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Solvability of the Class of Two-Dimensional Product-Type Systems of Difference Equations of Delay-Type (1, 3, 1, 1)

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Abstract: This paper essentially presents the last and important steps in the study of (practical) solvability of two-dimensional product-type systems of difference equations of the following form $z_n = \alpha z^a_{n-k} w^b_{n-l}$, $w_n = \beta w^c_{n-m} z^d_{n-s}$, $n \in \mathbb{N}_0$, where $k, l, m, s \in \mathbb{N}$, $a, b, c, d \in \mathbb{Z}$, and where α, β and the initial values are complex numbers. It is devoted to the most complex case which has not been considered so far (the case $k = l = s = 1$ and $m = 3$). Closed form formulas for solutions to the system are found in all possible cases. The structure of the solutions to the system is considered in detail. The following five cases: (1) $b = 0$; (2) $c = 0$; (3) $d = 0$; (4) $ac \neq 0$; (5) $a = 0$, $bcd \neq 0$, are considered separately. Some of the situations appear for the first time in the literature.

Keywords: system of difference equations; product-type system; system solvable in closed form

MSC: 39A20; 39A45

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1. Introduction

There has been a growing interest in difference equations and systems of difference equations (see, for example, [1–43]). Among several subfields of recent interest we mention here two, on whose intersection is the study in this paper. The first one is the classical subfield of finding solutions to the equations and systems in closed form. Books [6,12–16] contain some old information in the subfield. Some recent ones can be found, for example, in the following papers: [5,27,31–43] (see also numerous references therein). For some related results and applications of solvable difference equations and systems, see also [1–4]. The recent interest has been considerably motivated by the fact that some new interesting classes of nonlinear difference equations and systems have been solved by transforming them to known solvable ones. One of our transformations has had some impact in the recent interest. For more information, see, for example, [5,27,34], and the related references therein. Beside the line of investigation, there have been some other several ones which also use some related ideas. The reader can consult the representative paper [34] and find many other related ones in its list of references. Generally speaking, above mentioned lines of investigations use the method of transformation in solving the equations and systems therein. In many of these papers obtained formulas for the solutions to the equations and systems studied therein are used in describing their long-term behavior (for example, in [5,39]).

The second one is the subfield on concrete systems of difference equations. Some of the papers which have had some impact on the growing interest in the subfield are [19–21] by Papaschinopoulos and Schinas. One of their main ideas is to consider symmetric systems of difference equations obtained from the following scalar one

$$x_n = f(x_{n-k}, x_{n-l}), \quad n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $k \neq l$, for concrete values of function f, that is, to study some concrete systems of difference equations of one of the following forms

$$\begin{aligned} x_n &= f(x_{n-k}, y_{n-l}), \quad y_n = f(y_{n-k}, x_{n-l}), \\ x_n &= f(y_{n-k}, x_{n-l}), \quad y_n = f(x_{n-k}, y_{n-l}), \\ x_n &= f(y_{n-k}, y_{n-l}), \quad y_n = f(x_{n-k}, x_{n-l}), \quad n \in \mathbb{N}_0. \end{aligned}$$

This, among other things, has motivated us to study the solvability of some concrete systems of the form, such as the ones in [5,39,40] (see also the related references therein). The idea naturally evolved into the investigation of more general symmetric systems of difference equations. For example, a few symmetric systems with three variables were studied in [23], while in [22] were studied, among others, the invariants of the following system of nonlinear difference equations

$$\begin{aligned} x_{n+1} &= \frac{a + b(y_n + x_{n-1} + \cdots + y_{n-k+1} + x_{n-k})}{y_{n-k-1}} \\ y_{n+1} &= \frac{a + b(x_n + y_{n-1} + \cdots + x_{n-k+1} + y_{n-k})}{x_{n-k-1}} \end{aligned}$$

where k is an odd number and $a, b \in (0, \infty)$, which means that more complex symmetric systems were studied therein. In fact, the study of the invariants of some equations and systems can be regarded as a kind of the study of their solvability, so that paper [22], as well as [21] essentially belong to both subfields.

It is highly expected that the methods and ideas used in the study of symmetric systems of difference equations can produce the same or related results for the systems which are not symmetric, but are close to them. For example, the following max-type system of difference equations

$$x_{n+1} = \frac{\max\{a_n y_n, b_n\}}{y_n x_{n-1}}, \quad y_{n+1} = \frac{\max\{c_n x_n, d_n\}}{x_n y_{n-1}}, \quad n \in \mathbb{N}_0,$$

where a_n, b_n, c_n, d_n are sequences of positive numbers and x_{-1}, x_0, y_{-1}, y_0 are positive numbers, is such one, and was studied in [25] (for another max-type system see also [28]). Such system are called *close-to-symmetric* systems of difference equations and are frequently studied (see, for example, [8,17,24,29,31,33] and the related references therein). In paper [11] was initiated study of cyclic systems of difference equations, which naturally evolved into the study of some close to cyclic systems of difference equations (see, for example, [7,18]).

Studying positive solutions to some classes of equations and systems, such as the ones of the special cases of the following equation

$$x_n = \alpha + \frac{x_{n-k}^p}{x_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

where $\alpha, p, q \in (0, \infty)$, $k, l \in \mathbb{N}$, $k \neq l$ (see, for example, [30], as well as [9,10,26] and the related references therein), as well as of the corresponding two-dimensional symmetric systems of difference equations, such as

$$x_n = \alpha + \frac{y_{n-k}^p}{x_{n-l}^q}, \quad y_n = \alpha + \frac{x_{n-k}^p}{y_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

we came across some product-type equations and systems.

The solvability of the product-type equations and systems in this case is something which should be known to any expert. Namely, if all the initial values are positive a simple inductive argument shows that all other terms are also positive, so that by using the logarithm the equation/system is transformed to a linear one with constant coefficients, which is one of the most known solvable equation/system. If the domain is changed, then several problems occur which prevent using the standard method for solving the equations and systems for the case of positive solutions. Our study of the systems on the complex plane was set off in [39]. It turned out that finding solutions to some related systems is not so simple problem. The next product-type system was solved in [40], but without detailed analysis of the structure of its solutions. The forms of the systems studied in [39,40] are similar, which suggested an investigation of the extensions which include both of them. On the other hand, the occurrence of some multipliers in some cases of the one-dimensional equation in [34] has suggested an investigation of the related systems with additional constant multipliers, which has been done for the first time in [31].

Somewhat later, a detailed analysis has shown that complete lists of formulas can be given for some of concrete systems of difference equations of the following form

$$z_n = \alpha z_{n-k}^a w_{n-l}^b, \quad w_n = \beta w_{n-m}^c z_{n-s}^d, \quad n \in \mathbb{N}, \tag{1}$$

with "small" values of delays k, l, m and s, which means that they are solvable. Since we have studied so far a number of the systems of type (1), to facilitate classification and terminology, from now on we will say that the system is of *delay-type* (k, m, l, s).

The corresponding lists of formulas for solutions are given first for the systems in [33,43], unlike the systems in above mentioned paper [40] and for the system in [42]. For some systems such as the ones in [35,41] the solutions were obtained more easily, so the analysis was simpler and it was of a different character. Some technical problems in dealing with the systems of the sort in (1) lead us to devising another method for solving them in [32], which has been recently also used in [37]. Recently, we have done, for the first time a detailed analysis of the structure of solutions to a class of product-type systems with an associated polynomial (to the system) of the fourth order in [38], and quite recently in [36].

The main goal of the whole project is to classify solvable product-type systems of difference equations of the form in (1) and present their solutions in closed form in terms of the involved parameters and initial values. Here we continue the project. This paper is a natural continuation of our research in [31–33,35–43], and essentially presents the last and important steps in the finishing of the project.

Our task here is to show the solvability of system (1) of *delay-type* $(1, 3, 1, 1)$ (the case $k = l = s = 1$ and $m = 3$), that is, of

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-2}^c z_n^d, \quad n \in \mathbb{N}_0, \tag{2}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. The case when some of the quantities α, β, z_0, w_{-2}, w_{-1}, w_0 is zero we do not take into consideration because in the case are obtained solutions which are either not defined or trivial, so of not special interest.

The following five cases are considered separately in this paper: (1) $b = 0$; (2) $c = 0$; (3) $d = 0$; (4) $ac \neq 0$; (5) $a = 0$, $bcd \neq 0$. We would like to point out that the fifth case is not covered with the fourth one, since the condition $a = 0$ changes the order of an associated polynomial appearing in the study. If $k, l \in \mathbb{Z}$, then the notation $\overline{k, l}$, denotes the set of all $j \in \mathbb{Z}$ such that $k \leq j \leq l$, whereas we regard that $\sum_{j=l}^{l-1} c_j = 0$ for each $l \in \mathbb{Z}$.

2. Auxiliary Results

The following three lemmas are useful tools in our investigation and have been already used in some of our previous papers devoted to the project on product-type systems. The first one is a consequence of the Langrage formula applied to the functions $f_s(t) = t^s$, $s \in \mathbb{N}$ (see, for example, [13] or [44], as well as [40] for a proof based on complex analysis).

Lemma 1. *Let*

$$p_k(t) = c_k \prod_{j=1}^{k} (t - t_j),$$

$c_k \neq 0$ *and* $t_i \neq t_j$, $i \neq j$. *Then*

$$\sum_{j=1}^{k} \frac{t_j^s}{p_k'(t_j)} = 0, \quad 0 \leq s \leq k-2,$$

and

$$\sum_{j=1}^{k} \frac{t_j^{k-1}}{p_k'(t_j)} = \frac{1}{c_k}.$$

Further, we need several closed form formulas for some sums which can be found in numerous books (see, for example, [13] or [16]). For a general method for calculating this type of sums consult our recent paper [33] where a recurrent formula for this type of sums is presented, and by using it the sums can be calculated.

Lemma 2. *Let*

$$s_n^{(i)}(z) = \sum_{j=1}^{n} j^i z^{j-1}, \quad n \in \mathbb{N}, \tag{3}$$

where $i \in \mathbb{N}_0$ *and* $z \in \mathbb{C}$.

Then

$$s_n^{(0)}(z) = \frac{1 - z^n}{1 - z}, \tag{4}$$

$$s_n^{(1)}(z) = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}, \tag{5}$$

$$s_n^{(2)}(z) = \frac{1 + z - (n+1)^2 z^n + (2n^2 + 2n - 1)z^{n+1} - n^2 z^{n+2}}{(1-z)^3}, \tag{6}$$

$$s_n^{(3)}(z) = \frac{n^3 z^n (z-1)^3 - 3n^2 z^n (z-1)^2 + 3nz^n (z^2 - 1) - (z^n - 1)(z^2 + 4z + 1)}{(1-z)^4}, \tag{7}$$

for every $z \in \mathbb{C} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

The following lemma describes the nature/type of the zeros of an arbitrary fourth order polynomial equation. The results in the lemma are certainly folklore and were essentially obtained, for example, in [45] (the lemma formulates the results appearing therein in a unified way, although the notation and some quantities are different).

Lemma 3. *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e,$$

where b, c, d, e are real numbers,

$$\Delta_0 = c^2 - 3bd + 12e, \qquad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce,$$
$$\Delta = \frac{1}{27}(4\Delta_0^3 - \Delta_1^2), \qquad P = 8c - 3b^2,$$
$$Q = b^3 + 8d - 4bc, \qquad D = 64e - 16c^2 + 16b^2c - 16bd - 3b^4.$$

Then, the following statements hold.

(a) *If $\Delta < 0$, then two zeros of P_4 are real and different, and two are complex conjugate.*

(b) *If $\Delta > 0$, then all the zeros of P_4 are real or none is. More precisely,*

- 1° *if $P < 0$ and $D < 0$, then all four zeros of P_4 are real and different;*
- 2° *if $P > 0$ or $D > 0$, then there are two pairs of complex conjugate zeros of P_4.*

(c) *If $\Delta = 0$, then and only then P_4 has a multiple zero. The following cases can occur:*

- 1° *if $P < 0$, $D < 0$ and $\Delta_0 \neq 0$, then two zeros of P_4 are real and equal and two are real and simple;*
- 2° *if $D > 0$ or ($P > 0$ and ($D \neq 0$ or $Q \neq 0$)), then two zeros of P_4 are real and equal and two are complex conjugate;*
- 3° *if $\Delta_0 = 0$ and $D \neq 0$, there is a triple zero of P_4 and one simple, all real;*
- 4° *if $D = 0$, then*
  - 4.1° *if $P < 0$ there are two double real zeros of P_4;*
  - 4.2° *if $P > 0$ and $Q = 0$ there are two double complex conjugate zeros of P_4;*
  - 4.3° *if $\Delta_0 = 0$, then all four zeros of P_4 are real and equal to $-b/4$.*

3. Main Results

In this section we state and prove our main results and by using them and some further analysis we get several corollaries. Before this we give a list of first several members of the sequences z_n and w_n, whose values will be used in the proofs of some of the results.

