1}^{\infty}(1+zt^{2i})(1+t^i)\,.$$

Note that when $z=0$, we obtain Euler's identity 2.3.4. We present two bijective proofs, both of which introduce different intermediate set of partitions. The resulting correspondences are also different.

First, let us restate the identity in combinatorial language. Recall that for a partition λ, we denote $a(\lambda) = \lambda_1$ and $\ell(\lambda) = \lambda_1'$. Let $\mathcal{V}_{n,k}$ be the set of pairs of partitions (λ,μ), such that $\lambda,\mu \in \mathcal{D}$, $|\lambda|+|\mu| = n$, $\ell(\mu)=k$, and $a(\mu) \le \ell(\lambda)$. Let $\mathcal{E}_{n,k}$ be the set of pairs of partitions (σ,τ), such that $|\sigma|+|\tau|=n$, $\ell(\sigma)=k$, and σ is a partition into even parts.

▼ *Lebesgue identity is equivalent to the following partition theorem:* $|\mathcal{V}_{n,k}| = |\mathcal{E}_{n,k}|$, *for all* $n,k \ge 0$.

▶ Observe that adding Sylvester's triangle as in the proof of Euler's identity 2.3.4 combines $t^{\binom{r+1}{2}}$ and the denominator on the l.h.s. into a generating function for partitions λ into distinct parts, with $\ell(\lambda)=r$. The product in the numerator is a generating function for partitions μ into distinct parts with $a(\mu) \le r$. Summing over all $r \le k = \ell(\mu)$, we see that the l.h.s. is equal to $\sum_{n,k} |\mathcal{V}_{n,k}|\, t^n z^k$. For the r.h.s. of the identity, the result is straightforward. ■

4.2. First proof of Lebesgue identity.

4.2.1. *Bessenrodt's bijection.* We start by introducing an intermediate set of partitions. Let $\mathcal{R}_{n,k}$ be the set of all partitions $\pi \vdash n$, such that π has exactly k even parts, where these even parts are not repeated. We shall prove that

$$|\mathcal{V}_{n,k}| = |\mathcal{R}_{n,k}| = |\mathcal{E}_{n,k}|.$$

The second equality is straightforward. Start with $(\sigma,\tau) \in \mathcal{E}_{n,k}$. We have $\tau \in \mathcal{D}_m$, for some $m \le n$. By Euler's Theorem 3.1, $|\mathcal{D}_m| = |\mathcal{O}_m|$. Now let $\omega = \psi^{-1}(\tau)$, where $\psi : \mathcal{O}_m \to \mathcal{D}_m$ is Sylvester's bijection 3.4.1. Join the parts of ω and σ together, to form a partition π. Note that $\pi \in \mathcal{R}_{n,k}$ as desired.

For the first equality, we shall construct a map $\varrho : \mathcal{R}_{n,k} \to \mathcal{V}_{n,k}$, by using 2-modular diagrams of partitions. Start with the 2-modular diagram $[\pi]_2$ of a partition $\pi \in \mathcal{R}_{n,k}$. Mark the last squares in each row whenever it's a 2. Since even parts are not repeated in π, no two marked squares lie in the same column or row. For every marked square below the main diagonal $i=j$, remove the row of 2-s which contains it. For every marked square on or above the main diagonal $i=j$,

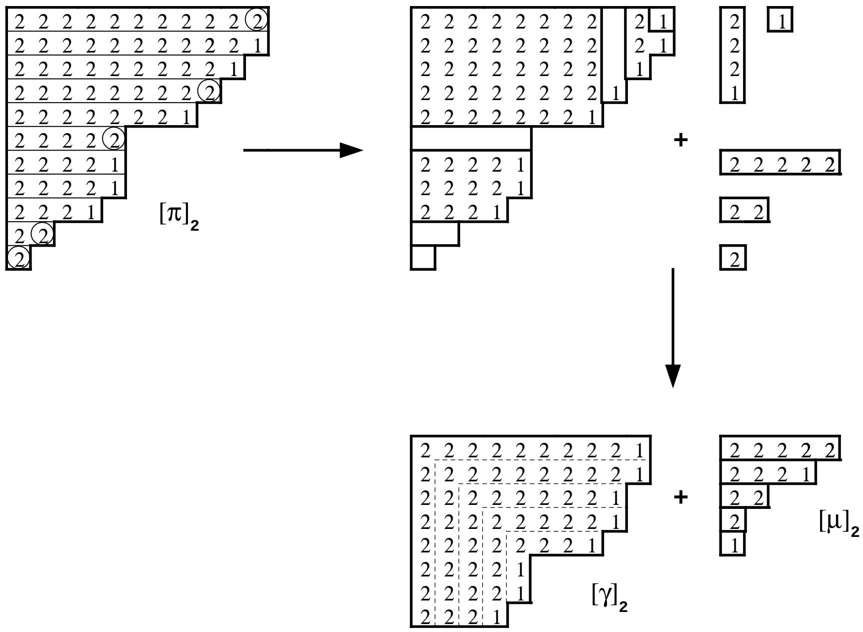

FIGURE 16. Example of the bijection ϱ.

remove the column of 2-s which is above it, replacing the marked square with a 1, and attaching one square with a 1 to the column (see Figure 16). Denote by $[\gamma]_2$ the remaining 2-modular diagram. Observe that $\gamma \in \mathcal{O}$. Now let $\lambda = \psi(\gamma) \in \mathcal{D}$.

Now conjugate all the removed columns and join them with the removed rows in a 2-modular diagram, which we denote by $[\mu]_2$. Define by $\varrho(\pi) = (\lambda, \mu)$. Note that $\lambda, \mu \in \mathcal{D}$, $|\lambda| + |\mu| = |\pi| = n$, and $\ell(\mu)$ is the number k of even parts in π. Finally, the geometry of the construction guarantees that the length of each removed row or column is at most the size of the Durfee square δ_r in a diagram $[\pi]_2$. This translates into $a(\mu) \le \ell(\lambda)$, and implies that $(\lambda, \mu) \in \mathcal{V}_{n,k}$.

▼ *The map $\varrho : \mathcal{R}_{n,k} \to \mathcal{V}_{n,k}$ is a bijection.*

4.2.2. ($\lozenge$) An example of the bijection $\varrho : \mathcal{R}_{n,k} \to \mathcal{V}_{n,k}$ is given in Figure 16. The 2-modular diagram $[\pi]_2$ of the partition $\pi = (22, 21, 19, 18, 15, 10, 9^2, 7, 4, 2) \in \mathcal{R}_{136,5}$ is mapped into $[\gamma]_2$ and $[\mu]_2$, where $\gamma = (19^2, 17^2, 15, 9^2, 7)$, $\mu = (10, 7, 4, 2, 1)$. The last step, corresponding to Sylvester's bijection ψ, is not drawn but the corresponding hooks are indicated on $[\gamma]_2$ by dashed lines (cf. Figure 14). Now $\lambda = \psi(\gamma) = (17, 16, 15, 14, 12, 11, 10, 8, 6, 3)$, and $(\lambda, \mu) = \varrho(\pi)$. Note that the Durfee square $\delta_5 \subset [\gamma]_2$ has 2 in its lower right corner (it always coincides with the corner of the last hook). This corresponds to $r = \ell(\lambda) = 10$, and $a(\mu) = 10 \le r$.

4.2.3. ($\lozenge$) Define a natural extension of Sylvester's statistic $\zeta(\lambda)$ to this case. Find the corresponding partition identity.

4.2.4. ($\circ$) Generalize the above construction to partitions of type (c, m).

4.3. **Göllnitz Theorem.**

4.3.1. The following result for partitions is not directly related to Euler Theorem, but has a similar flavor. The bijective proof will also be helpful in the next section in motivating the second proof of the Lebesgue identity.

Let $\mathcal{A}_n$ be the set of partitions $\lambda \vdash n$ into parts $\equiv 1, 5$ or $6 \bmod 8$. Let $\mathcal{H}_n$ be the set of partitions $\mu \vdash n$ into parts which differ by at least 2, and such that odd parts differ by at least 4.

▼ Göllnitz Theorem. $|\mathcal{A}_n| = |\mathcal{H}_n|$.

Denote by $\mathcal{B}_n \subset \mathcal{D}_n$ set of all partitions $\lambda \vdash n$ into distinct parts, such that $\lambda_i \equiv 0, 1$ or $2 \bmod 4$. Recall that $|\mathcal{A}_n| = |\mathcal{B}_n|$, as described in 3.2.2. Thus it remains to show that $|\mathcal{B}_n| = |\mathcal{H}_n|$.

4.3.2. *Bressoud's bijection.* We construct a bijection $\xi : \mathcal{B}_n \to \mathcal{H}_n$, again by using 2-modular diagrams and standard MacMahon diagrams.

Start with a partition $\lambda \in \mathcal{B}_n$. Let k be the number of parts $\equiv 1 \bmod 4$ in λ. Map it into a 2-modular diagram $[\lambda]_2$. Map $[\lambda]_2$ into a MacMahon diagram $\big[\nu, \boxplus\big]$, which has exactly k marked squares. Split $\big[\nu, \boxplus\big]$ into two Young diagrams $[\alpha]$, $[\beta]$ and a standard MacMahon diagram $\big[\gamma, \boxplus\big]$. Namely, let $\big[\gamma, \boxplus\big]$ contain all rows with marked squares (they are shaded in Figure 17), and let $[\alpha]$ and $[\beta]$ consist of parts $> k$ and $\leq k$, respectively (and no marked squares).

Now let $\big[\upsilon, \boxplus\big] = [\beta]' + \big[\gamma, \boxplus\big]$. Clearly, $\ell(\upsilon) = k$. Attach $\big[\upsilon, \boxplus\big]$ right below $[\alpha]$ and remove Sylvester's triangle $\big[\rho_m, \boxplus\big]$, where $m = \ell(\alpha) + k - 1 = \ell(\alpha + \upsilon)$. This is possible indeed since the smallest of $[\alpha]$ is $> k$. Rearrange the remaining m rows in a nonincreasing order, and add back to $\big[\rho_m, \boxplus\big]$. Convert the resulting standard MacMahon diagram $\big[\omega, \boxplus\big]$ into a 2-modular diagram, and then to a Young diagram $[\mu]$. Let $\mu = \xi(\lambda)$.

▼ *The map ξ defined above is a bijection between $\mathcal{B}_n$ and $\mathcal{H}_n$.*

▶ First, observe that ξ is well defined. Indeed, in the standard MacMahon diagram $\big[\gamma, \boxplus\big]$ all rows are marked and have distinct odd lengths. This is preserved in $\big[\omega, \boxplus\big]$. Thus, the standard MacMahon diagram $\big[\omega, \boxplus\big]$ has all parts of distinct length, and no two rows with marked squares are adjacent. This translates into $\mu \in \mathcal{H}_n$.

For the inverse map, start with a Young diagram $[\mu]$, and convert it into a MacMahon diagram $\big[\omega, \boxplus\big]$. Remove Sylvester's triangle ρ_m, where $m = \ell(\mu) - 1$. Reorder the remaining rows, so that all k rows with marked squares are on the bottom. Split the resulting diagram into $[\alpha]$ and $\big[\upsilon, \boxplus\big]$. Remove columns from $\big[\upsilon, \boxplus\big]$, depending on the parity of the distance between adjacent marked squares, to obtain $\big[\gamma, \boxplus\big]$. Collect the removed columns of distinct length $\leq k$ into a Young diagram $[\alpha]'$. Now let $\big[\nu, \boxplus\big] = [\alpha] \cup [\beta] \cup \big[\gamma, \boxplus\big]$. Now convert the MacMahon diagram $\big[\nu, \boxplus\big]$ into a partition λ. The rest of the proof is straightforward. ■

4.3.3. ($\Diamond$) Check that the number k of odd parts in λ is equal to the number of odd parts in $\mu = \xi(\lambda)$.

4.3.4. ($\Diamond$) Prove that the number of partitions $\lambda \vdash n$ into parts $\equiv 2, 3$ or $7 \bmod 8$ is equal to the number of partitions $\mu \vdash n$ into parts which differ by at least 2, such that the odd parts differ by at least 4, and the smallest part is at least 2.

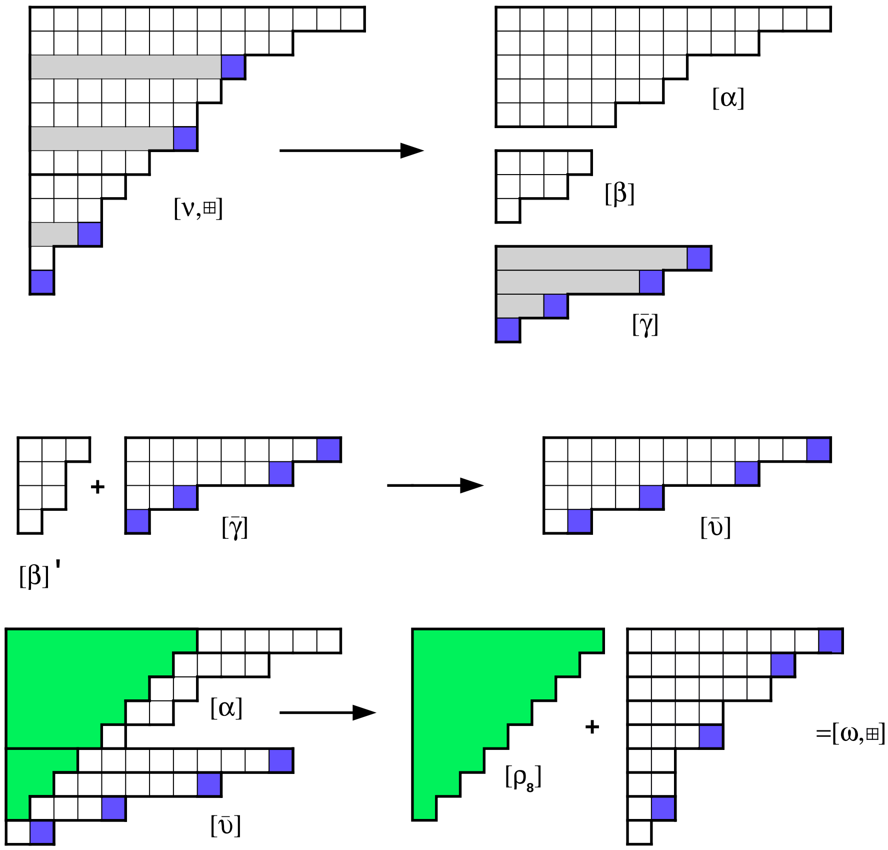

FIGURE 17. Intermediate steps of a bijection $\xi : \lambda \to \mu$, where $\lambda = (28, 22, 17, 16, 14, 13, 10, 8, 6, 5, 2, 1) \in \mathcal{B}_{142}$ and $\mu = (33, 27, 24, 18, 15, 10, 8, 5, 2) \in \mathcal{H}_{142}$, with $k = 4$.

4.3.5. *Bressoud's Theorem.* ($\Diamond$) Let $1 \le r < k$. Prove that the number of partitions $\lambda \vdash n$ into distinct parts $\equiv 0$, r, or k mod $2k$ is equal to the number of partitions $\mu \vdash n$ into parts $\equiv 0$ or r mod k, which differ by at least k, and such that parts $\equiv r$ mod $2k$ differ by at least 4.

Hint: translate both sets of partitions into the language of k-modular diagrams, then into standard MacMahon diagrams, and use the same bijection.

4.4. **Second proof of Lebesgue identity.**

4.4.1. *Alladi–Gordon's bijection.* We say that the i-th row in a diagram $[\lambda]$ has a *gap*, if $\lambda_i - \lambda_{i+1} \ge 2$. Let $\mathcal{G}_{n,k}$ be the set of all MacMahon diagrams with n squares, and k marked squares, such that all rows have distinct length, and every row with a marked square has a gap.

We shall prove that

$$|\mathcal{V}_{n,k}| = |\mathcal{G}_{n,k}| = |\mathcal{E}_{n,k}|,$$

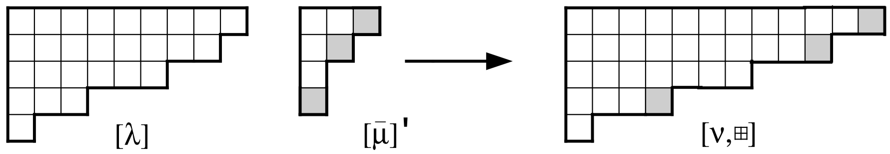

FIGURE 18. Example of a bijection $\varsigma : \mathcal{V}_{34,3} \to \mathcal{G}_{34,3}$.

which implies the Lebesgue identity. For this, construct two bijections $\varsigma : \mathcal{V}_{n,k} \to \mathcal{G}_{n,k}$, and $\kappa : \mathcal{E}_{n,k} \to \mathcal{G}_{n,k}$.

The first bijection is straightforward. Start with $(\lambda, \mu) \in \mathcal{V}_{n,k}$. Mark the last square in each of the k rows in $[\mu]$ to obtain a standard MacMahon diagram $[\overline{\mu}]$ with k marked squares. Now let $\left[\nu, \boxplus\right] = [\lambda] + \left[\overline{\mu}, \boxplus\right]'$, as in Figure 18. Finally, let $\varsigma(\lambda, \mu) = \left[\nu, \boxplus\right]$.

As for the second bijection, follow steps similar to that in 4.3.2. Start with $(\sigma, \tau) \in \mathcal{E}_{n,k}$. Mark the last square in each of the k rows in $[\sigma]$ (all of even length) to obtain a standard MacMahon diagram $[\overline{\sigma}]$ with k marked squares. Split $\tau = \alpha \cup \beta$ into parts $> k$ and $\leq k$, respectively. Consider $[\overline{\sigma}]+[\beta]'$ and attach it right below $[\alpha]$. Now remove Sylvester's triangle ρ_k, rearrange the rows into nonincreasing order, and reassemble them into a standard MacMahon diagram $\left[\nu, \boxplus\right]$. Let $\kappa(\sigma, \tau) = \left[\nu, \boxplus\right]$. In $\left[\nu, \boxplus\right]$, all rows have distinct length, and every row with a marked square (there are exactly k of them) has a gap.

> ▼ *The maps $\varsigma : \mathcal{V}_{n,k} \to \mathcal{G}_{n,k}$ and $\varkappa : \mathcal{E}_{n,k} \to \mathcal{G}_{n,k}$ defined above, are bijections.*

4.4.2. $(**)$ Find a combinatorial proof of the following identity:

$$1 + \sum_{n=1}^{\infty} q^{n^2} \frac{(1+q)(1+q^2)\cdots(1+q^{2n+1})}{(1-q)^2(1-q^2)^2\cdots(1-q^{2n})^2} \, = \, \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n}} \, ,$$

4.5. **Schur's Partition Theorem.**

4.5.1. Let $\mathcal{A}_n$ be the set of partitions of n into parts $\equiv 1$ or $5 \bmod 6$. Let $\mathcal{B}_n$ be the set of partitions of n into distinct parts $\equiv \pm 1 \bmod 3$. Finally, let $\mathcal{S}_n$ be the set of partitions of n with minimal difference 3 between parts, and no two parts which are consecutive multiples of 3. *Schur's Partition Theorem* states that

$$|\mathcal{A}_n| \, = \, |\mathcal{B}_n| \, = \, |\mathcal{S}_n|.$$

While the first equality is elementary and can be proved in a manner similar to Euler's Theorem 5.1.1 (see also 8.1.3), the second equality is more involved and will be proved here by an explicit bijection $\varphi : \mathcal{B}_n \to \mathcal{S}_n$.

