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Two Classes of Entropy Measures for Complex Fuzzy Sets

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Abstract: Complex fuzzy sets are characterized by complex-valued membership functions, whose range is extended from the traditional fuzzy range of $[0,1]$ to the unit circle in the complex plane. In this paper, we define two kinds of entropy measures for complex fuzzy sets, called type-A and type-B entropy measures, and analyze their rotational invariance properties. Among them, two formulas of type-A entropy measures possess the attribute of rotational invariance, whereas the other two formulas of type-B entropy measures lack this characteristic.

Keywords: complex fuzzy sets; entropy; rotational invariance

1. Introduction

A complex fuzzy set [1] (CFS) is an extension of the standard (i.e., type-1) fuzzy sets (FS) in which the range of the membership function was the unit disc of the complex plane. In the standard fuzzy sets and many other extensions of fuzzy sets, entropy measures are of high importance. Entropy measures of fuzzy sets [2–6], type-2 fuzzy sets [7,8], interval-valued fuzzy sets [9,10], Atanassov's intuitionistic fuzzy sets [11–13], hesitant fuzzy sets [14,15] and Pythagorean fuzzy sets [16] have been investigated and successfully used in various fields.

In considering the applications of CFSs, some measures and concepts have been introduced for CFSs, such as distance measures [17,18], linguistic variables [19], rotational invariance [20], parallelity and orthogonality relations [21–23]. However, for the entropy measures of complex fuzzy sets, as far as we know, no related research papers have been published yet. Generally speaking, complex fuzzy sets can be characterized by complex-valued membership functions containing an amplitude term and a phase term. While the amplitude term retains the traditional notion of “fuzziness”, the phase term is a completely novel parameter of membership function, that can essentially distinguish traditional fuzzy sets from complex fuzzy sets [24]. In this paper, we present two kinds of entropy measures of complex fuzzy sets, one of which depends on the amplitude of the complex-valued membership functions, which is closely related to entropy of traditional fuzzy sets while ignoring the phase term, while the other depends on both the amplitude and phase terms.

The rest of this paper is organized as follows. In Section 2, we first review some basic concepts of complex fuzzy sets. Sections 3 and 4 define two classes of entropy measures of complex fuzzy sets, called type-A and type-B entropy measures, respectively. Conclusions are presented in Section 5.

2. Preliminaries

Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse, a complex fuzzy set A on X may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

where membership function $\mu_A(x)$ is of the form $r_A(x) \cdot e^{j\theta_A(x)}$, $j = \sqrt{-1}$, the amplitude term $r_A(x)$ and the phase term $\theta_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. Because $e^{j\theta_A(x)}$ is a periodic function whose periodicity law is 2π , we only consider $\theta_A(x) \in [0, 2\pi)$ in this paper.

Several complex fuzzy complements of A are specified by functions

$$\begin{aligned} \mu_{\neg_1 A}(x) &= c(r_A(x)) \cdot e^{j(-\theta_A(x))} \\ \mu_{\neg_2 A}(x) &= c(r_A(x)) \cdot e^{j(\theta_A(x))} \\ \mu_{\neg_3 A}(x) &= c(r_A(x)) \cdot e^{j(\theta_A(x)+\pi)} \end{aligned}$$

where $c(r_A(x)) = 1 - r_A(x)$. In this paper, the phase term is confined to the interval $[0, 2\pi)$, so the phase terms of above \neg_1 and \neg_3 are functions with modulo 2π .

3. Type-A Entropy Measures of Complex Fuzzy Sets

(1) *A review of entropy of traditional fuzzy sets:* The following axiomatic-based definition for the entropy of a traditional fuzzy set can be found in De Luca and Termini [2], and is supported by many others (e.g., Refs. [5,9–11,15]).

Definition 1. (De Luca and Termini [2]) *Let A and B be two FSs. A mapping $e: FS(X) \rightarrow [0, 1]$ is called an entropy on $FS(X)$ if e satisfies the following axioms:*

- (a1) $e(A) = 0$ if and only if A is a crisp set;
- (a2) $e(A) = 1$ if $\mu_A(x) = 0.5$ for all $x \in X$;
- (a3) $e(A) \leq e(B)$ if A is less fuzzy than B , i.e., if $\mu_A(x) \leq \mu_B(x)$ when $\mu_B(x) \leq 0.5$ and $\mu_A(x) \geq \mu_B(x)$ when $\mu_B(x) \geq 0.5$;
- (a4) $e(A) = e(\neg A)$, $\neg A$ is the complement of A .