We have

$$\begin{aligned}
z_1 &= \alpha z_0^a w_0^b,\\
w_1 &= \beta w_{-2}^c z_0^d,\\
z_2 &= \alpha(\alpha z_0^a w_0^b)^a(\beta w_{-2}^c z_0^d)^b = \alpha^{1+a}\beta^b z_0^{a^2+bd} w_{-2}^{bc} w_0^{ab},\\
w_2 &= \beta w_{-1}^c z_1^d = \beta w_{-1}^c(\alpha z_0^a w_0^b)^d = \alpha^d \beta z_0^{ad} w_{-1}^c w_0^{bd}\\
z_3 &= \alpha z_2^a w_2^b = \alpha(\alpha^{1+a}\beta^b z_0^{a^2+bd} w_{-2}^{bc} w_0^{ab})^a(\alpha^d \beta z_0^{ad} w_{-1}^c w_0^{bd})^b\\
&= \alpha^{1+a+a^2+bd}\beta^{b+ab} z_0^{a^3+2abd} w_{-2}^{abc} w_{-1}^{bc} w_0^{b(a^2+bd)}.
\end{aligned} \tag{8}$$

As we have already mentioned we will consider the following five cases separately: (1) $b = 0$; (2) $c = 0$; (3) $d = 0$; (4) $ac \neq 0$; (5) $a = 0$, $bcd \neq 0$. Hence, we will prove five results on the solvability and by further analysis we will get several consequences from them.

Theorem 1. *Assume that $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (2) is solvable in closed form.*

Proof. Since $b = 0$, we have

$$z_{n+1} = \alpha z_n^a, \quad w_{n+1} = \beta w_{n-2}^c z_n^d, \quad n \in \mathbb{N}_0, \tag{9}$$

from which it follows that

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \tag{10}$$

for $n \in \mathbb{N}$.

From (10) we have that

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{11}$$

when $a \neq 1$, and

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}, \tag{12}$$

when $a = 1$.

Using (10) in the second equation in (9), is obtained

$$w_n = \alpha^{d\sum_{j=0}^{n-2} a^j} \beta z_0^{da^{n-1}} w_{n-3}^c, \quad n \geq 2,$$

which is equivalent to

$$w_{3m+i} = \alpha^{d\sum_{j=0}^{3m+i-2} a^j} \beta z_0^{da^{3m+i-1}} w_{3(m-1)+i}^c, \tag{13}$$

for $m \in \mathbb{N}$ and $i = -1, 0, 1$.

From (13) and by induction it is proved that

$$w_{3m+i} = \alpha^{d\sum_{l=0}^{k-1} c^l \sum_{j=0}^{3(m-l)+i-2} a^j} \beta^{\sum_{l=0}^{k-1} c^l} z_0^{d\sum_{l=0}^{k-1} c^l a^{3(m-l)+i-1}} w_{3(m-k)+i}^{c^k},$$

for $m \geq k$ and $i = -1, 0, 1$, from which for $k = m$ is obtained

$$w_{3m+i} = \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)+i-2} a^j} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l a^{3(m-l)+i-1}} w_i^{c^m}, \tag{14}$$

for $m \in \mathbb{N}$ and $i = -1, 0, 1$.

From (8) and (14), we have

$$w_{3m-1} = \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l-1)} a^j} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l a^{3(m-l)-2}} w_{-1}^{c^m}, \tag{15}$$

$$w_{3m} = \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)-2} a^j} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l a^{3(m-l)-1}} w_0^{c^m}, \tag{16}$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)-1} a^j} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l a^{3(m-l)}} (\beta w_{-2}^c z_0^d)^{c^m} \\ &= \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)-1} a^j} \beta^{\sum_{l=0}^{m} c^l} z_0^{d\sum_{l=0}^{m} c^l a^{3(m-l)}} w_{-2}^{c^{m+1}}, \end{aligned} \tag{17}$$

for $m \in \mathbb{N}$.

Now we use the formulas in (15)–(17) in five subcases separately.

Case $a \neq 1 \neq c \neq a^3$. From (15)–(17) and by Lemma 2, we obtain

$$\begin{aligned} w_{3m-1} &= \alpha^{d\sum_{l=0}^{m-1} c^l \frac{1-a^{3(m-l)-2}}{1-a}} \beta^{\frac{1-c^m}{1-c}} z_0^{ad\frac{a^{3m}-c^m}{a^3-c}} w_{-1}^{c^m} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-c^m}{1-c} - a\frac{a^{3m}-c^m}{a^3-c}\right)} \beta^{\frac{1-c^m}{1-c}} z_0^{ad\frac{a^{3m}-c^m}{a^3-c}} w_{-1}^{c^m} \\ &= \alpha^{\frac{d(a^3-c+(1-a)(a+a^2+c)c^m+(c-1)a^{3m+1})}{(1-a)(1-c)(a^3-c)}} \beta^{\frac{1-c^m}{1-c}} z_0^{ad\frac{a^{3m}-c^m}{a^3-c}} w_{-1}^{c^m}, \end{aligned} \tag{18}$$

$$\begin{aligned} w_{3m} &= \alpha^{d\sum_{l=0}^{m-1} c^l \frac{1-a^{3(m-l)-1}}{1-a}} \beta^{\frac{1-c^m}{1-c}} z_0^{da^2 \frac{a^{3m}-c^m}{a^3-c}} w_0^{c^m} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-c^m}{1-c} - a^2 \frac{a^{3m}-c^m}{a^3-c}\right)} \beta^{\frac{1-c^m}{1-c}} z_0^{da^2 \frac{a^{3m}-c^m}{a^3-c}} w_0^{c^m} \\ &= \alpha^{\frac{d(a^3-c+(1-a)(c+ca+a^2)c^m+(c-1)a^{3m+2})}{(1-a)(1-c)(a^3-c)}} \beta^{\frac{1-c^m}{1-c}} z_0^{da^2 \frac{a^{3m}-c^m}{a^3-c}} w_0^{c^m}, \end{aligned} \tag{19}$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1} c^l \frac{1-a^{3(m-l)}}{1-a}} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d \frac{a^{3m+3}-c^{m+1}}{a^3-c}} w_{-2}^{c^{m+1}} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-c^m}{1-c} - a^3 \frac{a^{3m}-c^m}{a^3-c}\right)} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d \frac{a^{3m+3}-c^{m+1}}{a^3-c}} w_{-2}^{c^{m+1}} \\ &= \alpha^{\frac{d(a^3-c+(1-a^3)c^{m+1}+(c-1)a^{3m+3})}{(1-a)(1-c)(a^3-c)}} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d \frac{a^{3m+3}-c^{m+1}}{a^3-c}} w_{-2}^{c^{m+1}}, \end{aligned} \tag{20}$$

for $m \in \mathbb{N}$.

Case $a \neq 1 \neq c = a^3$. From (15)–(17) and by Lemma 2, we obtain

$$\begin{aligned} w_{3m-1} &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \sum_{j=0}^{3(m-l-1)} a^j} \beta^{\sum_{l=0}^{m-1} a^{3l}} z_0^{d\sum_{l=0}^{m-1} a^{3l} a^{3(m-l)-2}} w_{-1}^{a^{3m}} \\ &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \frac{1-a^{3(m-l)-2}}{1-a}} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{dma^{3m-2}} w_{-1}^{a^{3m}} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-a^{3m}}{1-a^3} - ma^{3m-2}\right)} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{dma^{3m-2}} w_{-1}^{a^{3m}} \\ &= \alpha^{\frac{d(ma^{3m+1}-a^{3m}-ma^{3m-2}+1)}{(1-a)(1-a^3)}} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{dma^{3m-2}} w_{-1}^{a^{3m}}, \end{aligned} \tag{21}$$

$$\begin{aligned} w_{3m} &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \sum_{j=0}^{3(m-l)-2} a^j} \beta^{\sum_{l=0}^{m-1} a^{3l}} z_0^{d\sum_{l=0}^{m-1} a^{3l} a^{3(m-l)-1}} w_0^{a^{3m}} \\ &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \frac{1-a^{3(m-l)-1}}{1-a}} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{dma^{3m-1}} w_0^{a^{3m}} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-a^{3m}}{1-a^3} - ma^{3m-1}\right)} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{mda^{3m-1}} w_0^{a^{3m}} \\ &= \alpha^{\frac{d(ma^{3m+2}-a^{3m}-ma^{3m-1}+1)}{(1-a)(1-a^3)}} \beta^{\frac{1-a^{3m}}{1-a^3}} z_0^{dma^{3m-1}} w_0^{a^{3m}}, \end{aligned} \tag{22}$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \sum_{j=0}^{3(m-l)-1} a^j} \beta^{\sum_{l=0}^{m} a^{3l}} z_0^{d\sum_{l=0}^{m} a^{3l} a^{3(m-l)}} w_{-2}^{a^{3m+3}} \\ &= \alpha^{d\sum_{l=0}^{m-1} a^{3l} \frac{1-a^{3(m-l)}}{1-a}} \beta^{\frac{1-a^{3m+3}}{1-a^3}} z_0^{d(m+1)a^{3m}} w_{-2}^{a^{3m+3}} \\ &= \alpha^{\frac{d}{1-a}\left(\frac{1-a^{3m}}{1-a^3} - ma^{3m}\right)} \beta^{\frac{1-a^{3m+3}}{1-a^3}} z_0^{d(m+1)a^{3m}} w_{-2}^{a^{3m+3}} \\ &= \alpha^{\frac{d(ma^{3m+3}-(m+1)a^{3m}+1)}{(1-a)(1-a^3)}} \beta^{\frac{1-a^{3m+3}}{1-a^3}} z_0^{d(m+1)a^{3m}} w_{-2}^{a^{3m+3}}, \end{aligned} \tag{23}$$

for $m \in \mathbb{N}$.

Case $a \neq 1 = c$. From (15)–(17) and by Lemma 2, we obtain

$$\begin{aligned} w_{3m-1} &= \alpha^{d\sum_{l=0}^{m-1}\sum_{j=0}^{3(m-l-1)} a^j} \beta^{\sum_{l=0}^{m-1} 1} z_0^{d\sum_{l=0}^{m-1} a^{3(m-l)-2}} w_{-1} \\ &= \alpha^{d\sum_{l=0}^{m-1} \frac{1-a^{3(m-l)-2}}{1-a}} \beta^m z_0^{ad\frac{a^{3m}-1}{a^3-1}} w_{-1} \\ &= \alpha^{\frac{d}{1-a}\left(m - a\frac{a^{3m}-1}{a^3-1}\right)} \beta^m z_0^{ad\frac{a^{3m}-1}{a^3-1}} w_{-1} \\ &= \alpha^{\frac{d(a^{3m+1}+m(1-a^3)-a)}{(1-a)(1-a^3)}} \beta^m z_0^{ad\frac{a^{3m}-1}{a^3-1}} w_{-1}, \end{aligned} \tag{24}$$

$$\begin{aligned} w_{3m} &= \alpha^{d\sum_{l=0}^{m-1}\sum_{j=0}^{3(m-l)-2} a^j} \beta^{\sum_{l=0}^{m-1} 1} z_0^{d\sum_{l=0}^{m-1} a^{3(m-l)-1}} w_0 \\ &= \alpha^{d\sum_{l=0}^{m-1} \frac{1-a^{3(m-l)-1}}{1-a}} \beta^m z_0^{da^2\frac{a^{3m}-1}{a^3-1}} w_0 \\ &= \alpha^{\frac{d}{1-a}\left(m - a^2\frac{a^{3m}-1}{a^3-1}\right)} \beta^m z_0^{da^2\frac{a^{3m}-1}{a^3-1}} w_0 \\ &= \alpha^{\frac{d(a^{3m+2}+m(1-a^3)-a^2)}{(1-a)(1-a^3)}} \beta^m z_0^{da^2\frac{a^{3m}-1}{a^3-1}} w_0, \end{aligned} \tag{25}$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1}\sum_{j=0}^{3(m-l)-1} a^j} \beta^{\sum_{l=0}^{m} 1} z_0^{d\sum_{l=0}^{m} a^{3(m-l)}} w_{-2} \\ &= \alpha^{d\sum_{l=0}^{m-1} \frac{1-a^{3(m-l)}}{1-a}} \beta^{m+1} z_0^{d\frac{a^{3m+3}-1}{a^3-1}} w_{-2} \\ &= \alpha^{\frac{d}{1-a}\left(m - a^3\frac{a^{3m}-1}{a^3-1}\right)} \beta^{m+1} z_0^{d\frac{a^{3m+3}-1}{a^3-1}} w_{-2} \\ &= \alpha^{\frac{d(a^{3m+3}-(m+1)a^3+m)}{(1-a)(1-a^3)}} \beta^{m+1} z_0^{d\frac{a^{3m+3}-1}{a^3-1}} w_{-2}, \end{aligned} \tag{26}$$

for $m \in \mathbb{N}$.