Start with a partition $\lambda \in \mathcal{B}_n$. Consider a 3-modular diagram $[\lambda]_3$ (see 2.1.4). By the definition of $\mathcal{B}_n$, all rows in $[\lambda]_3$ are distinct and end with a 1 or 2 square. Working from the bottom to the top row, arrange rows into pairs and single rows by the following rules. Only rows which differ by 1 or 2 can form a pair. If this holds, and λ_i is not paired yet, pair the rows λ_{i-1} and λ_i if either $i = \ell(\lambda)$,

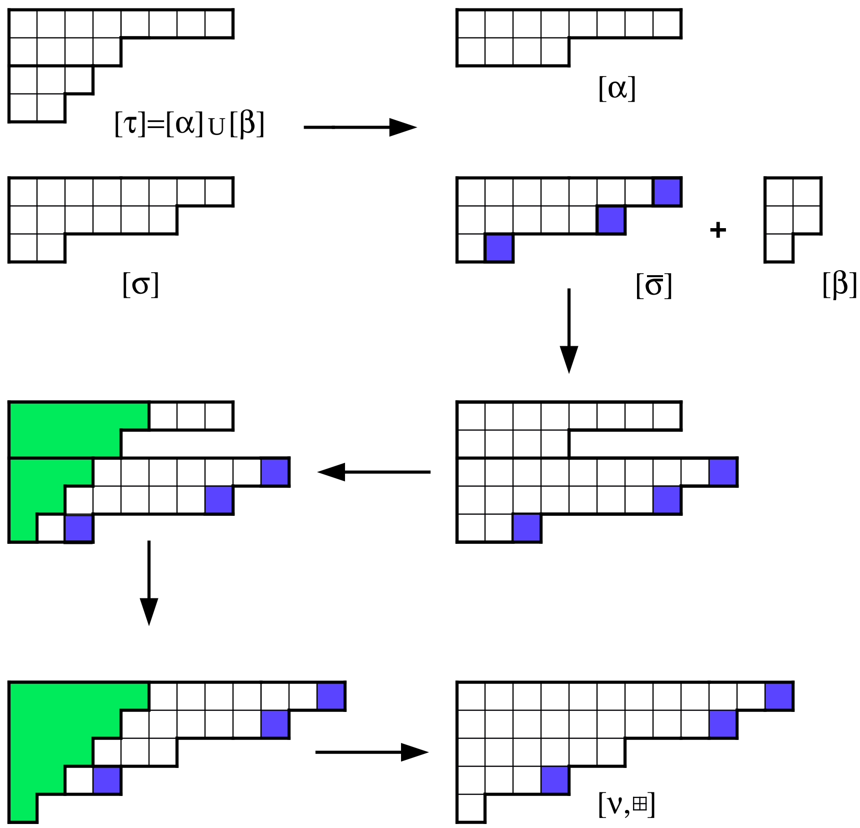

FIGURE 19. Example of a bijection $\kappa : \mathcal{V}_{33,3} \to \mathcal{G}_{33,3}$.

or $\lambda_i - \lambda_{i+1} \geq 3$, or the previous two rows λ_{i+1} and λ_{i+2} form a pair. Now add paired row to each other to form a single 3-modular diagram. Now remove Sylvester's triangle (see 2.3.4), rearrange the rows in nonincreasing order and add Sylvester's triangle back again. Denote the resulting 3-modular diagram by $[\mu]_3$, and let $\varphi(\lambda) = \mu$ (see Figure 20).

▼ *The map φ defined above is a bijection between $\mathcal{B}_n$ and $\mathcal{S}_n$.*

4.5.2. *Bressoud's Generalization.* ($\Diamond$) Fix integers r and m, such that $r < m/2$. Then the number of partitions of n into distinct parts $\equiv \pm r \bmod m$ is equal to the number of partitions of n into parts $\equiv 0, \pm r \bmod m$, with minimal difference m between parts, and no two parts are consecutive multiples of m. Prove this theorem by converting the 3-modular diagrams, used in the bijection above, into m-modular ones.

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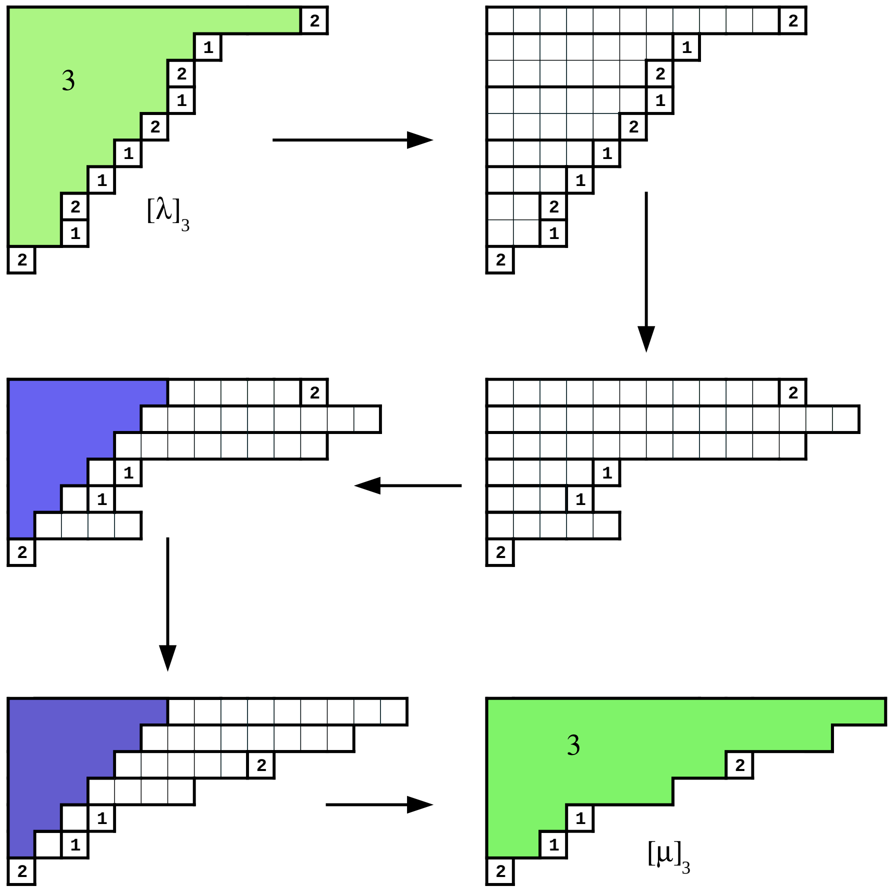

FIGURE 20. An example of bijection $\varphi : [\lambda]_3 \to [\mu]_3$, defined above. Here $\lambda = (35, 22, 20, 19, 17, 13, 10, 8, 7, 2) \in \mathcal{B}_{153}$, and $\mu = (45, 39, 29, 21, 10, 7, 2) \in \mathcal{S}_{153}$.

4.5.3. ($\circ\circ$) Denote by $\mathcal{E}_n$ the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 5, & \text{if } \lambda_i \equiv 0 \bmod 3, \\ 3, & \text{if } \lambda_i \equiv 1 \bmod 3, \\ 2, & \text{if } \lambda_i \equiv 2 \bmod 3. \end{cases}$$

Prove bijectively that $|\mathcal{E}_n| = |\mathcal{S}_n|$.

5. Euler's Pentagonal Theorem

5.1. The identity.

5.1.1. *Euler's Pentagonal Theorem* is the following identity:

$$\prod_{i=1}^{\infty} (1 - t^i) = \sum_{m=-\infty}^{\infty} (-1)^{m-1} t^{\frac{m(3m-1)}{2}}.$$

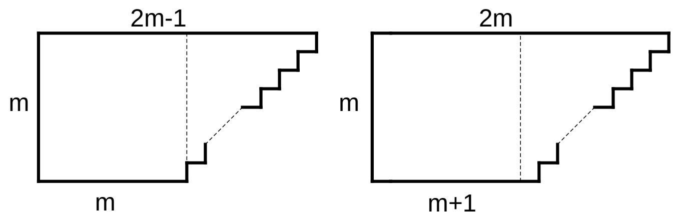

FIGURE 21. Young diagrams $[2m-1, 2m-2, \ldots, m+1, m]$ and $[2m, 2m-1, \ldots, m+2, m+1]$.

The integers $m(3m \pm 1)/2$ are called *Pentagonal numbers* following the ancient Greek tradition.

We present two bijective proofs of the identity in this section. Jacobi's triple product identity, generalizing Euler's Pentagonal Theorem is presented in the next section.

5.1.2. ($\Diamond$) Deduce from 5.1.1 *Euler's recurrence relation*:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \\ - \ldots + (-1)^m \left(n - \frac{m(3m-1)}{2}\right) + \left(n - \frac{m(3m+1)}{2}\right).$$

This formula was used by Euler to tabulate values of $p(n)$. Using asymptotic formula 9.6.1, estimate the complexity of Euler's algorithm for computing the first n values $p(1)$, $p(2)$, $\ldots$, $p(n)$.

5.2. **Franklin's involution.**

5.2.1. Let $\mathcal{D}_n = \mathcal{D}_n^+ \cup \mathcal{D}_n^-$ be the set of partitions into *distinct parts* 2.3.4, and $\mathcal{D}_n^+$, $\mathcal{D}_n^-$ be the subsets with an even and an odd number of parts, respectively. Let $\mathcal{F}$ be the set of pentagonal Young diagrams as in Figure 21. Let $\mathcal{F}_n = \mathcal{F} \cap \mathcal{D}_n$. Clearly, $|\mathcal{F}_n| = 0$ unless $n = m(3m \pm 1)/2$, in which case $|\mathcal{F}_n| = 1$. Thus Euler's Pentagonal Theorem 5.1.1 is *equivalent* to the identity

$$|\mathcal{D}_n^+| - |\mathcal{D}_n^-| = \pm |\mathcal{F}_n|,$$

where the sign is determined by the number of parts of a unique partition in $\mathcal{F}_n$.

Franklin's involution $\alpha : \mathcal{D}_n \to \mathcal{D}_n$ gives a bijective proof of Euler's Pentagonal Theorem. It is defined as follows. First, compare the sizes of *horizontal* and *diagonal* lines of squares in Young diagram $[\lambda]$: $(\ell, 1)$, $(\ell, 2)$, $\ldots$, $(\ell, \lambda_\ell) \in [\lambda]$, and $(1, k)$, $(2, k-1)$, $\ldots \in [\lambda]$, where $\ell = \ell(\lambda)$, $k = a(\lambda)$. Let $s = s(\lambda)$ and $g = g(\lambda)$, respectively, be the lengths of these lines. If $s > g$, move the diagonal line below the horizontal line. Otherwise, (if $s \leq g$), move the horizontal line to the right of the diagonal. If $s = g$, or $s = g + 1$, and the lines have a common square, stay put.

▼ *The above construction gives a sign-reversing involution α on $\mathcal{D}_n$ with the set $\mathcal{F}_n$ as the only possible fixed point.*

▶ Observe that unless we are at a fixed point, the involution changes the number of parts in a partition by one. Thus, α is sign-reversing. Clearly, the involution is well defined and has fixed points only when horizontal and diagonal lines intersect at a point which is to be moved. Thus the set of fixed points is exactly $\mathcal{F}_n$. ■

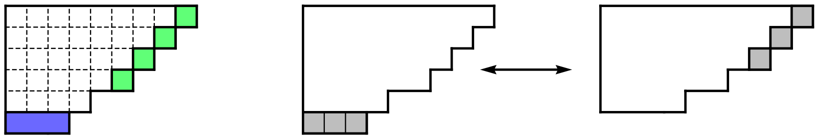

FIGURE 22. Young diagram $[\lambda] = [9, 8, 7, 6, 4, 3]$ with horizontal and diagonal lines of length $s(\lambda) = 3$ and $g(\lambda) = 4$, respectively. An example of Franklin's involution.

5.2.2. $(\diamondsuit)$ Let $\mathcal{D}_n^0$ and $\mathcal{D}_n^1$ be the sets of partitions λ of n into distinct parts, such that the largest part $a(\lambda)$ is even and odd, respectively. Check that Franklin's involution $\alpha : \lambda \to \mu$ satisfies $a(\mu) = a(\lambda) \pm 1$, unless $\lambda \in \mathcal{F}_n$. Conclude that

$$|\mathcal{D}_n^0| - |\mathcal{D}_n^1| = \begin{cases} 1, & \text{if } \ n = k(3k+1)/2, \\ -1, & \text{if } \ n = k(3k-1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

5.2.3. $(\circ)$ Modify Franklin's involution to prove the following version of 5.1.1 :

$$\prod_{i=2}^{\infty}(1-t^i) \, = \, \sum_{m=0}^{\infty} (-1)^{m-1} t^{\frac{m(3m+1)}{2}} (1 + t + t^2 + \ldots + t^{2m}) \, .$$

5.2.4. $(\circ)$ Extend Franklin's involution to obtain the following identity, refining Euler's Pentagonal Theorem:

$$1 + \sum_{k=1}^{m} (-1)^k \left(t^{\frac{k(3k-1)}{2}} + t^{\frac{k(3k+1)}{2}} \right) \, = \, \sum_{0 \le r \le m} (-1)^i t^{rm + \binom{r+1}{2}} \prod_{r < i \le m} (1-t^i) \, .$$

5.2.5. $(*)$ Do the same for the following identity:

$$\prod_{i=1}^{3m}(1-t^i) \, = \, 1 + \sum_{k=1}^{m} (-1)^k \left(t^{\frac{k(3k-1)}{2}} + t^{\frac{k(3k+1)}{2}} \right) \times$$
$$\times \prod_{j=1}^{k} \left(1 - t^{3m-3k+3j}\right) \prod_{r=1}^{m-k} \left(1 - t^{3m+3k+3r}\right) .$$

5.2.6. $(\circ\circ)$ Use Franklin's involution to prove the following identity:

$$\sum_{k=1}^{\infty} (-1)^k \left[(3k-1) t^{\frac{k(3k-1)}{2}} + (3k) t^{\frac{k(3k+1)}{2}} \right] \, = \, \sum_{n=0}^{\infty} \left[\prod_{i=1}^{\infty} \frac{1}{(1-t^i)} - \prod_{i=1}^{n} \frac{1}{(1-t^i)} \right]$$
$$- \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} \sum_{j=1}^{\infty} \frac{t^j}{(1-t^j)} \, .$$

5.2.7. $(**)$ Let $\mathcal{D}_n^\diamond$ be the set of partitions $\lambda \in \mathcal{D}_n$, with the smallest part $s(\lambda)$ being odd. Find an explicit involution to show that $|\mathcal{D}_n^\diamond|$ is odd if and only if n is a square.

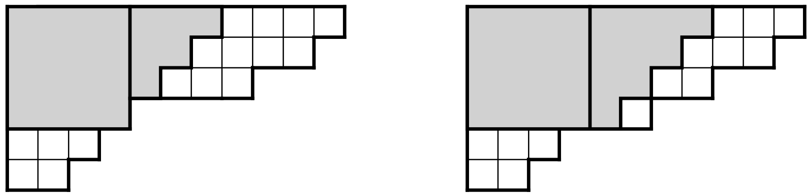

FIGURE 23. Young diagrams $[11, 10, 8, 4, 3, 2]$ and $[11, 10, 8, 6, 3, 2]$, which contain diagrams $[\theta_4]$ and $[\theta'_4]$, respectively.

5.2.8. $(**)$ Let $\mathcal{Q}_n^i$ denote the set of partitions λ, such that $\lambda, \lambda' \in \mathcal{O}_n$, and the number of parts $\ell(\lambda) \equiv i \bmod 4$. Prove combinatorially that

$$|\mathcal{Q}_n^1| - |\mathcal{Q}_n^3| = \begin{cases} (-1)^k, & \text{if } n = 12k^2 + 8k + 1 \text{ or } n = 12k^2 + 16k + 5, \\ 0, & \text{otherwise.} \end{cases}$$

5.3. **Sylvester's identity.**

5.3.1. Consider *Sylvester's identity*:

$$\sum_{n=0}^{\infty} (-1)^n x^n t^{\frac{n(3n+1)}{2}} \left(1 - xt^{2n+1}\right) \prod_{i=1}^{n} \frac{1}{(1-t^i)} \prod_{i=n+1}^{\infty} \frac{1}{(1-xt^n)} = 1\,.$$

Multiplying by $\prod_{n=1}^{\infty}(1-xt^n)$, and setting $z = -x$, we obtain:

$$\sum_{n=0}^{\infty} \left(z^n t^{\frac{n(3n+1)}{2}} + z^{(n+1)} t^{\frac{(n+1)(3(n+1)-1)}{2}}\right) \prod_{i=1}^{n} \frac{(1+zt^i)}{(1-t^i)} = \prod_{i=1}^{\infty} (1+zt^i).$$

Note that when $z = -1$ we obtain Euler's Pentagonal Theorem 5.1.1. On the other hand, this identity can be compared with Euler's identity in 2.3.1. The following bijective proof is based on a modification of the Durfee squares and is similar to that in 2.3.1.

Denote by θ_m, θ'_m pentagonal partitions, as in Figure 21. Suppose $\lambda \in \mathcal{D}_n$ is a partition with distinct parts, and δ_m be the Durfee square in $[\lambda]$. There are two possibilities to consider in this case.

If $\lambda_m = m$, consider $[\lambda] - [\theta_m]$, which is a disjoint union of two diagrams $[\mu]$ and $[\nu]$, such that $[\mu]$ has at most $(m-1)$ parts, while $[\nu]$ has distinct parts of size at most $(m-1)$ (see Figure 23). Take $\varphi(\lambda) = (\mu, \nu, \theta_m)$.

Similarly, if $\lambda_m > m$, consider $[\lambda] - [\theta'_m]$, which is a disjoint union of two diagrams $[\mu]$ and $[\nu]$, such that $[\mu]$ has at most m parts, while $[\nu]$ has distinct parts of size at most m. Take $\varphi(\lambda) = (\mu, \nu, \theta'_m)$

▼ *The map φ defined above is a bijection which proves Sylvester's identity.*

5.3.2. $(\diamondsuit)$ Set $z = -1$ in Sylvester's identity. The two products on the l.h.s. cancel. Instead of cancelling them analytically, use Vahlen's involution 2.3.7. Now, starting with partitions $\lambda \in \mathcal{D}_n$ corresponding to the r.h.s., obtain $\varphi(\lambda) = (\mu, \nu, \theta_m)$ and cancel triples with nonempty μ and ν. Check that the resulting involution is *identical* to Franklin's involution 5.2.1.

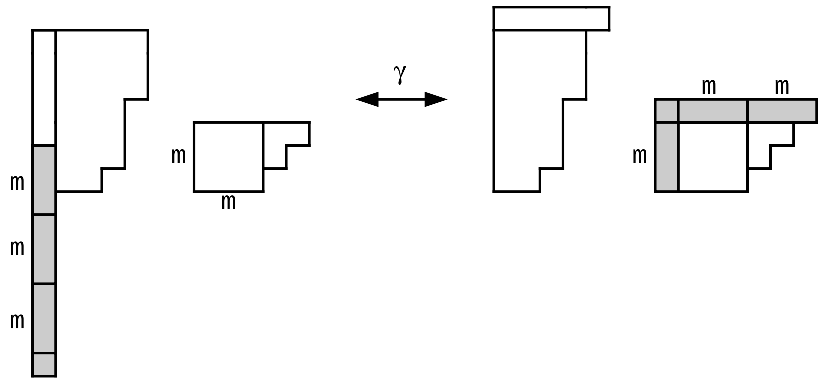

FIGURE 24. Bijection γ proving Euler's recurrence relation.

