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A (2 + 1)-Dimensional Integrable Breaking Soliton Equation and Its Algebro-Geometric Solutions

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Abstract: A new (2 + 1)-dimensional breaking soliton equation with the help of the nonisospectral Lax pair is presented. It is shown that the compatible solutions of the first two nontrivial equations in the (1 + 1)-dimensional Kaup–Newell soliton hierarchy provide solutions of the new breaking soliton equation. Then, the new breaking soliton equation is decomposed into the systems of solvable ordinary differential equations. Finally, a hyperelliptic Riemann surface and Abel–Jacobi coordinates are introduced to straighten the associated flow, from which the algebro-geometric solutions of the new (2 + 1)-dimensional integrable equation are constructed by means of the Riemann θ functions.

Keywords: breaking soliton equation; algebro-geometric solution; Abel–Jacobi coordinates; Riemann θ function

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1. Introduction

Algebro-geometric solutions are an important class among exact solutions to nonlinear equations, which were first developed as analogs of inverse scattering theory. They can be regarded as explicit solutions of the nonlinear integrable evolution equation and used to approximate more general solutions. Algebro-geometric solutions can not only reveal the intrinsic structure of solutions, but also characterize the quasi-periodic behavior of nonlinear phenomena. Various approaches have been developed to obtain algebro-geometric solutions of soliton equations, such as the algebro-geometric approach [1], the nonlinearization of Lax pairs [2], the finite-order expansion of the Lax matrix [3], and so on [4–14].

On the one hand, based on the nonlinearization technique of Lax pairs and direct methods, many algebro-geometric solutions of (1 + 1)-dimensional [4–6], (2 + 1)-dimensional [3,7], and differential-difference [7,8] soliton equations have been obtained [9,10]. On the other hand, algebro-geometric solutions are successfully extended from a single equation to a hierarchy [11–13]. Recently, the Riemann–Hilbert method was also provided to solve algebro-geometric solutions of the Korteweg-de Vries equation [15]. And the algebro-geometric solutions of the entire Sine–Gordon hierarchy are constructed by using the asymptotic properties of the meromorphic function [13]. Compared with algebro-geometric solutions of the above well-known soliton equations, the study of algebro-geometric solutions of breaking soliton equations has received comparatively less attention.

It is well known that breaking soliton equations are types of nonlinear evolution equations which can be used to describe the interaction of a Riemann wave along the y-axis and the wave along the x-axis [16,17]. So, the derivation of new integrable breaking soliton equations is an interesting topic. In Ref. [18], we have given the algebro-geometric solutions of the known (2 + 1)-dimensional breaking soliton equation associated with the Ablowitz–Kaup–Newell–Segur soliton hierarchy resorting to the direct method.

In this paper, based on the well-known (1 + 1)-dimensional soliton equations, we propose the following (2 + 1)-dimensional integrable breaking soliton equation:

$$\begin{cases} w_t = \frac{i}{2} w_{xy} + \frac{1}{2}[w\partial_x^{-1}(wv)_y]_x, \\ v_t = -\frac{i}{2} v_{xy} + \frac{1}{2}[v\partial_x^{-1}(wv)_y]_x, \end{cases} \tag{1}$$

where x and y are the scaled space coordinates, t is the scaled time coordinate, q and r are functions of (x, y, t) which represent the wave profiles. To the best of our knowledge, the above (2 + 1)-dimensional breaking soliton equation has not before appeared in the literature. It is shown that the algebro-geometric solutions of the new (2 + 1)-dimensional breaking soliton equation are reduced to solving two systems of solvable ordinary differential equations by the variable separation technique. Thus, with the help of a hyperelliptic Riemann surface, Abel–Jacobi coordinates, and the Riemann θ functions, the algebro-geometric solutions of the new (2 + 1)-dimensional integrable equation can be obtained.

In Section 2 that follows, we construct the new (2 + 1)-dimensional breaking soliton Equation (1) associated with the Kaup–Newell soliton hierarchy. A zero curvature representation for (1) is presented. Then, in Section 3, based on the the solutions of the (1 + 1)-dimensional soliton equations and the elliptic coordinates, the solutions of the new (2 + 1)-dimensional integrable equation are reduced to solving ordinary differential equations. In Section 4, a hyperelliptic Riemann surface and Abel–Jacobi coordinates are introduced to straighten the associated flow. Jacobi's inversion problem is discussed, from which the algebro-geometric solutions of the new (2 + 1)-dimensional integrable equation are constructed in terms of the Riemann θ functions. A short conclusion is presented in Section 5.

2. The New (2 + 1)-Dimensional Integrable Breaking Soliton Equation

In this section, we shall construct the new (2 + 1)-dimensional breaking soliton equation associated with the Kaup–Newell soliton hierarchy. It is well known that the Kaup–Newell equation is an important nonlinear derivative Schödinger equation. Some corresponding results for the Chen–Lee–Liu equation and Gerdjikov–Ivanov equation can be obtained from the Kaup–Newell equation in principle [4,19].

