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Approximating the Moments of Generalized Gaussian Distributions via Bell's Polynomials †

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† Dedicated to Prof. Hari M. Srivastava on the Occasion of His 82th Birthday.

Abstract: Bell's polynomials are used in many different fields of mathematics, ranging from number theory to operator theory. This paper shows a relevant application in probability theory aimed at computing the moments of generalized Gaussian distributions. To this end, a table containing the first values of the complete Bell's polynomials is provided. Furthermore, a dedicated code for approximating the moments of the general distributions in terms of complete Bell's polynomials is detailed. Several test cases concerning different nested functions are discussed.

Keywords: moments of a continuous distribution; Bell's polynomials; generalized Gaussian distributions

MSC: 05A10; 11B83; 60E05; 62E17

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1. Introduction

Bell's polynomials were initially introduced to represent successive derivatives of composite functions [1]. However, as it has been pointed out in [2], such polynomials, being related to partitions of integers [3], find immediate applications in combinatorial mathematics [4]. They also intervene in formulas for representing symmetric functions of the zeros of analytic functions, thus generalizing the classical Newton–Girard formulas. Using these results, the reduction formulas of the orthogonal invariants of a strictly positive compact operator were expressed. In this way, Robert's formulas [5] and other expressions related to those can be derived.

Generalized classes of Bell's polynomials have been proposed in the literature, e.g., by Fujiwara [6], Kim [7], and Rai, Singh [8]. Further generalizations, including the multi-dimensional case, can be found in papers by Bernardini, Natalini, Ricci and Natalini, and Ricci (see [9], and the references therein).

There are many areas in which the adoption of Bell's polynomials has allowed useful approximations to be obtained in applied mathematics, such as those concerning the Laplace transform of composite analytic functions (see [10], and the references therein).

In this paper, we discuss another application to the probability theory [11] by deriving some formulas useful to calculate the moments of generalized Gaussian distributions. To this end, judicious use is made of the extensions of Bell's polynomials presented in previous papers.

This article is organized as follows: In Section 2, Bell's polynomials and their main properties are briefly recalled. In Section 3, Bell's polynomials are used to approximate the primitives of composite exponential functions. Finally, after recalling, in Section 4, the generalization of Bell's polynomials to the case of multi-nested functions, the moments of generalized Gaussian distributions are approximated in Section 5. Some examples illustrated using version 13.2 of the computer algebra software Mathematic© are shown in Section 6.

2. Recalling the Bell'S Polynomials

Consider the composite function $\Phi(t) := f(g(t))$, where $x = g(t)$ and $y = f(x)$ are differentiable functions (up to a sufficiently large order), defined in suitable intervals of the real axis so that $\Phi(t)$ can be differentiated n times with respect to t by using the chain rule. Here, and in what follows, we use the following notation:

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then, the n-th derivative of $\Phi(t)$ is represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_n)\, f_k, \tag{1}$$

where Y_n denotes the general Bell's polynomial, and the coefficients $B_{n,k}$, for all $k = 1, \ldots, n$, are homogeneous polynomials of the variables $g_1, g_2, \ldots, g_n$, of degree k and *isobaric* of weight n (i.e., they are a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ having the same degree $k_1 + 2k_2 + \ldots + nk_n = n$).

Bell's polynomials satisfy the recurrence relation

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases} \tag{2}$$

and they can be expressed explicitly using the Faà di Bruno's formula [12,13]

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \ldots r_n!} f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n}, \tag{3}$$

where the sum runs over all the partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition [2].

The coefficients $B_{n,k}$ in Equation (1) depend only on the values $g_1, g_2, \ldots, g_{n-k+1}$, and satisfy the recursion

$$B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, g_{n-k-h+1})\, g_{h+1}\,. \tag{4}$$

In the particular case where $f(x) = e^x$, that is equivalent to considering the composite function $e^{g(t)}$, and upon assuming that $g(0) = 0$, we obtain the following simplified expression for the right-hand side of (1):

$$\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k = \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1}) = B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_n)\,, \tag{5}$$

where the B_n are the *complete Bell's polynomials*. It results in $B_0(g_0) := f(g(0))$, and the first few values of B_n, for $n = 1, 2, \ldots, 5$ are given by

$$B_1(g_1) = g_1, \quad B_2(g_1, g_2) = g_1^2 + g_2, \quad B_3(g_1, g_2, g_3) = g_1^3 + 3g_1 g_2 + g_3\,. \tag{6}$$

Further values are reported in Appendix A.

The values of the complete Bell's polynomials for particular choices of the parameter can be found in [14].

