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On Some Combinatorial Properties of Balancing Split Quaternions

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Abstract: Quaternions and split quaternions are used in quantum physics, computer science, and in many areas of mathematics. In this paper, we define and study two new classes of split quaternions, namely balancing split quaternions and Lucas-balancing split quaternions. Moreover, well-known properties, e.g., Catalan, d'Ocagne, and Vajda identities, for these quaternions are also presented. We give matrix generators for balancing split quaternions and Lucas-balancing split quaternions, too.

Keywords: balancing numbers; quaternions; split quaternions

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1. Introduction

Let $\mathbb{C}$ be a set of complex numbers. In 1843, W. R. Hamilton introduced an extension of complex numbers—the set of quaternions, denoted by $\mathbb{H}$. A quaternion q is defined as

$$q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\,, x_t \in \mathbb{R},\ t = 0,1,2,3,$$

where units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy the quaternion multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Multiplication of quaternions is non-commutative. The addition, the subtraction, and the multiplication by scalar $s \in \mathbb{R}$ for quaternions are defined in the following way:

Let $q_1 = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $q_2 = v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $s \in \mathbb{R}$. Then,

$$q_1 \pm q_2 = (x_0 \pm v_0) + (x_1 \pm v_1)\mathbf{i} + (x_2 \pm v_2)\mathbf{j} + (x_3 \pm v_3)\mathbf{k},$$

$$sq_1 = sx_0 + sx_1\mathbf{i} + sx_2\mathbf{j} + sx_3\mathbf{k}.$$

The quaternion $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ can be also represented by the square matrix of order 4 of the form

$$\begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix}.$$

Moreover, we can use the matrix of order 2 with complex number entries to define the quaternion q:

$$\begin{bmatrix} x_0 + x_1\mathbf{i} & x_2 + x_3\mathbf{i} \\ -x_2 + x_3\mathbf{i} & x_0 - x_1\mathbf{i} \end{bmatrix}.$$

Many authors have studied quaternion matrices (see [1,2]). By analogy with the theory of complex numbers, the conjugate of the quaternion $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is the quaternion $\overline{q} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. The norm of the quaternion q is defined as $N(q) = q \cdot \overline{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2$. If $q \neq 0$, then the quaternion has a multiplicative invers $q^{-1} = \frac{\overline{q}}{N(q)}$.

For basic quaternion concepts and some interesting properties of them, see, for example, [3,4].

The set of split quaternions (coquaternions), denoted by $\hat{\mathbb{H}}$, was introduced by J. Cockle in 1849 [5]. The split quaternion is defined as

$$p = y_0 + y_1 i + y_2 j + y_3 k\,, y_t \in \mathbb{R},\ t = 0, 1, 2, 3,$$

where units i, j, and k satisfy the non-commutative multiplication rules:

$$i^2 = -1,\ j^2 = k^2 = ijk = 1,$$

$$ij = k = -ji,\ jk = -i = -kj,\ ki = j = -ik.$$

We can write the split quaternion as follows:

$$p = (y_0 + y_1 i) + (y_2 + y_3 i)j = z_1 + z_2 j,\ z_1, z_2 \in \mathbb{C}.$$

The scalar and the vector part of a split quaternion are denoted by $S_p = y_0$ and $\vec{V}_p = y_1 i + y_2 j + y_3 k$, respectively. Hence, we can write a split quaternion as $p = S_p + \vec{V}_p$.

The set of split quaternions is four-dimensional and non-commutative, like the set of quaternions. The split quaternions contain nilpotent elements, nontrivial idempotents, and zero divisors. The conjugate of a split quaternion $p = y_0 + y_1 i + y_2 j + y_3 k$ is defined as $\overline{p} = y_0 - y_1 i - y_2 j - y_3 k$. The norm of p has the form

$$N(p) = p\overline{p} = y_0^2 + y_1^2 - y_2^2 - y_3^2. \tag{1}$$

For the basics of split quaternion theory, see [6]. Some interesting properties of split quaternions are presented in [7–11]; for example, De Moivre's formula and the roots of a split quaternion are given in [7]. In [8], split quaternion matrices are considered.

Quaternions are used in differential geometry, quantum physics, and in the synthesis of mechanisms and machines [12]. Split quaternions are used, among others, in color balance. The model refers to the Jordan algebra of symmetric matrices of order 2 with real entries; for details, see [13].

2. Balancing and Lucas-Balancing Numbers

Balancing numbers B_n were introduced by A. Behera and G. K. Panda in [14]. A positive integer n is called a balancing number with balancer r, if it is the solution of the following equation:

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

named a Diophantine equation. For each balancing number n, $\sqrt{8n^2+1}$ is called a Lucas-balancing number C_n (see [14]). Moreover, the balancing numbers and Lucas-balancing numbers are defined recursively:

$$B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 1, B_0 = 0, B_1 = 1, \tag{2}$$

$$C_{n+1} = 6C_n - C_{n-1} \text{ for } n \geq 1, C_0 = 1, C_1 = 3. \tag{3}$$

Table 1 includes eight terms of the sequences $\{B_n\}$ and $\{C_n\}$.