The Kaup–Newell soliton hierarchy is an isospectral evolution equation hierarchy associated with the spectral problem [20]

$$\psi_x = U\psi = \begin{pmatrix} -i\lambda^2 & q\lambda \\ r\lambda & i\lambda^2 \end{pmatrix}\psi, \quad \lambda_t = 0, \quad \psi = (\psi_1, \psi_2)^T, \tag{2}$$

where q and r are two scalar potentials, and λ is a constant spectral parameter. To derive the hierarchy, we first introduce the Lenard gradient sequence $\{S_j\}_{j=0}^{\infty}$ by the recursion relation

$$KS_{j-1} = JS_j, \quad S_j|_{(q,r)=(0,0)} = 0, \quad S_0 = (2r, 2q, -2i)^T, \tag{3}$$

where $S_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$ and

$$K = \begin{pmatrix} -\frac{i}{2}\partial & 0 & 0 \\ 0 & \frac{i}{2}\partial & 0 \\ -q & r & \partial \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & -ir \\ 0 & 1 & -iq \\ -q & r & \partial \end{pmatrix}.$$

It is easy to see that S_j is uniquely determined by the recursion relation (3). A direct calculation gives

$$S_1 = \begin{pmatrix} -ir_x + qr^2 \\ iq_x + q^2 r \\ -iqr \end{pmatrix}, \quad S_2 = \begin{pmatrix} -\frac{1}{2}r_{xx} - \frac{3i}{2}rr_x q + \frac{3}{4}q^2 r^3 \\ -\frac{1}{2}q_{xx} + \frac{3i}{2}qq_x r + \frac{3}{4}q^3 r^2 \\ \frac{1}{2}(q_x r - r_x q) - \frac{3i}{4}q^2 r^2 \end{pmatrix},$$

the auxiliary spectral of (2)

$$\psi_{tn} = V\psi = \begin{pmatrix} C & B \\ A & -C \end{pmatrix}\psi = \begin{pmatrix} \sum_{j=0}^{n} S_j^{(3)}\lambda^{2(n-j)+2} & \sum_{j=0}^{n} S_j^{(2)}\lambda^{2(n-j)+1} \\ \sum_{j=0}^{n} S_j^{(1)}\lambda^{2(n-j)+1} & -\sum_{j=0}^{n} S_j^{(3)}\lambda^{2(n-j)+2} \end{pmatrix}\psi, \quad (4)$$

The compatibility condition between (2) and (4) is the stationary zero curvature equation

$$U_{tn} - V_x^{(n)} + [U, V^{(n)}] = 0,$$

which is equivalent to the well-known Kaup–Newell hierarchy of soliton equations

$$X_n = \begin{pmatrix} q_{tn} \\ r_{tn} \end{pmatrix} = \begin{pmatrix} S_{nx}^{(2)} \\ S_{nx}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}\begin{pmatrix} S_n^{(1)} \\ S_n^{(2)} \end{pmatrix}. \ n = 1, 2, \ldots \quad (5)$$

The first two nontrivial members $n = 1$ and $n = 2$ in the hierarchy are

$$\begin{cases} q_{t_1} = iq_{xx} + (q^2 r)_x, \\ r_{t_1} = -ir_{xx} + (r^2 q)_x; \end{cases} \quad (6)$$

and

$$\begin{cases} q_{t_2} = -\frac{1}{2}q_{xxx} + \frac{3}{4}(2iqq_x r + q^3 r^2)_x, \\ r_{t_2} = -\frac{1}{2}r_{xxx} - \frac{3}{4}(2irr_x q - r^3 q^2)_x. \end{cases} \quad (7)$$

Let $t_1 = y$, $t_2 = t$ and $w(x,y,t) = q(x,y,t)$, $v(x,y,t) = r(x,y,t)$ in (6) and (7); then, we can obtain

$$(wv_x - vw_x + \frac{3i}{2}(w^2 v^2))_x = (iwv)_y. \quad (8)$$

We substitute (6) into (7) and note that (8) yields the (2 + 1)-dimensional Equation (1). Therefore, if q and r are the compatible solutions of (6) and (7), we can see that $w = q$ and $v = r$ are also the solutions of the (2 + 1)-dimensional equation (1), where ∂_x^{-1} represents an inverse operator of $\partial_x = \partial/\partial x$ with the condition $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$, which can be defined as $(\partial_x^{-1}f)(x) = \int_{-\infty}^{x} f(y)dy$ under the decaying condition at infinity.