The complete Bell's polynomials satisfy the identity (see e.g., [15])

$$B_{n+1}(g_1,\ldots,g_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(g_1,\ldots,g_{n-k})\, g_{k+1}. \tag{7}$$

3. Primitives of Exponential Functions

Let $f(g(t))$ be a composite function that is analytic in a neighborhood of the origin. The relevant Taylor's expansion is given by

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n = D_t^n[f(g(t))]_{t=0}, \tag{8}$$

where

$$\begin{aligned} a_0 &= f(\overset{\circ}{g}_0), \\ a_n &= D_t^n[f(g(t))]_{t=0} = \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k, \quad (n \geq 1), \end{aligned} \tag{9}$$

and

$$\overset{\circ}{f}_k := D_x^k f(x)|_{x=g(0)}, \qquad \overset{\circ}{g}_h := D_t^h g(t)|_{t=0}. \tag{10}$$

Then, the primitive of the composite function $f(g(t))$ writes

$$\int_0^x f(g(t))dt = xf(g(0)) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k \right) \frac{x^{n+1}}{(n+1)!}, \tag{11}$$

Remark 1. *Note that here and in what follows we avoid adding a constant to indefinite integrals, based on the assumption that the relevant primitives vanish at the origin.*

In the case of exponential functions, Equation (11) becomes

$$\int_0^x \exp(g(t))dt = \sum_{n=0}^{\infty} B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_n)\, \frac{x^{n+1}}{(n+1)!}, \tag{12}$$

where we have assumed, for convenience, $B_0 := 1$.

Then, in particular, it results

$$\int_0^x \exp(-t^2)dt = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} = \sum_{n=0}^{\infty} B_n(0,-2,0,\ldots,0)\, \frac{x^{n+1}}{(n+1)!}. \tag{13}$$

As a consequence, we find

$$B_{2k}(0,-2,0,\ldots,0) = (-1)^k, \quad B_{2k+1}(0,-2,0,\ldots,0) = 0, \quad (k \geq 0). \tag{14}$$

4. An Extension of Bell's Polynomials

We consider the second-order Bell's polynomials, $Y_n^{[2]}(f_1, g_1, h_1; f_1, g_1, h_1; \ldots; f_n, g_n, h_n)$, defined by the n-th derivative of the composite function $\Phi(t) := f[g(h(t))]$.

Consider the functions $x = h(t)$, $z = g(x)$, $y = f(z)$, and suppose that $h(t)$, $g(x)$, and $f(z)$ are n times differentiable with respect to their variables, so that the composite function $\Phi(t) := f[g(h(t))]$ can be differentiated n times with respect to t, by using the chain rule.

Similarly to the previous section, we use the following notation:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_y^h f(y)|_{y=g(x)}, \quad g_k := D_x^k g(x)|_{x=h(t)}, \quad h_r := D_t^r h(t).$$

Then, the n-th derivative of $\Phi(t)$ can be represented by

$$\Phi_n = Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \ldots; f_n, g_n, h_n) = Y_n^{[2]}([f, g, h]_n), \tag{15}$$

where $Y_n^{[2]}$ denotes the general second-order Bell's polynomial.

The first few polynomials are given by

$$\begin{aligned} &Y_1^{[2]}([f, g, h]_1) = f_1 g_1 h_1; \\ &Y_2^{[2]}([f, g, h]_2) = f_1 g_1 h_2 + f_1 g_2 h_1^2 + f_2 g_1^2 h_1^2; \\ &Y_3^{[2]}([f, g, h]_3) = f_1 g_1 h_3 + f_1 g_3 h_1^3 + 3 f_1 g_2 h_1 h_2 + 3 f_2 g_1 g_2 h_1^3 + f_3 g_1^3 h_1^3. \end{aligned} \tag{16}$$

A more extended table can be found in [10].

The connection between $Y_n^{[2]}$ and the ordinary Bell's polynomials is discussed below. In [16], the following theorem has been proven.

Theorem 1. *For every integer n, the polynomials $Y_n^{[2]}$ are represented in terms of the ordinary Bell's polynomials by the following equation:*

$$\begin{aligned} &Y_n^{[2]}(f_1, g_1, h_1; \ldots; f_n, g_n, h_n) = \\ &= Y_n(f_1, Y_1(g_1, h_1); f_2, Y_2(g_1, h_1; g_2, h_2); \ldots; f_n, Y_n(g_1, h_1; g_2, h_2; \ldots; g_n, h_n)) \end{aligned} \tag{17}$$

From here, it has been shown that the second-order Bell's polynomials can be computed recursively, avoiding the use of computationally cumbersome expressions such as the Faà di Bruno formula, on the basis of the following theorem.

Theorem 2. *The second-order Bell's polynomials verify the recursion*

$$\begin{aligned} &Y_0^{[2]} = f_1; \\ &Y_{n+1}^{[2]}(f_1, g_1, h_1; \ldots; f_{n+1}, g_{n+1}, h_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}^{[2]}(f_2, g_1, h_1; f_3, g_2, h_2; \ldots \\ &\qquad \ldots; f_{n-k+1}, g_{n-k}, h_{n-k}) Y_{k+1}(g_1, h_1; \ldots; g_{k+1}, h_{k+1}). \end{aligned} \tag{18}$$

The proof of this result is provided in [16].

