Post-Primary

Overview

Project Maths

NCCA

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Project Maths

Project Maths is an exciting, dynamic development in Irish education. It involves empowering students to develop essential problem-solving skills for higher education and the workplace by engaging teenagers with mathematics set in interesting and real-world contexts.

Watch the video *A fresh approach to maths*.

Syllabus

Find Junior Certificate and Leaving Certificate syllabuses here.

Information and FAQs

Common syllabus queries answered, research and evaluation reports about Project Maths.

Teachers

Learn about providing and assessing problem-solving opportunities in your classroom.

Students

Access tasks that will help you reason and make sense of mathematics.

Parents

Find out the real facts about the Project Maths initiative and how it will affect your child.

Syllabus

New syllabuses for 1st year and 5th year students from September 2013.

Information note sent to schools.

A DES information note was sent to schools in August 2013 drawing attention to the changes in the Junior and Leaving Certificates syllabuses.

- Junior Certificate note
- Leaving Certificate note

Junior Certificate

If you are doing your Junior Certificate Examination from 2016 this is your Syllabus.

Leaving Certificate

If you are doing your Leaving Certificate Examination from 2015 this is your Syllabus.

Bridging framework for mathematics

The bridging framework for mathematics highlights for students and their parents how the objectives of the Mathematics Curriculum at primary level are continued and progressed at post-primary level.

Information about Project Maths

FAQs

Think deeper about the learning outcome by reading these FAQs.

Background

Background to Project Maths.

Initial 24 schools involved in the project.

Reports

Research and evaluation reports about Project Maths.

FAQs

Think deeper about the learning outcome by reading these FAQs.

Functions

- A guide to Post-Primary Functions
  Read the Big Ideas about Functions, how they develop across the levels and connect across the strands
- Read the questions about Functions most frequently asked by teachers

Inferential Statistics

- A guide to Post-Primary Functions
  Read the Big Ideas about Inferential Statistics and how they develop across the levels

Algebra

- Algebra, the Big Ideas
  See an overview of how the concept of algebra develops from early childhood

Geometry

- Geometry for Post-primary schools
  Read the Big Ideas about synthetic geometry. Click here to go directly to the page which sets out the syllabus for each level, from JCOL – LCHL
- Read the questions most frequently asked about synthetic geometry.

Review of Mathematics

The National Council for Curriculum and Assessment conducted a review of post-primary mathematics education. This review came at a time when there was concern about the uptake of Higher level mathematics, particularly in the Leaving Certificate, and about the standards of mathematical achievement in state examinations and in international tests such as the Programme for International Student Assessment (PISA).

Since mathematics underpins many other disciplines, including science and technology, a decline in mathematical knowledge and skills among school leavers can affect the potential of our society for future economic growth and development.

The review also took place against the background of a revised curriculum in primary schools and revised Junior Certificate mathematics syllabuses. Mathematics forms a significant part of a student's education, with all students taking this subject to Leaving Certificate. It is critical, therefore, that issues related to post-primary mathematics education can be widely discussed, and that all voices can have the opportunity to contribute to shaping its future development.

Discussion paper

The discussion paper Review of Mathematics in Post-Primary Education sets out the issues related to mathematics education. It was published in October 2005.

A companion paper International Trends in Post-Primary Mathematics Education was also commissioned.

Everybody counts - consultation on mathematics education

A consultation was held in October-December 2005 on the issues raised in the discussion paper.

Over 300 responses were received from individuals and groups/organisations; these are being collated and analysed. Focus meetings to discuss the issues identified were held with a number of groups and organisations, including parents, teachers, lecturers, and education personnel.

Consultation report

- Review of Mathematics - Report on the Consultation (2006)

  The report presents a summary and analysis of the views submitted during the consultation and revisits the issues raised in the discussion paper. It also considers the steps that will be required to re-shape the teaching, learning and assessment of post-primary mathematics.

Documentation

- Review of Mathematics in Post-Primary Education
- Consultation Questionnaire
- International Trends in Post-Primary Mathematics Education

Links

- Project Maths
- Curriculum development at senior cycle
- Senior cycle review
- Junior cycle developments

Reports

A Review of School Textbooks for Project Maths

Lisa O'Keeffe and John O'Donoghue

National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL) 2012.

Final Report on the impact of Project Maths on student achievement, learning and motivation.

Jennifer Jeffes, Emily Jones, Máirín Wilson, Emily Lamont, Suzanne Straw, Rebecca Wheater, Anneka Dawson National Foundation for Educational Research 2013.

Teaching and Learning in Project Maths: Insights from Teachers who participated in PISA 2012

Jude Cosgrave, Rachel Perkins, Gerry Shiel, Rosemary Fish, and Lasairíona McGuinness Education Research Centre 2012,

NCCA response to debate about Project Maths

This paper was published in 2012, in response to public comment on Project Maths around that time. It revisits the origins of the curriculum and assessment initiative, and sets out the context for the review and the research evidence which informed the project.

Leaving Certificate Mathematics: a comparative analysis

This report, by Dr. Sue Pope from the University of Manchester, compares the Leaving Certificate Mathematics syllabus for examination from 2015 with five other countries/ jurisdictions. A table comparing the topics/content in these curricula with the Strands of the LC maths syllabus can be found here.

Interim Report on the impact of Project Maths on student achievement, learning and motivation.

Jennifer Jeffes, Emily Jones, Rachel Cunningham, Anneka Dawson, Louise Cooper, Suzanne Straw, Linda Sturman, Megan O'Kane.

National Foundation for Educational Research 2012

Maths in Practice: report and recommendations

Here you will find the report of a *Maths in Practice* group, which was formed to look at how teachers could be further supported in working with the new mathematics syllabuses. The report contains a number of recommendations for review and development work by the NCCA and other agencies, some of which is already under way.

Project Maths: Reviewing the project in the initial group of 24 schools - report on school visits

In 2011-2012, NCCA personnel met with teachers in the 24 initial schools to get feedback on their experiences with the revised maths syllabuses over the first three years of the project. This report presents the main themes and recommendations which emerged in these school visits.

Teachers

CIC

Learn more about the Common Introductory Course (CIC).

Communities of practice

Rescources to use with colleagues, A library of ideas that you can use in your own school community

- Investigating the commutative property
- Focus on learning
- Developing questioning

Synthesis and problem solving

Learn how to assess synthesis and problem solving skills.

- The Framework for Assessing synthesis and Problem Solving Skills
- Case Study-Using the Framework for Assessing synthesis and Problem Solving Skills

Reasoning Tasks

A library of tasks that encourage your students to reason and make sense of mathematics.

- Understanding Equality

Overview of Statistics

Read an overview of the post-primary statistics course.

Teachers helping teachers

This is the page where research meets practice. The videos and lesson reports have been designed to give you insights into how lesson ideas 'work' in real classrooms. Discuss the material with your colleagues and try out some ideas in your classrooms.

- Stories from the Classroom

  Watch videos and read teachers' personal experiences from the classroom
- Stories from Research

  Watch videos and read about current research in Mathematics education

Feedback

Please share your experiences if you try any of these resources in your classroom.

projectmaths@ncca.ie

Common Introductory Course

Many of the learning outcomes on the CIC relate to a number of **key concepts** in mathematics that learners need to develop in order to progress in mathematics.

Their development begins in early childhood and continues through primary school yet a significant number of learners present in secondary school with fragmented or incomplete conceptual understanding. The focus in the bridging period should be on providing learners with tasks that allow them to consolidate their prior learning and to further develop these key concepts.

Many learners will continue to experience difficulty with these concepts beyond the bridging period and teachers should use the developmental nature of the concepts to guide them as they select appropriate tasks for these learners, manage their expectations and the scaffolding they provide for them.

Subitising

Often referred to as *Trusting the Count* a student can subitise when they have developed flexible mental objects for the numbers 0-10 and can recognise collections of these numbers without counting. Most students enter First Year with this concept well –developed, a minority will need help to develop this if they are to progress.

Place-value

The ten for one trade structure of our number system is quite complex. Being able to label the tens place and the ones place, or even being able to count by tens, does not, necessarily signal an understanding that 1 ten is simultaneously 10 ones. Becoming mindful of this relationship between tens and ones, or staying mindful of it, is neither simple nor trivial. Using physical models and discussion in the context of computation may help students to come to deeper understandings of the one-for ten and the ten-for one trades that can be made at each pair of neighbouring places.

- Investigating student thinking: [Analysis of student work]
- Supporting the growth to abstract reasoning about number [Task]

Multiplicative thinking

Multiplicative thinking is a foundational concept and the key to understanding rational number and developing efficient mental and written computation strategies in later years. The transition from additive to multiplicative thinking is not straightforward and since access to multiplicative thinking represents a real and persistent barrier to many learners' mathematical progress in post primary school the bridging period ought to focus on empowering students to think multiplicatively.

- Supporting the shift from a 'groups of' way of thinking about multiplication to an array-based representation
- Case Study - Multiplicative thinking rational numbers
- Case study- assessing understanding (no link)

Partitioning

A deep understanding of how fractions are made, named and renamed. The connection between fractions and the sharing or partitative idea of division, and to multiplicative thinking more generally.

- Case study - Partitioning

Proportional reasoning

Proportional reasoning has been referred to as the capstone of the primary mathematics and the cornerstone of algebra and beyond.

- Investigating student thinking: [Case study]
- Making sense of absolute and relative comparison [Task]
- Reasoning multiplicatively about comparison [Task]
- Making sense of proportion in real life [Task]

Generalising

Students begin to develop this concept when the recognise and number properties and patterns. They firstly begin to describe these in words and finally use the complexities of algebraic text.

- Coherence and continuity (Case study)
- Focus on equivalance (Case Study)

Learning outcomes

See how the learning outcomes on the CIC relate to these **key concepts.**

Feedback

Please share your experiences if you try any of these resources in your classroom.

projectmaths@ncca.ie

Teachers helping teachers: Stories from the Classroom

Hear and read about how classroom teachers engaged with students during the learning process to improve their achievement. If you would like to share a story from *your* classroom contact us at projectmaths@ncca.ie

Una Caraher
"They sit there huddled in their groups......staring blankly at me!"

Geraldine O'Flynn
Assessing for learning in a maths class.

Aidan Breen
What goes on in their heads?

Lesley Byrne
Lesley discusses the new syllabus

Willie O'Gorman
Activities designed by teachers.

Experiences of maths teachers involved in the piloting of Project Maths:
Project Maths: Reviewing the project in the initial group of 24 schools - report on school visits

Teachers helping teachers: Stories from Research

The work of reseachers in mathematics education is vital. Keep up to date with their work.

Dr Aoibhinn Ní Shúilleabháin

Aoibhinn was a mathematics teacher in St Marks Community School in Tallaght, one of the initial schools involved in the *Project Maths* initiative. Currently she is a member of the School of Mathematical Sciences in University College Dublin researching and lecturing in Mathematics Education.

This report highlights the main ideas from her PhD research into Lesson Study as a form of Professional Development which she completed with the School of Education in Trinity College Dublin.

- Lesson Study as a form of in-School Professional Development: Case studies in two post-primary schools

Professor Jo Boaler

Dr Jo Boaler is a Professor of Mathematics Education at Stanford University. Former roles have included being the Marie Curie Professor of Mathematics Education at the University of Sussex, England, a mathematics teacher in London comprehensive schools and a researcher at King's College, London.

Papers written by Jo Boaler:

- How Complex Instruction led to high and Equitable Achievement
- "Opening our ideas": How a detracked mathematics approach promoted respect, responsibility and high achievement.

Videos

- Keynote Speech - Summer Maths 2012
- RTE News Interview

Students

This is your site, it contains presentations and tasks that will help you reason and make sense of mathematics.

Why not form a study group with your friends and work together through the materials?

Syllabus
Read the learning outcomes for both Junior and Leaving certificate mathematics.

Strand 1
The Statistics and Probability section.

Strand 2
The Geometry and Trigonometry section.

Strands 3 and 4
The Number and Algebra sections.

Strand 5
The Functions section.

Examination Info
Examination information and preparation advice.

Feedback
Project Maths is yours; please share your experiences if you try any of these resources.
projectmaths@ncca.ie

Strand 1

Strand 1 is the study of Statistics and Probability. Improve your understanding of concepts in this strand by working through these resources.

Remember, Leaving Certificate Mathematics builds on your understanding of Junior Certificate Mathematics. If you are finding a concept difficult, it will help if you review some of the Junior Certificate materials.

Resources: JC and LC Students

- Reviewing Statistics
- Strand 1 supported tasks
- Descriptive Statistics Questions
- Descriptive Statistics Student Answers
- Leaving Certificate Resouces for students booklet
- Junior Certificate Resouces for students booklet

Strand 2

Strand 2 is the study of Geometry and Trigonometry. Improve your understanding of concepts in this strand by working through these resources.

Remember, Leaving Certificate Mathematics builds on your understanding of Junior Certificate Mathematics. If you are finding a concept difficult, it will help if you review some of the Junior Certificate materials

Resources: JC and LC Students

- Review Questions
- Student solutions to review questions
- Leaving Certificate Resouces for students
- Junior Certificate Resouces for students
- Developing the concept of slope (Slide presentation)
- Using geometry and trigonometry in real life (Slide presentation)
- Using geometry and trigonometry to solve problems (Slide presentation)

Strand 3 and 4

Strands 3 and 4 are the study of Number and Algebra. Improve your understanding of concepts in these strands by working through these resources.

Remember, Leaving Certificate Mathematics builds on your understanding of Junior Certificate Mathematics. If you are finding a concept difficult, it will help if you review some of the Junior Certificate materials.

Resources: JC and LC Students

- Tasks
- Application of geometric patterns [Financial mathematics]
- Leaving Certificate style questions [Financial mathematics]
- Student answers [Financial mathematics]

Strand 5

Strand 5 is the study of Functions. Improve your understanding of concepts in this strand by working through these resources.

Remember, Leaving Certificate Mathematics builds on your understanding of Junior Certificate Mathematics. If you are finding a concept difficult, it will help if you review some of the Junior Certificate materials

Resources: JC and LC Students

- The derivative- Making sense of differentiation [LCOL/HL]
- Finding the domain and range of a function [LC HL task]
- Making connections ...Concept tasks [LCHL]

Examination Info

Read advice from the SEC, the people who prepare your examination papers. Get familiar with the types of questions you will be expected to answer in your examination.

Remember, the examination is a place where you must show evidence of your mathematical understanding. Write down all the steps to your solution and be prepared to justify and explain your thinking.

Resources: JC and LC Students

- Review material JC strands 1 and 2 [Part A]
- Review material JC strands 1 and 2 [Part B]
- Examination advice from the SEC
- Examination style questions JCHL [Strands 1 and 2]
- Examination style questions JCOL [Strands 1 and 2]
- Examination style questions JCFL [Strands 1 and 2]
- Examination style questions LCHL
- Student solutions to examination style questions LCHL
- Examination style questions LCOL
- Student solutions to examination style questions LCOL

Parents

Watch the *Project Maths* Video

The new syllabus

Maths teachers talking about the new syllabus.

- Lesley Byrne
- Jennifer Kelly

Common queries

Listen to a maths teacher, working in the NCCA, answering some common queries from parents:

- What is *Project Maths* all about?
- What's on the new course?
- Who designed the new course?
- What are the main changes to the exam?
- CAO Points
- Calculus

Ideas to help your child suceed in maths and enjoy it

This video is a really useful resource for helping your son/daughter with maths.

Why maths matters?

Watch Prof. Bill Barton talk about the importance of giving every child the tools to use maths.

What students are saying...

Research published this year showed that students are saying the following about the new course:

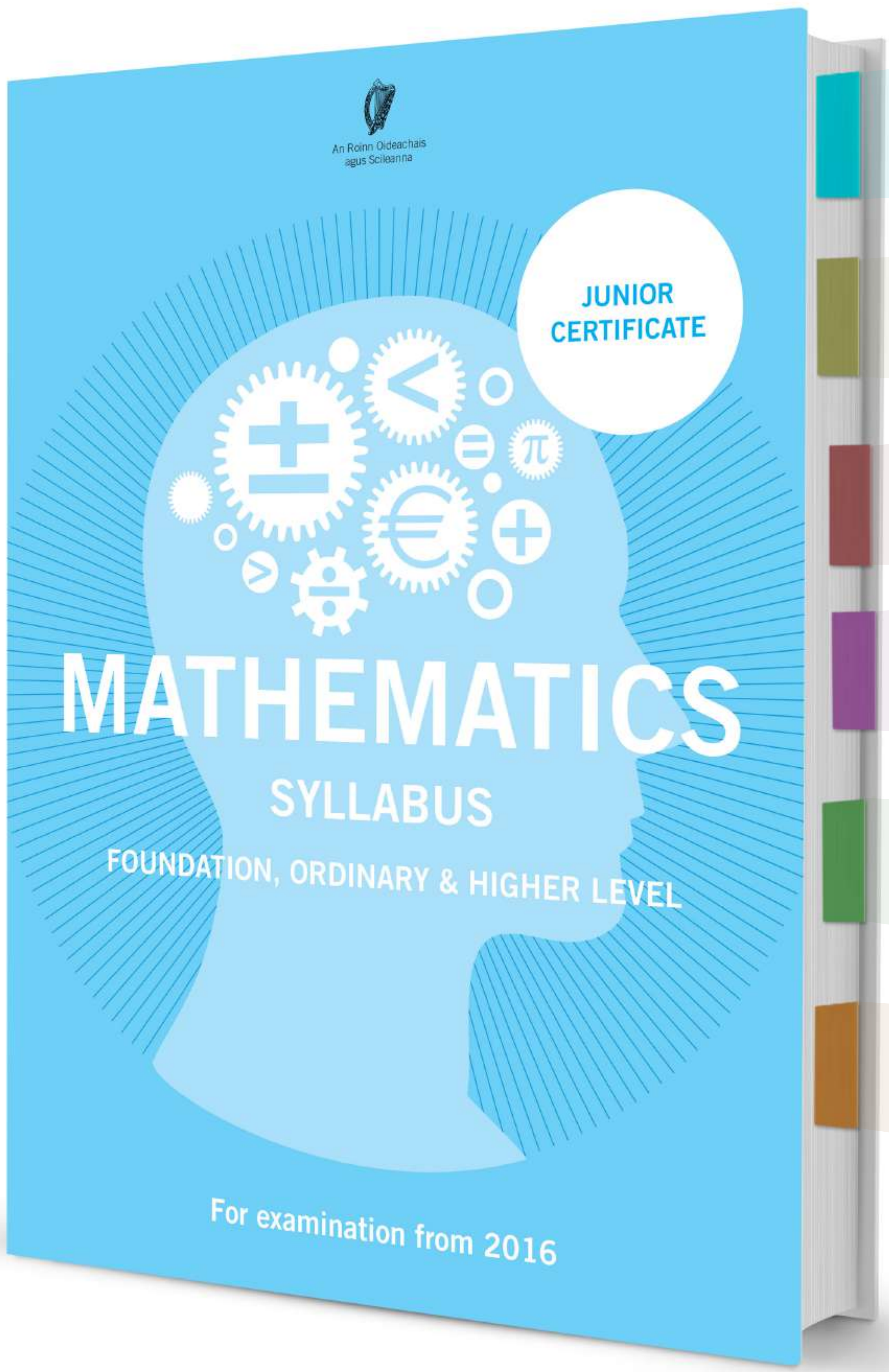
- they are more positive about maths
- they are more interested in maths class and are doing more active learning
- they are getting used to unexpected problems
- they are more willing to stick with a difficult problem.

Read the complete report here.

Contact us

f there's anything else you need to know, drop us an email at projectmaths@ncca.ie

JC MATHEMATICS SYLLABUS CHANGES



Page 6 Read the objectives. They have been reworded to set out very clearly what it means to be mathematically proficient.

Page 10 A new section explains what teachers and students can expect in a problem-solving environment.

Page 16 Read the reworded first learning outcome in section 1.6.

Page 20 Read the reworded second learning out come in section 2.4.

Page 22 Read the new fifth learning outcome in section 3.1.

Page 32 Read about expectations for learners at Foundation level.

GUIDELINES AVAILABLE: WWW.NCCA.IE/PROJECTMATHS

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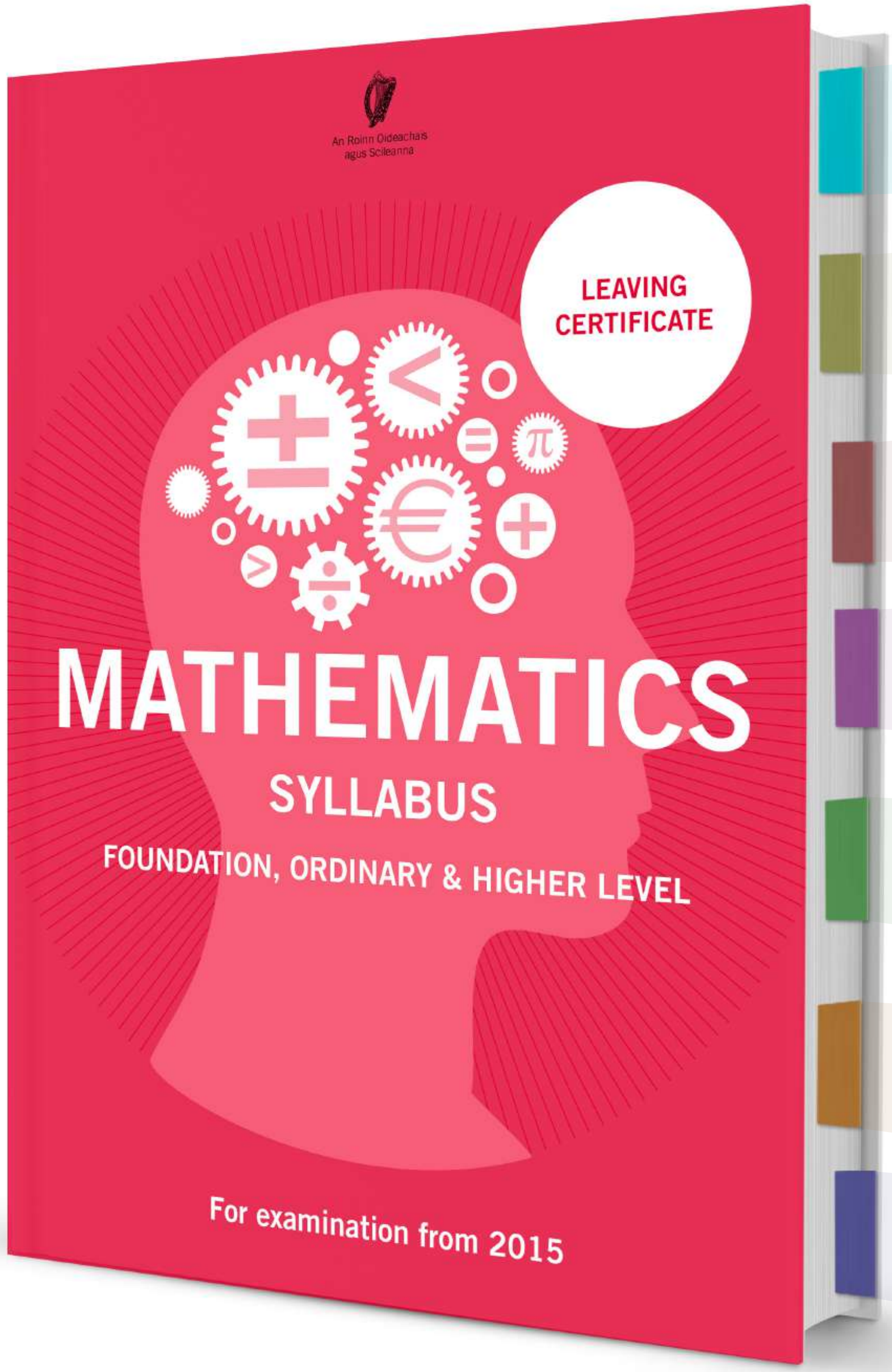
NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

LC MATHEMATICS SYLLABUS CHANGES



Page 6 Read the objectives. They have been reworded to set out very clearly what it means to be mathematically proficient.

Page 10 A new section explains what teachers and students can expect in a problem-solving environment.

Page 16 FL is no longer a subset of the OL and HL syllabuses. FL learning outcomes begin here.

Pages 17 and 43 Read new second learning outcome in sections 1.1 and 1.2 and new fifth learning out come in section 5.2.

Pages 15, 23, 27, 35 and 41 Read the learning outcomes related to synthesis and problem-solving skills.

Page 21 Interim arrangements have ended. Read the learning outcomes for OL and HL inferential statistics.

Page 44 Read about the changes to assessment at FL.

GUIDELINES AVAILABLE: WWW.NCCA.IE/PROJECTMATHS

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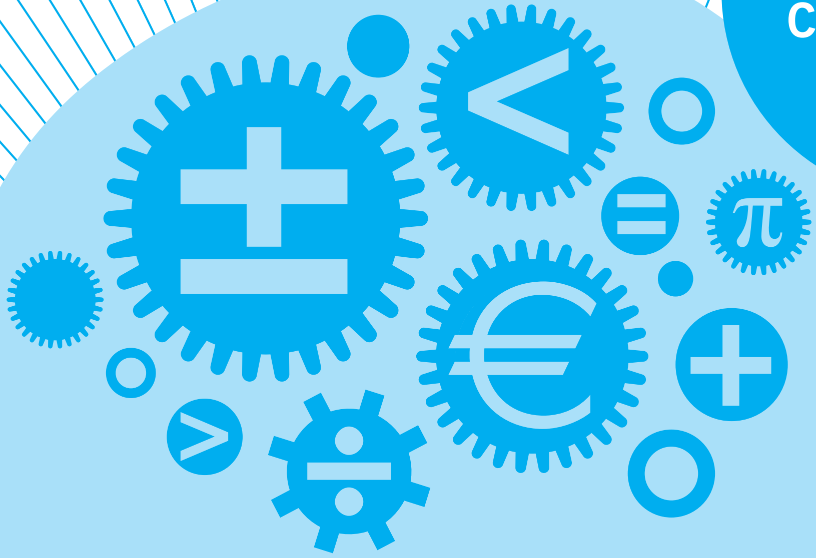
NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

An Roinn Oideachais
agus Scileanna

JUNIOR CERTIFICATE



MATHEMATICS

SYLLABUS

FOUNDATION, ORDINARY & HIGHER LEVEL

For examination from 2016

MATHEMATICS

Junior Certificate Mathematics

Introduction

Mathematics may be seen as the study of quantity, structure, space and change. What does that mean in the context of learning mathematics in post-primary school? In the first instance the learner needs essential skills in numeracy, statistics, basic algebra, shape and space, and technology to be able to function in society. These skills allow learners to make calculations and informed decisions based on information presented and to solve problems they encounter in their everyday lives. The learner also needs to develop the skills to become a good mathematician. Someone who is a good mathematician will be able to compute and then evaluate a calculation, follow logical arguments, generalise and justify conclusions, problem solve and apply mathematical concepts learned in a real life situation.

Mathematical knowledge and skills are held in high esteem and are seen to have a significant role to play in the development of the knowledge society and the culture of enterprise and innovation associated with it. Mathematics education should be appropriate to the abilities, needs and interests of learners and should reflect the broad nature of the subject and its potential for enhancing their development.

The elementary aspects of mathematics, use of arithmetic and the display of information by means of a graph, are an everyday occurrence. Advanced mathematics is also widely used, but often in an unseen and unadvertised way. The mathematics of error-correcting codes is applied to CD players and to computers. The stunning pictures of far away planets and nebulae sent by Voyager II and Hubble could not have had their crispness and quality without such mathematics. Statistics not only provides the theory and methodology for the analysis of wide varieties of data but is essential in medicine for analysing data on the causes of illness and on the utility of new drugs. Travel by aeroplane would not be possible without the mathematics of airflow and of control systems. Body scanners are the expression of subtle mathematics discovered in the 19th century, which makes it possible to construct an image of the inside of an object from information on a number of single X-ray views of it.

Through its application to the simple and the everyday, as well as to the complex and remote, it is true to say that mathematics is involved in almost all aspects of life and living.

Aims

Junior Certificate Mathematics aims to

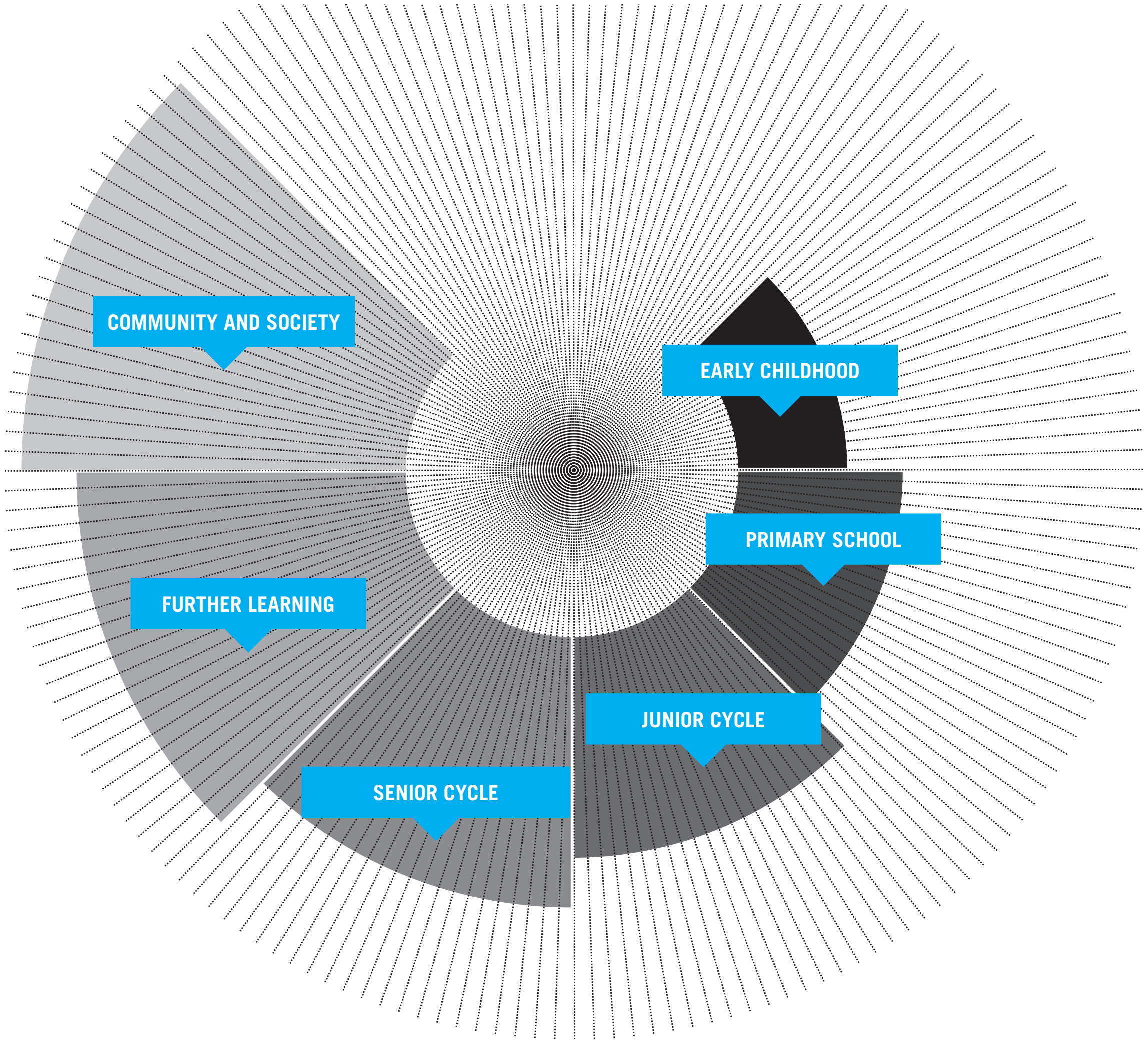
- develop the mathematical knowledge, skills and understanding needed for continuing education, for life and for work
- develop the skills of dealing with mathematical concepts in context and applications, as well as in solving problems
- support the development of literacy and numeracy skills
- foster a positive attitude to mathematics in the learner.

Objectives

The objectives of Junior Certificate Mathematics are that learners develop mathematical proficiency, characterised as

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems in both familiar and unfamiliar contexts
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, justification and communication
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence, perseverance and one's own efficacy.

Related learning



Mathematical learning is cumulative with work at each level building on and deepening what students have learned at the previous level to foster the overall development of understanding. The study of Junior Certificate Mathematics encourages the learner to use the numeracy and problem solving skills developed in early childhood education and primary mathematics. The emphasis is on building connected and integrated mathematical understanding. As learners progress through their education, mathematical skills, concepts and knowledge are developed when they work in more demanding contexts and develop more sophisticated approaches to problem solving.

Mathematics is not learned in isolation. It has significant connections with other curriculum subjects. Many elements of Science have a quantitative basis and learners are expected to be able to work with data, produce graphs, and interpret patterns and trends. In Technical Graphics, drawings are used in the analysis and solution of 2D and 3D problems through the rigorous application of geometric principles. In Geography, learners use ratio to determine scale and in everyday life people use timetables, clocks and currency conversions to make life easier. Consumers need basic financial awareness and in Home Economics learners use mathematics when budgeting and making value for money judgements. In Business Studies learners see how mathematics can be used by business organisations in budgeting, consumer education, financial services, enterprise, and reporting on accounts.