5.4. **Bijective proof of Euler's recurrence relation.**

5.4.1. Recall Euler's recurrence relation 5.1.2:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \ldots$$

We present an explicit bijection proving it in the following form:

$$\gamma \,:\, \bigcup_{m \,\text{even}} \mathcal{P}_{n-m(3m-1)/2} \;\longleftrightarrow\; \bigcup_{m \,\text{odd}} \mathcal{P}_{n-m(3m-1)/2}\,,$$

where $m \in \mathbb{Z}$ on both sides is allowed to be negative, and the map γ is defined by the following rule:

$$\text{for } \lambda \in \mathcal{P}_{n-m(3m-1)/2}\,, \quad \gamma(\lambda) \,=\, \begin{cases} \psi_{-3m-1}(\lambda), & \text{if } \; r(\lambda)+3m \le 0, \\ \psi^{-1}_{-3m+2}(\lambda), & \text{if } \; r(\lambda)+3m > 0, \end{cases}$$

where ψ_r is Dyson's map 2.5.1.

In Figure 24 we exhibit a pentagonal number by one of the diagrams as in Figure 21. So when m changes, we show how where the change comes from. By definition, γ is a sign-reversing involution.

▼ *The map γ defined above is a bijection.*

5.4.2. ($\Diamond$) Recall the following identity in 2.5.4: $P(t) = 1 + G_0(t) + G_1(t)$, where

$$G_r(t) \,=\, \sum_{m=1}^{\infty} (-1)^{m-1} t^{\frac{m(3m-1)}{2}+rm}\, P(t)\,, \quad \text{and} \quad P(t) \,=\, \prod_{i=1}^{\infty} \frac{1}{(1-t^i)}\,.$$

Deduce from here Euler's Pentagonal Theorem.

5.4.3. ($\circ$) Combine the two involutions in 2.5.5 for $r = 0$ and $r = 1$ to give an involution proving Euler's recurrence relation. Check that this involution is identical to that in 5.4.1.

5.5. **Gauss identity.**

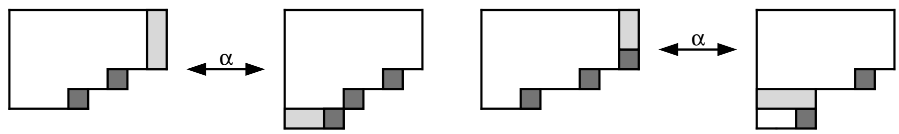

FIGURE 25. Two examples of the involution α.

5.5.1. The following classical *Gauss identity* has an involutive proof:

$$\prod_{m=1}^{\infty} \frac{(1-t^m)}{(1+t^m)} \; = \; \sum_{r=-\infty}^{\infty} (-1)^r \, t^{r^2} \, .$$

First, interpret the coefficient of t^n on the l.h.s. as the sum of $(-1)^{\ell(\lambda)}$, over all standard MacMahon diagrams $\bigl[\lambda, \boxplus\bigr]$ of shape $\lambda \vdash n$. We shall define a sign-reversing involution on $\bigl[\lambda, \boxplus\bigr]$ with no fixed points unless n is a square.

In a Young diagram $[\lambda]$ define a *horizontal line* and a *vertical line* to be the bottom row and the rightmost column. As before, let $s = s(\lambda)$ be the length of the horizontal line. Similarly, let $f = f(\lambda) = m_{a(\lambda)}$ be the length, and let $q = q\bigl[\lambda, \boxplus\bigr]$ be the number of unmarked squares of the vertical line in $[\lambda]$. We say that a column (row) is *marked* if it contains a marked square. By the definition of a standard MacMahon diagram, $f = q + 1$ if the vertical line is a marked column, and $f = q$ otherwise.

Now, if $f < s$, or $q < f = s$, attach a row of length f to the horizontal line; make it marked if the vertical line was unmarked, or vice versa (see Figure 25). Conversely, if $s < f$, or $s = q = f$, attach a column of length s to the vertical line and make it marked if the horizontal line was unmarked, or vice versa. Denote by α the involution we obtain.

There are four exceptional cases when α is undefined: when $\bigl[\lambda, \boxplus\bigr]$ is an $r \times (r+1)$ rectangle with no marked squares, an $(r+1) \times r$ rectangle with one marked square, and an $r \times r$ rectangle with or without a marked square. The first two cases cancel each other, while the last two give the terms on the r.h.s.

> ▼ *The map α defined above gives a sign-reversing involution with square shaped standard MacMahon diagrams as fixed points.*

5.5.2. ($\Diamond$) Deduce from the proof:

$$1 - 2\sum_{m=1}^{\infty} \frac{(1-zt)(1-zt^2)\cdots(1-zt^{m-1})\, z^{m+1}\, t^m}{(1+zt)(1+zt^2)\cdots(1+zt^m)} \; = \; 1 - 2\sum_{r=1}^{\infty} z^{2r}\, (-1)^r \, t^{r^2}$$

5.5.3. ($\circ$) Modify the previous argument to prove another *Gauss identity*:

$$\prod_{n=0}^{\infty} \frac{(1-t^{2m})}{(1-t^{2m-1})} \; = \; \sum_{n=0}^{\infty} t^{\frac{n(n+1)}{2}} \, .$$

5.5.4. ($\circ$) Deduce both Gauss identities above and identity 5.5.2 from the Rogers-Fine identity 2.3.3.

6. Jacobi's triple product identity

6.1. Variations on the theme.

6.1.1. The following summation is known as the *Jacobi identity* or the *triple product identity*:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} \, = \, \prod_{k=1}^{\infty}(1-q^{2k})(1+zq^{2k-1})(1+z^{-1}q^{2k-1})\,.$$

6.1.2. ($\Diamond$) Deduce Euler's Pentagonal Theorem from the Jacobi identity.

6.1.3. ($\Diamond$) Deduce Gauss identities 5.5.1, 5.5.3 from the Jacobi identity.

6.1.4. ($\circ$) Deduce the following *Gauss identity*:

$$\prod_{i=1}^{\infty}(1-t^i)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)\,t^{\frac{n(n+1)}{2}}\,,$$

6.1.5. ($\circ$) *Vahlen's Theorem.* Let $\epsilon(m)$ be the integer $i \in \{-1,0,1\}$, such that $m \equiv i \bmod 3$. Let $\epsilon(\lambda) = \epsilon(\lambda_1) + \epsilon(\lambda_2) + \ldots \in \mathbb{Z}$. Define $\mathcal{O}_{n,k} = \{(\lambda_1,\lambda_2,\ldots) \in \mathcal{O}_n : \epsilon(\lambda) = k\}$. Then for all $k \in \mathbb{Z}$:

$$\sum_{\lambda \in \mathcal{O}_{n,k}} (-1)^{\ell(\lambda)} \; = \; \begin{cases} (-1)^k\,, & \text{if } \; n = \dfrac{k(3k-1)}{2} \\ 0\,, & \text{otherwise.} \end{cases}$$

6.1.6. ($**$) Prove combinatorially the *quintuple product identity*:

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}\, z^{3n}\,(1+zq^n)$$
$$= \prod_{n=0}^{\infty}(1+z^{-1}q^{n+1})(1+zq^n)(1-z^{-2}q^{2n+1})(1-z^2q^{2n+1})(1-q^{n+1}).$$

6.1.7. ($**$) Find a combinatorial proof of the following *Ramanujan's identity*:

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-aq)(1-aq^2)\cdots(1-aq^n)\,(1-bq)(1-bq^2)\cdots(1-bq^n)}$$
$$= (1-a^{-1})\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2} b^n a^{-n}}{(1-bq)(1-bq^2)\cdots(1-bq^n)}\right)$$
$$+\, a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} b^n a^{-n} \prod_{k=1}^{\infty}\frac{1}{(1-aq^k)(1-bq^k)}\,.$$

6.2. Direct bijection.

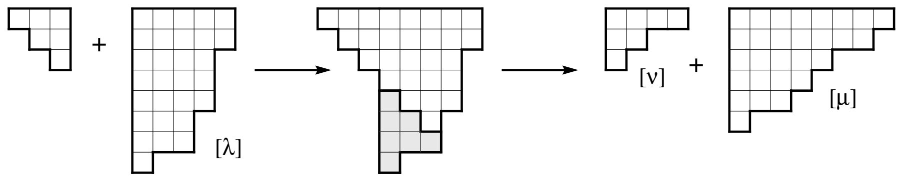

FIGURE 26. The case $k = 3$, $n = 35$. The bijection $\varphi : \lambda \to (\mu, \nu)$, where $\lambda = (5^2 4^3 3^2 1) \vdash 29 = 35 - \binom{4}{2}$, $\mu = (8, 7, 5, 4, 3, 1)$, $\nu = (4, 2, 1)$. Note that $\ell(\mu) - \ell(\nu) - k = 0$.

6.2.1. We start with the following equivalent form of the Jacobi identity:

$$\sum_{k=-\infty}^{\infty} s^k t^{\frac{k(k+1)}{2}} \prod_{i=1}^{\infty} \frac{1}{1 - t^i} = \prod_{i=1}^{\infty} (1 + st^i) \prod_{j=0}^{\infty} (1 + s^{-1} t^j).$$

The coefficient of $s^r t^n$ on the l.h.s. can be interpreted as the number of partitions in $\mathcal{P}_{n-k(k+1)/2}$. On the r.h.s. we have $|\mathcal{W}_{n,k}|$, where

$$\mathcal{W}_{n,k} = \big\{(\mu, \nu) : \ \mu \in \mathcal{D}', \nu \in \mathcal{D}, \, |\mu| + |\nu| = n, \, \ell(\mu) - \ell(\nu) = k\big\}.$$

We define a map $\varphi = \varphi_{n,k} : \mathcal{P}_{n-k(k+1)/2} \to \mathcal{W}_{n,k}$ as in Figure 26. We start with $\lambda \in \mathcal{P}_{n-k(k+1)/2}$ and the integer k. First, arrange $\binom{k+1}{2}$ squares into a rotated Sylvester's triangle, and attach it sideways to diagram $[\lambda]$. When $k < 0$, attach the triangle on the other side of $[\lambda]$. Then split the obtained diagram along the $i - j = k$ diagonal, and read columns below the diagonal, and rows on or above the diagonal. This gives us two partitions: μ into distinct parts, and ν into nonnegative distinct parts.

▼ *The map φ defined above is a bijection between $\mathcal{P}_{n-k(k+1)/2}$ and $\mathcal{W}_{n,k}$.*

Note that when $k = 0$, the above bijection φ give Frobenius coordinates 2.3.5.

6.2.2. Let us present here another direct bijection to prove the Jacobi identity in essentially the same form as in 6.2.1. In fact, we present here two different bijections defining the same correspondence.

The first bijection is essentially the same as the bijection ϕ in 6.2.1, with a substitution $t = q^2$ and $s = z/q$:

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} \prod_{r=1}^{\infty} \frac{1}{1 - q^{2r}} = \prod_{i=1}^{\infty} (1 + zq^{2i-1}) \prod_{i=1}^{\infty} (1 + z^{-1} q^{2i-1}) \, .$$

Define $\mathcal{D}^\circ = \mathcal{D} \cap \mathcal{O}$ to be the set of partitions into distinct odd parts. Let $\mathcal{V}_{n,k} = \{(\mu, \nu) : \ \mu, \nu \in \mathcal{D}^\circ, \, |\mu| + |\nu| = n, \, \ell(\nu) - \ell(\mu) = k\}$. We present map $\phi = \phi_{n,k} : \mathcal{P}_{(n-k^2)/2} \to \mathcal{V}_{n,k}$ in Figure 27. Here the squares of the intermediate diagrams are divided into two triangles, to account for the length of the resulting partitions being odd.

The second bijection is rather unusual, in a sense that we allow diagrams to overlap to our advantage. We shall use Sylvester's idea for representing partitions with odd parts as a stack of hooks (see 3.4.1, Figure 12.) Start with a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ into even parts and convert it into a partition $\lambda/2 = (\lambda_1/2, \lambda_2/2, \ldots)$,

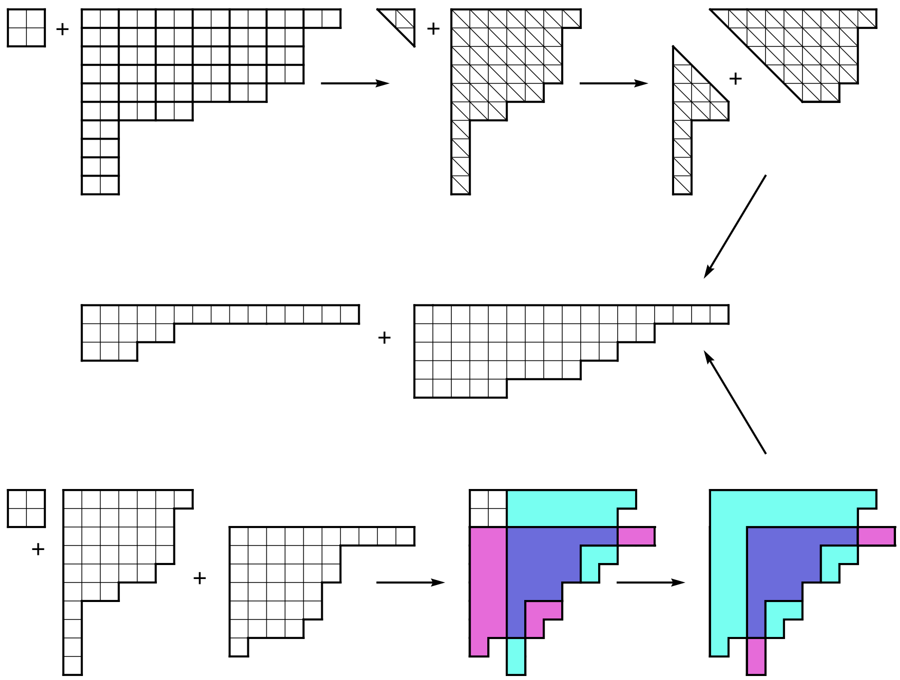

FIGURE 27. The case $k = 2$, $n = 78$. Bijections $\phi, \eta : \lambda \to (\mu, \nu)$, where $\lambda = (14, 12^3, 10, 6, 2^4) = 2 \cdot (7, 6^3, 5, 3, 1^4) \vdash 74 = 78 - 2^2$, $\mu = (17, 13, 11, 9, 5)$, $\nu = (15, 5, 3)$. Note that $\ell(\mu) - \ell(\nu) - k = 0$.

and its conjugate $(\lambda/2)'$. Now overlap their Young diagrams so that a k-square fits in the upper left corner. View the resulting arrangement of squares (counted with multiplicity) as a superimposed picture of two stacks of hooks, corresponding to partitions $\mu, \nu \in \mathcal{D}^\circ$. Denote this map by η.

▼ *The maps ϕ, and η are identical bijections between $\mathcal{P}_{(n-k^2)/2}$ and $\mathcal{V}_{n,k}$.*

6.2.3. ($\lozenge$) Deduce from the proof the following *MacMahon's identity*:

$$\prod_{i=1}^{m}(1 + zq^{2i-1}) \prod_{j=1}^{n}(1 + z^{-1}q^{2j-1}) = \sum_{k=-n}^{m} z^k\, q^{k^2} \binom{m+n}{k+n}_{q^2}.$$

Check that as $m, n \to \infty$ we obtain Jacobi identity.

6.3. **Involutive proof.** We present a sign-reversing involution proving the Jacobi identity in the following form:

$$\prod_{n=1}^{\infty}(1-u^n v^{n-1})(1-u^{n-1}v^n)(1-u^n v^n) = 1 + \sum_{k=1}^{\infty}(-1)^n \left(u^{\binom{k+1}{2}} v^{\binom{k}{2}} + u^{\binom{k}{2}} v^{\binom{k+1}{2}}\right).$$

Setting $q^2 = uv$, $z = -u/v$, we obtain Jacobi identity as in 6.1.1.

Let Λ_-, Λ_0, and Λ_+ be the sets of pairs of partitions $\{(\lambda, \lambda^-)\}$, $\{(\lambda, \lambda)\}$, and $\{(\lambda^-, \lambda)\}$, respectively, where $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{D}$, and $\lambda^- = (\lambda_1 - 1, \lambda_2 - 1, \dots)$. We use the notation (ν, ν^+) for elements of Λ_+. Let $\Lambda = \Lambda_- \times \Lambda_0 \times \Lambda_+$, and let

$$\mathcal{A}_{m,n} := \left\{ M = \begin{pmatrix} \lambda,\, \mu,\, \nu \\ \lambda^-, \mu, \nu^+ \end{pmatrix} \in \Lambda : \ |\lambda| + |\mu| + |\nu| = m,\, |\lambda^-| + |\mu| + |\nu^+| = n \right\}$$

be the set of triples of pairs of partitions. The sign of such a triple is defined by

$$\varepsilon(M) = \varepsilon\begin{pmatrix} \lambda,\, \mu,\, \nu \\ \lambda^-, \mu, \nu^+ \end{pmatrix} := (-1)^{\ell(\lambda)+\ell(\mu)+\ell(\nu^+)} \in \{\pm 1\}.$$

Now the Jacobi identity is reduced to the following summation:

$$\sum_{M \in \mathcal{A}_{m,n}} \varepsilon(M) = \begin{cases} (-1)^k, & \text{if } m = \binom{k+1}{2}, n = \binom{k}{2}, \text{ or } m = \binom{k}{2}, n = \binom{k+1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the sum on the l.h.s. is symmetric, so it suffices to calculate it for $m \geq n$. In this case we shall cancel all terms except for $F_k = \binom{\rho_{k+1}, 0, 0}{\rho_k, 0, 0}$, where $m = \binom{k+1}{2}$, $n = \binom{k}{2}$, and $\rho_r = (r, r-1, \dots, 2, 1)$ for all $r > 0$. We say that λ is *triangular*, if $\lambda = \rho_k$ for some $k > 0$.

The proof follows the same idea as Franklin's proof 5.2.1. Let $s(\lambda)$ be the length of the horizontal line in $[\lambda]$, and let $g(\lambda)$ be the length of the diagonal line defined as in 5.2.1. Clearly, $g(\lambda^-) = g(\lambda)$ unless λ is triangular.

Let $M = \binom{\lambda, \mu, \nu}{\lambda^-, \mu, \nu^+} \in \mathcal{A}_{m,n}$, and $m \geq n$. Consider two cases: $g(\lambda) \geq s(\mu)$, and $g(\lambda) < s(\mu)$, with λ not triangular. Move the diagonal line from $[\lambda]$ to the horizontal line in $[\mu]$, or vice versa; and the same for $[\lambda^-]$ and $[\mu]$.

Now suppose $\lambda = \rho_k$ and $k = g(\lambda) < s(\mu)$. Consider another two cases: $s(\mu) > k + s(\nu)$, and $s(\mu) \leq k + s(\nu)$, with $\nu \neq \emptyset$. Move the largest part of $[\lambda]$ and the horizontal line in $[\nu]$ to combine into the horizontal line in $[\mu]$, or vice versa; repeat the same for $[\lambda^-]$ and $[\nu^+]$.

We demonstrate the map φ defined above in both cases in Figure 28, where we present only partitions (λ, μ, ν), omitting the matching triple (λ^-, μ, ν^+). Note that φ changes parity in $\ell(\lambda) + \ell(\mu) + \ell(\nu)$.