Primitives of Nested Functions

Considering the second-order nested functions of the type $f[g(h(t))]$, the preceding Equation (11) writes

$$\int_0^x f[g(h(t))]dt = x f[g(h(0))] + \sum_{n=1}^{\infty} Y_n^{[2]}(\mathring{f}_1, \mathring{g}_1, \mathring{h}_1; \mathring{f}_2, \mathring{g}_2, \mathring{h}_2; \ldots; \mathring{f}_n, \mathring{g}_n, \mathring{h}_n) \, \frac{x^{n+1}}{(n+1)!}, \tag{19}$$

and in the case of a nested exponential function, with $g(h(0)) = 0$,

$$\int_0^x \exp[g(h(t))]dt = x + \sum_{n=1}^{\infty} Y_n^{[2]}(1, \mathring{g}_1, \mathring{h}_1; 1, \mathring{g}_2, \mathring{h}_2; \ldots; 1, \mathring{g}_n, \mathring{h}_n) \, \frac{x^{n+1}}{(n+1)!}. \tag{20}$$

5. Moments of Generalized Gaussian Distributions

Consider the analytic monotonically increasing function $h(t)$ and consider the nested exponential function $\exp[-(h(t))^2]$.

The normalization constant of the generalized Gaussian distribution writes

$$A = \int_{-\infty}^{+\infty} \exp[-(h(t))^2]dt \,, \tag{21}$$

the mean value is

$$\mu = \mu_1 = \frac{1}{A}\int_{-\infty}^{+\infty} t \, \exp[-(h(t))^2]dt \,, \tag{22}$$

and the second moment

$$\mu_2 = \frac{1}{A}\int_{-\infty}^{+\infty} t^2 \, \exp[-(h(t))^2]dt \,, \tag{23}$$

so that we find the variance

$$\sigma^2 = \mu_2 - \mu^2 \,. \tag{24}$$

Note that, thanks to the rapid decay of the exponential function, the integrals in (21)–(23) can be estimated with a pretty negligible numerical error using a suitable Gaussian quadrature rule on a bounded interval of the real axis.

Then, assuming $y = f(z) = \exp(z)$; $z = g(x) = -x^2$; $x = h(t)$, the nth moment is given by

$$\mu_n = \frac{1}{A}\int_{-\infty}^{+\infty} t^n \, \exp[-(h(t))^2]dt \simeq \frac{1}{A}\int_{\mu-3\sigma}^{\mu+3\sigma} t^n \, \exp[-(h(t))^2]dt \,, \tag{25}$$

and using the second order Bell's polynomials (17), and assuming $Y_0^{[2]} := 1$, we find

$$\begin{aligned} \mu_n \simeq \frac{1}{A} \sum_{k=0}^{\infty} Y_k^{[2]}((1,0,\overset{\circ}{h}_1;1,-2,\overset{\circ}{h}_2;1,0,\overset{\circ}{h}_3;\ldots;\ldots;1,0,\overset{\circ}{h}_k)) \cdot \\ \cdot \frac{(\mu+3\sigma)^{n+k+1} - (\mu-3\sigma)^{n+k+1}}{(n+k+1)\,k!} \,. \end{aligned} \tag{26}$$

6. Particular Examples

6.1. Example 1

Consider the generalized Gaussian distribution defined by the function

$$F(t) = \exp[-(\sinh(t))^2] \,. \tag{27}$$

We find $\mu = \mu_1 = 0$ and $\sigma^2 = 0.55296$.

Its graph is shown in Figure 1.

The percentage deviation between $F(t)$ and the relevant expansion $\tilde{F}(t)$ in terms of Bell's polynomials is shown in Figure 2. As can be seen, $F(t)$ is nicely approximated by $\tilde{F}(t)$ in the domain where $F(t)$ does not assume negligibly small values.

The table containing the first few moments of the considered function $F(t)$, as computed using the Gauss–Kronrod quadrature rule (G-KQR), μ_k, and the proposed Bell's method, $\tilde{\mu}_k$, is reported in Figure 3.

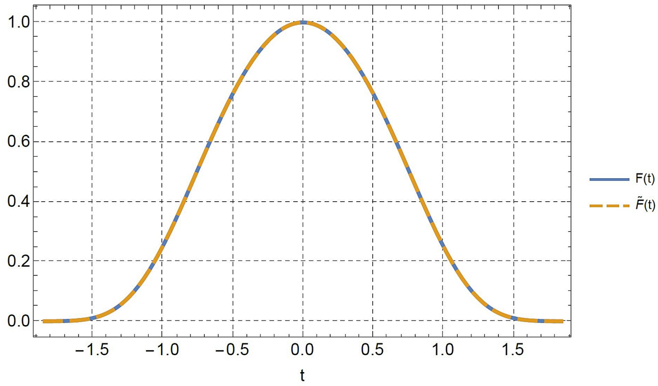

Figure 1. Graph of the function $F(t) = \exp[-(\sinh(t))^2]$.