Mathematics, Music and Art have a long historical relationship. As early as the fifth century B.C., Pythagoras uncovered mathematical relationships in music; many works of art are rich in mathematical structure. The modern mathematics of fractal geometry continues to inform composers and artists.

Senior cycle and junior cycle mathematics have been developed simultaneously to allow for strong links to be established between the two. The strands structure allows a smooth transition from junior cycle to a similar structure in senior cycle mathematics. The pathways in each strand are continued, allowing the learner to see ahead and appreciate the connectivity between junior and senior cycle mathematics.

Bridging Framework for Mathematics

Post-primary mathematics education builds on and progresses the learner's experience of mathematics in the Primary School Curriculum. This is achieved with reference not only to the content of the syllabuses but also to the teaching and learning approaches used.

Mathematics in the Primary School Curriculum is studied by all children from junior infants to sixth class. Content is presented in two-year blocks but with each class level clearly delineated. The Mathematics Curriculum is presented in two distinct sections.

It includes a skills development section which describes the skills that children should acquire as they develop mathematically. These skills include

- applying and problem solving
- communicating and expressing
- integrating and connecting
- reasoning
- implementing
- understanding and recalling.

It also includes a number of strands which outline content that is to be included in the mathematics programme at each level. Each strand includes a number of strand units. Depending on the class level, strands can include

- early mathematical activities
- number
- algebra
- shape and space
- measures
- data.

The adoption of a strands structure in Junior Certificate Mathematics continues the pathways which different topics of mathematics follow as the learner progresses from primary school. To facilitate a smooth transition between mathematics in the primary school and in junior cycle a Bridging Framework has been developed. This contains three elements, a *Common Introductory Course*, a *bridging content document* and a *bridging glossary*.

The *Common Introductory Course*, will be studied by all learners as a minimum (see page 33). It is designed so that all of the strands are engaged with to some extent in the first year, so ensuring that the range of topics which have been studied in fifth and sixth classes are revisited.

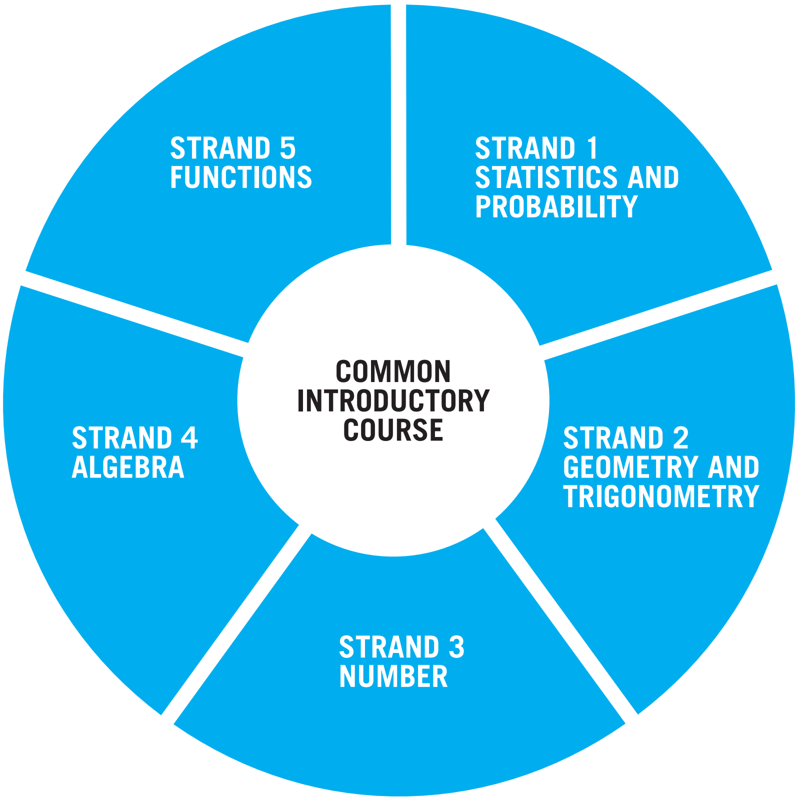
The *bridging content* document has been developed to illustrate to both primary and post-primary teachers the pathways for learners in each strand. Another element of the Bridging Framework is a *bridging glossary* of common terminology for use in upper primary school and early junior cycle. Sample bridging activities have also been developed to assist teachers of fifth and sixth classes in primary school in their planning. These can be used by post-primary mathematics teachers to support learners in the transition to junior cycle mathematics. These documents can be viewed at www.ncca.ie/projectmaths.

The Bridging Framework for Mathematics provides a lens through which teachers in primary school can view post-primary mathematics syllabuses and post-primary teachers can also view mathematics in the Primary School Curriculum. It facilitates improved continuity between mathematics in primary and post-primary schools.

SYLLABUS OVERVIEW

Syllabus overview

Junior Certificate Mathematics



Structure

The Junior Certificate Mathematics syllabus comprises five strands:

1. Statistics and Probability
2. Geometry and Trigonometry
3. Number
4. Algebra
5. Functions

The selection of topics and learning outcomes in each strand is presented in tabular form, and Ordinary level is a subset of Higher level **(HL)**. **Material for Higher level only is shown in bold text**.

The strand structure of the syllabus should not be taken to imply that topics are to be studied in isolation. Where appropriate, connections should be made within and across the strands and with other areas of learning.

Time allocation

The Junior Certificate Mathematics syllabus is designed as a 240 hour course of study.

Problem solving

Problem solving means engaging in a task for which the solution is not immediately obvious. Problem solving is integral to mathematical learning. In day-to-day life and in the workplace the ability to problem solve is a highly advantageous skill.

In the mathematics classroom problem solving should not be met in isolation, but should permeate all aspects of the teaching and learning experience. Problems may concern purely mathematical matters or some applied context. In a mathematics problem-solving environment it is recognised that there are three things learners need to do:

- make sense of the problem
- make sense of the mathematics they can learn and use when doing the problem
- arrive at a correct solution to the problem.

However, in the mathematics classroom, the focus is on the mathematical knowledge and skills that can be learned in the process of obtaining an answer, rather than on the answer itself. The emphasis, therefore, is on generating discussion and on the reasoning and sense-making opportunities the problem affords the learners as they engage with the mathematics involved. They learn to analyse the problem and break it down into manageable steps, to reflect on their strategies and those of others and to adjust their own approaches where necessary.

Teachers play an important role in helping students develop these kinds of skills. By encouraging learners to share, explain and justify their solution strategies, those that work as well as those that don't work, teachers can help learners to develop robust and deep mathematical understanding as well as confidence in their mathematical ability.

The quality of the tasks that learners engage with play an important role in a problem-solving environment. A task must engage learners and present them with a challenge that requires exploration. Problem-solving tasks activate creative mathematical thinking processes as opposed to imitative thinking processes activated by routine tasks. Reasoning mathematically about tasks empowers learners to make connections within mathematics and to develop deep conceptual understanding

Teaching and learning

In each strand, and at each syllabus level, emphasis should be placed on making connections between the strands and on appropriate contexts and applications of mathematics so that learners can appreciate its relevance to current and future life. The focus should be on the learner understanding the concepts involved, building from the concrete to the abstract and from the informal to the formal. As outlined in the syllabus objectives and learning outcomes, the learner's experiences in the study of mathematics should contribute to the development of problem-solving skills through the application of mathematical knowledge and skills.

The use of context-based tasks and a collaborative approach to problem solving can support learners in developing their literacy and numeracy skills. Through discussing ideas about the tasks and their solutions, learners develop the ability to explain and justify their thinking and so gain confidence in their ability to communicate mathematical ideas.

The learner builds on knowledge constructed initially through exploration of mathematics in primary school. This is facilitated by the study of the *Common Introductory Course* at the start of post-primary schooling, which facilitates both continuity and progression in mathematics. Particular emphasis is placed on promoting the learner's confidence in themselves (that they can 'do' mathematics) and in the subject (that mathematics makes sense). Through the use of meaningful contexts, opportunities are presented for the learner to achieve success.

The variety of activities engaged in enables learners to take charge of their own learning by setting goals, developing action plans and receiving and responding to assessment feedback. As well as varied teaching strategies, varied assessment strategies will provide information that can be used as feedback so that teaching and learning activities can be modified in ways which best suit individual learners.

Careful attention must be paid to the learner who may still be experiencing difficulty with some of the material covered at primary level. The experience of post-primary mathematics must therefore help the learner to construct a clearer knowledge of, and to develop improved skills in basic mathematics and to develop an awareness of its usefulness. Appropriate new material should also be introduced, so that the learner can feel that progress is being made. At junior cycle, the course pays attention to consolidating the foundation laid in the primary school and to addressing practical issues; but it should also cover new topics and underpin progress to the further study of mathematics in each of the strands.

Differentiation

Students learn at different rates and in different ways. Differentiation in teaching and learning and in the related assessment arrangements is essential in order to meet the needs of all learners. In junior cycle syllabuses, differentiation is primarily addressed in three areas: the content and learning outcomes of the syllabus; the process of teaching and learning; the assessment arrangements associated with examinations. For exceptionally able students, differentiation may mean extending and/or enriching some of the topics or learning outcomes. This should supplement, but not replace, the core work being undertaken. For students with general learning difficulties, differentiation may mean teaching at a different pace, having varied teaching methodologies or having a variety of ways of assessing them.

Junior Certificate Mathematics is offered at two levels, Ordinary and Higher level. There is no separate course for Foundation level. The Higher level learning outcomes are indicated in bold in each strand. Learners at Higher level will engage with all of the learning outcomes for Ordinary level as well as those designated for Higher level only.

In each strand, learners at Foundation level will engage appropriately with all of the learning outcomes at Ordinary level. This allows them to have a broad experience of mathematics. More importantly, it will also allow them to follow the Ordinary level course at senior cycle, should they choose to do so.

Mathematics at Ordinary level is geared to the needs of learners who are beginning to deal with abstract ideas. However, learners may go on to use and apply mathematics in their future careers, and will meet the subject to a greater or lesser degree in daily life.

The Ordinary level, therefore, must start by offering mathematics that is meaningful and accessible to learners at their present stage of development. It should also provide for the gradual introduction of more abstract ideas, leading learners towards the use of academic mathematics in the context of further study at senior cycle.

Mathematics at Ordinary level places particular emphasis on the development of mathematics as a body of knowledge and skills that makes sense, and that can be used in many different ways as an efficient system for solving problems and finding answers. Alongside this, adequate attention must be paid to the acquisition and consolidation of fundamental mathematical skills, in the absence of which the learner's development and progress will be hindered. The Ordinary level is intended to equip learners with the knowledge and skills required in everyday life, and it is also intended to lay the groundwork for those who may proceed to further studies in areas in which specialist mathematics is not required.

Mathematics at Higher level is geared to the needs of learners who will proceed with their study of mathematics at Leaving Certificate and beyond. However, not all learners taking the course are future specialists or even future users of academic mathematics. Moreover, when they start to study the material, some of them are only beginning to deal with abstract concepts.

Junior Certificate Mathematics is designed for the wide variety and diversity of abilities and learners. On the one hand it focuses on material that underlies academic mathematical studies, ensuring that learners have a chance to develop their mathematical ability and interest to a high level. On the other, it addresses the practical and obviously applicable topics that learners meet in life outside of school. At Higher level, particular emphasis can be placed on the development of powers of abstraction and generalisation and on the idea of rigorous proof, hence giving learners a feeling for the great mathematical concepts that span many centuries and cultures. Problem solving can be addressed in both mathematical and applied contexts.

STRANDS OF STUDY

Strand 1: Statistics and Probability

In Junior Certificate Mathematics, learners build on their primary school experience and continue to develop their understanding of data analysis by collecting, representing, describing, and interpreting numerical data. By carrying out a complete investigation, from formulating a question through to drawing conclusions from data, learners gain an understanding of data analysis as a tool for learning about the world. Work in this strand focuses on engaging learners in this process of data investigation: posing questions, collecting data, analysing and interpreting this data in order to answer questions.

Learners advance in their knowledge of chance from primary school to deal more formally with probability. The *Common Introductory Course* (see appendix), which draws on a selection of learning outcomes from this strand, enables learners to begin the process of engaging in a more formal manner with the concepts and processes involved.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- use a variety of methods to represent their data
- explore concepts that relate to ways of describing data
- develop a variety of strategies for comparing data sets
- complete a data investigation of their own
- encounter the language and concepts of probability.

Strand 1: Statistics and Probability

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
1.1 Counting	Listing outcomes of experiments in a systematic way, such as in a table, using sample spaces, tree diagrams.	– list all possible outcomes of an experiment – apply the fundamental principle of counting
1.2 Concepts of probability	The probability of an event occurring: students progress from informal to formal descriptions of probability. Predicting and determining probabilities. Difference between experimental and theoretical probability.	– decide whether an everyday event is likely or unlikely to occur – recognise that probability is a measure on a scale of 0-1 of how likely an event is to occur **– use set theory to discuss experiments, outcomes, sample spaces** – use the language of probability to discuss events, including those with equally likely outcomes – estimate probabilities from experimental data – recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – associate the probability of an event with its long-run, relative frequency
1.3 Outcomes of simple random processes	Finding the probability of equally likely outcomes.	– construct sample spaces for two independent events – apply the principle that, in the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, containers with different coloured objects, playing cards, sports results, etc.) **– use binary / counting methods to solve problems involving successive random events where only two possible outcomes apply to each event**
1.4 Statistical reasoning with an aim to becoming a statistically aware consumer	Situations where statistics are misused and learn to evaluate the reliability and quality of data and data sources. Different types of data.	– engage in discussions about the purpose of statistics and recognise misconceptions and misuses of statistics – work with different types of data: categorical: nominal or ordinal numerical: discrete or continuous in order to clarify the problem at hand **– evaluate reliability of data and data sources**
1.5 Finding, collecting and organising data	The use of statistics to gather information from a selection of the population with the intention of making generalisations about the whole population. Formulating a statistics question based on data that vary, allowing for distinction between different types of data.	– clarify the problem at hand – formulate questions that can be answered with data – explore different ways of collecting data – generate data, or source data from other sources including the internet **– select a sample from a population (Simple Random Sample)** **– recognise the importance of representativeness so as to avoid biased samples** – design a plan and collect data on the basis of above knowledge – summarise data in diagrammatic form including data presented in spreadsheets

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
1.6 Representing data graphically and numerically	Methods of representing data. Students develop a sense that data can convey information and that organising data in different ways can help clarify what the data have to tell us. They see a data set as a whole and so are able to use proportions and measures of centre to describe the data.	Graphical – select appropriate methods to represent and describe the sample (univariate data only) – evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others – use pie charts, bar charts, line plots, histograms (equal intervals), stem and leaf plots **and back-to-back stem and leaf plots** to display data – use appropriate graphical displays to compare data sets
	Mean of a grouped frequency distribution.	Numerical – use a variety of summary statistics to describe the data: central tendency – mean, median, mode variability – range, **quartiles and inter-quartile range** **– recognise the existence of outliers**
1.7 Analysing, interpreting and drawing conclusions from data	Drawing conclusions from data; limitations of conclusions.	– interpret graphical summaries of data – relate the interpretation to the original question **– recognise how sampling variability influences the use of sample information to make statements about the population** **– draw conclusions from graphical and numerical summaries of data, recognising assumptions and limitations**

Students learn about	Students should be able to
1.8 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 2: Geometry and Trigonometry

The synthetic geometry covered in Junior Certificate Mathematics is selected from *Geometry for Post-primary School Mathematics*, including terms, definitions, axioms, propositions, theorems, converses and corollaries. The formal underpinning for the system of post-primary geometry is that described by Barry (2001)[1].

The geometrical results listed in the following pages should first be encountered by learners through investigation and discovery. The *Common Introductory Course* will enable learners to link formal geometrical results to their study of space and shape in primary mathematics. Learners are asked to accept these results as true for the purpose of applying them to various contextualised and abstract problems. They should come to appreciate that certain features of shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features or results can be established in a formal manner through logical proof. Even at the investigative stage, ideas involved in mathematical proof can be developed. Learners should become familiar with the formal proofs of the specified theorems (some of which are examinable at Higher level).

1 P.D. Barry. *Geometry with Trigonometry*, Horwood, Chicester (2001)

It is envisaged that learners will engage with dynamic geometry software, paper folding and other active investigative methods.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- recall basic facts related to geometry and trigonometry
- construct a variety of geometric shapes and establish their specific properties or characteristics
- solve geometrical problems and in some cases present logical proofs
- interpret information presented in graphical and pictorial form
- analyse and process information presented in unfamiliar contexts
- select appropriate formulae and techniques to solve problems.

Strand 2: Geometry and Trigonometry

<table>
<tr><th>Topic</th><th>Description of topic
Students learn about</th><th>Learning outcomes
Students should be able to</th></tr>
<tr><td>2.1 Synthetic geometry</td><td>Concepts (see Geometry Course section 9.1 for OL and 10.1 for HL)
Axioms (see Geometry Course section 9.3 for OL and 10.3 for HL):
1. [Two points axiom] There is exactly one line through any two given points.
2. [Ruler axiom] The properties of the distance between points
3. [Protractor Axiom] The properties of the degree measure of an angle
4. Congruent triangles (SAS, ASA and SSS)
5. [Axiom of Parallels] Given any line l and a point P, there is exactly one line through P that is parallel to l.

Theorems: [Formal proofs are not examinable at OL. Formal proofs of theorems 4, 6, 9, 14 and 19 are examinable at HL.]
1. Vertically opposite angles are equal in measure.
2. In an isosceles triangle the angles opposite the equal sides are equal. Conversely, if two angles are equal, then the triangle is isosceles.
3. If a transversal makes equal alternate angles on two lines then the lines are parallel, (and converse).
4. The angles in any triangle add to 180˚.
5. Two lines are parallel if and only if, for any transversal, the corresponding angles are equal.
6. Each exterior angle of a triangle is equal to the sum of the interior opposite angles.
9. In a parallelogram, opposite sides are equal and opposite angles are equal (and converses).
10. The diagonals of a parallelogram bisect each other.
11. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.
12. Let ABC be a triangle. If a line l is parallel to BC and cuts [AB] in the ratio s:t, then it also cuts [AC] in the same ratio (and converse).
13. If two triangles are similar, then their sides are proportional, in order (and converse).
14. [Theorem of Pythagoras] In a right-angled triangle the square of the hypotenuse is the sum of the squares of the other two sides.
15. If the square of one side of a triangle is the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.
19. The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.</td><td>– recall the axioms and use them in the solution of problems
– use the terms: theorem, proof, axiom, corollary, converse and implies

– apply the results of all theorems, converses and corollaries to solve problems
– prove the specified theorems</td></tr>
</table>

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
	Corollaries: **1. A diagonal divides a parallelogram into 2 congruent triangles.** **2. All angles at points of a circle, standing on the same arc, are equal, (and converse).** 3. Each angle in a semi-circle is a right angle. 4. If the angle standing on a chord [BC] at some point of the circle is a right-angle, then [BC] is a diameter. **5. If ABCD is a cyclic quadrilateral, then opposite angles sum to 180˚, (and converse).**	
	Constructions: 1. Bisector of a given angle, using only compass and straight edge. 2. Perpendicular bisector of a segment, using only compass and straight edge. **3. Line perpendicular to a given line *l*, passing through a given point not on *l*.** 4. Line perpendicular to a given line *l*, passing through a given point on *l*. 5. Line parallel to a given line, through a given point. 6. Division of a line segment into 2 or 3 equal segments, without measuring it. **7. Division of a line segment into any number of equal segments, without measuring it.** 8. Line segment of a given length on a given ray. 9. Angle of a given number of degrees with a given ray as one arm. 10. Triangle, given lengths of three sides 11. Triangle, given SAS data 12. Triangle, given ASA data 13. Right-angled triangle, given the length of the hypotenuse and one other side. 14. Right-angled triangle, given one side and one of the acute angles (several cases). 15. Rectangle, given side lengths.	– complete the constructions specified

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
2.2 Co-ordinate geometry	Co-ordinating the plane. Properties of lines and line segments including midpoint, slope, distance and the equation of a line in the form. $y - y_1 = m(x - x_1)$. $y = mx + c$. $\mathbf{ax + by + c = 0}$ where a, b, c, are integers and m is the slope of the line. Intersection of lines. **Parallel and perpendicular lines and the relationships between the slopes.**	– explore the properties of points, lines and line segments including the equation of a line – find the point of intersection of two lines **– find the slopes of parallel and perpendicular lines**
2.3 Trigonometry	Right-angled triangles. Trigonometric ratios. **Working with trigonometric ratios in surd form for angles of 30˚, 45˚ and 60˚** **Right-angled triangles.** **Decimal and DMS values of angles.**	– apply the theorem of Pythagoras to solve right-angled triangle problems of a simple nature involving heights and distances – use trigonometric ratios to solve problems involving angles (integer values) between 0˚ and 90˚ **– solve problems involving surds** **– solve problems involving right-angled triangles** **– manipulate measure of angles in both decimal and DMS forms**
2.4 Transformation geometry	Translations, central symmetry, axial symmetry and rotations.	– locate axes of symmetry in simple shapes – recognise images of points and objects under translation, central symmetry, axial symmetry and rotations

Students learn about	Students should be able to
2.5 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 3: Number

This strand builds on the ideas about number that learners developed in primary school and facilitates the transition between arithmetic and algebra; the *Common Introductory Course* provides appropriate continuity with, and progression from, primary school mathematics. Within this strand, in the context of learning about numbers and computation, learners explore and investigate some generalisations that are central to our number system, the properties and relationships of binary operations, and the results of operating on particular kinds of numbers. Learners are introduced to the notion of justification or proof. They extend their work with ratios to develop an understanding of proportionality which can be applied to solve single and multi-step problems in numerous contexts. Learners are expected to be able to use calculators appropriately and accurately, as well as carrying out calculations by hand and mentally.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- revisit previous learning on number and number operations
- develop a meaningful understanding of different number types, their use and properties
- engage in applications of numeracy to solve real life problems
- apply set theory as a strategy for solving problems in arithmetic.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.1 Number systems **N**: the set of natural numbers, **N** = {1,2,3,4….} **Z**: the set of integers, including 0 **Q**: the set of rational numbers **R**: the set of real numbers **R/Q**: the set of irrational numbers	The binary operations of addition, subtraction, multiplication and division and the relationships between these operations, beginning with whole numbers and integers. They explore some of the laws that govern these operations and use mathematical models to reinforce the algorithms they commonly use. Later, they revisit these operations in the context of rational numbers **and irrational numbers (R/Q)** and refine, revise and consolidate their ideas. Students learn strategies for computation that can be applied to any numbers; implicit in such computational methods are generalisations about numerical relationships involving the operations being used. Students articulate the generalisation that underlies their strategy, firstly in the vernacular and then in symbolic language. Problems set in context, using diagrams to solve the problems so they can appreciate how the mathematical concepts are related to real life. Algorithms used to solve problems involving fractional amounts.	– investigate models such as decomposition, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, subtraction, multiplication and division in **N** where the answer is in **N,** including the inverse operations – investigate the properties of arithmetic: commutative, associative and distributive laws and the relationships between them – appreciate the order of operations, including the use of brackets – investigate models such as the number line to illustrate the operations of addition, subtraction, multiplication and division in **Z** – use the number line to order numbers in **N, Z, Q (and R for HL)** – generalise and articulate observations of arithmetic operations – investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – analyse solution strategies to problems – engage with the idea of mathematical proof – calculate percentages – use the equivalence of fractions, decimals and percentages to compare proportions – consolidate their understanding and their learning of factors, multiples and prime numbers in **N** – consolidate their understanding of the relationship between ratio and proportion – check a result by considering whether it is of the right order of magnitude – check a result by working the problem backwards – justify approximations and estimates of calculations – present numerical answers to the degree of accuracy specified

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.2 Indices	Binary operations of addition, subtraction, multiplication and division in the context of numbers in index form.	– use and apply the rules for indices (where $a \in \mathbf{Z}$, $a \neq 0$; $p, q \in \mathbf{N}$): • $a^p a^q = a^{p+q}$ • $\frac{a^p}{a^q} = a^{p-q}$ $p > q$ • $(a^p)^q = a^{pq}$ **– use and apply rules for indices (where $a, b \in \mathbf{R}$, $a, b \neq 0$; $p, q \in \mathbf{Q}$; $a^p, a^q, \in \mathbf{R}$; complex numbers not included):** • $a^p a^q = a^{p+q}$ • $\frac{a^p}{a^q} = a^{p-q}$ • $a^0 = 1$ • $(a^p)^q = a^{pq}$ • $a^{1/q} = \sqrt[q]{a}$, $q \in Z$, $q \neq 0$, $a > 0$ • $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$ $p, q \in Z$, $q \neq 0$, $a > 0$ • $a^{-p} = \frac{1}{a^p}$ • $(ab)^p = a^p b^p$ • $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$ **– operate on the set of irrational numbers R \ Q** – use the notation $a^{1/2}$, $a \in \mathbf{N}$ – express rational numbers ≥1 in the approximate form a x10^n, where a is in decimal form correct to a specified number of places and where $n = 0$ or $n \in \mathbf{N}$ **– express non-zero positive rational numbers in the approximate form a x10^n, where $n \in \mathbf{Z}$ and $1 \leq a < 10$** – compute reciprocals
3.3 Applied arithmetic	Solving problems involving, e.g., mobile phone tariffs, currency transactions, shopping, VAT and meter readings. Making value for money calculations and judgments. Using ratio and proportionality.	– solve problems that involve finding profit or loss, % profit or loss (on the cost price), discount, % discount, selling price, compound interest for not more than 3 years, income tax (standard rate only), net pay (including other deductions of specified amounts) **– solve problems that involve cost price, selling price, loss, discount, mark up (profit as a % of cost price), margin (profit as a % of selling price) compound interest, income tax and net pay (including other deductions)**

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.4 Applied measure	Measure and time. 2D shapes and 3D solids, including nets of solids (two-dimensional representations of three-dimensional objects). Using nets to analyse figures and to distinguish between surface area and volume. Problems involving perimeter, surface area and volume. Modelling real-world situations and solve a variety of problems (including multi-step problems) involving surface areas, and volumes of cylinders **and prisms.** The circle and develop an understanding of the relationship between its circumference, diameter and π.	– calculate, interpret and apply units of measure and time – solve problems that involve calculating average speed, distance and time – investigate the nets of rectangular solids – find the volume of rectangular solids and cylinders – find the surface area of rectangular solids – identify the necessary information to solve a problem – select and use suitable strategies to find length of the perimeter and the area of the following plane figures: disc, triangle, rectangle, square, and figures made from combinations of these – draw and interpret scaled diagrams **– investigate nets of prisms (polygonal bases) cylinders and cones** **– solve problems involving surface area of triangular base prisms (right angle, isosceles, equilateral), cylinders and cones** **– solve problems involving curved surface area of cylinders, cones and spheres** **– perform calculations to solve problems involving the volume of rectangular solids, cylinders, cones, triangular base prisms (right angle, isosceles, equilateral), spheres and combinations of these**

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.5 Sets	Set language as an international symbolic mathematical tool; the concept of a set as being a well-defined collection of objects or elements. They are introduced to the concept of the universal set, null set, subset, cardinal number; the union, intersection, set difference operators, and Venn diagrams. They investigate the properties of arithmetic as related to sets and solve problems involving sets.	– use suitable set notation and terminology – list elements of a finite set – describe the rule that defines a set – consolidate the idea that equality of sets is a relationship in which two equal sets have the same elements – perform the operations of intersection, union (for two sets), set difference and complement – investigate the commutative property for intersection, union and difference **– explore the operations of intersection, union (for three sets), set difference and complement** **– investigate the associative property in relation to intersection, union and difference** **– investigate the distributive property of union over intersection and intersection over union.**

Students learn about	Students should be able to
3.6 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 4: Algebra

Algebra builds on the proficiency that learners have been developing in Strand 3. Two aspects of algebra that underlie all others are algebra as a systematic way of expressing generality and abstraction, including algebra as generalised arithmetic, and algebra as syntactically guided transformations of symbols. These two main aspects of algebra have led to the categorisation of three types of activities that learners of school algebra should engage in: representational activities, transformational activities and activities involving generalising and justifying.

In this strand the approaches to teaching and learning should promote inquiry, build on prior knowledge, and enable learners to have a deep understanding of algebra which allows easy movement between equations, graphs, and tables. The *Common Introductory Course* provides the initial engagement with patterns, relationships and expressions, laying the groundwork for progression to symbolic representation, equations and formulae.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- make use of letter symbols for numeric quantities
- emphasise relationship-based algebra
- connect graphical and symbolic representations of algebraic concepts
- use real life problems as vehicles to motivate the use of algebra and algebraic thinking
- use appropriate graphing technologies (calculators, computer software) throughout the strand activities.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.1 Generating arithmetic expressions from repeating patterns	Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.	– use tables to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence
4.2 Representing situations with tables, diagrams and graphs	Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next.	– use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling) – develop and use their own generalising strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.3 Finding formulae	Ways to express a general relationship arising from a pattern or context.	– find the underlying formula written in words from which the data are derived (linear relations) **– find the underlying formula algebraically from which the data are derived (linear, quadratic relations)**
4.4 Examining algebraic relationships	Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships. Non-constant rate of change: quadratic relationships. Proportional relationships.	– show that relations have features that can be represented in a variety of ways – distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulas expressed in words, and algebraically – use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others – recognise that a distinguishing feature of quadratic relations is the way change varies – discuss rate of change and the y-intercept; consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula – decide if two linear relations have a common value – investigate relations of the form $y=mx$ and $y=mx+c$ – recognise problems involving direct proportion and identify the necessary information to solve them

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.5 Relations without formulae	Using graphs to represent phenomena quantitatively.	– explore graphs of motion – make sense of quantitative graphs and draw conclusions from them – make connections between the shape of a graph and the story of a phenomenon – describe both quantity and change of quantity on a graph
4.6 Expressions	Using letters to represent quantities that are variable. Arithmetic operations on expressions; applications to real life contexts. Transformational activities: collecting like terms, simplifying expressions, substituting, expanding and factoring.	– evaluate expressions of the form • $ax + by$ • $a(x + y)$ • $x^2 + bx + c$ • $\frac{ax + by}{cx + dy}$ • axy where $a, b, c, d, x, y \in \mathbf{Z}$ • **$ax^2 + bx + c$** • **$x^3 + bx^2 + cx + d$** **where $a, b, c, d, x, y \in \mathbf{Q}$** – add and subtract simple algebraic expressions of forms such as: • $(ax + by + c) \pm (dx + ey + f)$ • $(ax^2 + bx + c) \pm (dx^2 + ex + f)$ • $\frac{ax + b}{c} \pm \frac{dx + e}{f}$ where $a, b, c, d, e, f \in \mathbf{Z}$ • **$\frac{ax + b}{c} \pm \ldots \pm \frac{dx + e}{f}$** • **$(ax + by + c) \pm \ldots \pm (dx + ey + f)$** • **$(ax^2 + bx + c) \pm \ldots \pm (dx^2 + ex + f)$** **where $a, b, c, d, e, f \in Z$** • **$\frac{a}{bx + c} \pm \frac{p}{qx + r}$ where $a, b, c, p, q, r \in Z$.** – use the associative and distributive property to simplify such expressions as: • $a(bx + cy + d) + e(fx + gy + h)$ • **$a(bx + cy + d) + \ldots + e(fx + gy + h)$** • $a(bx^2 + cx + d)$ • $ax(bx^2 + c)$ where $a, b, c, d, e, f, g, h \in Z$ • **$(x+y)(x+y)$; $(x-y)(x-y)$** – multiply expressions of the form: • $(ax + b)(cx + d)$ • **$(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \mathbf{Z}$** – divide expressions of the form: • $ax^2 + bx + c \div dx + e$, where $a, b, c, d, e \in \mathbf{Z}$ • **$ax^3 + bx^2 + cx + d \div ex + f$, where $a, b, c, d, e \in \mathbf{Z}$** – factorise expressions such as ax, axy where $a \in \mathbf{Z}$ $abxy + ay$, where $a, b \in \mathbf{Z}$ $sx - ty + tx - sy$, where s, t, x, y are variable $ax^2 + bx$, where $a, b, c \in \mathbf{Z}$ $x^2 + bx + c$, where $b, c \in \mathbf{Z}$ $x^2 - a^2$ **$ax^2 + bx + c$, $a \in \mathbf{N}$ $b, c \in \mathbf{Z}$** **difference of two squares $a^2x^2 - b^2y^2$ where $a, b \in \mathbf{N}$** **– rearrange formulae**

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.7 Equations and inequalities	Selecting and using suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations and inequalities. They identify the necessary information, represent problems mathematically, making correct use of symbols, words, diagrams, tables and graphs.	– consolidate their understanding of the concept of equality – solve first degree equations in one or two variables, with coefficients elements of **Z** and solutions also elements of **Z** – **solve first degree equations in one or two variables with coefficients elements of Q and solutions also in Q** – solve quadratic equations of the form $x^2 + bx + c = 0$ where $b, c \in \mathbf{Z}$ and $x^2 + bx + c$ is factorisable **$ax^2 + bx + c = 0$ where $a, b, c \in \mathbf{Q}$ $x \in \mathbf{R}$** – **form quadratic equations given whole number roots** – solve simple problems leading to quadratic equations – solve equations of the form $\frac{ax+b}{c} \pm \frac{dx+e}{f} = \frac{g}{h}$, where $a, b, c, d, e, f, g, h \in \mathbf{Z}$ – solve linear inequalities in one variable of the form $g(x) \leq k$ where $g(x) = ax+b$, $a \in \mathbf{N}$ and $b, k \in \mathbf{Z}$; **$k \leq g(x) \leq h$ where $g(x) = ax+b$, and $k, a, b, h, \in Z$ and $x \in R$**

Students should learn about	Students should be able to
4.8 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 5: Functions

This strand seeks to make explicit the connections and relationships already encountered in strand 3 and strand 4. Learners revisit and consolidate the learning outcomes of the previous strands.