Table 1. The values of balancing and Lucas-balancing numbers.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40,391
C_n	1	3	17	99	577	3363	19,601	114,243

Balancing numbers and Lucas-balancing numbers are given by Binet formulas:

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}}, \tag{4}$$

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}, \tag{5}$$

where

$$\lambda_1 = 3 + 2\sqrt{2}, \ \ \lambda_2 = \lambda_1^{-1} = 3 - 2\sqrt{2}.$$

Note that

$$\begin{aligned} \lambda_1 + \lambda_2 &= 6, \\ \lambda_1 - \lambda_2 &= 4\sqrt{2}, \\ \lambda_1\lambda_2 &= 1. \end{aligned} \tag{6}$$

Balancing numbers have a negative extension $B_{-n} = -B_n$. Hence, the sequence of balancing numbers $\ldots, -35, -6, -1, 0, 1, 6, 35, \ldots$ has a symmetry property.

Some properties of balancing numbers and Lucas-balancing numbers are given in [14–17]. We recall some of them:

$$\begin{array}{rcll} C_n^2 &=& 8B_n^2 + 1 & \\ C_{2n} &=& 16B_n^2 + 1 & \\ B_{n+m} &=& B_nC_m + C_nB_m & \\ B_{n-m} &=& B_nC_m - C_nB_m & \\ C_{n-m} &=& C_nC_m - 8B_nB_m & \\ C_{n+m} &=& C_nC_m + 8B_nB_m & \\ B_{n-r}B_{n+r} - B_n^2 &=& -B_r^2 & \text{(Catalan identity)} \\ C_{n-r}C_{n+r} - C_n^2 &=& C_r^2 - 1 & \text{(Catalan identity)} \\ B_{n-1}B_{n+1} - B_n^2 &=& -1 & \text{(Cassini identity)} \\ C_{n-1}C_{n+1} - C_n^2 &=& 8 & \text{(Cassini identity)} \\ B_mB_{n+1} - B_{m+1}B_n &=& B_{m-n} & \text{(d'Ocagne identity)} \\ C_mC_{n+1} - C_{m+1}C_n &=& -8B_{m-n} & \text{(d'Ocagne identity)} \end{array}$$

$$3B_n - B_{n-1} = C_n \tag{7}$$

$$B_{n+2} - B_{n-2} = 12C_n \tag{8}$$

$$\sum_{l=0}^{n} B_l = \frac{B_{n+1} - B_n - 1}{4} \tag{9}$$

$$\sum_{l=0}^{n} C_l = \frac{C_{n+1} - C_n + 2}{4}. \tag{10}$$

3. The Balancing Split Quaternions and Lucas-Balancing Split Quaternions

In the literature, the quaternions and split quaternions of the well-known sequences have been considered. In [18], Horadam considered Fibonacci and Lucas quaternions, defined in the following way:

$$\begin{aligned} FQ_n &= F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3}, \\ LQ_n &= L_n + \mathbf{i}L_{n+1} + \mathbf{j}L_{n+2} + \mathbf{k}L_{n+3}, \end{aligned}$$

where F_n is the nth Fibonacci number and L_n is the nth Lucas number, and $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the standard basis of quaternions.

In [19], the split Fibonacci quaternion Q_n and split Lucas quaternion T_n were introduced by the following relations:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$
$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where $\{1, i, j, k\}$ is the standard basis of split quaternions. In the literature, there are many generalizations of the Fibonacci and Lucas sequences; among others, the k-Fibonacci sequence $\{F_{k,n}\}$ and the k-Lucas sequence $\{L_{k,n}\}$ are defined for $k \in \mathbb{N}$ in the following way:

$$F_{k,0} = 0, F_{k,1} = 1, \; F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } n \geq 2,$$

$$L_{k,0} = 2, L_{k,1} = k, \; L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } n \geq 2.$$

Some new results for the split k-Fibonacci and split k-Lucas quaternions can be found in [20]. In [21], the authors studied split Pell quaternions SP_n and split Pell–Lucas quaternions SPL_n, defined by

$$SP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$
$$SPL_n = PL_n + iPL_{n+1} + jPL_{n+2} + kPL_{n+3},$$

where P_n and PL_n are the nth Pell and Pell–Lucas number, respectively. In [22,23], balancing quaternions, Lucas-balancing quaternions, and some generalizations of these quaternions were considered. Inspired by these results, we introduce balancing split quaternions and Lucas-balancing split quaternions and present some properties of these split quaternions.