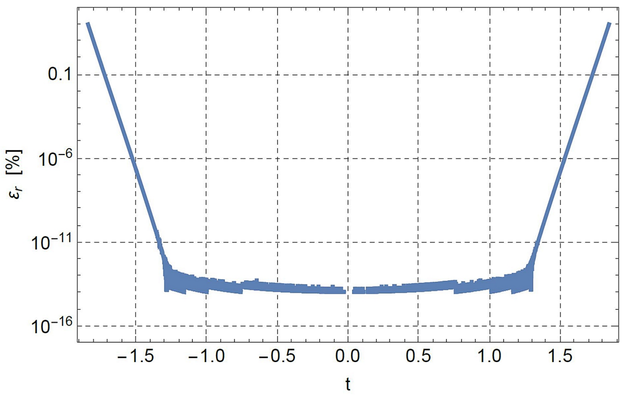

Figure 2. Relative percentage error of the Bell's approximation of $F(t)$ in (27).

μ[1] = 0	μ̃[1] = 0.
μ[2] = 0.305765	μ̃[2] = 0.30574
μ[3] = 0	μ̃[3] = 0.
μ[4] = 0.23272	μ̃[4] = 0.232633
μ[5] = 0	μ̃[5] = 0.
μ[6] = 0.256353	μ̃[6] = 0.256048
μ[7] = 0	μ̃[7] = 0.
μ[8] = 0.352805	μ̃[8] = 0.351738
μ[9] = 0	μ̃[9] = 0.
μ[10] = 0.567349	μ̃[10] = 0.56361

Figure 3. Moments of $F(t)$ in (27) as computed using the G-KQR (**left**) and the proposed Bell's polynomial approximation (**right**) methods.

6.2. Example 2

Consider the generalized Gaussian distribution defined by the function

$$F(t) = \exp[-(t + t^3/4\sqrt{3} + t^5/240)^2]\,. \tag{28}$$

We find $\mu = \mu_1 = 0$ and $\sigma^2 = 0.566431$.
Its graph is shown in Figure 4.

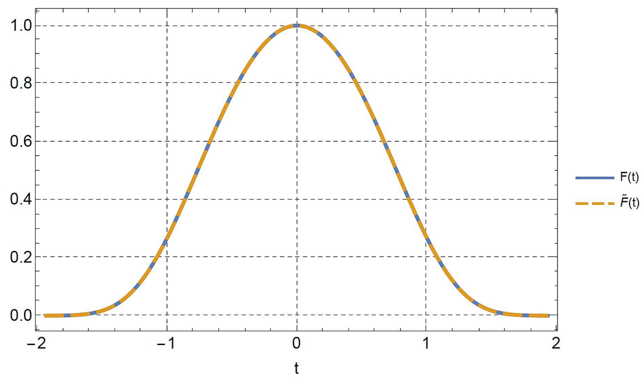

Figure 4. Graph of the function $F(t) = \exp[-(t + t^3/4\sqrt{3} + t^5/240)^2]$.

The percentage deviation between $F(t)$ and the relevant expansion $\tilde{F}(t)$ in terms of Bell's polynomials is shown in Figure 5. As can be seen, $F(t)$ is nicely approximated by $\tilde{F}(t)$ in the domain where $F(t)$ does not assume negligibly small values.

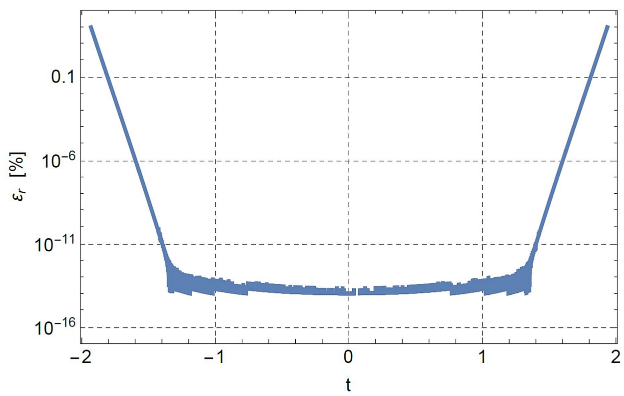

Figure 5. Relative percentage error of the Bell's approximation of $F(t)$ in (28).

The table containing the first few moments of the considered function $F(t)$, as computed using the Gauss–Kronrod quadrature rule (G-KQR), μ_k, and the proposed Bell's method, $\tilde{\mu}_k$, is reported in Figure 6.

$\mu[1] = 0$	$\tilde{\mu}[1] = 0.$
$\mu[2] = 0.320844$	$\tilde{\mu}[2] = 0.320833$
$\mu[3] = 0$	$\tilde{\mu}[3] = 0.$
$\mu[4] = 0.259794$	$\tilde{\mu}[4] = 0.25975$
$\mu[5] = 0$	$\tilde{\mu}[5] = 0.$
$\mu[6] = 0.307347$	$\tilde{\mu}[6] = 0.307162$
$\mu[7] = 0$	$\tilde{\mu}[7] = 0.$
$\mu[8] = 0.457552$	$\tilde{\mu}[8] = 0.456789$
$\mu[9] = 0$	$\tilde{\mu}[9] = 0.$
$\mu[10] = 0.800564$	$\tilde{\mu}[10] = 0.797407$

Figure 6. Moments of $F(t)$ in (28) as computed using the G-KQR (**left**) and the proposed Bell's polynomial approximation (**right**) methods.