N.B. Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- engage with the concept of a function (that which involves a set of inputs, a set of possible outputs and a rule that assigns one output to each input)
- emphasise the relationship between functions and algebra
- connect graphical and symbolic representations of functions
- use real life problems as motivation for the study and application of functions
- use appropriate graphing technologies.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
5.1 Functions	The meaning and notation associated with functions.	– engage with the concept of a function, domain, co-domain and range – make use of function notation $f(x) =$, $f : x \rightarrow$, and $y =$
5.2 Graphing functions	Interpreting and representing linear, quadratic and exponential functions in graphical form.	– interpret simple graphs – plot points and lines – draw graphs of the following functions and interpret equations of the form $f(x) = g(x)$ as a comparison of functions • $f(x) = ax + b$, where $a, b \in \mathbf{Z}$ • $f(x) = ax^2 + bx + c$, where $a \in \mathbf{N}$; $b, c \in \mathbf{Z}$; $x \in \mathbf{R}$ • **$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$** • **$f(x) = a2^x$ and $f(x) = a3^x$, where $a \in \mathbf{N}$, $x \in \mathbf{R}$** – use graphical methods to find approximate solutions where $f(x) = g(x)$ and interpret the results – **find maximum and minimum values of quadratic functions from a graph** – **interpret inequalities of the form $f(x) \leq g(x)$ as a comparison of functions of the above form; use graphical methods to find approximate solution sets of these inequalities and interpret the results** – graph solution sets on the number line for linear inequalities in one variable

Students learn about	*Students should be able to*
5.3 Synthesis and problem-solving skills	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Assessment

General principles

Assessment in education involves gathering, interpreting and using information about the processes and outcomes of learning. It takes different forms and can be used in a variety of ways, such as to test and certify achievement, to determine the appropriate route for learners to take through a differentiated curriculum, or to identify specific areas of difficulty (or strength) for a given learner. While different techniques may be employed for formative, diagnostic and certification purposes, assessment of any kind can improve learning by exerting a positive influence on the curriculum at all levels. To do this it must reflect the full range of curriculum goals.

Assessment should be used as a continuous part of the teaching-learning process and involve learners, wherever possible, as well as teachers in identifying next steps. In this context, the most valuable assessment takes place at the site of learning. Assessment also provides an effective basis for communication with parents in a way that helps them to support their children's learning. Assessment must be valid, reliable and equitable. These aspects of assessment are particularly relevant for national assessment for certification purposes.

Assessment for certification

Junior Certificate Mathematics is assessed at Foundation, Ordinary and Higher levels. At Foundation level there is one examination paper. There are two assessment components at Ordinary and Higher level

- Mathematics Paper 1
- Mathematics Paper 2

Differentiation at the point of assessment is achieved through the language level in the examination questions, the stimulus material presented, and the amount of structured support given in the questions, especially for candidates at Foundation level.

The learner's understanding of mathematics will be assessed through a focus on *concepts and skills* and *contexts and applications.* Learners will be asked to engage with mathematical and real life problems and to explain and justify conclusions. In this regard some assessment items will differ from those traditionally presented in examination papers.

Learners at Foundation level can expect to engage with a variety of tasks, including word problems, but in language that is appropriate to this level. They will be expected to deal with concepts at a concrete level and will not be expected to engage in more formal abstraction. There will be structured support within tasks to assist in progression through a problem. Learners will be expected to give an opinion and to justify and explain their reasoning in some answers. The assessment will reflect the changed methodology and active nature of teaching and learning in the classroom.

The tasks for learners at Ordinary level will be more challenging than Foundation level tasks and candidates may not receive the same level of structured support in a problem. They will be expected to deal with problem solving in real world contexts and to draw conclusions from answers. The quality of the answering expected will be higher than that at Foundation level.

Learners at Higher level will be expected to deal with more complex and challenging problems than those at Ordinary level. They will be asked to demonstrate a deeper understanding of concepts and an ability to employ a variety of strategies to solve problems as well as to apply mathematical knowledge. Learners at this level can expect to be tested on Ordinary level learning outcomes but their tasks will be, to an appropriate degree, more complex and difficult.

Appendix: Common Introductory Course for Junior Cycle Mathematics

The *Common Introductory Course* is the minimum course to be covered by all learners at the start of junior cycle. It is intended that the experience of this course will lay the foundation for conceptual understanding which learners can build on subsequently. The order in which topics are introduced is left to the discretion of the teacher. The topics and strands should not be treated in isolation; where appropriate, connections should be made between them. Classroom strategies should be adopted which will encourage students to develop their synthesis and problem-solving skills.

Once the introductory course has been completed, teachers can decide which topics to extend or explore to a greater depth, depending on the progress being made by the class group.

The following table, when read in conjunction with the section on the Bridging Framework for Mathematics (see page 8), may help teachers to prepare teaching and learning plans for the *Common Introductory Course* in order to facilitate a smooth transition for learners from their mathematics education in the primary school.

Strand /Topic Title	**Learning outcomes** Students should be able to
Strand 1: 1.1 Counting	– list all possible outcomes of an experiment – apply the fundamental principle of counting
Strand 1: 1.2 Concepts of probability It is expected that the conduct of experiments (including simulations), both individually and in groups, will form the primary vehicle through which the knowledge, understanding and skills in probability are developed.	– decide whether an everyday event is likely or unlikely to occur – recognise that probability is a measure on a scale of 0 - 1 of how likely an event is to occur
Strand 1: 1.5 Finding, collecting and organising data	– explore different ways of collecting data – plan an investigation involving statistics and conduct the investigation – summarise data in diagrammatic form – reflect on the question(s) posed in light of data collected
Strand 1: 1.6 Representing data graphically and numerically	– select appropriate graphical or numerical methods to represent and describe the sample (univariate data only) – use stem and leaf plots, line plots and bar charts to display data
Strand 2: 2.1 Synthetic geometry *(see Geometry for Post-primary School Mathematics)* The geometrical results should be first encountered through discovery and investigation.	– convince themselves through investigation that theorems 1-6 appear to be true – construct 1. the bisector of a given angle, using only compass and straight edge 2. the perpendicular bisector of a segment, using only compass and straight edge 4. a line perpendicular to a given line l, passing through a given point on l 5. a line parallel to a given line l, through a given point 6. divide a line segment into 2, 3 equal segments, without measuring it 8. a line segment of given length on a given ray
Strand 2: 2.2 Co-ordinate geometry	– coordinate the plane – locate points on the plane using coordinates
Strand 2: 2.4 Transformation geometry	– use drawings to show central symmetry, axial symmetry and rotations

Strand /Topic Title	Learning outcomes Students should be able to
Strand 3: 3.1: Number systems Students explore the operations of addition, subtraction, multiplication and division and the relationships between these operations – in the first instance with whole numbers and integers. They will explore some of the laws that govern these operations and use mathematical models to reinforce the algorithms they commonly use. Later, they revisit these operations in the contexts of rational numbers and refine and revise their ideas. Students will devise strategies for computation that can be applied to any number. Implicit in such computational methods are generalisations about numerical relationships involving the operations being used. Students will articulate the generalisation that underlies their strategy, firstly in common language and then in symbolic language.	– investigate models such as decomposition, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, subtraction, multiplication, and division in **N** where the answer is in **N**, including the inverse operations – investigate the properties of arithmetic: commutative, associative and distributive laws and the relationships between them – appreciate the order of operations, including use of brackets – investigate models, such as the number line, to illustrate the operations of addition, subtraction, multiplication and division in **Z** – use the number line to order numbers in **N, Z, Q (and R for HL)** – generalise and articulate observations of arithmetic operations – investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – analyse solution strategies to problems – begin to look at the idea of mathematical proof – calculate percentages – use the equivalence of fractions, decimals and percentages to compare proportions – consolidate their understanding and their learning of factors, multiples and prime numbers in **N** – consolidate their understanding of the relationship between ratio and proportion – check a result by considering whether it is of the right order of magnitude – check a result by working the problem backwards – justify approximations and estimates of calculations – present numerical answers to degree of accuracy specified
Strand 3: 3.5 Sets Students learn the concept of a set as being a collection of well-defined objects or *elements*. They are introduced to the concept of the universal set, null set, sub-set; the union and intersection operators and to Venn diagrams: simple closed bounded curves that contain the elements of a set. They investigate the properties of arithmetic as related to sets and solve problems involving sets.	– list elements of a set – describe the rule that defines a finite set – consolidate the idea that equality of sets is a relationship in which two equal sets have the same elements – use the cardinal number terminology when referring to set membership – perform the operations of intersection, union (for two sets) – investigate the commutative property for intersection and union – illustrate sets using Venn diagrams

Strand /Topic Title	Learning outcomes Students should be able to
Strand 4: 4.1 Generating arithmetic expressions from repeating patterns Students examine patterns and the rules that govern them and so construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.	– use tables and diagrams to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence
Strand 4: 4.2 Representing situations with tables diagrams and graphs Students examine relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks.They look at various patterns and make predictions about what comes next.	– use tables, diagrams and graphs as a tool for analysing relations – develop and use their own mathematical strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
All Strands **Synthesis and problem-solving skills**	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

SECTION B

Geometry for Post-primary School Mathematics

This section sets out the course in geometry for both Junior Certificate Mathematics and Leaving Certificate Mathematics. Strand 2 of the relevant syllabus document specifies the learning outcomes at the different syllabus levels.

Geometry for Post-primary School Mathematics

1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It was prepared by a working group of the NCCA course committees for mathematics and, following minor amendments, was adopted by both committees for inclusion in the syllabus documents. Readers should refer to Strand 2 of the syllabus documents for Junior Certificate and Leaving Certificate mathematics for the range and depth of material to be studied at the different levels. A summary of these is given in sections 9–13 of this document.

The preparation and presentation of this document was undertaken principally by Anthony O'Farrell, with assistance from Ian Short. Helpful criticism from Stefan Bechluft-Sachs, Ann O'Shea, Richard Watson and Stephen Buckley is also acknowledged.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane, point, line, $<_l$ (precedes on a line), (open) half-plane, distance, and degree-measure**, and seven axioms: A_1: about incidence, A_2: about order on lines, A_3: about how lines separate the plane, A_4: about distance, A_5: about degree measure, A_6: about congruence of triangles, A_7: about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as "the point is/lies on the line", "the line passes through the point", etc.

- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).

- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.

- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.

- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word "reflex" precedes or follows.

- We make no reference to results such as Pasch's property and the "crossbar theorem". (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)

- We refer to "the number of degrees" in an angle, whereas Barry treats this more correctly as "the degree-measure" of an angle.

- We take it that the definitions of parallelism, perpendicularity and "sidedness" are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).

- We do not refer explicitly to triangles being **congruent** "under the correspondence $(A, B, C) \to (D, E, F)$", taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say "ΔABC is congruent to ΔDEF" we mean, using Barry's terminology, "Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence $(A, B, C) \to (D, E, F)$".

- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle[1]. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: "opposite sides of a parallelogram are equal", or refer to "a circle of radius r". Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A, then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry's geometry that are retained in our less formal version:

[1] In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.
- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry[2].
- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.
- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4 Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

[2] Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).

3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.

4. Some guidance on teaching (Section 8).

5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle, sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line[3] is short for straight line. Take a fixed **plane**[4], once and for all, and consider just lines that lie in it. The plane and the lines are **sets**[5] of **points**[6]. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.

Axioms are statements we will accept as true[7].

Axiom 1 (Two Points Axiom)**.** *There is exactly one line through any two given points. (We denote the line through A and B by AB.)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all

[3]Line is undefined.
[4]Undefined term
[5]Undefined term
[6]Undefined term
[7] An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A, B and C be points that are not collinear. The **triangle** ΔABC is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**[8] by $\mathbb{R}$.

Definition 4. We denote the **distance**[9] between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a, b, and c. The usual thing for a triangle ΔABC is to take $a = |BC|$, i.e. the length of the side opposite the vertex A, and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom[10])**.** *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. *$|AB| = |BA|$;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

[8]Undefined term

[9]Undefined term

[10] Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with [11]

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A, called its **vertex**;

2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;

3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

[11] Students may notice that the first equality implies the second.

Definition 12. Suppose that A, B, and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A, B, C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α,β,γ, etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle \alpha|$, as the case may be.

Axiom 3 (Protractor Axiom)**.** *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$*. The number of degrees of an ordinary angle is less than* $180°$*. It has these properties:*

1. *A straight angle has* $180°$*.*
2. *Given a ray $[AB$, and a number d between* 0 *and* 180*, there is exactly one ray from A on each side of the line AB that makes an (ordinary) angle having d degrees with the ray $[AB$.*
3. *If D is a point inside an angle $\angle BAC$, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ measures $360° - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is 'an angle of' (for instance) $45°$, if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly $90°$.

Definition 15. An angle is **acute** if it has less than 90°, and **obtuse** if it has more than 90°.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180°.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

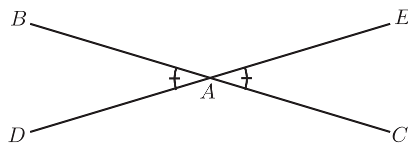


Figure 1.

Theorem 1 (Vertically-opposite Angles)**.**
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180°. In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180°, \\ |\angle CAE| + |\angle BAE| &= 180°, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0°, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

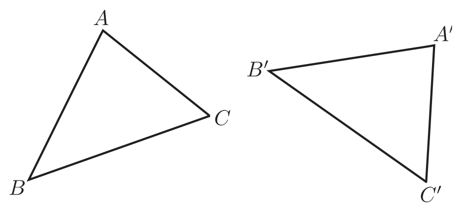


Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12])**.**
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, *and* $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$
then the triangles ΔABC *and* $\Delta A'B'C'$ *are congruent.*

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to $90°$, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS]
$\therefore \angle B = \angle C$.

[12]It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.
[13] The simple "equal" is preferred to "of equal length"

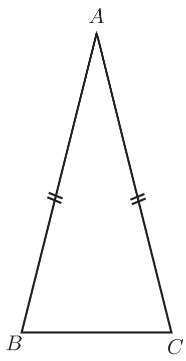


Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|, \Delta ABC$ is isosceles. □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SAS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels)**.** *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

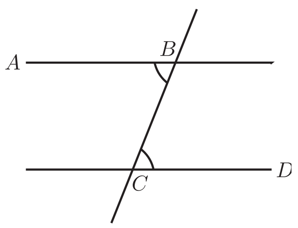


Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.

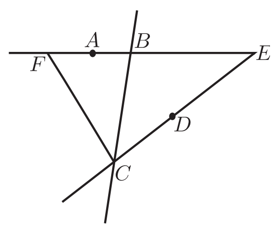


Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

[14]Fuller detail: There are three cases:
1°: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
2°: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then ΔBCE is congruent to ΔCBF. [SAS]
Thus
$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that F lies on DC. [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

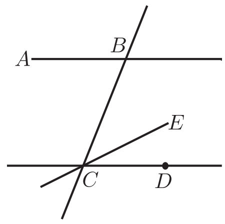


Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. □

Theorem 4 (Angle Sum 180)**.** *The angles in any triangle add to* 180°.

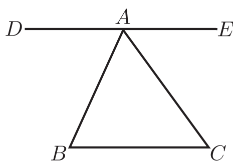


Figure 7.

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so that F lies on DC. [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

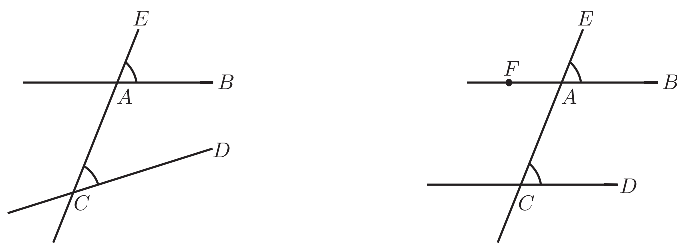
$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ = & |\angle DAB| + |\angle CAE| + |\angle BAC| \\ = & |\angle DAE| = 180°, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].



(a) (b)

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]

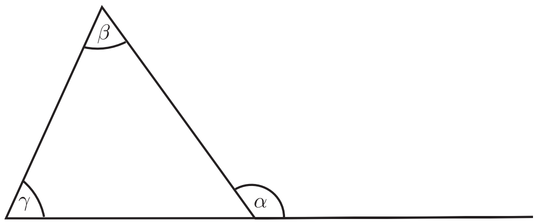


Figure 9.

Theorem 6 (Exterior Angle)**.** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then

$|\alpha| + |\angle A| = 180°$ [Supplementary angles]

and

$|\angle B| + |\angle C| + |\angle A| = 180°$. [Angle sum 180°]

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. □

[16]The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

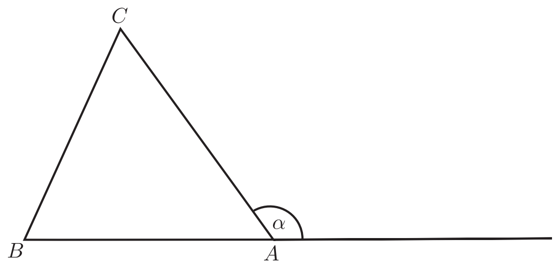


Figure 10.

Theorem 7.
(1) In ΔABC, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.
(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
$|AD| = |AB|$. [Ruler Axiom]

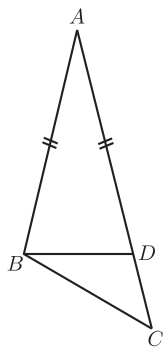


Figure 11.

See Figure 11. Then ΔABD is isosceles, so

$$\begin{aligned} |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\ &= |\angle ABD| && \text{[Isosceles Triangle]} \\ &< |\angle ABC|. \end{aligned}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)
Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

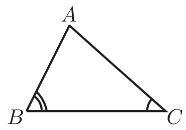


Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. □

Theorem 8 (Triangle Inequality)**.**
Two sides of a triangle are together greater than the third.

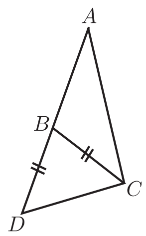


Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$[17] we have

$$|\angle BCD| < |\angle ACD|.$$

[17] B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180°$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to ΔADC we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. [18] *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing though a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB.

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

[18] In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

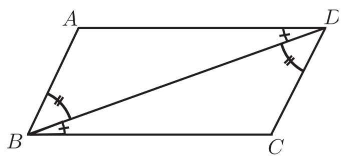


Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB||CD$ and $AD||BC$. Then

$|\angle ABD| = |\angle BDC|$ [Alternate Angle Theorem]
$|\angle ADB| = |\angle DBC|$ [Alternate Angle Theorem]
ΔDAB is congruent to ΔBCD. [ASA]

$$\therefore |AB| = |CD|,\ |AD| = |CB|,\ \text{and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360°. It follows that adjacent angles add to 180°. Theorem 3 then yields the result. □

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. □

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

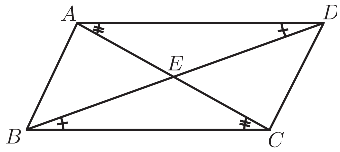


Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of ΔADE and ΔCBE.

In detail: Let AC cut BD in E. Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \Delta ADE$ is congruent to ΔCBE. [ASA] □

Proposition 6 (Converse)**.** *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of ΔABE and ΔCDE. Then use Alternate Angles. □

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180°, so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

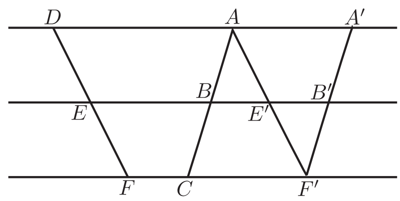


Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD||BE||CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE'||DE$, cutting EB at E' and CF at F'.
Draw $F'B'||AB$, cutting EB at B'. See Figure 16.

Then

$\lvert B'F'\rvert$	$=$	$\lvert BC\rvert$	[Theorem 9]
	$=$	$\lvert AB\rvert$.	[by Assumption]
$\lvert\angle BAE'\rvert$	$=$	$\lvert\angle E'F'B'\rvert$.	[Alternate Angle Theorem]
$\lvert\angle AE'B\rvert$	$=$	$\lvert\angle F'E'B'\rvert$.	[Vertically Opposite Angles]
$\therefore \Delta ABE'$	is congruent to	$\Delta F'B'E'$.	[ASA]
$\therefore \lvert AE'\rvert$	$=$	$\lvert F'E'\rvert$.	

But
$|AE'| = |DE|$ and $|F'E'| = |FE|$. [Theorem 9]
$\therefore |DE| = |EF|$. □

Definition 36. Let s and t be positive real numbers. We say that a point C **divides the segment** $[AB]$ **in the ratio** $s : t$ if C lies on the line AB, and is between A and B, and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l **cuts** $[AB]$ **in the ratio** $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B, and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. *Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.*

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n. Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \ldots, D_{m-1}, D_m = D, D_{m+1}, \ldots, D_{m+n-1}, D_{m+n} = B,$$

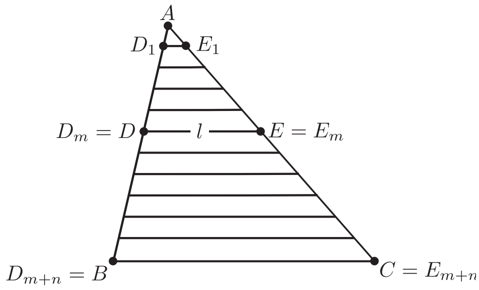


Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \dots [D_iD_{i+1}], \dots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines $D_1E_1, D_2E_2, \dots$ parallel to BC with $E_1, E_2, \dots$ on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \dots, [E_{m+n-1}C]$$

have the same length, [Theorem 11]
and $E_m = E$ is the point where l cuts $[AC]$. [Axiom of Parallels]
Hence E divides $[AC]$ in the ratio $m : n$. □

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB$ and C'' on $[AC$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D. Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so ΔABC is similar to $\Delta A'B'C'$. [Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:
Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC.

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. *If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

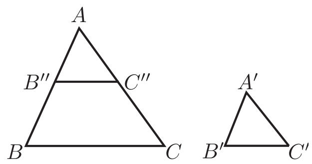


Figure 18.

Proof. We may suppose $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$\Delta AB''C''$	is congruent to	$\Delta A'B'C'$	[SAS]
$\therefore |\angle AB''C''|$	$=$	$|\angle ABC|$	
$\therefore B''C''$	$\parallel$	BC	[Corresponding Angles]
$\therefore \dfrac{|A'B'|}{|A'C'|}$	$=$	$\dfrac{|AB''|}{|AC''|}$	[Choice of B'', C'']
	$=$	$\dfrac{|AB|}{|AC|}$	[Theorem 12]
$\dfrac{|AC|}{|A'C'|}$	$=$	$\dfrac{|AB|}{|A'B'|}$	[Re-arrange]

Similarly, $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$ □

Proposition 8 (Converse)**.** *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC *and* $\Delta A'B'C'$ *are similar.*

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC. □

6.9 Pythagoras

Theorem 14 (Pythagoras)**.** *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

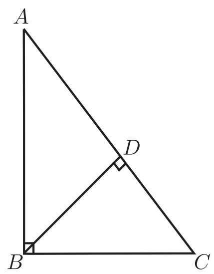


Figure 19.

Proof. Let ΔABC have a right angle at B. Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A. $\therefore$ ΔABC is similar to ΔADB.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| \, (|AD| + |DC|) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras)**.** *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

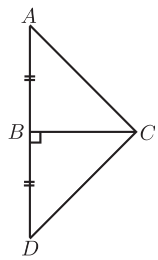


Figure 20.

Proof. (Idea: Construct a second triangle on the other side of $[BC]$, and use Pythagoras and SSS to show it congruent to the original.)
In detail: We wish to show that $|\angle ABC| = 90°$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned}
|DC| &= \sqrt{|DC|^2} \\
&= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\
&= \sqrt{|AB|^2 + |BC|^2} && [\ |AB| = |BD|\ ] \\
&= \sqrt{|AC|^2} && \text{[Hypothesis]} \\
&= |AC|.
\end{aligned}$$

$\therefore \Delta ABC$ is congruent to ΔDBC. [SSS]
$\therefore |\angle ABC| = |\angle DBC| = 90°$. □

Proposition 9 (RHS)**.** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B', and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment* $[AB]$ *is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C, hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

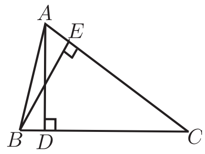


Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by "area of ΔABC"[19].

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

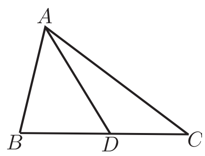


Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h, so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

[19] $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles[20].

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas[21].

Proposition 14. *The area of a rectangle having sides of length a and b is ab.*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC.

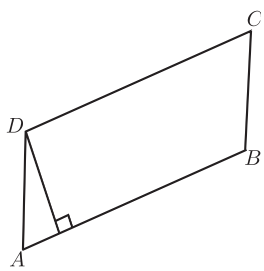


Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD, and as the length of the perpendicular segment from D onto AB.*

[20] If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

[21] Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, ΔABD and ΔCDB. These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. □

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called **a** radius. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the** diameter of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B, and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π.

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

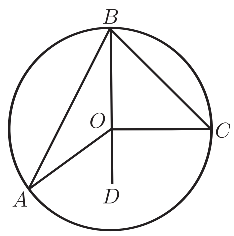


Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D. Then

$$\begin{aligned} |OA| &= |OB|. && \text{[Definition of circle]} \\ \therefore |\angle BAO| &= |\angle ABO|. && \text{[Isosceles triangle]} \\ \therefore |\angle AOD| &= |\angle BAO| + |\angle ABO| && \text{[Exterior Angle]} \\ &= 2 \cdot |\angle ABO|. \end{aligned}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{aligned} |\angle AOC| &= |\angle AOD| + |\angle COD| \\ &= 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\ &= 2 \cdot |\angle ABC|. \end{aligned}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC, then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC:

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC, and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C. If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s. We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. □

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90°$.*

Proof. The angle at the centre is a straight angle, measuring 180°, and half of that is 90°. □

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180°, so is straight, and so the line BC passes through the centre. □

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to* 180°.

Proof. The two angles at the centre standing on the same arcs add to 360°, so the two halves add to 180°. □

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to* 180°, *then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. □

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2, where r is its radius. For this reason, we say that the area of the disc is πr^2.

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. □

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.
(1) Each tangent is perpendicular to the radius that goes to the point of contact.
(2) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is tangent to s.

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP.

Let the perpendicular to the tangent from the centre O meet it at Q. Pick R on PQ, on the other side of Q from P, with $|QR| = |PQ|$ (as in Figure 25).

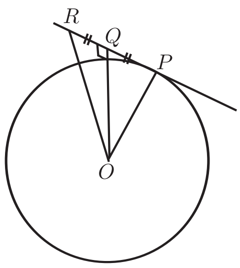


Figure 25.

Then ΔOQR is congruent to ΔOQP. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP, as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P, and hence is not on the circle.)

In detail: Let Q be any point on l, other than P. See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 \quad \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

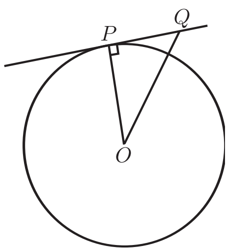


Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
$\therefore P$ is the only point of l on the circle.
$\therefore l$ is a tangent. [Definition of tangent]
□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

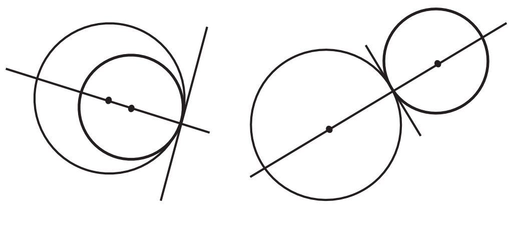


Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.
(1) The perpendicular from the centre to a chord bisects the chord.
(2) The perpendicular bisector of a chord passes through the centre.

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.) See Figure 28.

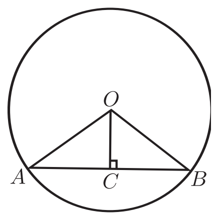


Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| \\ \\ |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \Delta OAC$ is congruent to ΔOBC. [SSS]
$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.
Let C be the foot of the perpendicular from O on AB.
By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.
Thus CO is the perpendicular bisector of AB.
Hence the perpendicular bisector of AB passes through O. □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A, B, and C, then its centre lies on the perpendicular bisector of each side of the triangle ΔABC.*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

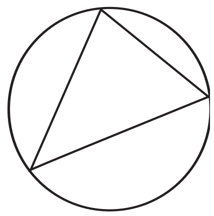


Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

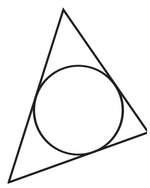


Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

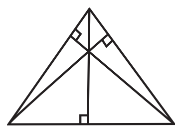


Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A, such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of $30°$, $60°$, and $45°$. It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l, passing through a given point not on l.

4. Line perpendicular to a given line l, passing through a given point on l.
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of 60°, without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

8 Teaching Approaches

8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).
2. Ideas from Technical Drawing.
3. Material in [3].

8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that $n^2 + n + 41$ is prime for all $n \in \mathbb{N}$). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or “cuts” involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras’ theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word “prove” means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a “proof” is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based “proofs” of fallacies and then probe a dissection-based proof of Pythagoras’ theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

9 Syllabus for JCOL

9.1 Concepts

Set, plane, point, line, ray, angle, real number, length, degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area, tangent to a circle, subset, segment, collinear points, distance, midpoint of a segment, reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle, angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio, isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate angles, corresponding angles, polygon, quadrilateral, convex quadrilateral, rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line, circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, point of contact of a tangent, vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral, centre of a circle.

9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

9.3 Axioms and Proofs

The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 13 (statement only), 14, 15; and direct proofs of Corollaries 3, 4.

10 Syllabus for JCHL

10.1 Concepts

Those for JCOL, and concurrent lines.

10.2 Constructions

Students will study all the constructions prescribed for JC-OL, and also constructions 3 and 7.

10.3 Logic, Axioms and Theorems

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 1, 2, 3, 4*, 5, 6*, 9*, 10, 11, 12, 13, 14*, 15, 19*, Corollaries 1, 2, 3, 4, 5, and their converses. Those marked with a * may be asked in examination.

The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration.

11 Syllabus for LCFL

Students are expected to build on their mathematical experiences to date.

11.1 Constructions

Students revisit constructions 4, 5, 10, 13, 15, and learn how to apply these in real-life contexts.

12 Syllabus for LCOL

12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 16–21.

12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed.

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.

13 Syllabus for LCHL

13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry.

They will be asked to solve geometrical problems (so-called "cuts") and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

References

[1] Patrick D. Barry. *Geometry with Trigonometry.* Horwood. Chichester. 2001. ISBN 1-898563-69-1.

[2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers.* Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.

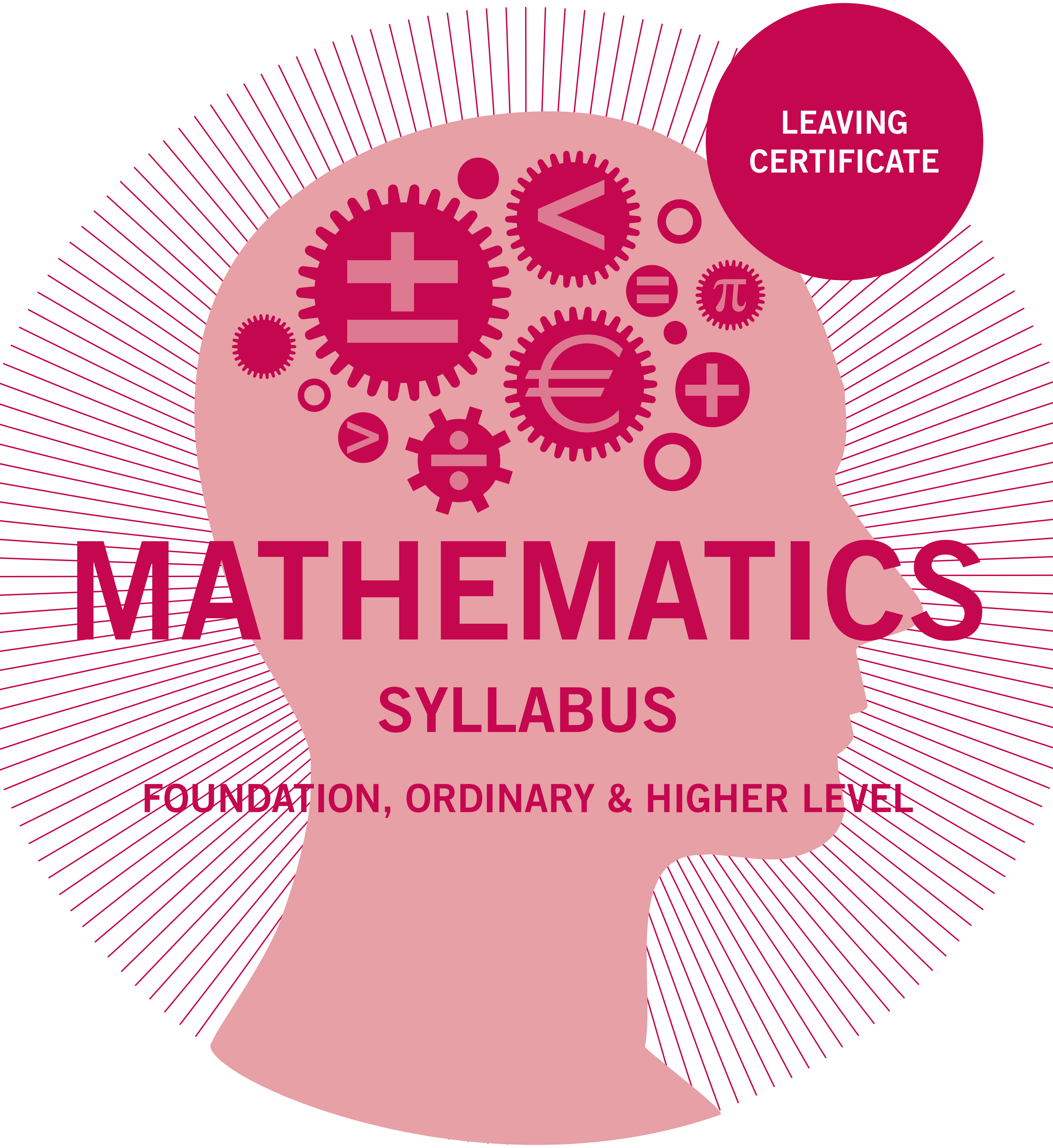
[3] Fiacre O'Cairbre, John McKeon, and Richard O. Watson. A Resource for Transition Year Mathematics Teachers. DES. Dublin. 2006.

