## §4 A first conclusion

We now prove that if M is uniquely normally iterable, then every normal iterate of M is normally iterable. We prove it in the slightly stronger form:

Thm 1 Let M be uniquely normally iterable. Let $I^* = \langle\langle M_i^*\rangle, \langle \nu_i^*\rangle, \langle \pi_{ij}^*\rangle, T^*\rangle$ be a normal iteration of M of length $\eta^*+1$. Let $\sigma^*: N \xrightarrow[\Sigma^*]{} M_{\eta^*}^*$ $\min(\rho^*)$. Then N is normally iterable.

Proof.
We must show that N has a successful normal iteration strategy. Let $I = \langle\langle N_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a normal iteration of N. We define:

Def By a justification of I (wrt. $I^*, \sigma^*, \rho^*$) we mean a pair $\langle \mathcal{S}, I'\rangle$ s.t.

(a) $\mathcal{S} = \langle\langle I^i\rangle, \langle \nu^i\rangle, \langle e^{ij}\rangle, T\rangle$ is an insertion s.t. $I^0 = I^*$ and $lh(\mathcal{S}) = lh(I)$.
Let $I^i = \langle\langle M_h^i\rangle, \langle \nu_h^i\rangle, \langle \pi_{hj}^i\rangle, T^i\rangle$.
We let $\sigma_h^{ij}, \tilde{\sigma}_h^{ij}$ be the insertion maps induced by $e^{ij}, \tilde{e}^{ij}$.

(b) $I' = \langle\langle N'_i\rangle, \langle \pi'_{ij}\rangle, \langle \sigma_i\rangle, \langle \rho^i\rangle\rangle$ is a mirror of $I$ s.t.

(i) $N'_i = M^i_{\eta_i}$ for $i < lh(I)$, where

! $lh(I^i) = \eta_i + 1$.

(ii) $\rho^0 = \rho^*$ ; $\sigma_0 = \sigma^*$

(iii) Let $h = T(i+1)$. Then $\pi'_{h,i+1} = \tilde{\sigma}^{h,i+1}_{\eta}$

where, $\eta + 1 = lh(I^i_*)$ (hence $I^i_* = I^h | \eta+1$).

(iv) $\nu^i = \nu'_i = \sigma_i(\nu_i)$

<u>Note</u> Let $h = T(i+1)$. If $\tau^i$ is not a cardinal in $N'_h = M^h_{\eta_h}$, then it is not a cardinal in $M^h_{t_i}$ and $M^h_{\eta_h} \| \mu = M^h_{t_i} \| \mu$ where $\mu$ is maximal s.t. $\tau^i$ is a cardinal in $M^h_{\eta_h} \| \mu$.

Moreover:

$$\tilde{\sigma}^{h,i+1}_{t_i} = \pi'_{h,i+1} : M^h_{\eta_h} \| \mu \xrightarrow[E'_{\nu_i}]{*} M^{i+1}_{\eta_{i+1}}.$$

On the other hand, if $\tau^i$ is a cardinal in $N'_i$, then

$$\tilde{\sigma}^{h,i+1}_{\eta_h} = \pi'_{h,i+1} : N'_h \xrightarrow[\Sigma^*]{} N'_{i+1} \text{ and}$$

$$E^{N'_i}_{\nu^i} = \pi'_{h,i+1} \upharpoonright \mathbb{P}(\kappa^i).$$

<u>Note</u> $\langle S, I'\rangle$ is uniquely determined by $\langle I^*, \sigma^*, \rho^*\rangle$.

<u>Def</u> $I$ is <u>justifiable</u> (wrt. $I^*, \sigma^*, \rho^*$) iff it has a justification.

Lemma 1 Let $I$ be of length $\mu+1$. Let $E_\nu^{N_\mu} \neq \emptyset$ where $\nu > \nu_i$ for $i < \mu$. Then $I$ extends to a justifiable iteration of length $\mu+2$ s.t. $\nu = \nu_\mu$.

proof

Set $\nu' = \sigma_\mu(\nu)$. Then $\nu' > \nu^i = \nu_i'$ for $i < \eta$. Hence by II Lemma 3.3, $S$ extends to an insertion of length $\mu+2$ with $\nu^\mu = \nu'$. Extend $I$ to a potential iteration of length $\mu+2$ by appointing: $\nu_\mu = \nu$. Set $\nu_\mu' = \nu^\mu = \sigma_\mu(\nu)$. Note that, letting $N_{\mu+1}' = M^{\mu+1}_{\eta_{\mu+1}}$ in the extension of $S$ and letting $\pi' = \sigma^{h,\mu+1}_{\eta_{\mu+1}}$, then $\pi' : M_\mu'^* \xrightarrow[\Sigma^*]{} N_{\mu+1}'$ where $h = T(\mu+1)$ in the potential iteration extending $I$. Applying III Lemma 2 we obtained an extended mirror system $\langle \hat{I}, \hat{I}' \rangle$ with $\pi' = \pi_{h,\mu+1}'$ and $\pi_{h,\mu+1} : N_\mu^* \xrightarrow[E_{\nu_\mu}]{} N_{\mu+1}$ in $\langle \hat{I}, \hat{I}' \rangle$.

