WHAT IS SO EXOTIC ABOUT DIMENSION FOUR?

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Abstract. We give an overview about what goes wrong in higher dimensions, especially regarding smooth structures on 4-manifolds. References are given, when possible, but a lot the things discussed are still open problems and conjectures. We also try to show sketches of the constructions and the proofs. But because we don't want to oversize the notes and because this is very technical and difficult, we try to avoid too much detail.

First, a quick refresher about manifolds:

- A **topological** manifold is one whose atlas has continuous transition maps.
- A **smooth** one, on the contrary, has smooth transition maps.

Then, we say that a topological manifold is **smoothable** if its atlas is compatible with a smooth one (recall that two atlases $\mathscr{A}$ and $\mathscr{B}$ are compatible if $\mathscr{A} \cup \mathscr{B}$ is a continuous atlas). The underlying idea is that one can extend the topological structure into a smooth one.

- Two topological manifolds M and N are **homeomorphic** if there exists a bijection $f : M \to N$ such that both f and f^{-1} are continuous.
- Two smooth ones are **diffeomorphic** if f and f^{-1} are smooth.

Some facts:

(1) In dimensions $\leqslant 3$, all topological manifolds are smoothable, and in an essentially unique way (two smooth atlases on the same manifold are always compatible). In particular, two homeomorphic manifolds are diffeomorphic.
(2) This is no longer true in dimensions $\geqslant 4$.

Part of the problem is revealed when looking at the classification of manifolds:

(1) Classifying 1-manifolds is trivial.
(2) Classifying surfaces is easy: the triple (o, b, χ) is a complete invariant, with $o = 0$ or 1 depending on whether the surface is orientable, b the number of boundary components, and χ the Euler characteristic of the surface that has been capped with discs.
(3) For 3-manifolds, it is more difficult, but:
    - Prime decomposition, torus decompositions, presentation by Dehn surgery, *etc.*
    - Haken manifolds, prime/irreducible manifolds, Seifert-fibered manifolds, *etc.*
(4) For 4-manifolds and higher: this becomes undecidable!

To show that the problem is really undecidable, the idea is that one can build an n-manifold whose fundamental group is any given finitely presented group ($n \geqslant 4$). Indeed, consider such a grroup $G = \langle g_1, \dots, g_k | r_1, \dots, r_s \rangle$. Just like one defines a 2-dimensional CW-complex, start with $\natural^k(\mathbf{S}^1 \times \mathbf{B}^{n-1})$ (these are just thickened edges), and glue s copies of $\mathbf{B}^2 \times \mathbf{B}^{n-2}$, with attaching maps given by the relations. This describes a **handle decomposition** of the resulting manifold, and this manifold deformation retracts to the 2-dimensional CW-complex built analogously. In particular, its π_1 has presentation that of G. We know that the word problem (deciding if two finitely presented groups are isomorphic or not) is undecidable; see [Nov55] for the first proof of this fact. This implies that:

- There cannot be an algorithm to recognize n-manifolds, otherwise this could be applied to solve the word problem.
- The word problem is also undecidable for the trivial group; thus, there cannot be an algorithm to decide if an n-manifold is simply-connected or not.

Note that there may be a procedure that fully computes the fundamental group of a manifold (*e.g.* by Kirby calculus), but then one still needs to solve the word problem for that specific presentation.

Some further comments:

(1) Not every finitely presented group arises as the fundamental group of a 3-manifold (see [AFW15]).
(2) Simply-connected smooth manifolds in dimensions $\geqslant 5$ are classified (for instance, see [Sma62b] or [Bar65] for 5-manifolds). This comes mostly from the h-cobordism theorem (see [Sma62a]) and from surgery theory. In particular, in dimension five, it suffices to classify them up to homotopy equivalence.
(3) Simply-connected **topological** 4-manifolds have been classified by Freedman (see [Fre82]), by means of intersection theory and the Kirby–Siebenmann invariant.

