THE PIECEWISE-LINEAR STRUCTURE OF EUCLIDEAN SPACE

By JOHN STALLINGS†

Communicated by E. C. Zeeman

Received 19 December 1961

1. *Euclidean space R^n ought to have a unique piecewise-linear structure and a unique differentiable structure.*

It is known that, for $n \leqslant 3$, 'ought to have' can truthfully be replaced by 'has' (see (4), and (5), Cor. 6·6). In this paper, this conjecture will be proved for $n \geqslant 5$. The only unsolved case then will be in dimension four.

Another corollary of the main theorem of this paper is this. If M^n is the Cartesian product of two non-trivial, contractible, open, piecewise-linear (or differentiable) manifolds, and if $n \geqslant 5$, then M^n is piecewise-linearly (or differentiably) equivalent to R^n. This extends results of McMillan and Zeeman (3).

Here is a plan of the proof. It will be shown that each compact subset of M is contained in a piecewise-linear n-cell; it will then follow from a theorem of Newman and Gugenheim that M is piecewise-linearly R^n; if M is also a differentiable manifold, the piecewise-linear result will imply that M is differentiably R^n. In order to enclose each compact subset of M in a piecewise-linear cell, it is necessary to invoke a result of Zeeman (11) and the author to the effect that in a contractible M^n every $(n-3)$-dimensional compact polyhedron is contained in a cell; and then it will be necessary to show that (when $n \geqslant 5$) an infinite 2-dimensional polyhedron can be deformed to miss any given compact set; the construction of this deformation involves an assumption that neighbourhoods of infinity are simply connected.

A proof of the Engulfing Theorem (3·1 in this paper) is included because of the fact that Zeeman's corresponding theorem is not quite general enough to prove a fact which is needed here (Corollary 3·3). This Engulfing Theorem has many important corollaries which I hope to discuss in subsequent papers.

2. The word 'space' in this paper is reserved for metrizable, locally compact, locally connected spaces. A space X, or a pair (X, Y), is said to be *k-connected* if $\pi_i(X) = 0$, or $\pi_i(X, Y) = 0$, for all $i \leqslant k$.

The space X is said to *have one end*, if X is not compact, and if for every compact $C \subset X$, there is a compact D, where $C \subset D \subset X$, such that $X - D$ is 0-connected.

The space X is said to be 1*-connected at infinity*, if for every compact $C \subset X$, there is a compact D, where $C \subset D \subset X$, such that $X - D$ is 1-connected.

For example, the plane R^2 has one end but is not 1-connected at infinity.

2·1. Proposition. *If A and B are* 1*-connected spaces, each of which has one end, then $A \times B$ is* 1*-connected at infinity.*

† This research was supported by the Air Force Office of Scientific Research.

Proof. Let C be a compact subset of $A \times B$. Since A and B have one end, there are compacts D and E, where $D \subset A$ and $E \subset B$ and $C \subset D \times E$, such that $A - D$ and $B - E$ are 0-connected. The claim is that $A \times B - D \times E$ is 1-connected. For,

$$A \times B - D \times E = A \times (B-E) \cup (A-D) \times B$$

and
$$A \times (B-E) \cap (A-D) \times B = (A-D) \times (B-E).$$

The spaces appearing in these formulae are all open, 0-connected subsets of

$$A \times B - D \times E.$$

Hence van Kampen's Theorem ((2), (7)) may be applied; it says that

$$\pi_1(A \times B - D \times E)$$

is obtained from the free product of $\pi_1(A \times (B-E))$ and $\pi_1((A-D) \times B)$ by identifying the images in each of the elements of $\pi_1((A-D) \times (B-E))$. This is evidently the trivial group since A and B are 1-connected. Hence $A \times B - D \times E$ is 1-connected, and so $A \times B$ is 1-connected at infinity.

2·2. PROPOSITION. *If A is a 1-connected space with one end and R is the real line, then $A \times R$ is 1-connected at infinity.*

Proof. Let C be a compact subset of $A \times R$. Then there is $D \subset A$ such that $A - D$ is 0-connected, and there is $I = [-r, r] \subset R$, such that $C \subset D \times I$. The claim is that $A \times R - D \times I$ is 1-connected. Let

$$U = A \times (-\infty, r) - D \times I \quad \text{and} \quad V = A \times (-r, \infty) - D \times I;$$

then $U \cap V = (A-D) \times (-r, r)$. Since $U \cap V$ is an open, 0-connected subset of $U \cup V = A \times R - D \times I$, van Kampen's Theorem may be applied. Therefore

$$A \times R - D \times I$$

will be 1-connected if both U and V are 1-connected. Now,

$$U = A \times (-\infty, -r) \cup (A-D) \times (-\infty, r);$$

applying van Kampen's Theorem to this expression of U and noting that A is 1-connected, it follows that U is 1-connected. Similarly V is 1-connected. Hence $A \times R - D \times I$ is 1-connected, and so $A \times R$ is 1-connected at infinity.