As mentioned in Section 1, there are already many different real equations satisfying the axiomatic requirements of Definition 1. In this work, we takes the following two specific functions for discussion and comparison [2,5].

$$e_{LT}(A) = -\frac{1}{n} \sum_1^n [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))] \tag{1}$$

$$e_{PB}(A) = \frac{4}{n} \sum_1^n (\mu_A(x_i)(1 - \mu_A(x_i))) \tag{2}$$

(2) *Type-A entropy of complex fuzzy sets:* Since the amplitude term retains the traditional notion of "fuzziness" [24], the simpler method of defining the complex fuzzy sets entropy measure based on the amplitude term can be used to calculate the "fuzziness".

Definition 2. *Let A and B be two CFSs. A mapping $e: CFS(X) \rightarrow [0, 1]$ is called a type-A entropy on $CFS(X)$ if e satisfies the following axioms:*

- (a1') $e(A) = 0$ if and only if $|\mu_A(x)| = 0$ or $|\mu_A(x)| = 1$ for all $x \in X$;
- (a2') $e(A) = 1$ if $|\mu_A(x)| = 0.5$ for all $x \in X$;

- (a3') $e(A) \leq e(B)$ if $|\mu_A(x)| \leq |\mu_B(x)|$ when $|\mu_B(x)| \leq 0.5$ and $|\mu_A(x)| \geq |\mu_B(x)|$ when $|\mu_B(x)| \geq 0.5$;
- (a4') $e(A) = e(\neg A)$, $\neg A$ is a complex fuzzy complement of A .

Specifically, this definition of entropy complex fuzzy sets reduces to its traditional counterpart (Definition 1) when a real-valued membership function, i.e., one with $\theta_A(x) = 0$ for all x , is used.

Two type-A entropy formulas respectively corresponding to (1), (2) are proposed as follows:

$$e_1(A) = -\frac{1}{n} \sum_1^n \left[|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) \right], \tag{3}$$

$$e_2(A) = \frac{4}{n} \sum_1^n |\mu_A(x_i)| \cdot |1 - \mu_A(x_i)|. \tag{4}$$

When $\theta_A(x) = 0$ for all x , then we have $e_1(A) = e_{LT}(A)$ and $e_2(A) = e_{PB}(A)$.

Theorem 1. *The mappings e_1 and e_2 , defined by formulas (3) and (4) respectively, are type-A entropy measures for CFSs.*

Proof. In order for (3) to be qualified as an entropy measure for CFSs, it must satisfy the conditions (a1')–(a4') in Definition 2.

From $0 \leq |\mu_A(x_i)| \leq 1$, then $0 \leq |\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) \leq 1$, so we have $0 \leq e_1 \leq 1$.

(a1') Suppose $e_1(A) = 0$, it follows that $|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) = 0$ for all $x_i \in X$. Thus $|\mu_A(x_i)| = 0$ or $|\mu_A(x_i)| = 1$ for all $x_i \in X$. On the other hand, let $|\mu_A(x_i)| = 0$ or $|\mu_A(x_i)| = 1$ for all $x_i \in X$. Then $|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) = 0$ for all $x_i \in X$. Thus $e_1(A) = 0$.

(a2') Suppose $e_1(A) = 1$, it follows that $|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) = 1$ for all $x_i \in X$. Thus $|\mu_A(x_i)| = 1/2$ for all $x_i \in X$. On the other hand, let $|\mu_A(x_i)| = 1/2$ for all $x_i \in X$. Then $|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) = 1$ for all $x_i \in X$. Thus $e_1(A) = 1$.

(a3') Suppose that $|\mu_A(x_i)| \leq |\mu_B(x_i)|$ when $|\mu_B(x_i)| \leq 0.5$ and $|\mu_A(x_i)| \geq |\mu_B(x_i)|$ when $|\mu_B(x_i)| \geq 0.5$, then $|\mu_A(x_i)| \log |\mu_A(x_i)| + (1 - |\mu_A(x_i)|) \log (1 - |\mu_A(x_i)|) \leq |\mu_B(x_i)| \log |\mu_B(x_i)| + (1 - |\mu_B(x_i)|) \log (1 - |\mu_B(x_i)|)$ for all $x_i \in X$. Therefore, $e_1(A) \leq e_1(B)$.