In the following, we can check that the (2 + 1)-dimensional Equation (1) has non-isospectral zero curvature representation:

$$M_t - N_x + [M, N] - \lambda M_y = 0,$$

which can be deduced from the compatibility condition of the following equations:

$$\varphi_x = M\varphi, \quad \varphi_t = \lambda\varphi_y + N\varphi, \quad \lambda_t = \lambda\lambda_y,$$

where $\lambda = \lambda(y,t)$, and

$$M = \begin{pmatrix} -i\lambda & \lambda w \\ v & i\lambda \end{pmatrix}, \quad N = \begin{pmatrix} -\frac{i\lambda}{2}\partial_x^{-1}(wv)_y & \frac{\lambda}{2}(iw_y + w\partial_x^{-1}(wv)_y) \\ \frac{1}{2}(-iv_y + v\partial_x^{-1}(wv)_y) & \frac{i\lambda}{2}\partial_x^{-1}(wv)_y \end{pmatrix}. \quad (9)$$

Therefore, the new (2 + 1)-dimensional Equation (1) is integrable in the Lax sense. The parameter λ satisfies a Riemann equation $\lambda_t = \lambda\lambda_y$, so (1) is also a breaking soliton equation, which is not the same as that in Ref. [16]

3. Variable Separation

In this section, we shall show how the (1 + 1)-dimensional (6) and (7) are reduced to solvable ordinary differential equations. Assume that (2) and (4) have two basic solutions $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$. We define a matrix W of three functions f, g, h by

$$W = \frac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify by (2) and (4) that

$$W_x = [U, W], \quad W_{t_n} = [V^{(n)}, W], \tag{10}$$

which imply that the function $detW$ is a constant independent of x and t. Equation (10) can be written as

$$g_x = -2ig\lambda^2 - 2qf\lambda, \quad h_x = 2ih\lambda^2 + 2rf\lambda, \quad f_x = \lambda(qh - rg), \tag{11}$$

and

$$g_t = 2gA - 2fB, \quad h_t = 2fC - 2hA, \quad f_t = hB - gC. \tag{12}$$

Now, suppose that the functions f, g, and h are finite-order polynomials in λ:

$$f = \sum_{j=0}^{N} f_{2j+1}\lambda^{2(N-j)+1}, \quad g = \sum_{j=0}^{N} g_{2j}\lambda^{2(N-j)}, \quad h = \sum_{j=0}^{N} h_{2j}\lambda^{2(N-j)}. \tag{13}$$

Substituting (13) into (11) yields

$$KG_{j-1} = JG_j, \quad JG_0 = 0, \quad KG_N = 0, \quad G_j = (h_{2j}, g_{2j}, f_{2j+1})^T. \tag{14}$$

It is easy to see that $JG_0 = 0$ has the general solution

$$G_0 = \alpha_0 S_0, \tag{15}$$

where α_0 is a constant of integration. So, $KerJ = \{cS_0|\forall c\}$. Acting with the operator $(J^{-1}K)^{K+1}$ upon (15), we can obtain from (3) and (14) that

$$G_k = \sum_{j=0}^{k} \alpha_j S_{k-j}, \quad k = 0, 1, \ldots \tag{16}$$

where α_0, ..., α_k are integral constants. Substituting (16) into (14) obtains the following stationary evolution equation:

$$\alpha_0 KS_N + \ldots + \alpha_N KS_0 = 0. \tag{17}$$

This means that expression (13) is existent.

In what follows, we decompose (6) and (7) into systems of integrable ordinary differential equations. Without loss of generality, let $\alpha_0 = 1$. From (3) and (14), we have

$$\begin{aligned}
&g_0 = 2q, h_0 = 2r, f_1 = -2i, f_3 = -iqr + \alpha_1,\\
&g_2 = iq_x + q^2 r + iq\alpha_1, h_2 = -ir_x + qr^2 + ir\alpha_1,\\
&g_4 = -\frac{1}{2}q_{xx} + \frac{3i}{2}qq_x r + \frac{3}{4}q^3 r^2 + \frac{\alpha_1}{2}(iq^2 r - q_x) + i\alpha_2 q,\\
&h_4 = -\frac{1}{2}r_{xx} - \frac{3i}{2}rr_x q + \frac{3}{4}q^2 r^3 + \frac{\alpha_1}{2}(iq^2 r + r_x) + i\alpha_2 r,\\
&f_5 = \frac{1}{2}(q_x r - r_x q) - \frac{3i}{4}q^2 r^2 + \frac{\alpha_1}{2}(qr) + \alpha_2, \ldots
\end{aligned} \tag{18}$$