Let $n \geq 0$. We define the balancing split quaternion sequence $\{BSQ_n\}$ in the following way:

$$BSQ_n = B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}, \tag{11}$$

where B_n is the nth balancing number and $\{1, i, j, k\}$ is the basis of split quaternions. Similarly, we define the Lucas-balancing split quaternion sequence $\{CSQ_n\}$:

$$CSQ_n = C_n + iC_{n+1} + jC_{n+2} + kC_{n+3}, \tag{12}$$

where C_n is defined by (3).

Formulas (2), (3), (7), and (8) can be extended to the sequences $\{BSQ_n\}$ and $\{CSQ_n\}$.

Theorem 1. *Let $n \geq 2$ be an integer. Then,*

(i) $BSQ_n = 6BSQ_{n-1} - BSQ_{n-2}$,
(ii) $CSQ_n = 6CSQ_{n-1} - CSQ_{n-2}$,

where $BSQ_0 = i + 6j + 35k$, $BSQ_1 = 1 + 6i + 35j + 204k$, $CSQ_0 = 1 + 3i + 17j + 99k$, and $CSQ_1 = 3 + 17i + 99j + 577k$.

Proof. (i) By (11) and (2), we obtain

$$\begin{aligned}
& 6BSQ_{n-1} - BSQ_{n-2} \\
= \; & 6(B_{n-1} + iB_n + jB_{n+1} + kB_{n+2}) - (B_{n-2} + iB_{n-1} + jB_n + kB_{n+1}) \\
= \; & 6B_{n-1} - B_{n-2} + i(6B_n - B_{n-1}) + j(6B_{n+1} - B_n) + k(6B_{n+2} - B_{n+1}) \\
= \; & B_n + iB_{n+1} + jB_{n+2} + kB_{n+3} = BSQ_n.
\end{aligned}$$

We omit the proof of formula (ii). □

Theorem 2. *Let $n \geq 1$ be an integer. Then,*

$$3BSQ_n - BSQ_{n-1} = CSQ_n.$$

Proof. Using formulas (11) and (7), we have

$$\begin{aligned} 3BSQ_n - BSQ_{n-1} &= 3(B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}) \\ &\quad -B_{n-1} - iB_n - jB_{n+1} - kB_{n+2} \\ &= 3B_n - B_{n-1} + i(3B_{n+1} - B_n) \\ &\quad +j(3B_{n+2} - B_{n+1}) + k(3B_{n+3} - B_{n+2}) \\ &= C_n + iC_{n+1} + jC_{n+2} + kC_{n+3} = CSQ_n, \end{aligned}$$

which ends the proof. □

Corollary 1. *Let $n \geq 0$ be an integer. Then,*

$$BSQ_{n+1} - 3BSQ_n = CSQ_n.$$

Theorem 3. *Let $n \geq 2$ be an integer. Then,*

$$BSQ_{n+2} - BSQ_{n-2} = 12CSQ_n.$$

Proof. By (11) and (8), we have

$$\begin{aligned} BSQ_{n+2} - BSQ_{n-2} &= B_{n+2} + iB_{n+3} + jB_{n+4} + kB_{n+5} \\ &\quad -B_{n-2} - iB_{n-1} - jB_n - kB_{n+1} \\ &= B_{n+2} - B_{n-2} + i(B_{n+3} - B_{n-1}) \\ &\quad +j(B_{n+4} - B_n) + k(B_{n+5} - B_{n+1}) \\ &= 12(C_n + iC_{n+1} + jC_{n+2} + kC_{n+3}) = 12CSQ_n. \end{aligned}$$

This completes the proof. □

Now, we present some properties of the balancing and Lucas-balancing split quaternions. By simple calculations, we obtain the following results.

Theorem 4. *Assume that $n \geq 0$ is an integer. Then,*

$$BSQ_n + \overline{BSQ_n} = 2B_n,$$

$$CSQ_n + \overline{CSQ_n} = 2C_n.$$

Theorem 5. *Assume that $n \geq 0$ is an integer. Then,*

(i) $BSQ_n^2 + N(BSQ_n) = 2B_n BSQ_n$,

(ii) $CSQ_n^2 + N(CSQ_n) = 2C_n CSQ_n$.

Proof. By formulas (1) and (12), we have

$$\begin{aligned} CSQ_n^2 + N(CSQ_n) &= C_n^2 - C_{n+1}^2 + C_{n+2}^2 + C_{n+3}^2 \\ &\quad + 2iC_nC_{n+1} + 2jC_nC_{n+2} + 2kC_nC_{n+3} \\ &\quad + C_n^2 + C_{n+1}^2 - C_{n+2}^2 - C_{n+3}^2 \\ &= 2(C_n^2 + iC_nC_{n+1} + jC_nC_{n+2} + kC_nC_{n+3}) \\ &= 2C_n(C_n + iC_{n+1} + jC_{n+2} + kC_{n+3}) \\ &= 2C_nCSQ_n. \end{aligned}$$

The proof of (i) is similar. □

Now, we give the Binet formulas for the balancing split quaternions and Lucas-balancing split quaternions.