By a *manifold M of dimension n*, will be meant a triangulable space such that each point $x \in M$ has a neighbourhood homeomorphic to n-dimensional Euclidean space. Thus, M has no 'boundary'. The assumption of triangulability is for the sake of convenience at this point. Later, attention will be restricted to manifolds with especially nice triangulations.

$H_k(M)$ will denote the k-dimensional homology group of M with coefficient group the integers. $H_f^k(M)$ will denote the k-dimensional cohomology group of M *based on finite cochains* with integer coefficient group.

2·3. PROPOSITION. *A contractible manifold M^n of dimension $n \geqslant 2$ has one end.*

Proof. In fact, the only fact needed about M is that $H_{n-1}(M) = 0$. Then by the Poincaré Duality Theorem, $H_f^1(M) \approx H_{n-1}(M) = 0$. A result of Specker ((8), Satz IV) now implies that M has one end.

2·4. Proposition. *If $M^n = X \times Y$, where X and Y are manifolds, neither of which is a point, and if M is contractible and $n \geqslant 3$, then M is 1-connected at infinity.*

Proof. If X and Y are both of dimension $\geqslant 2$, then 2·3 shows X and Y each have one end, and 2·1 shows $X \times Y$ is 1-connected at infinity. If either X or Y has dimension 1, then it is homeomorphic to R, the real line; the other has dimension $\geqslant 2$; then 2·3 and 2·2 show $X \times Y$ to be 1-connected at infinity.

2·5. Proposition. *Let M^n be a contractible manifold which is 1-connected at infinity. Then for every compact $C \subset M$, there is a compact D, where $C \subset D \subset M$, such that $(M, M-D)$ is 2-connected.*

Proof. Take D to be such that $M-D$ is 1-connected. Then the exact homotopy sequence of $(M, M-D)$ plus the fact M is 2-connected imply that $(M, M-D)$ is 2-connected.

3. Now it is time for the most difficult part of the argument, the Engulfing Theorem. This says that an open subset of a piecewise-linear manifold can expand to engulf, like a piecewise-linear amoeba, any given subpolyhedron, provided certain dimension, connectivity, and finiteness conditions are met.

The proof is a long argument by a multiple induction. The steps of this induction are elementary arguments which should be included in any basic text on piecewise-linear topology. Unfortunately, such a text does not exist. Therefore the proof given here will contain gaps where these basic facts will be assumed but not themselves proved. Such facts are general position lemmas (see (11)), and the fact that a piecewise-linear cell can be linearly triangulated to be a collapsible simplicial complex (see the first few pages of (9)). Also, the tedious construction of certain maps which almost obviously exist will be omitted in favour of what I hope is a heuristic description. It is hoped that this abbreviation of the proof will make it easier to grasp. It is believed that the reader will be able to supply the omitted details, if he so wishes.

First, after some definitions, the Engulfing Theorem is stated. Then the two corollaries which will be used in this paper will be derived from it. Then a proof of the Engulfing Theorem is described.

Simplicial complexes K and L are *combinatorially equivalent* if there are rectilinear subdivisions of K and L which are isomorphic as simplicial complexes. K is a *combinatorial cell* if K is combinatorially equivalent to a closed simplex. A triangulation T of a manifold M is *piecewise-linear* if the closed star of each point is a combinatorial cell (there is no known triangulation of a manifold which is not piecewise-linear); a manifold M and a piecewise-linear triangulation of it define a *piecewise-linear manifold. A polyhedron* P in a simplicial complex K is a subset corresponding to a subcomplex of some rectilinear subdivision of K. A *piecewise-linear cell* is a polyhedron whose rectilinear triangulation is a combinatorial cell. A *piecewise-linear map* $f: K \to L$ is a continuous function whose graph is a polyhedron in $K \times L$.

By a homeomorphism $h: X \to Y$ will be meant a homeomorphism of X *onto* Y. $1|X$ will denote the restriction to X of the function defined by $1(x) = x$ for all x.