We can write g and h as the following finite products:

$$g = 2q\prod_{j=1}^{N}(\lambda^2 - \mu_j^2) = 2q\prod_{j=1}^{N}(\tilde{\lambda} - \tilde{\mu}_j), \quad h = 2r\prod_{j=1}^{N}(\lambda^2 - \nu_j^2) = 2r\prod_{j=1}^{N}(\tilde{\lambda} - \tilde{\nu}_j), \tag{19}$$

where $\lambda^2 = \tilde{\lambda}, \mu_j^2 = \tilde{\mu}_j, \nu_j^2 = \tilde{\nu}_j$. Comparing the coefficients of $\tilde{\lambda}^{N-1}$ and $\tilde{\lambda}^{N-2}$, we obtain

$$\begin{aligned}
&g_2 = -2q\sum_{j=1}^{N}\tilde{\mu}_j, h_2 = -2r\sum_{j=1}^{N}\tilde{\nu}_j,\\
&g_4 = 2q\sum_{i<j}\tilde{\mu}_i\tilde{\mu}_j, h_4 = 2r\sum_{i<j}\tilde{\nu}_i\tilde{\nu}_j.
\end{aligned} \tag{20}$$

Thus, from (18) and (20), we obtain

$$\begin{aligned}
&\partial_x \ln q - iqr + \alpha_1 = 2i\sum_{j=1}^{N}\tilde{\mu}_j,\\
&-\partial_x \ln r - iqr + \alpha_1 = 2i\sum_{j=1}^{N}\tilde{\nu}_j.
\end{aligned} \tag{21}$$

If using the third expression in (12) with $m = 1$, we have

$$(-iqr)_y = h_2 iq_x + h_4 2q + g_2 ir_x - g_4 2r, \tag{22}$$

which, together with (20), yields

$$\partial_y(\ln qr) = 2\sum_{j=1}^{N}\tilde{\nu}_j\partial_x \ln q + 4i\sum_{i<j}(\tilde{\mu}_i\tilde{\mu}_j - \tilde{\nu}_i\tilde{\nu}_j) + 2\sum_{j=1}^{N}\tilde{\mu}_j\partial_x \ln r. \tag{23}$$

On the other hand, noticing (6) and (21), we obtain

$$\begin{aligned}
\partial_y(\ln qr) &= (i\partial_x \ln q + qr)_x + (-i\partial_x \ln r + qr)_x + i(\partial_x \ln q - iqr)^2 + i(i\partial_x \ln r + qr)^2 - \partial_x(qr)\\
&= -2\partial\sum_{j=1}^{N}(\tilde{\mu}_j + \tilde{\nu}_j) + 4i((\sum_{j=1}^{N}\tilde{\nu}_j)^2 - (\sum_{j=1}^{N}\tilde{\mu}_j)^2) + 4\alpha_1\sum_{j=1}^{N}(\tilde{\mu}_j - \tilde{\nu}_j) - \partial_x(qr),
\end{aligned}$$

which, together with (21) and (23), implies

$$\begin{aligned}
&\partial_x(qr) + 2i\sum_{j=1}^{N}(\tilde{\nu}_j - \tilde{\mu}_j)qr\\
&= -2\partial\sum_{j=1}^{N}(\tilde{\mu}_j + \tilde{\nu}_j) + 4i((\sum_{j=1}^{N}\tilde{\nu}_j)^2 - (\sum_{j=1}^{N}\tilde{\mu}_j)^2) + 2\alpha_1\sum_{j=1}^{N}(\tilde{\mu}_j - \tilde{\nu}_j) - 4i\sum_{i<j}(\tilde{\mu}_i\tilde{\mu}_j - \tilde{\nu}_i\tilde{\nu}_j).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
qr = exp[2i \int \sum_{j=1}^{N} (\tilde{\mu}_j - \tilde{\nu}_j) dx] \int [-2\partial \sum_{j=1}^{N} (\tilde{\mu}_j + \tilde{\nu}_j) + 6i((\sum_{j=1}^{N} \tilde{\nu}_j)^2 - (\sum_{j=1}^{N} \tilde{\mu}_j)^2) \\
+2\alpha_1 \sum_{j=1}^{N} (\tilde{\mu}_j - \tilde{\nu}_j) + 2i(\sum_{j=1}^{N} \tilde{\mu}_j{}^2 - \sum_{j=1}^{N} \tilde{\nu}_j{}^2)] exp[2i \int \sum_{j=1}^{N} (\tilde{\nu}_j - \tilde{\mu}_j) dx] dx,
\end{aligned}
\tag{24}
$$

in view of the equality

$$
2 \sum_{i<j}^{N} \xi_i \xi_j = (\sum_{i=1}^{N} \xi_i)^2 - \sum_{i=1}^{N} \xi_i^2. \tag{25}
$$

Let us consider the function $detW$, which is a $(2N+1)$-order polynomial in λ with constant coefficients of the x flow and t_n flow

$$
-detW = f^2 + gh = -4 \prod_{j=1}^{2N+1} (\lambda^2 - \lambda_j^2) = -4 \prod_{j=1}^{2N+1} (\tilde{\lambda} - \tilde{\lambda}_j) \equiv -4 \frac{R(\tilde{\lambda})}{\tilde{\lambda}}. \tag{26}
$$