Theorem 6. *Let $n \geq 0$ be an integer. Then,*

$$BSQ_n = \frac{\hat{\lambda}_1 \lambda_1^n - \hat{\lambda}_2 \lambda_2^n}{4\sqrt{2}}, \tag{13}$$

$$CSQ_n = \frac{\hat{\lambda}_1 \lambda_1^n + \hat{\lambda}_2 \lambda_2^n}{2}, \tag{14}$$

where

$$\lambda_1 = 3 + 2\sqrt{2}, \;\; \lambda_2 = 3 - 2\sqrt{2},$$

$$\hat{\lambda}_1 = 1 + i\lambda_1 + j\lambda_1^2 + k\lambda_1^3, \tag{15}$$

$$\hat{\lambda}_2 = 1 + i\lambda_2 + j\lambda_2^2 + k\lambda_2^3. \tag{16}$$

Proof. By formula (5), we have

$$\begin{aligned} CSQ_n &= C_n + iC_{n+1} + jC_{n+2} + kC_{n+3} \\ &= \frac{1}{2}[\lambda_1^n + \lambda_2^n + i(\lambda_1^{n+1} + \lambda_2^{n+1}) \\ &+ j(\lambda_1^{n+2} + \lambda_2^{n+2}) + k(\lambda_1^{n+3} + \lambda_2^{n+3})] \\ &= \frac{1}{2}[\lambda_1^n \left(1 + i\lambda_1 + j\lambda_1^2 + k\lambda_1^3\right) + \lambda_2^n \left(1 + i\lambda_2 + j\lambda_2^2 + k\lambda_2^3\right)] \\ &= \frac{\hat{\lambda}_1 \lambda_1^n + \hat{\lambda}_2 \lambda_2^n}{2}. \end{aligned}$$

We omit the proof of formula (13). □

4. Some Identities for the Balancing Split Quaternions and Lucas-Balancing Split Quaternions

In this section, we will present some identities for the balancing split quaternions and Lucas-balancing split quaternions. By simple calculations, using (6), (15), and (16), we have

$$\begin{aligned} \hat{\lambda}_1 \hat{\lambda}_2 &= 2 + (6 + 4\sqrt{2})i + (34 + 24\sqrt{2})j + (198 - 4\sqrt{2})k, \\ \hat{\lambda}_2 \hat{\lambda}_1 &= 2 + (6 - 4\sqrt{2})i + (34 - 24\sqrt{2})j + (198 + 4\sqrt{2})k. \end{aligned} \tag{17}$$

Moreover,

$$\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_2 \hat{\lambda}_1 = 4(1 + 3i + 17j + 99k) = 4CSQ_0. \tag{18}$$

Theorem 7. *Let $r \geq 0$, $s \geq 0$, $t \geq 0$, and $u \geq 0$ be integers such that $r + s = t + u$. Then,*

$$\begin{aligned} &BSQ_r \cdot BSQ_s - BSQ_t \cdot BSQ_u \\ &= \frac{1}{32}[\hat{\lambda}_1 \hat{\lambda}_2 (\lambda_1^r \lambda_2^s - \lambda_1^t \lambda_2^u) + \hat{\lambda}_2 \hat{\lambda}_1 (\lambda_2^r \lambda_1^s - \lambda_2^t \lambda_1^u)], \end{aligned} \tag{19}$$

$$\begin{aligned} &CSQ_r \cdot CSQ_s - CSQ_t \cdot CSQ_u \\ &= \frac{1}{4}[\hat{\lambda}_1 \hat{\lambda}_2 (\lambda_1^r \lambda_2^s - \lambda_1^t \lambda_2^u) + \hat{\lambda}_2 \hat{\lambda}_1 (\lambda_2^r \lambda_1^s - \lambda_2^t \lambda_1^u)], \end{aligned} \tag{20}$$

where $\hat{\lambda}_1 \hat{\lambda}_2$, and $\hat{\lambda}_2 \hat{\lambda}_1$ are given by (17).

Proof. By (13), we obtain

$$
\begin{aligned}
&BSQ_r \cdot BSQ_s - BSQ_t \cdot BSQ_u \\
&= \frac{1}{32}(\lambda_1^{r+s}(\hat{\lambda}_1)^2 + \lambda_1^r\lambda_2^s\hat{\lambda}_1\hat{\lambda}_2 + \lambda_1^s\lambda_2^r\hat{\lambda}_2\hat{\lambda}_1 + \lambda_2^{r+s}(\hat{\lambda}_2)^2 \\
&- \lambda_1^{t+u}(\hat{\lambda}_1)^2 - \lambda_1^t\lambda_2^u\hat{\lambda}_1\hat{\lambda}_2 - \lambda_1^u\lambda_2^t\hat{\lambda}_2\hat{\lambda}_1 - \lambda_2^{t+u}(\hat{\lambda}_2)^2).
\end{aligned}
$$

Since $r + s = t + u$, we obtain formula (19). We omit the proof of formula (20). □

Using Theorem 7, we have the well-known identities: Catalan-type identities, Cassini-type identities, d'Ocagne-type identities, and Vajda-type identities for balancing split quaternions and Lucas-balancing spit quaternions.