3·1. Engulfing Theorem (cf. Zeeman (11)). *Let M^n be a piecewise-linear manifold; U an open subset of M; P a polyhedron of M of dimension p. Let (M, U) be p-connected;*

let $P \cap (M-U)$ be compact; let $p \leqslant n-3$. Then there is a compact $E \subset M$, and there is a piecewise-linear homeomorphism $h: M \to M$, such that $P \subset h(U)$ and

$$h\,|(M-E) = 1|(M-E).$$

Before proving this, two corollaries are obtained.

3·2. COROLLARY. *Let M^n be a contractible piecewise-linear manifold; let P be a compact polyhedron in M; let dimension $P \leqslant n-3$. Then P is contained in the interior of a piecewise-linear n-cell $A \subset M$.*

Proof. Apply 3·1 where U is the interior of a piecewise-linear n-cell V in M. Then $A = h(V)$.

3·3. COROLLARY. *Let M^n be a piecewise-linear manifold which is contractible and 1-connected at infinity; let C be a compact subset of M; let T be a piecewise-linear triangulation of M; let T^2 denote its 2-skeleton. Let $n \geqslant 5$. Then there is a compact E_1, and there is a piecewise-linear homeomorphism $h_1: M \to M$ such that*

$$C \subset E_1 \subset M \quad \textit{and} \quad T^2 \subset h_1(M-C) \quad \textit{and} \quad h_1\,|(M-E_1) = 1|\,(M-E_1).$$

Proof. By 2·5, there is a compact D with $C \subset D \subset M$ and $(M, M-D)$ 2-connected. Apply 3·1, substituting $M-D$ for U and T^2 for P. The conclusion of 3·3 is immediate, taking h_1 to be h and E_1 to be $E \cup C$. (The condition $n \geqslant 5$ is used to obtain the hypothesis $2 = p \leqslant n-3$ of 3·1.)

3·4. *Proof of the Engulfing Theorem.* The proof is by a multiple induction. Thus it is convenient to restate the theorem as follows:

ET(q,r). *Let M^n be a piecewise-linear manifold; U an open subset of M; P a polyhedron of M of dimension $p \leqslant n-3$; let (M, U) be p-connected. Let Q be a subpolyhedron of P such that $Q \subset U$ and dimension $(P-Q) \leqslant q$. Let P have a triangulation into piecewise-linear cells of M, such that Q is a subcomplex and the number of cells of $P-Q$ is $r < \infty$. Then there is a compact $E \subset M$, and there is a piecewise-linear homeomorphism $h: M \to M$, such that $h\,|(M-E) = 1|\,(M-E)$ and $P \subset h(U)$.*

The truth of ET(q,r) for all q and r implies 3·1, since in the hypothesis of 3·1 is the condition that $P \cap (M-U)$ is compact, and therefore a polyhedron Q satisfying the hypothesis of ET(q,r), for some q and r, can be found. The proof of ET(q,r) is by induction first on q and then on r.

ET$(q,*)$ means 'for all r, ET(q,r)'. Similarly, ET$(*,r)$ means 'for all q, ET(q,r)'.

(A) ET$(-1,*)$ *and* ET$(*,0)$ *are true.* For in this case $P = Q \subset U$; take $E = \varnothing$ and $h = 1$.

(B) ET$(q,1)$ *and* ET$(q,r-1)$ *imply* ET(q,r). Suppose the hypothesis of ET(q,r) holds. Then in $P-Q$ there is a maximal-dimensional cell Δ. Let $P' = P-(\text{interior}\,\Delta)$. Then $Q \subset U \cap P'$. ET$(q,r-1)$ is now applicable to P'. There is a compact $E' \subset M$ and a piecewise-linear homeomorphism $h': M \to M$ such that

$$h'\,|(M-E') = 1|\,(M-E') \quad \text{and} \quad P' \subset h'(U).$$

Define $Q_1 = P'$, $P_1 = P$, $U_1 = h'(U)$. ET$(q, 1)$ is now applicable to P_1, Q_1, U_1. There is a compact $E_1 \subset M$ and a piecewise-linear homeomorphism h_1: $M \to M$ such that $h_1|(M-E_1) = 1|(M-E_1)$ and $P_1 \subset h_1(U_1)$.

The conclusion of ET(q,r) follows by taking $h = h_1 h'$ and $E = E' \cup (h')^{-1}(E_1)$.

