Combinatorial graph theory and connectivity

Keith Briggs

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`comb-graph-th2004jul02.tex` TYPESET 2004 JULY 13 10:20 IN PDFLaTeX ON A LINUX SYSTEM

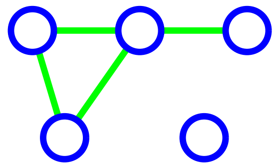
The inspiration

- ☞ Bruno Salvy: Phénomème d'Airy et combinatoire analytique des graphes connexes (INRIA seminar 2003 Dec 15)
- ☞ `algo.inria.fr/seminars/seminars.html`
- ☞ e.g. number of labelled (étiquetés) connected graphs: with excess (edges-vertices) $= k \geqslant 1$ is

$$A_k(1)\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)}+\frac{A_k'(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{2/n}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

- ☞ $A_k(1)$ given in terms of Airy functions: $A_1(1)=5/24, A_2(1)=5/16$ etc.
- ☞ Airy in Playford: `www.ast.cam.ac.uk/~ipswich/History/Airys_Country_Retreat.htm`

Definitions for graphs

☞ (simple unlabelled undirected) graph: 

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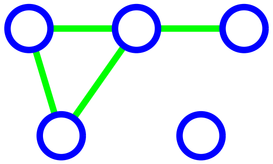
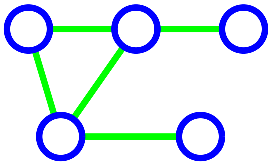
☞ (simple unlabelled undirected) graph: 

☞ (simple unlabelled undirected) connected graph: 

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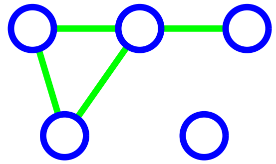
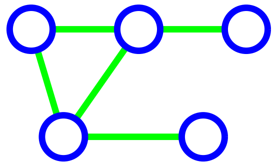
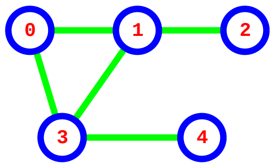
☞ (simple unlabelled undirected) graph: 

☞ (simple unlabelled undirected) connected graph: 

☞ (simple undirected) labelled graph: 

Definitions for generating functions

☞ generating function (gf):

$$\{a_1, a_2, a_3, \ldots\} \leftrightarrow \sum_{k=1}^{\infty} a_k x^k$$

☞ exponential generating function (egf):

$$\{a_1, a_2, a_3, \ldots\} \leftrightarrow \sum_{k=1}^{\infty} \frac{a_k}{k!} x^k$$

☞ Euler transform ($b = ET(a)$):

$$1+\sum_{k=1}^{\infty} b_k x^k = \prod_{i=1}^{\infty}(1-x^i)^{-a_i} \leftrightarrow \log(1+B(x)) = \sum_{k=1}^{\infty} A(x^k)/k$$

Exponential generating functions

☞ exponential generating function for all labelled graphs:

$$g(w,z)=\sum_{n=0}^{\infty}(1+w)^{\binom{n}{2}}z^n/n!$$

☞ exponential generating function for all connected labelled graphs:

$$\begin{aligned} c(w,z) &= \log(g(w,z)) \\ &= z+w\frac{z^2}{2}+(3w^2+w^3)\frac{z^3}{6}+(16w^3+15w^4+6w^5+w^6)\frac{z^4}{4!}+\dots \end{aligned}$$

egfs for labelled graphs [jan]

- rooted labelled trees

$$T(z) = z\exp(T(z)) = \sum_{n\geqslant 1} n^{n-1}\frac{z^n}{n!} = z + \tfrac{2}{2!}z^2 + \tfrac{9}{3!}z^3 + \cdots$$

- unrooted labelled trees

$$U(z) = T(z) - T(z)^2/2 = z + \tfrac{1}{2!}z^2 + \tfrac{3}{3!}z^3 + \tfrac{16}{4!}z^4 + \dots$$

- unicyclic labelled graphs

$$\widehat{V}(z) = \frac{1}{2}\log\left[\frac{1}{1-T(z)}\right] - \frac{1}{2}T(z) - \frac{1}{4}T(z)^2 = \tfrac{1}{3!}z^3 + \tfrac{15}{4!}z^4 + \tfrac{222}{5!}z^5 + \tfrac{3660}{6!}z^6 + \dots$$