Corollary 2. (*Catalan-type identities*) *Assume that* $n \geq 0, m \geq 0$ *are integers such that* $n \geq m$*. Then,*

$$BSQ_{n-m}BSQ_{n+m} - BSQ_n^2 = \frac{(\lambda_1^m - \lambda_2^m)(\hat{\lambda}_1\hat{\lambda}_2\lambda_2^m - \hat{\lambda}_2\hat{\lambda}_1\lambda_1^m)}{32},$$

$$CSQ_{n-m}CSQ_{n+m} - CSQ_n^2 = \frac{(\lambda_2^m - \lambda_1^m)(\hat{\lambda}_1\hat{\lambda}_2\lambda_2^m - \hat{\lambda}_2\hat{\lambda}_1\lambda_1^m)}{4}.$$

Corollary 3. (*Cassini-type identities*) *Let* $n \geq 1$*. Then,*

$$BSQ_{n-1}BSQ_{n+1} - BSQ_n^2 = \frac{\hat{\lambda}_1\hat{\lambda}_2\lambda_2 - \hat{\lambda}_2\hat{\lambda}_1\lambda_1}{4\sqrt{2}},$$

$$CSQ_{n-1}CSQ_{n+1} - CSQ_n^2 = -\sqrt{2}(\hat{\lambda}_1\hat{\lambda}_2\lambda_2 - \hat{\lambda}_2\hat{\lambda}_1\lambda_1).$$

Corollary 4. (*d'Ocagne-type identities*) *Assume that* $m \geq 0$ *and* $n \geq 0$ *are integers such that* $m \geq n$*. Then,*

$$BSQ_mBSQ_{n+1} - BSQ_{m+1}BSQ_n = \frac{\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{m-n} - \hat{\lambda}_2\hat{\lambda}_1\lambda_2^{m-n}}{4\sqrt{2}},$$

$$CSQ_mCSQ_{n+1} - CSQ_{m+1}CSQ_n = -\sqrt{2}(\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{m-n} - \hat{\lambda}_2\hat{\lambda}_1\lambda_2^{m-n}).$$

Corollary 5. (*Vajda-type identities*) *Assume that* $n \geq 0$*,* $m \geq 0$*, and* $k \geq 0$ *are integers such that* $n \geq k$*. Then,*

$$
\begin{aligned}
&BSQ_{m+k}BSQ_{n-k} - BSQ_mBSQ_n \\
&= \frac{1}{32}\Big[\hat{\lambda}_1\hat{\lambda}_2\,\lambda_1^m\lambda_2^n\Big(1 - (17 + 12\sqrt{2})^k\Big) + \hat{\lambda}_2\hat{\lambda}_1\,\lambda_1^n\lambda_2^m\Big(1 - (17 - 12\sqrt{2})^k\Big)\Big], \\
&CSQ_{m+k}CSQ_{n-k} - CSQ_mCSQ_n \\
&= \frac{1}{4}\Big[\hat{\lambda}_1\hat{\lambda}_2\,\lambda_1^m\lambda_2^n\Big((17 + 12\sqrt{2})^k - 1\Big) + \hat{\lambda}_2\hat{\lambda}_1\,\lambda_1^n\lambda_2^m((17 - 12\sqrt{2})^k - 1)\Big].
\end{aligned}
$$

In the next theorems, we present other identities for balancing split quaternions and for Lucas-balancing split quaternions. They show some dependencies between these split quaternions.

Theorem 8. *Assume that* $m \geq 0$ *and* $n \geq 0$ *are integers such that* $n \geq m$*. Then,*

$$BSQ_nCSQ_m - CSQ_nBSQ_m = \frac{\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{n-m} - \hat{\lambda}_2\hat{\lambda}_1\lambda_2^{n-m}}{4\sqrt{2}}.$$

Proof. By formulas (4) and (5), we have

$$\begin{aligned}
&BSQ_nCSQ_m - CSQ_nBSQ_m \\
&= \frac{1}{8\sqrt{2}}[(\hat{\lambda}_1\lambda_1^n - \hat{\lambda}_2\lambda_2^n)(\hat{\lambda}_1\lambda_1^m + \hat{\lambda}_2\lambda_2^m) - (\hat{\lambda}_1\lambda_1^n + \hat{\lambda}_2\lambda_2^n)(\hat{\lambda}_1\lambda_1^m - \hat{\lambda}_2\lambda_2^m)] \\
&= \frac{1}{8\sqrt{2}}[2\hat{\lambda}_1\hat{\lambda}_2\lambda_1^n\lambda_2^m - 2\hat{\lambda}_2\hat{\lambda}_1\lambda_1^m\lambda_2^n] \\
&= \frac{1}{4\sqrt{2}}[(\lambda_1\lambda_2)^n(\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{n-m} - \hat{\lambda}_2\hat{\lambda}_1\lambda_2^{n-m})] = \frac{\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{n-m} - \hat{\lambda}_2\hat{\lambda}_1\lambda_2^{n-m}}{4\sqrt{2}}.
\end{aligned}$$