(C) *Therefore the problem of proving* ET(q,r) *for all q and r has been reduced to showing that* ET$(q, 1)$ *follows from* ET$(q', *)$ *for some* $q' < q$.

From henceforth, the hypothesis of ET$(q, 1)$ will be assumed. Δ will denote the cell of $P-Q$. If X is a cell or a simplex, $\dot{X}$ will denote its *boundary*; juxtaposition will denote *join*; for example, $v\dot{X}$ denotes the join of the point v to the boundary of X. I denotes $[0,1]$.

(D) *There is a map* f: $\Delta \times I \to M$ *such that* $f|\Delta \times 0$ *is the embedding of* Δ *into* M *and such that* $f(\dot{\Delta} \times I \cup \Delta \times 1) \subset U$. This follows from $\dot{\Delta} \subset Q \subset U$, dimension $\Delta \leqslant p$, and the fact that (M, U) is p-connected.

Let K be a space with triangulation T. Then a map f: $K \to M^n$ is said to be in *general position with respect to* T, provided:

(1) For every simplex A of T, $f|A$ is a piecewise-linear embedding.

(2) If A and B are simplexes of T, then $A \cap f^{-1}f(B) = (A \cap B) \cup S(A,B)$ where $S(A,B)$ is a polyhedron of dimension $\leqslant$ dimension A + dimension $B - n$.

(E) *Let* K *be the union of* P *and* $\Delta \times I$, *identifying* Δ *with* $\Delta \times 0$. *Then there is a map* f: $K \to M$ *such that* $f|P$ *is the embedding of* P *into* M, *and* $f(\dot{\Delta} \times I \cup \Delta \times 1) \subset U$; *and there is a rectilinear triangulation* T *of* K *such that* f *is in general position with respect to* T. This follows from (D) and an approximation lemma which is omitted.

Let T be a triangulation of X in which Y occurs as a subcomplex. It is said that *T collapses onto Y* if there are closed simplexes $A_1, \ldots, A_s$ of T such that

(1) $X = Y \cup A_1 \cup \ldots \cup A_s$,

(2) Each A_i has a vertex v_i and face B_i such that $A_i = v_i B_i$ and

$$(Y \cup A_1 \cup \ldots \cup A_{i-1}) \cap A_i = v_i \dot{B}_i.$$

(F) *The statement* (E) *is true with the additional conclusion that* T *collapses onto* $Q \cup \dot{\Delta} \times I \cup \Delta \times 1$. This follows from the fact that any rectilinear triangulation of $\Delta \times I$ has a subdivision which collapses onto $\dot{\Delta} \times I \cup \Delta \times 1$ (Whitehead (9), Theorem 6), and from (E) and from an omitted approximation lemma.

Henceforth, $A_1, \ldots, A_s$ will denote closed simplexes of the triangulation T of (F) such that $(Q \cup \dot{\Delta} \times I \cup \Delta \times 1 \cup A_1 \cup \ldots \cup A_{i-1}) \cap A_i = v_i \dot{B}_i$, where $A_i = v_i B_i$, and $K = Q \cup \dot{\Delta} \times I \cup \Delta \times 1 \cup A_1 \cup \ldots \cup A_s$.

D_i will denote the part of the p-skeleton of T which is contained in

$$Q \cup \dot{\Delta} \times I \cup \Delta \times 1 \cup A_1 \cup \ldots \cup A_i.$$

D_i is then that whole set except in the extreme case when dimension $\Delta = p$ when it will omit the interiors of some of the simplexes A_j. Nevertheless, $D_{i-1} \cap A_i = v_i \dot{B}_i$.

(G) *There is a compact set* $E_i \subset M$ *and a piecewise-linear homeomorphism* h_i: $M \to M$ *such that* $f(D_i) \subset h_i(U)$ *and* $h_i|(M-E_i) = 1|(M-E_i)$, *provided* ET$(q-1, *)$ *is true.*

The proof is by induction on i. For $i = 0$, take $E_0 = \varnothing$ and $h_0 = 1$, since

$$D_0 = Q \cup \dot{\Delta} \times I \cup \Delta \times 1 \quad \text{and} \quad f(D_0) \subset U.$$

Let $\Sigma_i = \cup\{S(A_i, B)|B$ is a simplex of T such that $B \subset D_{i-1}\}$. For the definition of $S(A, B)$ see (D). Then Σ_i is a polyhedron in A_i of dimension $\leqslant$ dimension $A_i + (n-3) - n$; now, $A_i \subset \Delta \times I$ and dimension $\Delta \leqslant q$. Therefore,

$$\text{dimension}\, \Sigma_i \leqslant (q+1) + (n-3) - n = q-2.$$

Let Λ_i be the union of all the line segments in A_i which (1) are parallel to the line from v_i to the barycentre of B_i, and (2) intersect Σ_i. Then Λ_i is a polyhedron of dimension $\leqslant q-1$; and A_i has a rectilinear triangulation which collapses onto $v_i\dot{B}_i \cup \Lambda_i$.