Substituting (13) into (26), comparing the coefficient of $\tilde{\lambda}^{2N}$ and $\tilde{\lambda}^{2N-1}$, and considering (18), we can obtain

$$
2f_1 f_3 + g_0 h_0 = 4 \sum_{j=1}^{2N+1} \tilde{\lambda}_j, \quad 2f_1 f_5 + f_3^2 + g_0 h_2 + h_0 g_2 = -4 \sum_{i<j} \tilde{\lambda}_i \tilde{\lambda}_j. \tag{27}
$$

Together with (18), we have

$$
\alpha_1 = i \sum_{j=1}^{2N+1} \tilde{\lambda}_j, \quad \alpha_2 = -i \sum_{i<j} \tilde{\lambda}_i \tilde{\lambda}_j + \frac{i}{4} (\sum_{j=1}^{2N+1} \tilde{\lambda}_j)^2. \tag{28}
$$

From (26), we see that

$$
f|_{\tilde{\lambda} = \tilde{\mu}_k} = 2i \sqrt{R(\tilde{\mu}_k)/\tilde{\mu}_k}, \quad f|_{\tilde{\lambda} = \tilde{\nu}_k} = \sqrt{R(\tilde{\nu}_k)/\tilde{\nu}_k}. \tag{29}
$$

Using (11) and (19), we obtain

$$
\begin{aligned}
g_x|_{\tilde{\lambda} = \tilde{\mu}_k} = -2q\sqrt{\tilde{\lambda}} f|_{\lambda = \mu_k} = -2q\mu_{kx} \prod_{j=1, j\neq k}^{N} (\tilde{\mu}_k - \tilde{\mu}_j), \\
h_x|_{\tilde{\lambda} = \tilde{\nu}_k} = 2r\sqrt{\tilde{\lambda}} f|_{\tilde{\lambda} = \tilde{\nu}_k} = -2r\tilde{\nu}_{kx} \prod_{j=1, j\neq k}^{N} (\tilde{\nu}_k - \tilde{\nu}_j).
\end{aligned}
\tag{30}
$$

Together with (29), we obtain

$$
\tilde{\mu}_{kx} = \frac{2i\sqrt{R(\tilde{\mu}_k)}}{\prod_{j=1, j\neq k}^{N} (\tilde{\mu}_k - \tilde{\mu}_j)}, \quad \tilde{\nu}_{kx} = \frac{-2i\sqrt{R(\tilde{\nu}_k)}}{\prod_{j=1, j\neq k}^{N} (\tilde{\nu}_k - \tilde{\nu}_j)}. \tag{31}
$$

Similarly, using (4) (n = 1, n = 2), (13), (19), and (29), we obtain

$$
\begin{aligned}
g_{t_1}|_{\tilde{\lambda} = \tilde{\mu}_k} = -2q\tilde{\mu}_{kt_1} \prod_{j=1, j\neq k}^{N} (\tilde{\mu}_k - \tilde{\mu}_j) = -4iq\sqrt{R(\tilde{\mu}_k)} (2\tilde{\mu}_k - 2\sum_{j=1}^{N} \tilde{\mu}_j - i\alpha_1), \\
h_{t_1}|_{\tilde{\lambda} = \tilde{\nu}_k} = -2r\tilde{\nu}_{kt_1} \prod_{j=1, j\neq k}^{N} (\tilde{\nu}_k - \tilde{\nu}_j) = 4ir\sqrt{R(\tilde{\nu}_k)} (2\tilde{\nu}_k - 2\sum_{j=1}^{N} \tilde{\nu}_j - i\alpha_1),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
g_{t_2}|_{\tilde{\lambda}=\tilde{\mu}_k} &= -4iq\sqrt{R(\tilde{\mu}_k)}(2\tilde{\mu}_k^2 - (2\sum_{j=1}^{N}\tilde{\mu}_j + i\alpha_1)\tilde{\mu}_k + 2\sum_{i<j}\tilde{\mu}_i\tilde{\mu}_j + \frac{i\alpha_1}{2}(2\sum_{j=1}^{N}\tilde{\mu}_j + i\alpha_1) - i\alpha_2), \\
h_{t_2}|_{\tilde{\lambda}=\tilde{\nu}_k} &= 4ir\sqrt{R(\tilde{\nu}_k)}(2\tilde{\nu}_k^2 - (2\sum_{j=1}^{N}\tilde{\nu}_j + i\alpha_1)\tilde{\nu}_k + 2\sum_{i<j}\tilde{\nu}_i\tilde{\nu}_j + \frac{i\alpha_1}{2}(2\sum_{j=1}^{N}\tilde{\nu}_j + i\alpha_1) - i\alpha_2),
\end{aligned} \tag{33}
$$

