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Neighborhood Structures and Continuities via Cubic Sets

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Abstract: Cubic sets are a very useful generalization of fuzzy sets where one is allowed to extend the output through a subinterval of [0, 1] and a number from [0, 1]. In this article, we first highlight some of the claims made in the previous article about cubic sets. Then, the concept of semi-coincidence in cubic sets, cubic neighborhood system according to cubic topology, and cubic bases and subbases are introduced. This article deals with a cubic closure and a cubic interior and how to obtain their various properties. In addition, cubic compact spaces and their properties are defined and a useful example is given. We mainly focus on the concept of cubic continuities and deepen our research by finding its characterization. One of the most important discoveries of this paper is determining that there is a cubic product topology induced by the projection mappings, and discovering sufficient conditions for the projection mappings to be cubic open.

Keywords: cubic set; cubic quasi-coincidence; cubic topology; cubic neighborhood; cubic base and subbase; cubic closure and interior; cubic continuous mapping; cubic quotient mapping

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1. Introduction

In 2012, Jun et al. [1] defined a cubic set by combining with an interval-valued fuzzy set (see [2]) and a fuzzy set (see [3]). After that, Kang and Kim [4] and Chinnadurai et al. [5] discussed some properties of the image and the preimage of mappings of cubic sets under a mapping and under various characterizations of cubic sets, respectively. Kim et al. [6] studied various properties for cubic relations. Smarandache et al. [7] proposed the concept of neutrosophic cubic sets as the extension of cubic sets and obtained some of its properties. Rashid et al. [8] dealt with decision-making problems based on cubic sets. Furthermore, many researchers [9–18] applied cubic sets to various algebraic structures. Zeb et al. [19] investigated topological structures via cubic sets; however, they studied only some basic properties of topological structures based on a cubic set.

Neighborhood structures in a topological space is very important to study separation axioms, connectedness, nets, etc. As such, our research aims to study neighborhood structures, closures and interiors, and continuities based on cubic sets which play important roles in a topology. To do this, this paper includes the following: We recall basic concepts related to cubic sets in Section 2. We define a quasi-coincidence in cubic sets and obtain some of its properties in Section 3. In Section 4, first, we introduce the notion of cubic neighborhood systems with respect to a cubic topological space and find its some properties (see Theorems 5 and 6). Next, we define a cubic base and a cubic subbase and discuss some of their properties. We propose the concepts of cubic closures and cubic interiors, as well as deal with their various properties in Section 5. In Section 6, we define cubic continuous mapping and give its characterization (see Theorem 15). Moreover, we confirm that the concrete category $\mathbf{PCT_{op}}$ has an initial structure (see Remarks 11 and 13). In addition, we introduce the notion of cubic quotient mappings and find its some properties. Lastly, we provide the sufficient conditions for the projection mappings to be cubic open (see Proposition 16).

Throughout this paper, the classical unit interval $[0, 1]$ is denoted as I, J denotes a index set and X, Y, Z are nonempty sets.

2. Preliminaries

In this section, we list some basic definitions and one result needed in the next sections.

A mapping $A : X \to I$ is called a *fuzzy set* in X (See [3]). The *fuzzy empty set* and the *fuzzy whole set* in X, denoted by $\mathbf{0}$ and $\mathbf{1}$, is defined by, respectively: for each $x \in X$,

$$\mathbf{0}(x) = 0 \text{ and } \mathbf{1}(x) = 1.$$

I^X denotes the collection of all fuzzy sets in X.

Let $[I]$ be the set of all closed subintervals of I, i.e., $[I] = \{\widetilde{a} = [a^-, a^+] : 0 \leq a^- \leq a^+ \leq 1\}$. Each member of $[I]$ is called *interval number*. Refer to [1] for the definitions of the *order* ($\leq$) between two interval-valued numbers, the *inf* ($\bigwedge$) and the *sup* ($\bigvee$) of a family of interval-valued numbers, and the the *complement* (^{c}) of an interval-valued number.

A mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X (See [2,20,21]). The *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, denoted by $\widetilde{\mathbf{0}}$ and $\widetilde{\mathbf{1}}$, is defined by, respectively: for each $x \in X$,

$$\widetilde{\mathbf{0}}(x) = [0, 0] \text{ and } \widetilde{\mathbf{1}}(x) = [1, 1].$$

The set of all IVFSs is denoted by $IVFS(X)$.

Definition 1 ([22]). *Let $P(X)$ be the power set of X, let $\widetilde{A} = [A^-, A^+] \in IVS(X)$ and $A \in P(X)$. Then $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle$ is called a cubic crisp set (briefly, CCS) in X. We can consider special cubic crisp sets:*

$$\hat{X} = \left\langle \widetilde{X}, X \right\rangle,\ \hat{\emptyset} = \left\langle \widetilde{\emptyset}, \emptyset \right\rangle,\ \ddot{X} = \left\langle \widetilde{X}, \emptyset \right\rangle,\ \ddot{\emptyset} = \left\langle \widetilde{\emptyset}, X \right\rangle.$$

where $\widetilde{X} = [X, X]$, $\widetilde{\emptyset} = [\emptyset, \emptyset]$. $\hat{X}$ [resp. $\hat{\emptyset}$] is called a cubic crisp whole [resp. empty] set in X. $CCS(X)$ denotes the set of all CCSs in X.

We define two types of orders between two cubic numbers, and two types of infimums and supremums for any collection of cubic numbers: for any cubic number $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$ and each collection $(\widetilde{\widetilde{a}}_j)_{j \in J}$ of cubic numbers,

(i) (*P-order*) $\widetilde{\widetilde{a}} \leq_P \widetilde{\widetilde{b}} \Longleftrightarrow \widetilde{a} \leq \widetilde{b},\ a \leq b$,

(ii) (*R-order*) $\widetilde{\widetilde{a}} \leq_R \widetilde{\widetilde{b}} \Longleftrightarrow \widetilde{a} \leq \widetilde{b},\ a \geq b$,

(iii) (*P-inf*) $\bigwedge^P_{j \in J} \widetilde{\widetilde{a}}_j = \left\langle \bigwedge_{j \in J} \widetilde{a}_j, \bigwedge_{j \in J} a_j \right\rangle$,

(iv) (*P-sup*) $\bigvee^P_{j \in J} \widetilde{\widetilde{a}}_j = \left\langle \bigvee_{j \in J} \widetilde{a}_j, \bigvee_{j \in J} a_j \right\rangle$,

(v) (*R-inf*) $\bigwedge^R_{j \in J} \widetilde{\widetilde{a}}_j = \left\langle \bigwedge_{j \in J} \widetilde{a}_j, \bigvee_{j \in J} a_j \right\rangle$,

(vi) (*R-sup*) $\bigvee^R_{j \in J} \widetilde{\widetilde{a}}_j = \left\langle \bigvee_{j \in J} \widetilde{a}_j, \bigwedge_{j \in J} a_j \right\rangle$.

Definition 2 ([1]). *A mapping $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle : X \to [I] \times I$ is called a cubic set in X. Consider the following special cubic sets in X defined by:*

$$\ddot{0} = \langle \widetilde{\mathbf{0}}, \mathbf{1} \rangle,\ \ddot{1} = \left\langle \widetilde{\mathbf{1}}, \mathbf{0} \right\rangle,\ \hat{0} = \langle \widetilde{\mathbf{0}}, \mathbf{0} \rangle,\ \hat{1} = \left\langle \widetilde{\mathbf{1}}, \mathbf{1} \right\rangle.$$

Then $\hat{0}$ [resp. $\hat{1}$] is called the cubic empty [resp. whole] set in X. $CS(X)$ denotes the set of all cubic sets in X.

Remark 1. *For each $A \in P(X)$, let χ_A be the characteristic function of A. Then $\chi_{\mathcal{A}} = \{\langle[\chi_A, \chi_A], \chi_A\rangle\} \in CS(X)$. Thus, a cubic set is the generalization of a classical set. Moreover, it is well known [22] that a cubic crisp set is the generalization of a classical set and the special case of a cubic set.*

Definition 3 ([1])**.** *For $\mathcal{A}$, $\mathcal{B} \in CS(X)$, the following relations are defined:*

(i) *(Equality)* $\mathcal{A} = \mathcal{B} \Leftrightarrow \widetilde{A} = \widetilde{B}$ and $A = B$,

(ii) *(P-inclusion)* $\mathcal{A} \sqsubset \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B}$ and $A \subset B$,

(iii) *(R-inclusion)* $\mathcal{A} \Subset \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B}$ and $A \supset B$.

For $\mathcal{A} \in CS(X)$ and $(\mathcal{A}_j)_{j\in J} \subset CS(X)$, the definitions of the complement $\mathcal{A}^c$ of $\mathcal{A}$, and P-union $\sqcup_{j\in J}\mathcal{A}_j$, P-intersection $\sqcap_{j\in J}\mathcal{A}_j$, R-union $\Cup_{j\in J}\mathcal{A}_j$ and R-intersection $\Cap_{j\in J}\mathcal{A}_j$ of $(\mathcal{A}_j)_{j\in J}$ refer to [1].

Remark 2. *From Definitions 1 and 3, it is clear that $\hat{0} \sqsubset \mathcal{A} \sqsubset \hat{1}$ and $\ddot{0} \Subset \mathcal{A} \Subset \ddot{1}$ for each $\mathcal{A} \in CS(X)$. Furthermore, $\hat{0} \sqsubset \ddot{0}$, $\ddot{1} \sqsubset \hat{1}$ and $\ddot{0} \Subset \hat{0}$, $\hat{1} \Subset \ddot{1}$. Then we can see that $\hat{0}$ [resp. $\hat{1}$] is the smallest [resp. largest] element of $CS(X)$ in the sense of P-order and $\ddot{0}$ [resp. $\ddot{1}$] is the smallest [resp. largest] element of $CS(X)$ in the sense of R-order.*

It is well known [1] that the following hold:

(1) $\check{0}^c = \check{1}$, $\check{1}^c = \check{0}$, $\ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$,

(2) for each $\mathcal{A} \in ([I] \times I)^X$, $(\mathcal{A}^c)^c = \mathcal{A}$,

(3) for any $(\mathcal{A}_j)_{j\in J} \subset ([I] \times I)^X$,

$$(\sqcup_{j\in J}\mathcal{A}_j)^c = \sqcap_{j\in J}\mathcal{A}_j^c,\ (\sqcap_{j\in J}\mathcal{A}_j)^c = \sqcup_{j\in J}\mathcal{A}_j^c,$$

$$(\Cup_{j\in J}\mathcal{A}_j)^c = \Cap_{j\in J}\mathcal{A}_j^c,\ (\Cap_{j\in J}\mathcal{A}_j)^c = \Cup_{j\in J}\mathcal{A}_j^c.$$

Result 1 (Proposition 3.4, [6])**.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \in CS(X)$, let $(\mathcal{A}_j)_{j\in J} \subset CS(X)$.*

(1) $\mathcal{A} \sqcup \mathcal{A} = \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{A} = \mathcal{A}$, $\mathcal{A} \Cup \mathcal{A} = \mathcal{A}$, $\mathcal{A} \Cap \mathcal{A} = \mathcal{A}$.

(2) $\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \sqcup \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{B} = \mathcal{B} \sqcap \mathcal{A}$, $\mathcal{A} \Cup \mathcal{B} = \mathcal{B} \Cup \mathcal{A}$, $\mathcal{A} \Cap \mathcal{B} = \mathcal{B} \Cap \mathcal{A}$.

(3) $\mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{C}$, $\mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C}$,
$\mathcal{A} \Cup (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cup \mathcal{C}$, $\mathcal{A} \Cap (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cap \mathcal{C}$.

(4) $\mathcal{A} \sqcup (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C})$, $\mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C})$,
$\mathcal{A} \Cup (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cap (\mathcal{A} \Cup \mathcal{C})$, $\mathcal{A} \Cap (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cup (\mathcal{A} \Cap \mathcal{C})$.

(4)' $\mathcal{A} \sqcup (\sqcap_{j\in J}\mathcal{A}_j) = \sqcap_{j\in J}(\mathcal{A} \sqcup \mathcal{A}_j)$, $\mathcal{A} \sqcap (\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J}(\mathcal{A} \sqcap \mathcal{A}_j)$,
$\mathcal{A} \Cup (\Cap_{j\in J}\mathcal{A}_j) = \Cap_{j\in J}(\mathcal{A} \Cup \mathcal{A}_j)$, $\mathcal{A} \Cap (\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J}(\mathcal{A} \Cap \mathcal{A}_j)$.

3. P-Quasi-Coincidence and R-Quasi-Coincidence

In this section, first, we recall the notions of fuzzy points and interval-valued fuzzy points and define an interval-valued fuzzy quasi-coincidence, and find some of its properties. Next, we introduce the notions of P-quasi-coincidences and R-quasi-coincidences, and obtain some of their properties.

Definition 4 ([23])**.** *$x_a \in I^X$ is called a fuzzy point with the support $x \in X$ and the value $a \in I$ with $a > 0$, if for each $y \in X$,*

$$x_a = \begin{cases} a & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

$F_P(X)$ is the set of all fuzzy points in X. Let $x_a \in F_P(X)$ and A, $B \in I^X$. Then

(i) *x_a is said to belong to A, denoted by $x_a \in A$, if $A(x) \geq a$,*

(ii) *x_a is said to quasi-coincident with A, denoted by $x_a qA$, if $A(x) > A^c(x)$, i.e., $a + A(x) > 1$,*

(iii) *A is said to be quasi-coincident with B, denoted by AqB, if there is $x \in X$ such that $A(x) > B^c(x)$, i.e., $A(x) + B(x) > 1$.*

It is well-known [24] that $A = \bigcup_{x_a \in A} x_a$ for each $A \in I^X$.

Definition 5 ([21]). *$x_{\widetilde{a}} \in IVFS(X)$ is called an interval-valued fuzzy point (briefly, IVFP) with the support $x \in X$ and the value $\widetilde{a} \in [I]$ with $a^- > 0$, if for each $y \in X$,*

$$x_{\widetilde{a}} = \begin{cases} \widetilde{a} & \text{if } y = x \\ [0,0] & \text{otherwise.} \end{cases}$$

$IVF_P(X)$ is the set of all IVFPs in X. Let $x_{\widetilde{a}} \in IVF_P(X)$ and $\widetilde{A} \in IVFS(X)$. Then $x_{\widetilde{a}}$ is said to belong to $\widetilde{A}$, denoted by $x_{\widetilde{a}} \in \widetilde{A}$, if $\widetilde{a} \leq \widetilde{A}(x)$, i.e., $a^- \leq A^-(x)$, $a^+ \leq A^+(x)$.

It is well-known [21] that $\widetilde{A} = \bigcup_{x_{\widetilde{a}} \in \widetilde{A}} x_{\widetilde{a}}$ for each $\widetilde{A} \in [I]^X$.

