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Optimal Decision for Repairable Products Sale and Warranty under Two-Dimensional Deterioration with Consideration of Production Capacity and Customers' Heterogeneity

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Abstract: An effective warranty policy is not only an obligation for the manufacturer or vendor, but it also enhances the willingness of customers to purchase from them in the future. To earn more customers and increase sales, manufacturers or vendors should be inclined to prolong the service life of their products as an effort to gain more customers. Nevertheless, manufacturers or vendors will not provide a boundless warranty in order to dominate the market, since the related warranty costs will eventually exceed the profits in the end. Therefore, it is a question of weighing the advantage of extending the warranty term in order to earn the trust of new customers against the investment. In addition, since deterioration depends on both time and usage, the deterioration estimation for durable products may be incorrect when considering only one factor. For such problems, a two-dimensional deterioration model is suitable, and the failure times are drawn from a non-homogeneous Poisson process (NHPP). Moreover, customers' heterogeneity, manufacturers' production capacity, and preventive maintenance services are also considered in this study. A mathematical model with the corresponding solution algorithm is proposed to assist manufacturers in making systematic decisions about pricing, production, and warranty. Finally, managerial implications are also provided for refining related decision-making.

Keywords: NHPP; free replacement warranty; two-dimensional failure model; bivariate Weibull probability distribution; preventive maintenance

MSC: 62F15; 62N02; 62N05; 62C10; 65C20

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Citation: Chen, M.-N.; Fang, C.-C. Optimal Decision for Repairable Products Sale and Warranty under Two-Dimensional Deterioration with Consideration of Production Capacity and Customers' Heterogeneity. *Axioms* **2023**, *12*, 701. https://doi.org/10.3390/axioms12070701

Academic Editors: Behzad Djafari-Rouhani, Lechang Yang, Qingqing Zhai, Rui Peng and Aibo Zhang

Received: 15 May 2023
Revised: 17 July 2023
Accepted: 18 July 2023
Published: 19 July 2023

1. Introduction

A warranty is generally considered an obligation by the manufacturer or vendor of a product. In addition to enhancing consumers' purchase willingness, a good warranty policy can increase their satisfaction. Therefore, in order to earn more customers and increase sales, manufacturers or vendors should be responsible for extending the service life of their products. However, extending the warranty term will also increase the related costs. In this regard, a manufacturer will not provide an unlimited warranty in order to monopolize the market because the related warranty costs will eventually exceed the profit of the manufacturer. It would be fair to say that it is a trade-off question between the investment and the benefit of extending warranty terms in order to attract more customers.

Considering the above, in order to understand the nature of the warranty market and the behavior of consumers, it is important to study consumer behavior. A demand function regarding product warranty was proposed by Glickman and Berger [1] in order to identify the ideal warranty length and selling price to maximize the total profit for the product. In order to determine warranty prices for risk-averse customers of non-repairable goods, Ritchken and Tapiero [2] developed a model for measuring customers' attitudes.

During a study by Lassar et al. [3], an analysis was conducted to examine how the timing of product failure and the warranty coverage affect consumers' reaction to said product failure. In order to develop an optimal pricing and warranty strategy, Zhou et al. [4] studied the patterns of customers' purchase intention under different pricing strategies, and then discussed the best pricing and warranty strategy for possible scenarios. As a result, it is assumed that customers make heterogeneous risk assessments when it comes to uncertain repair costs incurred after the warranty period has expired. Yeh and Fang [5] proposed a demand function regarding pro-rata and free replacement warranty policies for dealing with manufacturers' durable product marking issues. Their study considered that if the repair cost just increases, it would be unwise to abridge the warranty term to save on related costs because the action would result in a loss of sales. According to Lee et al. [6], consumers are heterogeneous, and they are classified into weak and strong subpopulations based on their various characteristics. As a result of this study, it was found that products have relatively short and long lifetimes, respectively, for weak and strong subpopulations. In response to the decline in profit margins for most durable products, Bian et al. [7] proposed an optimal extended warranty strategy taking into account consumer's aversion to risks. They argued that manufacturers and retailers increasingly sell extended warranties to generate higher profits because of the decreasing profit margins for durable products. Various strategies for a complimentary extended warranty were proposed by Liu et al. [8] based on the different risk attitudes of consumers, and they concluded that risk-averse consumers might find the proposed warranty more profitable and that retailers' profits will be impacted by the degree to which consumers choose to take on risk. Huang et al. [9] conducted a study on a warranted consumer electronics product that degrades over time and experiences random shocks. Their study was based on the consideration of consumers' different attitudes toward risk as it pertains to their purchases. Thus, this means that all consumers should be heterogeneous, and their purchase intentions will be influenced by the different warranties available to them. According to the studies listed above, marketing strategies are taken into consideration no matter whether customers are heterogeneous or not, including pricing and warranty provisions. However, if manufacturers fail to consider the appropriate production quantity based upon their capacity, they may make an inappropriate decision, since their manufacturing system is not able to satisfy their marketing strategy. Therefore, it is important that warranty, marketing, and production decisions are integrated and not separated from one another. In this study, we made integrated decisions regarding warranty, pricing, production, and preventive maintenance with the help of a synthetic decision-making process that could lead the company to the best policy for the company in the long run.

Furthermore, since manufacturers understand that preventive maintenance (PM) actions within a period of warranty will have an influence on the sales of the products, they need to offer their customers a PM program, especially when dealing with large-scale equipment or facilities. As a general rule, PM policies are based on time intervals as the main determinant of their effectiveness. There are two common PM policies that are discussed in the related literature: The periodic policy and the sequential policy. As a whole, the periodic PM policy is characterized by the decision for the best time interval of PM actions that are taken on a regular basis. In the sequential PM policy, the task of determining how many PM actions should be undertaken during a given period of time is characterized by a search for the optimal number of actions to be performed in that period. It also searches for an optimal interval between two PM activities. According to Park et al. [10], the best PM period and the number of PM actions can be found based on a periodic PM policy and minimal repairs after breakdowns. According to Jung and Park [11], a periodic PM policy that is optimal following the expiration of a warranty is recommended. By minimizing the long-run maintenance costs, decision-makers are able to determine the optimal number and period of post-warranty maintenance policies. According to Yeh et al. [12], the effect of various PM cost functions on the periodic policy could be investigated in the case of a leased product with a Weibull lifetime distribution. As Schutz and Rezg [13] suggested, an optimal PM policy should be established for products to ensure that a minimum level of reliability is achieved in order to meet the requirements of customers. Using an NHPP assumption, Tsarouhas [14] developed PM models for the ice cream industry, which were based on NHPP assumptions. He used RAM analysis methods to monitor the status of a company's manufacturing systems, including the reliability, adequacy, and maintainability of the machines. As an approach to improving preventive maintenance, Lastra et al. [15] made the case that additive manuring processes could be used. In their opinion, spare parts may play an important role in the maintenance of machines if they are available. Fang et al. [16] proposed a statistical approach based on Bayesian theory as a solution to the issues associated with periodic preventive maintenance. The behavior of the system's deterioration is characterized using NHPP with power law failure intensity functions. As a result, this research has the potential to offer beneficial solutions to assist managers in making good decisions regarding the preventative maintenance of large-scale facilities. Diatte et al. [17] proposed that a method to improve brake systems of automobiles could be implemented by integrating the reliability, availability, and maintainability of the equipment into the system engineering and dependability analyses, as a means of reducing costs and increasing reliability in such systems. According to that mentioned above, it is understood that suitable PMs could effectively reduce the related costs regarding repairs and system availability. In order to raise customers' satisfaction and reduce the related warranty costs, a periodical PM policy is considered in this study.

In view of the fact that equipment or facility deterioration is not only dependent on the passage of time, but also on the usage of equipment or facilities, only considering one of them could lead to a distorted estimation of the equipment's deterioration in such a situation. Thus, it would be appropriate to address such problems by using a two-dimensional failure model that represents failures in two dimensions. As a result of two factors, age and usage of the system, Baik et al. [18] argued for two-dimensional failure modeling in the case of deteriorating systems. Using a bivariate Weibull model, they extended the concept of minimum repair for the one-dimensional case to the two-dimensional case and characterized failures utilizing the idea of minimal repair for the one-dimensional case. In a similar way, He et al. [19] used a similar concept when evaluating the reliability of piezoelectric micro-actuators by taking into account two factors: The driving voltage and the operating temperature of the actuator when it was operated. There were also some useful bivariate models that were recently proposed to allow for the consideration of different probability distributions. According to Huang and Yen [20], manufacturers have the option of offering two-dimensional warranty plans that include time and usage limits, in combination with documentation that explains how the warranty works. According to Shahanaghi et al. [21], the extended warranty contract for automobile manufacturers should be structured in two dimensions. Its model emphasizes the importance of optimizing preventive maintenance strategies based on the warranty's coverage length and usage and how long the warranty will last. In addition to this, Huang et al. [22] also used a bivariate Weibull distribution for the analysis of warranty costs associated with periodic preventive maintenance. According to Wang et al. [23], they proposed and applied two-dimensional deterioration preventive maintenance policies for the automobile industry, based on punctual and unpunctual preventive maintenance. An analysis was carried out by Fang et al. [24], who considered two dimensions of deterioration (time and use) for the purpose of developing mathematical models and an efficient calculating process for determining the optimal financial leasing decision for facilities. A maintenance scheme is also considered in their mathematical models in order to reduce the related costs during the lease term. Dong et al. [25] proposed a two-dimensional deterioration model for a multi-component system. They considered that under a specific utilization rate, the life of a parallel multi-component system can be estimated. Moreover, multi-component systems with two-dimensional deterioration were studied by Dong et al. [26]. The authors utilized particle swarm optimization to accomplish the objectives of reducing the cost of their proposed system and increasing its availability in order to solve their proposed model efficiently. In light of the above discussion, this study developed an analytical model for determining the optimal warranty policy for repairable products based on bivariate Weibull distributions considering two deterioration factors.

Taking into account the discussion above, it can be seen that there are some issues that may be worth exploring, and they are as follows: (1) In order to estimate the demand for a repairable product, the marketing department should take into account both price and warranty in determining the market demand for the product, and decision-makers should not rely only on one factor to determine demand. Moreover, a manufacturer should also consider its production capacity have to meet its marketing strategy. (2) In order to increase customer satisfaction and reduce warranty costs, a preventive maintenance program should be considered as part of the warranty conditions. Therefore, a manufacturer needs to evaluate the warranty cost considering the influence of various preventive maintenance programs. (3) In light of the fact that the deterioration of repairable products depends on both usage and time, considering only one of them could result in a distorted estimation of the products' deterioration. Therefore, a two-dimensional deterioration model would be appropriate to address such problems. (4) The above issues regarding warranty, marketing, production, preventive maintenance, and two-dimensional deterioration should be integrated, not separated. It is always better to make an integrated decision rather than a separate one. If a manufacturer does not fully consider all of the related influence factors, the decision may not reach its true optimum. Accordingly, this study applied a bivariate Weibull distribution by integrating the issues of warranty, marketing, production, preventive maintenance, and different customer usage rates to construct a decision model. Additionally, the successive failure times of deteriorating products are described by a non-homogeneous Poisson process for easily evaluating the expected value of product breakdowns. The remainder of this study is organized as follows: Section 2 introduces the mathematical models for two-dimensional deterioration models, the estimation of warranty costs, the stepped cost function with production capacity constraints, and the demand function for repairable products. Section 3 introduces the proposed mathematical programming model and the corresponding solution algorithm for an optimal decision about warranty, pricing, and production quantity. Section 4 presents the numerical applications and sensitivity analyses of the study. Finally, Section 5 presents the concluding remarks and future studies.