> ▼ *The map φ is a sign-reversing involution on $\mathcal{A}_{m,n}$, $m \geq n$, with no fixed points, except when $m = \binom{k+1}{2}$, $n = \binom{k}{2}$, and F_k is a unique fixed point.*

This completes the involutive proof of the Jacobi identity.

7. Rogers-Ramanujan identities

7.1. **Combinatorial Interpretations.**

7.1.1. The classical *Rogers-Ramanujan identities* are:

$$(\star) \qquad 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} = \prod_{i=0}^{\infty} \frac{1}{(1-t^{5i+1})(1-t^{5i+4})}\,,$$

$$(\star\star) \qquad 1 + \sum_{k=1}^{\infty} \frac{t^{k(k+1)}}{(1-t)(1-t^2)\cdots(1-t^k)} = \prod_{i=0}^{\infty} \frac{1}{(1-t^{5i+2})(1-t^{5i+3})}\,.$$

The two identities are similar in nature, so we concentrate only on $(\star)$.

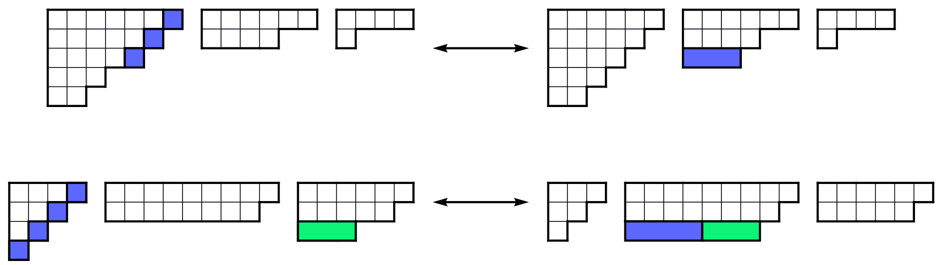

FIGURE 28. In the first case, we have:

$$\varphi : \begin{pmatrix} (7,6,5,3,2)\,,\,(6,4)\,,\,(4,1) \\ (6,5,4,2,1)\,,\,(6,4)\,,\,(5,2) \end{pmatrix} \to \begin{pmatrix} (6,5,4,3,2)\,,\,(6,4,3)\,,\,(4,1) \\ (5,4,3,2,1)\,,\,(6,4,3)\,,\,(5,2) \end{pmatrix}.$$

In the second case, we have:

$$\varphi : \begin{pmatrix} (4,3,2,1)\,,\,(9,8)\,,\,(6,5,3) \\ (3,2,1)\,,\,(9,8)\,,\,(7,6,4) \end{pmatrix} \to \begin{pmatrix} (3,2,1)\,,\,(9,8,7)\,,\,(6,5) \\ (2,1)\,,\,(9,8,7)\,,\,(7,6) \end{pmatrix}.$$

Let us start by giving combinatorial interpretations to the coefficients of t^n in $(\star)$. The r.h.s. is clear: this is the number of partitions of n into parts $\equiv \pm 1 \bmod 5$. We denote the set of such partitions by $\mathcal{A}_n$.

Let $\mathcal{B}_n$ be the number of partitions of n into parts which differ by at least 2. Let $\mathcal{C}_n$ be the number of partitions $\lambda \vdash n$ such that $s(\lambda) \geq \ell(\lambda)$. Then the coefficient of t^n on the l.h.s. in $(\star)$ is equal to $|\mathcal{B}_n| = |\mathcal{C}_n|$.

7.1.2. $(\Diamond)$ Use the Durfee square 2.3.1 to obtain the generating function for $\mathcal{C}_n$ and compare it with the l.h.s. of $(\star)$. Similarly, use the modified Sylvester's triangle 2.3.4 to obtain the generating function for $\mathcal{B}_n$ and compare it with the l.h.s. of $(\star)$. Finally, find a direct bijection $\pi : \mathcal{B}_n \to \mathcal{C}_n$.

7.1.3. $(\Diamond)$ Obtain similar combinatorial interpretations for $(\star\star)$.

7.1.4. $(\circ)$ Consider a Young diagram $[\lambda]$ and its Durfee square $[\delta_r]$. Consider the lower of the two Young diagrams in $[\lambda] \smallsetminus [\delta_r]$. Repeatedly take the Durfee square until an empty diagram is obtained. Let $\mathcal{C}_{n,k}$ be the set of partitions $\lambda \vdash n$ with at most $k-1$ successive Durfee squares. Let $\mathcal{A}_{n,k}$ be the set of partitions into parts $\equiv \pm k \bmod 2k+1$. Write the generating functions for $|\mathcal{C}_{n,k}|$ and $|\mathcal{A}_{n,k}|$. Their equality is called *Gordon's identity*.

7.2. **Schur's proof of Rogers-Ramanujan's identities.**

7.2.1. $(\Diamond)$ Apply Jacobi identity 6.1.1 to rewrite the r.h.s. of $(\star)$:

$$\prod_{r=0}^{\infty} \frac{1}{(1-t^{5r+1})(1-t^{5r+4})} = \sum_{m=-\infty}^{\infty} (-1)^m \, t^{\frac{m(5m-1)}{2}} \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} \,.$$

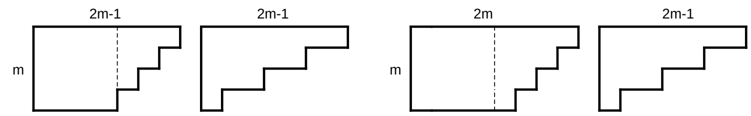

FIGURE 29. Fixed points of Schur's involution $\alpha : \mathcal{R}_n \to \mathcal{R}_n$.

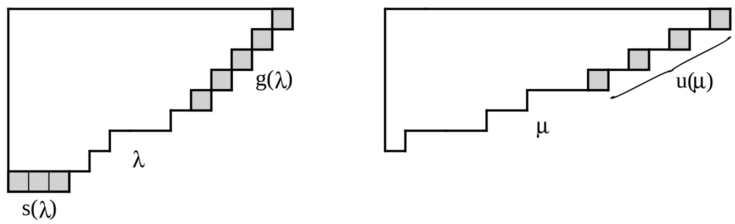

FIGURE 30. For a pair of partitions $(\lambda, \mu) \in \mathcal{R}$ as above, we have $s(\lambda) = 3$, $g(\lambda) = 5$, $u(\mu) = 4$.

7.2.2. *Schur's Involution.* From the above observation, rewrite $(\star)$ in the following equivalent form:

$$\prod_{i=1}^{\infty}(1-t^i)\left(1+\sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}\right) = \sum_{m=-\infty}^{\infty}(-1)^m\, t^{\frac{m(5m-1)}{2}}\,.$$

We shall prove this identity by an explicit sign-reversing involution, by combining elements of Vahlen's involution 2.3.7 and Franklin's involution 5.2.1. The construction we present is called *Schur's Involution*.

We give a combinatorial interpretation of the coefficient of t^n on both sides of the equation above. For the l.h.s. we have a set of pairs $\lambda \in \mathcal{D}$ and $\mu \in \mathcal{B}$, where $\mathcal{D}$ is a set of partitions into distinct parts, and $\mathcal{B} = \cup\mathcal{B}_n$ is a set of partitions μ with no equal or consecutive parts (see 7.1.1 above). Let $\mathcal{R} = \mathcal{D} \times \mathcal{B}$, and let $\mathcal{R}_n$ consist of pairs $(\lambda, \mu) \in \mathcal{R}$, such that $|\lambda| + |\mu| = n$. The *sign* of a pair (λ, μ) is a parity of $\ell(\lambda)$. We define an involution $\alpha : \mathcal{R}_n \to \mathcal{R}_n$ which is sign-reversing except for the fixed points, defined as in Figure 29. Observe that these fixed points give a combinatorial interpretation for the r.h.s. of the equation above.

Start with $(\lambda, \mu) \in \mathcal{R}_n \subset \mathcal{D}\times\mathcal{B}$. First, compare $a(\lambda)$ and $a(\mu)$. If $a(\lambda) \geq a(\mu)+2$, move part λ_1 to μ. If $a(\lambda) < a(\mu)$, move part μ_1 to λ. There remain the cases $a(\lambda) = a(\mu)$ and $a(\lambda) = a(\mu) + 1$. Denote these cases by $\mathcal{R}_n^1$ and $\mathcal{R}_n^2$, respectively.

As in 5.2.1, let $s(\lambda)$ be the length of the *horizontal line* in $[\lambda]$, let $g(\lambda)$ be the length of the *diagonal line*, and let $u(\mu)$ be the length of the *tangential line* defined as in Figure 30. Start with $(\lambda, \mu) \in \mathcal{R}_n^1$, and suppose this is not a fixed point. If $s(\lambda) \leq g(\lambda), u(\mu)$, remove the horizontal line and attach it to the diagonal line. Conversely, if $(\lambda, \mu) \in \mathcal{R}_n^2$ is not a fixed point, and $g(\lambda) < s(\lambda)$, $g(\lambda) \leq u(\mu)$, remove the diagonal line and attach it to the horizontal line (see Figure 31).

Suppose $(\lambda, \mu) \in \mathcal{R}_n^2$ with $s(\lambda) \leq g(\lambda), u(\mu)$. Then remove the horizontal line and attach it to the tangential line. Conversely, if $(\lambda, \mu) \in \mathcal{R}_n^1$ and $u(\mu) < s(\lambda)$, $u(\mu) \leq g(\lambda)$, then remove the tangential line and attach it to the horizontal line (see Figure 31).

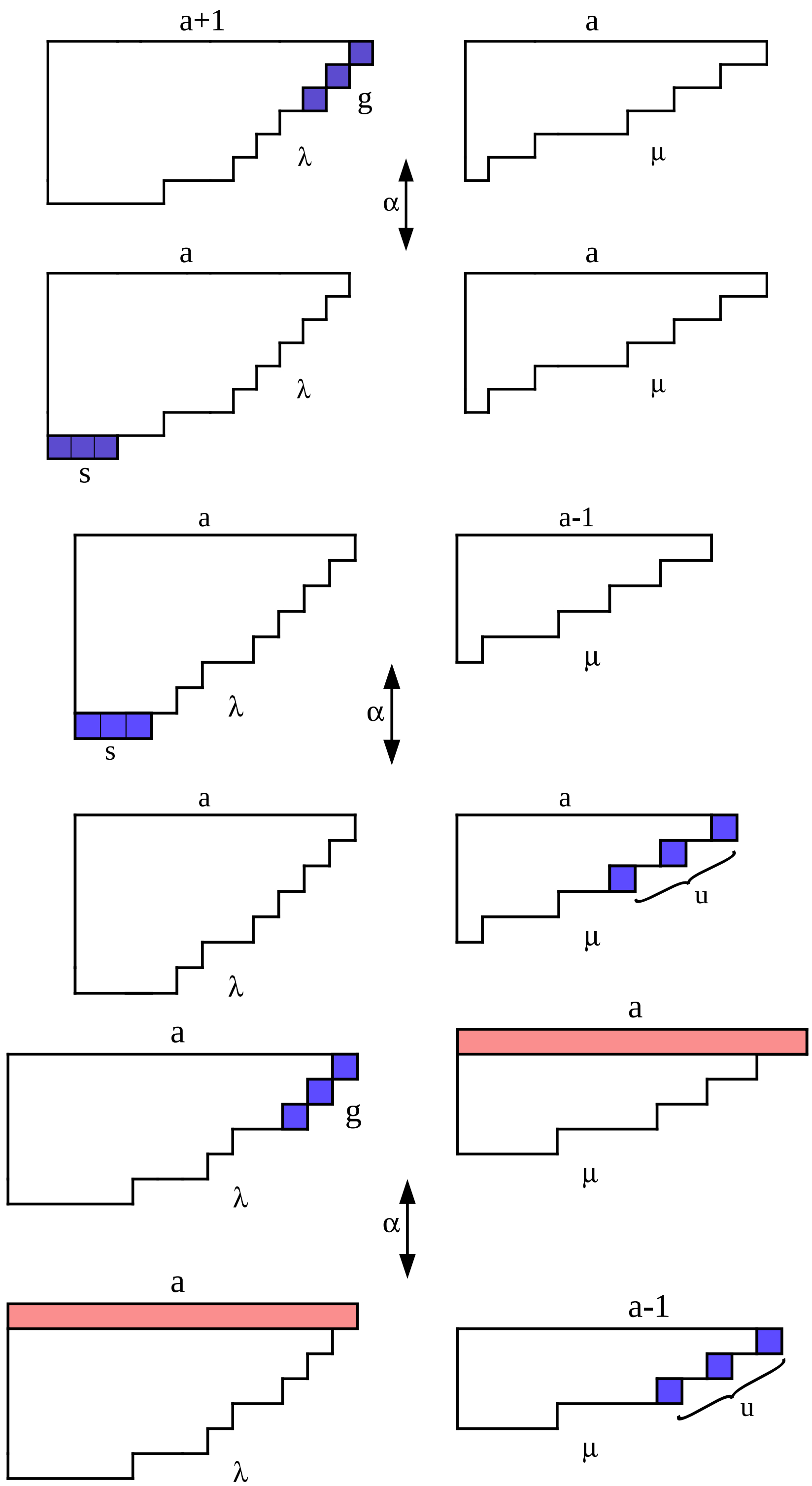

FIGURE 31. Three cases of Schur's involution α.

Finally, if $(\lambda,\mu)\in\mathcal{R}_n^1$ and $g(\lambda)<s(\lambda),u(\mu)$, then remove the largest part $a(\mu)$ and the diagonal line, and attach them to the largest part $a(\lambda)$ and the tangential line, respectively. Conversely, if $(\lambda,\mu)\in\mathcal{R}_n^2$ and $u(\lambda)<s(\lambda),u(\mu)$, then remove the largest part $a(\lambda)$ and the tangential line, and attach them to the largest part $a(\mu)$ and the diagonal line, respectively (see Figure 31).

▼ *The map $\alpha:\mathcal{R}_n\to\mathcal{R}_n$ defined above is a sign-reversing involution.*

▶ First, observe that α is defined for all (λ,μ) except for fixed points. Also, $\alpha^{-1}=\alpha$ by construction, in each of the four cases considered. Finally, the number of parts $\ell(\lambda)$ always changes by 1, so α is sign-reversing. ■

7.2.3. ($\circ$) Define polynomials $A_m(q)$ and $B_m(q)$ as follows:

$$A_m=A_{m-1}+q^mA_{m-2},\quad A_0=1,\quad A_1=1+q\,,$$
$$B_m=B_{m-1}+q^mB_{m-2},\quad B_0=1,\quad B_1=1\,.$$

Prove by induction that

$$A_m(q)=\sum_r(-1)^r q^{r(5r-3)/2}\binom{m-1}{\lfloor\frac{m+1-5r}{2}\rfloor}_q,$$

$$B_m(q)=\sum_r(-1)^r q^{r(5r+1)/2}\binom{m-1}{\lfloor\frac{m-1-5r}{2}\rfloor}_q,$$

where summation is over all r for which the q-binomial coefficient are defined (see 2.2.5). Compute *Schur's limits*:

$$A_\infty(q)\;=\;\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+1})(1-q^{5i+4})}\,,\quad B_\infty(q)\;=\;\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+2})(1-q^{5i+3})}\,.$$

7.2.4. ($\circ\circ$) Let $A_m(t)$, $B_m(t)$ be as in 7.2.3. Modify Schur's involution 7.2.2 to prove the following generalization of Rogers-Ramanujan's identities:

$$\begin{aligned}1+\sum_{k=1}^{\infty}&\frac{t^{k(k+m)}}{(1-t)(1-t^2)\cdots(1-t^k)}\\ &=\;(-1)^m t^{-\binom{m}{2}}B_{m-2}(t)\prod_{i=0}^{\infty}\frac{1}{(1-t^{5i+1})(1-t^{5i+4})}\\ &\quad-\;(-1)^m t^{-\binom{m}{2}}A_{m-2}(t)\prod_{i=0}^{\infty}\frac{1}{(1-t^{5i+2})(1-t^{5i+3})}\,.\end{aligned}$$

7.2.5. ($**$) Find a combinatorial proof of the *Farkas-Kra identity*:

$$\prod_{i=1}^{\infty}(1+t^{2i-1})\;=\;A_\infty(t)\,A_\infty(t^4)\;+\;t\,B_\infty(t)\,B_\infty(t^4)\,.$$

7.2.6. ($\circ\circ$) Recall Euler's recurrence relation 5.1.2 and its bijective proof 5.4.1. Modify Dyson's map to obtain a similar proof of the recurrence relation for the numbers $|\mathcal{C}_n|$, corresponding to the following equivalent form of ($\star$) :

$$1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} \; = \; \sum_{m=-\infty}^{\infty} (-1)^m \, t^{\frac{m(5m-1)}{2}} \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} \, .$$

7.3. **Ramanujan's Continued Fraction.**

7.3.1. Define

$$F(x,q) \; = \; 1 + \sum_{k=1}^{\infty} \frac{x^k \, q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \, ,$$

and let $c(x,q) = F(x,q)/F(xq,q)$. Observe that $F(x,q) = F(xq,q) + xqF(xq^2,q)$, and therefore

$$c(x,q) = 1 + \frac{xq}{F(xq,q)/F(xq^2,q)} = 1 + \frac{xq}{c(xq,q)} \, .$$

This immediately gives the *Ramanujan's continued fraction*:

$$c(x,q) \; = \; 1 + \cfrac{xq}{1 + \cfrac{xq^2}{1 + \cfrac{xq^3}{1 + \cfrac{xq^4}{1 + \dots}}}}$$

When $x = 1$, we have $c(1,q) = F(1,q)/F(q,q)$. Now identities ($\star$) and ($\star\star$) imply the famous Ramanujan's formula:

$$1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cfrac{q^4}{1 + \dots}}}} = \prod_{i=0}^{\infty} \frac{(1-q^{5i+2})(1-q^{5i+3})}{(1-q^{5i+1})(1-q^{5i+4})} \, .$$

7.3.2. ($\circ$) Let $f(z,t) = 1/c(-z,t)$. Define *Dyck words* to be 0-1 sequences with an equal number of 0's and 1's and such that the k-th 0 always precedes the k-th 1. Denote the set of such words by $\mathcal{W}$, and let $\ell(\omega)$ be half the length of the word. Define $a(\omega)$ to be the number of 0-1 pairs in a word such that 1 precedes 0. Note that $0 \le a(\omega) \le \binom{n}{2}$, for all $\ell(\omega) = n$. Use the recurrence relation for $c(z,t)$ to show that

$$f(z,t) \; = \; \sum_{\omega \in \mathcal{W}} z^{\ell(\omega)} \, t^{\binom{n}{2} - a(\omega)} \, .$$

7.3.3. ($\circ\circ$) Find a (infinite) subset $\mathcal{W}^\circ \subset \mathcal{W}$ of Dyck words, such that

$$c(-z,t) \; = \; \prod_{\omega \in \mathcal{W}^\circ} \left(1 - z^{\ell(\omega)} \, t^{\binom{n}{2} - a(\omega)}\right) \, .$$

8. Involution principle and partition identities

8.1. **Equivalent partition bricks.**

8.1.1. *Andrews' Theorem.* Fix a sequence $a = (a_1, a_2, \dots)$, where $a_i \in \mathbb{P} \cup \{\infty\}$. Let $\mathrm{supp}(a)$ be the set of all $i \in \mathbb{P}$ such that $a_i < \infty$. We say that two such sequences a and $b = (b_1, b_2, \dots)$ are equivalent, denoted $a \sim b$, if there exists a one-to-one correspondence $\pi : \mathrm{supp}(a) \to \mathrm{supp}(b)$ such that $i \cdot a_i = j \cdot b_j$, for all $j = \pi(i)$.