(a4') For each $\neg_k, k = 1, 2, 3$, we can easily obtain that $e_1(A) = e_1(\neg_k A)$.

For the mapping e_2 , the proof is similar to that of e_1 . \square

For convenience, we consider the case of $e_2(A)$, where A is a complex fuzzy set on $X = \{x\}$. The contour plot of $e_2(A)$ is depicted as in Figure 1. The points whose moduli are 0.5 have the degree of fuzziness 100%. The points whose moduli are 0 or 1 have the degree of fuzziness 0%, as shown in Figure 2.

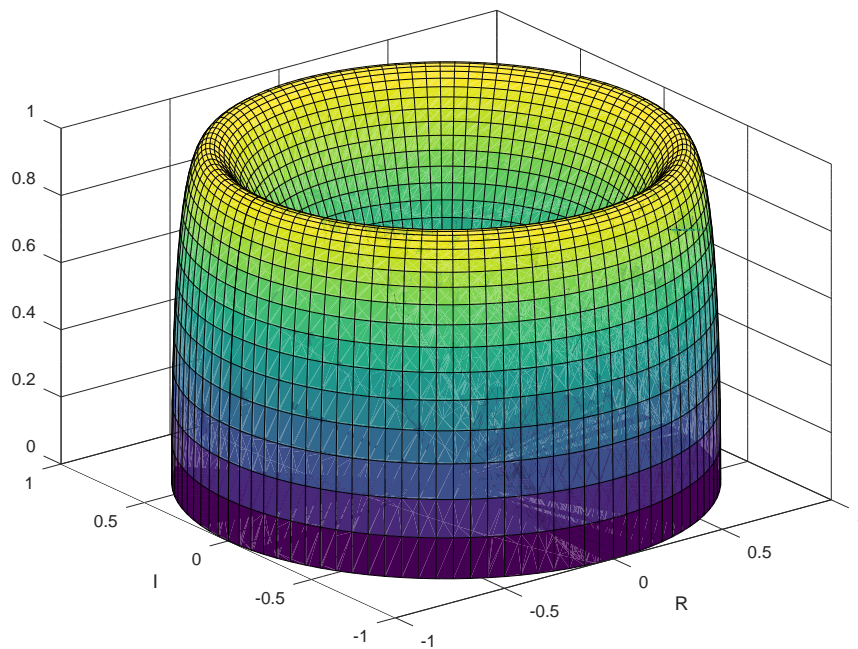


Figure 1. Contour plot of the type-A entropy calculated from Equation (4).

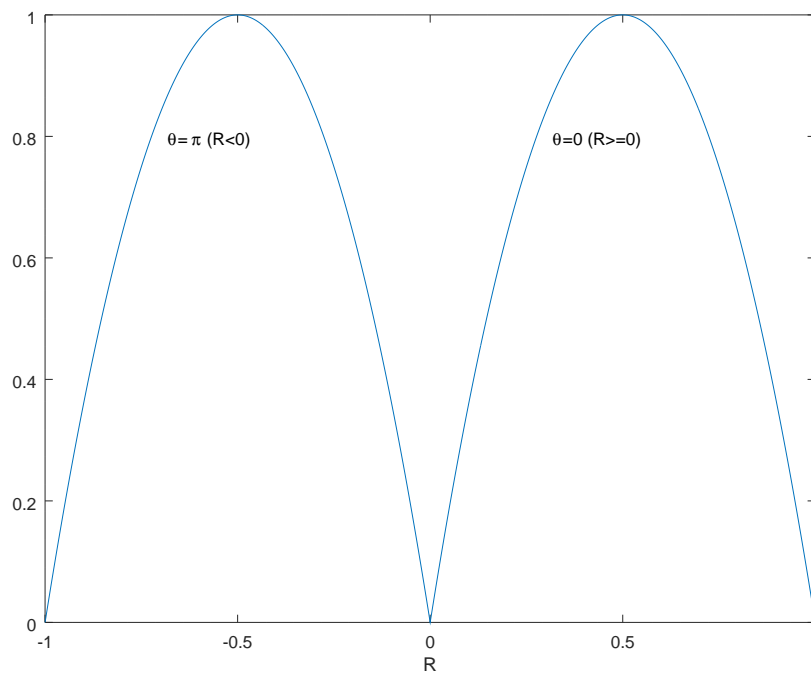


Figure 2. Vertical section of Figure 1.