- bicyclic labelled graphs

$$\widehat{W}(z) = \frac{T(z)^4\big(6-T(z)\big)}{24\big(1-T(z)\big)^3} = \tfrac{6}{4!}z^4 + \tfrac{205}{5!}z^5 + \tfrac{5700}{6!}z^6 + \dots$$

Unlabelled graphs with n nodes [slo]

- no simple exact formula available - use group theory
- $g = 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 31426485969804 3$

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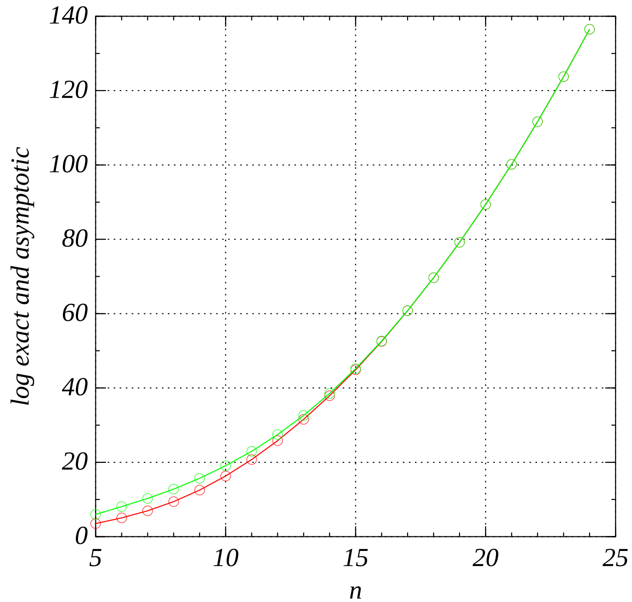
- $g_n = \frac{2^{\binom{n}{2}}}{n!}\left[1+\frac{n(n-1)}{2^{n-1}}+\frac{8n!}{2^{2n}(n-4)!}(3n-7)(3n-9)+\mathcal{O}\left(n^5/2^{5n/2}\right)\right]$



exact asymptotic

Unlabelled connected graphs with n nodes [slo03]