Then the statement ET$(q-1, *)$ applies to P_i, Q_i, U_i defined as follows:

$$P_i = f(D_{i-1} \cup \Lambda_i),$$
$$Q_i = f(D_{i-1}),$$
$$U_i = h_{i-1}(U).$$

Therefore, there is a compact $E \subset M$ and a piecewise-linear homeomorphism $h: M \to M$, such that $h\,|(M-E) = 1|\,(M-E)$ and $P_i \subset h(U_i)$.

Since A_i has a triangulation which collapses on to $v_i\dot{B}_i \cup \Lambda_i$, and since

$$D_{i-1} \cap A_i = v_i\dot{B}_i,$$

it follows that some triangulation of $D_{i-1} \cup A_i$ collapses onto $D_{i-1} \cup \Lambda_i$. Since $\Sigma_i \subset \Lambda_i$ there is no difficulty mirroring this collapsing in the image $f(D_{i-1} \cup A_i)$.

Hence $f(D_{i-1} \cup A_i)$ has a rectilinear triangulation which collapses onto $f(D_{i-1} \cup \Lambda_i)$. Since $f(D_{i-1} \cup \Lambda_i) \subset h(U_i)$, utilizing this collapsing, one may piecewise-linearly expand $h(U_i)$ until it covers $f(D_{i-1} \cup A_i)$. That is to say:

There is a compact $E_* \subset M$ and a piecewise-linear homeomorphism $h_*: M \to M$, such that $h_*\,|(M-E_*) = 1|\,(M-E_*)$ and $f(D_{i-1} \cup A_i) \subset h_* h(U_i)$.

One now defines $h_i = h_* h h_{i-1}$, and $E_i = E_{i-1} \cup h_{i-1}^{-1}(E) \cup h_{i-1}^{-1} h^{-1}(E_*)$. This clearly satisfies (G); and hence (G) has been proved.

(H) ET$(q-1, *)$ *implies* ET$(q, 1)$. This follows from (G), by taking $i = s$, noting that $P \subset D_s$ and $f|P$ is the embedding into M; define the h required in the conclusion of ET$(q, 1)$ to be h_s; and define E to be E_s.

Finally, (C) and (H) together prove ET(q, r) for all q and r. The proof of the Engulfing Theorem is over.

4. THEOREM. *Let M^n be a contractible piecewise-linear manifold which is* 1-*connected at infinity. If $n \geqslant 5$, then M is piecewise-linearly homeomorphic to Euclidean space R^n.*

Proof. (A) *If C is a compact subset of M, then $C \subset F \subset M$ where F is a piecewise-linear n-cell.*

Let T be a piecewise-linear triangulation of M, and apply 3·3. Let

$$K = T^2 \cup \{\text{all closed simplexes of } T \text{ which are contained in } M - E_1\}.$$

Then $K \subset h_1(M-C)$.

Now, if Y is a subcomplex of the simplicial complex X, the *complementary complex* $X \div Y$ is defined to be the subcomplex of the barycentric subdivision of X which is maximal with respect to the property of not intersecting Y.

Let $L = T \div K$. Then L is a compact polyhedron of dimension $\leqslant n-3$. Apply Lemma 3·2, with L in place of P. Then L is contained in the interior of a piecewise-linear n-cell $A \subset M$.

The barycentric subdivision of T is a subcomplex of the join KL. Since $K \subset h_1(M-C)$ and $L \subset$ interior of A, there exists a piecewise-linear homeomorphism h_2: $M \to M$ such that $M = h_1(M-C) \cup h_2$ (interior of A). Then $h_1^{-1}h_2(A) = F$ is a piecewise-linear n-cell containing C.

(B) It follows from (A) that M is the union of a sequence $\{F_i\}$ of piecewise-linear n-cells such that $F_i \subset$ interior of F_{i+1}.