Let $\mathcal{A}_n$ and $\mathcal{B}_n$ be the sets of partitions $\lambda = (1^{m_1} 2^{m_2} \dots) \vdash n$ such that $m_i < a_i$ and $m_j < b_j$, respectively. We refer to $\mathcal{A} = \cup_n \mathcal{A}_n$ and $\mathcal{B} = \cup_n \mathcal{B}_n$ as *equivalent partition bricks*.

▼ Andrews' Theorem. *If $a \sim b$, then $|\mathcal{A}_n| = |\mathcal{B}_n|$, for all $n > 0$.*

▶ $$1 + \sum_{n=1}^{\infty} |\mathcal{A}_n|\, t^n = \prod_{i=1}^{\infty} \frac{1 - t^{i\, a_i}}{1 - t^i} = \prod_{j=1}^{\infty} \frac{1 - t^{j\, b_j}}{1 - t^j} = 1 + \sum_{n=1}^{\infty} |\mathcal{B}_n|\, t^n, \text{ where } t^{\infty} = 0. \quad \blacksquare$$

8.1.2. ($\Diamond$) Let $a = (2, 2, \dots)$, $b = (\infty, 1, \infty, 1, \dots)$, $\pi : i \to 2i$. Then $\mathcal{A}_n = \mathcal{D}_n$ and $\mathcal{B}_n = \mathcal{O}_n$. In this case Andrews' Theorem becomes Euler's Theorem 3.1.

8.1.3. ($\Diamond$) Let $\mathcal{A}_n$ be the set of partitions of n into parts $\equiv 1$ or $5 \bmod 6$. Let $\mathcal{B}_n$ be the set of partitions of n into distinct parts $\equiv \pm 1 \bmod 3$. Let $\mathcal{C}_n$ be the set of partitions of n into odd parts none appearing more than twice. Deduce from Andrews' Theorem the equality $|\mathcal{A}_n| = |\mathcal{B}_n| = |\mathcal{C}_n|$.

8.1.4. ($\circ$) Let $\gamma_{\mathcal{A}} : \mathcal{P} \to \mathbb{N}$ be a statistic defined by $\gamma_{\mathcal{A}}(\lambda) = |\{i : m_i(\lambda) \geq a_i\}|$. Define $\gamma_{\mathcal{B}}$ analogously. Prove that statistics $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ are equidistributed on $\mathcal{P}_n$. Compare this result with Franklin's extension 3.3.1.

8.2. **O'Hara's Algorithm.**

8.2.1. We present here *O'Hara's Algorithm*, which defines a bijection $\varphi : \mathcal{A}_n \to \mathcal{B}_n$.

Start with $\lambda \in \mathcal{A}_n$. Set $\mu := \lambda$. While μ contains any part (j) at least b_j times (i.e. $m_j(\mu) \geq b_j$), remove b_j copies of part (j) from μ, add a_i copies of the part (i) to μ, where $i = \pi^{-1}(j)$. Repeat until $\mu \in \mathcal{B}_n$.

▼ *The map $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ is a well defined bijection, independent of the order of parts removed in the algorithm.*

8.2.2. ($\Diamond$) Show that in Example 8.1.2, the bijection $\varphi : \mathcal{D}_n \to \mathcal{O}_n$ coincides with the map $\phi : \mathcal{D}_n \to \mathcal{O}_n$ in 3.2.1, the inverse to the Glaisher's bijection.

8.2.3. ($\Diamond$) Use O'Hara's Algorithm to give a bijective proof of the equalities in 8.1.3. Convert these into explicit 'à la Glaisher' bijections between the sets.

8.2.4. ($\circ$) Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $\pi = (2, 1) \in S_2$. Define the *speedy version* of O'Hara's Algorithm by combining identical iterations into one. Find a connection to Euclid's Algorithm and continued fractions. Conclude that the new version takes $O(\log M)$ steps, where $M = \max\{a_i, b_j\}$.

8.2.5. ($*$) Let m be fixed, $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m)$, $\pi \in S_m$. Prove that the speedy version of O'Hara's Algorithm requires $O(\log M)$ steps.

8.3. **Geometric version.**

8.3.1. Let $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$, and $w = (w_1, \ldots, w_m)$, where $a_i, b_i, w_i \in \mathbb{R}_+$. For the rest of this section, let $[m] = \{1, 2, \ldots, m\}$. We write $a \sim_\omega b$ if there exists a bijection $\pi : [m] \to [m]$, such that $a_i w_i = b_j w_j$ for all $j = \pi(i)$.

Let $V = \mathbb{R}^m$, and consider a linear function $\omega \in V^*$, defined by $\omega(x_1, \ldots, x_m) := w_1 x_1 + \cdots + w_m x_m$. Consider two ω-*equivalent bricks*:

$$\mathcal{A} = \{(x_1, \ldots, x_m) \in V : 0 \le x_i \le a_i, i \in [m]\},$$
$$\mathcal{B} = \{(x_1, \ldots, x_m) \in V : 0 \le x_j \le b_j, j \in [m]\},$$

and let $\mathcal{A}_c = \mathcal{A} \cap \{x \in V : \omega(x) = c\}$, $\mathcal{B}_c = \mathcal{B} \cap \{x \in V : \omega(x) = c\}$, where $c \in \mathbb{R}_+$.

▼ *If $a \sim_\omega b$, then* $\mathrm{vol}(\mathcal{A}_c) = \mathrm{vol}(\mathcal{B}_c)$ *for all* $c > 0$.

8.3.2. ($\circ$) Let $e(Q)$ denote the number of integer points in the convex polytope Q, and let $N \cdot Q$ be the polytope Q extended by a factor of N in all directions. The *Ehrhart polynomial* $f_Q(t)$ is defined by $f_Q(N) = e(N \cdot Q)$, for all $N \in \mathbb{N}$. Extend the above result to an equality of Ehrhart polynomials of the polytopes $\mathcal{A}_c$ and $\mathcal{B}_c$. Deduce Andrews' Theorem 8.1.1 in this case.

8.3.3. ($\circ$) Extend O'Hara's Algorithm 8.2.1 to a map $\varphi : \mathcal{A} \to \mathcal{B}$. Prove that φ is a piece-wise linear and volume preserving map, such that $\omega(x) = \omega(\varphi(x))$. Observe that φ is a parallel translation almost everywhere. Give another proof of 8.3.2.

8.4. **General involution principle.**

8.4.1. *Garsia–Milne Theorem.* Let $\mathcal{A} = \mathcal{A}_+ \sqcup \mathcal{A}_-$ and $\mathcal{B} = \mathcal{B}_+ \sqcup \mathcal{B}_-$ be two sets with two subsets. Suppose $\alpha : \mathcal{A} \to \mathcal{A}$ and $\beta : \mathcal{B} \to \mathcal{B}$ be two involutions with fixed points $F_\alpha \subset \mathcal{A}_+$ and $F_\beta \subset \mathcal{B}_+$, such that $\alpha : \mathcal{A}_+ \smallsetminus F_\alpha \to \mathcal{A}_-$ and $\beta : \mathcal{B}_+ \smallsetminus F_\beta \to \mathcal{B}_-$ are bijections. Such involutions are called *sign-reversing*. Finally, suppose $\psi : \mathcal{A} \to \mathcal{B}$ is a bijection which maps $\mathcal{A}_+$ into $\mathcal{B}_+$, and $\mathcal{A}_-$ into $\mathcal{B}_-$. Clearly, $|F_\alpha| = |\mathcal{A}_+| - |\mathcal{A}_-| = |\mathcal{B}_+| - |\mathcal{B}_-| = |F_\beta|$.

The *involution principle* defines the following map $\varphi : F_\alpha \to F_\beta$. Start at $a \in F_\alpha \subset \mathcal{A}_+$. If $b := \psi(a) \in F_\beta \subset \mathcal{B}_+$, let $\varphi(a) = b$. Otherwise, consider $b' = \beta(\psi(\alpha(\psi^{-1}(b)))) \in \mathcal{B}_+$. Again, if $b' \in F_\beta$, let $\varphi(a) = b'$. Otherwise, let $b'' := \beta(\psi(\alpha(\psi^{-1}(b')))) \in \mathcal{B}_+$ and repeat.

▼ Garsia–Milne Theorem. *The map* $\varphi : F_\alpha \to F_\beta$ *is a bijection.*

8.4.2. ($\lozenge$) Let $D_\infty = \langle \alpha, \beta \rangle / (\alpha^2 = \beta^2 = 1)$ be an infinite dihedral group, $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}$. Let $\rho : D_\infty \to S_N$ be a permutation representation of D_∞ on $[N]$. Show that orbits of the action of D_∞ give a perfect matching on a set $F = [N]^{D_\infty}$ of fixed points of the action of D_∞ on $[N]$. Deduce the Garsia–Milne Theorem.

8.4.3. ($\lozenge$) Recall Franklin's involution 5.2.1 and Vahlen's involution 2.3.7. Apply the involution principle to obtain an involutive proof of Euler's recurrence relation 5.1.2. Compare the resulting involution with 5.4.1.

8.4.4. ($\lozenge$) Recall Schur's involution 7.2.2 and direct bijective proof of Jacobi identity 6.2.1. Combine the two with Vahlen's involution to obtain an involution principle bijective proof of the Rogers-Ramanujan's identity ($\star$) in 7.1.1.

8.4.5. $(**)$ Prove that in the worst case the number of steps in the resulting bijection $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ (in the notation of 7.1.1) is $> \exp(n^\alpha)$, for some $\alpha > 0$. Compare this with the total number of partitions $|\mathcal{A}_n|$.

8.4.6. $(\circ)$ Let p be a prime $\equiv 1 \bmod 4$. Euler proved that $p = x^2 + 4y^2$ for some integers x and y. Consider a set of triples $\mathrm{A} = \{(x, y, z) \in \mathbb{Z}^3 : x^2 + 4yz = p\}$. Note that $|\mathrm{A}| \geq 1$ since $a = (1, 1, \frac{p-1}{4}) \in \mathrm{A}$. Define two involutions on A: $\alpha(x, y, z) = (x, z, y)$, and

$$\beta(x,y,z) \, = \, \begin{cases} (x+2z, z, y-x-z), & \text{if} \ \ x < y-z, \\ (2y-x, y, x-y+z), & \text{if} \ \ y-z < x < 2y, \\ (x-2y, x-y+z, y), & \text{if} \ \ x > 2y. \end{cases}$$

Note that a is a unique fixed point of the involution β. Deduce from here Euler's result and present an algorithm for finding a solution algorithmically.

8.4.7. $(**)$ Estimate the complexity of the above algorithm.

8.5. **Remmel's bijection.** We present here a bijective proof of Andrews' Theorem 8.1.1 by means of the involution principle. The idea is to multiply by the common denominator both sides of the identity in the proof of Andrews' Theorem, and then cancel terms accordingly.

We use the notation of 8.1. For simplicity, assume that $\operatorname{supp}(a) = \operatorname{supp}(b) = [m]$, i.e. $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, with $a_i, b_j < \infty$. Then $\pi \in S_m$, with $a_i i = b_{\pi(i)} \pi(i)$. In this case we have:

$$\mathcal{A} = \{(1^{c_1} \ldots m^{c_m}) : \ 0 \leq c_i < a_i, \text{ for all } i \in [m]\},$$

$$\mathcal{B} = \{(1^{c_1} \ldots m^{c_m}) : \ 0 \leq c_j < b_j, \text{ for all } j \in [m]\}.$$

Let $\mathcal{P}$ be the set of all partitions λ, and let $\mathcal{X} = \mathcal{P} \times 2^{[m]}$. Define

$$\mathcal{X}_+ = \{(\lambda, S) : \lambda \in \mathcal{P}, S \subset [m], \text{ and } |S| \text{ is odd}\}, \quad \text{and let } \mathcal{X}_- = \mathcal{X} \smallsetminus \mathcal{X}_+.$$

Finally, let $F_\alpha = \mathcal{A} \times \{\emptyset\} \subset \mathcal{X}$, and $F_\beta = \mathcal{B} \times \{\emptyset\} \subset \mathcal{X}$. We shall define two sign-reversing involutions α, β on $\mathcal{X}$, with F_α, F_β as their fixed points.

Consider $(\lambda, S) \in \mathcal{X}$, where $\lambda = (1^{c_1}, \ldots, m^{c_m})$, $S \in [m]$. Take the smallest $i \in [m]$, such that either $i \in S$, or $c_i \geq a_i$, or both. Now let

$$\alpha(\lambda, S) = \begin{cases} \big((1^{c_1}, \ldots, i^{c_i + a_i}, \ldots, m^{c_m}), S \smallsetminus \{i\}\big), & \text{if } i \in S, \\ \big((1^{c_1}, \ldots, i^{c_i - a_i}, \ldots, m^{c_m}), S \cup \{i\}\big), & \text{if } i \notin S, \ c_i \geq a_i \end{cases}$$

Define $\beta = \beta(\mu, S)$ analogously. Now use the involution principle to construct a bijection $\varphi : F_\alpha \to F_\beta$, and thus gives *Remmel's bijection* $\varphi' : \mathcal{A} \to \mathcal{B}$.

> ▼ Remmel's bijection. *The above construction gives a bijection* φ' : $\mathcal{A}_n \to \mathcal{B}_n$. *This bijection coincides with the map* $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ *defined by O'Hara's Algorithm 8.2.1.*

8.6. **Cohen–Remmel Theorem.**

8.6.1. We now consider a different setup for partition identities. Let $R = \mathbb{N}^m$ be the free abelian semigroup. Fix a semigroup homomorphism $\omega : R \to \mathbb{N}$, defined by $\omega(\overline{a}) = a_1 w_1 + \cdots + a_m w_m$, where $\overline{a} = (a_1, \ldots, a_m)$. We also assume that $w_i > 0$. For every $\mathcal{C} \subset R$, let $\mathcal{C}_n = \mathcal{C} \cap \{\overline{a} \in R : \omega(\overline{a}) = n\}$. We call ω the *weight function*.

For elements $\overline{a} = (a_1, \ldots, a_m)$, $\overline{a}' = (a'_1, \ldots, a'_m)$, ..., define

$$\operatorname{lcm}(\overline{a}, \overline{a}', \ldots) = \big(\max\{a_1, a'_1, \ldots\}, \ldots, \max\{a_m, a'_m, \ldots\}\big)$$

Let $\mathrm{A} = \{\overline{a}_1, \ldots, \overline{a}_r\}$, $\mathrm{B} = \{\overline{b}_1, \ldots, \overline{b}_r\} \subset \mathbb{N}^m$ be two subsets of an integer lattice. We say that A and B are $\operatorname{lcm}_\omega$*-equivalent*, denoted $\mathrm{A} \sim_\omega \mathrm{B}$, if for all $I = \{i_1, i_2, \ldots\} \subseteq [r]$, we have

$$\omega\left(\operatorname{lcm}(\overline{a}_{i_1}, \overline{a}_{i_2}, \ldots)\right) = \omega\left(\operatorname{lcm}(\overline{b}_{i_1}, \overline{b}_{i_2}, \ldots)\right).$$

Consider two lattice ideals $\mathcal{A} = \mathbb{N}\langle \overline{a}_1, \ldots, \overline{a}_r\rangle$, $\mathcal{B} = \mathbb{N}\langle \overline{b}_1, \ldots, \overline{b}_r\rangle$, and let $\mathcal{A}' = R \smallsetminus \mathcal{A}$, $\mathcal{B}' = R - \mathcal{B}$.

▼ Cohen–Remmel Theorem. *If* $\mathrm{A} \sim_\omega \mathrm{B}$*, then* $\mathcal{A}'_n = \mathcal{B}'_n$.

▶ Let $G'(t)$, $H'(t)$ denote the generating series for the ω-statistic on $\mathcal{A}'$ and $\mathcal{B}'$:

$$G'(t) = \sum_n |\mathcal{A}'_n|\, t^n = \sum_{\overline{c} \in \mathcal{A}'} t^{\omega}(\overline{c}), \qquad H'(t) = \sum_n |\mathcal{B}'_n|\, t^n = \sum_{\overline{c} \in \mathcal{B}'} t^{\omega}(\overline{c}).$$

Also, let

$$W(t) := \sum_n |R_n|\, t^n = \sum_{\overline{c} \in R} t^{\omega(\overline{c})} = \prod_i \frac{1}{1 - t^{w_i}}.$$

For every subset $I = \{i_1, i_2, \ldots\} \subset [r]$, consider the intersection of the dual lattice ideals (also called *filters*): $M_I = \mathbb{N}\langle \overline{a}_{i_1}\rangle \cap \mathbb{N}\langle \overline{a}_{i_2}\rangle \cap \ldots$, and the generating series for the weight function: $P_I(t) = \sum_{\overline{c} \in M_I} t^{\omega(\overline{c})}$. Also, let $A_I = \{\overline{a}_{i_1}, \overline{a}_{i_2}, \ldots\}$, and $\operatorname{lcm}(\mathrm{A}_I) = \operatorname{lcm}(\overline{a}_{i_1}, \overline{a}_{i_2}, \ldots)$. For the lattice subset B, define the dual lattice ideal N_I, generating series for the weight function $Q_I(t)$, and $\operatorname{lcm}(\mathrm{B}_I)$, analogously. From $\mathrm{A} \sim_\omega \mathrm{B}$, we have:

$$P_I(t) = t^{\omega(\operatorname{lcm}(\mathrm{A}_I))} W(t) = t^{\omega(\operatorname{lcm}(\mathrm{B}_I))} W(t) = Q_I(t).$$

The inclusion-exclusion principle gives:

$$G'(t) = \sum_{I \subset [r]} (-1)^{|I|} P_I(t) = \sum_{I \subset [r]} (-1)^{|I|} Q_I(t) = H'(t).$$

■

8.6.2. (◊) Fix $w_i = i$, so that $\omega(c_1, c_2, \ldots) = c_1 \cdot 1 + c_2 \cdot 2 + \ldots$ Let $\overline{a}_1 = (a_1, 0, 0, \ldots, 0)$, $\overline{a}_2 = (0, a_2, 0, \ldots, 0)$, ... Then $\mathcal{A}'$, $\mathcal{B}'$ correspond to equivalent partition bricks $\mathcal{A}$, $\mathcal{B}$, as in the notation of 8.1. Thus the Cohen-Remmel Theorem implies Andrews Theorem 8.1.1.

8.6.3. (◊) Deduce Glaisher's Theorem 3.2.3 from Cohen–Remmel Theorem.

8.6.4. (∘) Prove that $\mathrm{A} \sim_\omega \mathrm{B}$ implies that the *lcm-lattices* $\mathcal{L}_\mathrm{A} = \{\operatorname{lcm}(\mathrm{A}_I), I \subset [r]\}$ and $\mathcal{L}_\mathrm{B} = \{\operatorname{lcm}(\mathrm{B}_I), I \subset [r]\}$ are isomorphic.

8.6.5. (∘) Generalize Remmel's bijection 8.5 to prove the Cohen–Remmel Theorem. Find an example of when different orderings on $[m]$ produce different bijections.

8.6.6. ($\circ$) Define analogs of statistics $\gamma_{\mathcal{A}}$, $\gamma_{\mathcal{B}}$ from 8.1.4. Extend the Cohen–Remmel Theorem to general values of these statistics.

8.7. **Gordon's bijection.**

8.7.1. We present a bijective proof of the Cohen-Remmel Theorem 8.6.1. The idea behind *Gordon's bijection* is to generalize the use of the inclusion-exclusion principle from the proof of the Cohen–Remmel Theorem.