Case $a = 1 \neq c$. From (15)–(17) and by Lemma 2, we obtain

$$\begin{aligned} w_{3m-1} &= \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l-1)} 1} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l} w_{-1}^{c^m} \\ &= \alpha^{d\sum_{l=0}^{m-1}(3(m-l-1)+1)c^l} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_{-1}^{c^m} \\ &= \alpha^{d\left((3m-2)\frac{1-c^m}{1-c} - 3c\frac{1-mc^{m-1}+(m-1)c^m}{(1-c)^2}\right)} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_{-1}^{c^m} \\ &= \alpha^{\frac{d(3m-2-(3m+1)c+2c^m+c^{m+1})}{(1-c)^2}} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_{-1}^{c^m}, \end{aligned} \tag{27}$$

$$\begin{aligned} w_{3m} &= \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)-2} 1} \beta^{\sum_{l=0}^{m-1} c^l} z_0^{d\sum_{l=0}^{m-1} c^l} w_0^{c^m} \\ &= \alpha^{d\sum_{l=0}^{m-1}(3(m-l)-1)c^l} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_0^{c^m} \\ &= \alpha^{d\left((3m-1)\frac{1-c^m}{1-c} - 3c\frac{1-mc^{m-1}+(m-1)c^m}{(1-c)^2}\right)} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_0^{c^m} \\ &= \alpha^{\frac{d(3m-1-(3m+2)c+c^m+2c^{m+1})}{(1-c)^2}} \beta^{\frac{1-c^m}{1-c}} z_0^{d\frac{1-c^m}{1-c}} w_0^{c^m}, \end{aligned} \tag{28}$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1} c^l \sum_{j=0}^{3(m-l)-1} 1} \beta^{\sum_{l=0}^{m} c^l} z_0^{d\sum_{l=0}^{m} c^l} w_{-2}^{c^{m+1}} \\ &= \alpha^{d\sum_{l=0}^{m-1} 3(m-l)c^l} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d\frac{1-c^{m+1}}{1-c}} w_{-2}^{c^{m+1}} \\ &= \alpha^{d\left(3m\frac{1-c^m}{1-c} - 3c\frac{1-mc^{m-1}+(m-1)c^m}{(1-c)^2}\right)} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d\frac{1-c^{m+1}}{1-c}} w_{-2}^{c^{m+1}} \\ &= \alpha^{\frac{3d(m-(m+1)c+c^{m+1})}{(1-c)^2}} \beta^{\frac{1-c^{m+1}}{1-c}} z_0^{d\frac{1-c^{m+1}}{1-c}} w_{-2}^{c^{m+1}}, \end{aligned} \quad (29)$$

for $m \in \mathbb{N}$.

Case $a = c = 1$. From (15)–(17), we obtain

$$\begin{aligned} w_{3m-1} &= \alpha^{d\sum_{l=0}^{m-1} \sum_{j=0}^{3(m-l-1)} 1} \beta^{\sum_{l=0}^{m-1} 1} z_0^{d\sum_{l=0}^{m-1} 1} w_{-1} \\ &= \alpha^{d\sum_{l=0}^{m-1}(3(m-l-1)+1)} \beta^m z_0^{dm} w_{-1} \\ &= \alpha^{d\frac{m(3m-1)}{2}} \beta^m z_0^{dm} w_{-1}, \end{aligned} \quad (30)$$

$$\begin{aligned} w_{3m} &= \alpha^{d\sum_{l=0}^{m-1} \sum_{j=0}^{3(m-l)-2} 1} \beta^{\sum_{l=0}^{m-1} 1} z_0^{d\sum_{l=0}^{m-1} 1} w_0 \\ &= \alpha^{d\sum_{l=0}^{m-1}(3(m-l)-1)} \beta^m z_0^{dm} w_0 \\ &= \alpha^{d\frac{m(3m+1)}{2}} \beta^m z_0^{dm} w_0, \end{aligned} \quad (31)$$

$$\begin{aligned} w_{3m+1} &= \alpha^{d\sum_{l=0}^{m-1} \sum_{j=0}^{3(m-l)-1} 1} \beta^{m+1} z_0^{d(m+1)} w_{-2} \\ &= \alpha^{3d\sum_{l=0}^{m-1}(m-l)} \beta^{m+1} z_0^{d(m+1)} w_{-2} \\ &= \alpha^{3d\frac{m(m+1)}{2}} \beta^{m+1} z_0^{d(m+1)} w_{-2}, \end{aligned} \quad (32)$$

for $m \in \mathbb{N}$.

From (11), (12), (18)–(32) the theorem follows. □

The following corollary follows from Theorem 1.

Corollary 1. *Assume that $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If $a \neq 1 \neq c \neq a^3$, then the general solution to system (2) is given by formulas (11), (18)–(20).*
(b) *If $a \neq 1 \neq c = a^3$, then the general solution to system (2) is given by formulas (11), (21)–(23).*
(c) *If $a \neq 1 = c$, then the general solution to system (2) is given by formulas (11), (24)–(26).*
(d) *If $a = 1 \neq c$, then the general solution to system (2) is given by formulas (12), (27)–(29).*
(e) *If $a = c = 1$, then the general solution to system (2) is given by formulas (12), (30)–(32).*

Theorem 2. *Assume that $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (2) is solvable in closed form.*

Proof. In this case we have

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-2}^c, \quad n \in \mathbb{N}_0. \quad (33)$$

Hence

$$w_{3n+i} = \beta w_{3(n-1)+i}^{c}, \tag{34}$$

for $n \in \mathbb{N}$ and $i = \overline{-2,0}$, and consequently

$$w_{3n+i} = \beta^{\sum_{j=0}^{n-1} c^j} w_i^{c^n}, \tag{35}$$

for $n \in \mathbb{N}$ and $i = \overline{-2,0}$.

When $c \neq 1$ relation (35) implies

$$w_{3n+i} = \beta^{\frac{1-c^n}{1-c}} w_i^{c^n}, \tag{36}$$

for $n \in \mathbb{N}$ and $i = \overline{-2,0}$, and

$$w_{3n+i} = \beta^n w_i, \tag{37}$$

for $n \in \mathbb{N}$ and $i = \overline{-2,0}$, when $c = 1$.

Further, we have

$$\begin{aligned} z_n &= \alpha(\alpha z_{n-2}^a w_{n-2}^b)^a w_{n-1}^b \\ &= \alpha^{1+a} z_{n-2}^{a^2} w_{n-2}^{ba} w_{n-1}^b \\ &= \alpha^{1+a} (\alpha z_{n-3}^a w_{n-3}^b)^{a^2} w_{n-2}^{ba} w_{n-1}^b \\ &= \alpha^{1+a+a^2} w_{n-3}^{ba^2} w_{n-2}^{ba} w_{n-1}^b z_{n-3}^{a^3}, \end{aligned}$$

for $n \geq 3$, which can be written as

$$z_{3n} = \alpha^{1+a+a^2} w_{3n-3}^{ba^2} w_{3n-2}^{ba} w_{3n-1}^b z_{3(n-1)}^{a^3} \tag{38}$$

$$z_{3n+1} = \alpha^{1+a+a^2} w_{3n-2}^{ba^2} w_{3n-1}^{ba} w_{3n}^b z_{3(n-1)+1}^{a^3} \tag{39}$$

$$z_{3n+2} = \alpha^{1+a+a^2} w_{3n-1}^{ba^2} w_{3n}^{ba} w_{3n+1}^b z_{3(n-1)+2}^{a^3}, \tag{40}$$

for $n \in \mathbb{N}$.

Employing the method of induction we get

$$z_{3n} = z_0^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} w_{3j}^{ba^2} w_{3j+1}^{ba} w_{3j+2}^b \right)^{a^{3(n-j-1)}} \tag{41}$$

$$z_{3n+1} = z_1^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} w_{3j+1}^{ba^2} w_{3j+2}^{ba} w_{3j+3}^b \right)^{a^{3(n-j-1)}} \tag{42}$$

$$z_{3n+2} = z_2^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} w_{3j+2}^{ba^2} w_{3j+3}^{ba} w_{3j+4}^b \right)^{a^{3(n-j-1)}}, \tag{43}$$

for $n \in \mathbb{N}_0$.

Using formula (35), along with the first equation in (33) with $n = 0, 1$, into (41)–(43), we get

$$
\begin{aligned}
z_{3n} =& z_0^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} \beta^{ba^2 \sum_{l=0}^{j-1} c^l} w_0^{ba^2 c^j} \beta^{ba \sum_{l=0}^{j} c^l} w_{-2}^{bac^{j+1}} \beta^{b \sum_{l=0}^{j} c^l} w_{-1}^{bc^{j+1}} \right)^{a^{3(n-j-1)}} \\
=& \alpha^{\sum_{j=0}^{3n-1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j-1} c^l + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l} \\
& \times z_0^{a^{3n}} w_{-2}^{bac \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_{-1}^{bc \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_0^{ba^2 \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}},
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
z_{3n+1} =& (\alpha z_0^a w_0^b)^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} \beta^{ba^2 \sum_{l=0}^{j} c^l} w_{-2}^{ba^2 c^{j+1}} \beta^{ba \sum_{l=0}^{j} c^l} w_{-1}^{bac^{j+1}} \beta^{b \sum_{l=0}^{j} c^l} w_0^{bc^{j+1}} \right)^{a^{3(n-j-1)}} \\
=& \alpha^{\sum_{j=0}^{3n} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l} \\
& \times z_0^{a^{3n+1}} w_{-2}^{ba^2 c \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_{-1}^{bac \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_0^{b \sum_{j=0}^{n} c^j a^{3(n-j)}},
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
z_{3n+2} =& (\alpha^{1+a} \beta^b z_0^{a^2} w_{-2}^{bc} w_0^{ab})^{a^{3n}} \\
& \times \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} \beta^{ba^2 \sum_{l=0}^{j} c^l} w_{-1}^{ba^2 c^{j+1}} \beta^{ba \sum_{l=0}^{j} c^l} w_0^{bac^{j+1}} \beta^{b \sum_{l=0}^{j+1} c^l} w_{-2}^{bc^{j+2}} \right)^{a^{3(n-j-1)}} \\
=& \alpha^{\sum_{j=0}^{3n+1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + b \sum_{j=0}^{n} a^{3(n-j)} \sum_{l=0}^{j} c^l} \\
& \times z_0^{a^{3n+2}} w_{-2}^{bc \sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-1}^{ba^2 c \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_0^{ab \sum_{j=0}^{n} c^j a^{3(n-j)}},
\end{aligned}
\tag{46}
$$

for $n \in \mathbb{N}_0$.

As in the proof of the previous theorem, there are five subcases, depending on the values of parameters a and c, for which we get closed form formulas for solutions to system (2).