2. Problem Description and Model Development

Considering a manufacturer's plans to launch new and durable products into the market, in order to increase sales of the forthcoming product, the manager needs to devise an attractive and adequate warranty policy as part of his sales strategy. This ought to keep in mind that a product's failure rates may impact the warranty cost. It is necessary to capture the characteristic of the failure process, and this process is often believed to be followed by an NHPP (non-homogenous Poisson process) that has a specific intensity function. Traditionally, one-dimensional deterioration has been used as an estimation for inspection and repair works on electronic and mechanical systems in many studies conducted in the past. However, in the real world, the deterioration of most electronic and mechanical systems is two-dimensional. This means that the deterioration of most electronic and mechanical systems is not solely determined by the passage of time, but also by the manner in which they are used. In light of this, this study proposes a two-dimensional deterioration model for estimating the breakdown and deterioration of new products. Furthermore, customers are heterogeneous, so their usage rates should be different. Such customer usage rates can be described as probability distributions. In order to estimate the value of the model parameters, historical data can be used as a basis for estimation. In addition, the manufacturer generally offers a preventive maintenance program during the warranty period in order to mitigate breakdowns and deterioration of the product, as well as reduce the associated operating costs. Therefore, preventive maintenance services for customers during a warranty period are also be considered in our model. Moreover, because of the step-type function used in the production unit cost calculation, if the scale of the production increases, so will the cost. Concerning the demand function of the new product, we refer to Glickman and Berger's [1] demand function to estimate the possible demand quantity, which is influenced by the warranty period and the product's price. The corresponding parameters' estimated value of the demand function can be obtained through data analysis and market surveys.

Based on the above discussion, for maximizing profits, a manufacturer should take into account the period of the warranty, preventive maintenance service, product deterioration, customer usage rate, scale of production, and price of the product at the same time. The following subsections introduce the deterioration models, preventive maintenance program, demand function regarding warranty and pricing, and related costs. Table 1 in this study includes the following notations and terminologies that are used throughout the analysis:

Table 1. The notations and terminologies.

t: The age of the product.
t_k^-: The deteriorating system's effective age before it gets its kth maintenance.
t_k^+: The deteriorating system's effective age after it gets its kth maintenance.
α: Bivariate Weibull's scale parameter for the age of the system.
β: Bivariate Weibull's shape parameter for the age of the system.
ω: Bivariate Weibull's scale parameter for the usage of the system.
κ: Bivariate Weibull's shape parameter for the usage of the system.
t_r: The time that is required for a minimal repair to be performed.
u: The usage of the system.
s: The usage rate ($s = t/u$), which can be described as a random variable that follows a specific distribution.
$\lambda(t, u)$: The function that measures the intensity of deterioration caused by the age and usage of the system.
$\lambda(t, s)$: The function that measures the intensity of deterioration caused by the age and usage rate of the system.
P: The price of a product.
W: The length of the warranty period.
Q: The sales quantity.
x: A period of time between two scheduled periodic PM actions that are performed on a regular basis.
$E[N_{br}\|W, x, \Psi_{\mathcal{D}}(s, \Theta)]$: The expected failure number of the deteriorating system ($\mathcal{D} = (G)amma, (L)og\ normal,\ or\ (U)niform\ distributions$).
Q_u: The binary variable for setting the production stage ($Q_u \in \{0, 1\}$, where $Q_u = 1$ presents the specific production stage being chosen u ($u = 1..S$); S denotes the number of the production stages that could be chosen).
TFC: The setup cost of the manufacturer's production line.
C_v^P: The production unit cost under the production stage v ($v = 1..S$).
Q_v^B: The upper limit or bound of the manufacturer's production stage v.
C_{mr}: The expected cost to proceed with a minimal repair.
C_{pl}: The penalty cost as long as the actual repair time exceeds the predetermined time limit φ.
φ: The predetermined time limit for performing a minimal repair action.
$G(t_r)$: The probability density function of the repair time.
$C_{pm_k}^h$: The cost for performing the kth preventive maintenance under the preventive maintenance alternative h.
τ_h: The periodically increasing rate under the preventive maintenance alternative h.
δ_{pm}^h: The value of the age reduction factor in effective age for the preventive maintenance alternative q due to the kth preventive maintenance, where $\delta_{pm}^h \in [0, 1]$.
C_F^h: Base cost of proceeding with preventive maintenance work, related to the degree of preventive maintenance.

2.1. Two-Dimensional Deterioration Model

In terms of the breakdown rate of a product, it is assumed that the breakdown process is a time-dependent NHPP. In relation to the two-dimensional deterioration model, we assume that the deteriorating process of a durable product follows the bivariate Weibull model as far as it relates to the breakdown process. The failure intensity function of the durable product can be given as follows:

$$\lambda(t,u,\Theta) = \left(\frac{\beta}{\alpha^{\beta}} t^{\beta-1}\right)\left(\frac{\kappa}{\omega^{\kappa}} u^{\kappa-1}\right). \quad (1)$$

There are four main parameters that influence the estimation of the intensity function $\lambda(t,u,\Theta)$. α and β denote the scale and shape parameters, respectively, in terms of deterioration over time. Similarly, ω and κ denote the scale and shape parameters in terms of deterioration with usage. According to the bivariate Weibull model, a deteriorating system will deteriorate with usage and time if both parameters β and κ are greater than one. Moreover, if the values of parameters α and ω increase, the deterioration will present an exponential increasing velocity. In the majority of cases of deteriorating systems, such a model will be more flexible and manageable to present the behavior of deteriorating systems than others. In spite of this, the NHPP will downgrade into a HPP with constant failure intensity α and ω if the shape parameters β and κ are equal to one. Moreover, in order to estimate the related parameters of the model, the manufacturer may have to conduct accelerated deterioration experiments to obtain these values.

Without taking into account any preventative maintenance policy throughout the warranty period, the following equation is used to estimate the expected number of breakdowns of the product:

$$E[N_{br}|W,\Theta] = \int_0^{t_i}\int_0^{u_i} \lambda(t,u,\Theta)dudt = \left(\frac{t_i}{\alpha}\right)^{\beta}\left(\frac{u_i}{\omega}\right)^{\kappa}. \quad (2)$$

Since a deteriorating system is usually modeled as an NHPP with the intensity function $\lambda(t,u,\Theta)$, the probability of the number of breakdowns N_0 in the intervals of time (t_i, t_{i+1}) can be determined by the following equation:

$$Pr\{N_{br}(t_{i+1},u_i) - N_{br}(t_i,u_i) = N_0\} = \frac{\left(\int_{t_i}^{t_{i+1}}\int_0^{u_i} \lambda(t,u,\Theta)dudt\right)^{N_0} e^{-\int_{t_i}^{t_{i+1}}\int_0^{u_i} \lambda(t,u,\Theta)dudt}}{N_0!} \quad (3)$$

Clearly, the reliability of a facility is going to decline over time and with usage, and therefore, it is possible to define the reliability function $R(t_i,u_i)$ as follows:

$$R(t_i,u_i) = Pr\{N_{br}(t_i,u_i) = 0\} = e^{-\int_0^{t_i}\int_0^{u_i} \lambda(t,u,\Theta)dudt} = e^{-(\frac{t_i}{\alpha})^{\beta}(\frac{u_i}{\omega})^{\kappa}}. \quad (4)$$

Although different customers have their own customs, individual needs, or usage styles, each customer's durable product will experience different types of deterioration under the same use conditions. For measuring the different deteriorations from the customers' durable product, a usage rate can be defined as the proportion of the use time to the usage ($s = t/u$). Thus, usage rates can be considered an indicator of the extent to which customers use the products. A two-variate model can be transformed into a univariate model for easier analysis. Moreover, it is reasonable to assume that usage rates follow a probability distribution with an appropriate region. Consequently, by examining marketing and consumer surveys, we can assume that the probability distribution of the usage rate will follow a gamma, lognormal, or uniform distribution. An illustration of the relationship between use time and usage is shown in Figure 1. It can be seen that s_i denotes a customer's usage rate, and the usage rate s_{i+1} is lower that s_{i+2}. Due to customers' usage habits, the usage rate can be presented in different distributions.

According to all customers' usage in the real world, the customer usage rate can be described as a random variable that approximates a gamma distribution based on the actual usage. Such distribution is often used for describing customers' usage habits in related studies due to its flexibility. As a result, it is possible to estimate the number of breakdowns that are expected in the future as follows:

$$\begin{aligned} E[N_{br}|W,\Psi_G(s|\Theta)] &= \int_0^W \int_0^\infty \lambda(t,s)\Psi_G(s|\theta,\gamma)dsdt \\ &= \int_0^W \int_0^\infty \left(\frac{\beta t^{\beta-1}}{\alpha^\beta}\right)\left(\frac{\kappa(st)^{\kappa-1}}{\omega^\kappa}\right)\left(\frac{s^{\theta-1}}{\Gamma(\theta)\gamma^\theta e^{s/\gamma}}\right)dsdt \\ &= \beta\kappa\left(\frac{\Gamma(\kappa+\theta-1)}{\Gamma(\theta)}\right)\left(\frac{\gamma^{\kappa-1}}{(\beta+\kappa-1)W}\right)\left(\frac{W^{\beta+\kappa}}{\alpha^\beta\omega^\kappa}\right), \end{aligned} \tag{5}$$

where $\Psi_G(s|\theta,\gamma)$ denotes a gamma probability function with the shape parameter θ and the scale parameter γ, and its mathematical form can be presented as follows:

$$\Psi_G(s|\theta,\gamma) = \left(\frac{s^{\theta-1}}{\Gamma(\theta)\gamma^\theta e^{s/\gamma}}\right).(\Gamma(\theta) = \int_0^\infty y^{\theta-1}e^{-y}dy) \tag{6}$$