Definition 6. *Let $\widetilde{A}$, $\widetilde{B} \in IVFS(X)$. Then $\widetilde{A}$ and $\widetilde{B}$ are said to be intersecting, if there is $x \in X$ such that $(\widetilde{A} \cap \widetilde{B})(x) \neq [0,0]$. In this case, we say that $\widetilde{A}$ and $\widetilde{B}$ intersect at x.*

Definition 7. *Let $x_{\widetilde{a}} \in IVF_P(X)$ and and let $\widetilde{A}$, $\widetilde{B} \in IVFS(X)$.*

(i) *$x_{\widetilde{a}}$ is said to be quasi-coincident with $\widetilde{A}$, denoted by $x_{\widetilde{a}} q \widetilde{A}$, if $\widetilde{a} > \widetilde{A}^c(x)$, i.e.,*

$$a^- + A^+(x) > 1,\ a^+ + A^-(x) > 1.$$

When $x_{\widetilde{a}}$ is not quasi-coincident with $\widetilde{A}$, we write $x_{\widetilde{a}} \neg q \widetilde{A}$.

(ii) *$\widetilde{A}$ is said to be quasi-coincident with $\widetilde{B}$, denoted by $\widetilde{A} q \widetilde{B}$, if there is $x \in X$ such that $\widetilde{A}(x) > \widetilde{B}^c(x)$, i.e.,*

$$A^-(x) + B^+(x) > 1,\ A^+(x) + B^-(x) > 1.$$

In this case, we say that $\widetilde{A}$ and $\widetilde{B}$ are quasi-coincident (with each other) at x. When $\widetilde{A}$ is not quasi-coincident with $\widetilde{B}$, we write $\widetilde{A} \neg q \widetilde{B}$.

It is obvious that, if $\widetilde{A}$ and $\widetilde{B}$ are quasi-coincident at x, then they are intersect at x. From Definitions 5 and 7, we have the following.

Lemma 1. *Let $x_{\widetilde{a}} \in IVF_P(X)$ and and let $\widetilde{A} \in IVFS(X)$. Then $x_{\widetilde{a}} \in \widetilde{A}$ if and only if $x_{\widetilde{a}} \neg q \widetilde{A}^c$.*

Also from Definition 5 and Lemma 1, we obtain the following.

Lemma 2. *Let $\widetilde{A}$, $\widetilde{B} \in IVFS(X)$. Then the following are equivalent:*

(1) $\widetilde{A} \subset \widetilde{B}$,
(2) $x_{\widetilde{a}} \in \widetilde{B}$ *for each* $x_{\widetilde{a}} \in \widetilde{A}$,
(3) $\widetilde{A} \neg q \widetilde{B}^c$.

Lemma 3. *Let $\widetilde{A}$, $\widetilde{B} \in IVFS(X)$, let $(\widetilde{A}_j)_{j \in J} \subset IVFS(X)$ and let $x_{\widetilde{a}} \in IVF_P(X)$.*

(1) *$x_{\widetilde{a}} q \bigcup_{j \in J} \widetilde{A}_j$ if and only if there is $j_0 \in J$ such that $x_{\widetilde{a}} q \widetilde{A}_{j_0}$.*
(2) *$x_{\widetilde{a}} q (\widetilde{A} \cap \widetilde{B})$ if and only if $x_{\widetilde{a}} q \widetilde{A}$ and $x_{\widetilde{a}} q \widetilde{B}$.*

Proof. (1) Suppose $x_{\widetilde{a}} q \bigcup_{j \in J} \widetilde{A}_j$. Then we have

$$\begin{aligned} \widetilde{a} &> (\textstyle\bigcup_{j \in J} \widetilde{A}_j)^c(x) \\ &= (\textstyle\bigcap_{j \in J} \widetilde{A}_j^c)(x) \\ &= [\textstyle\bigwedge_{j \in J}(1 - A_j^-(x)), \bigwedge_{j \in J}(1 - A_j^+(x))] \\ &= [(1 - \textstyle\bigvee_{j \in J} A_j^-(x), 1 - \bigvee_{j \in J} A_j^+(x)]. \end{aligned}$$

Thus, $a^- > 1 - \bigvee_{j\in J} A_j^-(x)$, $a^+ > 1 - \bigvee_{j\in J} A_j^+(x)$. So there is $j_0 \in J$ such that

$$a^- > 1 - A_{j_0}^-(x),\ a^+ > 1 - A_{j_0}^+(x).$$

Hence, $x_{\widetilde{a}} q \widetilde{A}_{j_0}$.
Conversely, suppose the necessary condition holds. Then clearly, $x_{\widetilde{a}} q \bigcup_{j\in J} \widetilde{A}_j$.
(2) Suppose $x_{\widetilde{a}} q(\widetilde{A} \cap \widetilde{B})$. Then we have

$$\begin{aligned}\widetilde{a} &> (\widetilde{A} \cap \widetilde{B})^c(x)\\ &= (\widetilde{A}^c \cup \widetilde{B}^c)(x)\\ &= \widetilde{A}^c(x) \vee \widetilde{B}^c(x).\end{aligned}$$

Thus, $\widetilde{a} > \widetilde{A}^c(x)$ and $\widetilde{a} > \widetilde{B}^c(x)$. So $x_{\widetilde{a}} q \widetilde{A}$ and $x_{\widetilde{a}} q \widetilde{A}$.
The proof of the converse is easy. □

Lemma 4. *Let $\widetilde{A} \in IVFS(X)$ and let $\widetilde{A}(x) = \widetilde{b} \neq [0,0]$. Then for any interval number $\widetilde{a}$ such that $[0,0] < \widetilde{a} < \widetilde{b}$, $x_{\widetilde{a}^c} q \widetilde{A}$.*

Proof. Since $\widetilde{a}^c = [1 - a^+, 1 - a^-]$, we get

$$\begin{aligned}1 - a^+ + A^+(x) &= 1 - a^+ + b^+ \text{ [Since Since } \widetilde{a} < \widetilde{b}]\\ &= 1 + (b^+ - a^+)\\ &> 1. \text{ [Since } \widetilde{a} < \widetilde{b}]\end{aligned}$$

Similarly, we have $1 - a^- + A^-(x) > 1$. Thus, $x_{\widetilde{a}^c} q \widetilde{A}$. □

Definition 8 ([6])**.** *Let $\widetilde{\widetilde{a}} = \langle \widetilde{a}, a \rangle$ be any cubic number such that $a^- > 0$ and $a > 0$. Then $x_{\widetilde{\widetilde{a}}} \in CS(X)$ is called a cubic point in X with the support $x \in X$ and the value $\widetilde{\widetilde{a}}$, if for each $y \in X$,*

$$x_{\widetilde{\widetilde{a}}} = \begin{cases} \widetilde{\widetilde{a}} & \text{if } y = x\\ \langle [0,0], 0 \rangle & \text{otherwise.}\end{cases}$$

$C_P(X)$ is set of all cubic points in X. It is obvious that $\mathcal{A} = \sqcup_{x_{\widetilde{\widetilde{a}}} \in_P \mathcal{A}} x_{\widetilde{\widetilde{a}}}$ for each $\mathcal{A} \in CS(X)$.

Definition 9 ([6])**.** *Let $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ and let $\mathcal{A} \in CS(X)$.*

(i) $x_{\widetilde{\widetilde{a}}}$ is said to P-belong to $\mathcal{A}$, denoted by $x_{\widetilde{\widetilde{a}}} \in_P \mathcal{A}$, if
$\quad\widetilde{a} \leq \widetilde{A}(x)$ *and* $a \leq A(x)$, *i.e.,* $x_{\widetilde{a}} \in \widetilde{A}$ *and* $x_a \in A$.

(ii) $x_{\widetilde{\widetilde{a}}}$ is said to R-belong to $\mathcal{A}$, denoted by $x_{\widetilde{\widetilde{a}}} \in_R \mathcal{A}$, if
$\quad\widetilde{a} \leq \widetilde{A}(x)$ *and* $a \geq A(x)$, *i.e.,* $x_{\widetilde{a}} \in \widetilde{A}$ *and* $x_{1-a} \in A^c$.

Definition 10. *Let $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ and and let $\mathcal{A}$, $\mathcal{B} \in CS(X)$.*

(i) $x_{\widetilde{\widetilde{a}}}$ is said to be P-quasi-coincident with $\mathcal{A}$, denoted by $x_{\widetilde{\widetilde{a}}} q_P \mathcal{A}$, if $\widetilde{\widetilde{a}} >_P \mathcal{A}^c(x)$, equivalently, $\widetilde{a} > \widetilde{A}^c(x)$, $a > A^c(x)$, i.e.,

$$a^- + A^+(x) > 1,\ a^+ + A^-(x) > 1,\ a + A(x) > 1.$$

When $x_{\widetilde{\widetilde{a}}}$ is not P-quasi-coincident with $\mathcal{A}$, we write $x_{\widetilde{\widetilde{a}}} \neg q_P \mathcal{A}$.

(ii) $\mathcal{A}$ is said to be P-quasi-coincident with $\mathcal{B}$, denoted by $\mathcal{A} q_P \mathcal{A}$, if there is $x \in X$ such that such that $\mathcal{A}(x) >_P \mathcal{B}(x)$, equivalently, $\widetilde{A}(x) > \widetilde{B}^c(x)$, $A(x) > B^c(x)$, i.e.,

$$A^-(x) + B^+(x) > 1,\ A^+(x) + B^-(x) > 1,\ A(x) + B(x) > 1.$$

In this case, we say that $\mathcal{A}$ and $\mathcal{A}$ are P-quasi-coincident (with each other) at x. When $\mathcal{A}$ is not P-quasi-coincident with $\mathcal{B}$, we write $\mathcal{A} \neg q_P \mathcal{B}$.

(iii) $x_{\widetilde{\widetilde{a}}}$ *is said to be R-quasi-coincident with* $\mathcal{A}$*, denoted by* $x_{\widetilde{\widetilde{a}}} q_R \mathcal{A}$*, if* $\widetilde{\widetilde{a}} >_R \mathcal{A}^c(x)$*, equivalently,* $\widetilde{a} > \widetilde{A}^c(x)$*,* $a < A^c(x)$*, i.e.,*

$$a^- + A^+(x) > 1,\ a^+ + A^-(x) > 1,\ a + A(x) < 1.$$

When $x_{\widetilde{\widetilde{a}}}$ *is not R-quasi-coincident with* $\mathcal{A}$*, we write* $x_{\widetilde{\widetilde{a}}} \neg q_R \mathcal{A}$.

(iv) $\mathcal{A}$ *is said to be R-quasi-coincident with* $\mathcal{B}$*, denoted by* $\mathcal{A} q_R \mathcal{B}$*, if there is* $x \in X$ *such that* $\mathcal{A}(x) >_R \mathcal{B}^c(x)$*, equivalently,* $\widetilde{A}(x) > \widetilde{B}^c(x)$*,* $A(x) < B^c(x)$*, i.e.,*

$$A^-(x) + B^+(x) > 1,\ A^+(x) + B^-(x) > 1,\ A(x) + B(x) < 1.$$

In this case, we say that $\mathcal{A}$ *and* $\mathcal{A}$ *are R-quasi-coincident (with each other) at* x*. When* $\mathcal{A}$ *is not R-quasi-coincident with* $\mathcal{B}$*, we write* $\mathcal{A} \neg q_R \mathcal{B}$.

Remark 3. *From Definition 10, we can easily obtain the following.*

(1) $x_{\widetilde{\widetilde{a}}} q_P \mathcal{A}$ *if and only if* $x_{\widetilde{a}} q \widetilde{A}$*,* $x_a q A$ *and* $x_{\widetilde{\widetilde{a}}} \neg q_P \mathcal{A}$ *if and only if* $x_{\widetilde{\widetilde{a}}} \in_P \mathcal{A}^c$.

(2) $x_{\widetilde{\widetilde{a}}} q_R \mathcal{A}$ *if and only if* $x_{\widetilde{a}} q \widetilde{A}$*,* $a + A(x) < 1$ *and* $x_{\widetilde{\widetilde{a}}} \neg q_P \mathcal{A}$ *if and only if* $x_{\widetilde{\widetilde{a}}} \in_R \mathcal{A}^c$.

(3) $\mathcal{A} \neg q_P \mathcal{B}$ *if and only if* $\widetilde{A} \sqsubset \widetilde{B}^c$ *and* $\mathcal{A} \neg q_R \mathcal{B}$ *if and only if* $\mathcal{A} \Subset \mathcal{B}^c$.

Theorem 1. *Let* $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ *and let* $\mathcal{A}, \mathcal{B} \in CS(X)$*. The the following are equivalent:*

(1) $\mathcal{A} \sqsubset \mathcal{B}$,

(2) $x_{\widetilde{\widetilde{a}}} \in_P \mathcal{B}$ *for each* $x_{\widetilde{\widetilde{a}}} \in_P \mathcal{A}$,

(3) $\mathcal{A} \neg q_P \mathcal{B}^c$.

Proof. From Theorem 3.7 (1) in [6], it is clear that (1) $\Longleftrightarrow$ (2) holds. From Remark 3 (3), it is obvious that (1) $\Longleftrightarrow$ (3) holds. □

Theorem 2. *Let* $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ *and let* $\mathcal{A}, \mathcal{B} \in CS(X)$*. The the following are equivalent:*

(1) $\mathcal{A} \Subset \mathcal{B}$,

(2) $x_{\widetilde{\widetilde{a}}} \in_R \mathcal{B}$ *for each* $x_{\widetilde{\widetilde{a}}} \in_R \mathcal{A}$,

(3) $\mathcal{A} \neg q_R \mathcal{B}^c$.

Proof. From Theorem 3.7 (2) in [6], it is obvious that (1) $\Longleftrightarrow$ (2) holds. From Remark 3 (3), it is clear that (1) $\Longleftrightarrow$ (3) holds. □

Definition 11. *Let* $\mathcal{A}, \mathcal{B} \in CS(X)$*. Then* $\mathcal{A}$ *and* $\mathcal{B}$ *are said to be intersecting, if there is* $x \in X$ *such that* $(\widetilde{A} \cap \widetilde{B})(x) \neq [0,0]$ *and* $(A \cap B)(x) \neq 0$*. In this case, we say that* $\widetilde{A}$ *and* $\widetilde{B}$ *intersect at* x *and* $(\mathcal{A} \sqcap \mathcal{B})(x) \neq \langle [0,0], 0 \rangle$.

From Definition 10, it is clear that if $\mathcal{A}$ and $\mathcal{B}$ are quasi-coincident at x, then $\mathcal{A}$ and $\mathcal{B}$ intersect at x.

4. P-Cubic Neighborhoods [Resp. Q-Neighborhoods]

In this section, we define a cubic neighborhood of a cubic point with respect to a cubic topology and find some of its properties. Next, we introduce the notion of cubic bases and cubic subbases and obtain some of their properties.