☞ $c = 1, 1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571, 1006700565,$
$164059830476, 50335907869219, 29003487462848061,$
$31397381142761241960, 63969560113225176176277,$
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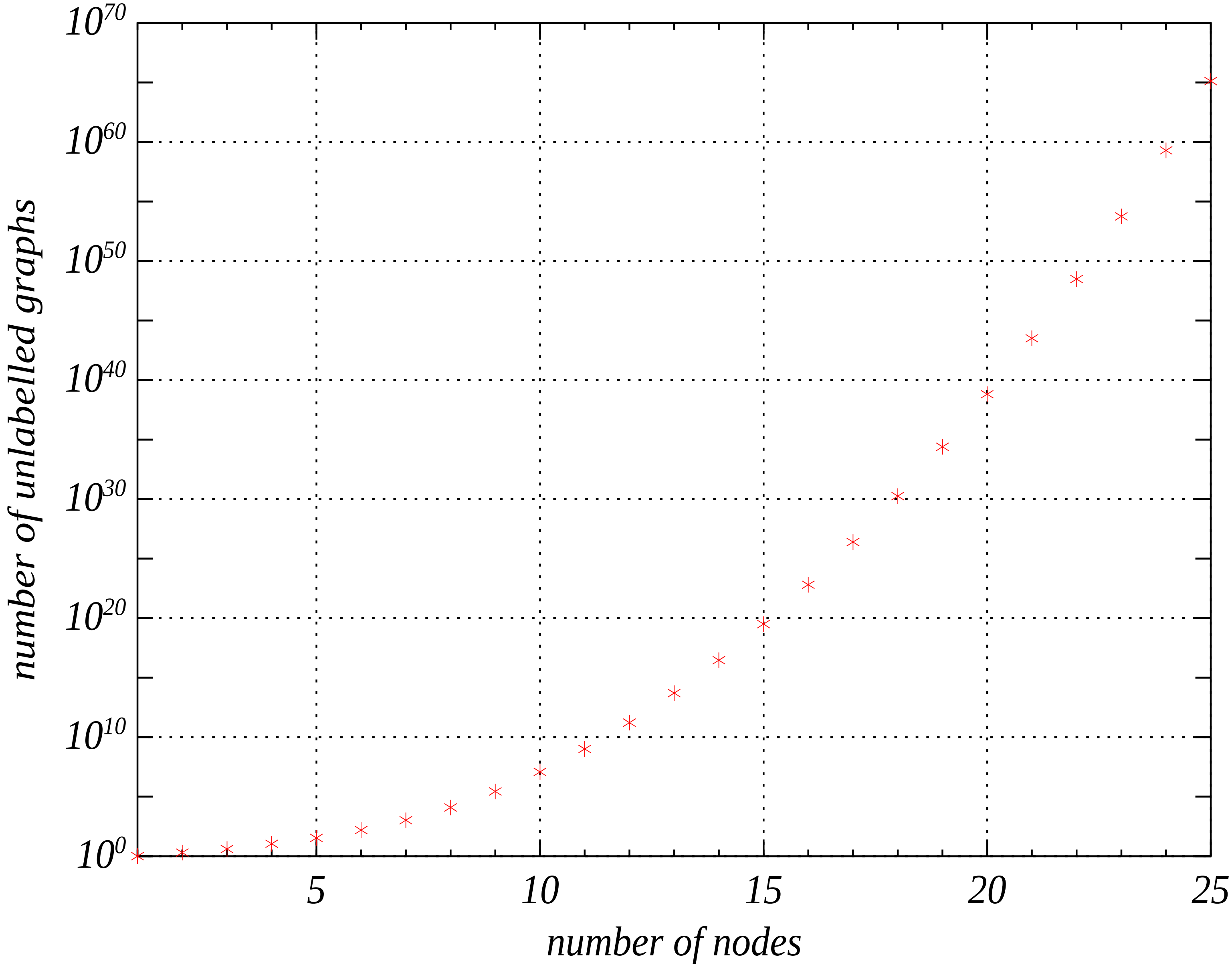
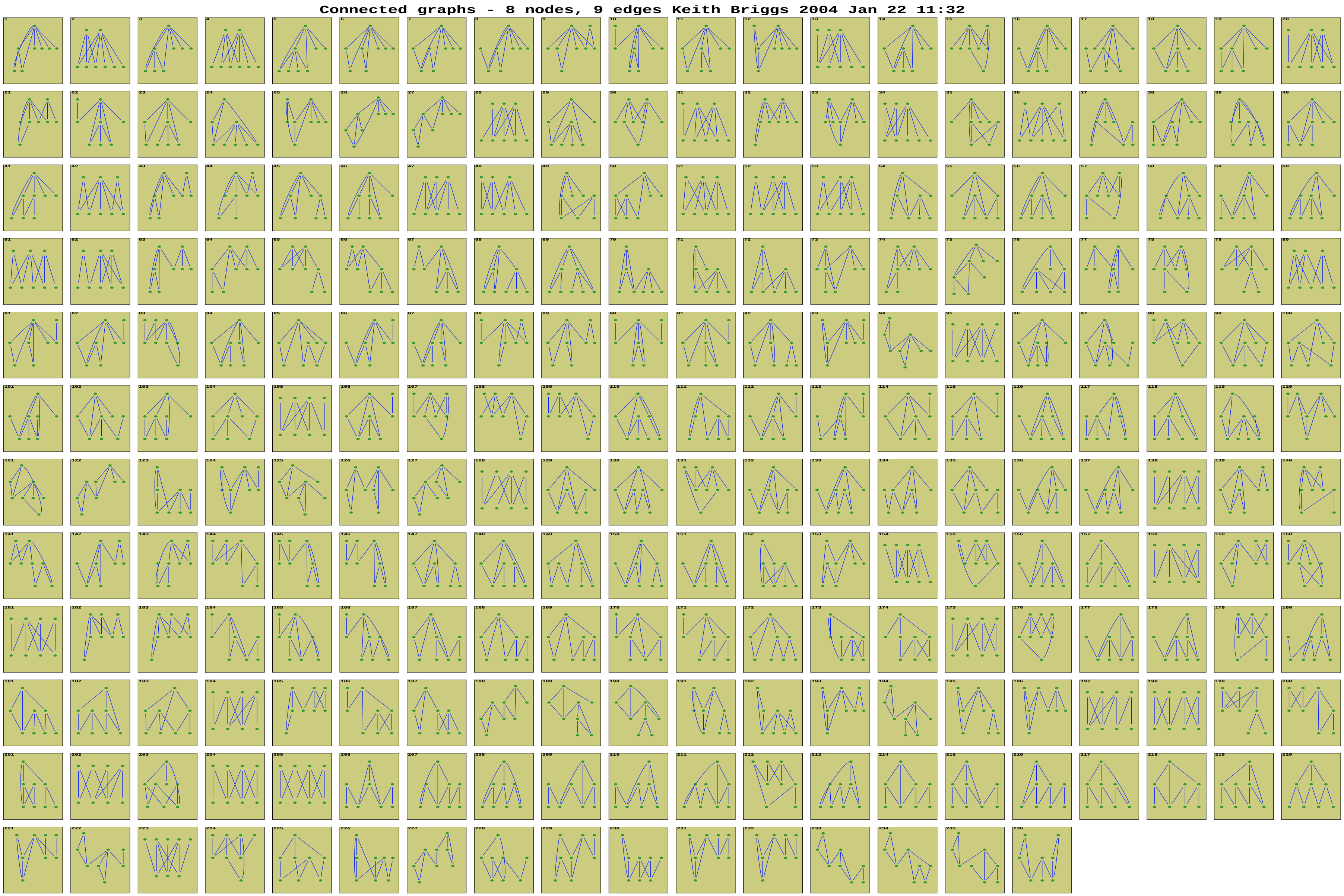
- $g = ET(c)$
- Edge generating functions for enumerating (not necessarily connected) graphs can be computed: e.g. for $n = 5$:

$$1+q+2q^2+4q^3+6q^4+6q^5+6q^6+4q^7+2q^8+q^9+q^{10}$$

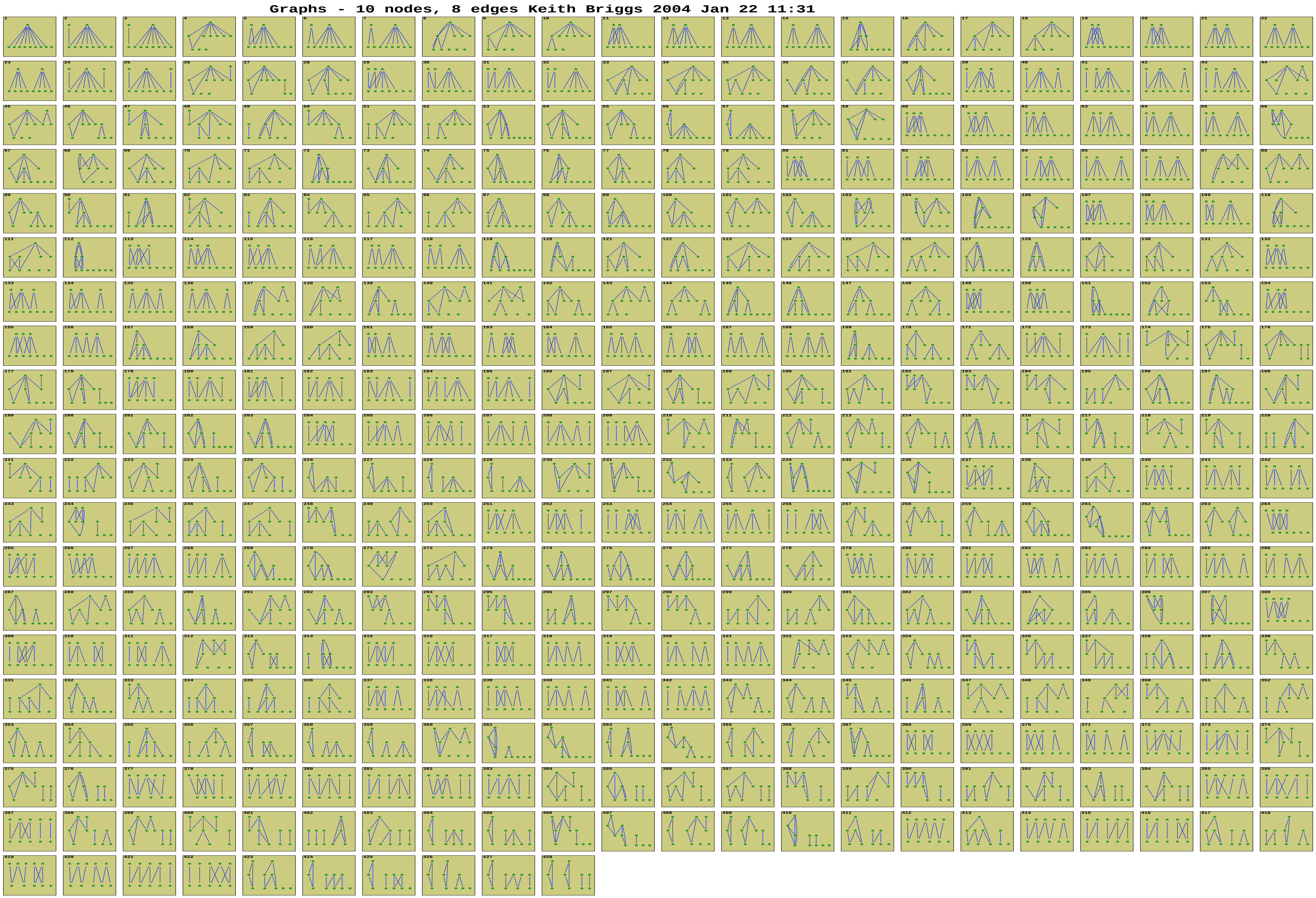
Total numbers of unlabelled graphs



Connected unlabelled graphs - 8 nodes and 9 edges



Unlabelled graphs - 10 nodes and 8 edges



Probability of connectivity 1 [gil]

☞ Bernoulli random graph model of Erdős and Rényi: edges appear independently with probability $p = 1 - q$. Let $P(n,p) = 1 - Q(n,p)$ be the probability that such a graph with n labelled nodes is connected.