The gamma distribution is a two-parameter family of continuous probability distributions, and it is often is used to describe a ratio's distribution. The exponential and chi-squared distributions can be regarded as special cases of the gamma distribution. However, if the customer usage rate approximates a lognormal distribution with the average μ and the standard deviation σ from the manufacturer's market surveys, there is a need to rewrite the estimated number of breakdowns in the following manner:

$$\begin{aligned} E[N_{br}|W,\Psi_L(s|\Theta)] &= \int_0^W \int_0^\infty \lambda(t,s)\Psi_L(s|\mu,\sigma)dsdt \\ &= \int_0^W \int_0^\infty \left(\frac{\beta t^{\beta-1}}{\alpha^\beta}\right)\left(\frac{\kappa(st)^{\kappa-1}}{\omega^\kappa}\right)\left(\frac{1}{\sqrt{2\pi}\sigma st}^{-\frac{1}{2}\left(\frac{ln[st]-\mu}{\sigma}\right)^2}\right)dsdt \\ &= \left(\frac{\left(\frac{W}{\alpha}\right)^\beta\left(\frac{W}{\omega}\right)^\kappa}{(\beta+\kappa-1)W}\right)^{(\mu+\frac{1}{2}(\kappa-1)\sigma^2)(\kappa-1)}, \end{aligned} \tag{7}$$

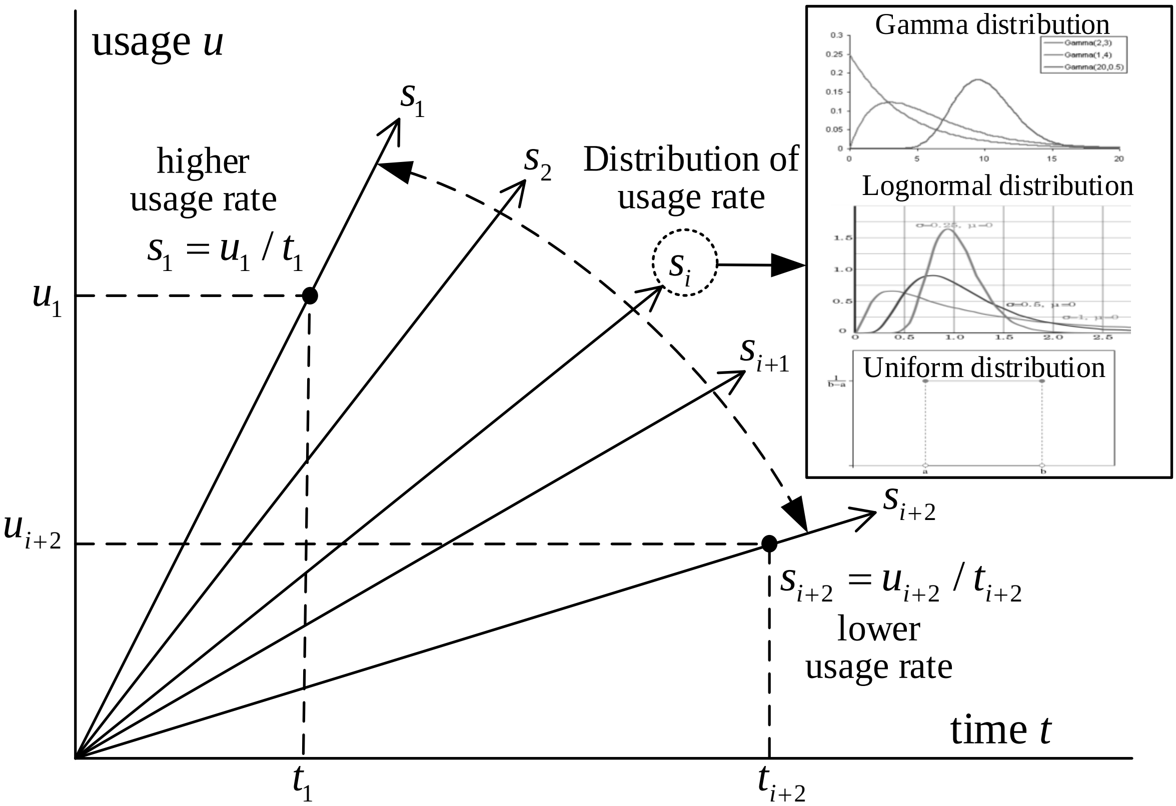

Figure 1. The relationship between use time t and usage u for the usage rate s.

Please note that a log-normal distribution can be translated into a normal distribution and vice versa using associated logarithmic calculations. Moreover, $\Psi_L(s|\mu,\sigma)$ is a lognormal probability function with the parameters μ and σ, and it can be denoted as:

$$\Psi_L(s|\mu,\sigma) = \left(\frac{1}{\sqrt{2\pi}\sigma st}^{-\frac{1}{2}\left(\frac{ln[st]-\mu}{\sigma}\right)^2} \right) \tag{8}$$

In contrast, if the customer usage rate has an almost equal probability of occurring within a given range $[U_1, U_2]$, then the estimated number of breakdowns can be presented as follows:

$$\begin{aligned} E[N_{br}|W, \Psi_U(s|\Theta)] &= \int_0^W \int_{U_1}^{U_2} \lambda(t,s)\Psi_U(s|U_1,U_2)dsdt \\ &= \int_0^W \int_{U_1}^{U_2} \left(\frac{\beta t^{\beta-1}}{\alpha^\beta}\right)\left(\frac{\kappa(st)^{\kappa-1}}{\omega^\kappa}\right)\left(\frac{1}{U_2-U_1}\right)dsdt \\ &= \frac{\beta\left(\frac{W}{\alpha}\right)^\beta\left(\frac{W}{\omega}\right)^\kappa({U_2}^\kappa-{U_2}^\kappa)}{(\beta+\kappa-1)(U_2-U_1)W}. \end{aligned} \tag{9}$$

where $\Psi_U(s|U_1,U_2)$ denotes a uniform probability density function within the range $[U_1, U_2]$, and its mathematical form can be expressed as follows:

$$\Psi_U(s|U_1,U_2) = \left(\frac{1}{U_2 - U_1}\right) \tag{10}$$

Before employing the study model, it is important to note the following assumptions:

1. An NHPP can be used to describe the failure or breakdown process that occurs as a result of time and usage of the product.
2. Once the failure or breakdown occurs within the warranty period, a minimal repair will be performed for the customers.
3. It is due to the imperfect nature of the PM activities that the product condition cannot be fully restored to its previous status after the PM process has been completed.
4. The manufacturer is responsible for paying the repair and PM costs involved.
5. It is assumed that the probability distribution of the usage rate of the products will be either gamma, lognormal, or uniform.

2.2. Estimation of the Related Costs under the Periodic Preventive Maintenance Policy

In order to ensure system safety and customer satisfaction, it is important that the reliability of durable products must be managed to an acceptable level. In order to reduce the frequency of product breakdowns or failures, adequate preventive maintenance (PM) can help prevent them from occurring. To improve the system's reliability, a PM policy can be implemented either on a periodic or non-periodic basis. A periodic PM policy was adopted in this study, since it would be more manageable for the manufacturer. In addition, it is believed that the failure times of the durable products adhere to an NHPP with an intensity function and that they would undergo $N-1$ PM tasks for the duration of the warranty period W, where the time interval between two PM tasks is specified to be x. The warranty service is terminated at the end of the warranty period, and an NHPP with a bivariate Weibull distribution is used to model the breakdown process of the deteriorating system. In order to evaluate the product's effective age, the symbols t_k^- and t_k^+ are used to denote the effective age before and after the kth PM task. It is assumed that a PM task can partially recover the deteriorating system, and the degree of recovery δ_{pm} can be measured through comparative deterioration experiments. Before the first PM action is taken, the effective age of the product should be $t_1^- = x$. However, once the PM action is performed on the product, the effective age is immediately recovered as $t_1^+ = x - \delta_{pm}x = \left(1-\delta_{pm}\right)x$ as a result. On the basis of this, it is possible to calculate the effective ages immediately before and after the kth PM action as follows:

$$t_k^- = kx - (k-1)\delta_{pm}x = (k-1)\left(1-\delta_{pm}\right)x + x = \left((k-1)\left(1-\delta_{pm}\right)+1\right)x \tag{11}$$

and

$$t_k^+ = t_k^- - \delta_{pm}x = kx - (k-1)\delta_{pm}x - \delta_{pm}x = k\big(1 - \delta_{pm}\big)x \tag{12}$$

respectively.

A product's deterioration under the imperfect PM model is illustrated in Figure 2, which presents the timeline and breakdowns over the period of warranty (W). In this period, the manufacturer provides N maintenance service with intervals (x) to reduce the possible breakdowns of products. The expected disbursements should include repair and maintenance costs during the warranty period of the product. In this study, the repair cost was composed of two items: (1) The average spent for carrying out a minimal repair (C_{mr}) and (2) the expected penalty cost (C_{pl}) if the actual repair time is longer than the tolerable waiting time φ. According to the manufacturers' responsibility, they have to pay the expenditure for any failure or breakdown of the durable product during the warranty period. For estimating the penalty cost, the probability of the overtime repair needs to be evaluated first. Suppose the repair time t_r is a random variable with a gamma distribution, the following equation denotes the probability function in which the repair time t_r exceeds the upper bound φ:

$$G(t_r) = \int_{\varphi}^{\infty} \frac{t_r^{\omega-1}\eta^{\omega}}{\Gamma(\omega)e^{\eta t_r}} dt_r \tag{13}$$

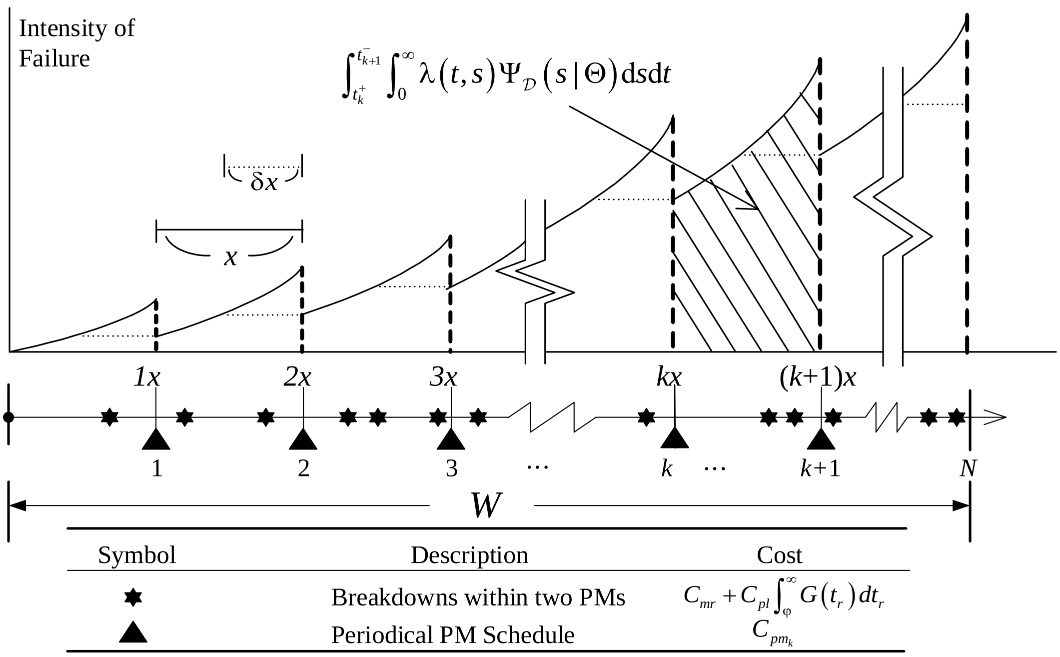

Figure 2. Timeline of a repairable product under the imperfect preventive maintenance model.