6.3. Example 3

Consider the generalized Gaussian distribution defined by the function

$$\psi(t) := \exp\left[-\left(\frac{1}{2}\left(t - \frac{\cos(\pi/8)}{8}\right)^2 \sin(\pi/8) + \frac{\cos(\pi/8)}{6!}\left(t - \frac{\sin(\pi/8)}{8}\right)^6 + \frac{t^8}{4!\,8!\,\sqrt{2}}\left(t - \frac{1}{8\sqrt{2}}\right)^4\right)^2\right]. \quad (29)$$

We find $\mu = \mu_1 = 0.096855$ and $\sigma^2 = 1.02151$.
Its graph is shown in Figure 7.

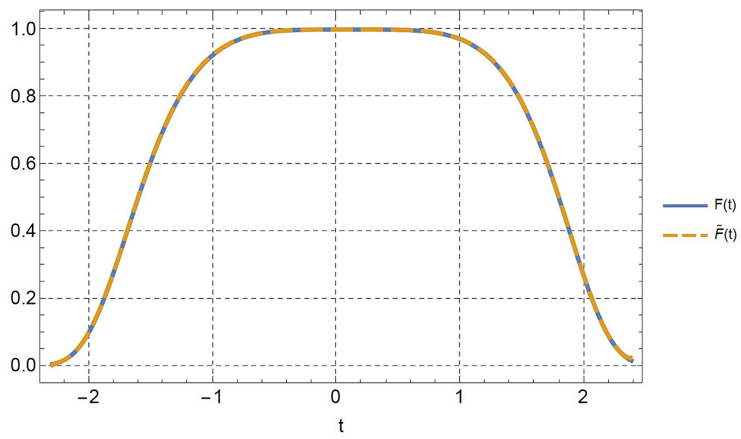

Figure 7. Graph of the function $\psi(t)$ in (29).

The percentage deviation between $\psi(t)$ and the relevant expansion $\tilde{\psi}(t)$ in terms of Bell's polynomials is shown in Figure 8. As can be seen, $\psi(t)$ is nicely approximated by $\tilde{\psi}(t)$ in the domain where $\psi(t)$ does not assume negligibly small values.

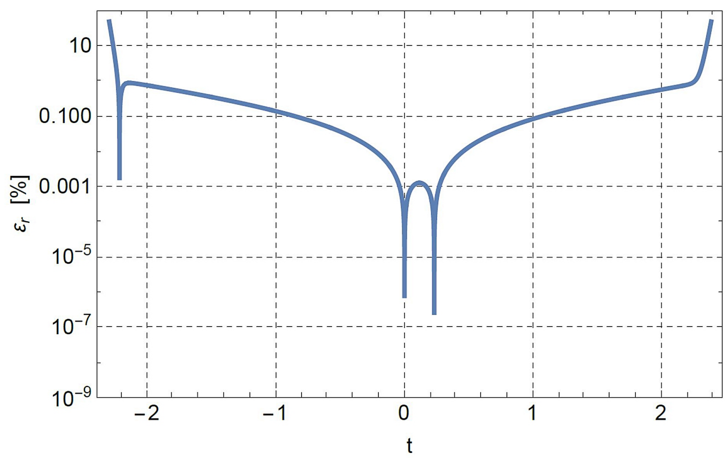

Figure 8. Relative percentage error of the Bell's approximation of $\psi(t)$ in (29).

The table containing the first few moments of the considered function $\psi(t)$, as computed using the Gauss–Kronrod quadrature rule (G-KQR), μ_k, and the proposed Bell's method, $\tilde{\mu}_k$, is reported in Figure 9.

μ[1] = 0.096855	$\tilde{\mu}$[1] = 0.0965639
μ[2] = 1.05286	$\tilde{\mu}$[2] = 1.05375
μ[3] = 0.298461	$\tilde{\mu}$[3] = 0.2967
μ[4] = 2.25481	$\tilde{\mu}$[4] = 2.25053
μ[5] = 1.03878	$\tilde{\mu}$[5] = 1.02607
μ[6] = 6.27311	$\tilde{\mu}$[6] = 6.22373
μ[7] = 3.94173	$\tilde{\mu}$[7] = 3.85297
μ[8] = 20.3358	$\tilde{\mu}$[8] = 19.9709
μ[9] = 15.9893	$\tilde{\mu}$[9] = 15.3913
μ[10] = 73.2833	$\tilde{\mu}$[10] = 70.8689

Figure 9. Moments of $\psi(t)$ in (29) as computed using the G-KQR (**left**) and the proposed Bell's polynomial approximation (**right**) methods.