The degree of fuzziness only relies on the modulus of a complex number, not its phase, as shown in Figure 3. This property is called rotational invariance, which can be formally expressed as follows.

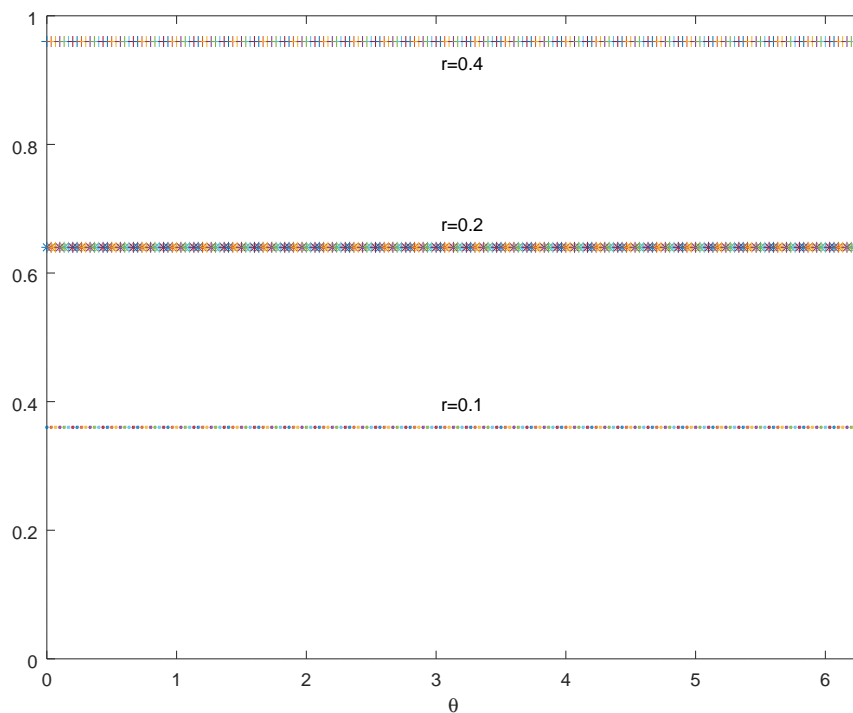


Figure 3. The relations between the phase term and type-A entropy measure of e_2 .

Rotational invariance, developed from Ref. [20], is an intuitively appealing property of function application. After a simple rotation, if the entropy measure of a CFS is invariant, it will makes a great deal of sense.

Let $A \in CFS(X)$ with membership function $\mu_A(x) = \nu_A(x) \cdot e^{j\omega_A(x)}$. The rotation of A by θ radians [24], denoted $Rot_\theta(A)$, is defined as $Rot_\theta(\mu_A(x)) = \nu_A(x) \cdot e^{j(\omega_A(x)+\theta)}$. The following is a definition of rotational invariance for entropy measures of complex fuzzy sets.

Definition 3. The entropy measure of CFSs $e : CFS(X) \rightarrow [0,1]$ is rotationally invariant,if and only if

$$e(Rot_\theta(A)) = e(A),$$

for any θ and CFS $A \in CFS(X)$.

Theorem 2. The mappings e_1 and e_2 are rotationally invariant.

Proof. Since $|\mu_A(x_i)| = |\mu_A(x_i) \cdot e^{j\theta}|$, we can easily obtain that $e_1(Rot_\theta(A)) = e_1(A)$ and $e_2(Rot_\theta(A)) = e_2(A)$. \square

Example 1. Assume a complex fuzzy sets A is given by

$$A = 0.1e^{j1.02\pi} / x_1 + 0.2e^{j0.95\pi} / x_2 + 0.3e^{j1.32\pi} / x_3 + 0.3e^{j0.75\pi} / x_4 + 0.1e^{j1.12\pi} / x_5$$