[4] Anthony G. O'Farrell. *School Geometry*. IMTA Newsletter 109 (2009) 21-28.

Schools participating in Project Maths

School Name		School Address	
Abbey Community College	Wicklow	Co Wicklow	
Abbey Vocational School	Donegal Town	Co Donegal	
Árdscoil na mBráithre	Clonmel	Co Tipperary	
Castleknock College	Castleknock	Dublin 15	
Coláiste Choilm	O'Moore Street	Tulach Mhór	Co Offaly
Coláiste Íosagáin	Portarlington	Co Laois	
Coláiste na Sceilge	Caherciveen	Co Kerry	
Coláiste Phobal Ros Cré	Corville Road	Roscrea	Co Tipperary
Loreto Abbey Secondary School	Dalkey	Co Dublin	
Meán Scoil an Chlochair	Kilbeggan	Mullingar	Co Westmeath
Moate Community School	Church Street	Moate	Co Westmeath
Our Lady's College	Greenhills	Drogheda	Co Louth
Pobalscoil Chorca Dhuibhne	Bóthar an Spá	An Daingean	Co. Chiarraí
Presentation Secondary School	Warrenmount	Dublin 8	
Presentation Secondary School	Listowel	Co Kerry	
Ratoath College	Jamestown	Ratoath	Co Meath
Sacred Heart School	Westport	Co Mayo	
Scoil Chonglais	Baltinglass	Co Wicklow	
Scoil Mhuire	Kanturk	Co Cork	
St Columba's Comprehensive Sch	Glenties	Co Donegal	
St Mark's Community School	Cookstown Road	Tallaght	Dublin 24
St Patrick's College	Gardiner's Hill	Cork	
Wesley College	Ballinteer	Dublin 16	
Wexford Vocational College	Westgate	Wexford	Co Wexford

An Roinn Oideachais
agus Scileanna



LEAVING CERTIFICATE

MATHEMATICS

SYLLABUS

FOUNDATION, ORDINARY & HIGHER LEVEL

For examination from 2015

MATHEMATICS

Leaving Certificate Mathematics

Introduction and rationale

Mathematics is a wide-ranging subject with many aspects. Most people are familiar with the fact that mathematics is an intellectual discipline that deals with abstractions, logical arguments, deduction and calculation. But mathematics is also an expression of the human mind reflecting the active will, the contemplative reason and the desire for aesthetic perfection. It is also about pattern, the mathematics of which can be used to explain and control natural happenings and situations. Increasingly, mathematics is the key to opportunity. No longer simply the language of science, mathematics contributes in direct and fundamental ways to business, finance, health and defence. For students it opens doors to careers. For citizens it enables informed decisions. For nations it provides knowledge to compete in a technological community. Participating fully in the world of the future involves tapping into the power of mathematics.

Mathematical knowledge and skills are held in high esteem and are seen to have a significant role to play in the development of the knowledge society and the culture of enterprise and innovation associated with it. Mathematics education should be appropriate to the abilities, needs and interests of learners and should reflect the broad nature of the subject and its potential for enhancing their development. The elementary aspects of mathematics, use of arithmetic and the display of information by means of a graph are an everyday occurrence. Advanced mathematics is also widely used, but often in an unseen and unadvertised way. The mathematics of error-correcting codes is applied to CD players and to computers. The stunning pictures of far away planets and nebulae sent by Voyager II and Hubble could not have had their crispness and quality without such mathematics. In fact, Voyager's journey to the planets could not have been planned without the mathematics of differential equations. In ecology, mathematics is used when studying the laws of population change. Statistics not only provides the theory and methodology for the analysis of wide varieties of data but is essential in medicine, for analysing data on the causes of illness and on the utility of new drugs. Travel by aeroplane would not be possible without the mathematics of airflow and of control systems. Body scanners are the expression of subtle mathematics discovered in the 19th century, which makes it possible to construct an image of the inside of an object from information on a number of single X-ray views of it. Thus, mathematics is often involved in matters of life and death.

Aim

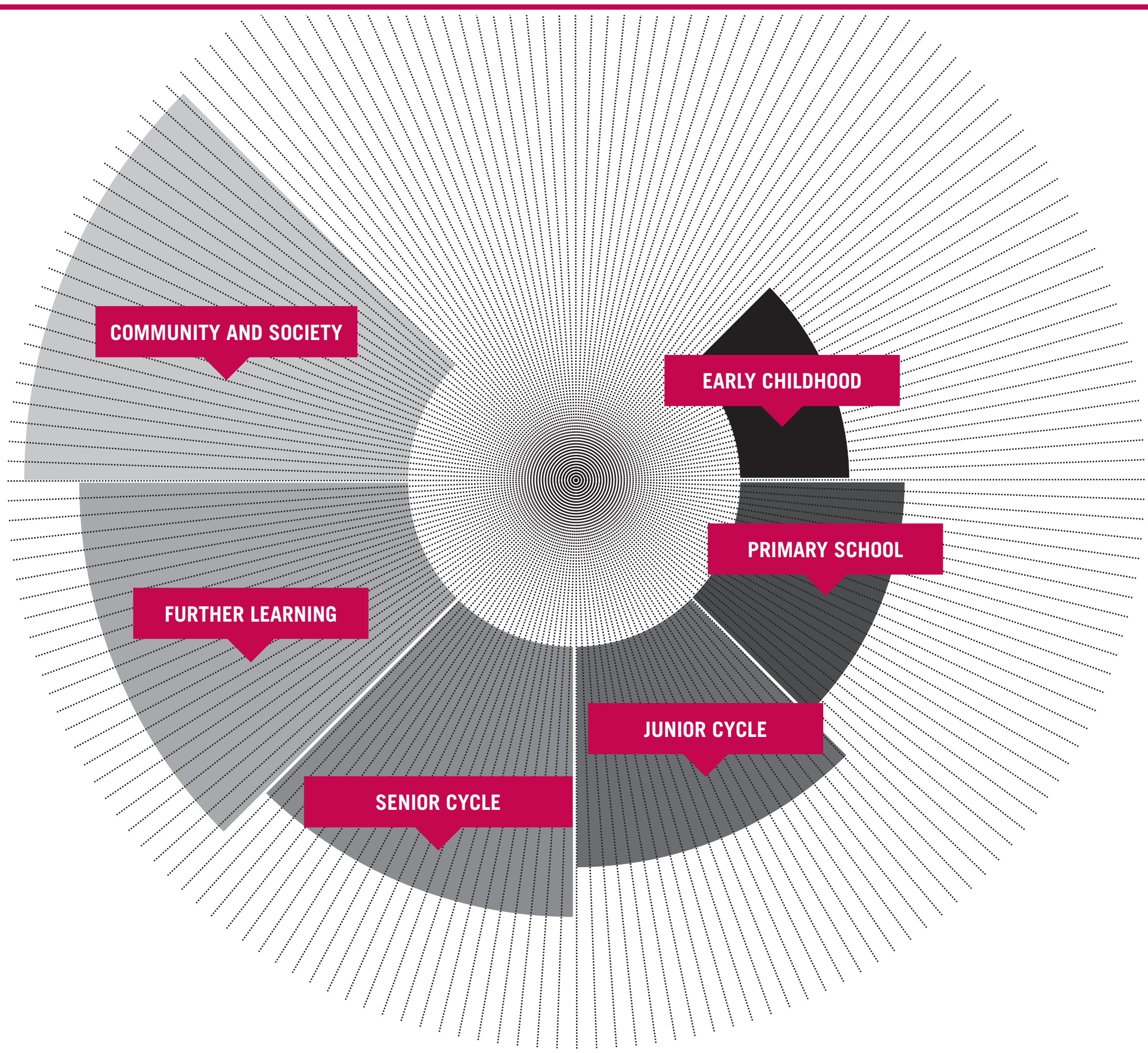
Leaving Certificate Mathematics aims to develop mathematical knowledge, skills and understanding needed for continuing education, life and work. By teaching mathematics in contexts that allow learners to see connections within mathematics, between mathematics and other subjects, and between mathematics and its applications to real life, it is envisaged that learners will develop a flexible, disciplined way of thinking and the enthusiasm to search for creative solutions.

Objectives

The objectives of Leaving Certificate Mathematics are that learners develop mathematical proficiency, characterised as

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems in both familiar and unfamiliar contexts
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, justification and communication
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence, perseverance and one's own efficacy.

Related learning

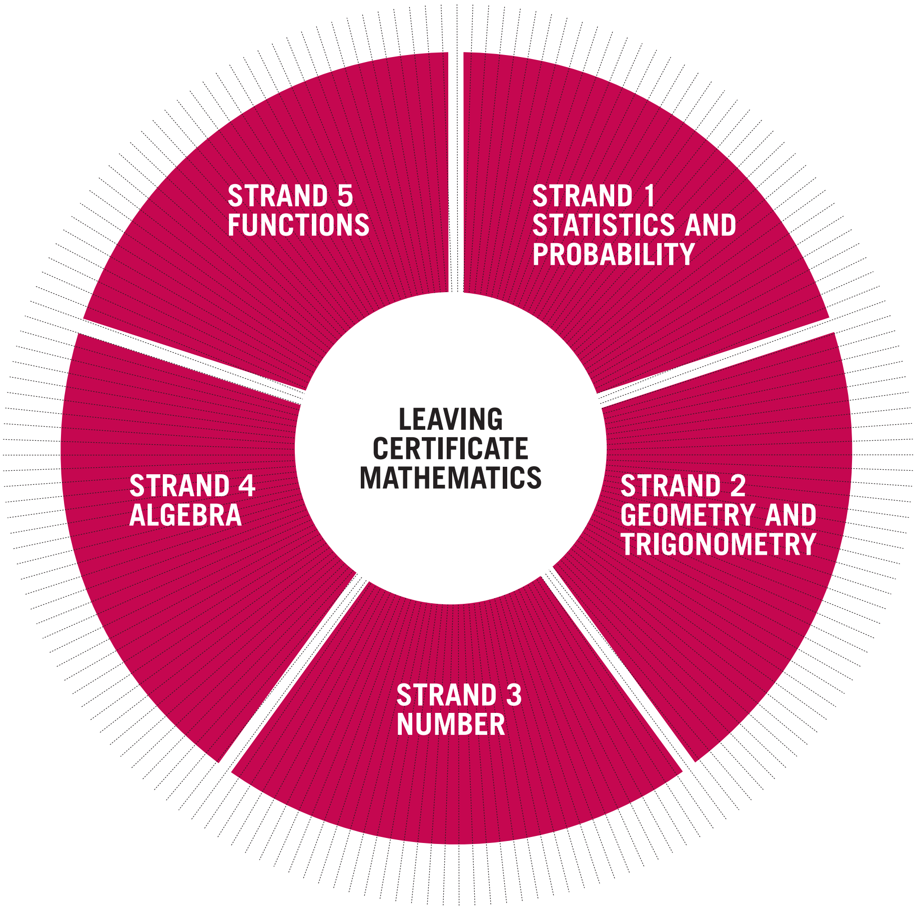


The way in which mathematics learnt at different stages links together is very important to the overall development of mathematical understanding. The study of Leaving Certificate Mathematics encourages learners to use the numeracy and problem solving skills developed in early childhood education, primary mathematics and junior cycle mathematics. The emphasis is on building connected and integrated mathematical understanding. As learners progress through their education, mathematical skills, concepts and knowledge are developed when they work in more demanding contexts and develop more sophisticated approaches to problem solving. In this way mathematical learning is cumulative, with work at each level building on and deepening what students have learned at the previous level.

Mathematics is not learned in isolation; it has significant connections with other curriculum subjects. Many science subjects are quantitative in nature and learners are expected to be able to work with data, produce graphs and interpret patterns and trends. Design and Communication Graphics uses drawings in the analysis and solution of two- and three-dimensional problems through the rigorous application of geometric principles. In Geography learners use ratio to determine scale. Every day, people use timetables, clocks and currency conversions to make life easier. Consumers need basic financial awareness and in Home Economics learners use mathematics when budgeting and making value for money judgements. Learners use mathematics in Economics for describing human behaviour. In Business Studies learners see how mathematics can be used by business organisations in accounting, marketing, inventory management, sales forecasting and financial analysis.

Mathematics, Music and Art have a long historical relationship. As early as the fifth century B.C., Pythagoras uncovered mathematical relationships in music, while many works of art are rich in mathematical structure. The modern mathematics of fractal geometry continues to inform composers and artists. Mathematics sharpens critical thinking skills, and by empowering learners to critically evaluate information and knowledge it promotes their development as statistically aware consumers.

Syllabus overview



Structure

The Leaving Certificate Mathematics syllabus comprises five strands:

1. Statistics and Probability
2. Geometry and Trigonometry
3. Number
4. Algebra
5. Functions

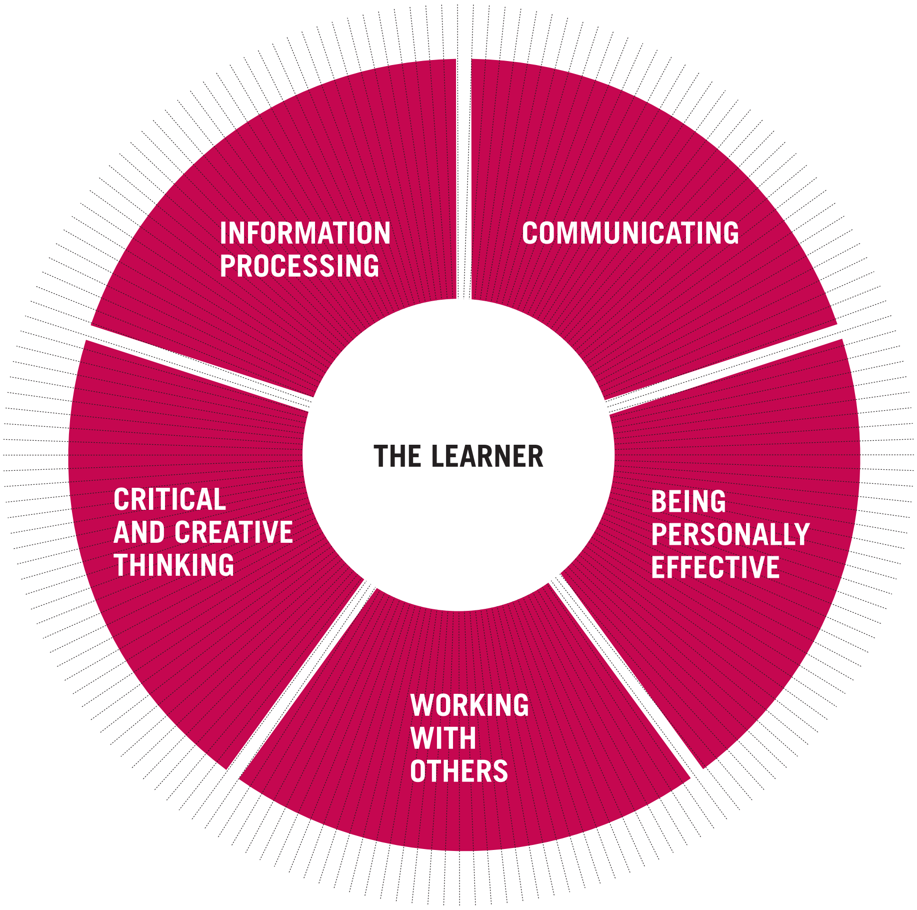
The strand structure of the syllabus should not be taken to imply that topics are to be studied in isolation. Where appropriate, connections should be made within and across the strands and with other areas of learning.

In each strand of this syllabus, learning outcomes specific to that strand are listed. The Foundation level learning outcomes are distinct from the Ordinary level and Higher level outcomes and are listed separately. The learning outcomes specified at Ordinary level are a subset of the learning outcomes for those studying at Higher level. At Ordinary level and Higher level, knowledge of the content and learning outcomes at the corresponding level in the Junior Certificate Mathematics syllabus is assumed.

Time allocation

The Leaving Certificate Mathematics syllabus is designed as a 180–hour course of study.

Key Skills



There are five key skills identified as central to teaching and learning across the curriculum at senior cycle. These are *information processing, being personally effective, communicating, critical and creative thinking* and *working with others*. These key skills are important for all learners to reach their full potential – both during their time in school and in the future – and to participate fully in society, including family life, the world of work and lifelong learning. By engaging with key skills learners enhance their ability to learn, broaden the scope of their learning and increase their capacity for learning.

Leaving Certificate Mathematics develops key skills in the following ways.

Information processing

Successful mathematics learning requires the efficient processing of the information that defines the mathematical tasks. Information is readily accessible from a variety of sources and information processing relates to the ways in which learners make sense of, or interpret, the information to which they are exposed.

Critical and creative thinking

There is a strong emphasis on investigation in mathematics and engaging in the investigative process requires learners to critically evaluate information and think creatively about it. Learners are encouraged to solve problems in a variety of ways and are required to evaluate methods and arguments and to justify their claims and results.

Communicating

In mathematics learners are encouraged to discuss approaches and solutions to problems and are expected to consider and listen to other viewpoints. Since mathematics emphasises investigation an important aspect of this is communicating findings to a variety of audiences in different ways.

Working with others

In mathematics learners are encouraged to work together in groups to generate ideas, problem solve and evaluate methods.

Being personally effective

Studying mathematics empowers learners to gain knowledge and skills that will benefit them directly in other aspects of their everyday lives. They participate in a learning environment that is open to new ideas and gain confidence in expressing their mathematical ideas and considering those of others.

While the Leaving Certificate Mathematics syllabus places particular emphasis on the development and use of information processing, logical thinking and problem-solving skills, the approach to teaching and learning involved gives prominence to learners being able to develop their skills in communicating and working with others. By adopting a variety of approaches and strategies for solving problems in mathematics, learners develop their self-confidence and personal effectiveness. The key skills are embedded within the learning outcomes and are assessed in the context of the learning outcomes.

In Leaving Certificate Mathematics students not only learn procedures and acquire reliable methods for producing correct solutions on paper-and-pencil exercises, but also learn mathematics with understanding. In particular, they should be able to explain why the procedures they apply are mathematically appropriate and justify why mathematical concepts have the properties that they do.

Problem Solving

Problem solving means engaging in a task for which the solution is not immediately obvious. Problem solving is integral to mathematical learning. In day-to-day life and in the workplace the ability to problem solve is a highly advantageous skill.

In the mathematics classroom problem solving should not be met in isolation, but should permeate all aspects of the teaching and learning experience. Problems may concern purely mathematical matters or some applied context.

In a mathematics problem-solving environment it is recognised that there are three things learners need to do:

- make sense of the problem
- make sense of the mathematics they can learn and use when doing the problem
- arrive at a correct solution to the problem.

However, in the mathematics classroom, the focus is on the mathematical knowledge and skills that can be learned in the process of obtaining an answer, rather than on the answer itself. The emphasis, therefore, is on generating discussion and on the reasoning and sense-making opportunities the problem affords the learners as they engage with the mathematics involved. They learn to analyse the problem and break it down into manageable steps, to reflect on their strategies and those of others and to adjust their own approaches where necessary.

Teachers play an important role in helping students develop these kinds of skills. By encouraging learners to share and explain their solution strategies, those that work as well as those that don't work, teachers can help learners to develop robust and deep mathematical understanding as well as confidence in their mathematical ability.

The quality of the tasks that learners engage with plays an important role in a problem-solving environment. A task must engage learners and present them with a challenge that requires exploration. Problem-solving tasks activate creative mathematical thinking processes as opposed to imitative thinking processes activated by routine tasks. Reasoning mathematically about tasks empowers learners to make connections within mathematics and to develop deep conceptual understanding.

Teaching and learning

In line with the syllabus objectives and learning outcomes, the experience of learners in the study of mathematics should contribute to the development of their problem-solving skills through the application of their mathematical knowledge and skills to appropriate contexts and situations. In each strand, at every syllabus level, emphasis is placed on appropriate contexts and applications of mathematics so that learners can appreciate its relevance to their current and future lives. The focus should be on learners understanding the concepts involved, building from the concrete to the abstract and from the informal to the formal.

Learners will build on their knowledge of mathematics constructed initially through their exploration of mathematics in the primary school and through their continuation of this exploration at junior cycle. Particular emphasis is placed on promoting learners' confidence in themselves (confidence that they can do mathematics) and in the subject (confidence that mathematics makes sense). Through the use of meaningful contexts, opportunities are presented for learners to achieve success.

Learners will integrate their knowledge and understanding of mathematics with economic and social applications of mathematics. By becoming statistically aware consumers, learners are able to critically evaluate knowledge claims and learn to interrogate and interpret data – a skill which has a value far beyond mathematics wherever data is used as evidence to support argument.

The variety of activities that learners engage in enables them to take charge of their own learning by setting goals, developing action plans and receiving and responding to assessment feedback. As well as varied teaching strategies, varied assessment strategies will provide information that can be used as feedback for teachers so that teaching and learning activities can be modified in ways which best suit individual learners. Results of assessments may also be used by teachers to reflect on their teaching practices so that instructional sequences and activities can be modified as required. Feedback to learners about their performance is critical to their learning and enables them to develop as learners. This formative assessment, when matched to the intended learning outcomes, helps to ensure consistency between the aim and objectives of the syllabus and its assessment. A wide range of assessment methods may be used, including investigations, class tests, investigation reports, oral explanation, etc.

Careful attention must be paid to learners who may still be experiencing difficulty with some of the material covered in the junior cycle. Nonetheless, they need to learn to cope with mathematics in everyday life and perhaps in further study. Their experience of Leaving Certificate Mathematics must therefore assist them in developing a clearer knowledge of and improved skills in, basic mathematics, and an awareness of its usefulness. Appropriate new material should also be introduced so that the learners can feel that they are progressing. At Leaving Certificate, the course followed should pay great attention to consolidating the foundation laid in the junior cycle and to addressing practical issues; but it should also cover new topics and lay a foundation for progress to the more traditional study of mathematics in the areas of algebra, geometry and functions.

Differentiation

Provision must be made not only for the academic student of the future, but also for the citizen of a society in which mathematics appears in, and is applied to, everyday life. The syllabus therefore focuses on material that underlies academic mathematical studies, ensuring that learners have a chance to develop their mathematical abilities and interests to a high level. It also covers the more practical and obviously applicable topics that learners meet in their lives outside school.

In each strand the learning outcomes are set out in terms of Foundation level, Ordinary level and Higher level. Ordinary level is a subset of Higher level. Therefore, learners studying at Higher level are expected to achieve the Ordinary level and Higher level learning outcomes. At Ordinary level and Higher level, knowledge of the content and learning outcomes at the corresponding level in the Junior Certificate Mathematics syllabus is assumed. In each strand, students are expected to use their mathematical knowledge and skills to solve appropriate problems, which can arise in both mathematical and applied contexts, and to make connections between topics and across strands.

Mathematics at Higher level is geared to the needs of learners who may proceed with their study of mathematics to third level. However, not all learners are future specialists or even future users of academic mathematics. Moreover, when they start to study the material, some of them are only beginning to deal with abstract concepts. For Higher level, particular emphasis can be placed on the development of powers of abstraction and generalisation and on the idea of rigorous proof, hence giving learners a feeling for the great mathematical concepts that span many centuries and cultures.

Mathematics at Ordinary level is geared to the needs of learners who are beginning to deal with abstract ideas. However, many of them may go on to use and apply mathematics in their future careers, and all of them will meet the subject to a greater or lesser degree in their daily lives. Ordinary level Mathematics, therefore, must start by offering mathematics that is meaningful and accessible to learners at their present stage of development. It should also provide for the gradual introduction of more abstract ideas, leading the learners towards the use of academic mathematics in the context of further study.

Mathematics at Foundation level places particular emphasis on the development of mathematics as a body of knowledge and skills that makes sense, and that can be used in many different ways as an efficient system for solving problems and finding answers. Alongside this, adequate attention must be paid to the acquisition and consolidation of fundamental skills, in the absence of which the learners' development and progress will be hindered. Foundation level Mathematics is intended to equip learners with the knowledge and skills required in everyday life, and it is also intended to lay the groundwork for learners who may proceed to further studies in areas in which specialist mathematics is not required.

Learners taking Foundation level mathematics are not required to deal with abstract mathematics. Thus, their experience of mathematics at Leaving Certificate should be approached in an exploratory and reflective manner, adopting a developmental and constructivist approach which allows them to make sense of their mathematical experiences to date and to solve the types of problems they may encounter in their daily lives. An appeal should be made to different interests and ways of learning, for example by paying attention to visual and spatial as well as to numerical aspects.

Differentiation will also apply in how Leaving Certificate Mathematics is assessed at Foundation, Ordinary and Higher levels. Ordinary level is a subset of Higher level and differentiation at the point of assessment will be reflected in the depth of treatment of the questions. It will be achieved also through the language level in the examination questions and the amount of structured support provided for examination candidates at different syllabus levels. Since, at Foundation level, learners have difficulty dealing with abstract ideas, at the point of assessment learners will be required to solve problems set in context relating to their daily lives. Information about the general assessment criteria applying to the examination of Leaving Certificate Mathematics is set out in the assessment section (page 44).

STRANDS OF STUDY

Strand 1: Statistics and Probability

The aim of the probability unit is two-fold: it provides certain understandings intrinsic to problem solving and it underpins the statistics unit. It is expected that the conduct of experiments (including simulations), both individually and in groups, will form the primary vehicle through which the knowledge, understanding and skills in probability are developed. References should be made to appropriate contexts and applications of probability.

It is envisaged that throughout the statistics course learners will be involved in identifying problems that can be explored by the use of appropriate data, designing investigations, collecting data, exploring and using patterns and relationships in data, solving problems, and communicating findings. This strand also involves interpreting statistical information, evaluating data-based arguments, and dealing with uncertainty and variation.

As they engage with this strand and make connections across other strands, learners develop and reinforce their synthesis and problem-solving skills.

At each syllabus level students should be able to

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 1: Statistics and Probability – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
1.1 Counting	Listing outcomes of experiments in a systematic way.	– list all possible outcomes of an experiment – apply the fundamental principle of counting
1.2 Concepts of probability	The probability of an event occurring: students progress from informal to formal descriptions of probability. Predicting and determining probabilities. Difference between experimental and theoretical probability.	– decide whether an everyday event is likely or unlikely to occur – recognise that probability is a measure on a scale of 0-1 of how likely an event is to occur – use the language of probability to discuss events, including those with equally likely outcomes – estimate probabilities from experimental data – recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – associate the probability of an event with its long-run, relative frequency
1.3 Outcomes of simple random processes	Finding the probability of equally likely outcomes.	– construct sample spaces for two independent events – apply the principle that, in the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, containers with different coloured objects, playing cards, sports results, etc.)
1.4 Statistical reasoning with an aim to becoming a statistically aware consumer	Situations where statistics are misused and learn to evaluate the reliability and quality of data and data sources. Different types of data.	– engage in discussions about the purpose of statistics and recognise misconceptions and misuses of statistics – discuss populations and samples – decide to what extent conclusions can be generalised – work with different types of data: • categorical: nominal or ordinal • numerical: discrete or continuous in order to clarify the problem at hand

Strand 1: Statistics and Probability – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
1.1 Counting	– count the arrangements of n distinct objects ($n!$) – count the number of ways of arranging r objects from n distinct objects	– count the number of ways of selecting r objects from n distinct objects – compute binomial coefficients
1.2 Concepts of probability	– use set theory to discuss experiments, outcomes, sample spaces – discuss basic rules of probability (AND/OR, mutually exclusive) through the use of Venn diagrams – calculate expected value and understand that this does not need to be one of the outcomes – recognise the role of expected value in decision making and explore the issue of fair games	– extend their understanding of the basic rules of probability (AND/OR, mutually exclusive) through the use of formulae – Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ – Multiplication Rule (Independent Events): $P(A \cap B) = P(A) \times P(B)$ – Multiplication Rule (General Case): $P(A \cap B) = P(A) \times P(B \mid A)$ – solve problems involving sampling, with or without replacement – appreciate that in general $P(A \mid B) \neq P(B \mid A)$ – examine the implications of $P(A \mid B) \neq P(B \mid A)$ in context
1.3 Outcomes of random processes	– find the probability that two independent events both occur – apply an understanding of Bernoulli trials* – solve problems involving up to 3 Bernoulli trials – calculate the probability that the 1st success occurs on the n^{th} Bernoulli trial where n is specified	– solve problems involving calculating the probability of k successes in n repeated Bernoulli trials (normal approximation not required) – calculate the probability that the k^{th} success occurs on the n^{th} Bernoulli trial – use simulations to explore the variability of sample statistics from a known population, to construct sampling distributions and to draw conclusions about the sampling distribution of the mean – solve problems involving reading probabilities from the normal distribution tables
1.4 Statistical reasoning with an aim to becoming a statistically aware consumer	– discuss populations and samples – decide to what extent conclusions can be generalised – work with different types of bivariate data	

* A Bernoulli trial is an experiment whose outcome is random and can be either of two possibilities: “success” or “failure”.

Strand 1: Statistics and Probability – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
1.5 Finding, collecting and organising data	The use of statistics to gather information from a selection of the population with the intention of making generalisations about the whole population. Formulating a statistics question based on data that vary, allowing for distinction between different types of data.	– clarify the problem at hand – formulate one (or more) questions that can be answered with data – explore different ways of collecting data – generate data, or source data from other sources including the internet – select a sample from a population (Simple Random Sample) – recognise the importance of representativeness so as to avoid biased samples – design a plan and collect data on the basis of above knowledge – summarise data in diagrammatic form, including data presented in spreadsheets
1.6 Representing data graphically and numerically	Methods of representing data. Students develop a sense that data can convey information and that organising data in different ways can help clarify what the data have to tell us. They see a data set as a whole and so are able to use proportions and measures of centre to describe the data.	**Graphical** – select appropriate methods to represent and describe the sample (univariate data only) – evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others – use pie charts, bar charts, line plots, histograms (equal intervals), stem and leaf plots to display data – use appropriate graphical displays to compare data sets **Numerical** – use a variety of summary statistics to describe the data: • central tendency mean, median, mode • variability – range

Strand 1: Statistics and Probability – Ordinary level and Higher level

<table>
<tr><th>Students learn about</th><th>Students working at OL should be able to</th><th>In addition, students working at HL should be able to</th></tr>
<tr><td>1.5 Finding, collecting and organising data</td><td>– select a sample (Simple Random Sample)
– recognise the importance of representativeness so as to avoid biased samples
– discuss different types of studies: sample surveys, observational studies and designed experiments
– design a plan and collect data on the basis of above knowledge</td><td>– recognise the importance of randomisation and the role of the control group in studies
– recognise biases, limitations and ethical issues of each type of study
– select a sample (stratified, cluster, quota – no formulae required, just definitions of these)
– design a plan and collect data on the basis of above knowledge</td></tr>
<tr><td>1.6 Representing data graphically and numerically</td><td>Graphical
– describe the sample (both univariate and bivariate data) by selecting appropriate graphical or numerical methods
– explore the distribution of data, including concepts of symmetry and skewness
– compare data sets using appropriate displays including back-to-back stem and leaf plots
– determine the relationship between variables using scatterplots
– recognise that correlation is a value from -1 to +1 and that it measures the extent of the linear relationship between two variables
– match correlation coefficient values to appropriate scatterplots
– understand that correlation does not imply causality

Numerical
– recognise standard deviation and interquartile range as measures of variability
– use a calculator to calculate standard deviation
– find quartiles and the interquartile range
– use the interquartile range appropriately when analysing data
– recognise the existence of outliers</td><td>Graphical
– analyse plots of the data to explain differences in measures of centre and spread
– draw the line of best fit by eye
– make predictions based on the line of best fit
– calculate the correlation coefficient by calculator

Numerical
– recognise the effect of outliers
– use percentiles to assign relative standing</td></tr>
</table>

Strand 1: Statistics and Probability – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
1.7 Analysing, interpreting and drawing conclusions from data	Drawing conclusions from data; limitations of conclusions.	– interpret graphical summaries of data – relate the interpretation to the original question – recognise how sampling variability influences the use of sample information to make statements about the population – use appropriate tools to describe variability, drawing inferences about the population from the sample – interpret the analysis – relate the interpretation to the original question

Strand 1: Statistics and Probability – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
1.7 Analysing, interpreting and drawing inferences from data	– recognise how sampling variability influences the use of sample information to make statements about the population – use appropriate tools to describe variability drawing inferences about the population from the sample – interpret the analysis and relate the interpretation to the original question – interpret a histogram in terms of distribution of data – make decisions based on the empirical rule – recognise the concept of a hypothesis test – calculate the margin of error ($\frac{1}{\sqrt{n}}$) for a population proportion* – conduct a hypothesis test on a population proportion using the margin of error	– build on the concept of margin of error and understand that increased confidence level implies wider intervals – construct 95% confidence intervals for the population mean from a large sample and for the population proportion, in both cases using z tables – use sampling distributions as the basis for informal inference – perform univariate large sample tests of the population mean (two-tailed z-test only) – use and interpret p-values

* The margin of error referred to here is the maximum value of the radius of the 95% confidence interval.