Now, Newman (6) and Gugenheim (1) (Theorem 3 of Gugenheim is the exact statement needed here) say that the piecewise-linear equivalence classes of embeddings of a piecewise-linear n-cell in the interior of a connected n-manifold are completely determined by the orientations involved.

So, let $G_i \subset$ interior of G_{i+1}, where G_i and G_{i+1} are piecewise-linear n-cells; let h_i: $F_i \to G_i$ be a piecewise-linear homeomorphism. There is a piecewise-linear homeomorphism f: $F_{i+1} \to G_{i+1}$ which is compatible with the map of orientations given by h_i. Hence by Gugenheim's Theorem, the embeddings h_i and $f|F_i$ in G_{i+1} are piecewise-linearly equivalent. That is, there is a piecewise-linear homeomorphism g: $G_{i+1} \to G_{i+1}$ such that $h_i = g(f|F_i)$. Then gf: $F_{i+1} \to G_{i+1}$ is a piecewise-linear homeomorphism h_{i+1} which extends h_i.

This shows that M is piecewise-linearly homeomorphic to any manifold which is the union of a sequence $\{G_i\}$ of piecewise-linear n-cells, where $G_i \subset$ interior of G_{i+1}. R^n is such a manifold. The proof is complete.

4·1. COROLLARY. *Euclidean space R^n, for $n \geqslant 5$, has a unique piecewise-linear structure.*

Proof. If M is homeomorphic to R^n, then it is contractible and 1-connected at infinity. By 4, M is piecewise-linearly homeomorphic to R^n.

4·2. COROLLARY. *If X and Y are contractible piecewise-linear manifolds of dimensions x and y, where $x \geqslant 1$ and $y \geqslant 1$ and $x+y \geqslant 5$, then $X \times Y$ is piecewise-linearly homeomorphic to R^{x+y}.*

Proof. Apply 2·4 and 4.

5. *The differentiable case.* Quoted here is Corollary 6·6 of Munkres (5):

'*Two differentiable manifolds homeomorphic to R^n are diffeomorphic if they are combinatorially equivalent.*'

Any differentiable manifold has a smooth piecewise-linear triangulation (10). These facts plus Theorem 4 and its corollaries immediately produce the following.

5·1. THEOREM. *Let M^n be a contractible differentiable manifold which is 1-connected at infinity. If $n \geqslant 5$, then M is diffeomorphic to Euclidean space R^n.*

5·2. COROLLARY. *Euclidean space R^n, for $n \geqslant 5$, has a unique differentiable structure.*

5·3. COROLLARY. *If X and Y are contractible differentiable manifolds of dimensions x and y, where $x \geqslant 1$ and $y \geqslant 1$ and $x+y \geqslant 5$, then $X \times Y$ is diffeomorphic to R^{x+y}.*

REFERENCES

(1) GUGENHEIM, V. K. A. M. Piecewise linear isotopy and embedding of elements and spheres (I). *Proc. London Math. Soc.* (3), 3 (1953), 29–53.
(2) VAN KAMPEN, E. On the connection between the fundamental groups of some related spaces. *American J. Math.* 55 (1933), 261–267.
(3) MCMILLAN, D. R. and ZEEMAN, E. C. On contractible open manifolds. *Proc. Cambridge Philos. Soc.* 58 (1962), 221–224.
(4) MOISE, E. E. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermuting. *Ann. of Math.* (2), 56 (1952), 96–114.
(5) MUNKRES, J. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Ann. of Math.* (2), 72 (1960), 521–554.
(6) NEWMAN, M. H. A. On the superposition of n-dimensional manifolds. *J. London Math. Soc.* 2 (1926), 56–64.
(7) OLUM, P. Non-abelian cohomology and van Kampen's theorem. *Ann. of Math.* (2), 68 (1958), 658–668.
(8) SPECKER, E. Die erste Cohomologiegruppe von Überlagerungen and Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten. *Comment. Math. Helv.* 23 (1949), 303–333.
(9) WHITEHEAD, J. H. C. Simplicial spaces, nuclei and m-groups. *Proc. London Math. Soc.* 45 (1939), 243–327.
(10) WHITEHEAD, J. H. C. On C^1-complexes. *Ann. of Math.* (2), 41 (1940), 809–824.
(11) ZEEMAN, E. C. On polyhedral manifolds (to appear).

INSTITUTE FOR ADVANCED STUDY
AND
PRINCETON UNIVERSITY