This completes the proof. □

Theorem 9. *Let $m \geq 0$ and $n \geq 0$ be integers. Then,*

$$BSQ_nCSQ_m + CSQ_nBSQ_m = \frac{(\hat{\lambda}_1)^2\lambda_1^{n+m} - (\hat{\lambda}_2)^2\lambda_2^{n+m}}{4\sqrt{2}}.$$

Theorem 10. *Assume that $n \geq 0$, $m \geq 0$, and $k \geq 0$ are integers such that $m \geq k$. Then,*

$$BSQ_{n+m}CSQ_{n+s} - BSQ_{n+s}CSQ_{n+m} = \frac{CSQ_0(\lambda_1^{m-k} - \lambda_2^{m-k})}{2\sqrt{2}}.$$

Proof. By formulas (13), (14), and (18), we have

$$\begin{aligned}
&BSQ_{n+m}CSQ_{n+k} - BSQ_{n+k}CSQ_{n+m} \\
&= \frac{1}{8\sqrt{2}}[(\hat{\lambda}_1\lambda_1^{n+m} - \hat{\lambda}_2\lambda_2^{n+m})(\hat{\lambda}_1\lambda_1^{n+k} + \hat{\lambda}_2\lambda_2^{n+k}) \\
&- (\hat{\lambda}_1\lambda_1^{n+k} - \hat{\lambda}_2\lambda_2^{n+k})(\hat{\lambda}_1\lambda_1^{n+m} + \hat{\lambda}_2\lambda_2^{n+m})] \\
&= \frac{1}{8\sqrt{2}}[\hat{\lambda}_1\hat{\lambda}_2\lambda_1^{n+m}\lambda_2^{n+k} - \hat{\lambda}_1\hat{\lambda}_2\lambda_1^{n+k}\lambda_2^{n+m} \\
&+ \hat{\lambda}_2\hat{\lambda}_1\lambda_1^{n+m}\lambda_2^{n+k} - \hat{\lambda}_2\hat{\lambda}_1\lambda_1^{n+k}\lambda_2^{n+m}] \\
&= \frac{1}{8\sqrt{2}}[(\lambda_1\lambda_2)^n(\hat{\lambda}_1\hat{\lambda}_2 + \hat{\lambda}_2\hat{\lambda}_1)(\lambda_1^m\lambda_2^k - \lambda_1^k\lambda_2^m)] \\
&= \frac{CSQ_0(\lambda_1^m\lambda_2^k - \lambda_1^k\lambda_2^m)}{2\sqrt{2}} = \frac{CSQ_0(\lambda_1^{m-k} - \lambda_2^{m-k})}{2\sqrt{2}}.
\end{aligned}$$

This completes the proof. □

Theorem 11. *Assume that $n \geq 0$ is an integer. Then,*

$$CSQ_n^2 - 8BSQ_n^2 = 2CSQ_0.$$

Proof. By simple calculations, using (18), we obtain

$$\begin{aligned}
CSQ_n^2 - 8BSQ_n^2 &= \left(\frac{\hat{\lambda}_1\lambda_1^n + \hat{\lambda}_2\lambda_2^n}{2}\right)^2 - 8\left(\frac{\hat{\lambda}_1\lambda_1^n - \hat{\lambda}_2\lambda_2^n}{4\sqrt{2}}\right)^2 \\
&= \frac{1}{4}[(\lambda_1\lambda_2)^n 2(\hat{\lambda}_1\hat{\lambda}_2 + \hat{\lambda}_2\hat{\lambda}_1)] \\
&= \frac{\hat{\lambda}_1\hat{\lambda}_2 + \hat{\lambda}_2\hat{\lambda}_1}{2} = 2CSQ_0,
\end{aligned}$$

which ends the proof. □

Theorem 12. *Assume that $n \geq 0$ is an integer. Then,*

$$CSQ_{2n} - 16BSQ_n^2 = \frac{1}{2}\left(\lambda_1{}^{2n}(\hat{\lambda}_1 - (\hat{\lambda}_1)^2) + \lambda_2{}^{2n}(\hat{\lambda}_2 - (\hat{\lambda}_2)^2\right) + 2CSQ_0.$$

Theorem 13. *Assume that n and m are integers such that $n \geq m$. Then,*

$$CSQ_nCSQ_m - 8BSQ_nBSQ_m = \frac{1}{2}(\lambda_1^{n-m}\hat{\lambda}_1\hat{\lambda}_2 + \lambda_2^{n-m}\hat{\lambda}_2\hat{\lambda}_1),$$

$$CSQ_nCSQ_m + 8BSQ_nBSQ_m = \frac{1}{2}\left(\lambda_1^{n+m}\hat{\lambda_1}^2 + \lambda_2^{n+m}\hat{\lambda_2}^2\right).$$

Now, we give summation formulas for the balancing split quaternions and Lucas-balancing split quaternions.