Case $a \neq 1 \neq c \neq a^3$. From (44)–(46) and by Lemma 2, we have

$$
\begin{aligned}
z_{3n} =& \alpha^{\sum_{j=0}^{3n-1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j-1} c^l + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} c^l} \\
& \times z_0^{a^{3n}} w_{-2}^{bac \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_{-1}^{bc \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_0^{ba^2 \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} \\
=& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-c^j}{1-c} + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-c^{j+1}}{1-c} + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-c^{j+1}}{1-c}} \\
& \times z_0^{a^{3n}} w_{-2}^{bac \frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bc \frac{a^{3n}-c^n}{a^3-c}} w_0^{ba^2 \frac{a^{3n}-c^n}{a^3-c}} \\
=& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{\frac{b(a^2+a+1)}{1-c} \frac{1-a^{3n}}{1-a^3} - \frac{b(a^2+ac+c)}{1-c} \frac{a^{3n}-c^n}{a^3-c}} z_0^{a^{3n}} w_{-2}^{bac \frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bc \frac{a^{3n}-c^n}{a^3-c}} w_0^{ba^2 \frac{a^{3n}-c^n}{a^3-c}} \\
=& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{\frac{b(a^3-c+(c-1)a^{3n+2}-(a^2+ac+c)(a-1)c^n)}{(1-c)(a^3-c)(1-a)}} z_0^{a^{3n}} w_{-2}^{bac \frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bc \frac{a^{3n}-c^n}{a^3-c}} w_0^{ba^2 \frac{a^{3n}-c^n}{a^3-c}},
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
z_{3n+1} =&\alpha^{\sum_{j=0}^{3n} a^j}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}c^l+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}c^l+b\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}c^l}\\
&\times z_0^{a^{3n+1}} w_{-2}^{ba^2c\sum_{j=0}^{n-1}c^j a^{3(n-j-1)}} w_{-1}^{bac\sum_{j=0}^{n-1}c^j a^{3(n-j-1)}} w_0^{b\sum_{j=0}^{n}c^j a^{3(n-j)}}\\
=&\alpha^{\frac{1-a^{3n+1}}{1-a}}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-c^{j+1}}{1-c}+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-c^{j+1}}{1-c}+b\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-c^{j+1}}{1-c}}\\
&\times z_0^{a^{3n+1}} w_{-2}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bac\frac{a^{3n}-c^n}{a^3-c}} w_0^{b\frac{a^{3n+3}-c^{n+1}}{a^3-c}}\\
=&\alpha^{\frac{1-a^{3n+1}}{1-a}}\beta^{\frac{b(a^2+a+1)}{1-c}\frac{1-a^{3n}}{1-a^3}-\frac{bc(a^2+a+1)}{1-c}\frac{a^{3n}-c^n}{a^3-c}} z_0^{a^{3n+1}} w_{-2}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bac\frac{a^{3n}-c^n}{a^3-c}} w_0^{b\frac{a^{3n+3}-c^{n+1}}{a^3-c}}\\
=&\alpha^{\frac{1-a^{3n+1}}{1-a}}\beta^{\frac{b(a^3-c+(c-1)a^{3n+3}+(1-a^3)c^{n+1})}{(1-c)(1-a)(a^3-c)}} z_0^{a^{3n+1}} w_{-2}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{bac\frac{a^{3n}-c^n}{a^3-c}} w_0^{b\frac{a^{3n+3}-c^{n+1}}{a^3-c}},
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
z_{3n+2} =&\alpha^{\sum_{j=0}^{3n+1} a^j}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}c^l+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}c^l+b\sum_{j=0}^{n}a^{3(n-j)}\sum_{l=0}^{j}c^l}\\
&\times z_0^{a^{3n+2}} w_{-2}^{bc\sum_{j=0}^{n}c^j a^{3(n-j)}} w_{-1}^{ba^2c\sum_{j=0}^{n-1}c^j a^{3(n-j-1)}} w_0^{ab\sum_{j=0}^{n}c^j a^{3(n-j)}}\\
=&\alpha^{\frac{1-a^{3n+2}}{1-a}}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-c^{j+1}}{1-c}+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-c^{j+1}}{1-c}+b\sum_{j=0}^{n}a^{3(n-j)}\frac{1-c^{j+1}}{1-c}}\\
&\times z_0^{a^{3n+2}} w_{-2}^{bc\frac{a^{3n+3}-c^{n+1}}{a^3-c}} w_{-1}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_0^{ab\frac{a^{3n+3}-c^{n+1}}{a^3-c}}\\
=&\alpha^{\frac{1-a^{3n+2}}{1-a}}\beta^{\frac{b(a^2+a)}{1-c}\frac{1-a^{3n}}{1-a^3}+\frac{b(1-a^{3n+3})}{(1-c)(1-a^3)}-\frac{bc(a^2+a)}{1-c}\frac{a^{3n}-c^n}{a^3-c}-\frac{bc}{1-c}\frac{a^{3n+3}-c^{n+1}}{a^3-c}}\\
&\times z_0^{a^{3n+2}} w_{-2}^{bc\frac{a^{3n+3}-c^{n+1}}{a^3-c}} w_{-1}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_0^{ab\frac{a^{3n+3}-c^{n+1}}{a^3-c}}\\
=&\alpha^{\frac{1-a^{3n+2}}{1-a}}\beta^{\frac{b(a^3-c+(c-1)a^{3n+4}+(1-a)(a^2+a+c)c^{n+1})}{(1-c)(1-a)(a^3-c)}} z_0^{a^{3n+2}}\\
&\times w_{-2}^{bc\frac{a^{3n+3}-c^{n+1}}{a^3-c}} w_{-1}^{ba^2c\frac{a^{3n}-c^n}{a^3-c}} w_0^{ab\frac{a^{3n+3}-c^{n+1}}{a^3-c}},
\end{aligned}
\tag{49}
$$

for $n \in \mathbb{N}_0$.

Case $a \neq 1 \neq c = a^3$. From (44)–(46) and by Lemma 2, we have

$$
\begin{aligned}
z_{3n} =&\alpha^{\sum_{j=0}^{3n-1} a^j}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j-1}a^{3l}+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}a^{3l}+b\sum_{j=0}^{n-1}a^{3(n-j-1)}\sum_{l=0}^{j}a^{3l}}\\
&\times z_0^{a^{3n}} w_{-2}^{ba^4\sum_{j=0}^{n-1}a^{3j}a^{3(n-j-1)}} w_{-1}^{ba^3\sum_{j=0}^{n-1}a^{3j}a^{3(n-j-1)}} w_0^{ba^2\sum_{j=0}^{n-1}a^{3j}a^{3(n-j-1)}}\\
=&\alpha^{\frac{1-a^{3n}}{1-a}}\beta^{ba^2\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-a^{3j}}{1-a^3}+ba\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-a^{3j+3}}{1-a^3}+b\sum_{j=0}^{n-1}a^{3(n-j-1)}\frac{1-a^{3j+3}}{1-a^3}}\\
&\times z_0^{a^{3n}} w_{-2}^{bna^{3n+1}} w_{-1}^{bna^{3n}} w_0^{bna^{3n-1}}\\
=&\alpha^{\frac{1-a^{3n}}{1-a}}\beta^{\frac{ba^2}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3}-na^{3n-3}\right)+\frac{ba}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3}-na^{3n}\right)+\frac{b}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3}-na^{3n}\right)}\\
&\times z_0^{a^{3n}} w_{-2}^{bna^{3n+1}} w_{-1}^{bna^{3n}} w_0^{bna^{3n-1}}\\
=&\alpha^{\frac{1-a^{3n}}{1-a}}\beta^{b\frac{na^{3n+2}-a^{3n}-na^{3n-1}+1}{(1-a)(1-a^3)}} z_0^{a^{3n}} w_{-2}^{bna^{3n+1}} w_{-1}^{bna^{3n}} w_0^{bna^{3n-1}},
\end{aligned}
\tag{50}
$$

$$\begin{aligned} z_{3n+1} =& \alpha^{\sum_{j=0}^{3n} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} a^{3l} + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} a^{3l} + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} a^{3l}} \\ & \times z_0^{a^{3n+1}} w_{-2}^{ba^5 \sum_{j=0}^{n-1} a^{3j} a^{3(n-j-1)}} w_{-1}^{ba^4 \sum_{j=0}^{n-1} a^{3j} a^{3(n-j-1)}} w_0^{b \sum_{j=0}^{n} a^{3j} a^{3(n-j)}} \\ =& \alpha^{\frac{1-a^{3n+1}}{1-a}} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-a^{3j+3}}{1-a^3} + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-a^{3j+3}}{1-a^3} + b \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-a^{3j+3}}{1-a^3}} \\ & \times z_0^{a^{3n+1}} w_{-2}^{bna^{3n+2}} w_{-1}^{bna^{3n+1}} w_0^{b(n+1)a^{3n}} \\ =& \alpha^{\frac{1-a^{3n+1}}{1-a}} \beta^{\frac{ba^2}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3} - na^{3n}\right) + \frac{ba}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3} - na^{3n}\right) + \frac{b}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3} - na^{3n}\right)} \\ & \times z_0^{a^{3n+1}} w_{-2}^{bna^{3n+2}} w_{-1}^{bna^{3n+1}} w_0^{b(n+1)a^{3n}} \\ =& \alpha^{\frac{1-a^{3n+1}}{1-a}} \beta^{\frac{b(na^{3n+3} - (n+1)a^{3n} + 1)}{(1-a)(1-a^3)}} z_0^{a^{3n+1}} w_{-2}^{bna^{3n+2}} w_{-1}^{bna^{3n+1}} w_0^{b(n+1)a^{3n}}, \end{aligned} \tag{51}$$

$$\begin{aligned} z_{3n+2} =& \alpha^{\sum_{j=0}^{3n+1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} a^{3l} + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{l=0}^{j} a^{3l} + b \sum_{j=0}^{n} a^{3(n-j)} \sum_{l=0}^{j} a^{3l}} \\ & \times z_0^{a^{3n+2}} w_{-2}^{ba^3 \sum_{j=0}^{n} a^{3j} a^{3(n-j)}} w_{-1}^{ba^5 \sum_{j=0}^{n-1} a^{3j} a^{3(n-j-1)}} w_0^{ab \sum_{j=0}^{n} a^{3j} a^{3(n-j)}} \\ =& \alpha^{\frac{1-a^{3n+2}}{1-a}} \beta^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-a^{3j+3}}{1-a^3} + ba \sum_{j=0}^{n-1} a^{3(n-j-1)} \frac{1-a^{3j+3}}{1-a^3} + b \sum_{j=0}^{n} a^{3(n-j)} \frac{1-a^{3j+3}}{1-a^3}} \\ & \times z_0^{a^{3n+2}} w_{-2}^{b(n+1)a^{3n+3}} w_{-1}^{bna^{3n+2}} w_0^{b(n+1)a^{3n+1}} \\ =& \alpha^{\frac{1-a^{3n+2}}{1-a}} \beta^{\frac{ba^2}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3} - na^{3n}\right) + \frac{ba}{1-a^3}\left(\frac{1-a^{3n}}{1-a^3} - na^{3n}\right) + \frac{b}{1-a^3}\left(\frac{1-a^{3n+3}}{1-a^3} - (n+1)a^{3n+3}\right)} \\ & \times z_0^{a^{3n+2}} w_{-2}^{b(n+1)a^{3n+3}} w_{-1}^{bna^{3n+2}} w_0^{b(n+1)a^{3n+1}} \\ =& \alpha^{\frac{1-a^{3n+2}}{1-a}} \beta^{\frac{b((n+1)a^{3n+4} - a^{3n+3} - (n+1)a^{3n+1} + 1)}{(1-a)(1-a^3)}} z_0^{a^{3n+2}} w_{-2}^{b(n+1)a^{3n+3}} w_{-1}^{bna^{3n+2}} w_0^{b(n+1)a^{3n+1}}, \end{aligned} \tag{52}$$

for $n \in \mathbb{N}_0$.

Case If $a \neq 1 = c$. From (44)–(46) and by Lemma 2, we have

$$\begin{aligned} z_{3n} =& \alpha^{\sum_{j=0}^{3n-1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1} ja^{3(n-j-1)} + ba \sum_{j=0}^{n-1} (j+1)a^{3(n-j-1)} + b \sum_{j=0}^{n-1} (j+1)a^{3(n-j-1)}} \\ & \times z_0^{a^{3n}} w_{-2}^{ba \sum_{j=0}^{n-1} a^{3(n-j-1)}} w_{-1}^{b \sum_{j=0}^{n-1} a^{3(n-j-1)}} w_0^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)}} \\ =& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{ba^2 \frac{a^{3n} - na^3 + n - 1}{(a^3-1)^2} + ba \frac{a^{3n+3} - (n+1)a^3 + n}{(a^3-1)^2} + b \frac{a^{3n+3} - (n+1)a^3 + n}{(a^3-1)^2}} \\ & \times z_0^{a^{3n}} w_{-2}^{ba \frac{1-a^{3n}}{1-a^3}} w_{-1}^{b \frac{1-a^{3n}}{1-a^3}} w_0^{ba^2 \frac{1-a^{3n}}{1-a^3}} \\ =& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{b \frac{(a^2+a+1)a^{3n+2} - na^5 - (n+1)a^4 - (n+1)a^3 + (n-1)a^2 + na + n}{(a^3-1)^2}} \\ & \times z_0^{a^{3n}} w_{-2}^{ba \frac{1-a^{3n}}{1-a^3}} w_{-1}^{b \frac{1-a^{3n}}{1-a^3}} w_0^{ba^2 \frac{1-a^{3n}}{1-a^3}} \\ =& \alpha^{\frac{1-a^{3n}}{1-a}} \beta^{b \frac{a^{3n+2} - na^3 + n - a^2}{(a-1)(a^3-1)}} z_0^{a^{3n}} w_{-2}^{ba \frac{1-a^{3n}}{1-a^3}} w_{-1}^{b \frac{1-a^{3n}}{1-a^3}} w_0^{ba^2 \frac{1-a^{3n}}{1-a^3}}, \end{aligned} \tag{53}$$

$$z_{3n+1} = \alpha^{\sum_{j=0}^{3n} a^j} \beta^{ba^2 \sum_{j=0}^{n-1}(j+1)a^{3(n-j-1)} + ba \sum_{j=0}^{n-1}(j+1)a^{3(n-j-1)} + b\sum_{j=0}^{n-1}(j+1)a^{3(n-j-1)}}$$
$$\times z_0^{a^{3n+1}} w_{-2}^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)}} w_{-1}^{ba \sum_{j=0}^{n-1} a^{3(n-j-1)}} w_0^{b\sum_{j=0}^{n} a^{3(n-j)}}$$
$$= \alpha^{\frac{1-a^{3n+1}}{1-a}} \beta^{b\frac{a^{3n+3}-(n+1)a^3+n}{(a-1)(a^3-1)}} z_0^{a^{3n+1}} w_{-2}^{ba^2\frac{1-a^{3n}}{1-a^3}} w_{-1}^{ba\frac{1-a^{3n}}{1-a^3}} w_0^{b\frac{1-a^{3n+3}}{1-a^3}}, \tag{54}$$

$$z_{3n+2} = \alpha^{\sum_{j=0}^{3n+1} a^j} \beta^{ba^2 \sum_{j=0}^{n-1}(j+1)a^{3(n-j-1)} + ba \sum_{j=0}^{n-1}(j+1)a^{3(n-j-1)} + b\sum_{j=0}^{n}(j+1)a^{3(n-j)}}$$
$$\times z_0^{a^{3n+2}} w_{-2}^{b\sum_{j=0}^{n} a^{3(n-j)}} w_{-1}^{ba^2 \sum_{j=0}^{n-1} a^{3(n-j-1)}} w_0^{ab\sum_{j=0}^{n} a^{3(n-j)}}$$
$$= \alpha^{\frac{1-a^{3n+2}}{1-a}} \beta^{ba^2\frac{a^{3n+3}-(n+1)a^3+n}{(a^3-1)^2} + ba\frac{a^{3n+3}-(n+1)a^3+n}{(a^3-1)^2} + b\frac{a^{3n+6}-(n+2)a^3+n+1}{(a^3-1)^2}}$$
$$\times z_0^{a^{3n+2}} w_{-2}^{b\frac{1-a^{3n+3}}{1-a^3}} w_{-1}^{ba^2\frac{1-a^{3n}}{1-a^3}} w_0^{ab\frac{1-a^{3n+3}}{1-a^3}}$$
$$= \alpha^{\frac{1-a^{3n+2}}{1-a}} \beta^{b\frac{a^{3n+4}+(1-a^3)n-(a^3+a-1)}{(a-1)(a^3-1)}} z_0^{a^{3n+2}} w_{-2}^{b\frac{1-a^{3n+3}}{1-a^3}} w_{-1}^{ba^2\frac{1-a^{3n}}{1-a^3}} w_0^{ab\frac{1-a^{3n+3}}{1-a^3}}, \tag{55}$$

for $n \in \mathbb{N}_0$.