Definition 12. $\mathcal{A} \in CS(X)$ *is called a constant cubic set in* X*, if there is* $\widetilde{\widetilde{a}} \in [I] \times I$ *such that* $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{a}}$ *for each* $x \in X$*. In this case, we will write* $\mathcal{A} = C_{\widetilde{\widetilde{a}}}$.

Definition 13. *Let* $\tau \subset CS(X)$*. Consider the following conditions:*

(PCO_1) $\ddot{0}, \ddot{1}, \hat{0}, \hat{1} \in \tau$,

($PCO_{1'}$) *for each* $\widetilde{\widetilde{a}} \in [I] \times I$, $C_{\widetilde{\widetilde{a}}} \in \tau$,

(PCO$_2$) $\mathcal{A} \sqcap \mathcal{B} \in \tau$ *for any* $\mathcal{A}$, $\mathcal{B} \in \tau$,
(RCO$_2$) $\mathcal{A} \Cap \mathcal{B} \in \tau$ *for any* $\mathcal{A}$, $\mathcal{B} \in \tau$,
(PCO$_3$) $\sqcup_{j\in J}\mathcal{A}_j \in \tau$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \tau$,
(RCO$_3$) $\Cup_{j\in J}\mathcal{A}_j \in \tau$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \tau$.

(i) τ *is called a P-cubic topology on X in Chang's sense, if it satisfies the conditions (PCO$_1$), (PCO$_2$) and (PCO$_3$) (see [19]).*
(ii) τ *is called a R-cubic topology on X in Chang's sense, if it satisfies the conditions (PCO$_1$), (RCO$_2$) and (RCO$_3$) (see [19]).*
(iii) τ *is called a P-cubic topology on X in Lowen's sense, if it satisfies the conditions (PCO$_{1'}$), (PCO$_2$) and (PCO$_3$).*
(iv) τ *is called a R-cubic topology on X in Lowen's sense, if it satisfies the conditions (PCO$_{1'}$), (RCO$_2$) and (RCO$_3$).*

In either case, the pair (X, τ) *is called a P-cubic topological space [resp. R-cubic topological space] and each member of* τ *is called a P-cubic open set (briefly, PCOS) [resp. R-cubic open set (briefly, RCOS)]. We will denote the set of all P-cubic topologies in Chang's sense [resp. Lowen's sense] on X as* $PCT(X)$ *[resp.* $PCT_L(X)$*]. Moreover, we will denote the set of all R-cubic topologies in Chang's sense [resp. Lowen's sense] on X as* $RCT(X)$ *[resp.* $RCT_L(X)$*]. A cubic set* $\mathcal{A}$ *is called a P-cubic closed set (briefly, PCCS) [resp. R-cubic closed set (briefly, RCCS)] in X, if* $\mathcal{A}^c \in \tau$*. For a P-cubic topological space X, we denote the set of all PCOs [resp. PCCSs] in X as* $PCO(X)$ *[resp.* $PCC(X)$*]. Furthermore, for a R-cubic topological space X, we denote the set of all RCOs [resp. RCCSs] in X as* $RCO(X)$ *[resp.* $RCC(X)$*].*

Example 1. *(1) Let* $X = \{x, y\}$ *and let* $\mathcal{A}_j \in CS(X)$ *defined as follows:*

$$\mathcal{A}_1(x) = \langle[0.6, 0.8], 0.8\rangle,\ \mathcal{A}_1(y) = \langle[0.5, 0.9], 0.7\rangle,\ \mathcal{A}_2(x) = \langle[0.4, 0.7], 0.4\rangle,$$
$$\mathcal{A}_2(y) = \langle[0.7, 0.8], 0.9\rangle,\ \mathcal{A}_3(x) = \langle[0.6, 0.9], 0.8\rangle,\ \mathcal{A}_3(y) = \langle[0.7, 0.9], 0.9\rangle,$$
$$\mathcal{A}_4(x) = \langle[0.4, 0.8], 0.4\rangle,\ \mathcal{A}_4(y) = \langle[0.5, 0.8], 0.7\rangle,\ \mathcal{A}_5(x) = \langle[0.6, 0.8], 1\rangle,$$
$$\mathcal{A}_5(y) = \langle[0.5, 0.9], 1\rangle,\ \mathcal{A}_6(x) = \langle[0.4, 09], 1\rangle,\ \mathcal{A}_6(y) = \langle[0.7, 0.8], 1\rangle,$$
$$\mathcal{A}_7(x) = \langle[1, 1], 0.8\rangle,\ \mathcal{A}_7(y) = \langle[1, 1], 0.7\rangle,\ \mathcal{A}_8(x) = \langle[1, 1], 0.4\rangle,$$
$$\mathcal{A}_8(y) = \langle[1, 1], 0.9\rangle,\ \mathcal{A}_9(x) = \langle[0, 0], 0.8\rangle,\ \mathcal{A}_9(y) = \langle[0, 0], 0.7\rangle,$$
$$\mathcal{A}_{10}(x) = \langle[0, 0], 0.4\rangle,\ \mathcal{A}_{10}(y) = \langle[0, 0], 0.9\rangle,\ \mathcal{A}_{11}(x) = \langle[0.6, 0.8], 0\rangle,$$
$$\mathcal{A}_{11}(y) = \langle[0.5, .90], 0\rangle,\ \mathcal{A}_{12}(x) = \langle[0.4, 0.9], 0\rangle,\ \mathcal{A}_{12}(y) = \langle[0.7, 0.8], 0\rangle.$$

Then we can easily check that the family $\tau = \{\ddot{0}, \ddot{1}, \hat{0}, \hat{1}\} \bigcup_{j=1}^{12} \mathcal{A}_j \in PCT(X)$.
(2) Let X be a nonempty set. Then we can easily see that

$$\tau = \{\ddot{0}, \ddot{1}, \hat{0}, \hat{1}, C_{\langle[0.4,0.7],0.6\rangle}, C_{\langle[0.4,0.7],1\rangle}, C_{\langle[0,0],0.6\rangle}, C_{\langle[1,1],0.6\rangle}, C_{\langle[0.4,0.7],0\rangle}\} \in PCR(X).$$

Definition 14 (See [22])**.** *Let* $\mathcal{T} \subset CCS(X)$*. Consider the following axioms:*
(PCCO$_1$) $\hat{\emptyset}$, $\hat{X}$, $\ddot{\emptyset}$, $\ddot{X} \in \mathcal{T}$,
(PCCO$_2$) $\mathcal{A} \cap_P \mathcal{B} \in \mathcal{T}$ *for any* $\mathcal{A}, \mathcal{B} \in \mathcal{T}$,
(RCCO$_2$) $\mathcal{A} \cap_R \mathcal{B} \in \mathcal{T}$ *for any* $\mathcal{A}, \mathcal{B} \in \mathcal{T}$,
(PCCO$_3$) $\bigcup_{P,\ j\in J} \mathcal{A}_j \in \mathcal{T}$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \mathcal{T}$,
(RCCO$_3$) $\bigcup_{R,\ j\in J} \mathcal{A}_j \in \mathcal{T}$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \mathcal{T}$.
Then $\mathcal{T}$ *is called a P-cubic crisp topology (briefly, PCCT) [resp. R-cubic crisp topology (briefly, RCCT)] on X, if it satisfies the axioms (PCCO$_1$), (PCCO$_2$) and (PCCO$_3$) [resp. (RCCO$_1$), (RCCO$_2$) and (RCCO$_3$)]. The pair* $(X, \mathcal{T})$ *is called a P-cubic crisp topological space (briefly, PCCTS) [resp. an R-cubic crisp topological space (briefly, RCCTS) and each member of T is called a P-cubic crisp open set (briefly, PCCOS) [resp. an R-cubic crisp open set (briefly, RCCOS)] in X. A CCS* $\mathcal{A}$ *is called a P-cubic crisp closed set (briefly, PCCCS) [resp. an R-cubic crisp closed set (briefly, RCCCS)] in X, if* $\mathcal{A}^c \in T$.

$PCCT(X)$ [resp. $RCCT(X)$] denotes the set of all PCCTs [resp. RCCTs] on X. For a PCCTS X, $PCCO(X)$ [resp. $PCCC(X)$] is the set of all PCCOs [resp. PCCCSs] in X. Moreover, for a RCCTS X, $RCCO(X)$ [resp. $RCCC(X)$] represents the set of all RCCOs [resp. RCCCSs] in X.

Remark 4. *(1) From Definition 13, it is clear that $\{\ddot{0}, \ddot{1}, \hat{0}, \hat{1}\}$ and $CS(X)$ are both PCTs and RCTs on X. In this case, $\{\ddot{0}, \ddot{1}, \hat{0}, \hat{1}\}$ [resp. $CS(X)$] is called the cubic indiscrete topology [resp. cubic discrete topology] on X and will be denoted by $\mathcal{I}$ [resp. $\mathcal{D}$]. The pair $(X, \mathcal{I})$ [resp. $(X, \mathcal{D})$] is called a cubic indiscrete space [resp. cubic discrete space]. It is obvious that that $\mathcal{I} \subset \tau \subset \mathcal{D}$ for each $\tau \in PCT(X)$ [resp. $\tau \in RCT(X)$]. Moreover, we can see that for each $\tau \in PCT(X)$ [resp. $\tau \in RCT(X)$], τ have the least element $\hat{0}$ [resp. $\ddot{0}$] and greatest element $\hat{1}$ [resp. $\ddot{1}$].*

(2) From Definition 13, it is clear that if $\tau \in PCT(X)$, then the family

$$\{\left\langle \widetilde{A}, A^c \right\rangle \in CS(X) : \mathcal{A} = \left\langle \widetilde{A}, A \right\rangle \in \tau\} \in RCT(X).$$

The converse holds.

(3) Let $IVFT(X)$ [resp. $FT(X)$] denote the set of all interval-valued fuzzy topologies (see [21]) [resp. all fuzzy topologies (see [25,26])] on X. Then

$$\tau \in PCT(X) \Longleftrightarrow \tau_{IVF} \in IVFT(X) \text{ and } \tau_F \in FT(X),$$

where $\tau_{IVF} = \{\widetilde{A} \in IVFS(X) : \mathcal{A} \in \tau\}$ and $\tau_F = \{A \in I^X : \mathcal{A} \in \tau\}$.

Additionally, we have

$$\tau \in RCT(X) \Longleftrightarrow \tau_{IVF} \in IVFT(X) \text{ and } \tau^c_F \in FT(X),$$

where $\tau^c_F = \{A^c \in I^X : A \in \tau_F\}$.

(4) (See Remarks 4.3 (1) and 4.17 (1) in [22]) Let $\mathcal{T} \in PCCT(X)$ [resp. $RCCT(X)$]. Then we can easily check that

$$\chi_{\mathcal{T}} = \{\langle \chi_{\mathbf{A}}, \chi_A \rangle \in CS(X) : \widetilde{A} = \langle \mathbf{A}, A \rangle \in \mathcal{T}\} \in PCT(X) \text{ [resp. } RCT(X)].$$

Then we have the relationships among classical topology, IVT, PCCT and PCT, RCCT and RCT:

Classical topology$\Longrightarrow$IVT$\Longrightarrow$PCCT$\Longrightarrow$PCT,

Classical topology$\Longrightarrow$IVT$\Longrightarrow$RCCT$\Longrightarrow$RCT.

Proposition 1. *Let $\tau \in PCT(X)$ [resp. $RCT(X)$] or $\tau \in PCT_L(X)$ [resp. $RCT_L(X)$] and let $\tau^c = \{\mathcal{A}^c : \mathcal{A} \in \tau\}$. Then τ^c satisfies the following conditions:*

(PCC_1) $\ddot{0}, \ddot{1}, \hat{0}, \hat{1} \in \tau^c$,

($PCC_{1'}$) *for each* $\widetilde{\tilde{a}} \in [I] \times I$, $C_{\widetilde{\tilde{a}}} \in \tau^c$,

(PCC_2) $\mathcal{A} \sqcup \mathcal{B} \in \tau^c$ [resp. (RCC_2) $\mathcal{A} \Cup \mathcal{B} \in \tau^c$] *for any* $\mathcal{A}, \mathcal{B} \in \tau^c$,

(PCC_3) $\sqcap_{j \in J} \mathcal{A}_j \in \tau^c$ [resp. (PCC_3) $\Cap_{j \in J} \mathcal{A}_j \in \tau^c$] *for each* $(\mathcal{A}_j)_{j \in J} \subset \tau^c$.

In this case, τ^c will be called a *P-cubic cotopology* [resp. *R-cubic cotopology*] on X.

Proof. The proof is straightforward from Definition 13. □

Now we will discuss P-cubic neighborhoods [resp. Q-neighborhoods].

Definition 15. *Let (X, τ) be a P-cubic topological space or an R-cubic topological space, let $\mathcal{A} \in CS(X)$ and let $x_{\widetilde{\tilde{a}}} \in C_P(X)$.*

(i) $\mathcal{A}$ is called a P-cubic neighborhood (briefly, PCN) of $x_{\widetilde{\tilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x_{\widetilde{\tilde{a}}} \in_P \mathcal{B} \sqsubset \mathcal{A}$. A PCN $\mathcal{A}$ is said to be P-cubic open, if $\mathcal{A} \in \tau$. The collection of all PCNs of $x_{\widetilde{\tilde{a}}}$ is called the system of P-cubic neighborhoods of $x_{\widetilde{\tilde{a}}}$ and will be denoted by $\mathcal{N}_P(x_{\widetilde{\tilde{a}}})$.

(ii) *$\mathcal{A}$ is called a P-cubic Q-neighborhood (briefly, PCQN) of $x_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x_{\widetilde{\widetilde{a}}} q_P \mathcal{B} \sqsubset \mathcal{A}$. The family of all PCQNs of $x_{\widetilde{\widetilde{a}}}$ is called the system of P-cubic Q-neighborhoods of $x_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$.*

(iii) *$\mathcal{A}$ is called a R-cubic neighborhood (briefly, RCN) of $x_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x_{\widetilde{\widetilde{a}}} \in_R \mathcal{B} \Subset \mathcal{A}$. A RCN $\mathcal{A}$ is said to be R-cubic open, if $\mathcal{A} \in \tau$. The collection of all RCNs of $x_{\widetilde{\widetilde{a}}}$ is called the system of R-cubic neighborhoods of $x_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_R(x_{\widetilde{\widetilde{a}}})$.*

(iv) *$\mathcal{A}$ is called a R-cubic Q-neighborhood (briefly, RCQN) of $x_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x_{\widetilde{\widetilde{a}}} q_R \mathcal{B} \Subset \mathcal{A}$. The family of all RCQNs of $x_{\widetilde{\widetilde{a}}}$ is called the system of R-cubic Q-neighborhoods of $x_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_{R,Q}(x_{\widetilde{\widetilde{a}}})$.*

Let $\widetilde{N}(x_{\widetilde{a}})$ be the set of all interval-valued fuzzy neighborhoods of an interval-valued fuzzy point $x_{\widetilde{a}}$ (see [21]) and let $N(x_a)$ [resp. $N_Q(x_a)$] denote the set of all fuzzy neighborhoods [resp. Q-neighborhoods] of a fuzzy point x_a (see [23]).