The parameters η and ω have to be estimated in advance. Due to the fact that the parameters η and ω are related to their mean and variance, we can obtain the parameters' values by $\eta = E(t_r)/\sigma(t_r)^2$ and $\omega = (E(t_r)/\sigma(t_r))^2$.

In order to evaluate the repair cost of a durable product, the manufacturer must estimate deterioration over the warranty period. Accordingly, based on the assumption that the failure process of a durable product is an NHPP with the intensity function $\lambda(t,s)$, the number of expected failures number the warranty period $[0, W]$ under PM interval x and age reduction δ can be denoted as $E\big[N_{br}\big|W, x, \delta_{pm}, \Psi_{\mathcal{D}}(s|\Theta)\big]$. According to the information stated above, the repair cost during the warranty period can be calculated as follows:

$$\left(C_{mr} + C_{pl}\int_{\varphi}^{\infty} G(t_r)dt_r\right) E\big[N_{br}\big|W, x, \delta_{pm}, \Psi_{\mathcal{D}}(s|\Theta)\big]. \tag{14}$$

where C_{mr} denotes the expected cost to proceed with a minimal repair; C_{pl} denotes the penalty cost as long as the actual repair time exceeds the predetermined time limit φ; $G(t_r)$ is the probability density function of the repair time. Moreover, if the manufacturers can provide preventive maintenance services to their customers, it will be helpful to reduce the repair expenses and increase their customers' satisfaction. Once the reduction in repair expenses exceeds the maintenance cost, the manufacturer should provide maintenance services for the business's consideration. Assume that a sequential PM policy is provided by the manufacturer during the warranty period, the PM costs will increase due to system aging. Based on the periodical PM policy with equal time intervals, the estimation of PM costs during the warranty period can be obtained as follows:

$$C_{pm_k} = (1 + \tau(k-1)x)C_F. \tag{15}$$

where C_F and τ denote the base cost and the increasing rate of each PM task, and therefore the total PM cost can be calculated as follows:

$$C_{pm}(W, x, C_F, \tau) = \sum_{k=1}^{\lfloor W/x \rfloor - 1} C_{pm_k} = \sum_{k=1}^{\lfloor W/x \rfloor - 1} (1 + \tau(k-1)x)C_F. \tag{16}$$

In addition, a manufacturer may have various PM alternatives for the warranty. Various PM alternatives can provide different levels of system recovery, but they also come with different PM costs associated with them. Suppose that the candidate list of PM alternatives $M_{pm} = \left\{ M_{pm}^1, M_{pm}^2, \ldots, M_{pm}^h, \ldots, M_{pm}^H \right\}$ can be chosen by the manufacturer, and the PM cost with PM alternative M_{pm}^h can be given as follows:

$$C_{pm}^h\left(W, x, C_F^h, \tau_h\right) = \sum_{k=1}^{\lfloor W/x \rfloor - 1} C_{pm_k}^h = \sum_{k=1}^{\lfloor W/x \rfloor - 1} (1 + \tau_h(k-1)x)C_F^h. \tag{17}$$

Moreover, the expected number of failures of a durable product considering PM services during the warranty W can be estimated as follows:

$$\begin{aligned} & E\left[N_{br}\middle| W, x, \delta_{pm}^h, \Psi_{\mathcal{D}}(s|\Theta)\right] \\ &= \int_{t_1^+}^{t_2^-} \left(\int_0^\infty \lambda(t,s)\Psi_{\mathcal{D}}(s|\Theta)ds\right)dt + \int_{t_2^+}^{t_3^-} \left(\int_0^\infty \lambda(t,s)\Psi_{\mathcal{D}}(s|\Theta)ds\right)dt \\ &\quad + \int_{t_3^+}^{t_4^-} \left(\int_0^\infty \lambda(t,s)\Psi_{\mathcal{D}}(s|\Theta)ds\right)dt + \ldots \\ &\quad + \int_{t_k^+}^{t_{k+1}^-} \left(\int_0^\infty \lambda(t,s)\Psi_{\mathcal{D}}(s|\Theta)ds\right)dt \\ &= \sum_{k=1}^{\lfloor W/x \rfloor - 1} \int_{t_k^+}^{t_{k+1}^-} \int_0^\infty \lambda(t,s)\Psi_{\mathcal{D}}(s|\Theta)dsdt \\ &= \sum_{k=1}^{\lfloor W/x \rfloor - 1} \int_{kx}^{(k+1)x} \int_0^\infty \lambda\left(t - k\delta_{pm}^h x, s\right)\Psi_{\mathcal{D}}(s|\Theta)dsdt. \end{aligned} \tag{18}$$

2.3. Estimation of the Stepped Production Cost

The cost per unit of production, in general, should be stepped-type in practice, which means that if the production scale is expanded at the present time, the cost will increase accordingly. Since manufacturers cannot adjust their long-term production capacity in a short period of time, overtime or outsourcing is required as a strategy to cope with the situation in case the market demand exceeds normal capacity. Figure 3 illustrates the manufacturer's stepped production unit costs. Generally, a manufacturer's production scale can be divided into the three stages.

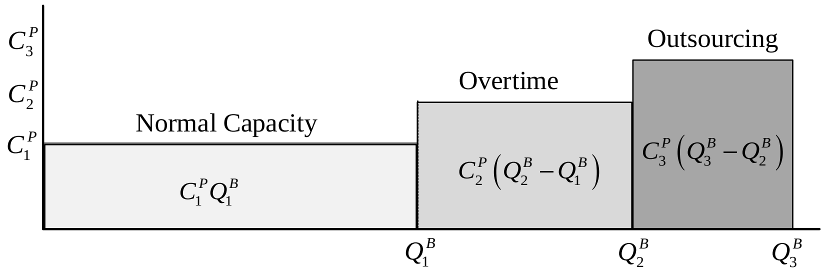

Figure 3. The stepped unit cost in each production stage.

In the first stage, the capacity is normal under the production volume Q_1^B, and the unit cost of production on this stage is C_1^P. As for the second stage, it is the capacity that is achieved by utilizing overtime strategies, and the production volume is between Q_1^B and Q_2^B, which has a higher production unit cost C_2^P than C_1^P, since the overtime wage is higher than the normal wage. The third stage is outsourcing, since the maximum capacity of the manufacturer, even with overtime, cannot satisfy market demand. The manufacturer outsources its products to others at production unit cost C_3^P. As a result, the total production cost will be $C_3^P(Q - Q_2^B) + C_2^P(Q_2^B - Q_1^B) + C_1^P Q_1^B$ if customers' demand is between Q_2^B and Q_3^B. To sum up, the total production cost can be formulated as follows:

$$\sum_{u=1}^{S}\left\{Q_u\left\{\sum_{v=1}^{u-1} C_v^P\left(Q_v^B - Q_{v-1}^B\right) + C_u^P\left(Q - Q_u^B\right)\right\}\right\}. \tag{19}$$

where Q_u denotes a binary variable ($Q_u \in \{0,1\}$), and $Q_u = 1$ means that production stage u is chosen ($u = 1..S$); C_v^P denotes the production unit cost under production stage v; Q_v^B denotes the upper limit of production stage v; S denotes the number of production stages that can be chosen. In general, the production capacity can be categorized into three stages, unless there is another strategy for other conditions that require other production plans to be implemented.

2.4. Demand Function for the Pricing and Warranty of Repairable Products

Since the different marketing strategies of manufacturers will influence customers' demand, measuring the effect of pricing and warranty on product sales is essential. Glickman and Berger [1] studied this issue and proposed the demand function of pricing and warranty. The demand function has been fairly validated in related studies in the past, and it is given by:

$$Q(P, W) = b_1 P^{-z_1}(b_2 + W)^{z_2}. \tag{20}$$

In this function, Q denotes the demand for durable products. The parameters P and W represent the product's price and the corresponding warranty, respectively. The coefficients b_1 and b_2 denote the amplitude factor and intercept, respectively. A manufacturer's marketing department can analyze marketing data to obtain the above parameters. It should be noted that as P decreases and/or W increases, the demand function increases as well. This is an indication that a lower price and/or a longer warranty period can achieve a higher number of sales. The parameters z_1 and z_2 denote the price and warranty elasticity, respectively. Customers will change a product due to severe damage or outdated functions, and therefore, the marginal demand will decline if the warranty period is overly prolonged. Figure 4 illustrates the contour of product sales by setting different prices and warranties. As can be seen from the iso-sales curves, the impact of the trade-off between price and warranty can be observed.

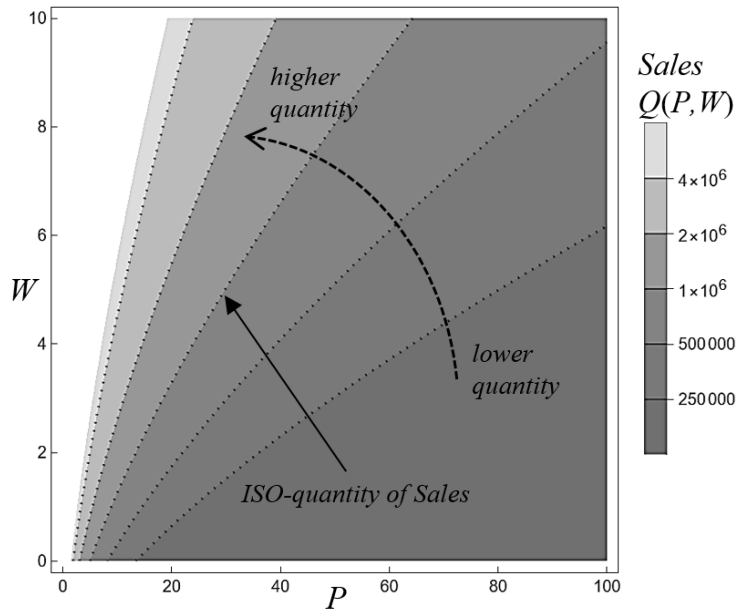

Figure 4. The contour plot of product sales by setting different prices and warranties.