In the notation of 8.6, for every $I = \{i_1, i_2, \dots\} \subset [r]$, consider *dual lattice ideals* $\mathcal{A}_I = \mathbb{N}\langle a_{i_1}, a_{i_2}, \dots\rangle$ and $\mathcal{A}_I^\bullet = \mathcal{A}_J \cap M_I \subset M_I$, where $J = [r] \smallsetminus I$. Finally, consider an ideal $\mathcal{A}'_I = M_I \smallsetminus \mathcal{A}_I^\bullet$. Clearly, $R = \oplus_{I \subset [r]} \mathcal{A}'_I$. Similarly, define dual lattice ideals $\mathcal{B}_I$, $\mathcal{B}_I^\bullet$, and an ideal $\mathcal{B}'_I$. Clearly, $\mathcal{A}' = \mathcal{A}'_\emptyset$ and $\mathcal{B}' = \mathcal{B}'_\emptyset$.

We shall construct by induction on $|I| \le r$ a family of bijections $\Phi_I : \mathcal{A}'_I \to \mathcal{B}'_I$, which preserve the statistic ω. By the inductive assumption, it suffices to construct only a bijection $\Phi := \Phi_\emptyset : \mathcal{A}' \to \mathcal{B}'$.

For every $I \subset [r]$ we have obvious bijections $\Psi_I : M_I \to N_I$, defined by

$$\Psi_I : \overline{c} \to \overline{c} - \mathrm{lcm}(\mathrm{A}_I) + \mathrm{lcm}(\mathrm{B}_I).$$

In particular, $\Psi := \Psi_\emptyset$ is an identity map.

For $r = 1$, we have $\mathrm{A} = \{\overline{a}\}$, $R_{\mathrm{A}} = R\langle x^{\overline{a}}\rangle$. In this case we have two maps $\Psi = \Psi_\emptyset$ and $\Psi_1 := \Psi_{\{1\}}$. Consider the following version of the involution principle 8.4. Start at $f = \overline{c} \in N'$ and consider $g_1 = \Psi(f)$. If $g_1 \in N'_{\mathrm{B}}$, let $\Phi(f) = g_1$. Otherwise, consider $g_2 = \Psi\big(\Psi_1^{-1}(g_1)\big)$. Again, if $g_2 \in N'_{\mathrm{B}}$, let $\Phi(f) = g_2$; otherwise, repeat.

In the general case $r \ge 2$ proceed analogously. Start at $f = \overline{c} \in N'$ and consider $g_1 = \Psi(f)$. Take a unique $I \subset [r]$ such that $g_1 \in N'_I$. If $I = \emptyset$, define $\Phi(f) = g_1$. Otherwise, $|I| \ge 1$ and the maps Φ_I are defined by the inductive assumption. Now let $g_2 = \Psi\big(\Psi_I^{-1}(g_1)\big)$ and repeat the procedure.

> ▼ *If* $\mathrm{A} \sim_\omega \mathrm{B}$, *then the map* $\Phi : \mathcal{A}' \to \mathcal{B}'$ *defined above is a weight preserving bijection.*

8.7.2. ($\lozenge$) Following Example 8.6.2, consider a special case when the Cohen–Remmel Theorem reduces to Andrews' Theorem. Prove that the bijection φ given by O'Hara's Algorithm 8.2.1 coincides with Gordon's bijection Φ in this case.

8.7.3. ($\circ$) Extend the Cohen-Remmel Theorem 8.6.1 to a geometric setting. Generalize Gordon's bijection to this case and prove the equality of the Ehrhart polynomials.

9. Miscellanea

9.1. **Plane Partitions.**

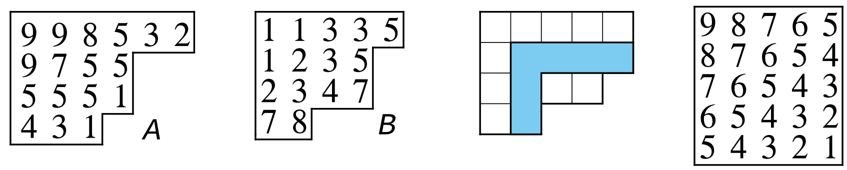

FIGURE 32. A plane partition A (zero entries are omitted) and a reverse plane partition B of shape $(5,4,4,2)$. Here $|A| = 95$ and $|B| = 55$. Two pictures on the right show the hook length $h(2,2) = 6$ in $[5,5,4,2]$, and hook lengths in $[5^5]$.

9.1.1. *MacMahon's Theorem.* A *plane partition* is a two-dimensional array of nonnegative integers $A = (\lambda_{i,j})$, such that $\lambda_{i,j} \geq \lambda_{i,j+1},\ \lambda_{i+1,j}$ for all $(i,j) \in \mathbb{Z}^2_{\geq 1}$. Denote by $\mathcal{M}$ the set of all plane partitions. Define $|A| = \sum_{i,j} \lambda_{i,j}$. Traditionally, plane partitions are represented by a function $\lambda_{i,j}$ written in squares (i,j) (see Figure 32).

The following formula is the classical *MacMahon's Theorem*:

$$\sum_{A \in \mathcal{M}} t^{|A|} \ = \ \prod_{r=1}^{\infty} \frac{1}{(1-t^r)^r}\,.$$

Define the *support* by $\operatorname{supp}(A) = \{(i,j) \in \mathbb{Z}^2 : \lambda_{i,j} > 0\}$. Let $\mathcal{M}_k$ be the set of plane partitions $A \in \mathcal{M}$, such that $\operatorname{supp}(A) \subset \{(i,j) : 1 \leq i,j \leq k\}$. The following formula is an extension of MacMahon's theorem:

$$\sum_{A \in \mathcal{M}_k} t^{|A|} \ = \ \prod_{r=1}^{k} \frac{1}{(1-t^r)^r} \ \prod_{i=1}^{k-1} \frac{1}{(1-t^{k+i})^{k-i}}\,.$$

Indeed, letting $k \to \infty$ gives the formula above.

9.1.2. *Stanley's Hook Content Formula.* A *reverse plane partition* of shape μ is an integer nonnegative function $f(i,j)$ on the squares $(i,j) \in [\mu]$ such that $f(i,j) \leq f(i+1,j)$ and $f(i,j) \leq f(i,j+1)$ whenever both squares are in $[\mu]$. Define $|B| = \sum_{(i,j)\in[\mu]} f(i,j)$. Denote by $\mathcal{R}(\mu)$ the set of reverse plane partitions $B = \{f(i,j)\}$ of shape μ. The following result is called *Stanley's hook content formula*:

$$\sum_{B \in \mathcal{R}(\mu)} t^{|B|} \ = \ \prod_{(i,j)\in[\mu]} \frac{1}{(1-t^{h(i,j)})}\,,$$

where $h(i,j) = \mu_i + \mu'_j - i - j + 1$ is the *hook length*, defined as the number of squares in $[\mu]$ to the right or below (i,j), including (i,j) (see Figure 32). When $[\mu] = [k^k]$ is a k-square, reverse plane partitions are centrally symmetric to (usual) plane partitions in $\mathcal{M}_k$, and Stanley's formula coincides with the extension of the MacMahon's Theorem as above (see hook lengths in $[5^5]$ in Figure 32).

9.1.3. *Bijective proof.* We present here a bijective proof of Stanley's formula by induction on $|\mu|$. Consider a set $\mathcal{C}(\mu)$ of nonnegative integer functions $C = \{g(i,j) : (i,j) \in [\mu]\}$, and define $\|C\| := \sum_{(i,j)\in[\mu]} h(i,j) g(i,j)$. We present a bijection $\xi_\mu : \mathcal{R}(\mu) \to \mathcal{C}(\mu)$, such that $|B| = \|C\|$ for all $C = \xi_\mu(B)$. The base of induction, when $|\mu| = 1$, is trivial.

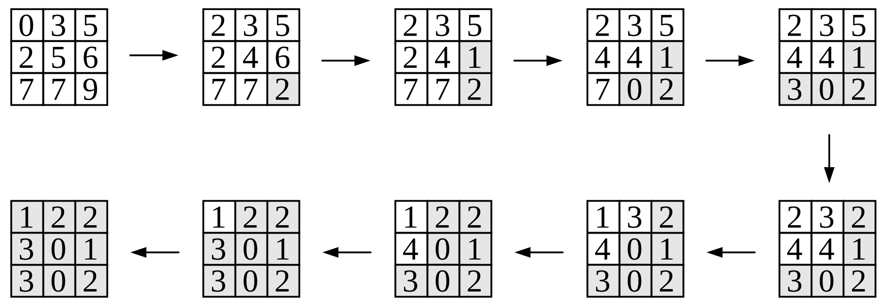

FIGURE 33. An example of bijection $\xi_\mu : B \to C$, for $\mu = (3^3)$. Here $|B| = \|C\| = 44$.

Now start with a plane partition B of shape μ defined by a function f on $[\mu]$. Let (p, q) be a corner in $[\mu]$, and $[\nu] = [\mu] - (p, q)$. By induction, assume that ξ_ν is already defined. Let $c = p - q$. Let us change the value of f on all squares $(i, j) \neq (p, q)$ on the diagonal $i - j = c$ by the following rule:

$$f'(i,j) \,=\, \max\{f(i-1,j), f(i,j-1)\} \,+\, \min\{f(i+1,j), f(i,j+1)\} \,-\, f(i,j)\,,$$

where we assume that $f(i, j) = 0$ whenever $i < 0$ or $j < 0$. Let $f'(i, j) = f(i, j)$ if $i - j \neq c$. Now define $\{g(i,j) : (i,j) \in \nu\} = \xi_\nu\big(\{f'(i,j)\}\big)$, and let $g(p, q) = f(p, q) - \max\{f(p-1, q), f(p, q-1)\}$ (see Figure 33).

▼ *The map* $\xi_\mu : \mathcal{R}(\mu) \to \mathcal{C}(\mu)$ *defined above is a bijection, such that* $|B| = \|C\|$ *for all* $C = \xi_\mu(B)$.

9.1.4. ($\Diamond$) A priori, the bijection ξ_μ may depend on the order of squares removed in the induction steps. Prove that ξ_μ is, in fact, independent of that order.

9.1.5. ($\circ$) Let $A = (\lambda_{i,j}) \in \mathcal{M}$ be a plane partition. Define $\mathrm{tr}(A) = \lambda_{1,1} + \lambda_{2,2} + \ldots$. Deduce from the proof a refinement of MacMahon's Theorem:

$$\sum_{A\in\mathcal{M}} t^{|A|}\, z^{\mathrm{tr}(A)} \,=\, \prod_{r=1}^{\infty} \frac{1}{(1-zt^r)(1-t^r)^{r-1}}\,.$$

9.1.6. ($\circ$) Show that ξ_μ is a continuous, piecewise linear, volume-preserving map from a cone of real reverse plane partitions of shape μ to a cone of nonnegative real functions on $[\mu]$. Extend the theorem to an equality of Ehrhart polynomials of convex polytopes.

9.1.7. ($\circ\circ\circ$) Let $B(m, n, \ell)$ be the number of plane partitions $\{\lambda_{i,j}\}$ with

$$\mathrm{supp}\{\lambda_{i,j}\} \subset \{(i,j) : 1 \le i \le m,\, 1 \le j \le n\}, \ \text{ and } \ \lambda_{i,j} \le \ell\,.$$

Prove combinatorially:

$$B(m,n,\ell) \,=\, \prod_{i=1}^{m}\prod_{j=1}^{n}\prod_{k=1}^{\ell} \frac{i+j+k-1}{i+j+k-2}\,.$$

9.2. **Bipartitions.**

9.2.1. *Carlitz's Theorem.* We say that $(\mu_1,\nu_1),(\mu_2,\nu_2),\ldots,(\mu_\ell,\nu_\ell)$ is a *bipartition* of (m,n) if μ_i,ν_i are nonnegative integers, $|\mu|=m$, $|\nu|=n$, and

$$\min\{\mu_i,\nu_i\}\geq\max\{\mu_{i+1},\nu_{i+1}\}\,.$$

Here we assume also that $(\mu_i,\nu_i)\neq(0,0)$, except when $(m,n)=(0,0)$. Clearly, the $\min-\max$ condition implies that both μ and ν are integer partitions.

Denote by $\mathcal{B}_{n,n}$ the set of all bipartitions of (m,n). The following result is called *Carlitz's Theorem*:

$$\sum_{(m,n)}\bigl|\mathcal{B}_{m,n}\bigr|\,x^my^n\;=\;\prod_{i=1}^{\infty}\frac{1}{(1-x^iy^{i-1})(1-x^{i-1}y^i)(1-x^{2i}y^{2i})}\,.$$

In other words, bipartitions are in bijection with all decompositions of a vector (m,n) into sum of vectors $(i,i-1)$, $(i-1,i)$ and $(2i,2i)$, with no regard to the order. Denote by $\mathcal{W}_{m,n}$ the set of such vector decompositions. Define a map $\varphi:\mathcal{B}_{m,n}\to\mathcal{W}_{m,n}$ as follows.

Start with $(\mu,\nu)\in\mathcal{B}_{m,n}$. Consider a sequence (μ_1',ν_1'), (μ_2',ν_2'), … Observe that the min–max condition now translates into $|\mu_i-\nu_i|\leq 1$, for all $i\geq 1$, and therefore (μ',ν') has a natural decomposition into vectors $(i,i-1)$, $(i-1,i)$, and (i,i). Now split each vector of type $(2r-1,2r-1)$ into two vectors $(r-1,r)$ and $(r,r-1)$, and leave all other vectors intact. Let $\varphi(\mu,\nu)\in\mathcal{W}_{m,n}$ be the resulting vector decomposition.

▼ *The map $\varphi:\mathcal{B}_{m,n}\to\mathcal{W}_{m,n}$ defined above is a bijection.*

▶ Note that (μ',ν') can contain either $(i,i-1)$ or $(i-1,i)$, but not both. Therefore, to define φ^{-1} one needs first to couple all pairs of vectors $(i,i-1)$ and $(i-1,i)$ into one vector $(2i-1,2i-1)$. Now collect all the remaining vectors into a pair of partitions, and take their conjugates. The rest of the proof is straightforward. ■

9.2.2. (♢) Let $\mathcal{B}^\ell_{m,n}$ be the set of $(\mu,\nu)\in\mathcal{B}_{m,n}$, such that $\ell(\mu),\ell(\nu)\leq\ell$. Prove the following refinement of Carlitz's Theorem:

$$\sum_{(m,n)}\bigl|\mathcal{B}^\ell_{m,n}\bigr|\,x^my^n\;=\;\prod_{i=1}^{\infty}\frac{(1-x^{2i-1}y^{2i-1})}{(1-x^iy^{i-1})(1-x^{i-1}y^i)(1-x^iy^i)}\,.$$

9.2.3. (○) Fix an integer $r\geq 1$, and let

$$\operatorname{smax}\{x_1,\ldots,x_r\}=(x_1+\ldots+x_r)\;-\;(r-1)\min\{x_1,\ldots,x_r\}\,.$$

Define *r-partitions* $\bigl(\mu^{(1)},\ldots,\mu^{(r)}\bigr)$ of $(m_1,\ldots,m_r)$ by the conditions $\mu^{(k)}=m_k$ for $1\leq k\leq r$, and $\min\bigl\{\mu_i^{(1)},\ldots,\mu_i^{(r)}\bigr\}\geq\operatorname{smax}\bigl\{\mu_i^{(1)},\ldots,\mu_i^{(r)}\bigr\}$ for all i. Compute the r-variable generating function for the number of r-partitions.

9.3. **Partitions and integral points in cones.**

9.3.1. *The setup.* Suppose a set of partitions can be defined as a set of integral points in a cone in a (finite dimensional) vector space $\mathbb{R}^d$. The cone $\mathcal{C}$ is called *unimodular* if it has d supporting rays spanned by integral vectors $v_1, \ldots, v_d$, such that $\det(v_1, \ldots, v_d) = \pm 1$. This implies that all integral points in $\mathcal{C} \cap \mathbb{Z}^d$ have a form $\alpha_1 v_1 + \ldots + \alpha_d v_d$, where $\alpha_i \in \mathbb{N}$. Now, given two such cones $\mathcal{C}$ and $\mathcal{C}'$ as above, one can obtain a bijection between the integral points in the cones by a unimodular map defined on a basis $\varphi : v_i' \to v_i$.

Usually, the set of partitions is given as a cone in an infinite dimensional vector space. In that case one has to define an increasing subsequence of finite dimensional vector spaces which converges to the desired partition space. Also, the vector space is usually provided with a weight function which has to be preserved under the bijection. The details are easy to understand in the following examples.

9.3.2. ($\Diamond$) Let $\mathcal{T}_n$ be the set of integer triples (a, b, c), such that $0 \le a \le b \le c \le a+b$, and the *perimeter* $a+b+c = n$. These triples are called *integer triangles*. Let us prove that $|\mathcal{T}_n|$ is equal to the number of partitions of n into parts 2, 3, and 4, an therefore

$$\sum_{n=0}^{\infty} |\mathcal{T}_n|\, t^n \;=\; \frac{1}{(1-t^2)(1-t^3)(1-t^4)}\,.$$

Indeed, the corresponding cone $\mathcal{C}$ is spanned by vectors $(0,1,1)$, $(1,1,1)$, and $(1,1,2)$ which have perimeter 2, 3, and 4, respectively. Similarly, partitions $\lambda = (2^x 3^y 4^z)$ are spanned by $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, which are partitions of 2, 3, and 4, respectively. Check that both cones are unimodular and that the corresponding map is given by $\varphi : \lambda = (2^x 3^y 4^z) \to (x+y, x+y+z, x+y+2z) \in \mathcal{T}_n$, for all $\lambda \vdash n$.

9.3.3. ($\Diamond$) Fix an integer $r \ge 1$. Consider a set $\mathcal{H}_n$ of partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, such that $\lambda_i \ge r\lambda_{i+1}$. Let us prove that $|\mathcal{H}_n|$ is equal to the number of partitions on n into parts $b_i = (r^i - 1)/(r-1)$.

First, restrict the problem to a finite dimensional vector space by considering only partitions with $\ell(\lambda) \le k$. Prove that the corresponding unimodular cone is spanned by vectors $v_i = (r^{i-1}, r^{i-2}, \ldots, r, 1, 0, \ldots, 0)$. Define an obvious map $\varphi : v_i \to (b_i)$ and check that the resulting linear map defines a bijection between $\{\lambda \in \mathcal{H}_n : \ell(\lambda) \le k\}$ and partitions into parts b_i, $1 \le i \le k$. Letting $k \to \infty$, obtain the result.

9.3.4. ($\circ$) Modify the previous example to partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ which satisfy *Fibonacci conditions*: $\lambda_i \ge \lambda_{i+1} + \lambda_{i+2}$.

9.3.5. ($\Diamond$) Consider a set $\mathcal{H}_n$ of partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \vdash n$ with nonnegative second differences $\Delta^2(\lambda) \ge 0$, i.e., such that $\lambda_i - 2\lambda_{i+1} + \lambda_{i+2} \ge 0$. Let us prove that $|\mathcal{H}_n|$ is equal to the number of partitions on n into parts $b_i = \binom{i}{2}$.

First, restrict the problem to a finite dimensional vector space by considering only partitions with $\ell(\lambda) \le k$. Prove that the corresponding unimodular cone is spanned by vectors $v_0 = (1, \ldots, 1)$ and $v_i = (i-1, i-2, \ldots, 2, 1, 0, \ldots, 0)$, for $1 \le i < k$. Define an obvious map $\varphi : v_i \to (b_i)$, $v_0 \to (k)$. Conclude that the number of partitions $\lambda \in \mathcal{H}_n$ with at most k parts is equal to the number of partitions into parts k and b_i, $1 \le i < k$. Letting $k \to \infty$, obtain the result.