Remark 5. *From Remark 4 (3), Definitions 9 and 15, it is obvious that*

$$\mathcal{A} \in \mathcal{N}_P(x_{\widetilde{\widetilde{a}}}) \Longleftrightarrow \widetilde{A} \in \widetilde{N}_{\tau_{IVF}}(x_{\widetilde{a}}),\ A \in N_{\tau_F}(x_a)$$

and

$$\mathcal{A} \in \mathcal{N}_R(x_{\widetilde{\widetilde{a}}}) \Longleftrightarrow \widetilde{A} \in \widetilde{N}_{\tau_{IVF}}(x_{\widetilde{a}}),\ A^c \in N_{\tau_F^c}(x_{1-a}).$$

From Remark 5, Theorem 7 in [21] and Proposition 1.8 in [24], we obtain the following.

Theorem 3. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_P(x_{\widetilde{\widetilde{a}}})$ for each $x_{\widetilde{\widetilde{a}}} \in_P \mathcal{A}$.*

Definition 16. *Let (X, τ) be an interval-valued fuzzy topological space (see [21]), let $x_{\widetilde{a}} \in IVF_P(X)$ and let $\widetilde{A} \in IVFS(X)$. Then $\widetilde{A}$ is called an interval-valued fuzzy Q-neighborhood (briefly, IVFN) of $x_{\widetilde{a}}$, denoted by $x_{\widetilde{a}} q \widetilde{A}$, if and only if there is $\widetilde{B} \in \tau$ such that $x_{\widetilde{a}} q \widetilde{B} \subset \widetilde{A}$. The set of all IVFNs of $x_{\widetilde{a}}$ is called the system of IVFNs of $x_{\widetilde{a}}$ and denoted by $\widetilde{N}_Q(x_{\widetilde{a}})$.*

Remark 6. *From Remark 4 (3), Definitions 9 and 15, it is clear that*

$$\mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}}) \Longleftrightarrow \widetilde{A} \in \widetilde{N}_{\tau_{IVF},Q}(x_{\widetilde{a}}),\ A \in N_{\tau_F,Q}(x_a)$$

and

$$\mathcal{A} \in \mathcal{N}_{R,Q}(x_{\widetilde{\widetilde{a}}}) \Longleftrightarrow \widetilde{A} \in \widetilde{N}_{\tau_{IVF},Q}(x_{\widetilde{a}}) \text{ and } A^c \in N_{\tau_F^c,Q}(x_{1-a}).$$

Lemma 5. *Let (X, τ) be an interval-valued fuzzy topological space and let $\widetilde{A} \in IVFS(X)$. Then $\widetilde{A} \in \tau$ if and only if $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$ for each $x_{\widetilde{a}} \in IVF_P(X)$ such that $[0,0] < \widetilde{a} < \widetilde{A}(x)$ and $x_{\widetilde{a}} q \widetilde{A}$.*

Proof. Suppose $\widetilde{A} \in \tau$ and let $x_{\widetilde{a}} \in IVF_P(X)$ such that $x_{\widetilde{a}} q \widetilde{A}$. Then clearly, $x_{\widetilde{a}} q \widetilde{A} \subset \widetilde{A}$. Thus, $\widetilde{A} \in \mathcal{N}_Q(x_{\widetilde{a}})$.

Conversely, suppose the necessary condition holds and let $x_{\widetilde{a}} \in \widetilde{A}$. Since $[0,0] < \widetilde{a} < \widetilde{A}(x)$. By Lemma 4, $x_{\widetilde{a}^c} q \widetilde{A}$. By the hypothesis, $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}^c})$. Then there is $\widetilde{U}_{x_{\widetilde{a}^c}} \in \tau$ such that $x_{\widetilde{a}^c} q \widetilde{U} \subset \widetilde{A}$. Thus, we have $\widetilde{a}^c > \widetilde{U}^c_{x_{\widetilde{a}^c}}(x)$, i.e., $U^+_{x_{\widetilde{a}^c}}(x) > a^+$, $U^-_{x_{\widetilde{a}^c}}(x) > a^-$. So $x_{\widetilde{a}} \in \widetilde{U}_{x_{\widetilde{a}^c}} \subset \widetilde{A}$. Hence, $\widetilde{A} = \bigcup_{x_{\widetilde{a}} \in \widetilde{A}} \widetilde{U}_{x_{\widetilde{a}^c}}$. Therefore, by ($PCO_3$), $\widetilde{A} \in \tau$. □

From Remark 6, Lemma 5 and Theorem 3.2 in [27], we have the following.

Theorem 4. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$ for each $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ such that $\widetilde{a} \neq \widetilde{A}(x)$, $a \neq A(x)$ and $x_{\widetilde{\widetilde{a}}} q_P \mathcal{A}$.*

Definition 17 ([21])**.** *Let $[0,0] \neq \widetilde{a},\ \widetilde{b} \in [I]$. Then $\widetilde{b}$ is said to be admissible with $\widetilde{a}$, if it satisfies the following conditions:*

(i) $b^- \geq 0,\ b^+ > 0$, (ii) $b^- = 0$ if and only if $a^- = 0$, (iii) $0 \leq a^- - b^- < a^+ - b^+$.

Result 2 (Theorems 8 and 9, [21])**.** *Let X be an interval-valued fuzzy topological space and let $x_{\widetilde{a}} \in IVF_P(X)$.*

(N1) *If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$, then $x_{\widetilde{a}} \in \widetilde{A}$.*

(N2) *If $\widetilde{A},\ \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$, then $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.*

(N3) *If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$ and $\widetilde{A} \subset \widetilde{B}$, then $\widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.*

(N4) *If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}-\widetilde{b}})$ for each admissible $\widetilde{b}$, then $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$,*

where $\widetilde{a} - \widetilde{b} = [a^- - b^-, a^+ - b^+]$.

(N5) *If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$, $\widetilde{B} \in \widetilde{N}(y_{\widetilde{a}})$, then $\widetilde{A} \cup \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}} \cup y_{\widetilde{a}})$.*

(N6) *If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$, then there is $\widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B} \in \widetilde{N}(y_{\widetilde{b}})$ for $y_{\widetilde{b}} \in \widetilde{B}$.*

Conversely, if for each $x_{\widetilde{a}} \in IVF_P(X)$, $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the conditions (N1)–(N5), then the family τ of cubic sets in X given by:

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \widetilde{N}_{x_{\widetilde{a}}} \text{ for each } x_{\widetilde{a}} \in \widetilde{A}\}$$

is an interval-valued fuzzy topology on X. Furthermore, if $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the condition (N6), then $\widetilde{N}_{x_{\widetilde{a}}}$ is exactly the system of interval-valued fuzzy neidhborhood of $x_{\widetilde{a}}$ with respect to τ, i.e., $\widetilde{N}_{x_{\widetilde{a}}} = \widetilde{N}(x_{\widetilde{a}})$.

The following is an immediate consequence of Remark 6, Proposition 2.2 in [23] and Result 2.

Theorem 5. *Let X be a P-cubic topological space and let $x_{\tilde{\tilde{a}}} \in C_P(X)$.*

(1) If $\mathcal{A} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$, then $x_{\tilde{\tilde{a}}} \in_P \mathcal{A}$.

(2) If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$, then $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$.

(3) If $\mathcal{A} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$ and $\mathcal{A} \sqsubset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$.

(4) If $\mathcal{A} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}_P(x_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}_P(y_{\tilde{\tilde{b}}})$ for $y_{\tilde{\tilde{b}}} \in_P \mathcal{B}$.

Conversely, if for each $x_{\tilde{\tilde{a}}} \in C_P(X)$, $\mathcal{N}_{P,x_{\tilde{\tilde{a}}}}$ satisfies the conditions (1)–(3), and for each $x_{\widetilde{a}} \in IVF_P(X)$, $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the conditions (N4) and (N5), then the family τ of cubic sets in X given by:

$$\tau = \{\ddot{0}, \ddot{1}\} \cup \{\mathcal{A} \in CS(X) : \mathcal{A} \in \mathcal{N}_{P,x_{\tilde{\tilde{a}}}} \text{ for each } x_{\tilde{\tilde{a}}} \in_P \mathcal{A}\}$$

is a P-cubic topology on X. Furthermore, if $\mathcal{N}_{P,x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{P,x_{\tilde{\tilde{a}}}}$ is exactly the system of P-cubic neidhborhood of $x_{\tilde{\tilde{a}}}$ with respect to τ, i.e., $\mathcal{N}_{P,x_{\tilde{\tilde{a}}}} = \mathcal{N}_P(x_{\tilde{\tilde{a}}})$.

Theorem 6. *Let X be an R-cubic topological space and let $x_{\tilde{\tilde{a}}} \in C_P(X)$.*

(1) If $\mathcal{A} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$, then $x_{\tilde{\tilde{a}}} \in_R \mathcal{A}$.

(2) If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$, then $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$.

(3) If $\mathcal{A} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$ and $\mathcal{A} \Subset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$.

(4) If $\mathcal{A} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}_R(x_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}_R(y_{\tilde{\tilde{b}}})$ for $y_{\tilde{\tilde{b}}} \in_R \mathcal{B}$.

Conversely, if for each $x_{\tilde{\tilde{a}}} \in C_P(X)$, $\mathcal{N}_{R,x_{\tilde{\tilde{a}}}}$ satisfies the conditions (1)–(3), and for each $x_{\widetilde{a}} \in IVF_P(X)$, $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the conditions (N4) and (N5), then the family τ of cubic sets in X given by:

$$\tau = \{\hat{0}, \hat{1}\} \cup \{\mathcal{A} \in CS(X) : \mathcal{A} \in \mathcal{N}_{R,x_{\tilde{\tilde{a}}}} \text{ for each } x_{\tilde{\tilde{a}}} \in_R \mathcal{A}\}$$

is an R-cubic topology on X. Furthermore, if $\mathcal{N}_{R,x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{R,x_{\tilde{\tilde{a}}}}$ is exactly the system of R-cubic neidhborhood of $x_{\tilde{\tilde{a}}}$ with respect to τ, i.e., $\mathcal{N}_{R,x_{\tilde{\tilde{a}}}} = \mathcal{N}_R(x_{\tilde{\tilde{a}}})$.

Proof. It can be proved similarly to Theorem 5 by using Remark 5. □

Lemma 6. *Let (X,τ) be an interval-valued fuzzy topological space and let $x_{\widetilde{a}} \in IVF_P(X)$.*

(1) If $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$, then $x_{\widetilde{a}} q \widetilde{A}$.
(2) If $\widetilde{A},\ \widetilde{B} \in \widetilde{N}_Q(x_{\widetilde{a}})$, then $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.
(3) If $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$ and $\widetilde{A} \subset \widetilde{B}$, then $\widetilde{B} \in \widetilde{N}_Q(x_{\widetilde{a}})$.
(4) If $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$, then there is $\widetilde{B} \in \widetilde{N}_Q(x_{\widetilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B} \in \widetilde{N}_Q(y_{\widetilde{b}})$ for $y_{\widetilde{b}} q \widetilde{B}$.

Conversely, if for each $x_{\widetilde{a}} \in IVF_P(X)$, $\widetilde{N}_{Q,x_{\widetilde{a}}}$ satisfies the conditions (1)–(3), then the family τ of cubic sets in X given by:

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \mathcal{N}_{Q,x_{\widetilde{a}}} \text{ for each } x_{\widetilde{a}} q \widetilde{A}\}$$

is an interval-valued fuzzy topology on X. Furthermore, if $\widetilde{N}_{Q,x_{\widetilde{a}}}$ satisfies the condition (4), then $\widetilde{N}_{Q,x_{\widetilde{a}}}$ is exactly the system of P-cubic Q-neighborhood of $x_{\widetilde{a}}$ with respect to τ, i.e., $\widetilde{N}_{Q,x_{\widetilde{a}}} = \widetilde{N}_Q(x_{\widetilde{a}})$.

Proof. (1) Suppose $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$. Then there is $\widetilde{U} \in \tau$ such that $x_{\widetilde{a}} \in \widetilde{U} \subset \widetilde{A}$. Thus,

$$\widetilde{a} > \widetilde{U}^c(x), \text{ i.e., } a^- + U^+(x) > 1,\ a^+ + U^+(x) > 1.$$

Since $\widetilde{U} \subset \widetilde{A}$, $a^- + A^+(x) > 1$, $a^+ + A^+(x) > 1$. So $\widetilde{a} > \widetilde{A}^c(x)$. Hence, $x_{\widetilde{a}} q \widetilde{A}$.

(2) $\widetilde{A},\ \widetilde{B} \in \widetilde{N}_Q(x_{\widetilde{a}})$. Then there are $\widetilde{U},\ \widetilde{V} \in \tau$ such that $x_{\widetilde{a}} q \widetilde{U} \subset \widetilde{A}$ and $x_{\widetilde{a}} q \widetilde{V} \subset \widetilde{B}$. Thus, $\widetilde{a} > \widetilde{U}^c(x)$ and $\widetilde{a} > \widetilde{V}^c(x)$. By Theorem 1 (ix) in [21], $\widetilde{a} > \widetilde{U}^c(x) \vee \widetilde{V}^c(x) = (\widetilde{U} \cap \widetilde{V})^c(x)$. So $x_{\widetilde{a}} q (\widetilde{U} \cap \widetilde{V})$. Moreover, $\widetilde{U} \cap \widetilde{V} \subset \widetilde{A} \cap \widetilde{B}$ and $\widetilde{U} \cap \widetilde{V} \in \tau$. Hence, $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.

(3) The proof is straightforward.

(4) Suppose $\widetilde{A} \in \widetilde{N}_Q(x_{\widetilde{a}})$. Then there is $\widetilde{B} \in \tau$ such that $x_{\widetilde{a}} q \widetilde{B} \subset \widetilde{A}$. Since $\widetilde{B} \subset \widetilde{B}$, $\widetilde{B} \in \widetilde{N}_Q(x_{\widetilde{a}})$ and moreover, $\widetilde{B} \in \widetilde{N}_Q(y_{\widetilde{b}})$ for each $y_{\widetilde{b}} q \widetilde{B}$.