which give

$$
\begin{aligned}
\mu_{kt_1} &= \frac{2i\sqrt{R(\tilde{\mu}_k)}}{\prod_{j=1,j\neq k}^{N}(\tilde{\mu}_k - \tilde{\mu}_j)}(2\tilde{\mu}_k - 2\sum_{j=1}^{N}\tilde{\mu}_j - i\alpha_1), \\
\nu_{kt_1} &= \frac{-2i\sqrt{R(\tilde{\nu}_k)}}{\prod_{j=1,j\neq k}^{N}(\tilde{\nu}_k - \tilde{\nu}_j)}(2\tilde{\nu}_k - 2\sum_{j=1}^{N}\tilde{\nu}_j - i\alpha_1),
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
\mu_{kt_2} &= \frac{2i\sqrt{R(\tilde{\mu}_k)}}{\prod_{j=1,j\neq k}^{N}(\tilde{\mu}_k - \tilde{\mu}_j)}(2\tilde{\mu}_k^2 - (2\sum_{j=1}^{N}\tilde{\mu}_j + i\alpha_1)\tilde{\mu}_k + 2\sum_{i<j}\tilde{\mu}_i\tilde{\mu}_j + \frac{i\alpha_1}{2}(2\sum_{j=1}^{N}\tilde{\mu}_j + i\alpha_1) - i\alpha_2), \\
\nu_{kt_2} &= \frac{2\sqrt{R(\nu_k)}}{\prod_{j=1,j\neq k}^{N}(\nu_k - \nu_j)}(2\tilde{\nu}_k^2 - (2\sum_{j=1}^{N}\tilde{\nu}_j + i\alpha_1)\tilde{\nu}_k + 2\sum_{i<j}\tilde{\nu}_i\tilde{\nu}_j + \frac{i\alpha_1}{2}(2\sum_{j=1}^{N}\tilde{\nu}_j + i\alpha_1) - i\alpha_2).
\end{aligned} \tag{35}
$$

In summary, if $\tilde{\lambda}_1, ..., \tilde{\lambda}_{2N+2}$ are $2N+2$ distinct parameters, $\tilde{\mu}_k, \tilde{\nu}_k (k = 1, \cdots, N)$ are compatible solutions of differential Equations (31), (34) and (35). Then, q and r determined by (21) and (24) is the compatible solution of (6) and (7), which means the (1 + 1)-dimensional soliton Equations (6) and (7) are decomposed into solvable ordinary differential equations with the help of the coordinates $\tilde{\mu}_k, \tilde{\nu}_k (k = 1, \cdots, N)$, so we can see that w and v is also a solution of the (2 + 1)-dimensional Equation (1).

4. Algebro-Geometric Solution

In this section, in order to obtain the algebro-geometric solutions of the (2 + 1)-dimensional breaking soliton Equation (1), we first introduce the hyperelliptic Riemann surface

$$
\Gamma : \xi^2 = R(\tilde{\lambda}), \quad R(\tilde{\lambda}) = \prod_{j=1}^{2N+2}(\tilde{\lambda} - \tilde{\lambda}_j),
$$

with genus $g = N$. On Γ, there are two infinite points, ∞_1 and ∞_2, which are not branch points of Γ. Equip Γ with the canonical basis of cycles $a_1, ..., a_N; b_1, ..., b_N$, and the holomorphic differentials

$$
\widetilde{\omega}_l = \frac{\tilde{\lambda}^{l-1} d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \quad l = 1, 2, ..., N.
$$

Then, the period matrices A and B, which are $N * N$ invertible matrices [21], are defined by

$$
A_{ij} = \int_{a_j} \widetilde{\omega}_i, \quad B_{ij} = \int_{b_j} \widetilde{\omega}_i.
$$

Using A and B, we can define matrices C and τ, where

$$
C = (C_{ij}) = A^{-1}, \quad \tau = (\tau_{ij}) = CB = A^{-1}B,
$$

then, matrix τ can be shown to be symmetric and it has a positive defined imaginary part. We normalize $\widetilde{\omega_j}$ into the new basis ω_j

$$\omega_j = \sum_{l=1}^{N} C_{jl}\widetilde{\omega_l}, \quad l = 1, 2, ..., N,$$

which satisfy

$$\int_{a_k} \omega_i = \sum_{l=1}^{N} C_{jl} \int_{a_k} \widetilde{\omega_l} = \sum_{l=1}^{N} C_{jl} A_{lk} = \delta_{jk}, \quad \int_{b_k} \omega_i = \sum_{l=1}^{N} C_{jl} \int_{b_k} \widetilde{\omega_l} = \sum_{l=1}^{N} C_{jl} B_{lk} = \tau_{jk}.$$