9.3.6. ($\circ$) Generalize the previous example to partitions with nonnegative r-th differences.

9.3.7. ($\circ\circ$) Let $\mathcal{L}_{n,k}$ be the set of *lecture hall partitions* $\lambda \vdash n$, $\ell(\lambda) \le k$, which are defined by conditions:

$$\frac{\lambda_1}{n} \ge \frac{\lambda_2}{n-1} \ge \cdots \ge \frac{\lambda_n}{1} \ge 0.$$

Prove that

$$1 + \sum_{n=1}^{\infty} |\mathcal{L}_{n,k}|\, t^n \;=\; \frac{1}{(1-t)(1-t^3)\cdots(1-t^{2k-1})}\,.$$

9.4. **Euler's recurrence for the sum of the divisors.**

9.4.1. Let $\zeta(n) = \sum_{d\,|\,n} d$ be the *sum of the divisors* of n. Note that $\zeta(p^k) = \frac{p^{k+1}-1}{p-1}$, if p is a prime. Also, $\zeta(m\,n) = \zeta(m)\zeta(n)$, if $\gcd(m,n)=1$.

▼ *For every $n > 0$, we have:*

$$\zeta(n) - \zeta(n-1) - \zeta(n-2) + \zeta(n-5) + \zeta(n-7) - \ldots + (-1)^r \zeta\left(n - \frac{r(3r\pm 1)}{2}\right) + \ldots$$
$$= \begin{cases} (-1)^{k-1} n\,, & \textit{if } \; n = \dfrac{k(3k\pm 1)}{2} \\ 0\,, & \textit{otherwise.} \end{cases}$$

The above result is called *Euler's identity.* While it can be easily deduced analytically from Euler's Pentagonal Theorem 5.1.1, the following proof is a nice example of "constructive arguments" in additive number theory.

▶ The proof is based on a double counting argument, and involves a sign-reversing involution. We start by defining a set

$$\Lambda_n = \{(\lambda, c, d) : \lambda \in \mathcal{D},\; c, d \ge 1,\; |\lambda| + c\,d = n\},$$

where $\mathcal{D}$ is a set of partitions with distinct parts. Now let $S_n = \sum_{(\lambda,c,d)\in\Lambda_n} (-1)^{\ell(\lambda)}\, d$. We will show that S_n is equal to both sides of Euler's identity above.
For the left hand side, from the proof of 5.1.1, we have:

$$\begin{aligned} S_n \;&=\; \sum_{m=1}^{n} \sum_{(\lambda,c,d)\in\Lambda_n :\, cd=m} (-1)^{\ell(\lambda)}\, d \;=\; \sum_{m=1}^{n} \left(\sum_{\lambda\in\mathcal{O}_{n-m}} (-1)^{\ell(\lambda)} \right) \sum_{d\,|\,m} d \\ &=\; \sum_{m=1}^{n} \zeta(m) \cdot \begin{cases} (-1)^r, \text{ if } n-m = r(3r\pm1)/2 \\ 0, \quad \text{otherwise} \end{cases} \\ &=\; \sum_r (-1)^r \zeta\left(n - \frac{r(3r-1)}{2}\right) \;+\; (-1)^r \zeta\left(n - \frac{r(3r+1)}{2}\right). \end{aligned}$$

For the right hand side, by definition of Λ_n, we have:

$$\begin{aligned} S_n \;=\; &\sum_{(\lambda,c,d):\; m_d(\lambda)>0} (-1)^{\ell(\lambda)}\, d \quad + \sum_{(\lambda,c,d):\; m_d(\lambda)=0,\; c>1} (-1)^{\ell(\lambda)}\, d \\ &+ \sum_{(\lambda,1,d):\; m_d(\lambda)=0} (-1)^{\ell(\lambda)}\, d\,, \end{aligned}$$

where $m_d(\lambda)$ is a multiplicity of part d in $\lambda \in \mathcal{D}$. Now, adding part d to λ maps triples (λ, c, d) with no part d in λ and $c > 1$ into triples (μ, c, d) with partition μ containing part d. Since $\ell(\mu) = \ell(\lambda) + 1$, this cancels the first two sums. For the third sum, consider again adding part d to λ. This gives a partition $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{D}_n$, from which one can subtract any of the parts. We obtain:

$$\sum_{(\lambda,1,d)\in\Lambda_n:\ m_d(\lambda)=0} (-1)^{\ell(\lambda)}\, d \ = \ -\sum_{\mu\in\mathcal{D}_n} (-1)^{\ell(\mu)} \big(\mu_1+\mu_2+\dots\big) \ = \ -n \sum_{\mu\in\mathcal{D}_n} (-1)^{\ell(\mu)}.$$

Now, from 5.1.1 this sum is equal to the right hand side in Euler's identity. This completes the proof. ∎

9.4.2. (∘) Prove in a similar manner the following identity:

$$\zeta(n)-3\zeta(n-1)+5\zeta(n-3)-7\zeta(n-6)+\dots+(-1)^r(2r+1)\,\zeta\left(n-\binom{r+1}{2}\right)+\dots$$

$$= \begin{cases} (-1)^{k-1}\dfrac{k(k+1)(2k+1)}{6}\,, & \text{if } \ n=\dfrac{k(k+1)}{2}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

9.4.3. (∗) Use the involution principle to give a sign-reversing involution which cancels terms in Euler's identity. Can one give a direct description of this map?

9.4.4. (◊) Prove combinatorially the following identity:

$$\sum_{n=1}^{\infty} \frac{t^n}{1-t^{2n}} \ = \ \sum_{n=1}^{\infty} \frac{t^{2n-1}}{1-t^{2n-1}}\,.$$

9.4.5. (∘) Let $\zeta_\circ(n)$ be the number of odd divisors of n. Prove combinatorially that $\zeta_\circ(n) = |\{(k,\ell) : n = \binom{k+1}{2} + k\ell,\ k,\ell \ge 0\}|$. Deduce from here another *Jacobi's identity*:

$$\frac{t}{1-t}+\frac{t^3}{1-t^3}+\frac{t^5}{1-t^5}+\frac{t^7}{1-t^7}+\dots = \frac{t}{1-t}+\frac{t^3}{1-t^2}+\frac{t^6}{1-t^3}+\frac{t^{10}}{1-t^4}+\dots$$

9.5. **Uchimura's formula for the number of divisors.**

9.5.1. Let $\sigma(n)$ be the *number of divisors*. Note that $\sigma(p^k) = k+1$, when p is a prime. Also, $\sigma(m\,n) = \sigma(m)\,\sigma(n)$, if $\gcd(m,n)=1$.

The following *Uchimura identity* gives an interpretation of $\sigma(n)$ in terms of partitions:

$$\sum_{d=1}^{\infty} \frac{t^d}{1-t^d} \ = \ \sum_{k=1}^{\infty} k\, t^k \prod_{i=k+1}^{\infty} (1-t^i).$$

Taking coefficients of t^n, this is equivalent to:

$$\sigma(n) \ = \ -\sum_{\lambda\in\mathcal{D}_n} (-1)^{\ell(\lambda)} s(\lambda),$$

where $\mathcal{D}_n$ is the set of partitions $\lambda \vdash n$ into distinct parts. We present here a bijective proof of the Uchimura identity in this form.

Let $\mathcal{C}(m) = \{\lambda \in \mathcal{D} : a(\lambda) \ge m > a(\lambda) - s(\lambda)\}$, and let $\mathcal{C}_n(m) = \{\lambda \vdash n : \lambda \in \mathcal{C}(m)\}$, where $a(\lambda) = \lambda_1$ is the largest part of λ. Clearly, for all $\lambda \in \mathcal{D}$ there exist exactly $s(\lambda)$ integers m such that $\lambda \in \mathcal{C}(m)$. This gives:

$$\sum_{\lambda\in\mathcal{D}_n} (-1)^{\ell(\lambda)} s(\lambda) \ = \ \sum_{m=1}^{n} \sum_{\lambda\in\mathcal{C}_n(m)} (-1)^{\ell(\lambda)}.$$

Now the Uchimura identity follows from:

$$\sum_{\lambda\in\mathcal{C}_n(m)} (-1)^{\ell(\lambda)} \, = \, \begin{cases} -1, \text{ if } m \,|\, n \\ \ \ 0, \text{ otherwise.} \end{cases}$$

Let $F_n(m)$ contain exactly one partition (n) when $m\,|\,n$, and let $F_n(m) = \emptyset$ otherwise. Let $\mathcal{A}_1 = \{\lambda \in \mathcal{C}_n(m) : \lambda_i \nmid n, \ \text{for all } 1 \le i \le \ell(\lambda)\}$, and let $\mathcal{A}_2 = \{\lambda \in \mathcal{C}_n(m) : \ell(\lambda) \ge 2, \ \text{and } \lambda_i \mid n, \ \text{for some } 1 \le i \le \ell(\lambda)\}$. Clearly, $\mathcal{C}_n(m) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup F_n(m)$.

We define a sign-reversing involution φ on the set of partitions $\mathcal{C}_n(m)$ with the set of fixed points $F_n(m)$. The map φ is also a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$, and will be defined as follows. Let $\lambda \in \mathcal{A}_2$, with part $\lambda_i \mid n$ and $\ell = \ell(\lambda) \ge 2$. Remove part $\lambda_i = c\,m$ from λ and add m to the smallest part $s(\widetilde{\lambda})$ of the remaining partition. Then add m to the smallest part of the obtained partition. Repeat this c times, until we obtain a partition $\mu \vdash n$. Now let $\varphi(\lambda) = \mu$. Note that $\ell(\mu) = \ell - 1$.

To reverse the procedure, start with $\mu \in \mathcal{A}_1$ and subtract m from the largest part μ_1. Then subtract m from the largest part $\widetilde{\mu}_1$ in the resulting partition μ. Repeat this until we reach $\widetilde{\lambda}$, and the total subtracted amount $c\,m$ satisfies $s(\widetilde{\lambda}) + m > c\,m > \widetilde{\lambda}_1 - m$. Then add part $(c\,m)$ to $\widetilde{\lambda}$, to obtain $\lambda = \varphi^{-1}(\lambda)$.

▼ *The map $\varphi : \mathcal{A}_2 \to \mathcal{A}_1$ defined above is a bijection.*

9.5.2. ($\Diamond$) Let $m = 7$, $\lambda = (16, 14, 13, 11) \in \mathcal{A}_2$. We have $\widetilde{\lambda} = (16, 13, 11)$, $c = 2$. Then the partition $\widetilde{\lambda}$ is successively transformed into $(18, 16, 13)$, and then into the partition $\varphi(\lambda) = \mu = (20, 18, 16) \in \mathcal{A}_1$.

9.5.3. ($\circ$) Let $\sigma_d(n) = \sum_{m|n} m^d$. Clearly, $\sigma_0(n) = \sigma(n)$, and $\sigma_1(n) = \zeta(n)$. Extend the above argument to show:

$$\sigma_d(n) \, = \, - \sum_{\lambda\in\mathcal{D}_n} (-1)^{\ell(\lambda)} \sum_{i=1}^{s(\lambda)} (\lambda_1 - s(\lambda) + i)^d.$$

9.5.4. ($*$) Can one describe the bijection φ by means of the involution principle?

9.5.5. ($*{*}*$) Prove combinatorially the following Jacobi formulas for the number $r_k(n)$ of decompositions of n as a sum of k squares of integers: $r_2(n) = 4\big(\delta_1(n) - \delta_3(n)\big)$, $r_4(n) = 8\,\zeta(n)$, when n is odd, and $r_4(n) = 24\,\zeta_o(n)$, when n is even. Here δ_i is the number of divisors $d\,|\,n$, such that $d \equiv i$ mod 4, and $\zeta_o(n)$ is the number of odd divisors of n (cf. 9.4.5).

9.5.6. ($**$) Prove combinatorially "*Liouville's Last Theorem*": For all integer $n > 0$, $\sigma_2(n) - n\sigma_0(n)$ is equal to the number of integer quintuples (w, x, y, z, u), such that $wx + xy + yz + zu = n$, and $w, x, z, u \ge 0$, $y > 0$.

9.5.7. ($**$) Prove combinatorially the *Dirichlet-Ramanujan identity*:

$$\sum_{a,b\in\mathbb{Z}} t^{a^2+ab+b^2} \, = \, 1 \, + \, 6 \sum_{n=0}^{\infty} \left(\frac{t^{3n+1}}{1 - t^{3n+1}} \, - \, \frac{t^{3n+2}}{1 - t^{3n+2}} \right).$$

9.6. **Asymptotic behavior of the partition function.**

9.6.1. There are very precise formulas for the asymptotic behavior of the partition function $p(n)$. The following formula of Hardy and Ramanujan is already too precise to be accessible by combinatorial methods:

$$p(n) \sim \frac{1}{4\sqrt{3}\,n}\, e^{\pi\sqrt{\frac{2}{3}n}}.$$

A much weaker result:

$$e^{a\sqrt{n}} < p(n) < e^{b\sqrt{n}} \quad \text{for some } b > a > 0$$

is not difficult to obtain, and we sketch two combinatorial proofs of both the lower and the upper bound. We shall use the notation in 2.2.1 and no analytic tools other than Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

9.6.2. ($\Diamond$) Write a partition $\lambda \in \mathcal{P}_{n,k}$ with at most k parts as a sum $n = \lambda_1 + \ldots + \lambda_k$, with $\lambda_i \geq 0$. Taking all permutations of the parts, deduce that

$$k!\, p_k(n) \geq \binom{n+k-1}{k-1} \geq \frac{n^{k-1}}{(k-1)!}.$$

Setting $k = \lfloor \sqrt{n} \rfloor$, obtain the lower bound.

9.6.3. ($\Diamond$) Let ρ_m be a partition $(m-1, m-2, \ldots, 1) \vdash \binom{m}{2}$. Take $m = 2k$, and consider all $\binom{2k}{k}$ Young diagrams obtained by adding k squares in the m outside corners of $[\rho_m]$. Setting $m = \lfloor \sqrt{2n} \rfloor$, obtain the lower bound.

9.6.4. ($\circ$) Define $q_k(n)$ by the following formula:

$$\sum_{n=0}^{\infty} q_k(n)\, t^n \;=\; \frac{1}{(1-t)^2(1-t^2)^2\ldots(1-t^k)^2}.$$

Deduce from here the recurrence relation:

$$q_k(n) \;=\; q_{k-1}(n) \,+\, 2\, q_{k-1}(n-k) \,+\, 3\, q_{k-1}(n-2k) \,+\, \ldots$$

Use induction to show that

$$q_k(n) \;\leq\; \frac{(n+k^2)^{2k-1}}{(2k-1)!(k!)^2}.$$

Rewrite Euler's identity 2.3.1 as follows:

$$p(n) \;=\; q_1(n-1) \,+\, q_2(n-4) \,+\, q_3(n-9) \,+\, \ldots$$

Therefore,

$$p(n) \;\leq\; \sum_{k=1}^{\infty} \frac{n^{2k-1}}{(2k-1)!\,(k!)^2}.$$

Use Stirling's formula to obtain the upper bound.

9.6.5. ($\Diamond$) Start with the following recurrence:

$$n\,p(n) \,=\, \sum_{r=1}^{n} r \sum_{\lambda\vdash n-r} m_r(\lambda) \,=\, \sum_{r=1}^{n} r \sum_{m=1}^{\lfloor n/r\rfloor} p(n-mr)\,.$$

The first equality can be obtained by the following double counting argument. Observe that $n\,p(n)$ is the total number of squares in all Young diagrams of partitions $\lambda \vdash n$. The middle term is a summation over all r of squares in all rows of length r, which occurs exactly $m = m_r$ times in $[\lambda]$. For the second equality, we have:

$$\begin{aligned}\sum_{\lambda\vdash n} m_r(\lambda) \,&=\, \bigl|\{\lambda\vdash n:\, m_r(\lambda)=1\}\bigr| \,+\, 2\,\bigl|\{\lambda\vdash n:\, m_r(\lambda)=2\}\bigr| \\ &\quad +\, 3\,\bigl|\{\lambda\vdash n:\, m_r(\lambda)=3\}\bigr| \,+\ldots \\ &=\, \bigl|\{\lambda\vdash n:\, m_r(\lambda)\ge 1\}\bigr| \,+\, \bigl|\{\lambda\vdash n:\, m_r(\lambda)\ge 2\}\bigr| \\ &\quad +\, \bigl|\{\lambda\vdash n:\, m_r(\lambda)\ge 3\}\bigr| \,+\ldots \\ &=\, p(n-r) \,+\, p(n-2r) \,+\, p(n-3r)\,\ldots\end{aligned}$$

Assume that $p(k) < e^{c\sqrt{k}}$ for al $k < n$, where $c = \pi\sqrt{\frac{2}{3}}$ is the same as in the Hardy-Ramanujan's formula. Now use the above formula in the induction step:

$$np(n) \,<\, \sum_{(i,m):\, im<n} r\, e^{c\sqrt{n-mr}} \,<\, e^{c\sqrt{n}} \sum_{r=1}^{\infty}\sum_{m=1}^{\infty} r e^{(-cm/2\sqrt{n})r}.$$

Note that $\sum_1^\infty r\,t^r \,=\, t/(1-t)^2$ and $e^{-x}/(1-e^{-x})^2 \,<\, \frac{1}{x^2}$, for all $x\in\mathbb{R}$. We conclude:

$$p(n) \,<\, \frac{e^{c\sqrt{n}}}{n} \sum_{m=1}^{\infty} \frac{e^{-cm/2\sqrt{n}}}{\bigl(1-e^{-cm/2\sqrt{n}}\bigr)^2} \,<\, \frac{e^{c\sqrt{n}}}{n} \sum_{m=1}^{\infty} \frac{4n}{c^2m^2} \,=\, e^{c\sqrt{n}}\frac{4}{c^2}\left(\frac{\pi^2}{6}\right) \,=\, e^{c\sqrt{n}}\,.$$

10. Final Remarks

1. Let us start by saying that the identities that appear in this survey seem to appear also in other subjects seemingly as remote as Statistical Physics, Algebra, Number Theory and Lie Theory [24]. Virtually none of the relevant results or references are presented here. For more on Partition Theory and q-series see [24, 78].

2. Traditionally, in the context of Partition Theory, partitions are usually represented by *Ferrers graphs* (named after Ferrers [121]), which are drawn with dots instead of squares (see e.g. [24, 4]). We chose to use Young diagrams for clarity and consistency.

Sylvester was also the one to name and use Ferrers' diagrams (see [121] p. 258). Interestingly, Sylvester agonized over the fact that he had to draw pictures. In [121], he tried several different versions and issued the following apologetic disclaimer:

> *The method is in its essence absolutely independent of graphical consideration, but as it becomes somewhat easier to apprehend by means of graphical description and nomenclature, I shall avail myself here of graphical terminology to express it.*

Despite obvious benefits to the reader, the use of Ferrers' graphs or Young diagrams to represent partitions became widespread only recently. Unfortunately, a number of older papers do not have any pictures. The following quote from [76] may explain the situation:

> *Combinatorial constructs involving partitions are most easily communicated by drawing suitably chosen pictures. This is not the style most often used in the literature. The reason may be that pictorial descriptions are sometimes thought to lack precision and rigor. On the other hand, mathematical language can be a rather imprecise medium at times...*

It seems m-modular diagrams and MacMahon diagrams go back to Frobenius and MacMahon, in one form or another. They were also rediscovered on many occasions afterwards and bear other names. We chose a name "MacMahon diagrams" in honor of the discoverer whose contributions were largely overlooked for so long. The standard MacMahon diagrams is a subclass of MacMahon diagrams invariant under conjugation; as the reader shall see this is the most useful notion.