Case If $a = 1 \neq c$. From (44)–(46) and by Lemma 2, we have

$$z_{3n} = \alpha^{3n} \beta^{b\sum_{j=0}^{n-1}\sum_{l=0}^{j-1} c^l + b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l + b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l} z_0 w_{-2}^{bc\sum_{j=0}^{n-1} c^j} w_{-1}^{bc\sum_{j=0}^{n-1} c^j} w_0^{b\sum_{j=0}^{n-1} c^j}$$
$$= \alpha^{3n} \beta^{b\sum_{j=0}^{n-1}\frac{1-c^j}{1-c} + b\sum_{j=0}^{n-1}\frac{1-c^{j+1}}{1-c} + b\sum_{j=0}^{n-1}\frac{1-c^{j+1}}{1-c}} z_0 w_{-2}^{bc\frac{1-c^n}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^n}{1-c}}$$
$$= \alpha^{3n} \beta^{\frac{b(2c^{n+1}+c^n-3(c-1)n-2c-1)}{(1-c)^2}} z_0 w_{-2}^{bc\frac{1-c^n}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^n}{1-c}}, \tag{56}$$

$$z_{3n+1} = \alpha^{3n+1} \beta^{b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l + b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l + b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l} z_0 w_{-2}^{bc\sum_{j=0}^{n-1} c^j} w_{-1}^{bc\sum_{j=0}^{n-1} c^j} w_0^{b\sum_{j=0}^{n} c^j}$$
$$= \alpha^{3n+1} \beta^{3b\sum_{j=0}^{n-1}\frac{1-c^{j+1}}{1-c}} z_0 w_{-2}^{bc\frac{1-c^n}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^{n+1}}{1-c}}$$
$$= \alpha^{3n+1} \beta^{3b\frac{c^{n+1}-(c-1)n-c}{(1-c)^2}} z_0 w_{-2}^{bc\frac{1-c^n}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^{n+1}}{1-c}}, \tag{57}$$

$$z_{3n+2} = \alpha^{3n+2} \beta^{b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l + b\sum_{j=0}^{n-1}\sum_{l=0}^{j} c^l + b\sum_{j=0}^{n}\sum_{l=0}^{j} c^l}$$
$$\times z_0 w_{-2}^{bc\sum_{j=0}^{n} c^j} w_{-1}^{bc\sum_{j=0}^{n-1} c^j} w_0^{b\sum_{j=0}^{n} c^j}$$
$$= \alpha^{3n+2} \beta^{b\sum_{j=0}^{n-1}\frac{1-c^{j+1}}{1-c} + b\sum_{j=0}^{n-1}\frac{1-c^{j+1}}{1-c} + b\sum_{j=0}^{n}\frac{1-c^{j+1}}{1-c}}$$
$$\times z_0 w_{-2}^{bc\frac{1-c^{n+1}}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^{n+1}}{1-c}}$$
$$= \alpha^{3n+2} \beta^{\frac{b(c^{n+2}+2c^{n+1}-3(c-1)n+1-4c)}{(1-c)^2}} z_0 w_{-2}^{bc\frac{1-c^{n+1}}{1-c}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^{n+1}}{1-c}}, \tag{58}$$

for $n \in \mathbb{N}_0$.

Case If $a = c = 1$. From (44)–(46), we have

$$z_{3n} = \alpha^{3n} \beta^{b\sum_{j=0}^{n-1} j + b\sum_{j=0}^{n-1}(j+1) + b\sum_{j=0}^{n-1}(j+1)} z_0 w_{-2}^{bn} w_{-1}^{bn} w_0^{bn}$$
$$= \alpha^{3n} \beta^{b\frac{n(3n+1)}{2}} z_0 w_{-2}^{bn} w_{-1}^{bn} w_0^{bn}, \tag{59}$$

$$
\begin{aligned}
z_{3n+1} =&\alpha^{3n+1}\beta^{b\sum_{j=0}^{n-1}(j+1)+b\sum_{j=0}^{n-1}(j+1)+b\sum_{j=0}^{n-1}(j+1)}z_0 w_{-2}^{bn}w_{-1}^{bn}w_0^{b(n+1)}\\
=&\alpha^{3n+1}\beta^{3b\frac{n(n+1)}{2}}z_0 w_{-2}^{bn}w_{-1}^{bn}w_0^{b(n+1)},
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
z_{3n+2} =&\alpha^{3n+2}\beta^{b\sum_{j=0}^{n-1}(j+1)+b\sum_{j=0}^{n-1}(j+1)+b\sum_{j=0}^{n}(j+1)}z_0 w_{-2}^{b(n+1)}w_{-1}^{bn}w_0^{b(n+1)}\\
=&\alpha^{3n+2}\beta^{b\frac{(n+1)(3n+2)}{2}}z_0 w_{-2}^{b(n+1)}w_{-1}^{bn}w_0^{b(n+1)},
\end{aligned}
\tag{61}
$$

for $n \in \mathbb{N}_0$.

From all above presented formulas we see that system of difference equations (2) is solvable in this case, as claimed. □

The statements in the following corollary are direct consequences of the formulas presented in the proof of Theorem 2.

Corollary 2. *Assume that $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

- *(a) If $a \neq 1 \neq c \neq a^3$, then the general solution to system (2) is given by (36), (47)–(49).*
- *(b) If $a \neq 1 \neq c = a^3$, then the general solution to system (2) is given by (36), (50)–(52).*
- *(c) If $a \neq 1 = c$, then the general solution to system (2) is given by (37), (53)–(55).*
- *(d) If $a = 1 \neq c$, then the general solution to system (2) is given by (36), (56)–(58).*
- *(e) If $a = c = 1$, then the general solution to system (2) is given by (37), (59)–(61).*

The following result concerns the case $c = 0$. Note that under the condition system (2) of difference equations is

$$
z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta z_n^d, \quad n \in \mathbb{N}_0,
\tag{62}
$$

which is system (3.15) in [33].

From (62) it follows that

$$
z_{n+1} = \alpha\beta^b z_n^a z_{n-1}^{bd},
$$

for $n \in \mathbb{N}$.

How this product-type difference equation can be solved was explained in the proof of Theorem 3.3 in [33]. From the closed-form formulas obtained in the proof of Theorem 3.3 therein it follows directly that the following result holds. Hence, to avoid repeating, the proof of the theorem is omitted here.

Theorem 3. *Assume that $a, b, d \in \mathbb{Z}$, $c = 0$, $\alpha, \beta, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) If $bd \neq 0$, $a^2 + 4bd \neq 0$ and $a + bd \neq 1$, then the general solution to system (2) is given by

$$
\begin{aligned}
z_n =&\alpha^{\frac{(\lambda_2-1)\lambda_1^{n+1}-(\lambda_1-1)\lambda_2^{n+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}\beta^{b\frac{(\lambda_2-1)\lambda_1^{n}-(\lambda_1-1)\lambda_2^{n}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}z_0^{\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}w_0^{b\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}},\\
w_n =&\alpha^{d\frac{(\lambda_2-1)\lambda_1^{n}-(\lambda_1-1)\lambda_2^{n}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}\beta^{1+bd\frac{(\lambda_2-1)\lambda_1^{n-1}-(\lambda_1-1)\lambda_2^{n-1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}}z_0^{d\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}}w_0^{bd\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}},
\end{aligned}
$$

for $n \in \mathbb{N}$, where

$$
\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4bd}}{2}.
$$

(b) *If $bd \neq 0$, $a^2 + 4bd = 0$ and $a + bd \neq 1$, then the general solution to system (2) is given by*

$$z_n = \alpha^{\frac{1-(n+1)\lambda_1^n + n\lambda_1^{n+1}}{(1-\lambda_1)^2}} \beta^{b\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}} z_0^{(n+1)\lambda_1^n} w_0^{bn\lambda_1^{n-1}}$$
$$w_n = \alpha^{d\frac{1-n\lambda_1^{n-1}+(n-1)\lambda_1^n}{(1-\lambda_1)^2}} \beta^{\frac{1-2\lambda_1+(n-1)\lambda_1^n-(n-2)\lambda_1^{n+1}}{(1-\lambda_1)^2}} z_0^{dn\lambda_1^{n-1}} w_0^{(1-n)\lambda_1^n}$$

for $n \in \mathbb{N}$, where

$$\lambda_1 = \frac{a}{2}.$$

(c) *If $bd \neq 0$, $a^2 + 4bd \neq 0$ and $a + bd = 1$, then the general solution to system (2) is given by*

$$z_n = \alpha^{\frac{(-bd)^{n+1}+(n+1)bd+n}{(1+bd)^2}} \beta^{b\frac{(-bd)^n+nbd+n-1}{(1+bd)^2}} z_0^{\frac{1-(-bd)^{n+1}}{1+bd}} w_0^{b\frac{1-(-bd)^n}{1+bd}},$$
$$w_n = \alpha^{d\frac{(-bd)^n+nbd+n-1}{(1+bd)^2}} \beta^{1+bd\frac{(-bd)^{n-1}+(n-1)bd+n-2}{(1+bd)^2}} z_0^{d\frac{1-(-bd)^n}{1+bd}} w_0^{bd\frac{1-(-bd)^{n-1}}{1+bd}},$$

for $n \in \mathbb{N}$.

(d) *If $bd \neq 0$, $a^2 + 4bd = 0$ and $a + bd = 1$, then the general solution to system (2) is given by*

$$z_n = \alpha^{\frac{n(n+1)}{2}} \beta^{b\frac{(n-1)n}{2}} z_0^{n+1} w_0^{bn}$$
$$w_n = \alpha^{d\frac{(n-1)n}{2}} \beta^{\frac{(3-n)n}{2}} z_0^{dn} w_0^{1-n}$$

for $n \in \mathbb{N}$.

(e) *If $bd = 0$ and $a \neq 1$, then the general solution to system (2) is given by*

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}} z_0^{a^n} w_0^{ba^{n-1}}$$
$$w_n = \alpha^{d\frac{1-a^{n-1}}{1-a}} \beta z_0^{da^{n-1}},$$

for $n \in \mathbb{N}$.

(f) *If $bd = 0$ and $a = 1$, then the general solution to system (2) is given by*

$$z_n = \alpha^n \beta^{b(n-1)} z_0 w_0^b$$
$$w_n = \alpha^{d(n-1)} \beta z_0^d,$$

for $n \in \mathbb{N}$.

Theorem 4. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (2) is solvable in closed form.*

Proof. To deal with the case we use and modify our method previously used, for example, in [40,43]. Since $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, we have $z_n w_n \neq 0$ for $n \in \mathbb{N}_0$. Hence,

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{63}$$

$$w_{n+1}^b = \beta^b w_{n-2}^{bc} z_n^{bd}, \quad n \in \mathbb{N}_0, \tag{64}$$

from which we get

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^a z_n^{bd} z_{n-1}^c z_{n-2}^{-ac}, \tag{65}$$

for $n \geq 2$.