3. Decision of the Optimal Warranty and Price of Repairable Products with Two-Dimensional Deterioration

The proposed model is implemented in this section in order to pursue the maximal estimated profit of the manufacturer based on the following conditions: (1) The product's healthy status after a repair can be restored to the previous healthy state immediately; (2) PM can reduce the probability of system failure and also prolong the product's lifetime; (3) the repair time does not affect the product's lifetime in any way.

3.1. Mathematical Programming Model for the Optimal Decision of Warranty, Pricing, and Production Quantity

During the project phase, reliability engineers evaluate and estimate the related parameters based on historical data before applying the proposed model. The system's parameters must be obtained in advance for engineers to estimate the expected related costs during the warranty period. It is also necessary to inspect the PM alternatives in advance, the cost structure of the production process, and the demand function to make sure that the manufacturer has all of the necessary information to handle the decision-making process properly. In order to solve the problem effectively, a mathematical model and algorithm are developed in this section. As a result of that mentioned above, the mathematical programming model can be given as follows:

$$\begin{aligned} &\underset{P,W_i,Q,Q_u,}{Max}\left\{\pi\left(P,Q,\{Q_1,..,Q_S\},W_i\middle|M_P^h\right)\right\} \\ &= PQ - TFC - \sum_{u=1}^{S}\left\{Q_u\left\{\sum_{v=1}^{u-1}C_v^P\left(Q_v^B - Q_{v-1}^B\right) + C_u^P\left(Q - Q_u^B\right)\right\}\right\} \\ &-\left(C_{mr} + C_{pl}\int_{\varphi}^{\infty}G(t_r)dt_r\right)E\left[N_{br}\middle|W_i,x,\delta_{pm}^h,\Psi_{\mathcal{D}}(s|\Theta)\right]Q \\ &-\sum_{k=1}^{\lfloor W_i/x\rfloor-1}(1+\tau_h kx)C_F^h Q \end{aligned} \tag{21}$$

Subject to:

$$Q = b_1 P^{-z_1}(b_2 + W_i)^{z_2} \tag{22}$$

$$Q_{v-1}^B + Q_S^B(Q_v - 1) \le Q < Q_v^B + Q_S^B(1 - Q_v),\ v = 1,..,S \tag{23}$$

$$Q_u \in \{0,1\},\ u = 1,..,S \tag{24}$$

$$\sum_{u=1}^{s} Q_u = 1 \tag{25}$$

$$P > 0,\ Q > 0 \tag{26}$$

The objective function (21) consists of five elements: The sales revenue, the setup cost of a production line, the production cost, the repair cost, and the PM cost. In order to follow the market convention in practice, the warranty length is usually set as a discrete value (e.g., 2, 2.5, or 3 years). Therefore, the candidate warranty length ($W_i, i = 1, ..., n$) is enumerated for deciding which is optimal. Constraint (22) is used to present the demand function of price and warranty. Constraints (23)–(25) are devised to verify one of the production scales selected by a manufacturer. Constraint (26) is to ensure the planned price and production quantity are positive.

3.2. Solution Algorithm

In order to solve such an integrated issue in an analytic manner, it needs to be simplified and decomposed. In order to facilitate the objective function, there are a few conditions that need to be met: (1) The production volume must be within the range of a specific production phase; (2) it must meet the demand function requirement; (3) the candidate warranty periods are devised as discrete values. Accordingly, it can be reconstructed as follows:

$$\begin{aligned}&\pi\left(Q\middle|Q_{u-1}^B < Q < Q_u^B, W_i, Q = Q(P,W_i), M_P^h\right) = \left(Q^{-\frac{1}{z_1}} {b_1}^{1/z_1}(b_2+W_i)^{z_2/z_1}\right)Q- \\ &TFC - \sum_{u=1}^{S}\left\{Q_u\left\{\sum_{v=1}^{u-1} C_v^P\left(Q_v^B - Q_{v-1}^B\right) + C_u^P\left(Q - Q_u^B\right)\right\}\right\} - (C_{mr}+ \\ &C_{pl}\int_{\varphi}^{\infty} G(t_r)dt_r E\left[N_{br}\middle|W_i, x, \delta_{pm}^h, \Psi_{\mathcal{D}}(s|\Theta)\right]Q - \sum_{k=1}^{\lfloor W_i/x\rfloor -1}(1+\tau_q kx)Q\end{aligned} \tag{27}$$

The objective function $\pi\left(Q\middle|Q_{u-1}^B < Q < Q_u^B, W_i, Q = Q(P,W_i)\right)$ is continuously differentiable in $[0,\infty)$. To verify whether the objective function is convex, the first and second order conditions must be examined in advance. The first order condition is to inspect the two inequalities ($\lim\limits_{Q\to 0}\frac{\partial\pi\left(Q|Q_{u-1}^B<Q<Q_u^B,W_i,Q=Q(P,W_i),M_P^h\right)}{\partial Q} > 0$ and $\lim\limits_{Q\to\infty}\frac{\partial\pi\left(Q|Q_{u-1}^B<Q<Q_u^B,W_i,Q=Q(P,W_i),M_P^h\right)}{\partial Q} < 0$ for $Q > 0$). The second order condition is to inspect $\frac{\partial^2\pi\left(Q|Q_{u-1}^B<Q<Q_u^B,W_i,Q=Q(P,W_i),M_P^h\right)}{\partial Q^2} < 0$ for $Q > 0$. If the above two conditions have been satisfied, the objective function $\pi\left(Q\middle|Q_{u-1}^B < Q < Q_u^B, W_i, Q = Q(P,W_i), M_P^h\right)$ is convex with respect to Q and has an unique root Q^{Root} (derived from the equation $\frac{\partial\pi\left(Q|Q_{u-1}^B<Q<Q_u^B,W_i,Q=Q(P,W_i),M_P^h\right)}{\partial Q} = 0$). In order to obtain the minimum value from a convex function, the volume of production has to be within a certain range defined by the production scale. Three independent cases are described as follows:

Case 1: In this case, the root Q^{Root} is within the range of a specific production scale ($Q_{u-1}^B < Q^{Root} < Q_u^B$), and the optimal production volume of the case should be set as

$$\begin{aligned}&Q^* = Q^{Root} = b_1(b_2+W_i)^{z_2} \\ &\left(\frac{z_1-1}{z_1}\left(C_u^P + \left(C_{mr} + C_{pl}\int_{\varphi}^{\infty} G(t_r)dt_r\right)E\left[N_{br}\middle|W_i, x, \delta_{pm}^h, \Psi_{\mathcal{D}}(s|\Theta)\right] + \sum_{k=1}^{\lfloor W_i/x\rfloor -1}(1+\tau_h kx)C_F^h\right)^{-z_1}\right)\end{aligned} \tag{28}$$

As a result, the optimal price for the case should be set as follows:

$$P^* = \left(\frac{b_1(b_2+W_i)^{z_2}}{Q^*}\right)^{\frac{1}{z_1}} = \frac{z_1-1}{z_1} \left(C_u^P + \left(C_{mr} + C_{pl}\int_{\varphi}^{\infty} G(t_r)dt_r\right)E\left[N_{br}\middle|W_i, x, \delta_{pm}^h, \Psi_{\mathcal{D}}(s|\Theta)\right] + \sum_{k=1}^{\lfloor W_i/x\rfloor-1}(1+\tau_h kx)C_F^h\right) \quad (29)$$

Case 2: In this case, the root Q^{Root} does not meet the minimum volume of the specific production scale. In other words, it means that Q^{Root} is smaller than the lower bound of the production scale Q_{u-1}^B. In order to comply with the constraint of the production scale, the optimal production volume in this case should be set as:

$$Q^* = Q_{u-1}^B \quad (30)$$

and the optimal price in this case should be:

$$P^* = \left(\frac{b_1(b_2+W_i)^{z_2}}{Q_{u-1}^B}\right)^{\frac{1}{z_1}} \quad (31)$$

Case 3: In this case, the root Q^{Root} is over the maximum volume of the specific production scale. This means that the production capacity cannot afford the ideal production volume, and therefore the manufacturer can only set the optimal production volume to the upper bound of this production scale as follows:

$$Q^* = Q_u^B \quad (32)$$

and the optimal price of the case should be:

$$P^* = \left(\frac{b_1(b_2+W_i)^{z_2}}{Q_u^B}\right)^{\frac{1}{z_1}} \quad (33)$$

In accordance with the mathematical analysis and discussion, the above inference can be used to develop a corresponding solution algorithm. The aforementioned cases did not completely consider all possible production scales and all candidate warranty terms. Accordingly, $P_{tmp}^{(i,u)}$, $Q_{tmp}^{(i,u)}$, and $\left\{Q_1^{(i,u)}, \ldots, Q_S^{(i,u)}\right\}$ are used as the temporary variables in developing the algorithm. In the case of each production scale Q_u, the warranty term W_i, price P, and quantity Q can be calculated according to Equations (28)–(33). Since the solution space for production scales $\{Q_1, \ldots, Q_S\}$ and candidate warranty terms $\{W_1, \ldots, W_n\}$ is finite, we can use an enumeration method to determine the optimal production scale and warranty term for a given price and production volume. Based on that mentioned above, an algorithm is proposed to obtain the optimal solutions of P^*, Q^*, W_i^*, and $\{Q_1{}^*, \ldots, Q_S{}^*\}$, which is given in Figure 5.

However, to deal with such complicated mathematical problems, a computerized application would be required. As can be seen in Figure 6, to increase the manageability of the whole application, it is possible to separate it into two independent systems (the model and information management system and the decision support system) so that they can be managed independently. In order to maintain failure datasets and estimate the progress of deteriorating model parameters, a model and information management system is primarily used. Additionally, cost or profit analyses need to include the different PM alternatives' parameter values, customer usage patterns, and associated probability distributions. Using a formalization mechanism might be different from a traditional relational database, since some data have a hierarchical structure, making it more convenient and efficient to store and retrieve information. Using the model and information management system, a firm's domain experts and engineers can store, retrieve, maintain, and analyze all of the required information. Furthermore, it is essential for the decision support system to be adopted in practice if the firm intends to provide useful information and suggestions to its managers. In order to analyze all possible alternatives, computation engines are needed for parameter estimation, solution algorithms, numerical integration, graphic presentation, and sensitivity analyses. Such computation engines can be developed by the R projects package, and system developers/programmers can write Java codes to apply them through an application programming interface (API).