Conversely, suppose $\widetilde{N}_{Q,x_{\widetilde{a}}}$ satisfies the conditions (1)–(3) for each $x_{\widetilde{a}} \in IVF_P(X)$. From the definition of τ, it is clear that $\widetilde{\mathbf{0}},\ \widetilde{\mathbf{1}} \in \tau$. Now let $\widetilde{A},\ \widetilde{B} \in \tau$ and let $x_{\widetilde{a}} q (\widetilde{A} \cap \widetilde{B})$. Then by Lemma 3 (2), $x_{\widetilde{a}} q \widetilde{A}$ and $x_{\widetilde{a}} q \widetilde{B}$. So by the definition of τ, $\widetilde{A} \in \widetilde{N}_{Q,x_{\widetilde{a}}}$ and $\widetilde{B} \in \widetilde{N}_{Q,x_{\widetilde{a}}}$. By the condition (2), $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}_{Q,x_{\widetilde{a}}}$. Hence, $\widetilde{A} \cap \widetilde{B} \in \tau$. Finally, let $(\widetilde{A}_j)_{j \in J} \subset \tau$, let $\widetilde{A} = \bigcup_{j \in J} \widetilde{A}_j$ and let $x_{\widetilde{a}} q \widetilde{A}$. By Lemma 3 (1), there is $j \in J$ such that $x_{\widetilde{a}} q \widetilde{A}_j$. Since $\widetilde{A}_j \in \tau$, $\widetilde{A}_j \in \widetilde{N}_{Q,x_{\widetilde{a}}}$. Since $\widetilde{A}_j \subset \widetilde{A}$, by the condition (3), $\widetilde{A} \in \tau$, i.e., $\bigcup_{j \in J} \widetilde{A}_j \in \tau$. Therefore, τ is an interval-valued fuzzy topology on X.

Now suppose $\widetilde{N}_{Q,x_{\widetilde{a}}}$ satisfies the conditions (4). Then we can easily show that $\widetilde{N}_{Q,x_{\widetilde{a}}} = \widetilde{N}_Q(x_{\widetilde{a}})$. □

The following is an immediate consequence of Lemma 6 and Proposition 2.2 in [23].

Theorem 7. *Let X be a P-cubic topological space and let $x_{\widetilde{\widetilde{a}}} \in C_P(X)$.*

(1) If $\mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$, then $x_{\widetilde{\widetilde{a}}} q_P \mathcal{A}$.
(2) If $\mathcal{A},\ \mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$, then $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$.
(3) If $\mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$ and $\mathcal{A} \sqsubset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$.
(4) If $\mathcal{A} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}_{P,Q}(x_{\widetilde{\widetilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}_{P,Q}(y_{\widetilde{\widetilde{b}}})$ for $y_{\widetilde{\widetilde{b}}} q_P \mathcal{B}$.

Conversely, if for each $x_{\widetilde{\widetilde{a}}} \in C_P(X)$, $\mathcal{N}_{P,Q,x_{\widetilde{\widetilde{a}}}}$ satisfies the conditions (1)–(3), then the family τ of cubic sets in X given by:

$$\tau = \{\ddot{0}, \ddot{1}\} \cup \{\mathcal{A} \in CS(X) : \mathcal{A} \in \mathcal{N}_{P,Q,x_{\widetilde{\widetilde{a}}}} \text{ for each } x_{\widetilde{\widetilde{a}}} q_P \mathcal{A}\}$$

is a P-cubic topology on X. Furthermore, if $\mathcal{N}_{P,Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{P,Q,x_{\tilde{\tilde{a}}}}$ *is exactly the system of P-cubic Q-neighborhood of* $x_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{P,Q,x_{\tilde{\tilde{a}}}} = \mathcal{N}_{P,Q}(x_{\tilde{\tilde{a}}})$.

Theorem 8. *Let X be an R-cubic topological space and let* $x_{\tilde{\tilde{a}}} \in C_P(X)$.

(1) *If* $\mathcal{A} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$*, then* $x_{\tilde{\tilde{a}}} q_R \mathcal{A}$.
(2) *If* $\mathcal{A}$, $\mathcal{A} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$*, then* $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$.
(3) *If* $\mathcal{A} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$ *and* $\mathcal{A} \Subset \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$.
(4) *If* $\mathcal{A} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$*, then there is* $\mathcal{B} \in \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$ *such that* $\mathcal{B} \Subset \mathcal{A}$ *and* $\mathcal{B} \in \mathcal{N}_{R,Q}(y_{\tilde{\tilde{b}}})$ *for* $y_{\tilde{\tilde{b}}} q_R \mathcal{B}$.

Conversely, if for each $x_{\tilde{\tilde{a}}} \in C_P(X)$, $\mathcal{N}_{R,Q,x_{\tilde{\tilde{a}}}}$ *satisfies the conditions (1)–(3), then the family* τ *of cubic sets in X given by:*

$$\tau = \{\hat{0}, \hat{1}\} \cup \{\mathcal{A} \in CS(X) : \mathcal{A} \in \mathcal{N}_{R,Q,x_{\tilde{\tilde{a}}}} \text{ for each } x_{\tilde{\tilde{a}}} q_R \mathcal{A}\}$$

is a P-cubic topology on X. Furthermore, if $\mathcal{N}_{R,Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{R,Q,x_{\tilde{\tilde{a}}}}$ *is exactly the system of R-cubic Q-neighborhood of* $x_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{R,Q,x_{\tilde{\tilde{a}}}} = \mathcal{N}_{R,Q}(x_{\tilde{\tilde{a}}})$.

Proof. It can be proved similarly to Theorem 7 by using Remark 6. □

From Lemma 3 (1) and Proposition 2.2 in [23], we get the following.

Theorem 9. *Let* $x_{\tilde{\tilde{a}}} \in C_P(X)$ *and let* $(\mathcal{A}_j)_{j\in J} \subset CS(X)$.

(1) $x_{\tilde{\tilde{a}}} q_P \sqcup_{j\in J} \mathcal{A}_j$ *if and only if there is* $j \in J$ *such that* $x_{\tilde{\tilde{a}}} q_P \mathcal{A}_j$.
(2) $x_{\tilde{\tilde{a}}} q_R \Cup_{j\in J} \mathcal{A}_j$ *if and only if there is* $j \in J$ *such that* $x_{\tilde{\tilde{a}}} q_R \mathcal{A}_j$.

Definition 18. *Let* (X, τ) *be a P-cubic topological space and let* $\beta \subset \tau$, $\sigma \subset \tau$.

(i) β *is called a P-cubic base for* τ*, if for each* $\mathcal{A} \in \tau$*, there is* $\beta_{\mathcal{A}} \in \beta$ *such that* $\mathcal{A} = \sqcup \beta_{\mathcal{A}}$.
(ii) σ *is called a P-cubic subbase for* τ*, if the family* $\beta = \{\sqcap \eta : \eta \text{ is a finite subset of } \sigma\}$ *is a P-cubic base for* τ.

Remark 7. *Let* (X, τ) *be a P-cubic topological space and let* β *[resp.* σ*] be a P-cubic base [resp. subbase] for* τ*. Then from Remark 4 (3), we can easily see that*

(1) $\beta_{IVF} = \{\widetilde{A} \in IVFS(X) : \mathcal{A} \in \beta\}$ *is an interval-valued fuzzy base for* τ_{IVF} *and* $\beta_F = \{A \in I^X : \mathcal{A} \in \beta\}$ *[resp.* $\beta^-_{IVF} = \{A^- \in I^X : \mathcal{A} \in \beta\}$ *and* $\beta^+_{IVF} = \{A^+ \in I^X : \mathcal{A} \in \beta\}$*] is a fuzzy base for* τ_F *[resp.* τ^-_{IVF} *and* τ^+_{IVF}*].*
(2) $\sigma_{IVF} = \{\widetilde{A} \in IVFS(X) : \mathcal{A} \in \sigma\}$ *is an interval-valued fuzzy subbase for* τ_{IVF} *and* $\sigma_F = \{A \in I^X : \mathcal{A} \in \sigma\}$ *[resp.* $\sigma^-_{IVF} = \{A^- \in I^X : \mathcal{A} \in \sigma\}$ *and* $\sigma^+_{IVF} = \{A^+ \in I^X : \mathcal{A} \in \sigma\}$*] is a fuzzy subbase for* τ_F *[resp.* τ^-_{IVF} *and* τ^+_{IVF}*].*

Lemma 7. *Let* (X, τ) *be an interval-valued fuzzy topological space and let* $\beta \subset \tau$*. Then* β *is an interval-valued fuzzy base for* τ *if and only if for each* $x_{\tilde{a}} \in IVF_P(X)$ *and for each interval-valued open Q-neighborhood* $\widetilde{A}$ *of* $x_{\tilde{a}}$*, there is* $\widetilde{B} \in \beta$ *such that* $x_{\tilde{a}} q \widetilde{B} \subset \widetilde{A}$.

Proof. Suppose β is an interval-valued fuzzy base for τ. Then from the definition of an interval-valued fuzzy base and Lemma 3 (1), it can be easily seen that the necessary condition holds.

Conversely, suppose the necessary condition holds. Assume that β is not an interval-valued fuzzy base for τ. Then there is $\widetilde{A} \in \tau$ such that $\widetilde{U} = \bigcup\{\widetilde{B} \in \beta : \widetilde{B} \subset \widetilde{A}\} \neq \widetilde{A}$. Thus, there is $x \in X$ such that $\widetilde{U}(x) < \widetilde{A}(x)$. Let $\tilde{a} = \widetilde{U}^c(x) = [1,1] - \widetilde{U}(x)$. Then clearly, $\tilde{a} > [0,0]$ and we obtain an interval-valued fuzzy point $x_{\tilde{a}}$. Moreover, $\widetilde{A}(x) + \tilde{a} > \widetilde{U}(x) + \tilde{a} > [1,1]$.

Thus, $x_{\tilde{a}} q \widetilde{A}$. On the other hand, let $\widetilde{B} \in \beta$ such that $\widetilde{B} \subset \widetilde{A}$. Then clearly, $\widetilde{B} \subset \widetilde{U}$. Thus, $\widetilde{B}(x) + \tilde{a} \leq \widetilde{U}(x) + \tilde{a} = [1,1]$. So $x_{\tilde{a}} \neg q \widetilde{B}$. This contradicts the hypothesis. □

From Theorem 9 and Lemma 7, and Proposition 2.4 in [23], we obtain the following.

Theorem 10. *Let (X, τ) be a P-cubic topological space and let $\beta \subset \tau$. Then β is a P-cubic base for τ if and only if for each $x_{\tilde{\tilde{a}}} \in C_P(X)$ and for each P-cubic open Q-neighborhood $\mathcal{A}$ of $x_{\tilde{\tilde{a}}}$, there is $\mathcal{B} \in \beta$ such that $x_{\tilde{\tilde{a}}} q_P \mathcal{B} \sqsubset \mathcal{A}$.*

The following gives a necessary and sufficient condition for a subset of $CS(X)$ to be a P-cubic base for a P-cubic topology on a set X.

Theorem 11. *Let X be a set and let $\beta \subset CS(X)$. Then β is a P-cubic base for some P-cubic topology τ if and only if the following hold:*

(1) $\hat{1}$, $\ddot{0}$, $\ddot{1} \in \beta$,

(2) *if $\mathcal{B}_1$, $\widetilde{\mathcal{B}}_2 \in \beta$ and $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqcap \mathcal{B}_2$, then there is $\mathcal{B} \in \beta$ such that*

$$x_{\tilde{\tilde{a}}} \in_P \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2.$$

In this case, τ is called the *P-cubic topology on X generated by β*.

Proof. Suppose β is a P-cubic base for a P-cubic topology τ. Since $\hat{1}$, $\ddot{0}$, $\ddot{1} \in \tau$, $\hat{1}$, $\ddot{0}$, $\ddot{1} \in \beta$. Then the condition (1) holds. Now suppose $\mathcal{B}_1$, $\widetilde{\mathcal{B}}_2 \in \beta$ and $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqcap \mathcal{B}_2$. Since $\beta \subset \tau$, $\mathcal{B}_1$, $\widetilde{\mathcal{B}}_2 \in \tau$. Then $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \tau$. Since $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \neq \hat{0}$. By the definition of a P-cubic base, there is $\beta' \subset \beta$ such that $\mathcal{B}_1 \sqcap \mathcal{B}_2 = \sqcup \beta'$. So there is $\mathcal{B} \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_P \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2$. Hence, the condition (2) holds.

Conversely, suppose the conditions (1) and (2) hold and let

$$\tau = \{\mathcal{U} \in CS(X) : \mathcal{U} = \hat{0} \text{ or there is } \beta' \subset \beta \text{ such that } \mathcal{U} = \sqcup \beta'\}.$$

Then clearly, $\hat{0}$, $\hat{1}$, $\ddot{0}$, $\ddot{1} \in \tau$. Thus, the condition (PCO_1) holds. Now suppose $\mathcal{U}_1$, $\mathcal{U}_2 \in \tau$ and $x_{\tilde{\tilde{a}}} \in_P \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then there are $\mathcal{B}_1$, $\mathcal{B}_2 \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_2 \sqsubset \mathcal{U}_2$. Thus, $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqcap \mathcal{B}_2 \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. By the condition (2), there is $\mathcal{B} \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_P \mathcal{B} \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. So $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$. Hence, the condition ($PCO_2$) holds. Since τ consists of all P-cubic unions of members of β, the P-cubic union of any family of members of τ is also a member of τ. Then (PCO_3) holds. □

Theorem 12. *Let X be a set and let σ be a cubic sets in X such that $\hat{1}$, $\ddot{0}$, $\ddot{1} \in \sigma$. Then there is a unique P-cubic topology τ on X such that σ is a P-cubic subbase for τ. In this case, τ is called the P-cubic topology on X generated by σ.*

Proof. Let $\beta = \{\sqcap \eta : \eta \text{ is a finite subset of } \sigma\}$ and let

$$\tau = \{\mathcal{U} \in CS(X) : \mathcal{U} = \hat{0} \text{ or there is } \beta' \subset \beta \text{ such that } \mathcal{U} = \sqcup \beta'\}.$$

Then clearly, $\hat{0}$, $\hat{1}$, $\ddot{1}$, $\ddot{0} \in \tau$ by the definition of τ. Thus, τ satisfies the condition (PCO_1). Let $\mathcal{U}_j \in \tau$ for each $j \in J$. Then there is $\beta_j \subset \beta$ such that $\mathcal{U}_j = \sqcup\{\mathcal{B} \in CS(X) : \mathcal{B} \in \beta_j\}$. Thus, $\sqcup_{j \in J} \mathcal{U}_j = \sqcup_{j \in J}(\sqcup_{\mathcal{B} \in \beta_j} \mathcal{B}$. So $\sqcup_{j \in J} \mathcal{U}_j \in \tau$. Hence, the condition ($PCO_3$) holds. Finally, suppose $\mathcal{U}_1$, $\mathcal{U}_2 \in \tau$ and $x_{\tilde{\tilde{a}}} \in_P \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then there are $\mathcal{B}_1$, $\mathcal{B}_2 \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_P \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $\mathcal{B}_2 \sqsubset \mathcal{U}_2$. Since each of $\mathcal{B}_1$ and $\mathcal{B}_2$ is the P-intersection of a finite number of members of σ, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \beta$. So there is $\beta' \subset \beta$ such that $\mathcal{U}_1 \sqcap \mathcal{U}_2 = \sqcup_{\mathcal{B} \in \beta'} \mathcal{B}$. Hence, $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$, i.e., the condition ($PCO_2$) holds. Therefore, $\tau \in PCT(X)$. It is obvious that τ is the unique P-cubic topology on X having σ as a P-cubic subbase. □

5. Cubic Interiors and Cubic Closures

In this section, we define a cubic closure and a cubic interior, and discuss some properties related to them.