Classifying **smooth** simply-connected 4-manifolds is still a widely open problem. This is mostly due to the following phenomena:

- A **wild** manifold is one that is not smoothable.
- An **exotic pair** is a pair of smooth manifolds that are homeomorphic but not diffeomorphic. Equivalently, a choice of a smooth structure on a smoothable topological manifold is a smooth equivalence class of atlases. In this regard, an exotic pair corresponds to a choice of two incompatible smooth structures.

Wild manifolds are bad, but not so much in terms of dealing with smooth manifolds (for they do not arise at all). The first such example is the E_8 manifold, which comes from intersection theory. We will roughly sketch how it was first discovered, but the two main ingredients' proofs are beyond this exposition.

Claim. *For any compact oriented topological manifold X^4, there exists a well-defined quadratic form $Q_X : H_2(X;\mathbf{Z}) \to \mathbf{Z}$, called the **intersection form** of X^4.*

We now mention the following theorem, due to Freedman:

Theorem. *Given any unimodular symmetric $Q \in \mathscr{M}_n(\mathbf{Z})$, there exists an oriented simply-connected closed topological manifold X with $Q_X = Q$. Moreover:*

(1) If $Q(x,x)$ is even for all x, then X is unique up to homeomorphism.
(2) If there exists x for which $Q(x,x)$ is odd, then there are exactly two non-homeomorphic satisfactory manifolds X and X', and at least one of them is wild. Those manifolds are distinguished by their Kirby–Siebenmann invariant.

In particular, two (closed oriented) smooth simply-connected 4-manifolds with the same intersection form are homeomorphic. Indeed, in the even case, there is only one possibility, and in the odd case, they cannot be the wild one of the two, for both are already smooth.

There is a form of converse of this theorem, due to Donaldson (see [Don83] and [Don87]), that restricts the possible intersection forms arising from smooth manifolds:

Theorem. *If X^4 is a smooth simply-connected 4-manifold that has a positive definite (or negative definite) intersection form, then this intersection form is diagonalizable over the integers.*

Take the following matrix:

$$E_8 = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}.$$

This is a unimodular symmetric bilinear form, and it is even. By Freedman's theorem, there exists a unique (up to homeomorphism) closed oriented simply-connected manifold, also denoted E_8, that has it as its intersection form. Now, this form is definite positive, but not it is not diagonalizable over the integers. By Donaldson's theorem, E_8 cannot be smooth; that is, E_8 is a wild manifold.

For exotic manifolds, this is a different story. One can construct an exotic structure on non-compact 4-manifolds by using the following theorem, due to Quinn:

Theorem. *Take any* ***non-compact*** *topological manifold X^4, and fix a smooth structure on ∂X. Then this structure extends to the whole X.*

For instance, on $\mathbf{R}^4$, one can think of $\partial\mathbf{R}^4$ as $\mathbf{S}^3$ or as a point. In any case, one obtains the standard structure on $\mathbf{R}^4$. Gompf used this theorem, together with the Trace Embedding Lemma, to obtain exotic $\mathbf{R}^4$'s. The following phenomenon occurs:

- For any $n \neq 4$, any smooth manifold homeomorphic to $\mathbf{R}^n$ is also diffeomorphic to it. That is, there are no exotic $\mathbf{R}^n$'s for $n \neq 4$. The proof for $n \geqslant 5$ is due to Stallings, see [Sta62].
- $\mathbf{R}^4$ has uncountably many smooth structures. This is due to Taubes, see [Tau87].
- Some exotic $\mathbf{R}^4$'s embed as an open subset of the standard $\mathbf{R}^4$, and some don't (those are respectively called **small** and **large** exotic $\mathbf{R}^4$'s). But there is a maximal exotic $\mathbf{R}^4$ in which all the others embed as open subsets (Freedman–Taylor, see [FT86]).