For a fixed point p_0, then we introduce an Abel–Jacobi coordinate as follows:

$$\rho_m = (\rho_m^{(1)}, \rho_m^{(2)}, \cdots, \rho_m^{(N)})^T, \quad m = 1, 2, \tag{36}$$

where

$$\rho_1^{(j)}(x,y,t) = \sum_{k=1}^{N} \int_{p_0}^{\tilde{\mu}_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{p_0}^{\tilde{\mu}_k(x,y,t)} C_{jl} \frac{\tilde{\lambda}^{l-1} d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \tag{37}$$

$$\rho_2^{(j)}(x,y,t) = \sum_{k=1}^{N} \int_{p_0}^{\tilde{\nu}_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{p_0}^{\tilde{\nu}_k(x,y,t)} C_{jl} \frac{\tilde{\lambda}^{l-1} d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}. \tag{38}$$

From (37) and the first expression of (31), we obtain

$$\partial_x \rho_1^{(j)} = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \frac{\tilde{\mu}_k^{l-1} \tilde{\mu}_{kx}}{\sqrt{R(\tilde{\lambda})}} = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \frac{2i\tilde{\mu}_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(\tilde{\mu}_k - \tilde{\mu}_j)} = 2iC_{jN} = \Omega_0^{(j)}, j = 1, \cdots, N, \tag{39}$$

with the help of the following equality:

$$\sum_{k=1}^{N} \frac{\tilde{\mu}_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(\tilde{\mu}_k - \tilde{\mu}_j)} = \delta_{jN}, \quad l = 1, \cdots, N.$$

In a similar way, we see from (31), (34), (35), (37), and (38) that

$$\partial_y \rho_1^{(j)} = 2i(C_{j,N-1} - i\alpha_1 C_{jN}) = \Omega_1^{(j)},$$
$$\partial_t \rho_1^{(j)} = 2i(C_{j,N-2} - \alpha_1 C_{j,N-1} + (\alpha_1^2 - \alpha_2)C_{jN}) = \Omega_2^{(j)},$$
$$\partial_x \rho_2^{(j)} = -\Omega_0^{(j)}, \quad \partial_y \rho_2^{(j)} = -\Omega_1^{(j)}, \quad \partial_t \rho_2^{(j)} = -\Omega_2^{(j)}.$$

On the basis of these results, we obtain the following:

$$\rho_1^{(j)}(x,y,t) = \Omega_0^{(j)} x + \Omega_1^{(j)} y + \Omega_2^{(j)} t + \gamma_0^{(j)}, \quad \rho_2^{(j)}(x,y,t) = -\Omega_0^{(j)} x - \Omega_1^{(j)} y - \Omega_2^{(j)} t + \gamma_1^{(j)},$$

where

$$\gamma_0^{(j)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} \omega_j, \quad \gamma_1^{(j)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} \omega_j.$$

An Abel map on Γ is defined as

$$A(p) = \int_{p_0}^{p} \omega, \omega = (\omega_1, \cdots, \omega_N)^T, \quad A(\sum n_k p_k) = \sum n_k A(p_k).$$

Consider two special divisors $\sum_{k=1}^{N} p_m^{(k)} (m = 1, 2)$, and we have

$$A(\sum_{k=1}^{N} p_1^{(k)}) = \sum_{k=1}^{N} A(p_1^{(k)}) = \sum_{k=1}^{N} \int_{p_0}^{\widetilde{\mu}_k} \omega = \rho_1,$$
$$A(\sum_{k=1}^{N} p_2^{(k)}) = \sum_{k=1}^{N} A(p_2^{(k)}) = \sum_{k=1}^{N} \int_{p_0}^{\widetilde{\nu}_k} \omega = \rho_2, \quad (40)$$

where $p_1^{(k)} = (\widetilde{\mu}_k, \xi(\widetilde{\mu}_k)), p_2^{(k)} = (\widetilde{\mu}_k, \xi(\widetilde{\mu}_k))$. The Riemann θ function of Γ is defined as

$$\theta(\zeta) = \sum_{z\in Z^N} exp(\pi i\langle \tau z, z\rangle + 2\pi i\langle \zeta, z\rangle), \quad \zeta \in C^N,$$

where $\zeta = (\zeta_1, \cdots, \zeta_N)^T, \langle \zeta, z\rangle = \sum_{j=1}^{N} \zeta_j z_j$. According to the Riemann theorem, there exist two constant vectors, $M_1, M_2 \in C^N$, such that

$$F_m = \theta(A(p) - \rho_m - M_m), \quad m = 1, 2$$

have exactly zeros at $\tilde{\mu}_1, \cdots, \tilde{\mu}_N$ for $m = 1$ or $\tilde{\nu}_1, \cdots, \tilde{\nu}_N$ for $m = 2$. To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. Notice the fact that the integrals