QED (Lemma 1)

By II Lemma 4 and III Lemma 1 we then easily get:

<u>Lemma 2</u> Let $I$ be of limit length $\mu$. Let $b$ be the unique cofinal well founded branch in $\mathbb{S}$. Then $b$ is a cofinal well founded branch in $I$ and $I$ can be extended to a justifiable iteration of length $\mu+1$ by setting: $T``\{\mu\}=b$.

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This gives us a successful iteration strategy $S$ for $N$: Let $I$ be a normal iteration of $N$ of limit length $\mu$. If $I$ is not justifiable, then $S(I)$ is undefined. If not, let $\langle \mathbb{S}, I' \rangle$ be the justification of $I$. Let $b$ be the unique cofinal well founded branch in $\mathbb{S}$. Then $S(I)=b$.

This proves the theorem.

Def By a <u>smooth iteration of M of finite degree</u> n we mean $\langle I_0, \ldots, I_n\rangle$ s.t.

- $I_i = \langle\langle M_h^i\rangle, \langle \nu_h^i\rangle, \langle \pi_{h\,l}^i\rangle, T^i\rangle$ is a normal iteration
- $M = M_0^0$
- If $i < n$, then $lh(I_i) = \eta_i + 1$, where $M_0^{i+1} = M_{\eta_i}^i$.

By an <u>iteration strategy</u> for such iterations we mean S s.t. S is defined only on $\langle I_0, \ldots, I_n\rangle$ s.t. $I_n$ is of limit length and $S(\langle I_0, \ldots, I_n\rangle)$, if defined, is a cofinal well founded branch in $I_n$.

We say that $\langle I_0, \ldots, I_n\rangle$ is S-<u>conforming</u> iff whenever $i \leq n$ and $\mu < lh(I_i)$ is a limit ordinal, then $T^{i\,\text{''}}\{\mu\} = S(\langle I_0, \ldots, I_i | \mu\rangle)$.

S is a <u>successful strategy</u> for <u>smooth iterations of M of finite degree</u> iff whenever $\langle I_0, \ldots, I_n\rangle$ is S-conforming and $I_n$ is of limit length, then $S(\langle I_0, \ldots, I_n\rangle)$ is defined.

Modifying the above proof slightly we get:

<u>Theorem 2</u> Let $M, I^*, \sigma^*, \rho^*, N$ be as above. Then $N$ has a successful iteration strategy for smooth iterations of finite degree.

proof (sketch)

Let $I = \langle I_0, \dots, I_n \rangle$ be a smooth iteration of finite degree.

<u>Def</u> By a <u>justification</u> of $I$ (wrt $I^*, \sigma^*, \rho^*$) we mean $\langle\langle \delta_0, I_1' \rangle, \dots, \langle \delta_n, I_n' \rangle\rangle$ s.t.

(a) $\langle \delta_0, I_0' \rangle$ is a justification of $I_0$ wrt $\langle I^*, \sigma^*, \rho^* \rangle$

(b) $\langle \delta_{i+1}, I_{i+1}' \rangle$ is a justification $I_{i+1}$ wrt $\langle I_{i+1}^*, \sigma_{i+1}^*, \rho_{i+1}^* \rangle$ where:

- $I_{i+1}^* =$ the final iteration in $\delta_i$
- $\sigma_{i+1}^* = \sigma^{i}_{\eta_i}$ where $I_i' = \langle \langle N_h^{i} \rangle, \langle \pi_{h j}^{i} \rangle, \langle \sigma_h^{i} \rangle, \langle \rho^{i,h} \rangle$ and $\eta_i \cancel{+} 1 = lh(I_i)$
- $\rho_{i+1}^* = \rho^{i,\eta_i}$

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With this machinery it is easily seen that $I$ is justifiable iff it is built accordingly to the strategy:

Let $\langle I_0, \dots, I_n \rangle$ have justification $\langle\langle \delta_0, I_0' \rangle, \dots, \langle \delta_n, I_n' \rangle\rangle$. ~~Let~~ $I_n$ be of limit length $\mu$. Let $b$ be the unique cofinal well founded branch in $\mathcal{S}_n$. Set $S(I) = b$.

The details are left to the reader.

QED (Thm 2)

Insertion invariance

Def Let $S$ be a normal iteration strategy for a premouse $M$. We call $S$ insertion invariant iff whenever $e$ inserts an iteration $I$ of $M$ into $I'$, and $I'$ is an $S$-iteration of $M$, then $I$ is an $S$-iteration of $M$. It is fairly easy to see that the assumption: "$M$ is uniquely normally iterable" can be replaced by: "$M$ has a successful normal iteration strategy which is insertion invariant." In particular, $M$ then has a successful normal iteration strategy for smooth iterations of finite degree. The proofs are virtually unchanged.