Let $\delta = \alpha^{1-c}\beta^{b}$,

$$a_1 = a, \quad b_1 = bd, \quad c_1 = c, \quad d_1 = -ac, \quad y_1 = 1, \tag{66}$$

then

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \geq 2. \tag{67}$$

From (67) is obtained

$$\begin{aligned} z_{n+2} &= \delta^{y_1} (\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1})^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \\ &= \delta^{y_1+a_1} z_n^{a_1 a_1 + b_1} z_{n-1}^{b_1 a_1 + c_1} z_{n-2}^{c_1 a_1 + d_1} z_{n-3}^{d_1 a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} z_{n-3}^{d_2}, \end{aligned} \tag{68}$$

for $n \geq 3$, where

$$a_2 := a_1 a_1 + b_1,\ b_2 := b_1 a_1 + c_1,\ c_2 := c_1 a_1 + d_1,\ d_2 := d_1 a_1,\ y_2 := y_1 + a_1. \tag{69}$$

Suppose

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k}, \tag{70}$$

for a $k \geq 2$ and $n \geq k+1$, and

$$\begin{aligned} a_k &= a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1}, \\ c_k &= c_1 a_{k-1} + d_{k-1}, \quad d_k = d_1 a_{k-1}, \end{aligned} \tag{71}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{72}$$

Using (67) in (70), we get

$$\begin{aligned} z_{n+2} &= \delta^{y_k} (\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1} z_{n-k-2}^{d_1})^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1 a_k + b_k} z_{n-k}^{b_1 a_k + c_k} z_{n-k-1}^{c_1 a_k + d_k} z_{n-k-2}^{d_1 a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} z_{n-k-2}^{d_{k+1}}, \end{aligned} \tag{73}$$

for $n \geq k+2$, where

$$\begin{aligned} a_{k+1} &:= a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k + c_k, \\ c_{k+1} &:= c_1 a_k + d_k, \quad d_{k+1} := d_1 a_k, \end{aligned} \tag{74}$$

$$y_{k+1} := y_k + a_k. \tag{75}$$

Hence, by the induction we get that (70)–(72) hold.

From (70) with $k = n - 1$ and (8), it follows that

$$\begin{aligned} z_{n+2} =& \delta^{y_{n-1}} z_3^{a_{n-1}} z_2^{b_{n-1}} z_1^{c_{n-1}} z_0^{d_{n-1}} \\ =& (\alpha^{1-c}\beta^b)^{y_{n-1}} (\alpha^{1+a+a^2+bd}\beta^{b+ab} z_0^{a^3+2abd} w_{-2}^{abc} w_{-1}^{bc} w_0^{b(a^2+bd)})^{a_{n-1}} \\ & \times (\alpha^{1+a}\beta^b z_0^{a^2+bd} w_{-2}^{bc} w_0^{ab})^{b_{n-1}} (\alpha z_0^a w_0^b)^{c_{n-1}} z_0^{d_{n-1}} \\ =& \alpha^{(1-c)y_{n-1}+(1+a+a^2+bd)a_{n-1}+(1+a)b_{n-1}+c_{n-1}} \beta^{by_{n-1}+b(1+a)a_{n-1}+bb_{n-1}} \\ & \times z_0^{(a^3+2abd)a_{n-1}+(a^2+bd)b_{n-1}+ac_{n-1}+d_{n-1}} w_{-2}^{abca_{n-1}+bcb_{n-1}} w_{-1}^{bca_{n-1}} \\ & \times w_0^{b(a^2+bd)a_{n-1}+abb_{n-1}+bc_{n-1}} \\ =& \alpha^{y_{n+2}-cy_{n-1}} \beta^{by_{n+1}} z_0^{a_{n+2}-ca_{n-1}} w_{-2}^{bca_n} w_{-1}^{bca_{n-1}} w_0^{ba_{n+1}}, \end{aligned} \tag{76}$$

for $n \in \mathbb{N}_0$.

From (71) we have

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}, \quad k \geq 5, \tag{77}$$

and that b_k, c_k and d_k also satisfy the equation, and using (74) and (75) for $k = 0, -1, -2, -3$ is obtained

$$a_{-3} = 0, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_0 = 1, \tag{78}$$

$$y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1, \tag{79}$$

and

$$y_k = \sum_{j=0}^{k-1} a_j, \tag{80}$$

(see, for example, [33] for more details).

The solvability of (77) is a classical thing. Hence, by finding closed form formula for a_k, employing it in (80), then using Lemma 2, is calculated y_k. These two formulas and (76) give a closed-form formula for solution to (65).

Now note that

$$z_n^d = \frac{w_{n+1}}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0, \tag{81}$$

$$z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0, \tag{82}$$

from which we get

$$w_{n+2} = \alpha^d \beta^{1-a} w_{n+1}^a w_n^{bd} w_{n-1}^c w_{n-2}^{-ac}, \quad n \in \mathbb{N}_0. \tag{83}$$

As above is get

$$w_{n+2} = \eta^{y_k} w_{n+2-k}^{a_k} w_{n+1-k}^{b_k} w_{n-k}^{c_k} w_{n-k-1}^{d_k}, \tag{84}$$

for every $k, n \in \mathbb{N}$, such that $n \geq k - 1$, where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are defined by (66) and (71), while $(y_k)_{k\in\mathbb{N}}$ is defined by (72) and (79).

From (84) with $k = n+1$ and (8), we have

$$\begin{aligned} w_{n+2} &= \eta^{y_{n+1}} w_1^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} w_{-2}^{d_{n+1}} \\ &= (\alpha^d \beta^{1-a})^{y_{n+1}} (\beta w_{-2}^c z_0^d)^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} w_{-2}^{d_{n+1}} \\ &= \alpha^{d y_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}} z_0^{d a_{n+1}} w_{-2}^{c a_{n+1}+d_{n+1}} w_{-1}^{c_{n+1}} w_0^{b_{n+1}} \\ &= \alpha^{d y_{n+1}} \beta^{y_{n+2}-a y_{n+1}} z_0^{d a_{n+1}} w_{-2}^{c(a_{n+1}-a a_n)} w_{-1}^{c(a_n - a a_{n-1})} w_0^{a_{n+2}-a a_{n+1}}, \end{aligned} \tag{85}$$

for $n \in \mathbb{N}_0$.

Recall that closed form formulas for a_k and y_k can be found. Applying them into (85) we show the solvability of equation (83). It can be checked that (76) and (85) present a solution to (2), from which the theorem follows. □

Corollary 3. *Consider system (2) with $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the general solution to (2) is given by (76) and (85), where $(a_k)_{k\in\mathbb{N}}$ is defined by (77) and (78), while $(y_k)_{k\in\mathbb{N}}$ is defined by (79) and (80).*

Detailed Form of Solutions Given in (76) and (85)

Equation (77) is not only theoretically but also practically solvable. The reason for this is that its characteristic polynomial

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - bd\lambda^2 - c\lambda + ac, \tag{86}$$

for the case $ac \neq 0$, is of the forth degree, thus, solvable by radicals.

Note that the equation $p_4(\lambda) = 0$ can be written as follows [44].

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\left(\frac{a^2}{4} + s + bd\right)\lambda^2 - \left(\frac{as}{2} - c\right)\lambda + \frac{s^2}{4} - ac\right) = 0, \tag{87}$$

Now, choose s so that the expression in the second bracket in (87) is a perfect square. Thus, it must be

$$(as - 2c)^2 = (a^2 + 4s + 4bd)(s^2 - 4ac),$$

that is,

$$s^3 + bds^2 - 3acs - a^3c - c^2 - 4abcd = 0. \tag{88}$$

Hence, (87) can be written as

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\frac{\sqrt{a^2+4s+4bd}}{2}\lambda - \frac{as-2c}{2\sqrt{a^2+4s+4bd}}\right)^2 = 0, \tag{89}$$

which is equivalent to

$$\lambda^2 - \left(\frac{a}{2} + \frac{\sqrt{a^2+4s+4bd}}{2}\right)\lambda + \frac{s}{2} + \frac{as-2c}{2\sqrt{a^2+4s+4bd}} = 0, \tag{90}$$

$$\lambda^2 - \left(\frac{a}{2} - \frac{\sqrt{a^2+4s+4bd}}{2}\right)\lambda + \frac{s}{2} - \frac{as-2c}{2\sqrt{a^2+4s+4bd}} = 0. \tag{91}$$

Using the change of variables $s = t - \frac{bd}{3}$ in (88), it follows that

$$t^3 - \left(\frac{b^2d^2}{3} + 3ac\right)t + \frac{2b^3d^3}{27} - a^3c - c^2 - 3abcd = 0. \tag{92}$$

Let

$$p = -\left(\frac{b^2d^2}{3} + 3ac\right) \quad \text{and} \quad q = -\frac{27a^3c + 27c^2 + 81abcd - 2b^3d^3}{27}.$$

As usual, a solution to (92) is found in the form $t = u + v$. Putting it into (92) and requesting $uv = -p/3$, is get $u^3 + v^3 = -q$ and $u^3v^3 = -p^3/27$. Hence, u^3 and v^3 are solutions to the equation $z^2 + qz - p^3/27 = 0$, so, they must be $(-q \pm \sqrt{q^2 + 4p^3/27})/2$.

Hence

$$t = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{93}$$

where any of the three possible values of the right-hand side can be chosen. If $p = -\Delta_0/3$ and $q = -\Delta_1/27$, then it can be written as

$$t = \frac{1}{3\sqrt[3]{2}}\left(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right). \tag{94}$$

For such chosen s, that is, t, (90) and (91) can be solved and by some calculation it is obtained that the zeros of p_4 are

$$\lambda_1 = \frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t} + \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4bd}{3} - t - \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t}}}, \tag{95}$$

$$\lambda_2 = \frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t} - \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4bd}{3} - t - \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t}}}, \tag{96}$$

$$\lambda_3 = \frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t} + \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4bd}{3} - t + \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t}}}, \tag{97}$$

$$\lambda_4 = \frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t} - \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4bd}{3} - t + \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2bd}{3} + t}}}, \tag{98}$$

where

$$Q := -a^3 - 8c - 4abd. \tag{99}$$

Recall, that the nature of these λ_j's depends on the sign of the discriminant

$$\Delta := \frac{1}{27}(4\Delta_0^3 - \Delta_1^2), \tag{100}$$

where

$$\Delta_0 := b^2d^2 + 9ac, \tag{101}$$

$$\Delta_1 := -2b^3d^3 + 81acbd + 27a^3c + 27c^2, \tag{102}$$

and the signs of

$$P := -8bd - 3a^2 \tag{103}$$

and

$$D := 48ac - 16b^2d^2 - 16a^2bd - 3a^4. \tag{104}$$

Zeros of p_4 are different and none of them is 1. If a, b, c and d are chosen such that

$$\Delta_0 = b^2d^2 + 9ac < 0,$$

then it will be $\Delta < 0$, from which by Lemma 3 we have in this case that p_4 has four different zeros. Moreover, since $\Delta < 0$ two zeros are real and two are complex-conjugate.

Zeros of p_4 are different and one of them is 1. Polynomial p_4 has a zero equal to 1 if $p_4(1) = 1 - a - bd - c + ac = 0$, that is, if

$$(a-1)(c-1) = bd, \tag{105}$$

so that

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - (a-1)(c-1)\lambda^2 - c\lambda + ac. \tag{106}$$

Thus, if we choose a and c such that

$$p_4'(1) = 4 - 3a - 2(a-1)(c-1) - c \neq 0,$$

that is, $(2a-1)(2c+1) \neq 3$, then p_4 will be such a polynomial if $\Delta \neq 0$. For example, if $a = 3$ and $c = 2$, then $bd = 2 \neq 0$, $\Delta \neq 0$, which means that polynomial p_4 has all zeros mutually different and exactly one of them is equal to 1

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 - 2\lambda^2 - 2\lambda + 6 = (\lambda - 1)(\lambda^3 - 2\lambda^2 - 4\lambda - 6). \tag{107}$$

Since in these two cases $\lambda_j \neq \lambda_i$, $i \neq j$, then the general solution to (77) is

$$a_n = \gamma_1\lambda_1^n + \gamma_2\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{108}$$

where γ_i, $i = \overline{1,4}$, are arbitrary constants.

Lemma 1 implies

$$\sum_{j=1}^{4} \frac{\lambda_j^l}{p_4'(\lambda_j)} = 0 \quad \text{for} \quad l = \overline{0,2}, \quad \text{and} \quad \sum_{j=1}^{4} \frac{\lambda_j^3}{p_4'(\lambda_j)} = 1. \tag{109}$$

From initial conditions (78) and (109), it is obtained

$$\begin{aligned} a_n = \sum_{j=1}^{4} \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} &= \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}, \end{aligned} \tag{110}$$

for $n \geq -3$ ([40]).

Combining (80) and (110), we get

$$y_n = \sum_{j=0}^{n-1}\sum_{i=1}^{4}\frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4}\frac{\lambda_i^3(\lambda_i^n-1)}{p_4'(\lambda_i)(\lambda_i-1)}, \quad n\in\mathbb{N}, \tag{111}$$

when $\lambda_i \neq 1$, $i=\overline{1,4}$, and

$$y_n = \frac{n}{2-a+c-2ac} + \sum_{i=2}^{4}\frac{\lambda_i^3(\lambda_i^n-1)}{p_4'(\lambda_i)(\lambda_i-1)}, \quad n\in\mathbb{N}, \tag{112}$$

when one of the zeros is 1, say λ_1, is equal to 1.