Strand 2: Geometry and Trigonometry

The synthetic geometry covered at Leaving Certificate is a continuation of that studied at junior cycle. It is based on the *Geometry for Post-primary School Mathematics*, including terms, definitions, axioms, propositions, theorems, converses and corollaries. The formal underpinning for the system of post-primary geometry is that described by Barry (2001) .[1]

At Ordinary and Higher level, knowledge of geometrical results from the corresponding syllabus level at Junior Certificate is assumed. It is also envisaged that, at all levels, learners will engage with a dynamic geometry software package.

In particular, at Foundation level and Ordinary level learners should first encounter the geometrical results below through investigation and discovery. Learners are asked to accept these results as true for the purpose of applying them to various contextualised and abstract problems. They should come to appreciate that certain features of shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features or results can be established in a formal manner through logical proof. Even at the investigative stage, ideas involved in mathematical proof can be developed. Learners should become familiar with the formal proofs of the specified theorems (some of which are examinable at Higher level). Learners will be assessed by means of problems that can be solved using the theory.

As they engage with this strand and make connections across other strands, learners develop and reinforce their synthesis and problem-solving skills.

At each syllabus level students should be able to

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

1 P.D. Barry. *Geometry with Trigonometry*, Horwood, Chicester (2001)

Strand 2: Geometry and Trigonometry – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
2.1 Synthetic geometry	Constructions and how to apply these in real-life situations. Dynamic geometry software. The instruments that are used to perform constructions with precision.	– revisit constructions 4,5,10,13 and 15 in real-life contexts – draw a circle of given radius – use the instruments: straight edge, compass, ruler, protractor and set square appropriately when drawing geometric diagrams
2.2 Co-ordinate geometry	Co-ordinating the plane. Linear relationships in real-life contexts and representing these relationships in tabular and graphical form. Equivalence of the slope of the graph and the rate of change of the relationship. Comparing linear relationships in real-life contexts, paying particular attention to the significance of the start value and the rate of change. The significance of the point of intersection of two linear relationships.	– select and use suitable strategies (graphic, numeric, mental) for finding solutions to real-life problems involving up to two linear relationships
2.3 Trigonometry	Right-angled triangles. Trigonometric ratios.	– apply the result of the theorem of Pythagoras to solve right-angled triangle problems of a simple nature involving heights and distances – use trigonometric ratios to solve real world problems involving angles
2.4 Transformation geometry, enlargements	Translations, central symmetry, axial symmetry and rotations. Enlargements.	– locate axes of symmetry in simple shapes – recognise images of points and objects under translation, central symmetry, axial symmetry and rotation – investigate enlargements and their effect on area, paying attention to • centre of enlargement • scale factor k where $0<k<1$, $k>1$ $k \in \mathbf{Q}$ – solve problems involving enlargements

Strand 2: Geometry and Trigonometry – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
2.1 Synthetic geometry	– perform constructions **16-21** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: theorem, proof, axiom, corollary, converse, implies – investigate theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21 and corollary 6 (see Geometry for Post-primary School Mathematics) and use them to solve problems	– perform construction **22** (see *Geometry for Post-primary School Mathematics*) – use the following terms related to logic and deductive reasoning: is equivalent to, if and only if, proof by contradiction – prove theorems 11,12,13, concerning ratios (see Geometry for Post-primary School Mathematics), which lay the proper foundation for the proof of the theorem of Pythagoras studied at junior cycle
2.2 Co-ordinate geometry	– use slopes to show that two lines are • parallel • perpendicular – recognise the fact that the relationship $ax + by + c = 0$ is linear – solve problems involving slopes of lines – calculate the area of a triangle – recognise that $(x\text{-}h)^2 + (y\text{-}k)^2 = r^2$ represents the relationship between the x and y co-ordinates of points on a circle with centre (h, k) and radius r – solve problems involving a line and a circle with centre (0, 0)	– solve problems involving • the perpendicular distance from a point to a line • the angle between two lines – divide a line segment internally in a given ratio $m: n$ – recognise that $x^2+y^2+2gx+2fy+c = 0$ represents the relationship between the x and y co-ordinates of points on a circle with centre $(-g,-f)$ and radius r where $r = \sqrt{(g^2+f^2-c)}$ – solve problems involving a line and a circle
2.3 Trigonometry	– use of the theorem of Pythagoras to solve problems (2D only) – use trigonometry to calculate the area of a triangle – solve problems using the sine and cosine rules (2D) – define sin θ and cos θ for all values of θ – define tan θ – solve problems involving the area of a sector of a circle and the length of an arc – work with trigonometric ratios in surd form	– use trigonometry to solve problems in 3D – graph the trigonometric functions sine, cosine, tangent – graph trigonometric functions of type • $f(\boldsymbol{\theta})= a+b\text{Sin } c\boldsymbol{\theta}$ • $g(\boldsymbol{\theta}) = a+b\text{Cos } c\boldsymbol{\theta}$ for $a,b,c \in \mathbf{R}$ – solve trigonometric equations such as Sin $n\boldsymbol{\theta}$=0 and Cos $n\boldsymbol{\theta}$= ½ giving all solutions – use the radian measure of angles – derive the trigonometric formulae 1, 2, 3, 4, 5, 6, 7, 9 (see appendix) – apply the trigonometric formulae 1-24 (see appendix)
2.4 Transformation geometry, enlargements	– investigate enlargements and their effect on area, paying attention to • centre of enlargement • scale factor k where $0<k<1$, $k>1$ $k \in \mathbf{Q}$ – solve problems involving enlargements	

Strand 3: Number

Strand 3 further develops the proficiency learners have gained through their study of strand 3 at junior cycle. Learners continue to make meaning of the operations of addition, subtraction, multiplication and division of whole and rational numbers and extend this sense-making to complex numbers.

They extend their work on proof and become more proficient at using algebraic notation and the laws of arithmetic and induction to show that something is always true. They utilise a number of tools: a sophisticated understanding of proportionality, rules of logarithms, rules of indices and 2D representations of 3D solids to solve single and multi-step problems in numerous contexts.

As they engage with this strand and make connections across other strands, learners develop and reinforce their synthesis and problem-solving skills.

At each syllabus level students should be able to

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 3: Number – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
3.1 Number systems **N:** the set of natural numbers, **N** = {1,2,3,4…} **Z:** the set of integers, including 0 **Q:** the set of rational numbers	Number: they develop a unified understanding of number, recognising fractions, decimals (that have a finite or a repeating decimal representation), and percentages as different representations of rational numbers. Addition, subtraction, multiplication, and division and extend their whole number understanding to rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. Explaining and interpreting the rules for addition, subtraction, multiplication and division with negative numbers by applying the properties of arithmetic, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero) Representing problems set in context, using diagrams to solve the problems so they can appreciate how the mathematical concepts are related to real life. Solve problems involving fractional amounts set in context.	– revisit the operations of addition, multiplication, subtraction and division in the following domains: • **N** of natural numbers • **Z** of integers • **Q** of rational numbers and use the number line to represent the order of these numbers – investigate models such as decomposition, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, subtraction, multiplication and division, in **N** where the answer is in **N** including their inverse operations – investigate the properties of arithmetic: commutative, associative and distributive laws and the relationships between them – appreciate the order of operations, including the use of brackets – investigate models, such as the number line, to illustrate the operations of addition, subtraction, multiplication and division in **Z** – generalise and articulate observations of arithmetic operations – investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers

Strand 3: Number – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
3.1 Number systems	– recognise irrational numbers and appreciate that **R** ≠ **Q** – work with irrational numbers – revisit the operations of addition, multiplication, subtraction and division in the following domains: • **N** of natural numbers • **Z** of integers • **Q** of rational numbers • **R** of real numbers and represent these numbers on a number line – investigate the operations of addition, multiplication, subtraction and division with complex numbers **C** in rectangular form a+ib – illustrate complex numbers on an Argand diagram – interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate – develop decimals as special equivalent fractions strengthening the connection between these numbers and fraction and place-value understanding – consolidate their understanding of factors, multiples, prime numbers in **N** – express numbers in terms of their prime factors – appreciate the order of operations, including brackets – express non-zero positive rational numbers in the form $a \times 10^n$, where $n \in \mathbf{Z}$ and $1 \le a < 10$ and perform arithmetic operations on numbers in this form	– geometrically construct √2 and √3 – prove that √2 is not rational – calculate conjugates of sums and products of complex numbers – verify and justify formulae from number patterns – investigate geometric sequences and series – prove by induction • simple identities such as the sum of the first n natural numbers and the sum of a finite geometric series • simple inequalities such as $n! > 2^n$, $2^n \ge n^2$ $(n \ge 4)$ $(1+x)^n \ge 1+nx$ $(x > -1)$ • factorisation results such as 3 is a factor of $4^n - 1$ – apply the rules for sums, products, quotients of limits – find by inspection the limits of sequences such as $\lim_{n\to\infty} \frac{n}{n+1}$; $\lim_{n\to\infty} r^n$, $\lvert r\rvert<1$ – solve problems involving finite and infinite geometric series including applications such as recurring decimals and financial applications, e.g. deriving the formula for a mortgage repayment – derive the formula for the sum to infinity of geometric series by considering the limit of a sequence of partial sums

Strand 3: Number – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
3.1 Number systems (continued)		– consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – analyse solution strategies to problems – calculate percentages – use the equivalence of fractions, decimals and percentages to compare proportions – consolidate their understanding and their learning of factors, multiples and prime numbers in **N** and the relationship between ratio and proportion – check a result by considering whether it is of the right order of magnitude and by working the problem backwards; round off a result – make and justify estimates and approximations of calculations – present numerical answers to the degree of accuracy specified – express non-zero positive rational numbers in the form $a \times 10^n$, where $n \in \mathbf{Z}$ and $1 \leq a < 10$
3.2 Indices	Representing numbers as squares, cubes, square roots, and reciprocals	– solve contextual problems involving numbers represented in the following ways: $\sqrt{a}, a^{\frac{1}{2}}, a^2, a^3, \frac{1}{a}$

Strand 3: Number – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
3.1 Number systems (continued)	– appreciate that processes can generate sequences of numbers or objects – investigate patterns among these sequences – use patterns to continue the sequence – generalise and explain patterns and relationships in algebraic form – recognise whether a sequence is arithmetic, geometric or neither – find the sum to n terms of an arithmetic series	
3.2 Indices	– solve problems using the rules for indices (where $a, b \in \mathbf{R}$; $p, q \in \mathbf{Q}$; $a^p, a^q \in \mathbf{Q}$; $a, b \neq \mathbf{0}$): • $a^p a^q = a^{p+q}$ • $\frac{a^p}{a^q} = a^{p-q}$ • $a^0 = 1$ • $(a^p)^q = a^{pq}$ • $a^{\frac{1}{q}} = \sqrt[q]{a}\quad q \in \mathbf{Z},\ q \neq 0, a > 0$ • $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p\ p, q \in \mathbf{Z},\ q \neq 0, a > 0$ • $a^{-p} = \frac{1}{a^p}$ • $(ab)^p = a^p b^p$ • $(\frac{a}{b})^p = \frac{a^p}{b^p}$	– solve problems using the rules of logarithms • $\log_a(xy) = \log_a x + \log_a y$ • $\log_a(\frac{x}{y}) = \log_a x - \log_a y$ • $\log_a x^q = q\log_a x$ • $\log_a a = 1$ and $\log_a 1 = 0$ • $\log_a x = \frac{\log_b x}{\log_b a}$

Strand 3: Number – Foundation level

Topic	**Description of topic** *Students learn about*	**Learning outcomes** *Students should be able to*
3.3 Arithmetic	Solving everyday problems, including problems involving mobile phone tariffs, currency transactions, shopping, VAT, meter readings, and timetables. Making value for money calculations and judgments. Using ratio and proportion. Measure and time.	– solve problems that involve finding profit or loss, % profit or loss (on the cost price), discount, % discount, selling price, compound interest for not more than 3 years, income tax (standard rate only), net pay (including other deductions of specified amounts) – calculate, interpret and apply units of measure and time – solve problems that involve calculating average speed, distance and time
3.4 Length, area and volume	2D shapes and 3D solids, including nets of solids. Using nets to analyse figures and to distinguish between surface area and volume. Problems involving perimeter, surface area and volume. Modelling real-world situations and solving a variety of problems (including multi-step problems) involving surface areas, and volumes of cylinders and rectangular solids. The circle, and develop an understanding of the relationship between its circumference, diameter and π.	– investigate the nets of rectangular solids and cylinders – select and use suitable strategies to find length of the perimeter and the area of the following plane figures: disc, triangle, rectangle, square, and figures made from combinations of these – select and use suitable strategies to estimate the area of a combination of regular and irregular shapes – select and use suitable strategies to find the volume and surface area of rectangular solids, cylinders and spheres – draw and interpret scaled diagrams

Strand 3: Number – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
3.3 Arithmetic	– check a result by considering whether it is of the right order of magnitude and by working the problem backwards; round off a result – accumulate error (by addition or subtraction only) – make and justify estimates and approximations of calculations; calculate percentage error and tolerance – calculate average rates of change (with respect to time) – solve problems that involve • calculating cost price, selling price, loss, discount, mark up (profit as a % of cost price), margin (profit as a % of selling price) • compound interest, depreciation (reducing balance method), income tax and net pay (including other deductions) • costing: materials, labour and wastage • metric system; change of units; everyday imperial units (conversion factors provided for imperial units) – make estimates of measures in the physical world around them	– use *present value* when solving problems involving loan repayments and investments
3.4 Length, area and volume	– investigate the nets of prisms, cylinders and cones – solve problems involving the length of the perimeter and the area of plane figures: disc, triangle, rectangle, square, parallelogram, trapezium, sectors of discs, and figures made from combinations of these – solve problems involving surface area and volume of the following solid figures: rectangular block, cylinder, right cone, triangular-based prism (right angle, isosceles and equilateral), sphere, hemisphere, and solids made from combinations of these – use the trapezoidal rule to approximate area	

Strand 4: Algebra

This strand builds on the relations-based approach of junior cycle where the five main objectives were

- to make use of letter symbols for numeric quantities
- to emphasise relationship based algebra
- to connect graphical and symbolic representations of algebraic concepts
- to use real life problems as vehicles to motivate the use of algebra and algebraic thinking
- to use appropriate graphing technologies (graphing calculators, computer software) throughout the strand activities.

Learners build on their proficiency in moving among equations, tables and graphs and become more adept at solving real-world problems.

As they engage with this strand and make connections across other strands, learners develop and reinforce their synthesis and problem-solving skills.

At each syllabus level students should be able to

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 4: Algebra – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
4.1 (a) Generating arithmetic expressions from repeating patterns	Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.	– use tables to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence
4.1 (b) Representing situations with tables, diagrams and graphs	Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next.	– use tables, diagrams and graphs as tools for representing and analysing linear patterns and relationships – develop and use their own generalising strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.1 (c) Finding formulae	Ways to express a general relationship arising from a pattern or context.	– find the underlying formula written in words from which the data is derived (linear relationships)
4.1 (d) Examining algebraic relationships	Features of a linear relationship and how these features appear in the different representations. Constant rate of change. Proportional relationships.	– show that relations have features that can be represented in a variety of ways – distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulae expressed in words – use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others – discuss rate of change and the y-intercept; consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula – decide if two linear relationships have a common value – recognise problems involving direct proportion and identify the necessary information to solve them
4.1 (e) Relations without formulae	Using graphs to represent phenomena quantitatively.	– explore graphs of motion – make sense of quantitative graphs and draw conclusions from them – make connections between the shape of a graph and the story of a phenomenon – describe both quantity and change of quantity on a graph
4.1 (f) Expressions	Evaluating expressions derived from real life contexts.	– evaluate expressions given the value of the variables

Strand 4: Algebra – Ordinary level and Higher level

<table>
<tr><th>Students learn about</th><th>Students working at OL should be able to</th><th>In addition, students working at HL should be able to</th></tr>
<tr><td>4.1 Expressions</td><td>– evaluate expressions given the value of the variables
– expand and re-group expressions
– factorise expressions of order 2
– add and subtract expressions of the form
• $(ax+by+c) \pm \ldots \pm (dx+ey+f)$
• $(ax^2+bx+c) \pm \ldots \pm (dx^2+ex+f)$
where $a,b,c,d,e,f \in \mathbf{Z}$
• $\frac{a}{bx+c} \pm \frac{p}{qx+r}$
where $a,b,c,p,q,r \in \mathbf{Z}$
– use the associative and distributive properties to simplify expressions of the form
• $a(bx \pm cy \pm d) \pm \ldots \pm e(fx \pm gy \pm h)$
where $a, b, c, d, e, f, g, h \in \mathbf{Z}$
• $(x \pm y)(w \pm z)$
– rearrange formulae</td><td>– perform the arithmetic operations of addition, subtraction, multiplication and division on polynomials and rational algebraic expressions paying attention to the use of brackets and surds
– apply the binomial theorem</td></tr>
</table>

Strand 4: Algebra – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
4.2 Solving equations	Solving linear equations set in context.	– select and use suitable strategies (graphic, numeric, mental) for finding solutions to equations of the form: $f(x) = g(x)$, with $f(x) = ax+b$, $g(x) = cx+d$, where $a, b, c, d \in \mathbf{Q}$ and interpret the results
4.3 Inequalities	Solving linear inequalities set in context.	– select and use suitable strategies (graphic, numeric, mental) for finding solutions to inequalities of the form: • $g(x) \leq k$, $g(x) \geq k$, • $g(x) < k$, $g(x) > k$, where $g(x) = ax + b$ and $a, b, k \in \mathbf{Q}$ and interpret the results

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
4.2 Solving equations	– select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form: • $f(x) = g(x)$, with $f(x) = ax+b$, $g(x) = cx+d$ where $a, b, c, d \in \mathbf{Q}$ • $f(x) = g(x)$ with $f(x) = \frac{a}{bx+c} \pm \frac{p}{qx+r}$; $g(x) = \frac{e}{f}$ where $a, b, c, e, f, p, q, r \in \mathbf{Z}$ • $f(x) = k$ with $f(x) = ax^2 + bx + c$ (and not necessarily factorisable) where $a, b, c \in \mathbf{Q}$ and interpret the results – select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to • simultaneous linear equations with two unknowns and interpret the results • one linear equation and one equation of order 2 with two unknowns (restricted to the case where either the coefficient of x or the coefficient of y is ± 1 in the linear equation) and interpret the results – form quadratic equations given whole number roots	– select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations of the form: $f(x) = g(x)$ with $f(x) = \frac{ax+b}{ex+f} \pm \frac{cx+d}{qx+r}$; $g(x) = k$ where $a, b, c, d, e, f, q, r \in \mathbf{Z}$ – use the Factor Theorem for polynomials – select and use suitable strategies (graphic, numeric, algebraic and mental) for finding solutions to • cubic equations with at least one integer root • simultaneous linear equations with three unknowns • one linear equation and one equation of order 2 with two unknowns and interpret the results
4.3 Inequalities	– select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to inequalities of the form: • $g(x) \leq k$, $g(x) \geq k$, • $g(x) < k$, $g(x) > k$, where $g(x) = ax + b$ and $a, b, k \in \mathbf{Q}$	– use notation $\mid x \mid$ – select and use suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to inequalities of the form: • $g(x) \leq k$, $g(x) \geq k$; • $g(x) < k$, $g(x) > k$, with $g(x) = ax^2+bx+c$ or $g(x) = \frac{ax+b}{cx+d}$ and $a, b, c, d, k \in \mathbf{Q}$, $x \in \mathbf{R}$ • $\mid x - a \mid < b$, $\mid x - a \mid > b$ and combinations of these, with $a, b, \in \mathbf{Q}$, $x \in \mathbf{R}$
4.4 Complex Numbers	See strand 3, section 3.1	– use the Conjugate Root Theorem to find the roots of polynomials – work with complex numbers in rectangular and polar form to solve quadratic and other equations including those in the form $z^n = a$, where $n \in \mathbf{Z}$ and $z = r(\text{Cos}\,\theta + i\text{Sin}\,\theta)$ – use De Moivre's Theorem – prove De Moivre's Theorem by induction for $n \in \mathbf{N}$ – use applications such as n^{th} roots of unity, $n \in \mathbf{N}$, and identities such as $\text{Cos}\,3\theta = 4\,\text{Cos}^3\,\theta - 3\,\text{Cos}\,\theta$

Strand 5: Functions

This strand builds on the learners' experience in junior cycle where they were formally introduced to the concept of a function as that which involves a set of inputs, a set of possible outputs and a rule that assigns one output to each input. The relationship between functions and algebra is further emphasised and learners continue to connect graphical and symbolic representations of functions. They are introduced to calculus as the study of how things change and use derivatives to solve various kinds of real-world problems. They learn how to go from the derivative of a function back to the function itself and use such methods to solve various geometric problems, such as computation of areas of specified regions.

As they engage with this strand and make connections across other strands, learners develop and reinforce their synthesis and problem-solving skills.

At each syllabus level students should be able to

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

Strand 5: Functions – Foundation level

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
5.1 Functions	Functions as a special type of relationship. Representing linear functions set in context graphically.	– recognise that a function assigns a unique output to a given input – graph functions of the form $ax+b$ where $a,b \in \mathbf{Q}$, $x \in \mathbf{R}$

Strand 5: Functions – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
5.1 Functions	– recognise that a function assigns a unique output to a given input – form composite functions – graph functions of the form • $ax+b$ where $a,b \in \mathbf{Q}$, $x \in \mathbf{R}$ • ax^2+bx+c where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$ • ax^3+bx^2+cx+d where $a,b,c,d \in \mathbf{Z}$, $x \in \mathbf{R}$ • ab^x where $a \in \mathbf{N}$, $b, x \in \mathbf{R}$ – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions – use graphical methods to find approximate solutions to • $f(x) = 0$ • $f(x) = k$ • $f(x) = g(x)$ where $f(x)$ and $g(x)$ are of the above form, or where graphs of $f(x)$ and $g(x)$ are provided – investigate the concept of the limit of a function	– recognise surjective, injective and bijective functions – find the inverse of a bijective function – given a graph of a function sketch the graph of its inverse – express quadratic functions in complete square form – use the complete square form of a quadratic function to • find the roots and turning points • sketch the function – graph functions of the form • ax^2+bx+c where $a,b,c \in \mathbf{Q}$, $x \in \mathbf{R}$ • ab^x where $a, b \in \mathbf{R}$ • logarithmic • exponential • trigonometric – interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions – informally explore limits and continuity of functions
5.2 Calculus	– find first and second derivatives of linear, quadratic and cubic functions by rule – associate derivatives with slopes and tangent lines – apply differentiation to • rates of change • maxima and minima • curve sketching	– differentiate linear and quadratic functions from first principles – differentiate the following functions • polynomial • exponential • trigonometric • rational powers • inverse functions • logarithms – find the derivatives of sums, differences, products, quotients and compositions of functions of the above form – apply the differentiation of above functions to solve problems – use differentiation to find the slope of a tangent to a circle – recognise integration as the reverse process of differentiation – use integration to find the average value of a function over an interval – integrate sums, differences and constant multiples of functions of the form • x^a where $a \in \mathbf{Q}$ • a^x where $a \in \mathbf{R}$, $a>0$ • Sin ax where $a \in \mathbf{R}$ • Cos ax where $a \in \mathbf{R}$ – determine areas of plane regions bounded by polynomial and exponential curves

Assessment

Assessment in Leaving Certificate Mathematics

Assessment for certification will be based on the aim, objectives and learning outcomes of the syllabus. Differentiation at the point of assessment will be achieved through examinations at three levels – Foundation level, Ordinary level, and Higher level. Ordinary level is a subset of Higher level; thus, learners at Higher level are expected to achieve the Ordinary level and Higher level learning outcomes. Differentiation will be achieved also through the language level in the examination questions, the stimulus material presented, and the amount of structured support given in the questions. It is accepted that, at Foundation level, learners engage with the mathematics at a concrete level.

Assessment components

At Ordinary level and Higher level there are two assessment components

- Mathematics Paper 1
- Mathematics Paper 2

Each paper will contain two sections – A and B.

- Section A will address core mathematics topics, with a focus on concepts and skills.
- Section B will include questions that are context-based applications of mathematics.

At Foundation level there is one assessment component, a written paper. Learners will be assessed by means of problems set in meaningful contexts.

General assessment criteria

A high level of achievement in Mathematics is characterised by a demonstration of a thorough knowledge and comprehensive understanding of mathematics as described by the learning outcomes associated with each strand. The learner is able to make deductions with insight even in unfamiliar contexts and can move confidently between different forms of representation. When investigating challenging problems, the learner recognises pattern structures, describes them as relationships or general rules, draws conclusions and provides justification or proof. The learner presents a concise, reasoned justification for the method and process and, where appropriate, considers the range of approaches which could have been used, including the use of technology.

A moderate level of achievement in Mathematics is characterised by a demonstration of a broad knowledge and good understanding of mathematics as described by the learning outcomes associated with each strand. The learner is able to make deductions with some insight even in unfamiliar contexts and can move between different forms of representation in most situations. When investigating problems of moderate complexity, the learner recognises pattern structures, describes them as relationships or general rules and draws conclusions consistent with findings. The learner successfully selects and applies skills and problem solving techniques. The learner presents a reasoned justification for the method and process and provides an evaluation of the significance and reliability of findings.

A low level of achievement in Mathematics is characterised by a demonstration of limited knowledge or understanding of mathematics as described by the learning outcomes associated with each strand. The learner recognises simple patterns or structures when investigating problems and applies basic problem solving techniques with some success. An attempt is made to justify the method used and to evaluate the reliability of findings.

Appendix: Trigonometric Formulae

It will be assumed that these formulae are established in the order listed here. In deriving any formula, use may be made of formulae that precede it.

1. $\cos^2 A + \sin^2 A = 1$
2. sine formula: $\dfrac{a}{\text{Sin } A} = \dfrac{b}{\text{Sin } B} = \dfrac{c}{\text{Sin } C}$
3. cosine formula: $a^2 = b^2 + c^2 - 2\,bc \cos A$
4. $\cos (A-B) = \cos A \cos B + \sin A \sin B$
5. $\cos (A+B) = \cos A \cos B - \sin A \sin B$
6. $\cos 2A = \cos^2 A - \sin^2 A$
7. $\sin (A+B) = \sin A \cos B + \cos A \sin B$
8. $\sin (A-B) = \sin A \cos B - \cos A \sin B$
9. $\tan (A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
10. $\tan (A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
11. $\sin 2A = 2 \sin A \cos A$
12. $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$
13. $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
14. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$
15. $\cos^2 A = \frac{1}{2} (1 + \cos 2A)$
16. $\sin^2 A = \frac{1}{2} (1 - \cos 2A)$
17. $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$
18. $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$
19. $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$
20. $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$
21. $\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
22. $\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$
23. $\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$
24. $\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

SECTION B

Geometry for Post-primary School Mathematics

This section sets out the course in geometry for both Junior Certificate Mathematics and Leaving Certificate Mathematics. Strand 2 of the relevant syllabus document specifies the learning outcomes at the different syllabus levels.

Geometry for Post-primary School Mathematics

1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It was prepared by a working group of the NCCA course committees for mathematics and, following minor amendments, was adopted by both committees for inclusion in the syllabus documents. Readers should refer to Strand 2 of the syllabus documents for Junior Certificate and Leaving Certificate mathematics for the range and depth of material to be studied at the different levels. A summary of these is given in sections 9–13 of this document.

The preparation and presentation of this document was undertaken principally by Anthony O'Farrell, with assistance from Ian Short. Helpful criticism from Stefan Bechluft-Sachs, Ann O'Shea, Richard Watson and Stephen Buckley is also acknowledged.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane, point, line, $<_l$ (precedes on a line), (open) half-plane, distance, and degree-measure**, and seven axioms: A_1: about incidence, A_2: about order on lines, A_3: about how lines separate the plane, A_4: about distance, A_5: about degree measure, A_6: about congruence of triangles, A_7: about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as "the point is/lies on the line", "the line passes through the point", etc.

- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).

- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.

- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.

- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word "reflex" precedes or follows.

- We make no reference to results such as Pasch's property and the "crossbar theorem". (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)

- We refer to "the number of degrees" in an angle, whereas Barry treats this more correctly as "the degree-measure" of an angle.

- We take it that the definitions of parallelism, perpendicularity and "sidedness" are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).

- We do not refer explicitly to triangles being **congruent** "under the correspondence $(A, B, C) \to (D, E, F)$", taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say "ΔABC is congruent to ΔDEF" we mean, using Barry's terminology, "Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence $(A, B, C) \to (D, E, F)$".

- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle[1]. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: "opposite sides of a parallelogram are equal", or refer to "a circle of radius r". Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A, then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry's geometry that are retained in our less formal version:

[1] In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.
- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry[2].
- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.
- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4 Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

[2] Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).

3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.

4. Some guidance on teaching (Section 8).

5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle, sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line[3] is short for straight line. Take a fixed **plane**[4], once and for all, and consider just lines that lie in it. The plane and the lines are **sets**[5] of **points**[6]. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.

Axioms are statements we will accept as true[7].

Axiom 1 (Two Points Axiom)**.** *There is exactly one line through any two given points. (We denote the line through A and B by AB.)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all

[3]Line is undefined.
[4]Undefined term
[5]Undefined term
[6]Undefined term
[7] An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A, B and C be points that are not collinear. The **triangle** ΔABC is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**[8] by $\mathbb{R}$.

Definition 4. We denote the **distance**[9] between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a, b, and c. The usual thing for a triangle ΔABC is to take $a = |BC|$, i.e. the length of the side opposite the vertex A, and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom[10])**.** *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. *$|AB| = |BA|$;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

[8] Undefined term

[9] Undefined term

[10] Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with [11]

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A, called its **vertex**;
2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;
3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

[11] Students may notice that the first equality implies the second.

Definition 12. Suppose that A, B, and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A, B, C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α,β,γ, etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle\alpha|$, as the case may be.

Axiom 3 (Protractor Axiom)**.** *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$*. The number of degrees of an ordinary angle is less than* $180°$*. It has these properties:*

1. *A straight angle has* $180°$*.*
2. *Given a ray* $[AB$*, and a number* d *between* 0 *and* 180*, there is exactly one ray from* A *on each side of the line* AB *that makes an (ordinary) angle having* d *degrees with the ray* $[AB$*.*
3. *If* D *is a point inside an angle* $\angle BAC$*, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ measures $360° - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is 'an angle of' (for instance) $45°$, if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly $90°$.

Definition 15. An angle is **acute** if it has less than 90°, and **obtuse** if it has more than 90°.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180°.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

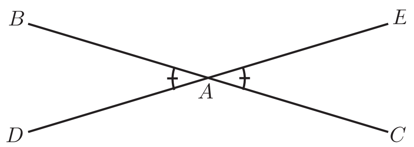


Figure 1.

Theorem 1 (Vertically-opposite Angles)**.**
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180°. In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180°, \\ |\angle CAE| + |\angle BAE| &= 180°, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0°, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

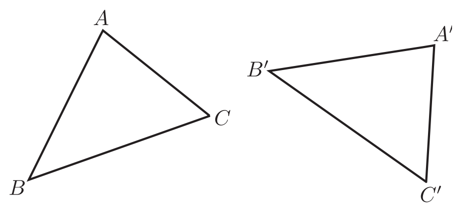


Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12])**.**
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, *and* $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$
then the triangles ΔABC *and* $\Delta A'B'C'$ *are congruent.*

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to 90°, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS]
$\therefore \angle B = \angle C$.

[12]It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.
[13] The simple "equal" is preferred to "of equal length"

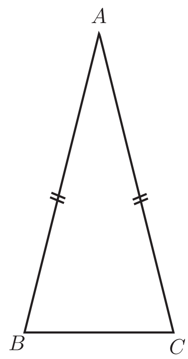


Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|, \Delta ABC$ is isosceles. □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SAS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels)**.** *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

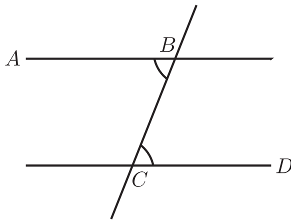


Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.

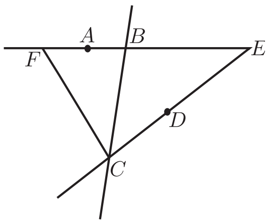


Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

[14]Fuller detail: There are three cases:
1°: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
2°: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then ΔBCE is congruent to ΔCBF. [SAS]
Thus
$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that F lies on DC. [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

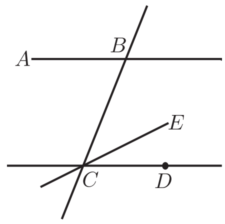


Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. □

Theorem 4 (Angle Sum 180)**.** *The angles in any triangle add to* 180°.