Then we have

$$\neg_1 A = 0.9e^{j0.98\pi} / x_1 + 0.8e^{j1.05\pi} / x_2 + 0.7e^{j0.68\pi} / x_3 + 0.7e^{j1.25\pi} / x_4 + 0.9e^{j0.88\pi} / x_5$$

$$\neg_2 A = 0.9e^{j1.02\pi} / x_1 + 0.8e^{j0.95\pi} / x_2 + 0.7e^{j1.32\pi} / x_3 + 0.7e^{j0.75\pi} / x_4 + 0.9e^{j1.12\pi} / x_5$$

$$\neg_3 A = 0.9e^{j0.02\pi} / x_1 + 0.8e^{j1.95\pi} / x_2 + 0.7e^{j0.32\pi} / x_3 + 0.7e^{j1.75\pi} / x_4 + 0.9e^{j0.12\pi} / x_5$$

It is easy to verify

$$e_1(A) = e_1(\neg_i A) \approx 0.25, \quad i = 1, 2, 3$$

$$e_2(A) = e_2(\neg_i A) = 0.608, \quad i = 1, 2, 3.$$

4. Type-B Entropy Measures of Complex Fuzzy Sets

We notice that that type-A entropy measures of a CFS ignored the phase term of this CFS. The phase term distinguishes between traditional and complex fuzzy sets [24]. Our new method is to define the entropy measures of complex fuzzy sets by combining the amplitude and phase terms.

Definition 4. Let A and B be two CFSs. A mapping $e: CFS(X) \rightarrow [0, 1]$ is called a type-B entropy on $CFS(X)$ if e satisfies the following axioms:

- (b1) $e(A) = 0$ if and only if $r_A(x) = 0$ or $r_A(x) = 1$, and $\theta_A(x) = 0$ for all $x \in X$;
- (b2) $e(A) = 1$ if $r_A(x) = 0.5$ and $\theta_A(x) = \pi$ for all $x \in X$;
- (b3) $e(A) \leq e(B)$ if $r_A(x) \leq r_B(x)$ and $\theta_A(x) \geq \theta_B(x)$ for $r_B(x) \leq \theta_B(x)/2\pi$ or $r_A(x) \geq r_B(x)$ and $\theta_A(x) \leq \theta_B(x)$ for $r_B(x) \geq \theta_B(x)/2\pi$;
- (b4) $e(A) = e(\neg A)$, $\neg A$ is a complex fuzzy complement of A .

Unfortunately, this definition of entropy complex fuzzy sets does not reduce to Definition 1 when $\theta_A(x) = 0$ for all x .

Two type-B entropy formulas respectively corresponding to (1), (2) are proposed as follows:

$$e_3(A) = -\frac{1}{2n} \sum_1^n \left[r_A(x_i) \log r_A(x_i) + (1 - r_A(x_i)) \log (1 - r_A(x_i)) + \frac{\theta_A(x_i)}{2\pi} \log \frac{\theta_A(x_i)}{2\pi} + \left(1 - \frac{\theta_A(x_i)}{2\pi}\right) \log \left(1 - \frac{\theta_A(x_i)}{2\pi}\right) \right]. \tag{5}$$

$$e_4(A) = \frac{2}{n} \sum_1^n \left[(r_A(x_i)(1 - r_A(x_i)) + \left(\frac{\theta_A(x_i)}{2\pi}\right) \left(1 - \frac{\theta_A(x_i)}{2\pi}\right) \right]. \tag{6}$$

Theorem 3. The mappings e_3 and e_4 , defined by formulas (5) and (6) respectively, are type-B entropy measures for CFSs.

Proof. Similar to Theorem 1. \square

Remark 1. In the above Theorem 3, the condition (b4) is hold with respect to \neg_1 and \neg_2 , i.e., $e_3(A) = e_3(\neg_1 A)$ and $e_4(A) = e_4(\neg_2 A)$, $i = 1, 2$. It does not hold with respect to \neg_3 .

For convenience, we consider the case of $e_4(A)$, where A is a complex fuzzy set on $X = \{x\}$. The contour plot of $e_4(A)$ is depicted as in Figure 4. The point whose modulus is 0.5 and phase is π has the degree of fuzziness 100%. The points whose modulus are 0 or whose phase is 0 have the degree of fuzziness 0%, as shown in Figure 5. When we fix the modulus of a complex number, several relations between the phase term and the degree of fuzziness are given in Figure 6.

The following results are obvious from Figure 6.