Definition 19. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then the cubic interior of $\mathcal{A}$, denoted by $\mathcal{A}^{\circ_P}$ or $int_P(\mathcal{A})$ or $int_{P,\tau}(\mathcal{A})$, is a cubic set in X defined as follows:*

$$int_P(\mathcal{A}) = \sqcup\{\mathcal{U} \in \tau : \mathcal{U} \sqsubset \mathcal{A}\}.$$

It is obvious that $int_P(\mathcal{A})$ is the largest cubic open set contained in $\mathcal{A}$.

Remark 8. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then, from Remark 4 (3) and Definition 19, we can easily see that*

$$int_{P,\tau}(\mathcal{A}) = \left\langle int_{\tau_{IVF}}(\widetilde{A}), int_{\tau_F}(A) \right\rangle,$$

where $int_{\tau_{IVF}}(\widetilde{A})$ denotes the interval-valued fuzzy interior of an interval-valued fuzzy set $\widetilde{A}$ in X and $int_{\tau_F}(A)$ denotes the fuzzy interior of a fuzzy set $\widetilde{A}$ in X (see [21,23] for each definition).

Proposition 2. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A}$, $\mathcal{B} \in CS(X)$. Then the following hold:*

(1) $int_P(\hat{1}) = \hat{1}$, $int_P(\ddot{1}) = \ddot{1}$,
(2) $int_P(\mathcal{A}) \sqsubset \mathcal{A}$,
(3) $int_P(int_P(\mathcal{A})) = int_P(\mathcal{A})$,
(4) $int_P(\mathcal{A} \sqcap \mathcal{A}) = int_P(\mathcal{A}) \sqcap int_P(\mathcal{A})$,
(5) $\mathcal{A} \in \tau$ *if and only if* $int_P(\mathcal{A}) = \mathcal{A}$.

Proof. The proofs are straightforward. □

Theorem 13. *Let (X, τ) be a P-cubic topological space, let $x_{\widetilde{\widetilde{a}}} \in C_P(X)$ and let $\mathcal{A} \in CS(X)$. Then $x_{\widetilde{\widetilde{a}}} \in int_{P,\tau}(\mathcal{A})$ if and only if $\widetilde{A} \in \widetilde{N}_{\tau_{IVF}}(x_{\widetilde{a}})$ and $A \in N_{\tau_F}(x_a)$, i.e., $x_{\widetilde{a}} \in int_{\tau_{IVF}}(\widetilde{A})$ and $x_a \in int_{\tau_F}(A)$.*

Proof. The proof follows from Definition 19, Remark 8, Definition 2.6 and Theorem 10 in [21], and Theorem 4.1 in [23]. □

Definition 20. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then the cubic closure of $\mathcal{A}$, denoted by $cl_P(\mathcal{A})$ or $cl_{P,\tau}(\mathcal{A})$, is a cubic set in X defined as follows:*

$$cl_P(\mathcal{A}) = \sqcap\{\mathcal{F} \in \tau^c : \mathcal{F} \sqsupset \mathcal{A}\}.$$

It is obvious that $cl_P(\mathcal{A})$ is the smallest cubic cubic set containing $\mathcal{A}$ and $cl_P(cl_P(\mathcal{A})) = cl_P(\mathcal{A})$.

Remark 9. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then, from Remark 4 (3) and Definition 20, we can easily see that*

$$cl_{P,\tau}(\mathcal{A}) = \left\langle cl_{\tau_{IVF}}(\widetilde{A}), cl_{\tau_F}(A) \right\rangle,$$

where $cl_{\tau_{IVF}}(\widetilde{A})$ denotes the interval-valued fuzzy closure of an interval-valued fuzzy set $\widetilde{A}$ in X and $cl_{\tau_F}(A)$ denotes the fuzzy closure of a fuzzy set $\widetilde{A}$ in X (see [21,23] for each definition).

Proposition 3. *Let (X, τ) be a P-cubic topological space and let $\mathcal{A}$, $\mathcal{B} \in CS(X)$. Then the following hold:*

(1) $cl_P(\hat{0}) = \hat{0}$, $cl_P(\ddot{0}) = \ddot{0}$,
(2) $\mathcal{A} \sqsubset cl_P(\mathcal{A})$,
(3) $cl_P(cl_P(\mathcal{A})) = cl_P(\mathcal{A})$,
(4) $cl_P(\mathcal{A} \sqcup \mathcal{B}) = cl_P(\mathcal{A}) \sqcup int_P(\mathcal{B})$,
(5) $\mathcal{A} \in \tau^c$ *if and only if* $cl_P(\mathcal{A}) = \mathcal{A}$.

Proof. The proofs are straightforward. □

Lemma 8. *Let (X, τ) be an interval-valued fuzzy topological space, let $x_{\tilde{a}} \in IVF_P(X)$ and let $\widetilde{A} \in IVFS(X)$. Then $x_{\tilde{a}} \in cl_\tau(\widetilde{A})$ if and only if for each $\widetilde{U} \in \widetilde{N}_Q(x_{\tilde{a}})$, $\widetilde{U}q\widetilde{A}$.*

Proof. Suppose $x_{\tilde{a}} \in cl_\tau(\widetilde{A})$ and assume that the necessary condition does not hold, i.e., there is $\widetilde{U} \in \widetilde{N}_Q(x_{\tilde{a}})$ such that $\widetilde{U}\neg q\widetilde{A}$. Then there is $\widetilde{B} \in \tau$ such that $x_{\tilde{a}}q\widetilde{B} \subset \widetilde{U}$. Since $\widetilde{U}\neg q\widetilde{A}$, by Lemma 2, $\widetilde{U} \subset \widetilde{A}^c$. By Lemma Theorem 2 (viii) in [21], $\widetilde{U}^c \supset \widetilde{A}$. Thus, $\widetilde{B}^c \supset \widetilde{A}$ and $\widetilde{B}^c \in \tau^c$. Since $x_{\tilde{a}}q\widetilde{B}$, $\tilde{a} > \widetilde{B}^c(x)$. So $\tilde{a} > \widetilde{A}(x)$. Hence, $x_{\tilde{a}} \notin cl_\tau(\widetilde{A})$. This contradicts the hypothesis. Therefore, the necessary condition holds.

The proof of the converse is similar. □

From Lemma 8 and Theorem 4.1$'$ in [23], we have the following.

Theorem 14. *Let (X, τ) be a P-cubic topological space, let $x_{\tilde{\tilde{a}}} \in C_P(X)$ and let $\mathcal{A} \in CS(X)$. Then $x_{\tilde{\tilde{a}}} \in cl_{P,\tau}(\mathcal{A})$ if and only if for each $\mathcal{U} \in \mathcal{N}_Q(x_{\tilde{\tilde{a}}})$, $\mathcal{U}q_P\mathcal{A}$.*

Definition 21. *Let X be an interval-valued fuzzy topological space, let $x_{\tilde{a}} \in IVF_P(X)$ and let $\widetilde{A} \in IVFS(X)$. Then $x_{\tilde{a}}$ is called an interval-valued fuzzy closure point or interval-valued fuzzy adherence point of $\widetilde{A}$, if for each $\widetilde{U} \in \widetilde{N}_Q(x_{\tilde{a}})$, $\widetilde{U}q\widetilde{A}$.*

Lemma 9. *Let X be an interval-valued fuzzy topological space and let $\widetilde{A} \in IVFS(X)$. Then $cl(\widetilde{A})$ is the union of all the interval-valued fuzzy closure points of $\widetilde{A}$, i.e., $cl(\widetilde{A}) = \bigcup\{x_{\tilde{a}} \in IVF_P(X) : x_{\tilde{a}}\ is\ an\ interval-valued\ fuzzy\ closure\ point\ of\ \widetilde{A}\}$.*

Proof. The proof is easy from Lemma 8 and Definition 21. □

Definition 22. *Let X be a P-cubic topological space, let $x_{\tilde{\tilde{a}}} \in C_P(X)$ and let $\mathcal{A} \in CS(X)$. Then $x_{\tilde{\tilde{a}}}$ is called an P-cubic closure point or P-cubic adherence point of $\mathcal{A}$, if for each $\mathcal{U} \in \mathcal{N}_Q(x_{\tilde{\tilde{a}}})$, $\mathcal{U}q_P\mathcal{A}$.*

Lemma 10. *Let X be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then $cl_P(\mathcal{A})$ is the union of all the P-cubic closure points of $\mathcal{A}$, i.e.,*

$$cl_P(\mathcal{A}) = \sqcup\{x_{\tilde{\tilde{a}}} \in C_P(X) : x_{\tilde{\tilde{a}}} \text{ is a P-cubic closure point of } \mathcal{A}\}.$$

Proof. The proof follows from Definition 20, Remark 9, Lemma 8 and Corollary in [23]. □

Proposition 4. *Let X be a P-cubic topological space and let $\mathcal{A} \in CS(X)$. Then $cl_P(\mathcal{A}) = [int_P(\mathcal{A}^c)]^c$.*

Proof. The proof follows from Remarks 8 and 9, and Theorems 12 in [21] and 4.2 in [23]. □

Definition 23. *A mapping $f : IVFS(X) \to IVFS(X)$ is called an interval-valued fuzzy closure operator on X, if it satisfies the following conditions: for any $\widetilde{A}$, $\widetilde{B} \in IVFS(X)$,*

[IVFK 1] $f(\widetilde{\mathbf{0}}) = \widetilde{\mathbf{0}}$,
[IVFK 2] $\widetilde{A} \subset f(\widetilde{A})$,

[IVFK 3] $f(f(\widetilde{A})) = f(\widetilde{A})$,
[IVFK 4] $f(\widetilde{A} \cup \widetilde{B}) = f(\widetilde{A}) \cup f(\widetilde{B})$.

The following Lemma shows that an interval-valued fuzzy closure operator determines completely an interval-valued fuzzy topology τ on X and that $f = cl_{\tau}$.

Lemma 11. *Let $f : IVFS(X) \to IVFS(X)$ be an interval-valued fuzzy closure operator on a nonempty set X. Let $\rho = \{\widetilde{F} \in IVFS(X) : f(\widetilde{F}) = \widetilde{F}\}$ and let $\tau = \{\widetilde{U} \in IVFS(X) : \widetilde{U}^c \in \rho\}$. Then τ is an interval-valued fuzzy topology on X. Furthermore, if cl_{τ} is the interval-valued fuzzy closure operator defined by τ, then $f(\widetilde{A}) = cl_{\tau}(\widetilde{A})$ for each $\widetilde{A} \in IVFS(X)$.*

Proof. It can be proved almost similarly to Theorem 3.2.3 in [28]. □

Definition 24. *A mapping $f : CS(X) \to CS(X)$ is called an P-cubic closure operator on X, if it satisfies the following conditions: for any $\mathcal{A}$, $\mathcal{B} \in IVFS(X)$,*
[CK 1] $f(\hat{0}) = \hat{0}$, $f(\ddot{0}) = \ddot{0}$,
[CK 2] $\mathcal{A} \sqsubset f(\mathcal{A})$,
[CK 3] $f(f(\mathcal{A})) = f(\mathcal{A})$,
[CK 4] $f(\mathcal{A} \sqcup \mathcal{B}) = f(\mathcal{A}) \sqcup f(\mathcal{B})$.

Moreover, we give a P-cubic topology on X by a P-cubic closure operator on X.

Proposition 5. *Let $f : CS(X) \to CS(X)$ be a P-cubic closure operator on a nonempty set X. Let $\rho = \{\mathcal{F} \in CS(X) : f(\mathcal{F}) = \mathcal{F}\}$ and let $\tau = \{\mathcal{U} \in CS(X) : \mathcal{U}^c \in \rho\}$. Then τ is a P-cubic topology on X. Furthermore, if $cl_{P,\tau}$ is the P-cubic closure operator defined by τ, then $f(\mathcal{A}) = cl_{P,\tau}(\mathcal{A})$ for each $\mathcal{A} \in CS(X)$.*

Proof. The proof is straightforward from Lemma 11 and Theorem 4.4 in [23]. □

6. Cubic Continuous Mappings

In this section, we introduce the notion of cubic continuities and give its characterization. Next, we propose the concept of cubic quotient mappings and deal with some of their properties. Furthermore, we find the sufficient condition for the projection mappings to be cubic open.

Definition 25 (See [4]). *Let $f : X \to Y$ be a mapping and let $\mathcal{A} \in CS(X)$, $\mathcal{B} \in CS(Y)$.*

(i) *The pre-image of $\mathcal{B}$ under f, denoted by denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{B}), f^{-1}(B) \right\rangle$, is a cubic set in X defined as follows: for each $x \in X$,*

$$f^{-1}(\mathcal{B})(x) = \left\langle f^{-1}(\widetilde{B})(x), f^{-1}(B)(x) \right\rangle = \left\langle \widetilde{B}(f(x)), B(f(x)) \right\rangle.$$

(ii) *The P-image and the R-image of $\mathcal{A}$ under f, denoted by $f^P(\mathcal{A})$ and $f^R(\mathcal{A})$, are cubic sets in Y respectively defined as follows: for each $y \in Y$,*

$$f^P(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x \in f^{-1}(y)} \widetilde{A}(x), \bigvee_{x \in f^{-1}(y)} A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \langle [0,0], 0 \rangle & \text{otherwise,} \end{cases}$$

$$f^R(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x \in f^{-1}(y)} \widetilde{A}(x), \bigwedge_{x \in f^{-1}(y)} A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \langle [0,0], 1 \rangle & \text{otherwise.} \end{cases}$$

It is obvious that $f^P(\mathcal{A}) = \left\langle f(\widetilde{A}), f(A) \right\rangle$ and $f^R(\mathcal{A}) = \left\langle f(\widetilde{A}), f^R(A) \right\rangle$, where $f(\widetilde{A})$ and $f(A)$ denote the image of $\widetilde{A}$ and A under f, respectively, and $f^R(A)$ is a fuzzy set in X defined as the second component of $f^R(\mathcal{A})$ (see [9] for the definition of $f^R(\mathcal{A})$.