6.4. Example 4

Consider the generalized Gaussian distribution defined by the function

$$F(t) = \exp[-(\sinh(2t))^2]\,. \tag{30}$$

We find $\mu = \mu_1 = 0$ and $\sigma^2 = 0.27648$.
Its graph is shown in Figure 10.

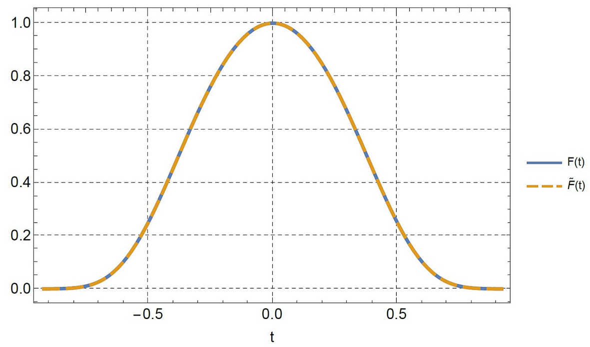

Figure 10. Graph of the function $F(t) = \exp[-(\sinh(2t))^2]$.

The percentage deviation between $F(t)$ and the relevant expansion $\tilde{F}(t)$ in terms of Bell's polynomials is shown in Figure 11. As can be seen, $F(t)$ is nicely approximated by $\tilde{F}(t)$ in the domain where $F(t)$ does not assume negligibly small values.

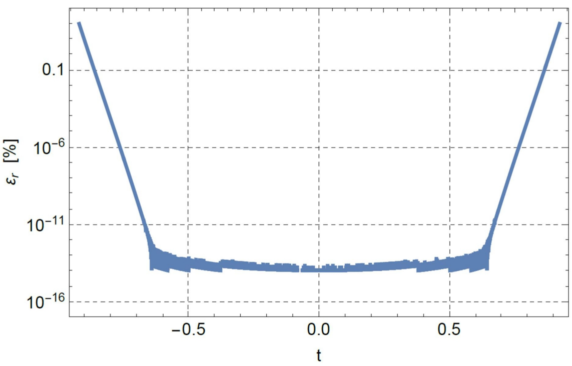

Figure 11. Relative percentage error of the Bell's approximation of $F(t)$ in (30).

The table containing the first few moments of the considered function $F(t)$, as computed using the Gauss–Kronrod quadrature rule (G-KQR), μ_k, and the proposed Bell's method, $\tilde{\mu}_k$, is reported in Figure 12.

μ[1] = 0	μ̃[1] = 0.
μ[2] = 0.0764412	μ̃[2] = 0.0764349
μ[3] = 0	μ̃[3] = 0.
μ[4] = 0.014545	μ̃[4] = 0.0145395
μ[5] = 0	μ̃[5] = 0.
μ[6] = 0.00400552	μ̃[6] = 0.00400075
μ[7] = 0	μ̃[7] = 0.
μ[8] = 0.00137814	μ̃[8] = 0.00137398
μ[9] = 0	μ̃[9] = 0.
μ[10] = 0.000554051	μ̃[10] = 0.0005504

Figure 12. Moments of $F(t)$ in (30) as computed using the G-KQR (**left**) and the proposed Bell's polynomial approximation (**right**) methods.

Different generalized Gaussian distributions can be evaluated using the approach illustrated in Section 5. The same applies to the multimodal distributions considered hereafter.

- Assuming $\exp[-(5x + (x - 1/2)^2 + x^6)^2]$, we find the graph in Figure 13.

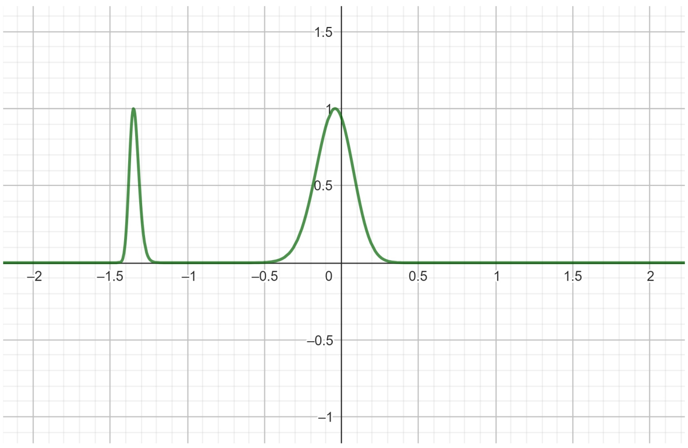

Figure 13. Graph of the function $\exp[-(5x + (x - 1/2)^2 + x^6)^2]$.

- Assuming $\exp[-\sinh^2(x^3 + 3x^2 - 2)]$, we find the graph in Figure 14.