Theorem 4. The mappings e_3 and e_4 are not rotationally invariant.

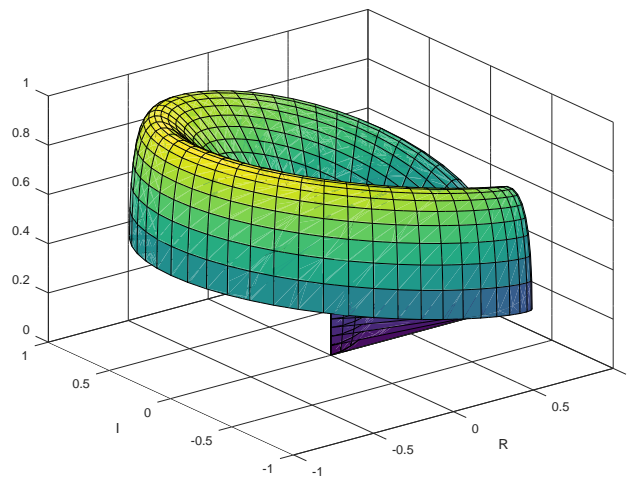


Figure 4. Contour plot of the entropy calculated from Equation (6).

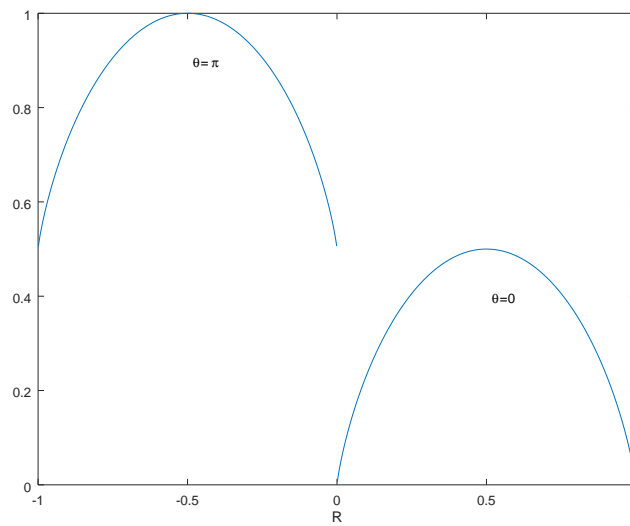


Figure 5. Vertical section of Figure 4.

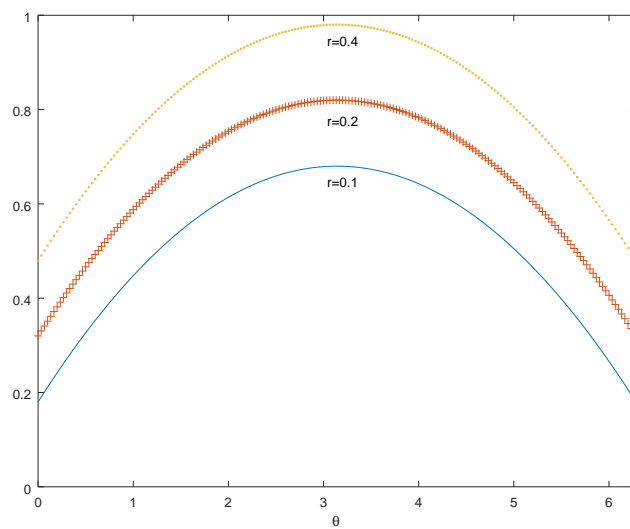


Figure 6. The relations between the phase term and entropy measure of e_4 .

Example 2. Let A be a complex fuzzy set,

$$A = 0.1e^{j1.5\pi} / x_1 + 0.2e^{j0.4\pi} / x_2 + 0.3e^{j1.3\pi} / x_3 + 0.3e^{j0.7\pi} / x_4 + 0.1e^{j1.1\pi} / x_5$$

then we have

$$\begin{aligned} \neg_1 A &= 0.9e^{j0.5\pi} / x_1 + 0.8e^{j1.6\pi} / x_2 + 0.7e^{j0.7\pi} / x_3 + 0.7e^{j1.3\pi} / x_4 + 0.9e^{j0.9\pi} / x_5 \\ \neg_2 A &= 0.9e^{j1.5\pi} / x_1 + 0.8e^{j0.4\pi} / x_2 + 0.7e^{j1.3\pi} / x_3 + 0.7e^{j0.7\pi} / x_4 + 0.9e^{j1.1\pi} / x_5 \\ \neg_3 A &= 0.9e^{j0.5\pi} / x_1 + 0.8e^{j1.4\pi} / x_2 + 0.7e^{j0.3\pi} / x_3 + 0.7e^{j1.7\pi} / x_4 + 0.9e^{j0.1\pi} / x_5 \end{aligned}$$