Proposition 6. *Let $f : X \to Y$ be a mapping, let $\mathcal{A}$, $\mathcal{A}_1, \mathcal{A}_2 \in CS(X)$, $\mathcal{B}$, $\mathcal{B}_1, \mathcal{B}_2 \in CS(Y)$ and let $(\mathcal{A}_j)_{j\in J} \subset CS(X)$, $(\mathcal{B}_j)_{j\in J} \subset CS(Y)$.*

(1) $f^{-1}(\mathcal{B}^c) = [f^{-1}(\mathcal{B})]^c$.
(2) $f^{-1}(\hat{0}) = \hat{0}$, $f^{-1}(\ddot{0}) = \ddot{0}$, $f^{-1}(\hat{1}) = \hat{1}$, $f^{-1}(\ddot{1}) = \ddot{1}$.
(3) $f^P(\mathcal{A}^c) \sqsupset [f^P(\mathcal{A})]^c$ *and* $f^R(\mathcal{A}^c) \Supset [f^R(\mathcal{A})]^c$.
(4) *If* $\mathcal{B}_1 \sqsubset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \sqsubset f^{-1}(\mathcal{B}_2)$.
(5) *If* $\mathcal{B}_1 \Subset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \Subset f^{-1}(\mathcal{B}_2)$.
(6) *If* $\mathcal{A}_1 \sqsubset \mathcal{A}_2$, *then* $f^P(\mathcal{A}_1) \sqsubset f^P(\mathcal{A}_2)$.
(7) *If* $\mathcal{A}_1 \Subset \mathcal{A}_2$, *then* $f^R(\mathcal{A}_1) \Subset f^R(\mathcal{A}_2)$.
(8) $f^P(f^{-1}(\mathcal{B})) \sqsubset \mathcal{B}$. *In particular, if f is surjective, then* $f^P(f^{-1}(\mathcal{B})) = \mathcal{B}$.
(9) $\mathcal{A} \sqsubset f^{-1}(f^P(\mathcal{A}))$. *In particular, if f is injective, then* $\mathcal{A} = f^{-1}(f^P(\mathcal{A}))$.
(10) $f^R(f^{-1}(\mathcal{B})) \Subset \mathcal{B}$. *In particular, if f is surjective, then* $f^R(f^{-1}(\mathcal{B})) = \mathcal{B}$.
(11) $\mathcal{A} \Subset f^{-1}(f^R(\mathcal{A}))$. *In particular, if f is injective, then* $\mathcal{A} = f^{-1}(f^R(\mathcal{A}))$.
(12) *If* $f^P(\mathcal{A}) \sqsubset \mathcal{B}$, *then* $\mathcal{A} \sqsubset f^{-1}(\mathcal{B})$.
(13) *If* $f^R(\mathcal{A}) \Subset \mathcal{B}$, *then* $\mathcal{A} \Subset f^{-1}(\mathcal{B})$.
(14) *For each* $x_{\tilde{\tilde{a}}} \in C_P(X)$, $f^P(x_{\tilde{\tilde{a}}}) \in C_P(Y)$ *and* $f^P(x_{\tilde{\tilde{a}}}) = [f(x)]_{\tilde{\tilde{a}}}$.
(15) *For each* $x_{\tilde{\tilde{a}}} \in C_P(X)$, *if* $x_{\tilde{\tilde{a}}} q\mathcal{A}$, *then* $f^P(x_{\tilde{\tilde{a}}}) q f^P(\mathcal{A})$.
(16) $f^P(\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J} f^P(\mathcal{A}_j)$.
(17) $f^R(\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J} f^R(\mathcal{A}_j)$.
(18) $f^{-1}(\sqcup_{j\in J}\mathcal{B}_j) = \sqcup_{j\in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cup_{j\in J}\mathcal{B}_j) = \Cup_{j\in J} f^{-1}(\mathcal{B}_j)$.
(19) $f^{-1}(\sqcap_{j\in J}\mathcal{B}_j) = \sqcap_{j\in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cap_{j\in J}\mathcal{B}_j) = \Cap_{j\in J} f^{-1}(\mathcal{B}_j)$.
(20) *If* $g : Y \to Z$ *is a mapping, then* $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$ *for each* $\mathcal{C} \in CS(Z)$, *where* $g \circ f$ *denotes the composition of* g *and* f.

Proof. The results related to the P-image are proved from Definition 25, (Theorem 2, [21]), (Theorem 4.1, [25]) and (Lemma 1.1, [29]).

(3) We prove only the second part. It is clear that $f(\widetilde{A}^c) \supset [f(\widetilde{A})]^c$ by Theorem 2 (ii) in [21]. Then it is sufficient to show that $f^R(A^c) \subset [f^R(A)]^c$. Let $y \in Y$ such that $f^{-1}(y) \neq \emptyset$. Then we have

$$\begin{aligned}[f^R(A^c)](y) &= \textstyle\bigwedge_{x\in f^{-1}(y)} A^c(x) \\ &= \textstyle\bigwedge_{x\in f^{-1}(y)}(1 - A(x)) \\ &= 1 - \textstyle\bigvee_{x\in f^{-1}(y)} A(x) \\ &\leq (1 - \textstyle\bigwedge_{x\in f^{-1}(y)} A(x)) \\ &= (1 - f^R(A)(y)) \\ &= [f^R(A)]^c(y).\end{aligned}$$

Thus, $f^R(A^c) \subset [f^R(A)]^c$. So $f^R(\mathcal{A}^c) \Supset [f^R(\mathcal{A})]^c$.

(10) Let $y \in Y$. Suppose $f^{-1}(y) \neq \emptyset$. Then we have

$$[f^R(f^{-1}(B))](y) = \bigwedge_{x\in f^{-1}(y)} f^{-1}(B)(x) = \bigwedge_{x\in f^{-1}(y)} B(f(x)) = B(y).$$

Thus, $f^R(f^{-1}(B)) = B$. Suppose $f^{-1}(y) = \emptyset$. Then clearly, $[f^R(f^{-1}(B))](y) = 1$. Thus, $f^R(f^{-1}(B)) \supset B$. So, by Definition 25 and Theorem 2 (v) in [21], $f^R(f^{-1}(\mathcal{B})) \Subset \mathcal{B}$. The proof of the second part is easy.

(11) Let $x \in X$. Then we have

$$[f^{-1}(f^R(A))](x) = [f^R(A))](f(x)) = \bigwedge_{y\in f^{-1}(f(x))} A(y) \leq A(x).$$

Thus, $f^{-1}(f^R(A)) \subset A$. So, by Definition 25 and Theorem 2 (vi) in [21], $\mathcal{A} \Subset f^{-1}(f^R(\mathcal{A}))$. The proof of the second part is straightforward.

(13) The proof follows from (11). □

Definition 26. *Let (X, τ), (Y, η) be P-cubic topological spaces and let $f : X \to Y$ be a mapping.*

(i) *f is said to be cubic continuous, if $f^{-1}(\mathcal{V}) \in \tau$ for each $\mathcal{V} \in \eta$.*

(ii) *f is called a cubic homeomorphism, if f is bijective, and f and f^{-1} is cubic continuous. In this case, (X, τ) and (Y, η) are said to be cubic homeomorphic and $\hat{1}$ is called the cubic homeomorphic image of X.*

Remark 10. *From Remark 4 (3), Definitions 25 and 26, it is obvious that*
$f : (X, \tau) \to (Y, \eta)$ is cubic continuous if and only if $f : (X, \tau_{IVF}) \to (Y, \eta_{IVF})$ is interval-valued fuzzy continuous and $f : (X, \tau_F) \to (Y, \eta_F)$ is fuzzy continuous.

Theorem 15. *Let (X, τ), $(Y\eta)$ be two P-cubic topological spaces and let $f : X \to Y$ be a mapping. Then the following are equivalent:*

(1) *f is cubic continuous,*

(2) *$f^{-1}(\mathcal{F}) \in \tau^c$ for each $\mathcal{F} \in \eta^c$,*

(3) *$f^{-1}(\mathcal{S}) \in \tau$ for each member $\mathcal{S}$ of a subbase σ for η,*

(4) *for each $x_{\tilde{\tilde{a}}} \in C_P(X)$ and each $\mathcal{V} \in \mathcal{N}_{P,\eta}(f^P(x_{\tilde{\tilde{a}}}))$, $f^{-1}(\mathcal{V}) \in \mathcal{N}_{P,\tau}(x_{\tilde{\tilde{a}}})$,*

(5) *for each $x_{\tilde{\tilde{a}}} \in C_P(X)$ and each $\mathcal{V} \in \mathcal{N}_{P,\eta}(f^P(x_{\tilde{\tilde{a}}}))$, there is $\mathcal{U} \in \mathcal{N}_{P,\tau}(x_{\tilde{\tilde{a}}})$ such that $f^P(x_{\tilde{\tilde{a}}}) \in f^P(\mathcal{U}) \sqsubset \mathcal{V}$,*

(6) *for each $x_{\tilde{\tilde{a}}} \in C_P(X)$ and each $\mathcal{V} \in \mathcal{N}_{Q,\eta}(f^P(x_{\tilde{\tilde{a}}}))$, $f^{-1}(\mathcal{V}) \in \mathcal{N}_{Q,\tau}(x_{\tilde{\tilde{a}}})$,*

(7) *for each $x_{\tilde{\tilde{a}}} \in C_P(X)$ and each $\mathcal{V} \in \mathcal{N}_{Q,\eta}(f^P(x_{\tilde{\tilde{a}}}))$, there is $\mathcal{U} \in \mathcal{N}_{Q,\tau}(x_{\tilde{\tilde{a}}})$ such that $f^P(x_{\tilde{\tilde{a}}}) q f^P(\mathcal{U}) \sqsubset \mathcal{V}$,*

(8) *$f^P(cl_{P,\tau}(\mathcal{A}) \sqsubset cl_{P,\eta}(f^P(\mathcal{A}))$ for each $\mathcal{A} \in CS(X)$,*

(9) *$cl_{P,\tau}(f^{-1}(\mathcal{B})) \sqsubset f^{-1}(cl_{P,\eta}(\mathcal{B}))$ for each $\mathcal{B} \in CS(Y)$.*

Proof. By using Theorem 13 in [21] and Theorem 1.1 in [29], they can be proved. □

Proposition 7. *Let (X, τ), (Y, η), (Z, γ) be P-cubic topological spaces and let $f : X \to Y$, $g : Y \to Z$ be mappings.*

(1) *The identity mapping $id : (X, \tau) \to (X, \tau)$ is cubic continuous.*

(2) *If f and g are cubic continuous, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is cubic continuous.*

Proof. The proofs are obvious from Definition 26 and Proposition 6 (17). □

Remark 11. *Let $\mathbf{PCT_{op}}$ be the class of all P-cubic topological spaces and cubic continuous mappings. Then, from Proposition 7, we can easily see that $\mathbf{PCT_{op}}$ forms a concrete category.*

Definition 27. *Let τ_1, $\tau_2 \in PCT(X)$. Then τ_1 is said to be coarser (weaker, smaller) than τ_2 or τ_2 is said to be finer (stronger, larger) than τ_1, if $\tau_1 \subset \tau_2$.*

Proposition 8. *Let $(\tau_j)_{j \in J} \subset PCT(X)$ [resp. $RCT(X)$]. Then $\bigcap_{j \in J} \tau_j \in PCT(X)$ [resp. $RCT(X)$].*

Proof. The proof is straightforward from Definition 13. □

Proposition 9. *Let $(\tau_j)_{j \in J} \subset PCT(X)$* [resp. *$RCT(X)$*]. *Then $(\tau_j)_{j \in J}$ forms a meet complete lattice with respect to set inclusion relation with the smallest element $\hat{0}$ [resp. $\ddot{0}$] and the largest element $\hat{1}$ [resp. $\ddot{1}$].*

Proof. The proof is straightforward from Definition 27 and Proposition 8. □

Definition 28. *Let (X, τ) and (Y, η) be P-cubic topological spaces and let $f : X \to Y$ be a mapping. Then f is said to be cubic open [resp. cubic closed], if $f(\mathcal{U}) \in \eta$ for each $\mathcal{U} \in \tau$ [resp. $f(\mathcal{F}) \in \eta^c$ for each $\mathcal{F} \in \tau^c$].*

Remark 12. *From Remark 4 (3), Definitions 25 and 26, it is obvious that*

$f : (X, \tau) \to (Y, \eta)$ is cubic open [resp. closed] if and only if $f : (X, \tau_{IVF}) \to (Y, \eta_{IVF})$ is interval-valued open [resp. closed] and $f : (X, \tau_F) \to (Y, \eta_F)$ is fuzzy open [resp. closed].

Proposition 10. *Let (X, τ), (Y, η) and (Z, γ) be P-cubic topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be mappings. If f and g are cubic open [res. closed], so is $g \circ f$.*

Proof. The proof is straightforward from Definition 28. □

The following is an immediate consequence of Definitions 26 (ii) and 28.

Theorem 16. *Let (X, τ) and (Y, η) be P-cubic topological spaces and let $f : X \to Y$ be a mapping. Then f is a cubic homeomorphism if and only if f is surjective, cubic continuous and cubic open*

Proposition 11. *Let (X, τ) be a P-cubic topological spaces, let Y be a set and let $f : X \to Y$ be a mapping. Let $f(\tau) = \{\mathcal{V} \in CS(Y) : f^{-1}(\mathcal{V}) \in \tau\}$. Then*

(1) $f(\tau) \in PCT(Y)$,
(2) $f : (X, \tau) \to (Y, f(\tau))$ is cubic continuous,
(3) if $\eta \in PCT(Y)$ such that $f : (X, \tau) \to (Y, \eta)$ is cubic continuous, then $f(\tau)$ is finer than η.

In this case, $f(\tau)$ is called the *final P-cubic topology on Y*.

Proof. (1) It can be easily proved from Proposition 6 (2), (15) and (16).

(2) The proof is straightforward from the definition of $f(\tau)$ and Definition 26.

(3) The proof is straightforward from the definition of $f(\tau)$ and Definitions 26 and 27. □

Definition 29. *Let (X, τ) be a P-cubic topological space, let Y be a set and let $f : X \to Y$ be a surjection. Then $f(\tau)$ is called the P-cubic quotient topology on Y induced by f. The pair $(Y, f(\tau))$ is called a P-cubic quotient space of X and f is called a P-cubic quotient mapping.*

By Proposition 11, the P-cubic quotient mapping f is not only cubic continuous but $f(\tau)$ is the finest P-cubic topology on Y for which f is cubic continuous. We can easily see that if $(Y, f(\tau))$ is a P-cubic quotient space with the P-cubic quotient mapping f, then a cubic set $\mathcal{F}$ in Y is P-cubic closed if and only if $f(\mathcal{F}) \in \tau^c$.

When (X, τ) and (Y, η) are P-cubic topological spaces, and $f : X \to Y$ is surjection, the following gives conditions on f make η equal to the P-cubic quotient topology $f(\tau)$ on Y induced by f.

Proposition 12. *Let (X, τ) and (Y, η) be P-cubic topological spaces, let $f : (X, \tau) \to (Y, \eta)$ be a cubic continuous surjection. If f is cubic open or cubic closed, then $\eta = (\tau)$ and thus f is a P-cubic quotient mapping.*

Proof. Suppose f is cubic open. By Proposition 11, $f(\tau)$ is the finest P-cubic topology on Y for which $f : (X, \tau) \to (Y, f(\tau))$ is cubic continuous. Then $\eta \subset f(\tau)$. Let $\mathcal{V} \in f(\tau)$. Then clearly, $f^{-1}(\mathcal{V}) \in \tau$. Thus, by the hypothesis, $f(f^{-1}(\mathcal{V})) \in \eta$. Since f is surjective, by Proposition 6 (8), $f(f^{-1}(\mathcal{V})) = \mathcal{V}$. So $\mathcal{V} \in \eta$, i.e., $f(\tau) \subset \eta$. Hence, $f(\tau) = \eta$.