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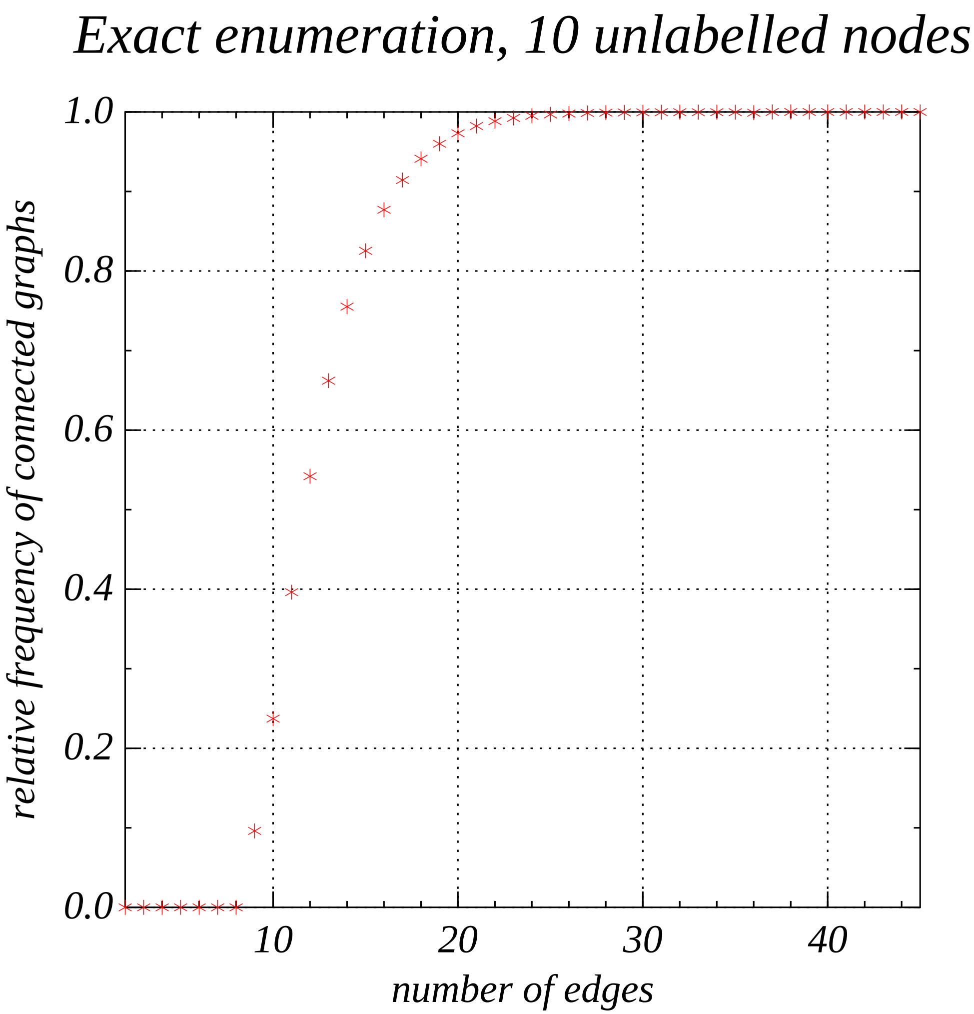
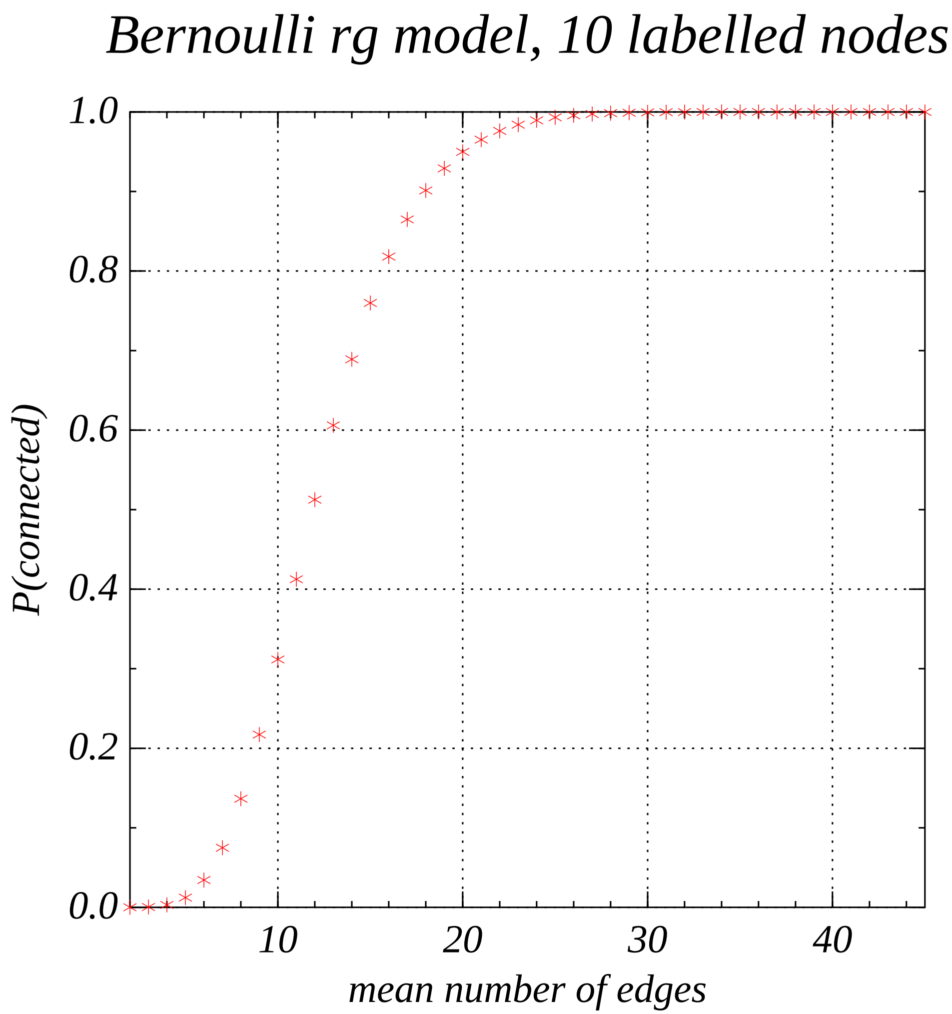
☞ $P(n,p)\sim 1-n\,q^{n-1}$ as $n\to\infty$

Probability of connectivity 2





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