$$\frac{1}{2\pi i}\int_\gamma \tilde{\lambda}^k dlnF_m(\tilde{\lambda}) = I_k(\Gamma), \quad k \geq 1,$$

are constants independent of ρ_1, ρ_2 with $I = I(\Gamma) = \sum_{j=1}^{N} \int_{a_j} \tilde{\lambda}^k \omega_j$. By the residue theorem, we have

$$I_k(\Gamma) = \sum_{l=1}^{N} \tilde{\mu}_l^k + \sum_{s=1}^{2} Res_{\tilde{\lambda}=\infty_s} \tilde{\lambda}^k dlnF_1(\tilde{\lambda}),$$
$$I_k(\Gamma) = \sum_{l=1}^{N} \tilde{\nu}_l^k + \sum_{s=1}^{2} Res_{\tilde{\lambda}=\infty_s} \tilde{\lambda}^k dlnF_2(\tilde{\lambda}). \quad (41)$$

Here, we need only compute the residues in (41) for $k = 1, 2$. In a way, this is similar to calculations [3,4]. So, we finally obtain

$$Res_{\lambda=\infty_s} \lambda d\ln F_m(\lambda) = -\frac{i}{2}(-1)^{s-1}\partial_x \ln\theta_s^{(m)}, \quad s = 1, 2; m = 1, 2,$$
$$Res_{\lambda=\infty_s} \lambda^2 d\ln F_m(\lambda) = -\frac{i}{4}(-1)^{s-1}\partial_y \ln\theta_s^{(m)} + \frac{1}{4}\partial_x^2 \ln\theta_s^{(m)}, \quad s = 1, 2; m = 1, 2. \quad (42)$$

where $\theta_s^{(1)} = \theta(\Omega_0 x + \Omega_1 y + \Omega_2 t + \pi_s), \theta_s^{(2)} = \theta(-\Omega_0 x - \Omega_1 y - \Omega_2 t + \eta_s)$, π_s, and η_s are constants. Thus, from (41) and (42), we arrive at

$$\sum_{j=1}^{N} \mu_j = I_1 - \frac{i}{2}\partial_x \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}, \quad \sum_{j=1}^{N} \nu_j = I_1 - \frac{i}{2}\partial_x \ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}, \quad (43)$$

$$\sum_{j=1}^{N} \mu_j^2 = I_2 - \frac{i}{4}\partial_y \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{4}\partial_x^2 \ln\theta_1^{(1)}\theta_2^{(1)},$$
$$\sum_{j=1}^{N} \nu_j^2 = I_2 - \frac{i}{4}\partial_y \ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}} - \frac{1}{4}\partial_x^2 \ln\theta_1^{(2)}\theta_2^{(2)}. \quad (44)$$

Substituting (43) and (44) into (21), (24) and considering (25), then we obtain the algebro-geometric solutions for (1):

$$\begin{aligned} w = q &= \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\exp[\int (i(2I+\Theta) - \alpha_1)dx], \\ v = r &= \frac{\theta_2^{(2)}}{\theta_1^{(2)}}\exp[\int (-i(2I+\Theta) + \alpha_1)dx], \end{aligned} \tag{45}$$

where

$$\begin{aligned} \Theta = & \frac{\theta_2^{(1)}}{\theta_1^{(1)}}\frac{\theta_2^{(2)}}{\theta_1^{(2)}}\int [-4\partial_x I_1 + i\partial_x^2 \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\frac{\theta_1^{(2)}}{\theta_2^{(2)}} + (\frac{1}{2}\partial_y - i\alpha_1\partial_x - 6\partial_x)\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\frac{\theta_2^{(2)}}{\theta_1^{(2)}} \\ & - \frac{3i}{2}\partial_x \ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\partial_x \ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\frac{\theta_1^{(1)}}{\theta_2^{(1)}} - \frac{i}{2}\partial_x^2 \ln\frac{\theta_1^{(1)}}{\theta_1^{(2)}}\frac{\theta_2^{(1)}}{\theta_2^{(2)}}]\frac{\theta_1^{(1)}}{\theta_2^{(1)}}\frac{\theta_1^{(2)}}{\theta_2^{(2)}}dx. \end{aligned} \tag{46}$$

Expression (45) is the algebro-geometric solution of the new (2 + 1)-dimensional integrable breaking soliton Equation (1).

5. Conclusions

A new (2 + 1)-dimensional integrable breaking soliton equation is presented with the help of (1 + 1)-dimensional soliton equations associated with the Kaup–Newell soliton hierarchy. The (2 + 1)-dimensional integrable equation is reduced into solvable ordinary differential equations. By introducing the hyperelliptic Riemann surface and Abel–Jacobi coordinates, the associated flow is straightened. Then, the algebro-geometric solutions of the new (2 + 1)-dimensional integrable equation are constructed by means of the Riemann θ functions. It should also be pointed out that the method used here is suitable for other soliton hierarchies.

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