It is easy to verify

$$\begin{aligned} e_3(A) &= e_3(\neg_i A) \approx 0.35, \quad i = 1, 2 \\ e_4(A) &= e_4(\neg_i A) = 0.724, \quad i = 1, 2. \end{aligned}$$

But

$$\begin{aligned} e_3(\neg_3 A) &\approx 0.204, \\ e_4(\neg_3 A) &\approx 0.584. \end{aligned}$$

Several differences of the above four entropy measures can be summarized as in Table 1. The symbol “ \times ” means “dissatisfy the corresponding property”, “ \checkmark ” means “satisfy the corresponding property”. Here, $e(A) = e(Rot_\theta(A))$ means the property of rotational invariance.

Table 1. Comparison of these entropy measures.

	$e(A) = e(\neg_1 A)$	$e(A) = e(\neg_2 A)$	$e(A) = e(\neg_3 A)$	$e(A) = e(Rot_\theta(A))$
e_1	\checkmark	\checkmark	\checkmark	\checkmark
e_2	\checkmark	\checkmark	\checkmark	\checkmark
e_3	\checkmark	\checkmark	\times	\times
e_4	\checkmark	\checkmark	\times	\times

Example 3. We consider a selection problem in complex fuzzy environment. Assume that there are five complex fuzzy values as follows

$$(0.94 \cdot e^{j1.5\pi}, 0.95 \cdot e^{j1.4\pi}, 0.89 \cdot e^{j1.5\pi}, 0.91 \cdot e^{j1.2\pi}, 0.89 \cdot e^{j0.9\pi}).$$

Then we have the following order

$$|0.89 \cdot e^{j1.5\pi}| = |0.89 \cdot e^{j0.9\pi}| < |0.91 \cdot e^{j1.2\pi}| < |0.94 \cdot e^{j1.5\pi}| < |0.95 \cdot e^{j1.4\pi}|.$$

Thus we can choose $0.89 \cdot e^{j1.5\pi}$ or $0.89 \cdot e^{j0.9\pi}$ based on the modulus of complex numbers. However, as we can see, depending on the modulus of complex numbers used, the ordering of these two complex fuzzy values is the same. Moreover, we have $e_i(0.89 \cdot e^{j1.5\pi}) = e_i(0.89 \cdot e^{j0.9\pi})$, ($i = 1, 2$) based on two formulas of type-A entropy measures. So we can use the formulas of type-B entropy measures and get $e_i(0.89 \cdot e^{j1.5\pi}) < e_i(0.89 \cdot e^{j0.9\pi})$, ($i = 3, 4$). Then we can choose $0.89 \cdot e^{j1.5\pi}$ based on minimum entropy criteria. Our definition of entropy measure gives a new index for the selection of complex fuzzy values.

5. Conclusions

In this paper, we propose two definitions of entropy measures for complex fuzzy sets: type-A and type-B. Among then, type-A depends on the amplitude of membership functions, and can reduce to its traditional counterpart, while type-B depends on both the amplitude and the phase of membership

function. Their differences can refer to Figures 1, 4 and Table 1. In addition, the idea of rotational invariance for entropy measures of complex fuzzy sets was analyzed. It was proved that the two formulas of type-A entropy measures satisfy the property of rotational invariance.

In this paper, we only presented entropy measures of complex fuzzy sets and their properties. Naturally, a discussion of their practical applications in signal processing, decision making and image processing will be both necessary and interesting.

We should note that the entropy measures presented in this paper are conservative in a certain sense. Membership grades usually represent truth degrees in a “direct” way. However, in complex fuzzy sets, how to understand the membership degree of an object to a set is interpreted by a complex number. There are different understandings of complex fuzzy grade from different viewpoints. In the future, we will further study entropy measures for complex fuzzy sets with other interpretations of complex fuzzy sets. This will be useful for applying complex fuzzy sets.

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