Question ([Kir78, Problem 4.77])**.** *An exotic $\mathbf{R}^4$ crossed with $\mathbf{R}^1$ is diffeomorphic to $\mathbf{R}^5$. How can we usefully see the exotic $\mathbf{R}^4$ in $\mathbf{R}^5$?*

Now, what happens in the compact case?

(1) Capell and Shaneson constructed exotic $\mathbf{RP}^4$'s, see [CS76].
(2) There are exotic Kummer surfaces, called the $K3$ surfaces, see [Kod70].
(3) In higher dimensions, there are exotic spheres. Here is a table for the OEIS entry `A001676`:

n	5	6	7	8	9	10	11	12	13	14	15	16	17
Number of exotic $\mathbf{S}^n$'s	1	1	28	2	8	6	992	1	3	2	16256	2	16

Note that this counts the number of **oriented** diffeomorphism classes of smooth structures on the sphere. The first exotic spheres were exotic $\mathbf{S}^7$'s constructed by Milnor, see [Mil56].

One may formulate the following conjecture, known as the Poincaré conjectures:

Conjecture. *Let X^n be a topological (resp. smooth) manifold homotopy equivalent to $\mathbf{S}^n$. Then, X is homeomorphic (resp. diffeomorphic) to $\mathbf{S}^n$.*

In fact, we have:

(1) For $n \leqslant 2$, this is given by the classification of manifolds.
(2) For $n = 3$, this is true, and the is due to Perelman.
(3) For $n \geqslant 5$, this is true in the topological case (by work of Smale and the h-cobordism theorem again), and false in most dimensions in the smooth case (see the table above, where it holds in dimensions 5, 6, 12, *etc.*).
(4) For $n = 4$, this is true in the topological case (due to Freedman), and still open in the smooth case. This is one of the most difficult (if not the most) problems in topology today. Not everyone trusts it is true, but all attempts at finding exotic $\mathbf{S}^4$'s have failed so far.

Some more "fun facts" about dimension four and exotic topology in general:

(1) The Schoenflies conjecture, which states that if $\mathbf{S}^{n-1}$ is smoothly embedded in $\mathbf{S}^n$, then it bounds a smothly embedded $\mathbf{B}^n$, is true for all $n \neq 4$. For $n = 2$, this is the Jordan curve theorem. For $n = 3$, this is the Schoenflies theorem. For $n \geqslant 5$, this comes from the h-cobordism theorem. The problem is still open for $n = 4$.
(2) By work from Akhmedov and Park (see for instance [AP10a] and [AP10b]), one has infinitely many smooth structures on manifolds such as $3\mathbf{CP}^2\#7\overline{\mathbf{CP}}^2$ or $\mathbf{S}^2 \times \mathbf{S}^2$.
(3) By work from Kirby and Siebenmann (see [KS77]), we know that any n-manifold with $n \geqslant 5$ has only finitely (eventually none) many smooth structures. On the other hand, all examples of smooth 4-manifolds so far have infinitely many. The question whether there exists a smooth 4-manifold with only finitely many smooth structures is still open.
(4) Wall has proven (see [Wal64]) that given any exotic pair (X^4, Y^4) of closed oriented smooth 4-manifolds, there exists $k \in \mathbf{N}^\star$ such that $X\#k(\mathbf{S}^2 \times \mathbf{S}^2)$ and $Y\#k(\mathbf{S}^2 \times \mathbf{S}^2)$ become diffeomorphic.
(5) A 4-manifold is **geometrically simply-connected** if it admits a handle decomposition with no 1-handles. It is conjectured ([Kir78, Problem 4.18]) that a simply-connected 4-manifold is geometrically simply-connected. However, in [HMP21], the authors prove that there are infinitely many pairs of topological 4-manifolds that are homeomorphic but one is geometrically simply-connected and the other is not. This demonstrates that geometric simple-connectedness can depend on the smooth structure.
(6) The Hauptvermutung is not true anymore in dimension 4 (two triangulations of the same 4-manifold need not be combinatorially equivalent). This was first shown by Freedman.

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