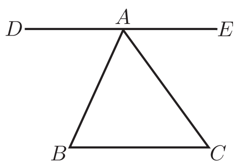


Figure 7.

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so that F lies on DC. [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ = & |\angle DAB| + |\angle CAE| + |\angle BAC| \\ = & |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].

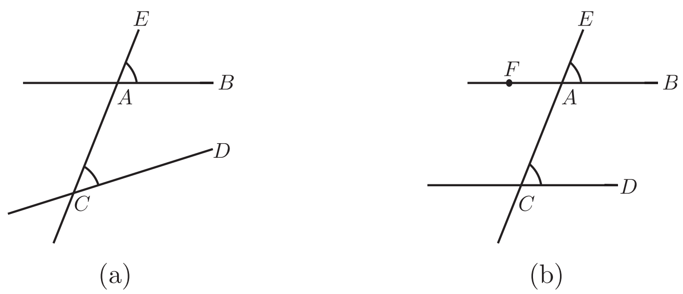

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]

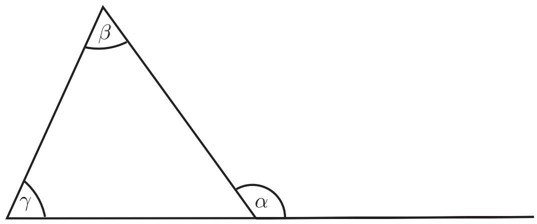

Figure 9.

Theorem 6 (Exterior Angle)**.** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then

$|\alpha| + |\angle A| = 180°$ [Supplementary angles]

and

$|\angle B| + |\angle C| + |\angle A| = 180°$. [Angle sum 180°]

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. □

[16]The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

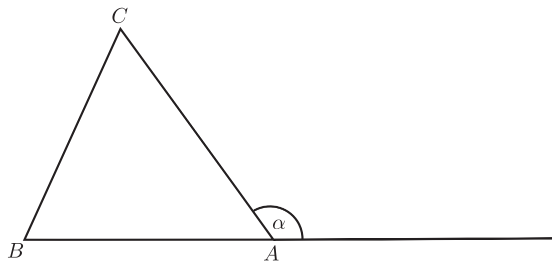

Figure 10.

Theorem 7.
(1) In ΔABC, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.
(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
$|AD| = |AB|$. [Ruler Axiom]

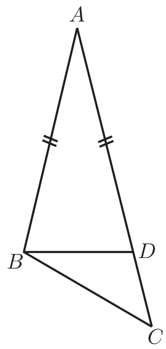

Figure 11.

See Figure 11. Then ΔABD is isosceles, so

$$\begin{aligned} |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\ &= |\angle ABD| && \text{[Isosceles Triangle]} \\ &< |\angle ABC|. \end{aligned}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)
Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

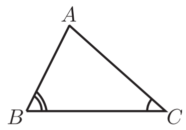

Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. □

Theorem 8 (Triangle Inequality)**.**
Two sides of a triangle are together greater than the third.

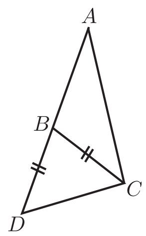

Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$[17] we have

$$|\angle BCD| < |\angle ACD|.$$

[17] B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180°$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to ΔADC we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. [18] *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing though a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB.

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

[18] In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

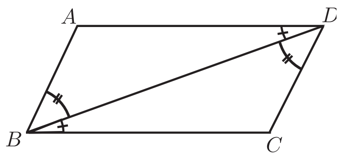

Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB||CD$ and $AD||BC$. Then

$|\angle ABD| = |\angle BDC|$ [Alternate Angle Theorem]
$|\angle ADB| = |\angle DBC|$ [Alternate Angle Theorem]
ΔDAB is congruent to ΔBCD. [ASA]

$$\therefore |AB| = |CD|, \ |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360°. It follows that adjacent angles add to 180°. Theorem 3 then yields the result. □

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. □

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

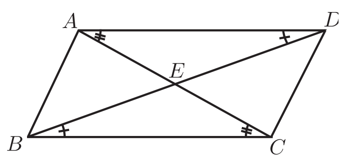

Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of ΔADE and ΔCBE.

In detail: Let AC cut BD in E. Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \Delta ADE$ is congruent to ΔCBE. [ASA] □

Proposition 6 (Converse)**.** *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of ΔABE and ΔCDE. Then use Alternate Angles. □

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180°, so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

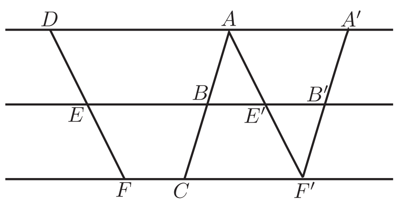

Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD||BE||CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE'||DE$, cutting EB at E' and CF at F'.
Draw $F'B'||AB$, cutting EB at B'. See Figure 16.

Then

$\vert B'F'\vert$	$=$	$\vert BC\vert$	[Theorem 9]
	$=$	$\vert AB\vert$.	[by Assumption]
$\vert\angle BAE'\vert$	$=$	$\vert\angle E'F'B'\vert$.	[Alternate Angle Theorem]
$\vert\angle AE'B\vert$	$=$	$\vert\angle F'E'B'\vert$.	[Vertically Opposite Angles]
$\therefore \Delta ABE'$	is congruent to	$\Delta F'B'E'$.	[ASA]
$\therefore \vert AE'\vert$	$=$	$\vert F'E'\vert$.	

But
$|AE'| = |DE|$ and $|F'E'| = |FE|$. [Theorem 9]
$\therefore |DE| = |EF|$. □

Definition 36. Let s and t be positive real numbers. We say that a point C **divides the segment** $[AB]$ **in the ratio** $s : t$ if C lies on the line AB, and is between A and B, and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l **cuts** $[AB]$ **in the ratio** $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B, and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. *Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.*

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n. Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \ldots, D_{m-1}, D_m = D, D_{m+1}, \ldots, D_{m+n-1}, D_{m+n} = B,$$

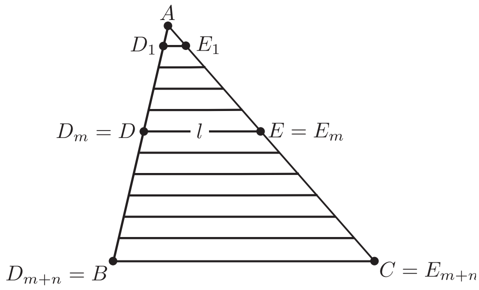

Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \dots [D_iD_{i+1}], \dots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines $D_1E_1, D_2E_2, \dots$ parallel to BC with $E_1, E_2, \dots$ on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \dots, [E_{m+n-1}C]$$

have the same length, [Theorem 11]
and $E_m = E$ is the point where l cuts $[AC]$. [Axiom of Parallels]
Hence E divides $[AC]$ in the ratio $m : n$. □

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB$ and C'' on $[AC$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D. Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so ΔABC is similar to $\Delta A'B'C'$. [Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:
Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC.

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. *If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

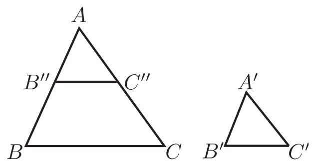

Figure 18.

Proof. We may suppose $|A'B'| \le |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$\Delta AB''C''$	is congruent to	$\Delta A'B'C'$	[SAS]
$\therefore |\angle AB''C''|$	$=$	$|\angle ABC|$	
$\therefore B''C''$	$\parallel$	BC	[Corresponding Angles]
$\therefore \frac{|A'B'|}{|A'C'|}$	$=$	$\frac{|AB''|}{|AC''|}$	[Choice of B'', C'']
	$=$	$\frac{|AB|}{|AC|}$	[Theorem 12]
$\frac{|AC|}{|A'C'|}$	$=$	$\frac{|AB|}{|A'B'|}$	[Re-arrange]

Similarly, $\frac{|BC|}{|B'C'|} = \frac{|AB|}{|A'B'|}$ □

Proposition 8 (Converse)**.** *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC. □

6.9 Pythagoras

Theorem 14 (Pythagoras)**.** *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

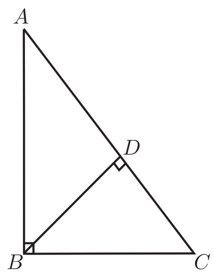

Figure 19.

Proof. Let ΔABC have a right angle at B. Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A. $\therefore$ ΔABC is similar to ΔADB.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| \left(|AD| + |DC|\right) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras)**.** *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

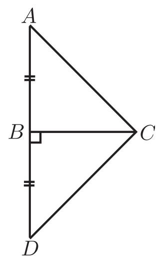

Figure 20.

Proof. (Idea: Construct a second triangle on the other side of [BC], and use Pythagoras and SSS to show it congruent to the original.)
In detail: We wish to show that $|\angle ABC| = 90°$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$
\begin{aligned}
|DC| &= \sqrt{|DC|^2} \\
&= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\
&= \sqrt{|AB|^2 + |BC|^2} && [\ |AB| = |BD|\ ] \\
&= \sqrt{|AC|^2} && \text{[Hypothesis]} \\
&= |AC|.
\end{aligned}
$$

$\therefore \Delta ABC$ is congruent to ΔDBC. [SSS]
$\therefore |\angle ABC| = |\angle DBC| = 90°$. □

Proposition 9 (RHS)**.** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B', and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment* $[AB]$ *is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C, hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

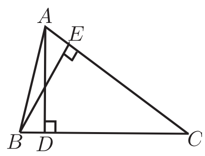

Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by "area of ΔABC"[19].

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

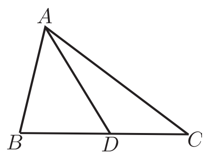

Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h, so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

[19] $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles[20].

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas[21].

Proposition 14. *The area of a rectangle having sides of length a and b is ab.*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC.

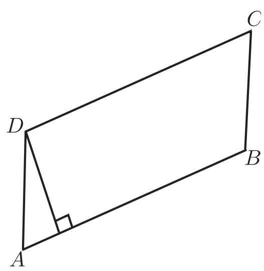

Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD, and as the length of the perpendicular segment from D onto AB.*

[20] If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

[21] Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, ΔABD and ΔCDB. These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. □

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called **a** radius. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the** diameter of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B, and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π.

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

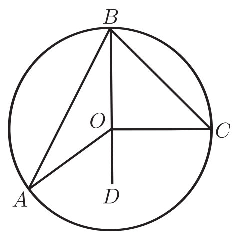

Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D. Then

$$\begin{array}{rcll} |OA| &=& |OB|. & \text{[Definition of circle]} \\ \therefore |\angle BAO| &=& |\angle ABO|. & \text{[Isosceles triangle]} \\ \therefore |\angle AOD| &=& |\angle BAO| + |\angle ABO| & \text{[Exterior Angle]} \\ &=& 2 \cdot |\angle ABO|. & \end{array}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{array}{rcl} |\angle AOC| &=& |\angle AOD| + |\angle COD| \\ &=& 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\ &=& 2 \cdot |\angle ABC|. \end{array}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC, then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC:

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC, and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C. If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s. We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. □

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90°$.*

Proof. The angle at the centre is a straight angle, measuring 180°, and half of that is 90°. □

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180°, so is straight, and so the line BC passes through the centre. □

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to* 180°.

Proof. The two angles at the centre standing on the same arcs add to 360°, so the two halves add to 180°. □

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to* 180°, *then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. □

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2, where r is its radius. For this reason, we say that the area of the disc is πr^2.

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. □

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.
(1) Each tangent is perpendicular to the radius that goes to the point of contact.
(2) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is tangent to s.

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP.

Let the perpendicular to the tangent from the centre O meet it at Q. Pick R on PQ, on the other side of Q from P, with $|QR| = |PQ|$ (as in Figure 25).

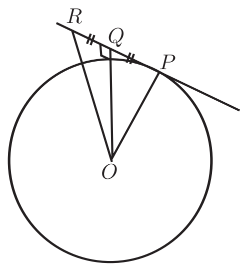

Figure 25.

Then ΔOQR is congruent to ΔOQP. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP, as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P, and hence is not on the circle.)

In detail: Let Q be any point on l, other than P. See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 \quad \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

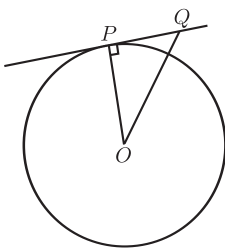

Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
$\therefore P$ is the only point of l on the circle.
$\therefore l$ is a tangent. [Definition of tangent]
□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

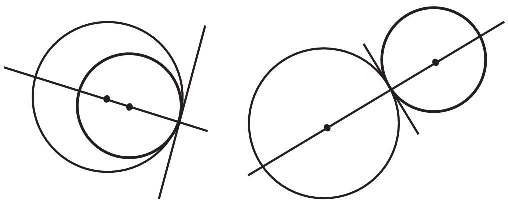

Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.
(1) The perpendicular from the centre to a chord bisects the chord.
(2) The perpendicular bisector of a chord passes through the centre.

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.) See Figure 28.

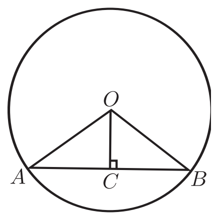

Figure 28.

In detail:

$$\begin{array}{rcll} |OA| &=& |OB| & \text{[Definition of circle]} \\ |OC| &=& |OC| & \\ & & & \\ |AC| &=& \sqrt{|OA|^2 - |OC|^2} & \text{[Pythagoras]} \\ &=& \sqrt{|OB|^2 - |OC|^2} & \\ &=& |CB|. & \text{[Pythagoras]} \end{array}$$

$\therefore \Delta OAC$ is congruent to ΔOBC. [SSS]
$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.
Let C be the foot of the perpendicular from O on AB.
By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.
Thus CO is the perpendicular bisector of AB.
Hence the perpendicular bisector of AB passes through O. □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A, B, and C, then its centre lies on the perpendicular bisector of each side of the triangle ΔABC.*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

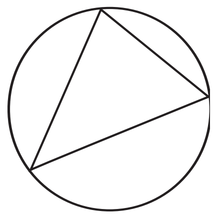

Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

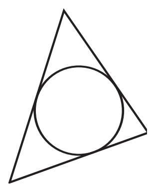

Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

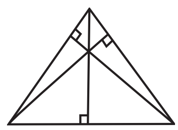

Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A, such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of $30°$, $60°$, and $45°$. It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.

2. Perpendicular bisector of a segment, using only compass and straight edge.

3. Line perpendicular to a given line l, passing through a given point not on l.

4. Line perpendicular to a given line l, passing through a given point on l.
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of 60°, without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

8 Teaching Approaches

8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).

2. Ideas from Technical Drawing.

3. Material in [3].

8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that $n^2 + n + 41$ is prime for all $n \in \mathbb{N}$). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or “cuts” involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras’ theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word “prove” means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a “proof” is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based “proofs” of fallacies and then probe a dissection-based proof of Pythagoras’ theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

9 Syllabus for JCOL

9.1 Concepts

Set, plane, point, line, ray, angle, real number, length, degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area, tangent to a circle, subset, segment, collinear points, distance, midpoint of a segment, reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle, angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio, isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate angles, corresponding angles, polygon, quadrilateral, convex quadrilateral, rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line, circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, point of contact of a tangent, vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral, centre of a circle.

9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

9.3 Axioms and Proofs

The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 13 (statement only), 14, 15; and direct proofs of Corollaries 3, 4.

10 Syllabus for JCHL

10.1 Concepts

Those for JCOL, and concurrent lines.

10.2 Constructions

Students will study all the constructions prescribed for JC-OL, and also constructions 3 and 7.

10.3 Logic, Axioms and Theorems

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 1, 2, 3, 4*, 5, 6*, 9*, 10, 11, 12, 13, 14*, 15, 19*, Corollaries 1, 2, 3, 4, 5, and their converses. Those marked with a * may be asked in examination.

The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration.

11 Syllabus for LCFL

Students are expected to build on their mathematical experiences to date.

11.1 Constructions

Students revisit constructions 4, 5, 10, 13, 15, and learn how to apply these in real-life contexts.

12 Syllabus for LCOL

12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 16–21.

12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed.

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.

13 Syllabus for LCHL

13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry.

They will be asked to solve geometrical problems (so-called "cuts") and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

References

[1] Patrick D. Barry. *Geometry with Trigonometry.* Horwood. Chichester. 2001. ISBN 1-898563-69-1.

[2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers.* Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.

[3] Fiacre O'Cairbre, John McKeon, and Richard O. Watson. A Resource for Transition Year Mathematics Teachers. DES. Dublin. 2006.

[4] Anthony G. O'Farrell. *School Geometry*. IMTA Newsletter 109 (2009) 21-28.

A Guide to the study of ***Functions*** at Post –Primary school

Children begin to develop the function concept in early childhood when they observe and continue patterns of objects in everyday life. They continue the preparation for functions in primary school by exploring patterns in numbers and looking for regularities. It is, however at ***JCOL*** when they are formally introduced to the notion of a function. They learn about *the meaning and notation associated with functions* and

- engage with the concept of a function, domain , co-domain and range
- make use of function notation f(x)= ; f:x $\rightarrow$, and y=

At this level students make connections between their study of relationships in *strand* ***4 algebra sections 4.1 to 4.5***, sets in ***strand 3 number section 3.5*** and data types in ***strand 1 section 1.4.*** They learn that ***function*** is a mathematical term that refers to a ***certain kind*** of relationship between two sets. One set called the ***domain*** and the other the ***range***. A function is a correspondence between these sets; from each element in the ***domain*** to exactly one element in the ***range.***

At this level students understand the ***domain*** as the set of what they have hitherto referred to as ***inputs*** or ***what goes into the function***, and the set of ***outputs*** as the ***range*** or what ***actually*** comes out of the function. They identify another set, the ***co-domain*** as the set of what ***possibly*** comes out of the function and view ***the range*** as a subset of this set. Having initially engaged in ***sections 4.1-4.5*** with a variety of relationships derived from familiar, everyday experiences the more formal definitions are gradually introduced. When examining a ***money box*** situation, for example, in which a box has €5 to start with and €2 added every day students at this level students would identify from the situation the ***domain*** as the set of whole numbers, the ***co-domain*** as the set of whole numbers or maybe even the set of whole numbers greater than or equal to 5 and the ***range*** as the set of odd whole numbers 5,7,9…... In addition they would be able to identify the ***domain*** and ***range*** in each of the other representations; tabular, ordered pair, and graphical.

The exploration of the ***co-domain*** and decision making around outcomes that are possible and those that are not presents an ideal opportunity to make connections with data types from ***section 1.4*** and to reinforce the concept of ***discrete*** and ***continuous*** data. By considering the possible outcomes of a function and how they should be represented graphically the students are presented with ***discrete*** and ***continuous*** data in a context other than statistics and can begin to categorise situations that produce each type of data.

At this level students use function notation as *shorthand*, for describing the correspondence in terms of input and output. Initially the notation is used in conjunction with the situation and students should recognise that the correspondence is built into the notation. In the moneybox situation described above ***f*(2)** should be interpreted as

the amount of money in the money box on day 2 and ***f*(x)** as the amount of money in the money box on any given day. They should recognise the **x** as a place holder and realise that ***f(b)*** would describe exactly the same situation. The students' understanding of the notation should be explored in relation to the context so that, for example, they understand the difference between ***f(x+2***), ***f(x)+2*** and ***2 f(x) .*** They should be able to explain that ***f(x+2***) is the amount of money in the money box 2 days after any given day; ***f(x)+2*** is the amount of money in the money box on any given day plus €2, and ***2 f(x)*** is twice the amount of money on any given day.
Later, students consolidate their learning from ***sections 4.3 ,5.1 and 5.2*** as they work to develop fluency in moving between the different representations (notation, graphical and the context), and use this fluency to solve purely mathematical problems or those set in a context. ***JCOL*** students explore the notation with linear functions and quadratic functions with whole number coefficients whilst at ***JCHL*** students extend their exploration of function notation to include quadratic functions with integer coefficients and simple exponential functions. At this level students should understand that a function can also be described in terms of its behaviour, for example, *over what input values is it increasing, decreasing or constant? For what input values are the output values positive, negative or zero?* This focus on function behaviour offers an ideal opportunity to reinforce the concept of domain and appreciate the usefulness of the graphical representation, since these behaviours are easily ***seen*** in a graph.
A discussion of function behaviour offers an ideal segue to ***LC*** ; by this level students should be developing ways of thinking that are general and which allow them to approach any type of function, work with it, and understand how it behaves, rather than regarding each function as a completely different ***thing*** to study. With a basis of experiences in building specific functions from scratch, beginning with an exploration of relations in ***sections 4.1- 4.5*** and progressing to generalisation, first in words – ***amount of money in the box = 2 (day number) + 5*** – and later using algebraic notation – ***y* = 2*x*+5** – a well-developed concept of equality allows students to make sense of the notation ***f*(*x*) = 2*x*+5** , interpreting it as ***the output is 2x +5*** when ***the input is x***. In addition, they develop their understanding of the equivalence of ***y*** and ***f(x),*** not only in this algebraic representation but also in the tabular and graphical representations. Now students should start to develop a notion of naturally occurring families of functions that deserve particular attention. For example, they should see linear and exponential functions as arising out of growth principles. Similarly, they should see quadratic, polynomial, and rational functions as belonging to the one system. Developing this notion takes time and students can start getting a feel for the effects of different parameters by playing around with the effect of the input and output variables on the graph of simple algebraic transformations. Quadratic (***LCOL***) and absolute value

functions (***LCHL***) are good contexts for getting a sense of the effects of many of these transformations.

Proficient mathematicians will make use of structure to help solve problems and at all levels students should be encouraged to look for and make use of structure. Consequently, students should develop the practice of writing expressions for functions in ways that reveal the key features of the function. At ***LCHL***, exploring quadratic functions provides an ideal opportunity for developing this ability, since the three principal representations for a quadratic expression – expanded, factored, and completed square – each give insight into different aspects of the function.

At ***LCHL***, students extend the idea of the co-domain introduced at ***JCOL*** when they begin to categorise functions as ***surjective***, ***injective*** or ***bijective***. Exploring the effects of limiting the domain and co-domain on the function 'status' reinforces the difference between them and also helps students to make sense of the categorisation.

At ***LC*** by examining contexts where change occurs at discrete intervals (such as payments of interest on a bank balance) or where the input variable is a whole number ***section 3.1*** they come to recognise that a ***sequence*** is a function whose domain is a subset of the set of integers. For example, when considering the sequence 5, 8, 11,14 by choosing an ***index*** that indicates which term they are talking about and which serves as the input value to the function, a student could make a table showing the correspondence and describe the sequence using function notation
$f(x) = 3x + 2, x \geq 1$ with the domain included in the description. Students are faced once again with the concept of ***discrete*** and ***continuous*** data when they attempt to represent a sequence graphically.

LCHL students begin engaging with the concept of ***the inverse*** of a function by first getting to grips with the idea of ***going backwards*** from output to input. They can get this sense of determining the input when the output is known by using a table or a graph of the function under examination. To reinforce this idea, correspondences between equations giving specific values of the functions, table entries, and points on the graph can be noted. Eventually students need to generalise the process for finding the input given a particular output and are required to generalise the process for ***bijective*** functions only. A well-developed concept of notation and equality is required if students are to make sense of the generalisation.

To help students develop the concept that "inverse" is a relationship between two functions rather than a new type of function the classroom focus should be on "inverses of functions". Questions such as, "*What is the inverse of this function*?" and "*Does this function have an inverse?*" are useful in keeping the focus on the relationship idea.

Connections can be made with the notion of ***function composition*** by examining the relationship between the composition of f^{-1} with ***f.*** This relationship, the ***identity function***, which assigns each function to itself allows students to deepen their understanding of inverses in general since it behaves with respect to composition of functions the way the multiplicative identity, 1, behaves with multiplication of real numbers. Now students can verify by composition (in both directions) that given functions are inverses of each other. They can also refine their informal "going backwards" idea, as they consider inverses of functions given by graphs or tables. They get a sense that "going backwards" interchanges the input and output and therefore the stereotypical roles of the letters ***x*** and ***y*** and can reason why the graph of $y=f^{-1}(x)$ will be the reflection across the line $y=x$ of the graph $y=f(x)$.

Section 3.2 provides ***LCHL*** students with further opportunity to reinforce the concept that "inverse" is a relationship between two functions, here students are required to not only understand logarithms as functions but also as inverses of exponential functions. Students can think of the logarithms as unknown exponents in expressions with base 10 and use the properties of exponents when explaining logarithmic identities and the laws of logarithms.

LC section 5.2 introduces students to the concept of a ***limit,*** a powerful tool for them as they start bringing together their ability to use graphs to reason about rates of change ***(JC sections4.1-4.5)*** and start thinking about the slope of a tangent line to a curve. Students can develop their understanding of differentiation and ***why*** the rules work by examining differentiation of linear and quadratic functions from fist principles although students at ***LCHL*** only will be examined in this process.

At ***LCHL*** students build on their ability to approximate area in ***section 3.4*** by investigating the area under a function. Starting by finding the area between a given linear function and the x-axis and progressing to *finding* the upper boundary function themselves they come to in ***section 5.2*** to recognise integration as the reverse process of differentiation***.*** By examining the problem of finding the average value of a function over a given interval they progress to a deeper understanding of the process of integration. The ability to determine areas of plane regions bounded by polynomial and exponential curves eases the transition from computing discrete probabilities to continuous ones ***section 1.3***. By understanding the ***Normal distribution*** as a ***probability density function*** students can understand why it is used to find probabilities for continuous random variables.

Question ***Exponential equations***

..can you clarify whether exponential equations (equations with variables in the index) are on the syllabus for Junior Cert and Leaving Cert, and at what levels please?..

.. Again I wonder if you could clarify if exponential graphs are on the syllabus for the present 2 nd years ie doing their junior cert in 2014 ?...

Answer
Yes, solving exponential equations arises for both JC and LC students. In order to give a more complete answer, I have drawn on the following sections of the syllabus:

- JC: *3.2, Indices; 4.7, Equations and inequalities* ; *5.2, Graphing Functions*
- LC: *3.2, Indices; 4.2, Solving equations; and 5.1, Functions*

At **JCOL** students will meet *exponential relationships* when they

> – use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling)

At this level, students should be engaged in activities that require them to informally solve exponential equations arising from a doubling or tripling context in order to answer questions such as…After how many days will there be 64 bacteria in the tray?...How many sections will there be if I fold the paper in half 4 times? ….When will I have €213 in my account?...

At **JCHL**, students build on this experience when they

> – draw graphs of the following functions and interpret equations of the form $f(x) = g(x)$ as a comparison of functions
> - $f(x) = ax + b$, where $a, b \in \mathbf{Z}$
> - $f(x) = ax^2 + bx + c$, where $a \in \mathbf{N}$; $b, c \in \mathbf{Z}$; $x \in \mathbf{R}$
> - **$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$**
> - **$f(x) = a2^x$ and $f(x) = a3^x$, where $a \in \mathbf{N}$, $x \in \mathbf{R}$**
>
> – use graphical methods to find approximate solutions where $f(x) = g(x)$ and interpret the results

At this level, students should use graphical methods to get approximate solutions to exponential equations, whether expressed informally or written formally using the standard notation. The exponential graphs could be given to the students or they could be asked to draw them themselves.

At **LCOL**, students are expected to work with more advanced exponential equations:

– graph functions of the form
- $ax+b$ where $a, b \in \mathbf{Q}$, $x \in \mathbf{R}$
- ax^2+bx+c where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$
- ax^3+bx^2+cx+d where $a,b,c,d \in \mathbf{Z}$, $x \in \mathbf{R}$
- ab^x where $a \in \mathbf{N}$, $b, x \in \mathbf{R}$

– interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions

– use graphical methods to find approximate solutions to
- $f(x) = 0$
- $f(x) = k$
- $f(x) = g(x)$

where $f(x)$ and $g(x)$ are of the above form, or where graphs of $f(x)$ and $g(x)$ are provided

And at **LCHL**, more sophisticated again:

– graph functions of the form
- ax^2+bx+c where $a,b,c \in \mathbf{Q}$, $x \in \mathbf{R}$
- ab^x where $a, b \in \mathbf{R}$
- logarithmic
- exponential
- trigonometric

– interpret equations of the form $f(x) = g(x)$ as a comparison of the above functions

Both **JCHL** and **LCOL** students should be able to

– use and apply rules for indices (where $a, b \in R$, $a, b \neq 0$; $p, q \in Q$; $a^p, a^q \in R$; complex numbers not included):
- $a^p a^q = a^{p+q}$
- $\frac{a^p}{a^q} = a^{p-q}$
- $a^0 = 1$
- $(a^p)^q = a^{pq}$
- $a^{\frac{1}{q}} = \sqrt[q]{a}$, $q \in Z$, $q \neq 0$, $a > 0$
- $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$, $p, q \in Z$, $q \neq 0$, $a > 0$
- $a^{-p} = \frac{1}{a^p}$
- $(ab)^p = a^p b^p$
- $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

– solve problems using the rules for indices (where $a, b \in \mathbf{R}$; $p, q \in \mathbf{Q}$; $a^p, a^q \in \mathbf{Q}$; $a, b \neq 0$):
- $a^p a^q = a^{p+q}$
- $\frac{a^p}{a^q} = a^{p-q}$
- $a^0 = 1$
- $(a^p)^q = a^{pq}$
- $a^{\frac{1}{q}} = \sqrt[q]{a}$ $q \in \mathbf{Z}$, $q \neq 0$, $a > 0$
- $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$ $p, q \in \mathbf{Z}$, $q \neq 0$, $a > 0$
- $a^{-p} = \frac{1}{a^p}$
- $(ab)^p = a^p b^p$
- $(\frac{a}{b})^p = \frac{a^p}{b^p}$

– solve problems using the rules of logarithms
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
- $\log_a x^q = q\log_a x$
- $\log_a a = 1$ and $\log_a 1 = 0$
- $\log_a x = \frac{\log_b x}{\log_b a}$

Proficient mathematicians will make use of structure to help solve problems and, at all levels, students should be encouraged to look for and make use of structure. Consequently, students should develop the practice of writing expressions for functions and equations in ways that reveal their key features. Students, therefore, should explore how the rules of indices can be used to rewrite simple exponential equations such as $8^x = 64$ in ways that allow them to see an algebraic solution to the equation. At **LCHL** students take this further; they are required to not only understand logarithms as functions but also as inverses of exponential functions. At this level, students can think of the logarithms as unknown exponents in expressions with base 10 and use the properties of exponents when explaining logarithmic identities and the laws of logarithms. They should be encouraged to explore algebraic solutions to appropriate exponential equations.

Question *Transformations of Functions*

I attended a workshop where we looked at scaling and shifting graphs. I have been unable to find this on the syllabus for Leaving Certificate Higher Level 2014. How many types of graphs do the students need to be able to scale? It's not clear in the syllabus.

Could you clarify are Transformatons of Linear, Quadratic, cubie and exponential functions on the syllabus for examination in 2014.If so can you tell me where this is on the syllabus or where it ins inferred on the syllabus.- page number etc

Answer

For students to develop a good understanding of functions, it is important that they are able to move fluidly between different representations of functions (equations, tables and graphs) and to use graphical representations as a way of solving equations. The activities in Workshop 7 dealing with scaling and shifting of graphs are designed to help develop students' conceptual understanding of functions and calculus, and it is really important that students engage in these types of learning experiences.

In section 5.1 of the syllabus you will see that students at all levels are required to be able to interpret equations of the form f(x) = g(x) as a comparison of functions (the functions at each level are outlined; remember also that learning outcomes at LCOL are a subset of LCHL).

At higher level you will see the following two learning outcomes:

– graph functions of the form

- $ax^2+bx + c$ where $a,b,c \in Q$, $x \in R$
- ab^x where $a,b \in R$
- logarithmic
- exponential
- trigonometric

– interpret equations of the form f(x) = g(x) as a comparison of the above functions.

This means that students are expected to be able to interpret equations and make comparisons between functions. Therefore, looking at the effect of, say, adding a constant or changing the coefficients on any of those functions is essential so that students will be able to make these comparisons.

For example, a student should be able to compare the functions

$f(x) = 3x^2 + 2x$ and $g(x) = 3x^2 + 2x + 5$

and describe the impact which adding the constant to produce g(x) has on the graph of f(x).

Question *Implicit differentiation*

Differentiation of the circle is stated on the syllabus but there is no mention of implicit differentiation which is needed to do it. Also it is not mentioned in the text book. Also it doesn't mention whether it is all circles or just circles with centre (0,0). Would be grateful for any clarification.
Implicit differentiation seems to be unavoidable given the sentence in Strand 5 about tangents to circles??

Answer

This specific learning outcome: students should be able to

- use differentiation to find the slope of a tangent to a circle

was included at **HL** to build on the **LCOL** learning outcome: students should be able to

- associate derivatives with slopes and tangent lines.

Students at **HL** should, therefore, first encounter this specific learning outcome with a circle of centre (0,0) and explore how differentiation can be used to find the slope of the tangent line. They can re-arrange the expression to present *y* as a function of *x* and then differentiate this, or the chain rule may be used to differentiate the y^2 component and thus obtain a dy/dx element in the result. Hence, they should be able to calculate the slope (of the tangent) at any desired point on the circle.
'***What if***…?' questions could then lead students to wonder how this might change if the circle did not have its centre at (0,0). …How would the differentiation problem change? What challenge might this change present? How could it be overcome?
There is no requirement to develop the concept further or to formally deal with implicit differentiation other than in this specific context.