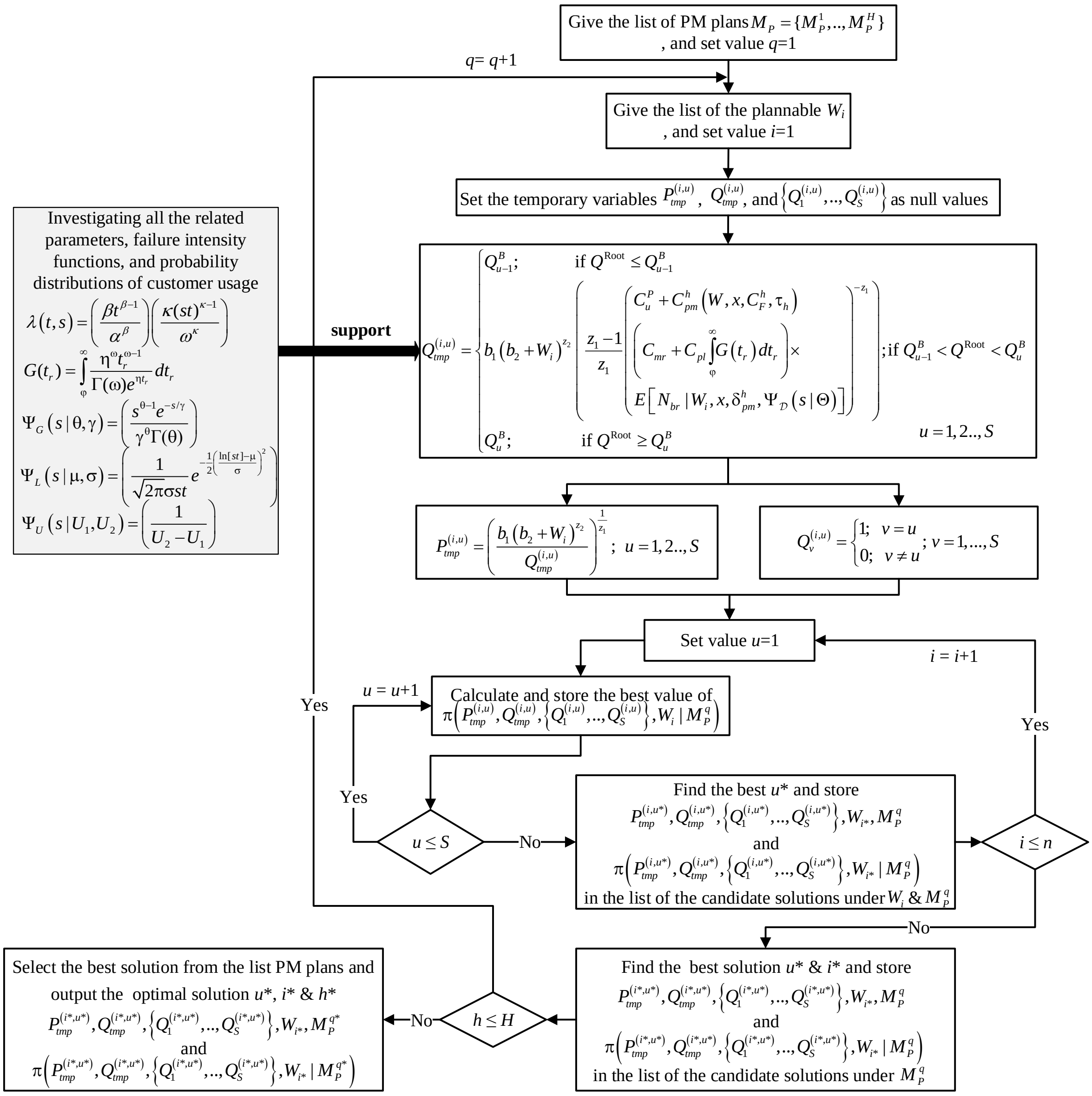

Figure 5. Flowchart for the solution algorithm.

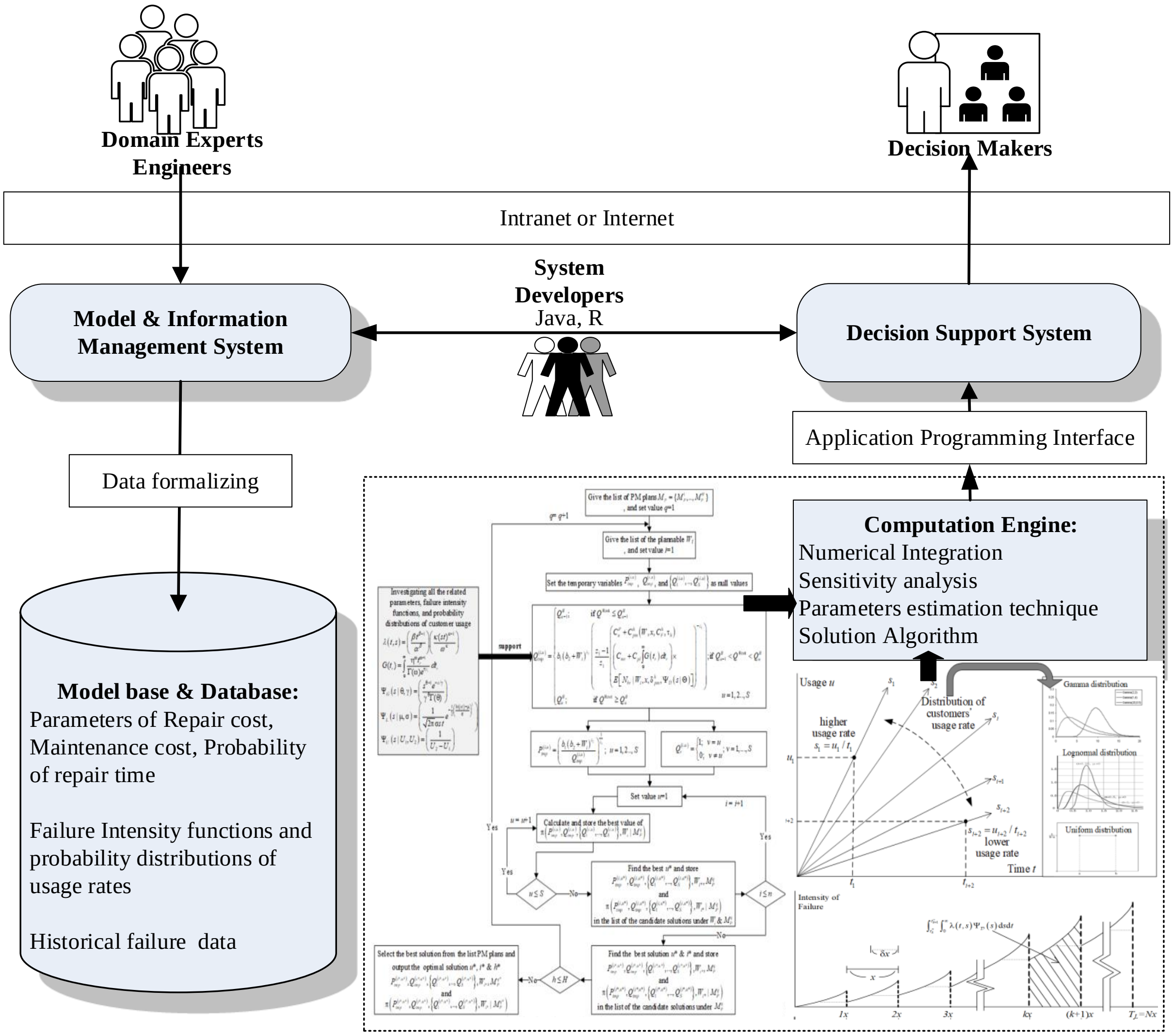

Figure 6. Design of the computerized information system.

4. Application and Sensitivity Analysis

4.1. Application

In order to illustrate the effectiveness of the proposed model, we present a practical example in this section. Assume that a manufacturer is going to sell new electronic and mechanical equipment on the market, and a free replacement warranty policy (FRW) will be adopted in accordance with the needs of the customers. Considering that equipment degradation is not only affected by time, but also by usage, domain experts and engineers are going to evaluate the characteristics of said equipment deterioration in order to make the right decision. Based on the evaluation conducted by domain experts and engineers, it is assumed that the equipment deterioration will follow the bivariate Weibull process. The parameters of the deterioration model can be estimated by analyzing the historical data and/or new data from accelerated deterioration experiments. Additionally, the customers are heterogeneous due to their customs and usage. Accordingly, it is assumed that the usage rates s of customers may follow a gamma distribution, where $E(s) = 1.5$ and $\sigma(s)^2 = 0.7$. In order to reduce the frequency of repairs to the equipment, the manufacturer provides a periodic PM policy with a maintenance interval of six months. In this case, there are five distinct PM alternatives recommended by reliability engineers, each of which result in different expenditures, as well as different recovery periods for the systems during the warranty period. However, the age reduction factor of some alternatives is greater than those of the others, but their PM costs and increasing rates are also higher than those of the others. Therefore, it is not easy for the manufacturer to determine which PM alternative is preferable. Due to the fact that the warranty length has an impact on the revenue and associated costs, the manufacturer should carefully choose a PM alternative and a warranty length to optimize their average profit as much as possible. There is also the fact that since the repair time is not constant and is affected by the degree of equipment breakdown, it is necessary to estimate the repair time at the beginning of the process. However, due to serious breakdowns, some repair works may take longer than expected, which will result in an increase in the penalty charge as a result of the extended time. Reliability engineers can assess whether there is a possibility that the repair time will exceed the tolerable limit. According to the results of the evaluation, the expected value $E(t_r)$ and the standard deviation $\sigma(t_r)$ for the time taken to perform a repair were determined to be 9 and 5 h, respectively. Moreover, since the different marketing strategies of manufacturers influence the market demand, the corresponding demand function needs to be estimated in advance. Based on the results of the market survey and analysis, the demand function can be given by $Q(P, W_i) \approx 236 \times 10^9 P^{-2.4}(3 + W_i)^{1.8}$. Moreover, the production condition plays a crucial role in the decision-making process, and this study classified the production scale into three categories: Normal, overtime, and outsourcing capacities. In terms of the production scale, the boundaries were 5500, 8500, and 12,000, respectively, with the unit production costs corresponding to \$1600, \$2400, and \$3200, for each range. The summary of the all relevant information about the case can be seen in Table 2.

Table 2. The relevant information of the case.