Suppose f is cubic close. Then we can prove similarly that $f(\tau) = \eta$. □

Proposition 13. *The composition of two P-cubic quotient mappings is a P-cubic quotient mapping.*

Proof. The proof is easy from Definition 29. □

The following is a basic result for P-cubic quotient spaces.

Theorem 17. *Let (X, τ), (Z, γ) be P-cubic topological spaces, let Y be a set and let $f : X \to Y$ be sujective. Then $g : (Y, f(\tau)) \to (Z, \gamma)$ is cubic continuous if and only if $g \circ f : (X, \tau) \to (Z, \gamma)$ is cubic continuous.*

Proof. The proof is straightforward. □

Theorem 18. *Let (X, τ), (Y, η) be P-cubic topological spaces and let $p : X \to Y$ be a cubic continuous sujection. Then p is a P-cubic quotient mapping if and only if for each P-cubic topological space (Z, γ) and each mapping $g : Y \to Z$, the cubic continuity of $g \circ f$ implies that of g.*

Proof. Suppose p is a P-cubic quotient mapping. Then, from Theorem 17, we obtain the desired result.

Conversely, suppose the necessary condition holds. Let $p^{'}$ denote p considered as a mapping from (X, τ) into $(Y, f(\tau))$ and let $id : (Y, \eta) \to (Y, f(\tau))$ be the identity mapping. Then clearly, $p^{'}$ is cubic continuous and $id \circ p = p^{'}$. Thus, by the hypothesis, id is cubic continuous. Since $id^{-1} \circ p^{'} = p$ is cubic continuous and $p^{'}$ is a P-cubic quotient mapping, by the hypothesis, id^{-1} is cubic continuous. So id^{-1} is a cubic homeomorphism. Hence, p is a P-cubic quotient mapping. □

Theorem 19. *Let (X, τ), (Y, η), (Z, γ) be P-cubic topological spaces, let $p : X \to Y$ be a P-cubic quotient mapping and let $g : Y \to Z$ be a surjection. Then $g \circ p$ is a P-cubic quotient mapping if and only if g is a P-cubic quotient mapping.*

Proof. The proof is straightforward from Definition 29 and Theorem 18. □

The following is the dual of Proposition 11.

Proposition 14. *Let X be a set, let (Y, η) be a P-cubic topological space and let $f : X \to Y$ be a mapping. Let $f^{-1}(\eta) = \{f^{-1}(\mathcal{V}) \in CS(X) : \mathcal{V} \in \eta\}$. Then*

(1) $f^{-1}(\eta) \in PCT(X)$,

(2) $f : (X, f^{-1}(\eta)) \to (Y, \eta)$ *is cubic continuous,*

(3) *if $\zeta \in PCT(X)$ such that $f : (X, \zeta) \to (Y, \eta)$ is cubic continuous, then $f^{-1}(\eta)$ is coarser than ζ.*

In this case, $f^{-1}(\eta)$ is called the *initial P-cubic topology* on X.

Proof. The proofs are similar to Proposition 11. □

The following is a generalization of Proposition 14.

Proposition 15. *Let X be a set, let $((Y_j, \eta_j))_{j \in J}$ be a family of P-cubic topological spaces, $(f_j : X \to (Y_j, \eta_j))_{j \in J}$ be a family of mappings and let $\sigma = \{f_j^{-1}(\mathcal{V}) \in CS(X) : \mathcal{V} \in \eta_j,\ j \in J\}$. Then there is a coarsest P-cubic topology τ on X such that $f_j : (X, \tau) \to (Y_j, \eta_j)$ is cubic continuous for each $j \in J$.*

In this case, τ is called the initial P-cubic topology on X induced by $(f_j)_{j \in J}$. In fact, τ is a P-cubic topology on X having σ as its P-cubic subbase.

Proof. Since η_j is a P-cubic topology on Y_j or each $j \in J$, $\hat{1},\ \ddot{0}, \ddot{1} \in \eta_j$. Then by Proposition 6 (2), $\hat{1},\ \ddot{0}, \ddot{1} \in \sigma$. Thus, by Theorem 10, there is P-cubic topology τ on X such that σ is a P-cubic subbase for τ. We can easily prove that τ is the coarsest P-cubic topology on X such that $f_j : (X, \tau) \to (Y_j, \eta_j)$ is cubic continuous for each $j \in J$. □

Now we will discuss the P-cubic product space of an arbitrary family of P-cubic topological spaces. Let $(X_j)_{j \in J}$ be a family of sets and let $k \in J$. Consider the mapping

$$\pi_k : \Pi_{j \in J} X_j \to X_k$$

defined by $\pi_k(\langle x_j \rangle) = x_k$ for each $\langle x_j \rangle \in \Pi_{j\in J}X_j$. Then π_k is called the k-th *projection mapping*.

By Proposition 15, we have the following.

Corollary 1. *Let $((X_j, \tau_j))_{j\in J}$ be a family of P-cubic topological spaces, let $X = \Pi_{j\in J}X_j$, let $(pi_j : X \to (X_j, \tau_j))_{j\in J}$ be a family of projection mappings and let $\sigma = \{pi_j^{-1}(\mathcal{V}) \in CS(X) : \mathcal{V} \in \tau_j,\ j \in J\}$. Then there is a coarsest P-cubic topology τ on X such that $pi_k : (X, \tau) \to (X_k, \tau_k)$ is cubic continuous for each $k \in J$.*

In this case, τ is called the cubic product topology on X of $(\tau_j)_{j\in J}$ and denoted by $\Pi_{j\in J}\tau_j$. From Remark 4 (3), it is clear that

$$\Pi_{j\in J}\tau_j = \langle \Pi_{j\in J}(\tau_j)_{IVF}, \Pi_{j\in J}(\tau_j)_F \rangle,$$

where $\Pi_{j\in J}(\tau_j)_{IVF}$ is the interval-valued fuzzy product topology (see [21]) and $\Pi_{j\in J}(\tau_j)_F$ is the fuzzy product topology (see [30]).

Theorem 20. *Let $((X_j, \tau_j))_{j\in J}$ be a family of P-cubic topological spaces, let τ be the P-cubic product topology on $X = \Pi_{j\in J}X_j$, let (Y, η) be a P-cubic topological space and let $f : Y \to X$ be a mapping. Then $f : (Y, \eta) \to (X, \tau)$ is cubic continuous if and only if $\pi_j \circ f : (Y, \eta) \to (X_j, \tau_j)$ is cubic continuous for each $j \in J$.*

Proof. The proof is clear from Theorems 16 in [21] and 3.1 (iii) in [30]. □

Remark 13. *(1) From Remark 11 and Theorem 20, it is obvious that the category* $\mathbf{PCT_{op}}$ *has an initial structures.*

(2) The projection mappings are not cubic open in general (see Example 2).

Example 2. *Let $X_1 = \{a, b\} = X_2$ and let τ_1, τ_2 be the P-cubic topology on X_1 and X_2 respectively given by:*

$$\tau_1 = \{\hat{0}, \hat{1}, \ddot{0}, \ddot{1}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\},\ \tau_2 = \{\hat{0}, \hat{1}, \ddot{0}, \ddot{1}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\},$$

where $\mathcal{A}_1(a) = \langle [0.4, 0.4], 0.3 \rangle,\ \mathcal{A}_1(b) = \langle [0.6, 0.6], 0.7 \rangle,$
$\mathcal{A}_2(a) = \langle [0, 0], 0.3 \rangle,\ \mathcal{A}_2(b) = \langle [0, 0], 0.7 \rangle,$
$\mathcal{A}_3(a) = \langle [04, 0.4], 1 \rangle,\ \mathcal{A}_2(b) = \langle [0.6, 0.6], 1 \rangle,$
$\mathcal{A}_4(a) = \langle [0.4, 0.4], 0 \rangle,\ \mathcal{A}_2(b) = \langle [0.6, 0.6], 0 \rangle,$
$\mathcal{A}_5(a) = \langle [1, 1], 0.3 \rangle,\ \mathcal{A}_2(b) = \langle [1, 1], 0.7 \rangle,$
$\mathcal{B}_1(a) = \langle [0.8, 0.8], 0.6 \rangle,\ \mathcal{B}_1(b) = \langle [0.3, 0.3], 0.2 \rangle,$
$\mathcal{B}_2(a) = \langle [0, 0], 0.6 \rangle,\ \mathcal{B}_2(b) = \langle [0, 0], 0.2 \rangle,$
$\mathcal{B}_3(a) = \langle [0.8, 0.8], 1 \rangle,\ \mathcal{B}_3(b) = \langle [0.3, 0.3], 1 \rangle,$
$\mathcal{B}_4(a) = \langle [0.8, 0.8], 0 \rangle,\ \mathcal{B}_4(b) = \langle [0.3, 0.3], 0 \rangle,$
$\mathcal{B}_5(a) = \langle [1, 1], 0.6 \rangle,\ \mathcal{B}_5(b) = \langle [1, 1], 0.2 \rangle.$

Let $\sigma = \{\pi_1^{-1}(\mathcal{U}) \in CS(X_1) : \mathcal{U} \in \tau_1\} \cup \{\pi_2^{-1}(\mathcal{V}) \in CS(X_1) : \mathcal{V} \in \tau_2\}$. Then clearly, σ is the P-cubic subbase for the P-cubic product topology $\tau_1 \times \tau_2$ on $X_1 \times X_2$. Let $\mathcal{U} = \pi_1^{-1}(\mathcal{A}_1)$, $\mathcal{V} = \pi_2^{-1}(\mathcal{B}_1)$. Then it is obvious that $\mathcal{U}$, $\mathcal{V} \in \tau_1 \times \tau_2$ but $\pi_1(\mathcal{V}) \notin \tau_1$ and $\pi_2(\mathcal{U}) \notin \tau_2$. Thus, π_1 and π_2 are not P-cubic open.

We give the sufficient condition for the projection mappings to be P-cubic open.

Proposition 16. *Let $((X_j, \tau_j))_{j\in J}$ be a family of P-cubic topological spaces and let $\tau = \Pi_{j\in J}\tau_j$ be the cubic product topology on the product set $X = \Pi_{j\in J}X_j$. If $\tau_j \in PCT_L(X)$ for each $j \in J$, then the projection mapping $\pi_j : (X, \tau) \to (X_j, \tau_j)$ is P-cubic open.*

Proof. From Remark 4 (3), Theorems 17 in [21] and 2.2 (2) in [23], the proof is easy. □

Definition 30. *Let (X, τ) be a P-cubic topological space and let δ be a family of cubic sets in X. Then δ is called a P-cubic cover of X, if $\hat{1} \sqsubset \sqcup_{\mathcal{U}\in\delta}\mathcal{U}$. It is called a P-cubic open cover of X, if it is a P-cubic cover of X and $\mathcal{U} \in \tau$ for each $\mathcal{U} \in \delta$. A P-cubic subcover of δ is a subfamily of δ which is also a P-cubic cover.*

Definition 31. *Let (X, τ) be a P-cubic topological space. Then X is said to be P-cubic compact, if each P-cubic open cover of X has a finite P-cubic subcover.*

The following is an immediate consequence of Remark 4 (3), Theorems 19 in [21] and 3.4 in [30].

Proposition 17. *Let (X_j, τ_j), $j = 1, 2, \cdots n$, be a family of P-cubic compact space and let $\tau = \Pi_{j=1}^{n}\tau_j$ be the P-cubic product topology on $\tau = \Pi_{j=1}^{n}X_j$. Then (X, τ) is P-cubic compact.*

When the numbers of spaces are infinite, the above proposition is not true in general (see Example 3).

Example 3. *Let $X_n = [0,1]$ ($n \in \mathbb{N}$) and for each $n \in \mathbb{N}$, let*

$$\tau_n = \{\hat{0}, \hat{1}, \ddot{0}, \ddot{1}, \mathcal{U}_n, \langle \widetilde{\mathbf{0}}, U_n \rangle, \left\langle \widetilde{U}_n, \mathbf{1} \right\rangle, \left\langle \widetilde{U}_n, \mathbf{0} \right\rangle, \left\langle \widetilde{\mathbf{1}}, U_n \right\rangle\},$$

where $\mathcal{U}_n(x) = \left\langle \widetilde{U}_n, U_n \right\rangle(x) = \langle [\frac{n}{n+1}, \frac{n}{n+1}], \frac{n}{n+1} \rangle$ for each $x \in [0,1]$. Then clearly, (X_n, τ_n) is a P-cubic compact space for each $n \in \mathbb{N}$. Now let $X = \Pi_{n=1}^{\infty}X_n$, $\tau = \Pi_{n=1}^{\infty}\tau_n$ and let $x \in X$. Then we have

$$[\sqcup_{n=1}^{\infty}\pi_n^{-1}(\mathcal{U}_n)](x) = \left\langle [\bigvee_{n=1}^{\infty} \frac{n}{n+1}, \bigvee_{n=1}^{\infty} \frac{n}{n+1}], \bigvee_{n=1}^{\infty} \frac{n}{n+1} \right\rangle = \langle [1,1], 1 \rangle.$$

Thus, $[\sqcup_{n=1}^{\infty}\pi_n^{-1}(\mathcal{U}_n)] = \hat{1}$. So $(\pi_n^{-1}(\mathcal{U}_n))_{n=1}^{\infty}$ is a P-cubic open cover of X. On the other hand, let J be a finite subset of $\mathbb{N}$ and let $x \in X$. Then clearly, $[\sqcup_{n\in J}\pi_n^{-1}(\mathcal{U}_n)](x) < \langle [1,1], 1 \rangle$. Thus, $(\pi_n^{-1}(\mathcal{U}_n))_{n=1}^{\infty}$ has no finite P-cubic subcover. So (X, τ) is not P-cubic compact.

7. Conclusions

In this paper, we have presented a detailed theoretical study of operations on cubic sets. Mainly, this study shows how the concept of two types of the cubic neighborhood is and proves that the cubic neighborhood system is an extension of the classical neighborhood system (see Theorems 4.16, 4.17, 4.19 and 4.20). Finally, the theory has been applied to cubic continuous mapping and has discussed the sufficient condition for the projection mappings to be cubic open, by proving its various properties. In order to deepen our work, cubic compact spaces were defined for the first time and an application of cubic compact spaces was developed. A natural extension of this still valid theory is to develop separation axioms and connectedness in cubic topological spaces. An important application of this type of transformation is to analyze cubic sets in category theory, graph theory, decision problem, etc. to find the best transformation that fits the experimental data as a model.

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