A guide to Post-Primary statistical inference

Strand 1 Section 1.7 lists learning outcomes related to ***statistical inference*** which deals with the principles involved in generalising observations from a ***sample*** to ***the whole population***. Such ***generalisations*** are valid only if the data are ***representative*** of that larger group.

> *A representative sample is one in which the relevant characteristics of the sample members are generally the same as those of the population.*
>
> *An improper or **biased** sample tends to systematically favour certain outcomes and can produce misleading results and erroneous conclusions.*

Randon sampling is a way to remove ***bias*** in sample selection, and tends to produce representative samples. At ***JC HL*** and ***all levels*** at ***LC***, students are required to

- recognise how sampling variability influences the use of sample information to make statements about the population

Whilst ***LC HL*** students are required to go beyond this and

- *use simulations to explore the variability of sample statistics from a known population, to construct sampling distributions and to draw conclusions about the sampling distribution of the mean*

At ***JC HL***, and ***LC FL*** and ***LC OL***, students should experience the consequences of non-random selection and develop a basic understanding of the principles involved in random selection procedures. At ***LC HL***, learners extend this understanding; they explore simulations that produce frequency distributions of sample means and conclude from these explorations that when we take a large number of random samples of the same size and get a frequency distribution of the sample means, this distribution – called ***the sampling distribution of the mean*** – tends to become a normal distribution and

- If the sample size is large (n≥ 30) then for any population, no matter what its distribution, the sampling distribution of the mean will be approximately normal
- This normal distribution will have a mean equal to the population mean with standard deviation $\frac{\sigma}{\sqrt{n}}$. This is calld the ***standard error of the mean***.

Suppose a group of students was investigating the sporting preferences of students in their school. At ***JC FL*** and ***JC OL***, students might survey the whole class; students at this level are ***not*** required to *look beyond the data* and no generalisation is required. At ***JC HL*** and at ***all levels*** at ***LC***, students begin to acknowledge that it is possible to *look beyond the data*. They would gather data from a ***sample*** and ***generalise*** to a larger group. In order to be able to ***generalise*** to all students at the school a ***representative sample*** of students from the school is needed. This can be done by selecting a ***simple random sample*** of students from the school.

At each of the levels ***JC HL***, ***LC FL, LC OL*** and ***LC HL***, students are required to deal with ***sampling variability*** in increasingly sophisticated ways.

Consider the data below gathered from a ***simple random sample*** of 50 students.

		Do You Like Soccer?		Row Total
		Yes	**No**	
Do You like Rugby?	**Yes**	25	4	29
	No	6	15	21
Column Total		31	19	**50**

Suppose, before the study began, a teacher ***hypothesised***: *I think that more than 50% of students in this school like Rugby*. Because 58% ($\frac{29}{50}$ = 58%) of the sample like rugby there is ***evidence*** to support the teachers claim. However, because we have only a sample of 50 students, it is ***possible*** that 50% of ***all*** the students like rugby but the variation due to random sampling might produce 58% or even more who like rugby.The statistical question, then, is whether the sample result of 58% is reasonable from the variation we expect to occur when selecting a random sample from a population with 50% successes? Or, in simple terms, ***What is a possible value for the true population proportion based on the sample evidence?***

At ***JCHL*** and ***LCFL*** it is sufficient for students to acknowledge sampling variability; a typical response at this level would be ….*although 58% of this sample reported that they like rugby, it is possible that a larger or smaller proportion would like rugby if a different sample was chosen. 58% is close to 50% and it is possible that 50% of all the students like rugby….* At this level, the acknowledgement of variability is more evident in the planning stage with students deciding to choose a large sample or perhaps several small samples and average the findings in order to reduce the sampling error. [If this cohort were deaing with numerical data and were looking for a set of posible values for the ***population mean*** the possible set of values could be determined by looking at the distribution of the data with respect to the ***sample mean*** and the ***range***.]

Building on this understanding, a more sophisticated approach to inference involves finding a set of possible values by using the **margin of error.**

The true population proportion = The sample proportion $\pm$ *Margin of Error*

The margin of error is estimated as $\frac{1}{\sqrt{n}}$ where ***n*** is the sample size and refers to the maximum value of the radius of the 95% confidence interval.

This is the level of inference required by ***OL*** students at Leaving Certificate. A ***LC OL*** student might therefore conclude

…there is evidence to support the teachers claim that more than 50% of students in the school like rugby because, based on the sample data, any values in the range 44% - 72% are possible values for the proportion of students in the school who like rugby…

[If this cohort were deaing with numerical data and were looking for a set of posible values for the ***population mean*** the possible set could be determined by engaging with the ***empirical rule.*** The empirical rule formalises the undertanding students get from examining the spread of the distribution with respect to the mean. Knowing the proportion of values that lie within approx 1,2 or 3 standard deviations from the mean allows students to determine what is a ***possible set of values for the population mean***.]

LC HL students are required to build further on these ideas and make more accurate estimates of the ***possible values*** of the ***true population proportion in the case of categorical data*** *or* ***the population mean in the case of numerical data.*** To do this they

- *construct 95% confidence intervals for the population mean from a large sample and for the population proportion, in both cases using z tables*

Constructing ***confidence intervals*** brings two ideas together:

- sampling variability and the idea of the ***standard error of the population proportion/mean***
- the ***empirical rule*** – 95% of the data lies within 1.96 standard deviations of the mean.

The set of possible values, or the ***confidence interval***, is

Sample mean/proportion ± 1.96 standard error

In the case being examined, the set of possible values for the ***true population proportion*** would be given by

$$\text{Sample proportion} \pm 1.96 \text{ standard error} = .58 \pm 1.96\sqrt{\frac{.58(1-.58)}{50}}$$

$$= .7168 \text{ or } .4432$$

So, the ***true population proportion*** lies beween 44.32% and 71.68%.

Compare this with the set of values obtained using the margin of error. ***LCHL*** students can examine the effect of increasing the sample size on the ***precision*** of the estimate.

LC OL students should understand a hypothesis as a ***theory*** or ***statement*** whose truth has yet to be proven. However, ***LC HL*** students must develop this idea and deal formally with ***hypothesis testing***. They

- perform univariate large sample tets of the population mean (two-tailed z-test only)
- use and interpret p-values

The ***p-value*** represents the chance of observing the result obtained in the sample, or a value more extreme, when the hypothesised value is in fact correct. A small p-value would support the teacher's claim because this would have indicated that, if the population proportion was 0.50 (50%), it would be very unlikely that an observation of 0.58 (58%) would be observed.

A large sample hypothesis test of the population ***mean*** has 4 components:

1. **A test statistic:** This is a standard normal ***z score*** that is the difference between the value we have ***observed for the sample*** and the ***hypothesised value for the poulation*** divided by the ***standard error of the mean.***

2. **A decision rule:** Reject the hypothesised value if ***z > 1.96*** or ***z < -1.96***

3. **A rejection zone:** ***z > 1.96 or z <-1.96***

4. **Critical values**: ***z = 1.96 , z = -1.96*** since we are using the 5% level of significance.

Conceptual progression to Algebra

Early Childhood →		Primary →	Bridging Period →			Junior Cycle →		
	Represent and solve problems involving addition and subtraction		Represent and solve problems involving multiplication and division Section3.1		Understand the place value system Section 3.1	Apply and extend previous understandings of multiplication and division to divide fractions by fractions Section 3.1	Apply and extend previous understandings of operations with fractions to add , subtract , multiply and divide rational numbers Section 3.1	Work with radical and integer exponents Sections 2.3, 3.2
Know number names and the count sequence			Understand properties of operations and the relationship between multiplication and division Section3.1 Multiply and divide within 100 Section3.1	Generalise place value understanding for multidigit whole numbers	Perform operations with multi-digit whole numbers and decimals to hundredths Section3.1	Apply and extend previous understandings to the system of rational numbers Section 3.1		Understand the connections between proportional relationships, lines and linear equations Sections 2.2,3.1,4.4
Count to tell the number of objects	Understand and apply properties of operations and the relationship between addition and subtraction Add and subtract within20			Use place value understanding and properties of operations to perform multi-digit arithmetic Section3.1	Use equivalent fractions as a strategy to add and subtract fractions Section3.1	Understand ratio concepts and use ratio reasoning to solve problems Sections 3.1,4.4	Analyse proportional relationships and use them to solve real-world and mathematical problems Sections 2.2, 3.1, 4.4, 4.7 Use properties of operations to generate equivalent expressions Section 3.1	
Compare numbers			Solve problems involving the four operations , and identify and explain patterns in arithmetic Section3.1	Extend understanding of fraction equivalence and ordering Section3.1	Apply and extend previous understandings of multiplication and division to multiply and divide fractions Section3.1	Apply and extend previous understandings of arithmetic to algebraic expressions Section3.1, 4.3, 4.6		Analyse and solve linear equations and pairs of simultaneous linear equations Sections 2.2, 3.1,4.4, 4.7,
Understand addition as *putting together* and *adding to* and understand subtraction as *taking apart* and *taking from*	Work with addition and subtraction equations. Extend the counting sequence	Use place value understanding and properties of operations to add and subtract measure and estimate lengths in standard units	Develop an understanding of fractions as numbers Section 3.1 Solve problems involving measurement and estimation of intervals of time, liquid volumes and masses of objects Section 3.4 understand concepts of area and relate area to multiplication and addition Section 3.4	Build fractions from unit fractions by applying and extending previous understandings of operations Section 3.1 Understand decimal notation for fractions, and compare decimal fractions Section 3.1	Geometric measurement: understand concepts of volume and relate volume to multiplication and addition Section 3.4	Reason about and solve one variable equations and inequalities Section 4.4, 4.7		Define evaluate and compare functions Sections 5.1,5.2
Work with numbers 11-19 to gain foundations for place value	Understand place value Use place value understanding and properties of operations to add and subtract measure lengths indirectly and by iterating length units	Relate addition and subtraction to length			Graph points on the co-ordinate plane to solve real world and mathematical problems Sections 2.2,	Represent and analyse quantitative relationships between dependent and independent variables Section 4.1	Solve real world and mathematical problems using numerical and algebraic expressions and equations Sections 4.2,4.3, 4.6, 4.7	Use functions to model relationships between quantities Section 4.4

Conceptual progression to Algebra

Question ***Acceptable proof of Pythagoras***

Please find a PDF attached, which shows two different methods for proving Pythagoras' Theorem. Are both methods acceptable at Junior Cert higher level? And if not, which one is not acceptable and why is it not acceptable? Thanking you in advance.

Answer

When the selection of theorems for post-primary maths was being made, the ordering of the theorems was deliberate. The syllabus says that, when it comes to formal proof of any theorem, axioms or theorems which come later in the logical sequence cannot be used to establish its proof. The particular issue of an 'area' proof for the Theorem of Pythagoras (theorem 14) is a case in point. The definition of area of a triangle (Defn 38, in Section 6.10) follows Theorem 16 which established that base times height for a triangle does not depend on the choice of base. This follows both Theorem 14 and its converse and so arises later in the sequence as set out in the syllabus.
The syllabus has been set out so that the order of proofs encourages the development of logical thinking. Therefore proofs should be based on the preceding theorems, so a proof for Pythagoras is one which draws on similar triangles (theorems 11, 12, 13 lay the foundations for the proof of Pythagoras's theorem), the first in the attached pdf which you sent.
Kind Regards,

Question ***Acceptable Proofs required for examination***

In relation to the proofs of theorems at JC and LC higher level, I am wondering if the proofs given in the Geometry for Post Primary Mathematics section of the syllabus are the only acceptable versions?

Answer

The geometry course is based on what's called "level 2", the semi-formal level as opposed to level 1 which is the fully rigorous level. The syllabus has been set out so that the order of proofs encourages the development of logical thinking. Therefore any proofs should be based on the preceding theorems
As outlined on page 42, no proof should be allowed at level 2 that uses axioms or theorems that come later in the logical sequence.

Question ***Proofs for examination***

I am confused as to which theorems on the JC Higher Maths Course (Exam 2013) students may be required to prove.
The syllabus mentions nos. 4,6,9,14 and 19.
However there is a reference also to 11,12,13 and 19 (in bold print) as being examinable.

Answer

Students at **JCHL** are expected to study the proofs of theorems 1,2,3,4,5,6,9,10,11,12,13,14,15,19 but they may be examined on the formal proofs of theorems 4,6,9,14,19. (You'll see this in the Geometry Course section on page 81.)

In other words at higher level they should be familiar with the proofs of the other theorems 11,12, 13 and be able to apply the knowledge of these theorems where necessary. It is envisaged that students would engage with the proof of the theorems in class, they should be able to follow the proofs and understand them and use the results to solve problems but the only proofs examinable are at **HL** and are the ones mentioned above.

NCCA

National Council for Curriculum and Assessment
An Chomhairle Náisiúnta Curaclaim agus Measúnachta

Review of Mathematics in Post-Primary Education

a discussion paper

October 2005

REVIEW OF MATHEMATICS IN POST-PRIMARY EDUCATION

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Acknowledgement

The NCCA wishes to gratefully acknowledge the work of Elizabeth Oldham in the initial drafting of this discussion paper, and for her comments and suggestions in subsequent refinements.
The contribution of Diane Birnie to the initial draft of this paper is also gratefully acknowledged.

CONTENTS

1 Introduction

2005 marks the bi-centenary of the birth of William Rowan Hamilton, one of Ireland's leading mathematicians and scientists. One of his most important discoveries was general methods in dynamics, which virtually predicted modern wave mechanics (which has many and varied applications). Computer users might appreciate his contribution to developments in graphics technology through his discovery of quaternions (while walking along the banks of the Royal Canal!). It is appropriate, therefore, that mathematics in post-primary education should come in for close scrutiny in the course of this year. Coincidentally, the United Nations declared 2005 as the International Year of Physics, noting that it is also the centenary of important scientific discoveries by Albert Einstein. In the UK, the year is being celebrated as 'Einstein Year'.

Mathematics matters. And it matters for different reasons. On the one hand, in its manifestations in terms of counting, measurement, pattern and geometry it permeates the natural and constructed world about us, providing basic language and techniques for handling many aspects of everyday and scientific life. On the other hand, it deals with abstractions, logical arguments, and fundamental ideas of truth and beauty—an intellectual discipline and a source of aesthetic satisfaction. Its role in education reflects this dual nature: it is both practical and theoretical—geared to applications and of intrinsic interest—with the two elements firmly interlinked.

Mathematics has traditionally formed a substantial part of the education of young people in Ireland throughout their schooldays. Its value as a component of general education, for employment, and for further and higher education is recognised by the community at large. The development of mathematical skills impinges on the individual's opportunities for development, with consequent economic implications in a society increasingly reliant on and influenced by advances in science and technology, which have a high dependency on mathematical principles. Accordingly, it is of particular importance that the mathematical education offered to and experienced by students should be appropriate to their abilities, needs and interests, and should fully and appositely reflect the broad nature of the subject and its potential for enhancing the students' development.

A recent UK report (Hoyles et al., 2002) concluded that mathematical literacy[1] can contribute to business success in an increasingly competitive and technologically based world-wide economy and that there is an inter-dependency of mathematical literacy and the use of information technology in the workplace. Of significance in this study is the fact that mathematical skills cannot be considered in isolation, but rather in the context of the work. The use of information (and communications) technology has 'changed the nature of the mathematical skills required, while not reducing the need for mathematics' itself (ibid, p.10).

In undertaking this review of mathematics education, the NCCA seeks to address a range of issues surrounding mathematics at post-primary level in Ireland. This paper presents an overview of these issues, outlines current trends in mathematics education, and provides data on uptake in mathematics at post-primary level. The paper also considers the performance of candidates in the state examinations and in international tests of achievement. Finally, it identifies particular areas of concern that will need to be addressed in any plan for revision of mathematics syllabuses that may arise as a result of this review. These are intended to act as a stimulus for discussion on the nature and role of mathematics education in Irish schools, while keeping in mind developments that are currently under consideration at both junior and senior cycle.

1 Mathematical literacy is defined in the OECD Programme for International Student Assessment (PISA) as "an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen" (OECD, 1999).

2 Context of the review

The review is not simply an exercise in syllabus revision—although this may be an outcome of the review—but rather a more fundamental evaluation of the appropriateness of the mathematics that students engage with in school and its relevance to their needs. It must take into consideration broader reviews that are currently taking place (the implementation of the primary school curriculum; junior cycle) and the proposals being developed for senior cycle education.

2.1 Concerns regarding mathematics

Internationally, there is concern about the low level of mathematical skills of students emerging from second-level education and, in particular, of those proceeding to third-level education (Tickly and Wolf, 2000). These uneven and inadequate mathematical skills affect not only the individual's development and career prospects, but also have more general implications for society.

Issues in relation to mathematics education in Ireland have been highlighted in a number of studies in recent years (Smyth et al., 2004; Lyons et al. 2003; Elwood and Carlisle, 2003; Smyth and Hannan, 2002). These include the provision and take-up of Higher level mathematics and the gender differences that exist in this take-up; the performance of students in state mathematics examinations and in international tests; and the teaching and learning practices that prevail in mathematics classrooms in Ireland.

Over recent years, growing concern has been expressed regarding mathematics in the senior cycle of post-primary education, especially in relation to the numbers of candidates achieving low grades in the Leaving Certificate Ordinary level mathematics examination papers. However, there has also been concern at the low level of mathematical knowledge and skills shown by some students proceeding to further and higher education, and their inability to cope with basic concepts and skill requirements in the mathematical aspects of their courses. O'Donoghue (2002), in particular, noted observations by university lecturers regarding the lack of fluency in fundamental arithmetic and algebraic skills, gaps in basic knowledge in important areas such as trigonometry and complex numbers, and an inability to use or apply mathematics except in the simplest or most practised way.

2.2 Recent developments in mathematics curriculum and assessment in Ireland

Mathematics in the primary school

A revised primary school curriculum was introduced in 1999 and is being implemented on a phased basis; mathematics was among the first group of subjects to be implemented. The 2003-2004 academic year was designated a year of consolidation and review of the Primary School Curriculum. It was also the first year of the NCCA's Primary Curriculum Review, which focused on teachers' and children's experiences with the English curriculum, the Visual Arts Curriculum and the Mathematics Curriculum.

In case studies conducted as part of that review, teachers reported a perceived improvement in motivation for mathematics learning among children, particularly where everyday, real-world materials and contexts were used. Children found mathematical games, puzzles and interesting problems a good motivational influence in their mathematics learning. However, data are not yet available to indicate whether increased engagement has resulted in improved performance.

Teachers reported doing practical (hands-on) work as their greatest success with the mathematics curriculum. There was also an awareness among teachers themselves of the need to integrate mathematics with other areas of the curriculum. Among the challenges in implementing the mathematics curriculum, almost half of the teachers identified catering for the range of children's mathematical abilities as the greatest challenge. In their ongoing implementation of the mathematics curriculum, teachers prioritised focusing more on specific curriculum content, increasing their use of practical work and giving more attention to the use of mathematical language.

Mathematics in the junior cycle

When the Junior Certificate was introduced in 1989, the syllabuses in mathematics were not revised, having been introduced (as syllabuses A, B and C) in 1987, but were renamed Higher, Ordinary and Foundation level syllabuses and first examined under their new titles in 1992. A revised syllabus covering all three levels was introduced in 2000 and first examined in 2003. This was accompanied by in-career development for teachers of mathematics through a dedicated support service. A particular focus of this support was the type of teaching methodology that might best facilitate the aims and objectives of the revised syllabus, thereby leading to improved mathematical understanding on the part of students (rather than learning mathematics by rote, which had tended to predominate previously).

A review of the curriculum at junior cycle is currently under way and part of the present focus involves the re-balancing of syllabuses and their presentation in a common format. Consideration is also being given to the role that *assessment for learning* can play in improving teaching and learning across a range of subjects. Mathematics is one of the subjects to be included in the second phase of both of these review elements.

Mathematics in the senior cycle

The current Leaving Certificate mathematics syllabuses at Ordinary and Higher level were introduced in 1992 and first examined in 1994. The Ordinary Alternative syllabus, introduced in 1990 for first examination in 1992, was re-designated as Foundation level in 1995. The proportion of the student cohort taking each of the three syllabus levels in mathematics (approximately 11% at Foundation level, 72% at Ordinary level, and 17% at Higher level) does not match the expected pattern of uptake when these syllabuses were being developed (20-25%, 50-60% and 20-25% respectively).

Mathematics is one of only two subjects at Leaving Certificate (Irish is the other) which is offered at three syllabus levels. Significantly, when the English syllabus was being revised, the course committee discussed the desirability of providing a third (Foundation level) English course. However, following consideration of the issues involved, it was decided to continue with just two syllabus levels. The aims and objectives of the English syllabus are the same for all learners, and the skills being fostered are the same. Unlike mathematics, differentiation in Leaving Certificate English is not achieved by reference to content, nor is there a specific intention to target the Higher level English syllabus at 'specialists'.

Proposals are being developed for a major re-structuring of the senior cycle of post-primary education. These developments are aimed at improving the rate and quality of participation, at sustaining excellence, at creating greater flexibility and choice for learners, and at meeting educational, social and economic needs. The proposals involve the restructuring of the curriculum to include subjects, short courses and transition units, balancing content and skills, a greater variety and frequency of assessment methods, and a focus on independent learning.

Mathematics is seen as a significant subject for all students in the senior cycle, both as a subject in its own right and as a support for the teaching and learning of other subjects and courses where mathematical competence is a pre-requisite. It is one of only two subjects (English was the other) which the vast majority (84%) of all respondents to the NCCA online questionnaire survey considered should be compulsory for all students. Significantly, 88% of employers who responded to the survey were of this view, as were 79% of students.

2.3 Current trends in mathematics education

Internationally, current trends in mathematics education include emphasis on *problem-solving, modelling* and so-called *'realistic mathematics education'*. It is worthwhile exploring each of these a little and considering how mathematics education here in Ireland compares with the developing international scene.

Problem-solving has always been an important mathematical activity, but it has been given special emphasis in some national curricula (or guidelines, or their equivalents) in the last twenty years. In the USA, the 1980s were designated as the decade of problem-solving. At the end of that decade, the highly influential *Standards* document produced by the National Council of Teachers of Mathematics (NCTM) in the USA placed problem-solving as the first of its list of 'standards' (NCTM, 1989). The newer version, *Principles and Standards for School Mathematics* (NCTM, 2000), may be a little more realistic in its assessment of what can be achieved and what works in the classroom, but fundamentally advocates a similar approach.

By solving mathematical problems, students acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that serve them well outside the mathematics classroom. In England & Wales and Northern Ireland, emphasis was placed in the 1990s on *'investigation'* more than on problem-solving. The two concepts are related, but investigation is perhaps more geared towards exploratory work rather than the solution of clearly-defined problems.

In Ireland, Leaving Certificate mathematics courses were revised in the early 1990s. Higher level mathematics was aimed at the more able students, including those who might not proceed to further study of mathematics or related subjects, and it placed particular emphasis on syllabus aims concerned with problem-solving, abstracting, generalising and proving. Ordinary level mathematics (and to a lesser extent the Ordinary Alternative course, which later became Foundation level) on the other hand, was designed to move gradually from the relatively concrete and practical to more abstract and general concepts, with particular emphasis on syllabus aims concerned with the use of mathematics. Thus, it was designed essentially as a service subject, providing knowledge and techniques needed for students' future study of science, business and technical subjects.

The exploratory, open-ended style associated with investigations does not seem to fit Irish teachers' and students' views of mathematics. Possible reasons for this may lie in the culture of mathematics teaching in this country (see Section 6), in the demands that this approach would make on teacher knowledge, skills and attitudes, and in the fact that such work is not currently subject to assessment in the examination. Elwood and Carlisle (2003) suggest that there is a very narrow view of achievement in mathematics promoted by the examinations, ' … one that does not sit comfortably with the aims and objectives outlined in the syllabuses on which the courses of mathematics … are based' (p.111).

Modelling is an approach traditionally associated with applied mathematics or applications of mathematics. It involves analysing a problem, translating it into mathematical form, solving it in that form, and translating back to the original (real-life or other) situation—and checking that the solution is plausible. It is time-consuming, and typically requires a very different approach from the explain-and-drill one associated with emphasis on basic skills and routine procedures.

Realistic mathematics education (RME) stems from the Netherlands. Developing from a reaction against the 'modern mathematics' movement, it emphasises the solution of problems set in contexts which engage students' interest. It thus combines elements of the problem-solving and modelling approaches. It is probably the most 'fashionable' approach among mathematics educators at present, and underpins the OECD Programme for International Student Assessment (PISA)[2] (Shiel et al., 2001).

While there are dangers in following fashion unquestioningly, our lack of opportunity to engage seriously with the issues—in the context of a radical critique of our junior cycle syllabus—has been an unfortunate accident of history. Post-primary mathematics syllabuses in Ireland do not currently make reference to the modelling or RME approaches.

The adoption of an underpinning philosophy along the lines of RME is not a step to be taken lightly, nor could it be expected that such a change would be successful, or have measurable effects, within a short period of time. A change of culture is required, together with a change in practice. Past experience, nationally and internationally, tells us that a longer-term strategy of implementation and support is required. There is also a need to consider the pre-service education of mathematics teachers, whose own experience of mathematics education (particularly at post-primary level) has been very much along the traditional lines identified by Lyons et al (2003) and whose 'comfort zone' may not extend to encompass more modern approaches in the teaching and learning of mathematics.

Increasingly, more students with special educational needs are being included in mainstream education. In common with other teachers, mathematics teachers will need to be able to adapt their teaching methodology so that these students can develop their mathematical knowledge and skill appropriately. Ireland is also seeing an increase in the numbers of students from other countries whose early mathematics education differs significantly from that of Irish students. Teachers will need to be able to make the connections for such students, and this requires some degree of familiarity with alternative approaches and methodologies.

The revised Primary School Curriculum is more in line with the RME philosophy and, in particular, with the problem-solving approaches to mathematics education. In time, it may eventually permeate second level education 'from the bottom up' according as students transferring to post-primary schools have had longer experience of such approaches at primary school and teaching and learning in mathematics at junior cycle adopt the changed approach advocated by the syllabus revisions implemented in 2000.

Although it is not compulsory in the senior cycle, almost all students in Ireland study mathematics to Leaving Certificate. Elsewhere, national requirements or cultural pressures to take mathematics lead to different patterns in uptake. In the context of the Second International Mathematics Study (SIMS) in the early 1980s, education systems were identified in which mathematics in the senior cycle is

- compulsory
- effectively compulsory (i.e. needed for further study/job purposes, so taken by almost all students)
- taken only by those in certain tracks
- genuinely optional [as for A-level GCE] (Travers and Westbury, 1990).

2 In PISA 2000, reading literacy formed the major domain of assessment, with mathematical literacy and scientific literacy as minor domains. Mathematical literacy formed the major domain in PISA 2003 (problem-solving was introduced as an additional minor domain) and scientific literacy will be the major domain of assessment in PISA 2006.

Other factors to be considered in relation to subject uptake include the number of subjects that students take, the amount of time given to each subject, and how students are allocated to classes/levels. It should be borne in mind that decisions on the time given to individual subjects and the allocation of students to classes are taken at school level. Thus, for example, it is perfectly acceptable to give reduced weight to mathematics in order to allocate time to other subjects, if this truly reflects the goals that are set for the education provided in the school. Where this occurs in a large number of schools, it effectively becomes the national norm. Under these circumstances, however, Irish students cannot be expected to reach the same standards in mathematics as do students in countries giving appreciably more time to the subject.

2.4 Mathematics in relation to other subjects

While mathematics is a discipline in its own right, it also plays an important role in a variety of other subjects, such as business, geography and, most notably, the science and technology subjects (mathematics has sometimes been called the language of science).

> 'Serious concern about the mathematical competence of students in schools and in higher education permeates the debate on the declining uptake in the (physical) sciences.' (Report of the Task Force on the Physical Sciences, 2001)

In his report on the inquiry into post-14 mathematics in the United Kingdom, Smith (2004) pointed to the need for teachers to be aware of the links between mathematics and other subjects, as well as the links within mathematics itself. He also drew attention to the need for continuing professional development in respect of mathematics for teachers of other subjects, something seen as important for integrating the teaching and learning of mathematical skills in other subjects and areas of the curriculum.

The concern that third-level institutions have expressed regarding the standard of mathematical knowledge and skills among their first year intake is not solely related to mathematics courses, but extends to other courses where mathematics provides an important basis for progression (O'Donoghue, 2002). Inadequate mathematical skills were also noted as an issue affecting science, social science, and technology courses in British universities, threatening the quality of degrees in a wide range of key disciplines (Tickly and Wolf, 2000). In this regard, the ability to apply mathematics in what, at first glance, might appear to be non-mathematical contexts is a significant consideration. Part of the problem may lie with the perception that students at second level have of individual subjects being self-contained areas of study, unconnected to other subjects or curriculum areas (and unrelated to real life).

Given the relatedness of mathematics to a variety of subjects, it may seem obvious that, in other subjects involving some level of mathematics, teachers should be able to cross-reference their work with what happens in the mathematics class. The findings of the primary curriculum review have indicated that integration is proving difficult for teachers to achieve. If this is the case where teachers have the same class for a range of areas of the curriculum, then it is likely to be more problematic at second level where, given the subject-specific nature of staffing and timetabling, teachers are likely to have even less opportunity (or need) to look beyond the boundaries of their individual subject to consider its relatedness to other subjects or programmes.

3 Provision and uptake of mathematics

3.1 Mathematics in the primary school

In the primary school, all pupils study mathematics, which is concerned with the acquisition, understanding and application of mathematical knowledge and skills. The curriculum emphasises that mathematics is both a creative activity and a process of managing and communicating information. In the Teacher Guidelines for mathematics, mathematical literacy is noted as being of central importance in providing the child with the necessary skills to live a full life as a child, and later as an adult. It is seen as necessary to make sense of data encountered in the media, to be competent in terms of vocational mathematical literacy and to use appropriate technology to support such applications. Mathematics is used in everyday life: in science, in industry, in business and in leisure activities. Society needs people who can think and communicate quantitatively and who can recognise situations where mathematics can be applied to solve problems. [Teacher Guidelines (1999), p. 2]

The areas of content in the primary school mathematics curriculum are presented as strands that form a network of related and interdependent units: *number, algebra, shape and space, measures, and data.* These are further developed as strand units, which range across four groupings of classes from infants up to sixth class. These strands do not form a hierarchy, but rather are seen as interrelated units in which understanding in one area is dependent on, and supportive of, ideas and concepts in other strands; integration opportunities are indicated in some strand units. *Number* is an integral component of all of the strands.

3.2 Mathematics in the junior cycle

In common with other subjects, mathematics education is seen as contributing to the personal development of the students, helping to provide them with the mathematical knowledge, skills and understanding needed for continuing their education, and eventually for life and work. Thus, students should be able to recall basic facts and demonstrate instrumental understanding; they should acquire relational understanding (appropriate to the syllabus being followed), be able to apply their knowledge and skills in analysing and communicating mathematical information, and develop an appreciation of mathematics—including its history—and its role in their lives.

Almost all students study mathematics, which is one of only two subjects in the junior cycle that are provided at three syllabus levels: Foundation, Ordinary and Higher. However, unlike most other subjects where Higher level is intended for the majority of students, the Higher level mathematics syllabus states that it is targeted at students of above average mathematical ability. Thus, the cohort of students who study the Higher level course is much smaller than is the case for many other subjects.

Table1.1 gives the numbers of Junior Certificate mathematics candidates taking the examination at different levels in the period 2002-2004. These figures show that over 41% of the candidates took the Higher level paper in 2004 (a slight increase on previous years), 47% took the Ordinary level paper (a slight decrease), and less than 12% took the Foundation level paper (also a slight decrease). This is in contrast with other subjects (except Irish), where considerably more than half of the examination candidates take the Higher level paper.

Table 1.1 JC Examination candidates taking mathematics at each level

Year	Total number of examination candidates	Maths (FL)	Maths (OL)	Maths (HL)	Total Maths
2002	60,439	7,886	29,588	21,821	59,295
2003	59,633	7,324	27,383	23,734	58,441
2004	57,074	6,584	26,345	23,006	55,935

Detailed results show that 44.5% of the candidates at Higher level achieved an A or a B grade, while 6.4% obtained less than a D grade at this level. At Ordinary level, a similar proportion achieved the top two grades, with 7.3% failing to get at least a D grade. At Foundation level, 56.8% of the candidates achieved an A or B grade, with just 2.1 % obtaining less than a D. Overall, 6.3% of the candidates who sat the mathematics examination in 2004 obtained less than a grade D at any level.

As mentioned previously, consideration needs to be given to how students choose, or are allocated to mathematics classes, especially in their first year of post-primary education. Class allocation and timetabling processes should facilitate as many students as possible having the opportunity to study the higher-level course and, particularly, that they are not locked into a level due to either late development of their mathematical knowledge and skills or initial under-achievement.

3.3 Mathematics at Leaving Certificate

Leaving Certificate mathematics forms part of a broad educational experience for students in the senior cycle as they complete their post-primary education, preparing them for further education, for the world of work and for citizenship. In addition, Leaving Certificate mathematics plays a significant role in terms of entry to courses at third level, something which is not always understood by second-level students.