The parameter estimation of the two-dimensional deterioration	$\alpha = 1.2, \beta = 1.8, \omega = 1.5, \kappa = 2.8$
Interval time of the scheduled preventive maintenance	$x = 6$ months
The distribution of the usage rate	$\Psi_G(s\|\theta,\gamma) = \left(\frac{s^{\theta-1}e^{-s/\gamma}}{\gamma^\theta \Gamma(\theta)}\right)$ with $E(s) = 1.5$ and $\sigma(s)^2 = 0.7$
The age reduction factors for the alternatives (no. 1..5)	$\delta^h_{pm} = \{0.6, 0.65, 0.7, 0.75, 0.8\}$
The base cost for preventive maintenance (no. 1..5)	$C^h_F = \{\$30, \$35, \$40, \$45, \$50\}$
The increasing rates of preventive maintenance cost (no. 1..5)	$\tau_h = \{0.05, 0.07, 0.08, 0.09, 0.12\}$
The demand function	$Q(P, W_i) \approx 236 \times 10^9 P^{-2.4}(3 + W_i)^{1.8}$
The parameters of the production function	Q^B_1 = 5500, Q^B_2 = 8500, Q^B_3 = 12,000, C^P_1 = \$1600, C^P_2 = \$2400, C^P_3 = \$3200
The candidates of the planned warranty term	$W_i = \{2,\ 2.5,\ 3, \ldots,\ 7\}$
The setup cost of the production line (fixed cost)	$TFC = \$5,500,000$
The average cost for a minimal repair	C_{mr}= \$50
The penalty base cost if the repair time exceeds the time limit φ	$C_{pl} = \$30$
The statistical characteristics concerning a minimal repair	$E(t_r) = 9$ h, $\sigma(t_r) = 5$ h
Tolerable waiting time limit for a minimal repair	$\varphi = 4.5$ h

With the help of the algorithm proposed in Section 3.2, the model can be run in order to obtain the optimal solution, and the results of the calculation can be viewed as follows: (1) The expected revenue and expected profit are \$47,345,345 and \$18,627,200, respectively; (2) the equipment's price should be set at \$6587 per unit; (3) it is recommended that the warranty length be set at 5.5 years; (4) the best PM alternative is no.5; (5) in order to achieve the lower unit costs, the second production scale would be an appropriate choice, which has a unit cost of \$1600 for 5500 units and \$2400 for 1405 units; (6) the production cost is \$12,171,189, and the maintenance cost and the repair cost is \$4,936,833 and \$6,110,095, respectively. Figure 7 presents the search path for the optimal solution. This figure shows that the contour outlines are unsmooth, since the stepped production cost function is considered. Furthermore, in this case, slightly extending the warranty term can effectively increase the sales of electronic equipment on the market. Because the increment in revenue is greater than the bearing cost of extending the warranty term, the manufacturer would intend to raise the price and extend the warranty term simultaneously. It can be seen that the left side of Figure 8 indicates that moderately manipulating pricing and warranty strategies together will help the manufacturer earn more profit. However, increasing the production quantity may not be advantageous to the manufacturer (right side of Figure 8); decreasing the production quantity will benefit the manufacturer when the warranty term is over 3.5 years. Table 3 presents the optimal profit under different prices, quantities, and warranty terms. Concerning the relationship between the optimal price and production quantity, if the price rises to over $5552, the manufacturer has to decrease the production quantity to prevent oversupply.

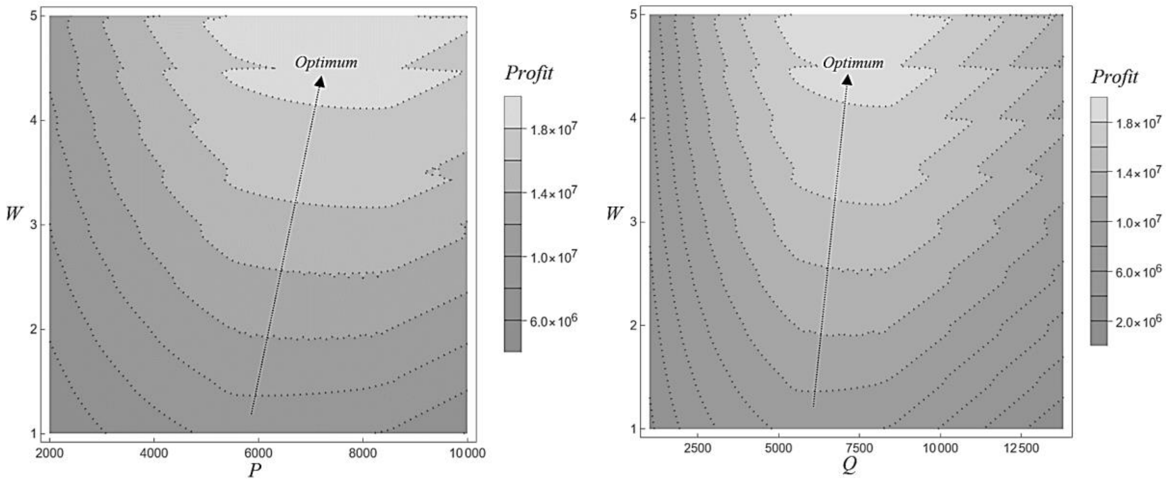

Figure 7. Contour plot of the search path for the optimal solution.

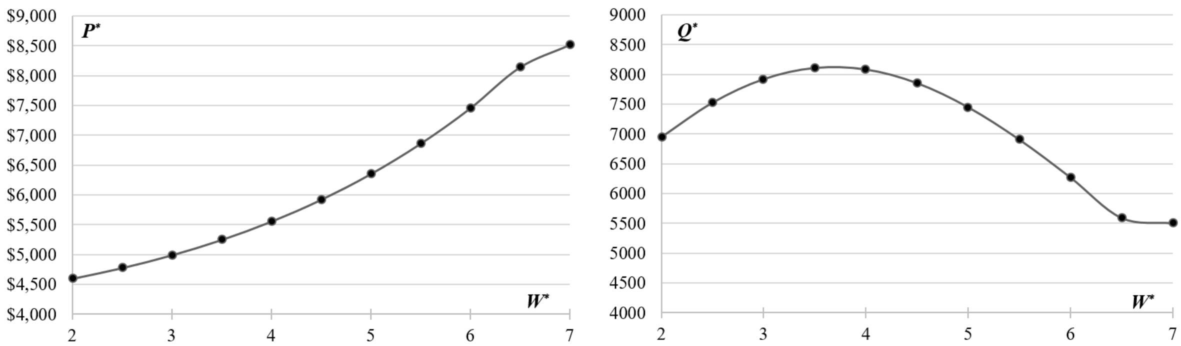

Figure 8. Optimal decisions under different combinations between prices and warranty terms or quantities and warranty terms.

Table 3. The expected profit under different price, quantity, and warranty term decisions.

W	*P**	*Q**	*Profit*
2	$4595	6944	$12,194,400
2.5	$4774	7521	$13,860,200
3	$4988	7916	$15,352,700
3.5	$5245	8104	$16,612,200
4	$5552	8079	$17,590,600
4.5	$5917	7851	$18,256,800
5	$6349	7446	$18,599,300
5.5	**$6857**	**6905**	**$18,627,200**
6	$7450	6271	$18,367,100
6.5	$8139	5591	$17,858,900
7	$8516	5500	$17,075,800

A comparison analysis was carried out to determine how the expected profit changes with the sale quantity under different warranty terms, so that the variation of the expected profit can be determined. It can be seen in Figure 9 that the profit curves present a concave shape, and they indicate that increasing production quantities will cause the profit eventually decline due to such increasing stepped production cost structure. Extending the warranty term is beneficial for earning customers. However, it may not be helpful to increase profits. According to Figure 9, we can see that the profit curve can move up until the warranty term is 5.5 years. Once the warranty term is over 5.5 years, the profit curve will move down. This indicates that the related repair and preventive maintenance costs will exponentially increase with extending the warranty term. Therefore, the manufacturer has to evaluate the most appropriate warranty term that can effectively earn more customers and can control and prevent the burden cost from drastically increasing.

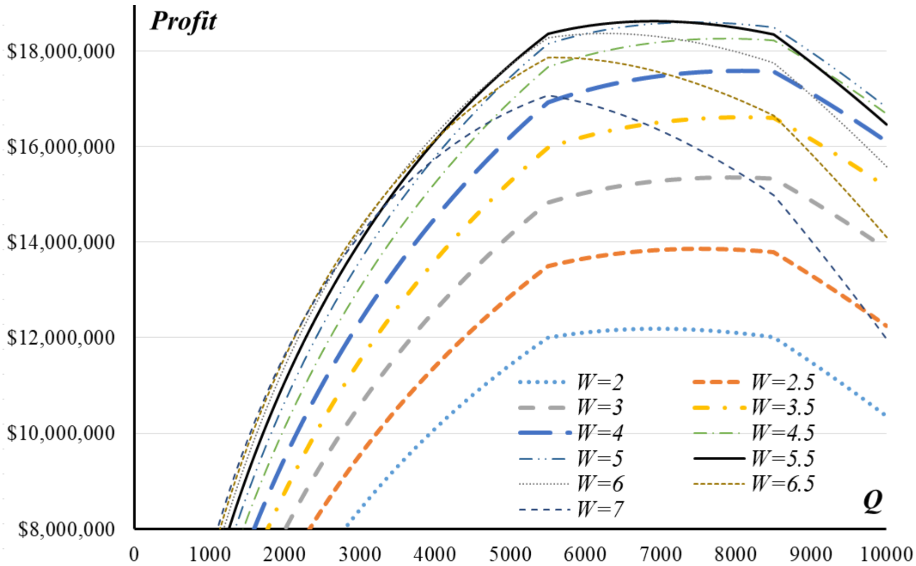

Figure 9. The influence of different quantities and warranty terms on the expected profits.

4.2. Sensitivity Analysis

As a result of an insufficient number of measurements collected from deterioration experiments, the estimation of the scale and shape parameters (α, ω, β, and κ) is subject to error. In order to prevent the prediction of an incorrect profit estimate from negatively impacting the forecasting of the average profit and the costs that are related to it, the manufacturer should be aware of any changes to the estimations as soon as they occur. Due to this, sensitivity analysis can be used as a method to assess the differences in the average profit estimates as a result. As a result, if we underestimate these parameters' values, we should assume that an incorrect estimate of the cost of repair will also be produced. In this scenario, people are prone to making incorrect decisions, which can result in incorrect warranty extensions or shortenings. The influence that the scale and form parameters (α, ω, β, and, κ) have on the anticipated amount of profit can be seen in Figure 10. In Figure 10, if the scale parameters are estimated to be larger, then the profits are also estimated to be amplified, since the breakdowns are estimated to be underestimated if the scale parameters are larger. On the contrary, overestimating the parameters for estimating the shape would lead to a decrease in the magnitude of the estimated profit, due to underestimating the shape parameters. A further point to note is that the parameters that have an impact on usage have a greater impact than those that have an impact on time. It may be said that the degradation of the use of a product is greater than the degradation of time, and the manufacturer should pay special attention to improvement of the usage components that are related to the degradation of the product in order to mitigate this degradation.

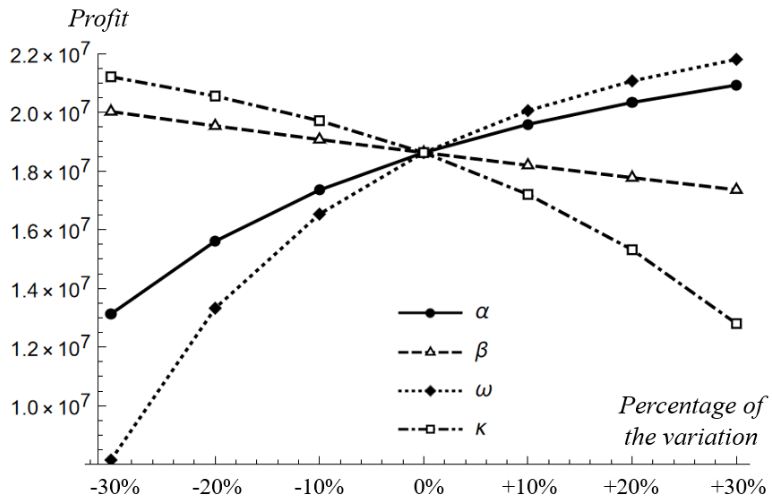

Figure 10. The influence of parameters α, ω, β, and κ on the expected profit.