Note that if one of the zeros is equal to 1, then we have

$$\begin{aligned} p_4(\lambda) =& \lambda^4 - a\lambda^3 - (a-1)(c-1)\lambda^2 - c\lambda + ac \\ =& (\lambda-1)(\lambda^3 + (1-a)\lambda^2 + (1-a)c\lambda - ac). \end{aligned} \tag{113}$$

By using the change of variables $\lambda = t + \frac{a-1}{3}$ in (113) we get that the equation

$$\lambda^3 + (1-a)\lambda^2 + (1-a)c\lambda - ac = 0, \tag{114}$$

is transformed to the following one

$$t^3 + \tilde{p}t + \tilde{q} = 0,$$

with

$$\tilde{p} = -\frac{(a-1)(a+3c-1)}{3} \quad \text{and} \quad \tilde{q} = -\left(\frac{2(a-1)^3}{27} + \frac{c(a-1)^2}{3} + ac\right), \tag{115}$$

whose solutions are given by

$$t_j = \varepsilon^{j-1}\sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}} + \varepsilon^{j-1}\sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}}, \quad j = \overline{1,3},$$

where $\varepsilon^3 = 1$ and $\varepsilon \neq 1$.

Hence,

$$\lambda_j = \frac{a-1}{3} + \varepsilon^{j-2}\sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}} + \varepsilon^{j-2}\sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}}, \tag{116}$$

for $j = \overline{2,4}$, are the other three zeros of the equation $p_4(\lambda) = 0$, in this case.

A simple calculation along with (109) shows that (111) holds also for $n = -j$, $j = \overline{0,3}$.

The previous analysis along with Corollary 3 implies the following corollary.

Corollary 4. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C}\setminus\{0\}$ and $\Delta \neq 0$. Then the following statements are true.*

- *(a) If $(a-1)(c-1) \neq bd$, then the general solution to (2) is given by (76) and (85), where $(a_n)_{n\geq -3}$ is given by (110), $(y_n)_{n\geq -3}$ is given by (111), while λ_j's, $j = \overline{1,4}$, are given by (95)–(98).*
- *(b) If $(a-1)(c-1) = bd$ and $(2a-1)(2c+1) \neq 3$, then the general solution to (2) is given by (76) and (85), where $(a_n)_{n\geq -3}$ is given by (110) with $\lambda_1 = 1$, $(y_n)_{n\geq -3}$ is given by (112), $\lambda_1 = 1$, while λ_j's, $j = \overline{2,4}$, are given by (116) and (115).*

p_4 has only one double zero which is equal to 1. Polynomial (86) has a double zero equal to 1 if (105) holds and

$$(2a-1)(2c+1) = 3. \tag{117}$$

From (117) we have that it must be $a = 2$ and $c = 0$, or $a = 1$ and $c = 1$, or $a = 0$ and $c = -2$, or $a = -1$ and $c = -1$. If $a = 0$ or $c = 0$, then $ac = 0$, which contradicts to the assumption $ac \neq 0$.

If $a = c = 1$, then $bd = 0$, from which it follows that

$$p_4(\lambda) = \lambda^4 - \lambda^3 - \lambda + 1 = (\lambda-1)^2(\lambda^2+\lambda+1),$$

and consequently

$$\lambda_{3,4} = \frac{-1 \pm i\sqrt{3}}{2}, \tag{118}$$

are the non-unit zeros of polynomial p_4 in the case.

If $a = c = -1$, then $bd = 4$, from which it follows that

$$p_4(\lambda) = \lambda^4 + \lambda^3 - 4\lambda^2 + \lambda + 1 = (\lambda-1)^2(\lambda^2+3\lambda+1),$$

and consequently

$$\lambda_{3,4} = \frac{-3 \pm \sqrt{5}}{2}, \tag{119}$$

are the non-unit zeros of polynomial p_4 in the case.

From this, we have proved in passing, that there are no $a, c \in \mathbb{Z} \setminus \{0\}$, such that 1 is a triple zero of p_4 or that it has two pairs of double zeros one of which is 1.

In these two cases we have (see, for example, [38])

$$a_n = \frac{n(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^{n+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)}, \tag{120}$$

and

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1}\left(\frac{j(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^{j+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)}\right) \\ &= \frac{(n-1)n}{2(1-\lambda_3)(1-\lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1-\lambda_3)^2(1-\lambda_4)^2} + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3-1)^3(\lambda_3-\lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4-1)^3(\lambda_4-\lambda_3)}. \end{aligned} \tag{121}$$

p_4 has a double zero different from 1. Let $bd = 0$, then

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - c\lambda + ac = (\lambda - a)(\lambda^3 - c).$$

If we take $c = a^3$, then it is obtained

$$p_4(\lambda) = (\lambda - a)^2(\lambda^2 + a\lambda + a^2),$$

which for $a \in \mathbb{Z} \setminus \{0, 1\}$ is a polynomial with a double zero different from 1 and two non-real complex-conjugate zeros.

Since, in the case $\lambda_1 = \lambda_2$, $\lambda_i \neq \lambda_j$, $2 \leq i, j \leq 4$, we have

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{122}$$

where γ_i, $i = \overline{1,4}$, are arbitrary constants, and the solution satisfying (78) can be obtained, for example, by letting $\lambda_1 \to \lambda_2$ in (110) [38]

$$\begin{aligned} a_n =& \frac{\lambda_2^{n+2}((n+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4) - \lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}. \end{aligned} \tag{123}$$

From (80), (123) and by Lemma 2, we get

$$\begin{aligned} y_n =& \sum_{j=0}^{n-1}\Bigg(\frac{\lambda_2^{j+2}((j+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4) - \lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2} \\ &+ \frac{\lambda_3^{j+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}\Bigg) \\ =& \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(1-\lambda_2)^2} + \frac{(\lambda_2^4 - 2\lambda_2^3\lambda_3 - 2\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n - 1)}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2(\lambda_2-1)} \\ &+ \frac{\lambda_3^3(\lambda_3^n-1)}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)(\lambda_3-1)} + \frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)(\lambda_4-1)}. \end{aligned} \tag{124}$$

From the previous analysis and Corollary 3 we obtain the following result.

Corollary 5. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$ and $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

- *(a) If only one of the zeros of p_4 is double and different from 1, then the general solution to (2) is given by (76) and (85), where $(a_n)_{n\geq -3}$ is given by (123), while $(y_n)_{n\geq -3}$ is given by (124).*
- *(b) If 1 is a unique double zero of polynomial p_4, say $\lambda_1 = \lambda_2 = 1$, then the general solution to (2) is given by (76) and (85), where $(a_n)_{n\geq -3}$ is given by (120), $(y_n)_{n\geq -3}$ is given by (121), while $\lambda_{3,4}$ are given by (118) if $a = c = 1$ or by (119) if $a = c = -1$.*

Two pairs of different double zeros. In this case it must be $D = 0$ which implies that

$$16t^2 + 16a^2t + 3a^4 - 48ac = 0, \tag{125}$$

where $t = bd$. On the other hand, it must be $\Delta = 0$, which is equivalent to $4\Delta_0^3 = \Delta_1^2$, that is,

$$(-2b^3d^3 + 81acbd + 27a^3c + 27c^2)^2 = 4(b^2d^2 + 9ac)^3,$$

from which it follows that

$$c\big(16at^4 + 4(a^3+c)t^3 - 207a^2ct^2 - 162ac(a^3+c)t - 27c(a^3-c)^2\big)\big) = 0. \tag{126}$$

The problem of the existence of a joint zero of the polynomials in (125) and (126) for some integers a, b, c and d, such that $ac \neq 0$, seems quite technical, so we leave it to the reader.

Solutions to (77) in this case are

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_1^n + (\gamma_3 + \gamma_4 n)\lambda_3^n, \quad n \in \mathbb{N}, \tag{127}$$

where γ_i, $i = \overline{1,4}$ are constants. The solution to (127) such that (78) holds is [38]

$$a_n = \frac{\lambda_2^{n+2}(n(\lambda_2-\lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2-\lambda_4)^4} + \frac{\lambda_4^{n+2}(n(\lambda_4-\lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4-\lambda_2)^4}. \tag{128}$$

From (80), (128) and Lemma 2, we get

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}(j(\lambda_2-\lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2-\lambda_4)^4} + \frac{\lambda_4^{j+2}(j(\lambda_4-\lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4-\lambda_2)^4}\right) \\ &= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_4)^2(1-\lambda_2)^2} + \frac{(\lambda_2^4 - 4\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_4^2)(\lambda_2^n - 1)}{(\lambda_2-\lambda_4)^4(\lambda_2-1)} \\ &\quad + \frac{\lambda_4^3 - n\lambda_4^{n+2} + (n-1)\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(1-\lambda_4)^2} + \frac{(\lambda_4^4 - 4\lambda_2\lambda_4^3 + 3\lambda_2^2\lambda_4^2)(\lambda_4^n - 1)}{(\lambda_4-\lambda_2)^4(\lambda_4-1)}. \end{aligned} \tag{129}$$

Corollary 6. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$ and $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If polynomial p_4 has two pairs of double zeros both different from 1, then the general solution to (2) is given by (76) and (85), where $(a_n)_{n \geq -3}$ is given by (128), while $(y_n)_{n \geq -3}$ is given by (129).*

(b) *The polynomial in (86) cannot have two pairs of double zeros such that one of them is equal to 1.*

Triple zero case. In this case it must be $\Delta = \Delta_0 = 0$, or equivalently, $\Delta_0 = \Delta_1 = 0$. Hence,

$$ac = -b^2d^2/9 \quad \text{and} \quad 2b^3d^3 - 81acbd - 27a^3c - 27c^2 = 0,$$

from which it follows that

$$11(bd)^3 + 3a^2(bd)^2 - \frac{(bd)^4}{3a^2} = 0. \tag{130}$$

Since $bd = 0$ implies $ac = 0$, which contradicts to the assumption $ac \neq 0$, from (130) it follows that

$$s^2 - 33s - 9 = 0, \tag{131}$$

where $s = bd/a^2$. Hence, it must be

$$bd/a^2 = \frac{33 + \sqrt{1125}}{2} \quad \text{or} \quad bd/a^2 = \frac{33 - \sqrt{1125}}{2},$$

which is not possible since bd/a^2 is a rational number, whereas $(33 \pm \sqrt{1125})/2$ are both irrational numbers. Hence, p_4 cannot have a triple, and consequently cannot have a quadruple zero.

Corollary 7. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$ and $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then polynomial (86) cannot have a triple zero.*

Theorem 5. *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $bcd \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (2) is solvable in closed form.*

Proof. In this case (2) is

$$z_{n+1} = \alpha w_n^b, \quad w_{n+1} = \beta w_{n-2}^c z_n^d, \quad n \in \mathbb{N}_0. \tag{132}$$

Using the first equation in (132) into the second one it follows that

$$w_{n+1} = \alpha^d \beta w_{n-1}^{bd} w_{n-2}^c, \quad n \in \mathbb{N}. \tag{133}$$

Let $\eta = \alpha^d \beta$,

$$\beta_1 = bd, \quad \gamma_1 = c, \quad \delta_1 = 0, \quad x_1 = 1, \tag{134}$$

then, we have

$$w_{n+1} = \eta w_{n-1}^{\beta_1} w_{n-2}^{\gamma_1} w_{n-3}^{\delta_1}, \quad n \in \mathbb{N}. \tag{135}$$

Hence

$$\begin{aligned} w_{n+1} =& \eta^{x_1} (\eta w_{n-3}^{\beta_1} w_{n-4}^{\gamma_1})^{\beta_1} w_{n-2}^{\gamma_1} w_{n-3}^{\delta_1} \\ =& \eta^{x_1+\beta_1} w_{n-2}^{\gamma_1} w_{n-3}^{\beta_1\beta_1+\delta_1} w_{n-4}^{\gamma_1\beta_1} \\ =& \eta^{x_2} w_{n-2}^{\beta_2} w_{n-3}^{\gamma_2} w_{n-4}^{\delta_2}, \end{aligned}$$

for $n \geq 3$, where

$$x_2 := x_1 + \beta_1, \quad \beta_2 := \gamma_1, \quad \gamma_2 := \beta_1\beta_1 + \delta_1, \quad \delta_2 := \gamma_1\beta_1.$$

Suppose that

$$w_{n+1} = \eta^{x_k} w_{n-k}^{\beta_k} w_{n-k-1}^{\gamma_k} w_{n-k-2}^{\delta_k}, \tag{136}$$

for a $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq k+1$, and

$$\beta_k = \gamma_{k-1}, \quad \gamma_k = \beta_1\beta_{k-1} + \delta_{k-1}, \quad \delta_k = \gamma_1\beta_{k-1}, \tag{137}$$

$$x_k = x_{k-1} + \beta_{k-1}. \tag{138}$$

Using (135) in (136), we have

$$\begin{aligned} w_{n+1} =& \eta^{x_k} (\eta w_{n-k-2}^{\beta_1} w_{n-k-3}^{\gamma_1})^{\beta_k} w_{n-k-1}^{\gamma_k} w_{n-k-2}^{\delta_k} \\ =& \eta^{x_k+\beta_k} w_{n-k-1}^{\gamma_k} w_{n-k-2}^{\beta_1\beta_k+\delta_k} w_{n-k-3}^{\gamma_1\beta_k} \\ =& \eta^{x_{k+1}} w_{n-k-1}^{\beta_{k+1}} w_{n-k-2}^{\gamma_{k+1}} w_{n-k-3}^{\delta_{k+1}}, \end{aligned}$$

for a $k \geq 2$ and $n \geq k+2$, and where

$$\beta_{k+1} := \gamma_k, \quad \gamma_{k+1} := \beta_1\beta_k + \delta_k, \quad \delta_{k+1} := \gamma_1\beta_k,$$

$$x_{k+1} := x_k + \beta_k.$$

From this we see that hypotheses (136)–(138) are true.