Theorem 14.

$$\sum_{l=0}^{n} BSQ_l = \frac{BSQ_{n+1} - BSQ_n - 1 - i - 5j - 29k}{4}, \tag{21}$$

$$\sum_{l=0}^{n} CSQ_l = \frac{CSQ_{n+1} - CSQ_n + 2 + i - 2j - 19k}{4}. \tag{22}$$

Proof. By formula (9), we have

$$\begin{aligned}
\sum_{l=0}^{n} BSQ_l &= \sum_{l=0}^{n}(B_l + iB_{l+1} + jB_{l+2} + kB_{l+3}) \\
&= \sum_{l=0}^{n} B_l + i\sum_{l=0}^{n} B_{l+1} + j\sum_{l=0}^{n} B_{l+2} + k\sum_{l=0}^{n} B_{l+3} \\
&= \frac{1}{4}(B_{n+1} - B_n - 1) + i(\frac{1}{4}(B_{n+2} - B_{n+1} - 1) - B_0) \\
&+ j(\frac{1}{4}(B_{n+3} - B_{n+2} - 1) - B_0 - B_1) \\
&+ k(\frac{1}{4}(B_{n+4} - B_{n+3} - 1) - B_0 - B_1 - B_2) \\
&= \frac{1}{4}(B_{n+1} + iB_{n+2} + jB_{n+3} + kB_{n+4} - (B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}) \\
&- (1 + i + j + k)) - iB_0 - j(B_0 + B_1) - k(B_0 + B_1 + B_2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{l=0}^{n} BSQ_l &= \frac{BSQ_{n+1} - BSQ_n - (1 + i + j + k) - (4j + 28k)}{4} \\
&= \frac{BSQ_{n+1} - BSQ_n - 1 - i - 5j - 29k}{4}.
\end{aligned}$$

Using formula (10), we can prove formula (22). □

5. Generating Functions and Matrix Representations

In this section, we will present the generating functions and matrix generators for the balancing split quaternions and Lucas-balancing split quaternions. We recall known results for sequences $\{B_n\}$ and $\{C_n\}$.

Theorem 15 ([14])**.** *The generating function of the balancing sequence $\{B_n\}$ is*

$$G(B_n; x) = \frac{x}{1 - 6x + x^2}.$$

Theorem 16 ([24])**.** *The generating function of the Lucas-balancing sequence* $\{C_n\}$ *is*

$$G(C_n; x) = \frac{1-3x}{1-6x+x^2}.$$

Theorem 17. *The generating function of the sequence* $\{CSQ_n\}$ *is*

$$f(t) = \frac{1-3t+(3-t)i+(17-3t)j+(99-17t)k}{1-6t+t^2}.$$

Proof. Let

$$f(t) = CSQ_0 + CSQ_1 t + CSQ_2 t^2 + \ldots + CSQ_n t^n + \ldots.$$

By the recurrence $CSQ_n = 6CSQ_{n-1} - CSQ_{n-2}$, we obtain

$$6tf(t) = 6CSQ_0 t + 6CSQ_1 t^2 + 6CSQ_2 t^3 + \ldots + 6CSQ_{n-1} t^n + \ldots$$
$$t^2 f(t) = CSQ_0 t^2 + CSQ_1 t^3 + CSQ_2 t^4 + \ldots + CSQ_{n-2} t^n + \ldots.$$

Hence,

$$\begin{aligned} & f(t) - 6tf(t) + t^2 f(t) \\ & CSQ_0 + (CSQ_1 - 6CSQ_0)t + (CSQ_2 - 6CSQ_1 + CSQ_0)t^2 + \ldots \\ & = CSQ_0 + (CSQ_1 - 6CSQ_0)t. \end{aligned}$$

Thus,

$$f(t) = \frac{CSQ_0 + (CSQ_1 - 6CSQ_0)t}{1-6t+t^2}.$$

Since $CSQ_0 = 1 + 3i + 17j + 99k$ and $CSQ_1 = 3 + 17i + 99j + 577k$, after simple calculations we have

$$f(t) = \frac{1-3t+(3-t)i+(17-3t)j+(99-17t)k}{1-6t+t^2},$$

which completes the proof. □

Theorem 18. *The generating function of the sequence* $\{BSQ_n\}$ *is*

$$g(t) = \frac{t+i+(6-t)j+(35-6t)k}{1-6t+t^2}.$$

In [17], a matrix generator for numbers B_n was given, balancing the Q-matrix, denoted by Q_B. The following theorem was presented:

Theorem 19 ([17])**.** *Let* $Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$. *Then, for* $n \geq 1$,

$$Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

Analogously, the following result for the Lucas-balancing numbers was proved.