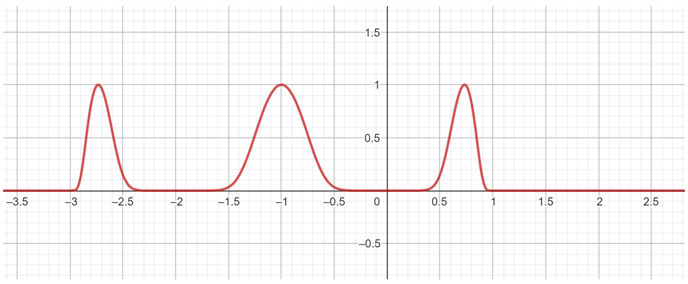

Figure 14. Graph of the function $\exp[-\sinh^2(x^3 + 3x^2 - 2)]$.

In the next example, we evaluate the moments of a bi-modal distribution. In this and other similar cases, one can verify that achieving the desired precision requires a drastic increase in the number of expansion terms, thus affecting the overall computational times.

6.5. Example 5

Consider the generalized Gaussian distribution defined by the function

$$F(t) = \exp[-(\sinh(t^4 - 1/2))^2]\,. \tag{31}$$

We find $\mu = \mu_1 = 0$ and $\sigma^2 = 0.64156$.
Its graph is shown in Figure 15.

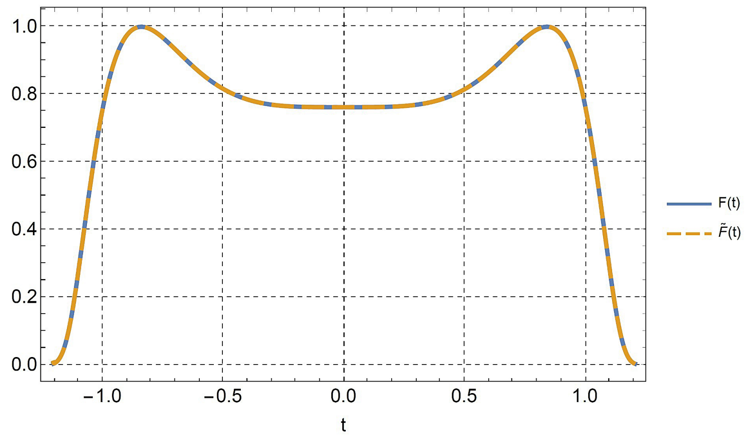

Figure 15. Graph of the function $F(t) = \exp[-(\sinh(t^4 - 1/2))^2$.

The percentage deviation between $F(t)$ and the relevant expansion $\tilde{F}(t)$ in terms of Bell's polynomials is shown in Figure 16. As can be seen, $F(t)$ is nicely approximated by $\tilde{F}(t)$ in the domain where $F(t)$ does not assume negligibly small values.

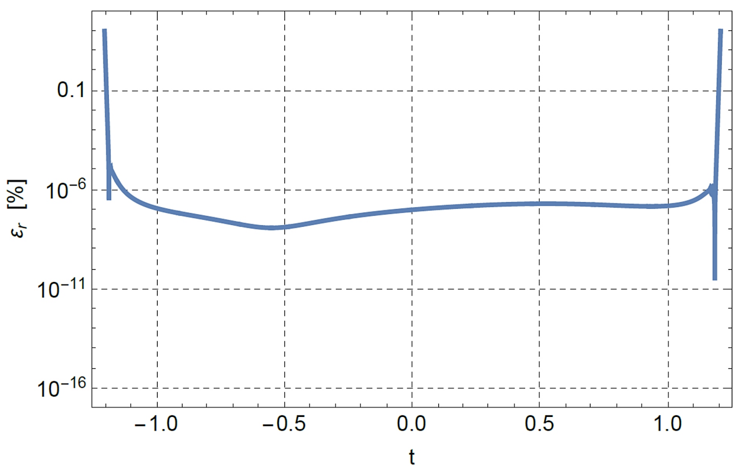

Figure 16. Relative percentage error of the Bell's approximation of $F(t)$ in (31).

The table containing the first few moments of the considered function $F(t)$, as computed using the Gauss–Kronrod quadrature rule (G-KQR), μ_k, and the proposed Bell's method, $\tilde{\mu}_k$, is reported in Figure 17.

μ[1] = 0	$\tilde{\mu}$[1] = 0
μ[2] = 0.411599	$\tilde{\mu}$[2] = 0.411555
μ[3] = 0	$\tilde{\mu}$[3] = 2.3822×10^{-10}
μ[4] = 0.291238	$\tilde{\mu}$[4] = 0.291172
μ[5] = 0	$\tilde{\mu}$[5] = 4.61546×10^{-10}
μ[6] = 0.244599	$\tilde{\mu}$[6] = 0.244502
μ[7] = 0	$\tilde{\mu}$[7] = 7.23248×10^{-10}
μ[8] = 0.225444	$\tilde{\mu}$[8] = 0.225301
μ[9] = 0	$\tilde{\mu}$[9] = 1.07339×10^{-9}
μ[10] = 0.22094	$\tilde{\mu}$[10] = 0.220728

Figure 17. Moments of $F(t)$ in (31) as computed using the G-KQR (**left**) and the proposed Bell's polynomial approximation (**right**) methods

7. Conclusions

We have shown how Bell's polynomials can be adopted to evaluate the moments of generalized Gaussian distributions originated by the composition of the classical Gauss exponential. The possibility of computing integrals of composite and higher-order nested analytic functions by means of Bell's polynomials allows for tackling the considered application in a very effective way.