The numerical results that were obtained after the proposed heuristic algorithm presented in Figure 5 had been applied are shown in Figure 11. According to Table 4, the optimal price and production quantity for the five PM alternatives should be $9540 and 3125, $8137 and 4579, $7539 and 5500, $7471 and 5621, and $6857 and 6905, with expected profits $6,923,870, $10,024,200, $13,530,400, $16,396,100, and $18,627,200, respectively. A less intensive PM alternative will lead to the decision of setting a lower production quantity and a higher price. It is obvious that PM alternative 5 is the most appropriate option for the manufacturer. As a general rule, higher intensive PM alternatives are able to reduce failure rates while reducing the expense of repairs. It is also important to note that the savings from repairs may not be enough to compensate for the increases in PM costs. Thus, it is not easy to judge which PM alternative is the best on the basis of intuition alone. Even though a lower-intensive PM alternative will result in significant failures during the post-phase, the manufacturer has the option to avoid this disadvantage by committing to a shorter warranty term if the manufacturer is forced to accept a lower intensive PM alternative because the critical components cannot be replaced. Moreover, in some cases, customers may not accept a long-term warranty contract, and therefore, the manufacturer should make an appropriate contract to different customers through negotiation. PM alternative 1 (lowest intensive PM) may be an appropriate option, for instance, if the manufacturer decides to shorten the warranty period for some reason in the future.

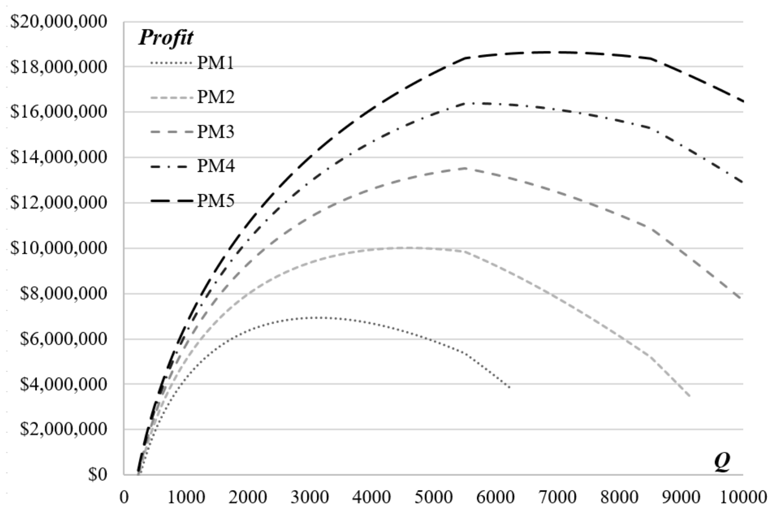

Figure 11. The expected profit versus production quantity for different PM alternatives.

Table 4. The optimal solutions under different prices and production quantities.

PM	*P**	*Q**	*Profit*
PM1	$9540	3125	$6,923,870
PM2	$8137	4579	$10,024,200
PM3	$7539	5500	$13,530,400
PM4	$7471	5621	$16,396,100
PM5	$6857	6905	$18,627,200

In view of the fact that each customer has different needs or ways of using their electronic and mechanical equipment, their electronic and mechanical equipment will present different levels of deterioration and different numbers of failures throughout the same warranty period. In order to evaluate the influence of different customers on equipment deterioration, the usage rate can be an appropriate indicator for calculating such different deteriorations. Figure 12 illustrates the influence of the variation of the expected value and variance of the usage rate on the manufacturer's expected profit, price, and quantity. In this case, this model is based on the premise that the usage rates for the system will follow a gamma distribution with $E(s) = 1.5$ and $V(s) = 0.7$. According to the information presented in Table 5, we can also see that the variation of $E(s)$ is more sensitive than that of $V(s)$. In this case, the sensitive result implies that the degree of usage degradation is much higher than the degree of time degradation. If the customer usage rate is higher than the manufacturer's original estimation, the optimal price should be higher, but the optimal production quantity should be decreased, since the higher usage rate causes the manufacturer have to bear more warranty-related expenditures. On the contrary, if the customer usage rate is lower, the manufacturer can use a lower-price strategy to earn more sales, since the marginal profit of the increment of sales would be more than the increment of the bearing costs. Additionally, due to the possibility that repair costs could increase or decrease in the future, it is important for the manufacturer to be aware of their effect on the profit or the strategy of sale price and production quantity. According to Figure 12 and Table 5, it can be seen that the manufacturer should take a price reduction strategy if the repair cost is going down in the future. This is because the lower repair cost can provide the manufacturer more space in the marginal profit to manipulate the price to earn more sales in the market.

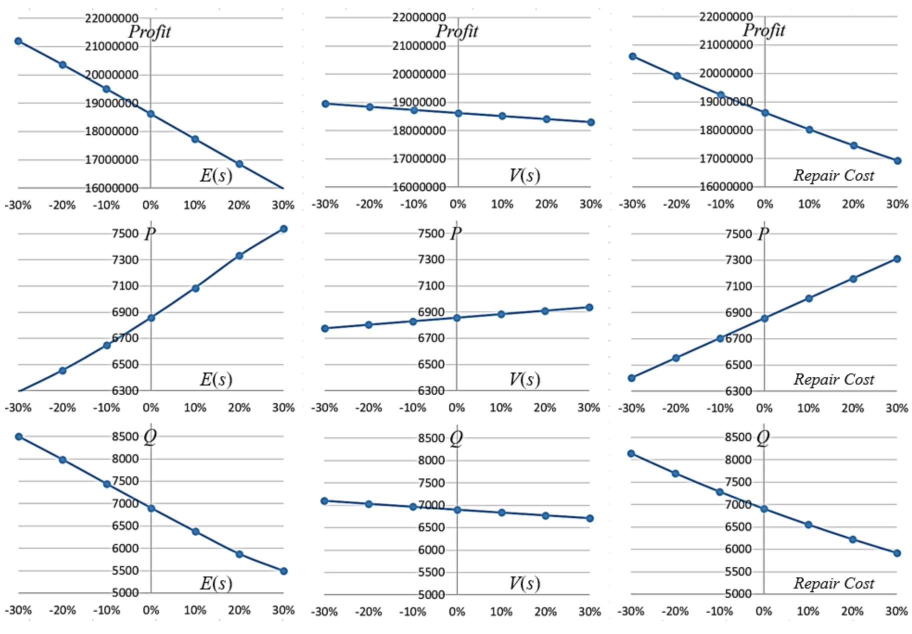

Figure 12. The influence of the variation of $E(s)$, $V(s)$, and repair costs on the expected profit, price, and quantity.

Table 5. The influence of the variation of $E(s)$, $V(s)$, and repair costs on the expected profit, price, and quantity.

Variation of E(s)	*P*	*Q*	*Profit*	*Variation of V(s)*	*P*	*Q*	*Profit*	*Variation of Repair Cost*	*P*	*Q*	*Profit*
−30%	6288	8500	21,201,600	−30%	6776	7103	18,956,600	−30%	6402	8142	20,618,100
−20%	6455	7983	20,369,900	−20%	6803	7036	18,845,300	−20%	6554	7697	19,917,600
−10%	6647	7441	19,507,300	−10%	6830	6970	18,735,500	−10%	6705	7286	19,254,900
0	6857	6905	18,627,200	0	6857	6905	18,627,200	0	6857	6905	18,627,200
10%	7086	6382	17,741,700	10%	6884	6841	18,520,300	10%	7009	6551	18,032,000
20%	7332	5879	16,860,700	20%	6910	6778	18,414,900	20%	7160	6223	17,467,000
30%	7539	5500	15,990,900	30%	6937	6716	18,310,700	30%	7312	5918	16,930,000

5. Conclusions

This study aimed to propose mathematical models and an efficient solution algorithm for dealing with the two-dimensional warranty issue by taking into consideration the various usage patterns of consumers, the stepped production cost function, and the periodic preventive maintenance service. There is a need to provide these things as part of the two-dimensional warranty issue that was addressed by this study. A non-homogeneous Poisson process was utilized for the goal of describing the sequential failure times of deteriorating products in order to meet the aforementioned purpose. In the past, the majority of research focused on how the conditions of the warranty affect the expenses related to the product, but only rarely did they explore how these factors could affect the strategic decision-making process in regard to marketing or production. The purpose of this article was to incorporate crucial choice variables into the analysis of decision variables. Some of these crucial decision variables are the product price, the length of the warranty, and the production capacity. A combination of considerations regarding price, manufacturing, and the manufacturer's warranty, in addition to the supply of the service of providing preventative maintenance, could perhaps lead the manufacturer to the most efficient plan. However, there are some limitations of this study. The failure process of a durable product must include a non-homogeneous Poisson process with a bivariate Weibull model. Moreover, all of the related deteriorating parameters, warranty expenditures, and marketing surveys can be reasonably estimated by engineering and marketing departments so that the manufacturer can utilize such information to make appropriate decisions.

Nevertheless, there are still some issues that have not been solved. Because the method of accelerated life testing might not be applicable to certain recently developed products, it may be impossible to acquire sufficient historical data to provide an accurate estimation of the rate of deterioration of products. Bayesian analysis is a method that may be useful in the absence of adequate historical data in order to solve the problem, because it can estimate the parameters using expert judgement and/or a limited number of relevant data points. Bayesian analysis is an approach that could be useful in the absence of sufficient historical data in order to solve the problem. Future work will concentrate on integrating Bayesian analysis and mathematical models in order to make decision-making more efficient and effective as a result of the research of the integration of these two methods. Bayesian analysis and mathematical models will be integrated in order to make decision-making more efficient and effective.

Author Contributions: Conceptualization, M.-N.C. and C.-C.F.; data curation, M.-N.C. and C.-C.F.; formal analysis, M.-N.C. and C.-C.F.; funding acquisition, M.-N.C.; investigation, C.-C.F.; methodology, M.-N.C. and C.-C.F.; project administration, M.-N.C. and C.-C.F.; resources, M.-N.C. and C.-C.F.; supervision, M.-N.C.; writing—review and editing, M.-N.C. and C.-C.F. All authors read and agreed to the published version of the manuscript.

Funding: This work was sponsored by Zhaoqing University's Science and Technology Foundation.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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