Theorem 20 ([17])**.** *Let* $R_B = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$. *Then, for* $n \geq 1$,

$$R_B Q_B^n = \begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}.$$

Using these concepts, we can prove the following theorems.

Theorem 21. *Let $n \geq 1$ be an integer. Then,*

$$\left[\begin{array}{cc} BSQ_{n+1} & -BSQ_n \\ BSQ_n & -BSQ_{n-1} \end{array}\right] = \left[\begin{array}{cc} BSQ_2 & -BSQ_1 \\ BSQ_1 & -BSQ_0 \end{array}\right] \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right]^{n-1}. \tag{23}$$

Proof. (By induction on n). For $n = 1$, the result is obvious. Assume that formula (23) holds for n. We will prove it for $n+1$. By the induction's hypothesis, we have

$$\left[\begin{array}{cc} BSQ_2 & -BSQ_1 \\ BSQ_1 & -BSQ_0 \end{array}\right] \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right]^{n-1} \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc} BSQ_{n+1} & -BSQ_n \\ BSQ_n & -BSQ_{n-1} \end{array}\right] \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc} 6BSQ_{n+1} - BSQ_n & -BSQ_{n+1} \\ 6BSQ_n - BSQ_{n-1} & -BSQ_n \end{array}\right].$$

Since $BSQ_n = 6BSQ_{n-1} - BSQ_{n-2}$, we obtain

$$\left[\begin{array}{cc} BSQ_{n+1} & -BSQ_n \\ BSQ_n & -BSQ_{n-1} \end{array}\right] \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} BSQ_{n+2} & -BSQ_{n+1} \\ BSQ_{n+1} & -BSQ_n \end{array}\right],$$

which ends the proof. □

In the same way, using Theorem 2 and Corollary 1, we can prove Theorem 22.

Theorem 22. *Let $n \geq 1$ be an integer. Then,*

$$\left[\begin{array}{cc} CSQ_{n+1} & -CSQ_n \\ CSQ_n & -CSQ_{n-1} \end{array}\right] = \left[\begin{array}{cc} 3 & -1 \\ 1 & -3 \end{array}\right] \cdot \left[\begin{array}{cc} BSQ_2 & -BSQ_1 \\ BSQ_1 & -BSQ_0 \end{array}\right] \cdot \left[\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right]^{n-1}.$$

Matrix generators are useful tools for obtaining new identities and algebraic representation.

6. Conclusions

In the literature, many authors have studied quaternions and split quaternions with coefficients that are terms of special integer sequences, among others Fibonacci numbers and their generalizations. There are many generalizations of balancing numbers and Lucas-balancing numbers. The second-order recurrences $B_n = 6B_{n-1} - B_{n-2}$ with $B_0 = 0$ and $B_1 = 1$ and $C_n = 6C_{n-1} - C_{n-2}$ with $C_0 = 1$ and $C_1 = 3$ have mainly been generalized in two ways: first by preserving the initial conditions and second by preserving the recurrence relations. In [25–27], the authors considered k-balancing numbers B_n^k and k-Lucas balancing numbers C_n^k, defined as follows: $B_n^k = 6kB_{n-1}^k - B_{n-2}^k$ for an integer $k \geq 1$ and $n \geq 2$ with initial conditions $B_0^k = 0$ and $B_1^k = 1$; $C_n^k = 6kC_{n-1}^k - C_{n-2}^k$ for an integer $k \geq 1$ and $n \geq 2$ with initial conditions $C_0^k = 1$ and $C_1^k = 3$. Another generalization of the Lucas-balancing numbers was presented in [28]. The authors introduced numbers $C_{k,n}$ defined by the recurrence $C_{k,n} = 6kC_{k,n-1} - C_{k,n-2}$ for an integer $k \geq 1$ and $n \geq 2$ with initial conditions $C_{k,0} = 1$ and $C_{k,1} = 3k$. In [16], the authors studied cobalancing numbers b_n and Lucas-cobalancing numbers c_n defined in the following way: $b_0 = 0, b_1 = 0$, $b_n = 6b_{n-1} - b_{n-2} + 2$ for $n \geq 2$; $c_0 = -1, c_1 = 1$, $c_n = 6c_{n-1} - c_{n-2}$ for $n \geq 2$. We can find other interesting generalizations of balancing numbers in [29–34]. Based on these concepts, it is natural to consider generalizations of balancing split quaternions and Lucas-balancing split quaternions.

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