By using the generalization of the Gaussian distribution, as proposed in this research study and illustrated with several examples, one can define a wide variety of curves starting from arbitrary coefficients of series expansions. Different bell-shaped functions can thus be obtained that can be adapted to represent the most diverse probabilistic distributions. Numerical calculations using the computer algebra program Mathematica have shown that our approach for approximating moments is accurate and efficient.

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Appendix A

$$B_1 = g_1,$$

$$B_2 = g_1^2 + g_2,$$

$$B_3 = g_1^3 + 3g_1g_2 + g_3,$$

$$B_4 = g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4,$$

$$B_5 = g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5,$$

$$B_6 = g_1^6 + 15g_1^4g_2 + 45g_1^2g_2^2 + 15g_2^3 + 20g_1^3g_3 + 60g_1g_2g_3 + 10g_3^2 + 15g_1^2g_4 + 15g_2g_4 + 6g_1g_5 + g_6,$$

$$B_7 = g_1^7 + 21g_1^5g_2 + 105g_1^3g_2^2 + 105g_1g_2^3 + 35g_1^4g_3 + 210g_1^2g_2g_3 + 105g_2^2g_3 + 70g_1g_3^2 + 35g_1^3g_4 + 105g_1g_2g_4 + 35g_3g_4 + 21g_1^2g_5 + 21g_2g_5 + 7g_1g_6 + g_7,$$

$$B_8 = g_1^8 + 28g_1^6g_2 + 210g_1^4g_2^2 + 420g_1^2g_2^3 + 105g_2^4 + 56g_1^5g_3 + 560g_1^3g_2g_3 + 840g_1g_2^2g_3 + 280g_1^2g_3^2 + 280g_2g_3^2 + 70g_1^4g_4 + 420g_1^2g_2g_4 + 210g_2^2g_4 + 280g_1g_3g_4 + 35g_4^2 + 56g_1^3g_5 + 168g_1g_2g_5 + 56g_3g_5 + 28g_1^2g_6 + 28g_2g_6 + 8g_1g_7 + g_8,$$

$$B_9 = g_1^9 + 36g_1^7g_2 + 378g_1^5g_2^2 + 1260g_1^3g_2^3 + 945g_1g_2^4 + 84g_1^6g_3 + 1260g_1^4g_2g_3 + 3780g_1^2g_2^2g_3 + 1260g_2^3g_3 + 840g_1^3g_3^2 + 2520g_1g_2g_3^2 + 280g_3^3 + 126g_1^5g_4 + 1260g_1^3g_2g_4 + 1890g_1g_2^2g_4 + 1260g_1^2g_3g_4 + 1260g_2g_3g_4 + 315g_1g_4^2 + 126g_1^4g_5 + 756g_1^2g_2g_5 + 378g_2^2g_5 + 504g_1g_3g_5 + 126g_4g_5 + 84g_1^3g_6 + 252g_1g_2g_6 + 84g_3g_6 + 36g_1^2g_7 + 36g_2g_7 + 9g_1g_8 + g_9,$$

$$B_{10} = g_1^{10} + 45g_1^8g_2 + 630g_1^6g_2^2 + 3150g_1^4g_2^3 + 4725g_1^2g_2^4 + 945g_2^5 + 120g_1^7g_3 + 2520g_1^5g_2g_3 + 12600g_1^3g_2^2g_3 + 12600g_1g_2^3g_3 + 2100g_1^4g_3^2 + 12600g_1^2g_2g_3^2 + 6300g_2^2g_3^2 + 2800g_1g_3^3 + 210g_1^6g_4 + 3150g_1^4g_2g_4 + 9450g_1^2g_2^2g_4 + 3150g_2^3g_4 + 4200g_1^3g_3g_4 + 12600g_1g_2g_3g_4 + 2100g_3^2g_4 + 1575g_1^2g_4^2 + 1575g_2g_4^2 + 252g_1^5g_5 + 2520g_1^3g_2g_5 + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 + 2520g_2g_3g_5 + 1260g_1g_4g_5 + 126g_5^2 + 210g_1^4g_6 + 1260g_1^2g_2g_6 + 630g_2^2g_6 + 840g_1g_3g_6 + 210g_4g_6 + 120g_1^3g_7 + 360g_1g_2g_7 + 120g_3g_7 + 45g_1^2g_8 + 45g_2g_8 + 10g_1g_9 + g_{10}\,.$$

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