For a bijection proving 2.1.5, consider the first step of Sylvester's bijection ψ defined in 3.4.1. This result seems to be due to Durfee (see [121]).

The terms of the summation on the l.h.s. in 2.1.6 are the probabilities that a random permutation $\sigma \in S_n$ has cyclic type $\lambda \vdash n$ (see e.g. [115] § 1.3). For 2.1.7 see [89]. The relationship between these two identities is puzzling.

The identity in 2.3.5 is taken from [118]. For a combinatorial proof of the Rogers-Fine identity, its history and applications see [14] (see also [59] for a proof in the language of MacMahon diagrams). For Vahlen's involution, see [123, 120]. Exercise 2.4.2 implies the *Regev–Vershik Conjecture*, as presented in [39] (see also [34] for the generating function). The identity 2.2.4 is given in [117].

Ramanujan's identity 2.2.2 implies one of the celebrated Ramanujan's congruence $p(5k-1) \equiv 0 \bmod 5$. Ramanujan also found congruences modulo 7 and 11, and now many other congruences are known (see e.g. [2, 24]). The rank of a partition (see 2.5.1) was defined by Dyson in [60] for the purposes of giving a combinatorial interpretation 2.5.6 of the congruences. He reminisced in [62] on his discovery: *"I gave thanks to Ramanujan for two things, for discovering congruence properties of partitions and for not discovering the criterion for dividing them into equal classes."* Dyson conjectured in [60] that his rank statistic gives a combinatorial interpretation of Ramanujan's congruences modulo 5 and 7, but found it errs modulo 11. These conjectures were later proved in [32]. In Dyson's own words, *"I think this should be enough to disillusion anyone who takes Professor Littlewood's innocent views of the difficulties of algebra"* (see the quote in the introduction).

Dyson also conjectured the existence of a hypothetical statistic he called "crank" which would give a combinatorial interpretation of all three congruences. He summarized his "guesses" in [60] and remarked that

> *Whatever these guesses are warranted by evidence, I leave to the reader to decide. Whatever the final verdict may be, I believe the "crank" is unique among arithmetic functions in having been named before it was discovered.*

Building on Garvan's work [77] for triples of partitions 2.5.7 the crank was eventually found by Andrews and Garvan in [29], where they proved 2.5.9 analytically. For a story of a famous phone call, see [31].

The Fine-Dyson symmetry relation was given in [61, 70]. Fine's relations 2.5.2 appeared in [69, 70] (see [105] for a historical account). The generating function derivation 2.5.4 follows [61] (see also [62, 37]). Combinatorial proofs of 2.5.8, 2.5.7, and 2.5.9 were given in [63].

The q-binomial identity was found by Rothe and was rediscovered by Cauchy and others (see [1] p. 5). For analytic proofs of the q-binomial identity and the Heine transformation see [24] p. 17, 20. Our proofs of the q-binomial theorem 2.6.1 and the Heine transformation 2.7.1 are loosely based on double counting arguments given in [12] (see also [5]). Hardy described Ramanujan's ${}_1\psi_1$-summation 2.7.6 as *"a remarkable formula with many parameters"* [80]. It was observed by Ismail [90] that the summation can be derived from the q-binomial identity 2.6.1, after substituting $b = q^m$ for the integer m, and then using analyticity (cf. [58]). The identity 2.6.3 was given in [5] and is a special case of Ramanujan's ${}_1\psi_1$-summation.

3. Euler's Theorem **??** was probably the starting point of Partition Theory [66]. See [4] for more on the history of Euler's and Glaisher's Theorems. Franklin's extension 3.3 (together with a generalization 3.3.2) was given in [73] and does not appear in modern literature. In Franklin's words, "[Glaisher's Theorem and Franklin's extension] *are very easily obtained either by the constructive proof or by generating function"* (see [121], p. 268). Most recently, these results were rediscovered in [127].

Sylvester's bijection 3.4.1 is presented in [121], and is sometimes called a *fish-hook construction* (see [24, 21, 23]). Sylvester [121] p. 287, gives an acute observation when comparing two correspondences:

> [Glaisher's] *correspondence is eminently arithmetic and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each integer with reference to the number 2.*
>
> *Very different is* [Sylvester's correspondence] *which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but separated each other of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated.*

Extensions 3.4.2 and 3.5 were stated by Fine in [69] and proved by analytic means in [70]; the proofs were published about four decades after their discovery. Both results were noted to follow from Sylvester's bijection and the Fine-Dyson map by Andrews [10, 20]. We refer to [24] for references and other proofs. Exercise 3.4.3, combined with 5.2.2 follows [105] (see also [130]). Together they prove two other results of Fine, related to certain identities of Ramanujan, which were proved analytically in [10, 19]. The iterated Dyson's map and a full historical account of Fine's partition results were given in [105].

Variation 3.2.4 goes back to Glaisher and Lehmer. A combinatorial proof is given in [79]. Vector partitions originated in [121, 83]. Generalization 3.2.5 is proved in [55] by means of generating functions.

The second presentation ζ of the Sylvester's bijection 3.4.1 follows [108]. It is essentially the same as that in [25], where the bijection was defined in the language of Frobenius coordinates of a partition. Finding a different presentation of Sylvester's bijection was justified in [21] by the fact that *"the reversal of Sylvester's* *algorithm is quite cumbersome".* Sylvester's bijection was extended to partitions of type (c, m) in [108] (see also [130]).

The third presentation η follows [38]. It is somewhat midway between the two and can be used to prove that all three maps define the same bijection [130]. Interestingly, yet a different version (of a reversed bijection) has recently appeared in [93] A refinement 3.4.5 was given in [41] (see also [93, 130]).

4. The intermediate sets of partitions $\mathcal{R}_{n,k}$ and $\mathcal{G}_{n,k}$ in both proofs of Lebesgue's identity 4.1 go back to Andrews [11]. Our presentation in 4.2.1 follows closely Bessenrodt's original paper [38]. The rest of the section uses the language of MacMahon diagrams, and we modify the constructions appropriately. The bijection 4.3.2 is due to Bressoud [43], where it was generalized to prove 4.3.5. While Bressoud's presentation may appear different, it is essentially equivalent to the bijection we give (see also [6]).

The second proof of the Lebesgue identity we present in 4.4.1 is a modified version of [7]. It is built heavily upon [43]. In the original paper [29], the authors formulate it as a double counting proof in the spirit of [11]. This explains the claim in [38] that [29] does not contain a direct proof. In fact, the proof 4.2.1 is indirect as it uses Euler's Theorem as the first step of a bijection. Partition identity 4.4.2 is taken from *Ramanujan's "Lost" Notebook* (see [16] p. 18).

Schur's Partition Theorem 4.5.1 was given in [112]. Our proof is a modified version of [44], which also contains 4.5.2. For various extensions, generalizations, a bijective proof of 4.5.3, and recent references, see [8] (cf. [27]).

5. Euler's Pentagonal Theorem is implicit in Euler [66]. The corresponding recurrence relation was, in fact, used for centuries to tabulate values of $p(n)$. Hardy and Ramanujan used such a table for $n \leq 200$, which was provided to them by MacMahon [80, 81].

Franklin's proof 5.2.1 was published in [72]. A modified version 5.2.3 is presented in [100]. The first refinement 5.2.4 is due to Shanks [113] who proved it by induction, and thus obtained a simple proof of Euler's Pentagonal Theorem. The identity was also proved in [95, 119] by Franklin's involution.

Formula 5.2.6 is due to Zagier; it was proved using Franklin's involution in [54]. The identity in 5.2.5 is taken from [86]. Results 5.2.2 and 5.2.7 are due to Fine [69, 70] (see 3.4.3 and [105]). Theorem 5.2.8 is equivalent to an identity of Ramanujan (see [16] p. 100).

Our proof of Sylvester's identity 5.3.1 follows the original generalization of Durfee squares by Sylvester [121] p. 268. For other generalizations of Durfee squares see [18]. Exercise 5.3.2 is perhaps the most natural explanation of the nature of Franklin's involution.

The bijective proof in 5.4.1 was found in [47]. The proof of 5.4.2 is due to Dyson [61]. For the history of the subject and the solution to 5.4.3 see [105]. The involution in 5.5.1 is a modified version of a bijection in [14]. For the rest of the section 5.5, references, and details, see [14].

6. Jacobi's triple product identity was first found by Gauss in an unpublished manuscript, and became famous after its rediscovery by Jacobi (see [30], §4.) The history of a direct bijection is quite involved and somewhat educational. We believe it deserves to be told in full as it is symptomatic of the subject.

The first bijective proof of the Jacobi identity is due to Sylvester [121] (see below), just one year after Franklin's proof was published. Eighty years later it was rediscovered by Wright [128] in a short note, with nice pictures and clear presentation. Soon thereafter Sudler realized that Wright's bijection is equivalent to that of Sylvester. He wrote:

> *I discovered that Sylvester had already given a proof of* [the Jacobi identity] *of the required type. However, because of his somewhat verbose and somewhat unclear style, his work on this topic has apparently been almost completely ignored in recent times except by MacMahon, who gave* [102] §323*, a generalization of Sylvester's idea.*

Naturally, Sudler decided to improve the rigor and exposition of Wright's and Sylvester's papers; his effort [120] did not contain a single picture. As J. Roberts put it in the AMS Review article on [120]: *"To read the paper one needs to have a copy of Wright's paper* [128] *at hand."*

A subsequent quest for a better exposition of Sylvester's bijection is perplexing. A series of papers [55, 57, 76, 97, 99, 120, 124] described a number of bijections, all of which are either equivalent or give *exactly the same* correspondence as Sylvester's. Since the authors seemed to be aware of the previous work, they emphasized the notation and the qualities of their presentation. For example, Leibenzon writes that his description *"seems the most elementary and explicit"* [97].

Our presentation in 6.2.1 follows Wright [128]. Two versions of a bijection in 6.2.2 follow Vershik [124] and Lewis [99]. The latter paper also acknowledges that the correspondence is identical to that of Sylvester. To quote Lewis: "[Sylvester's] *description of this correspondence is fairly obscure as the diligent reader will discover"* [99]. Most recently, an equivalent version has appeared in [57], where it was attributed to Itzykson and Viennot.

The involutive proof we present here follows Zolnowsky [131]. In fact, it can also be found in Sylvester's paper [121]. The following quote from [95] puts a new spin on the issue:

> *The literature contains several incorrect references to the history of Sylvester's construction. Sudler* [120] *says that the approach taken by Wright* [128] *is essentially that of Sylvester; but in fact it is essentially the same as another construction due to Arthur S. Hathaway, quoted by Sylvester* [121] § *62. Zolnowsky independently rediscovered Sylvester's rules* [...]
>
> *Sylvester's original treatment has apparently never been cited by anyone else, possibly because it comes at the end of a very long paper; furthermore, his notation was rather obscure and he made numerous errors that a puzzled reader must rectify.*

So who is the real author of the direct proof of Jacobi identity? Our brief historical investigation showed that both Hathaway and Sylvester are the authors of two different albeit equivalent versions. It seems Sudler is referring to Sylvester's proof in [121], § 38–40, while [95] is alluding to a full two page quote of Hathaway's paper [83] in Sylvester's "Exodion" [121], § 62. Sylvester himself did not seem to notice the relationship. Thus attributing the proof to both Hathaway and Sylvester (as done in [23]) is quite appropriate.

Perhaps, the shortest and the most elementary analytic proof of the Jacobi identity is due to Andrews [9], who deduced it directly from Euler's two identities in 2.2.3 and 2.3.4. The quintuple product identity is a classical result in analysis, going back to G. N. Watson and Karl Weierstrass. We refer to [52] for the history of the quintuple product identity, references and a simple proof. Another simple proof, extensions and more recent references can be found in [71]. Vahlen's Theorem 6.1.5 has appeared in [123], and in this form was presented in [125] p. 165. When $m = n$, MacMahon's identity 6.2.3 is called the *Cauchy identity* [48]. For a simple inductive proof, see [85]. Identity 6.1.7 is taken from [22] p. 99.

7. The Rogers-Ramanujan identities ($\star$) and ($\star\star$) are due to Rogers and were later rediscovered by Schur, Ramanujan, and others. There are numerous analytic proofs known, as well as proofs by means of Lie Theory, but not a single direct bijective proof. We refer to [24] for many generalizations and further references.

The following two quotes were highly influential in the subject. According to Hardy, *"None of the proofs of* [($\star$) and ($\star\star$)] *can be called "simple" and "straightforward, since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof"* [80]. Forty years later, Andrews concurred with this sentiment: *"Hardy's comments about the nonexistence of a really easy proof of the Rogers-Ramanujan identities are still true today"* [24].

In his lecture notes [75], Garsia challenges the above assessment. He starts by saying:

> *Schur independently discovers the Rogers identities* [($\star$) and ($\star\star$)] *and (unlike Ramanujan) is also able to provide a proof. We may add that it is really a great historical injustice (mostly due to the tabloid sensationalism of G. H. Hardy) to refer to* [($\star$) and ($\star\star$)] *as the Rogers-Ramanujan identities.*

He then continues to criticize the above Hardy's quote:

> *Hardy must have not given a close look at Schur's paper, otherwise such a judgement can only be a result of Hardy's lack of knowledge of 19th century "Partition" literature. Schur's proof is not only quite simple, but a straightforward extension of Franklin's proof of the Euler Pentagonal Theorem* [5.1.1]. *Moreover, as such it is substantially different from any innumerable other proofs* [of ($\star$) and ($\star\star$)], *that have been given in the more than 100 years since they have been discovered.*

In Hardy's defense, he did seem to know everything there was to know about "partition literature." In the very same book [80] he presents *"F. Franklin's beautiful proof"* (see pp. 83–85), and writes, *"About the same time* [of Ramanujan's rediscovery of ($\star$) and ($\star\star$) published earlier by Rogers] *I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs, one of which is "combinatorial" and is quite unlike any other proof known"* (see p. 92). Hardy then proceeds to restate Rogers-Ramanujan's identities as combinatorial results and concludes with the following passage:

> *These forms of the theorems are MacMahon's (or Schur's); neither Rogers nor Ramanujan ever considered their combinatorial aspect. It is natural to ask for a proof in which we set up, by "combinatorial" arguments, a direct correspondence between the two sets of partitions, but no such proof is known. Schur's "combinatorial" proof is based not on* [identity ($\star$)] *itself, but on a transformation of the formula* [...] *It is not unlike Franklin's proof of* [Euler's Pentagonal Theorem 5.1.1] *but a good deal more complicated.*

It is natural to assume that the preponderance of analysis over combinatorics in those days led Hardy to believe that Schur's proof 7.2.2 is quite complicated, a view not shared in modern times. Other than this evaluation, both authors seem to be in accordance with each other. Injustice or not, the name "Rogers-Ramanujan's identities" has long been accepted as standard in the field. See [92] § 7.11 for an independent literary account of how Ramanujan "rediscovered" and published identities ($\star$) and ($\star\star$) after previously seeing them in Rogers' paper, since, in Ramanujan's words, "[the identities] *had entirely slipped from my memory."*

The interpretation 7.1.4 is given in [18]. In our presentation of Schur's proof [111] we follow [76], which used a rather different language. Our Figure 31 is based upon pictures in [75]. Generalizations 7.2.4 (see also 7.2.3) are given in [74]. For 7.2.6, see [46]. The Farkas-Kra identity 7.2.5 was given in [68] p. 521. For 7.3.1, see e.g. [82] § 19.15 (see also [24]).

8. The involution principle was introduced by Garsia and Milne [76] as a tool to give a bijective proof of the Rogers–Ramanujan's identities (see 8.4.4). Although the authors claimed to have *"an algorithm for the construction of bijections in a wider combinatorial setting than that of the theory of partitions,"* the involution principle has rarely been used outside of the field.

The first equality in 8.1.3 is due to Schur [112], while the second is due to Andrews [27] (cf 4.5.1). The exercise 8.4.2 is explicit in [76].

Exercise 8.4.6 is based on Zagier's "one-sentence proof" [129]. For missing sentences see [28]. The involution β goes back to Heath-Brown. We dispute the assertion that Zagier's proof is ineffective, which was made in [129] and repeated in [3]. In fact, in view of the involution principle it is effective indeed, albeit the corresponding algorithm is probably very inefficient (see [114] for the analysis).

A few words about the history of the problem. In 1747 Euler showed that the decomposition $p = x^2 + 4y^2$ is unique, proving a conjecture of Fermat (see e.g. [64] §2.4). Fermat himself claimed to have such a proof. In a letter to Pascal he asks for a general rule for finding such a decomposition (ibid. §2.6). For efficient polynomial time algorithms see [33].

For Andrews Theorem 8.1.1 see [24], where the equivalent partition bricks are called *simple classes of partitions.* O'Hara's Algorithm 8.2.1 was given in [104]. Historically, O'Hara's paper was based upon Remmel's and Gordon's work and has appeared later. She proves in [104] that the bijection she defines coincides with Remmel's and Gordon's bijection in a special case.

The Cohen-Remmel Theorem 8.6.1 was found by Cohen [56] and then extended by Remmel [109] by removing a technical *disjointness condition.* Our presentation of the Cohen-Remmel Theorem follows the recent paper [103]. We use here a very different, slightly less general and more structured language. For other presentations see [126] (see also [115] § 2). The exercises 8.1.4 and 8.6.6 are taken from [127]. They are direct generalizations of Franklin's extension 3.3.1 of Euler's Theorem 3.1.

9. For MacMahon's Theorem 9.1.1 see [102], (see also [24]). For Stanley's formula, extension 9.1.5, other generalizations, a connection to symmetric functions and references, see [115], Chapter 7. The first bijective proof of MacMahon's Theorem is found in [36]. Our presentation follows [107]. Both proofs are related to the Robinson-Schensted-Knuth correspondence (see e.g. [24, 107, 115]). Formula 9.1.7 is also due to MacMahon; the only direct bijective proof we know [96] uses an assortment of 'bijective technology' not covered in this paper.

The bipartitions 9.2.1 were introduced in [50, 51]. The extension 9.2.3 is given in [15]. Our proof follows [110] (cf. [40]).

Our presentation of 9.3 follows [106]. For more on integer points in cones and polytopes see [115]. For integer triangles 9.3.2, see [17] (see also [26]). A bijection in 9.3.3 is given in [84]. Partitions with nonnegative r-th differences 9.3.6 were introduced in [26]; the first bijective proof was given in [49]. The lecture hall partitions 9.3.7 were introduced in [41] and further studied in [42].

For a thorough treatment of arithmetic functions and their properties, see [82]. Our proof of Euler's identity 9.4.1 follows [125], p. 161 (see also [68] p. 472). Identity 9.4.5 follows from the Gauss identity 6.1.4 [67]. The Uchimura identity 9.5.1 was obtained in [122]. Our proof follows closely [45]. Simple proofs of the Jacobi formulas 9.5.5 are given in [87, 88] where the author deduces them from the triple product identity (see also [3]). A proof of Liouville's Last Theorem can be found in [35] (see also [25] for historical context and recent references).

Our two lower bounds in 9.6 are probably folklore. The first upper bound proof 9.6.4 follows [81], while the second upper bound proof 9.6.5 follows [65]. The recurrence relation used in 9.6.5 was given in [81]. Note that the lower bounds, while simpler, give weaker estimates than the second proof of the upper bound (cf. [65]).

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