Let $k = n - 1$, then from (136)–(138), it follows that

$$\begin{aligned}w_{n+1} =&\eta^{x_{n-1}} w_1^{\beta_{n-1}} w_0^{\gamma_{n-1}} w_{-1}^{\delta_{n-1}}\\ =&(\alpha^d \beta)^{x_{n-1}} (\beta w_{-2}^c z_0^d)^{\beta_{n-1}} w_0^{\gamma_{n-1}} w_{-1}^{\delta_{n-1}}\\ =&\alpha^{dx_{n-1}} \beta^{x_{n-1}+\beta_{n-1}} z_0^{d\beta_{n-1}} w_{-2}^{c\beta_{n-1}} w_{-1}^{\delta_{n-1}} w_0^{\gamma_{n-1}}\\ =&\alpha^{dx_{n-1}} \beta^{x_n} z_0^{d\beta_{n-1}} w_{-2}^{c\beta_{n-1}} w_{-1}^{c\beta_{n-2}} w_0^{\beta_n},\end{aligned} \tag{139}$$

for $n \geq 3$.

From (137) it follows that

$$\beta_k = \beta_1 \beta_{k-2} + \gamma_1 \beta_{k-3}, \quad k \geq 4, \tag{140}$$

and that γ_k and δ_k also satisfy (140).

From (140) and since $\gamma_1 = c \neq 0$, we have that

$$\beta_{k-3} = \frac{\beta_k - \beta_1 \beta_{k-2}}{\gamma_1}, \tag{141}$$

From (138) and (141) with $k = 3, 2, 1$ and some calculation is obtained

$$\beta_{-3} = 0, \quad \beta_{-2} = 0, \quad \beta_{-1} = 1, \quad \beta_0 = 0, \tag{142}$$

$$x_{-3} = x_{-2} = x_{-1} = 0, \quad x_0 = x_1 = 1, \tag{143}$$

and

$$x_k = 1 + \sum_{j=1}^{k-1} \beta_j. \tag{144}$$

Since equation (140) is solvable, we can calculate β_k, from which along with (144) and Lemma 2, y_k is calculated. These facts along with (139) gives a closed form formula for (133).

Using (139) in the first equation in (132) we get

$$z_{n+1} = \alpha^{1+bdx_{n-2}} \beta^{bx_{n-1}} z_0^{bd\beta_{n-2}} w_{-2}^{bc\beta_{n-2}} w_{-1}^{bc\beta_{n-3}} w_0^{b\beta_{n-1}}, \quad n \in \mathbb{N}_0. \tag{145}$$

It is not difficult to see that (139) and (145) are solutions to (2) in the case. □

Theorem 5 solves theoretically system (2) when $a = 0$ and $bcd \neq 0$. Now we will practically solve it in terms of the parameters and initial values. The following polynomial

$$p_3(\lambda) = \lambda^3 - bd\lambda - c, \tag{146}$$

is the characteristic one associated to Equation (140), and its solutions are

$$\lambda_j = \frac{1}{3\sqrt[3]{2}} \left(\varepsilon^j \sqrt[3]{\widehat{\Delta}_1 - \sqrt{\widehat{\Delta}_1^2 - 4\widehat{\Delta}_0^3}} + \bar{\varepsilon}^j \sqrt[3]{\widehat{\Delta}_1 + \sqrt{\widehat{\Delta}_1^2 - 4\widehat{\Delta}_0^3}} \right), \ j = \overline{0,2}, \tag{147}$$

where

$$\widehat{\Delta}_0 = 3bd =: -3p \quad \text{and} \quad \widehat{\Delta}_1 = 27c =: -27q, \tag{148}$$

and $\varepsilon^3 = 1$, $\varepsilon \neq 1$.

Zeros of p_3 are different and none of them is 1. In this case it must be $\widehat{\Delta}_1^2 \neq 4\widehat{\Delta}_0^3$, which can written as

$$27c^2 \neq 4(bd)^3.$$

Hence, in the case all the zeros of p_3 are different. If additionally $p_3(1) \neq 1$, that is, $c + bd \neq 1$, then none of the zeros is 1. For example, if $c = bd = k \in \mathbb{N}$, then we have such a situation.

Zeros of p_3 are different and one of them is 1. Polynomial p_3 has a zero equal to 1 if $c + bd = 1$. Then

$$p_3(\lambda) = \lambda^3 + (c-1)\lambda - c = (\lambda - 1)(\lambda^2 + \lambda + c).$$

Hence

$$\lambda_1 = 1, \quad \lambda_{2,3} = \frac{1 \pm \sqrt{1-4c}}{2}, \tag{149}$$

are the zeros of the equation $p_3(\lambda) = 0$, in this case. Since $p_3'(1) = 2 + c$, it follows that p_3 can have 1 as a double zero only if $c = -2$.

The general solution to (140) in these two cases is

$$\beta_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \quad n \in \mathbb{N}, \tag{150}$$

for some constants α_i, $i = \overline{1,3}$, which due to $\gamma_1 = c \neq 0$ can be prolonged for every non-positive index.

From Lemma 1 with $Q(t) = p_3(t) = \prod_{j=1}^3 (t - \lambda_j)$, we have

$$\sum_{j=1}^{3} \frac{\lambda_j^l}{p_3'(\lambda_j)} = 0, \quad \text{for } l = 0, 1, \quad \text{and} \quad \sum_{j=1}^{3} \frac{\lambda_j^2}{p_3'(\lambda_j)} = 1. \tag{151}$$

From (142), (150) and (151), we get

$$\beta_n = \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \tag{152}$$

for $n \geq -2$.

From (144) and (152), it follows that

$$x_n = 1 + \sum_{k=1}^{n-1} \sum_{j=1}^{3} \frac{\lambda_j^{k+3}}{p_3'(\lambda_j)}, \tag{153}$$

for $n \in \mathbb{N}$.

If $\lambda_j \neq 1$, $j = \overline{1,3}$, then from formula (153), it follows that

$$\begin{aligned} x_n =& 1 + \frac{\lambda_1^4(\lambda_1^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-1)} + \frac{\lambda_2^4(\lambda_2^{n-1}-1)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-1)} \\ &+ \frac{\lambda_3^4(\lambda_3^{n-1}-1)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-1)}, \end{aligned} \tag{154}$$

for $n \in \mathbb{N}$, moreover, (154) holds for every $n \geq -2$.

If one of the zeros is 1, say λ_3, then $1 \neq \lambda_1 \neq \lambda_2 \neq 1$, and we have

$$x_n = 1 + \frac{\lambda_1^4(\lambda_1^{n-1}-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)^2} + \frac{\lambda_2^4(\lambda_2^{n-1}-1)}{(\lambda_2-\lambda_1)(\lambda_2-1)^2} + \frac{n-1}{(\lambda_1-1)(\lambda_2-1)}. \tag{155}$$

for $n \in \mathbb{N}$. Moreover, due to (151), (155) holds for every $n \geq -2$.

Corollary 8. *Assume that $a, b, c, d \in \mathbb{Z}$, $a = 0$, $bcd \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\widehat{\Delta}_1^2 \neq 4\widehat{\Delta}_0^3$. Then the following statements are true.*

(a) If $c + bd \neq 1$, then the general solution to (2) is given by (139) and (145), where $(\beta_n)_{n \geq -2}$ is given by (152), $(x_n)_{n \geq -2}$ is given by (154), while λ_j's, $j = \overline{1,3}$ are given by (147) and (148).

(b) If $c + bd = 1$, then p_3 has a unique zero equal to 1, say λ_3, and the general solution to (2) is given by formulas (139) and (145), where $(\beta_n)_{n \geq -2}$ is given by (152) with $\lambda_3 = 1$, $(x_n)_{n \geq -2}$ is given by (155), while λ_j's, $j = \overline{1,3}$ are given by (149).

One of the zeros is double. In this case it must be $\widehat{\Delta}_1^2 = 4\widehat{\Delta}_0^3$, that is, $(bd)^3 = 27c^2/4$. Assume that m is a double zero of p_3, then it must be

$$m^3 - bdm - c = 0 \quad \text{and} \quad 3m^2 - bd = 0,$$

from which it follows that

$$p_3(\lambda) = \lambda^3 - 3m^2\lambda + 2m^3 = (\lambda - m)^2(\lambda + 2m). \tag{156}$$

Since $bd \neq 0$, we have $m \neq 0$. From this and since $c \neq 0$, from (156) we see that p_3 cannot have a triple zero. It also cannot have a unique zero equal to 1, since otherwise we would have $2m = -1$, from which it would be $bd = 3/4 \notin \mathbb{Z}$, which would be a contradiction. Note also that the polynomial can have 1 as a double zero when $bd = 3$ and $c = -2$.

If $\lambda_1 \neq \lambda_2 = \lambda_3$, then the general solution to (140) has the following form

$$\beta_n = \hat{\alpha}_1\lambda_1^n + (\hat{\alpha}_2 + \hat{\alpha}_3 n)\lambda_2^n, \quad n \in \mathbb{N}, \tag{157}$$

where $\hat{\alpha}_i$, $i = \overline{1,3}$ are constants. Since, in our case condition (142) must be satisfied, the solution $(\beta_n)_{n \geq -2}$ to (140) can be found by letting $\lambda_3 \to \lambda_2$ in (152), so that

$$\beta_n = \frac{\lambda_1^{n+3} + \big(2\lambda_2 - 3\lambda_1 + n(\lambda_2 - \lambda_1)\big)\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)^2}, \tag{158}$$

for $n \geq -2$.

From (144) and (158), we have

$$x_n = 1 + \sum_{j=1}^{n-1}\beta_j = 1 + \sum_{j=1}^{n-1}\frac{\lambda_1^{j+3} + \big(2\lambda_2 - 3\lambda_1 + j(\lambda_2 - \lambda_1)\big)\lambda_2^{j+2}}{(\lambda_2 - \lambda_1)^2}, \tag{159}$$

for every $n \in \mathbb{N}$.

From (159) and Lemma 2, we get

$$x_n = 1 + \frac{\lambda_1^4(\lambda_1^{n-1} - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(2\lambda_2 - 3\lambda_1)\lambda_2^3(\lambda_2^{n-1} - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} + \frac{\lambda_2^3(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2}, \tag{160}$$

for $n \in \mathbb{N}$ (in fact, (160) hold also for every $n \geq -2$).

If we assume that $\lambda_1 \neq 1$ and $\lambda_2 = \lambda_3 = 1$, then from (159) it follows that

$$x_n = 1 + \frac{\lambda_1^4(\lambda_1^{n-1} - 1)}{(\lambda_1 - 1)^3} + \frac{(2 - 3\lambda_1)(n-1)}{(\lambda_1 - 1)^2} + \frac{(n-1)n}{2(1 - \lambda_1)}. \tag{161}$$

A direct calculation shows that (161) holds also for every $n \geq -2$.

Corollary 9. *Assume that* $b, c, d \in \mathbb{Z}$, $a = 0$, $bcd \neq 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ *and* $\widehat{\Delta}_1^2 = 4\widehat{\Delta}_0^3$. *Then the following statements are true.*

(a) *If* $c + bd \neq 1$, *then the general solution to* (2) *is given by* (139) *and* (145), *where* $(\beta_n)_{n\geq -2}$ *is given by* (158), *while* $(x_n)_{n\geq -2}$ *is given by* (160).

(b) *If* $bd = 3$ *and* $c = -2$, *then 1 is a double zero of* p_3, *say,* $\lambda_2 = \lambda_3 = 1$, *then the general solution to system* (2) *is given by* (139) *and* (145), *where* $(\beta_n)_{n\geq -2}$ *is given by* (158) *with* $\lambda_2 = 1$, $(x_n)_{n\geq -2}$ *is given by* (161), *while* $\lambda_3 = -2$.

(c) *It is not possible that 1 is a simple zero of* p_3.

Triple zero case. Since in this case p_3 must have the form in (156), we see that the only possibility that this polynomial has a triple zero is if $m = 0$, which is impossible due to the condition $c \neq 0$.

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