Three levels of mathematics course—perhaps sufficiently different to be considered as three distinct courses—are currently provided in the established Leaving Certificate: Foundation, Ordinary and Higher. This has been the case since 1992, as indicated above, and is in contrast to most subjects where there are just two levels (Ordinary and Higher), but similar to the provision in mathematics in the Junior Certificate. A course in applied mathematics is also provided (at two levels; uptake of the Ordinary level is very low).

In other countries, the provision of a range of courses for the senior cycle cohort is not unusual (however, the provision of a course in 'Applied Mathematics' is unusual, and may be restricted to countries that were influenced by practice in England). There appears to be a strong link between having different strata of educational provision (e.g. secondary, vocational) and having different kinds of mathematics courses. Also, internationally, the proportion of students who study mathematics in upper second-level education is comparatively lower than is the case in Ireland, where almost all students study mathematics to Leaving Certificate level.

The need to provide a range of syllabus levels, and in particular to provide what might be called 'general' as well as 'specialist' courses, is likely to be greater when a high proportion of the age cohort is retained in school and is required to take—or opts to take—mathematics as a subject in the senior cycle. However, it should be noted that the revised syllabus for Leaving Certificate English is a common one, with differentiation between Higher level and Ordinary level being achieved by specifying different texts as well as having separate examination papers for the two levels.

Table 1.2 gives the numbers of Leaving Certificate mathematics candidates taking the examinations at different levels in the period 2002-2004. These figures show that just under 18% of mathematics candidates took the Higher level paper in 2004 (similar to previous years), over 71% took the Ordinary level paper (a slight decrease), and 11% took the Foundation level paper (a slight increase). By contrast, in English (also taken by the vast majority of Leaving Certificate students, but with only two syllabus levels) over 60% of candidates take the Higher level examination paper. Students seem willing to take the Higher level paper in other subjects, but drop to Ordinary level in mathematics. This can be attributed in part to the perceived difficulty of mathematics, but also to attitudes and beliefs about mathematics (see section 6.3) and the 'elitist' status that Higher level mathematics can sometimes have in schools among students and teachers.

Table 1.2 LC Examination candidates taking mathematics at each level

Year	Total number of examination candidates	Maths (FL)	Maths (OL)	Maths (HL)	Total Maths
2002	55,496	5,296	38,932	9,430	53,658
2003	56,237	5,702	39,101	9,453	54,256
2004	55,254	5,832	37,796	9,429	53,057

The proportion of candidates taking mathematics at Leaving Certificate Higher level is less than half of those achieving an A or B grade on the Junior Certificate Higher level mathematics examination (and less than a quarter of those who achieved a grade C or higher). A compounding factor here is the comparatively smaller number base of Junior Certificate students in Higher level mathematics, as previously mentioned. The relatively poor take-up of Higher level mathematics rightly gives cause for concern, since it has implications for the follow-on study of mathematics to degree level.

As indicated already, mathematics is effectively compulsory in Ireland; it must also be noted that Irish students take a larger number of subjects in the Leaving Certificate than their counterparts in other countries. A consequence of this lack of specialisation is that the total time available has to be shared among many subjects, so the time allocated to any one subject at senior cycle is low in international terms. Evidence from international studies indicates that the proportion of time allocated in Ireland to mathematics in the junior cycle is also low by international comparison, and that the actual amount of time, taking into account the length of the school day and year, is likewise low (Travers and Westbury, 1990; Lapointe et al., 1992).

Anecdotally, comments from teachers during the in-career development programme that supported the implementation of the revised Junior Certificate mathematics syllabus over the period 2000-2004 point to a further erosion of the time allocated to mathematics in some schools.

In the Leaving Certificate Applied, students have an opportunity to consolidate and improve their conceptual understanding, knowledge and skills in mathematics through the practical, analytical, problem-solving approaches of the Mathematical Applications modules, as well as though integration of mathematics in other modules. The four Mathematical Applications modules, which reflect the applied nature of the Leaving Certificate Applied programme are:

- mathematics for living
- enterprise mathematics
- mathematics for leisure and civic affairs
- mathematics for working life.

4 Syllabus style and standard of examination papers

4.1 Syllabus style

The style of the present Leaving Certificate syllabus was set in the 1960s at the time of the 'modern mathematics revolution'. This emphasised abstraction, rigorous argument and use of precise terminology. The 'modern' emphasis has been diluted in subsequent revisions, and a more eclectic philosophy has taken its place. There have been minor 'trouble-shooting' revisions but no genuinely radical critique of the aims of mathematics education in the junior cycle or of the style of content, pedagogy and assessment that is appropriate for the cohort served by the programme. [This is documented in the *Guidelines for Teachers* that accompany the present Junior Certificate mathematics syllabus (Department of Education and Science/National Council for Curriculum and Assessment, 2002).] Perhaps because of the absence of such a root-and-branch revision, more recent trends in mathematics education did not permeate discussions in Ireland.

4.2 Examination papers

Examination papers still reflect the formal language and rigorous specification of questions that typify the 'modern mathematics' era. Most questions are presented as mathematical tasks (for example, 'solve the equation…') without being set in a context. Contextualised questions tend to involve a great deal of reading and/or some imprecision in specifying aspects of the problems. Also, individual contexts may appeal to some students while failing to engage others. The Irish examination papers in mathematics have aspired to fairness with regard to students' ability to read the questions and to answer them without the need for prior knowledge of a non-mathematical nature: hence, to test mathematical rather than other skills. However, the de-contextualised nature of questions has contributed to increased emphasis on recall and on the application of routine procedures.

Mathematics examination papers from some other countries, at least for lower second-level students, appear less technical than do the Higher and Ordinary level papers in Ireland and may not cover such advanced or formal mathematics. However, there is a greater emphasis in some countries on solving problems set in everyday contexts.

The Higher level Leaving Certificate mathematics examination papers up to 1993 probably over-emphasised problem-solving, in that candidates were not given adequate opportunities to display the more routine skills they possessed. By contrast, examination papers at Ordinary level were very routine.

From 1994 onwards, Leaving Certificate questions have displayed a 'gradient of difficulty' with a problem-solving section at the end. This reflects a more balanced emphasis on a fuller range of objectives as listed in the current syllabus. However, by placing the problem-solving material at the end of each question and allocating it approximately 40% of the marks, it does allow teachers or students who are targeting a safe 'C' grade to focus on the lower-order objectives at the expense of the problem-solving ones.

A further difference between mathematics examinations in Ireland and elsewhere is the absence of any form of coursework as part of the final assessment for certification (this is also true for many other subjects). As a consequence, the likelihood of achieving some of the syllabus aims and objectives, which do not lend themselves to being assessed by externally set terminal examinations, is diminished. Experience has shown that, where objectives are not assessed, they tend not to be emphasised in teaching and learning. Furthermore, in the absence of coursework, there is little opportunity or encouragement for students or teachers to engage in a more extended investigation of any one area of mathematics. However, there has been no pressure from the teaching body for coursework assessment of mathematics. Genuinely complementary, rather than supplementary, forms of assessment are probably outside the experience of almost all mathematics teachers.

Additionally, as noted in the Junior Certificate mathematics syllabus and the *Guidelines for Teachers*, while the syllabus aims and general objectives together provide a framework for all three syllabus levels, *level-specific* aims are identified for Foundation, Ordinary and Higher levels. That the general syllabus objectives are not all assessed (or assessable) by the terminal examination is acknowledged by the separate identification of assessment objectives which, although the same for all three syllabus levels, are meant to be interpreted in the context of the level-specific aims. The examination-focused teaching and the rote learning that appear to characterise mathematics classrooms in Ireland (Lyons et al, 2003) could mean that objectives which are not assessed are not likely to be addressed in class. While this is also true of many other subjects, the absence of a second mode of assessment, which could address additional objectives, means that the problem is more acute for mathematics.

It is noted in the guidelines that,

> 'Given the exclusion of some of the objectives from the summative assessment process, it is all the more important to ensure that these objectives are addressed during the students' mathematical education.' (page 91)

Difficulty level

It remains to comment on the general level of difficulty of the examination papers. Before the first examination of the revised Leaving Certificate syllabus in 1994, the NCCA course committee provided specimen questions that duly informed the production of sample papers. However, in subsequent years, anecdotal evidence from meetings where the mathematics examination papers were reviewed indicates that teachers believe there has been escalation in difficulty level of examination papers, by comparison with the sample papers. This perception may have contributed to the lower than expected increase in the proportion of candidates taking the Higher level course (an increase from 10% to almost 18% over the period since 1994, although still not achieving the 20-25% target aspired to).

While it must be borne in mind that a significant function of the examinations is to differentiate between candidates' levels of achievement, this could be managed through more rigid application of marking schemes rather than through more difficult questions.

5 Student achievement in mathematics

5.1 Leaving Certificate examination results

The level of low grades obtained in Leaving Certificate mathematics has given cause for concern. In particular, media attention has focused on the mathematics performance of Ordinary level candidates, with an average failure rate[3] of slightly more than 13% in recent years (see table of results below). When combined with the number of candidates who take the Foundation level mathematics examination (which, as already mentioned, is not accepted for entry to a range of third-level courses), this represents a sizeable proportion of Leaving Certificate candidates who 'fail' to get places at third level institutions. (Of course, such comment does not take into account the possibility that some of these students may not have had aspirations to progress to further or higher education in the first place.)

The tables below show the performance of candidates in the Leaving Certificate mathematics examinations since 2000. These are the overall grades obtained by candidates on the two examination papers at each level. Apart from 2001 when a greater percentage of candidates at Higher level achieved the top grades, the distribution of grades is fairly consistent over the five years shown.

LC HL Mathematics, percentage of candidates achieving the various grades

	A	B	C	D	E	F	NG
2000	14.0	28.3	31.9	20.7	3.8	0.9	0.2
2001	21.2	32.6	26.9	15.4	3.1	0.8	0.1
2002	13.2	28.3	33.5	20.7	3.4	0.9	0.1
2003	13.2	30.1	32.7	19.6	3.7	0.6	0.1
2004	16.1	30.0	31.2	18.4	3.3	0.9	0.1

LC OL Mathematics, percentage of candidates achieving the various grades

	A	B	C	D	E	F	NG
2000	14.4	26.0	25.5	21.4	8.2	4.0	0.5
2001	14.1	24.9	23.1	21.3	10.2	5.6	0.8
2002	13.6	24.4	24.6	23.0	9.4	4.4	0.6
2003	10.9	26.6	26.6	24.2	8.3	3.1	0.3
2004	15.7	28.6	24.9	19.3	7.7	3.3	0.5

3 'Failure' is used for convenience; it is more correct to speak of grades lower than a 'D'.

LC FL Mathematics, percentage of candidates achieving the various grades

	A	B	C	D	E	F	NG
2000	8.0	33.1	33.4	18.8	4.8	1.6	0.3
2001	7.8	31.8	33.3	19.8	5.2	1.8	0.2
2002	9.0	31.7	32.7	19.4	4.9	1.9	0.3
2003	12.2	34.9	30.5	16.6	4.3	1.2	0.2
2004	10.0	33.4	32.0	18.2	4.9	1.4	0.2

Unacceptable though the 'failure' rate is, it has been worse in the past. The figures in the late 1980s—prior to the introduction of a third Leaving Certificate course in mathematics—show that more than one-fifth of the Ordinary level mathematics examination candidates 'failed' . Moreover, at that time, the Higher level examination was taken by only 13% of the examination cohort (some 7000 candidates; the figure subsequently fell below 6000, or around 10% of the examination cohort).

The acute problems were somewhat alleviated by the introduction of the Ordinary Alternative syllabus in 1990 (for first examination in 1992) and the revision of the Higher and Ordinary courses in 1992 (for first examination in 1994):

- Percentages of candidates taking the Higher level examination rose considerably over the following years, but levelled out before reaching the aspirational range of 20-25%; at present around 17.5% of the cohort take the Higher level examination.
- Numbers taking the Ordinary Alternative/ Foundation level examination remained much lower than the Course Committee had expected—under 10%, rather than the 20-25% for whom the syllabus was designed.
- The Leaving Certificate Applied programme accounts for approximately 5% of the student cohort at this level. (The Ordinary Alternative course was not originally targeted at students who now take the Leaving Certificate Applied; they were to be served by a 'Senior Certificate' course.)

5.2 Junior Certificate examination results

In the Junior Certificate, mathematics is also assessed at three syllabus levels. As already indicated, a revised syllabus was introduced in 2000 and first examined in 2003. Thus there are only two years of examinations results relevant to the current syllabus. Also, a style was adopted for the examination papers that indicated to candidates where their working of solutions was required to be shown (at the risk of losing marks where this was not complied with).

The table opposite shows the number and performance of candidates taking the three syllabus levels in Junior Certificate mathematics examinations for 2003 and 2004. In the case of Higher level and Ordinary level, these are the overall grades obtained by candidates on the two examination papers (there is only one examination paper at Foundation level).

JC Mathematics, percentage of candidates achieving the various grades at each level

		A	B	C	D	E	F	NG
HL	2003	17.2	33.6	28.6	17.0	3.1	0.5	0.0
	2004	16.1	28.4	28.9	20.3	5.2	1.1	0.1
OL	2003	9.2	31.0	31.3	20.8	5.8	1.8	0.1
	2004	10.1	34.3	30.8	17.7	5.2	1.9	0.2
FL	2003	15.4	37.8	29.5	13.6	3.2	0.4	0.0
	2004	16.4	40.4	29.1	12.0	1.8	0.3	0.0

5.3 Evidence from cross-national studies

International studies of achievement have to be interpreted with great care because, all too often, they do not compare like with like. Nonetheless, when due account is taken of the context, they can provide helpful pointers to strengths and weaknesses in student achievement.

Ireland has not participated in studies of mathematics achievement at Leaving Certificate level, but those for younger students provide interesting information.

- In the first (1988) and second (1991) International Assessments of Educational Progress (IAEP I and II), considering 13-year-old students, Irish performance was decidedly moderate. In IAEP II, in particular, the test content was well matched to the Irish curriculum; average Irish performance was similar to that of Scotland, but the Irish results showed a worrying 'tail' (Lapointe et al., 1989; Lapointe et al., 1992).
- The mathematics tests for the Third International Mathematics and Science Study (TIMSS) (1994) were also well matched to the Irish syllabus content; on this occasion, performance of Irish second-year students was better than that of the comparable cohort in a number of countries with similar cultural and developmental level (Beaton et al., 1996).
- By contrast, students in Ireland achieved a score in mathematical literacy not significantly different from the OECD average on both the first (2000) and second (2003) cycles of the Programme for International Student Assessment (PISA). Mathematics was the major domain of assessment in 2003, and the average performance of Irish students was below that of several countries that might be deemed 'comparable'. While the mathematical concepts underlying the majority of PISA items in 2003 would be generally familiar to Irish students (although somewhat less familiar for Foundation level students), the situating of mathematics problems in a context (e.g. embedded in a real-life setting) was recognised as unfamiliar for the majority of items at all three syllabus levels (Shiel et al., 2001; Cosgrove et al., 2004)

Altogether, therefore, the message from the studies is somewhat mixed; but they provide evidence that the performance of some Irish students at junior cycle gives cause for concern. This suggests that the seeds of a least part of the problem at senior cycle may be sown during the junior cycle, or even earlier.

5.4 Evidence from Chief Examiners' reports

Chief Examiners' Reports for Leaving Certificate mathematics were produced in 2000 and again (in response to the poor Ordinary level results) the following year. These very valuable documents highlight specific areas of strength and weakness in students' answering and relate them to the objectives of the syllabus.

- For Ordinary level students, weaknesses include poor execution of basic skills in some areas and an apparent lack of *relational understanding* (understanding of 'why' rather than just 'how'—hence, the basis for applying knowledge in even slightly unfamiliar circumstances). The implications of this are considered below.
- The strengths of Ordinary level candidates are seen to lie in the area of competent execution of routine procedures in familiar contexts.
- The report points out that it 'is clear, both from the continuing relatively high failure rate and from the type of work presented by the candidates who are failing, that there are significant numbers of candidates who are wholly unsuited to taking this examination'.

The Examiners' Report for the Junior Certificate in 1996 also highlights basic weaknesses especially among students taking the Ordinary and Foundation level examination papers. (It helped to counteract any undue optimism from the comparatively 'good' results from TIMSS.) Thus, again, there is evidence that the problems observed at Leaving Certificate level start further down.

5.5 Other evidence

One measure (not, of course, the only measure) of the effectiveness of the Leaving Certificate mathematics course is the extent to which it prepares students for study at third level. Of particular interest here is the role of the Ordinary level course in equipping students for further and higher education courses in science, technology, other technical subjects, and other subjects requiring a good grasp of mathematics.

Evidence is accumulating that the incoming level of mathematical expertise—hence, the expertise of students who achieved a D grade or higher in the Leaving Certificate Ordinary level examination—is insufficient, and does not match expectations created by the objectives and content of the syllabus and by the standard of the examination papers.

The report by Morgan (2001) points to difficulties leading to failure and dropout from Institutes of Technology (ITs); this quantifies anecdotal evidence from lecturers in ITs that poor mathematical competency contributes to non-completion of courses. Some third-level institutions have identified problematic areas in students' mathematical knowledge and skills and have put in place successful interventions to address these (O'Donoghue, 2002).

6 Teaching and learning

6.1 Focus of teaching

The results of TIMSS provided insights into the approaches that Irish teachers feel are best for ensuring success in school mathematics.

- In international terms, memorising and routine performance were given exceptionally high emphasis in Ireland, while logic, creativity and applications were given very low emphasis (Ireland came last as regards applications) (Beaton et al., 1996).
- This could be attributed to the style of the examination papers. However, any move to make the papers less routine and more 'applied' are met with considerable opposition, for example through feedback via the Irish Mathematics Teachers' Association (IMTA). This may indicate that teachers are philosophically comfortable with the current style: that they see mathematics (or mathematics for school students) as being about fairly routine performance in no particular contexts. Alternatively, it may be that this style fits the classroom methodologies that they know and with which they feel secure.

IMTA meetings devoted to 'post-mortems' on the examination papers—these are regular events and are often among the better attended meetings, especially with regard to the Higher level Leaving Certificate papers—are at times obsessively devoted to 'what will get marks' rather than 'what may improve students' learning' or 'what might be good mathematics education'. This is understandable in the immediate lead-up to examinations, but increasingly seems to start a long way before the examinations.

The Chief Examiners' Reports, as mentioned earlier, emphasised students' lack of *relational understanding*. Research suggests that relational understanding, appropriately complemented by *instrumental understanding* (knowing 'what to do'), is important for successful learning that can be retained and applied (National Council of Teachers of Mathematics, 2000). The examination-focused teaching described above is not conducive to the development of relational understanding, which tends to require an emphasis variously described in the literature as 'sense-making' or 'meaning making' (Hiebert et al., 1997). It is facilitated by somewhat 'progressive' teaching, allowing for constructivist approaches in which concepts are explored, individuals' imperfect concepts and procedures are reflected on and 'de-bugged', and expository teaching is appropriately complemented by activities such as discussion and journal-writing.

Evidence from the international studies suggests that Irish classrooms are largely 'traditional,' involving teacher exposition and (probably, followed by) individual pupil work (Lapointe et al., 1989; Lapointe et al., 1992; Beaton et al., 1996). Of course this too can be used to facilitate relational understanding, but is not such a natural format for its development. The study by Lyons et al. (2003), involving collection of videotape evidence in a small number of classrooms, tends to support the idea that mathematics teaching tends to be unduly instrumental (see 6.2 below).

In considering the 'short-cuts' that some teachers are taking, the shortened and decreasing time allocated to mathematics should be borne in mind. Not all teachers *want* to teach in that way… and of course some, despite the shortage of time, do not. A further problem may be teachers' own knowledge base. This is considered in more detail below.

6.2 Focus of learning

The findings of research (Lyons et al., 2003) into the teaching and learning of mathematics in second-level schools in Ireland suggest a high level of uniformity in terms of how mathematics lessons are organised and presented. There is a concentration of class time on the two interrelated activities of teacher demonstration of mathematical procedures and skills, and student practice of these. The practice exercises were typically set by the teacher (in the majority of cases from the textbook) and undertaken by the students during class time or as homework. A procedural rather than a conceptual or problem-solving approach to mathematics prevails in the predominantly 'traditional' mathematics classroom. However, observations of English classes indicate that the use of traditional approaches to teaching is not confined to mathematics.

Research (carried out in England, but there is evidence of a similar phenomenon here) with students in teacher education courses indicates that some have gone through their undergraduate career, even perhaps in mathematics or related degree courses, without gaining a truly relational understanding of the subject (Suggate et al., 1999).

Teachers, of course, cannot do all the work themselves. Students' approaches may be unhelpful in this respect. Some students may be too inclined to sit back and expect the teachers to do the work so that they (the students) learn painlessly. Other students may be prepared to work very hard, but may put their hard work into inappropriate learning strategies: ones that do not promote meaningful learning. (Examples would include learning the proof of a geometrical theorem by 'learning it off by heart' without reference to a diagram, and therefore being entirely unable to carry out the proof if the diagram is labelled differently.) In fact, students may have a *learnt helplessness* that suggests to them that they cannot tackle even slightly unfamiliar work. Students who have suffered from a 'tell and drill' or 'busywork' approach (bereft of meaning) may already have learnt this helplessness before they enter second level school. However, the revised mathematics curriculum introduced in 1999 places increased emphasis on a practical, hands-on approach to the learning of mathematics, which is reported as promoting greater engagement in, and enjoyment of, mathematics learning on the part of the children (see 2.2 above).

Of relevance here is the culture of the classroom and especially the *didactical contract* implicitly made between students and their teacher (Nickson, 2000). This may be of the form: 'I am here to get my exams., you are here to teach me to do it.' Evidence of such a pragmatic approach is found in students' comments on being faced with a more meaning-related or discursive approach at third level.

6.3 Attitudes to and beliefs about mathematics

Implicit in much of what has been said above is the issue of *attitudes to mathematics* and the related issues of *beliefs, perceptions or conceptions* about mathematics.

Consideration can be given first to teachers. Research suggests that there is a connection between teachers' views of mathematics and their approach to teaching it (Thompson, 1992). A teacher who believes that mathematics is a bag of useful but unconnected tricks is likely to emphasise different things than will a teacher who believes that mathematics is a body of knowledge as near to absolute truth as we can get, a web of beautiful relationships, or an activity involving the formulation and solution of problems. Standard research on the characteristics of a good teacher indicates that one such characteristic is enthusiasm for the subject being taught.

For students, several issues arise. Research indicates that attitudes and achievement are correlated, albeit not particularly strongly. Notably, not all successful students like the subject (McLeod, 1992). Moreover, the within-country associations between attitudes and achievement do not necessarily hold across countries. In international studies, some of the highest-scoring countries had the most negative attitudes to mathematics and *vice versa* (Robitaille and Garden, 1989; Lapointe et al., 1989; Lapointe et al., 1992). However, these may be a reflection of cultural tendencies (with regard to it being 'OK to say you like schoolwork') rather than being related to approaches specifically to mathematics.

In PISA 2003, students were asked about four aspects of their approaches to learning in mathematics: motivation, self-related beliefs, anxiety, and learning strategies. The study found that interest in and enjoyment of mathematics is closely associated with performance in all OECD countries. Students who believe in their own abilities and efficacy, and who are not anxious about mathematics, are particularly likely to do well in the subject (OECD, 2004).

More generally, some findings noted in the context of the review of research for the *Cockcroft Report* (Committee of Inquiry into the Teaching of Mathematics in Schools, 1982) suggest that students like the simple, routine aspects of mathematics that are of limited educational value and have limited application to industrial prosperity (they can be mechanised), and that they dislike the aspects which highlight problem-solving—a rather depressing situation for mathematics educators.

Students see mathematics as effortful—'hard work' and 'natural ability' are required to do well in the subject. The issue of 'mathophobia', or fear of mathematics, is important (and this is not confined to students; many adults are uncomfortable when faced with numerical data and even relatively straight-forward number operations). Students for whom mathematics does not make sense might be expected to experience failure and to be scared of the subject. However, qualitative research dealing with rather gentle problem-solving approaches with weaker students suggest ways forward which might combine *appropriate courses* (from an educational and social point of view) with *appropriate pedagogy*.

In their study of Irish mathematics classrooms, Smyth et al (2004) found that students typically saw mathematics at second level as the same or 'harder' than in the primary school, and more than was the case for either Irish or English. This was particularly so in respect of students in the higher stream classes. Almost all of the students in the case study schools considered mathematics useful. Of those who had not received extra help or learning support in school, approximately one-third indicated that they would have liked to receive help with mathematics. Mathematics was the second least popular subject (after Irish) identified by the students. However, over 70% of the students considered that the time spent doing mathematics was about right, whereas about half of the students thought this was not the case for Irish and other languages (too much time), or the 'practical' subjects such as P.E., information technology, art, or materials technology wood (too little time).

An interesting aspect of the study undertaken by Lyons et al. (2003) was the decision to include interviews with parents of students observed in the mathematics classes, and their classification into three types: 'insiders', 'outsiders', and 'intermediaries' in terms of their knowledge of the education system, education level, and levels of intervention with their child's school. This approach offered a unique insight into the connections between school and home, with a particular focus on mathematics.

The insider parents had extensive experience of the education system, with most having obtained a third-level qualification. These parents held positive views of mathematics and monitored their children's progress in the subject. While believing that good teaching and hard work was needed for success in mathematics, 'most (of these parents) also thought that success at mathematics was dependant on having 'natural ability' in the first instance' (Lyons et al. 2003; p.342).

Outsider parents, while having reasonably good levels of education, had much less knowledge than insiders about how the education system works, or what was required to succeed in formal education. They had more negative attitudes to schooling and mathematics, based on their own experiences. However, like insider parents, they also had a strong belief that innate ability was crucial for successful learning in mathematics.

Parents were classified as intermediaries on the basis of being somewhere between the insiders and the outsiders. They had some knowledge of what they should do to ensure their children's educational success, but were concerned about the adequacy of this knowledge or their capacity to act in supporting their children's education. While they had concerns about their children's performance in, and attitudes to, mathematics, they regarded these as in some way linked to their own negative experiences in the subject.

6.4 Teacher competencies

Mention has already been made of problems with teachers' knowledge base. This is likely to be true of some primary teachers and some second-level teachers for whom mathematics is their second or third teaching subject—teachers who may have (at best) limited mathematics in their degrees.

A considerable amount of research points to limitations in student-teachers' content knowledge in mathematics (Brown and Borko, 1992). In particular, their knowledge of concepts may be poor (hence, they may have weak relational understanding—much the same notion).

The situation may differ in different countries. A small-scale piece of work emphasises the difference in 'profound understanding of fundamental mathematics' (PUFM) between a group of Chinese and a group of American primary level teachers (Ma, 1999). The Chinese displayed both relational and instrumental understanding; they could do elementary computations in different ways, giving reasons, and get them right. The Americans were less likely to understand the method they used and some made errors. It would be wrong to read too much into such a small study, but it has highlighted important issues relating to the teacher's own knowledge base in mathematics and their approach to teaching the subject.

It is likely (though perhaps not as well established by research) that some teachers and prospective teachers of mathematics may not have adequate pedagogical content knowledge, i.e. knowledge of (*inter alia*) how to develop relational understanding in their students. Much of the discussion of post-primary mathematics education was dominated by consideration of syllabus content and assessment issues (English and Oldham, 2004). Thus, even if their general knowledge and skills in teaching are good, teachers may not be able to use it suitably in teaching mathematics.

The recent in-career development programme for mathematics teachers at junior cycle addressed pedagogical content knowledge and other issues that may enhance teachers' and students' enjoyment of mathematics—and may even lead to the establishment of a different didactical contract (see 6.2 above). There is a need for teachers to recognise the emotional dimension to learning that, in light of the comments above regarding student attitudes, has particular relevance to mathematics. The sense of failure (and, possibly, of frustration) that some students feel at an early stage in relation to mathematics must be acknowledged and addressed if these students are to engage successfully with later learning in this subject.

There is no formal provision that facilitates teachers in routinely updating their skills, other than when new or revised courses are being implemented. Thus, for example, where developments emerge through particular computer applications or in our understanding of the different ways in which students learn, there are no established structures whereby the general body of teachers can become familiar with these in the context of mathematics teaching. This absence of a culture or provision of ongoing professional development impacts on all teachers; that in-career support is provided only when there is syllabus change communicates a message of change as event rather than process and suggests a role for the teacher as the recipient of change rather than its agent. The work of the subject associations—the Irish Mathematics Teachers Association in this case—is extremely important in challenging this prevailing culture.

6.5 The culture of the classroom

Typical classrooms have not facilitated the 'concrete' approach to mathematics education recommended in the in-career development programme for the revised Junior Certificate mathematics syllabus. They are often set up in such a way as to reinforce the 'expository plus seatwork' style referred to earlier.

The fact that many schools do not have designated 'mathematics classrooms' not only adds to difficulties in providing concrete materials; it means, for example, that classrooms are not decorated with posters that create a lively, interesting environment for the learning of mathematics.

The main classroom 'aid' is the textbook. Again, Irish textbooks are somewhat functional by comparison to those in some other countries (the small population base militating against large, glossy texts with many discussion points and suggested activities). As evidenced by inspection visits, teaching is highly dependent on the class textbook (which tends to reinforce the 'drill and practice' style) and the examinations, and there is frequently a very close relationship between these two. Lyons et al (2003) found that students were generally not given insights into the applications of mathematics in everyday life; learning was a matter of memorising mathematical procedures and facts:

> 'Mathematics was presented to students generally as a subject a) that had a fixed body of knowledge; b) that was abstract in character; c) that required demonstration of procedures rather than explanation; and d) that comprised discrete elements.' (Lyons et al., 2003; p. 143).

Elwood and Carlisle (2003) suggest that the better performance of girls in both the Junior Certificate and Leaving Certificate mathematics examinations (particularly at Ordinary level) may well be explained by them being better prepared and organised, more familiar with the conventions and requirements of the topics covered, and better able to recall the learnt rules and formulae as required by the questions asked. The 'traditional' mathematics classroom, as exemplified above by Lyons et al., facilitates such learning.

The image of mathematics as linked to 'real life' may not have been enhanced by the late and slight adoption of information and communications technology (ICT) in mathematics teaching. The use of calculators has been introduced for fifth and sixth classes in the primary school, and their use in the junior cycle has increased following the introduction of the revised syllabus in 2000 (recent research suggests that only limited use was made of them prior to their being allowed in the examinations). Computers are not regularly used in mathematics classrooms, although the Schools IT2000 initiative has resulted in more teachers making use of ICT in a range of subjects.

7 System and cultural influences

7.1 Backwash effect from institutions beyond school

Third level colleges have played a considerable part in the uptake or non-uptake of the different Leaving Certificate syllabuses. This has been the case particularly for Ordinary Alternative and Foundation level.

Initially, the universities accepted Ordinary Alternative as a course for matriculation purposes for entry to courses that would not require mathematics (if 'mathematics' was a requirement at all for such courses; for some, it is not). Naturally, they did not accept Ordinary Alternative for entry to subjects requiring a substantial level of mathematics.

However, the (then) Regional Technical Colleges, while naturally requiring at least the Ordinary level for entry to technological and scientific courses, also required this level for entry to a number of courses for which its content (including co-ordinate geometry and calculus) seemed irrelevant—in fact, for which a *good* knowledge of the Ordinary Alternative material would have been highly preferable to a rote-learnt and very imperfect knowledge of the Ordinary level material.

High passing grades represent worthwhile knowledge and skill. Acceptance of such grades might be conducive to more meaningful learning than is the case at Ordinary level at present for weaker students.

Outside the third level sector, other training bodies have also been unwilling to accept the 'third course' (with the exception of the Gardai where a grade B at Foundation level is accepted as an alternative to a grade D at Ordinary level). Again, it would probably be appropriate for such bodies to accept a high passing grade on that course.

7.2 CAO points

It is worth noting that the growth in numbers taking the Higher level examination has occurred despite the discontinuation of 'double points' for Higher level mathematics. The re-scaling of points that equated a C3 on the Higher level examination with an A1 on the Ordinary level examination has probably had a significant, but negative influence on the uptake at Higher level. While a very high mark at Ordinary level is not, perhaps, as easy to obtain as it was before the syllabus was revised, the current standard of the Higher level examination papers has meant that 'good, but not very good' candidates may not feel sufficiently confident of getting the C grade, and so take the 'easier' option of Ordinary level. These candidates feel that they can obtain the top Ordinary level grade with much less work than is required for the points-equivalent grade at Higher level. This reinforces the perception that, for mathematics, the points mapping between the grades is mismatched.

Worse effects emerged following the decision to award no points at all to the Ordinary Alternative, and subsequently to the Foundation level examination. The result was extremely damaging to the perceived status of the 'third course' and hence to the self-esteem of students taking it. The non-acceptance of Foundation level grades discourages students from taking this course, with a consequent increase in the number of low grades at Ordinary level.

7.3 Education as a passport to a career

Recalling research done in the early 1970s, and corroborated in TIMSS, it is perhaps reasonable to conjecture that students—even more now than then—perceive education in terms of access to third level or to careers rather than 'a love of the learning' (Beaton et al., 1996). In fact the very high level of study (by at least some students) that was noted—in newspaper articles on 'pressures of the points race' and so forth—in the comparatively hungry early 1990s was produced by extrinsic motivation, not by love of the subject.

A very pragmatic approach, in which students target the points they need *now* rather than the knowledge and skills they may need *later*, seems to be prevalent. This could mean that some low passing grades were fairly intentionally targeted (as being sufficient for, say, matriculation purposes)—and perhaps even some failing grades were student misjudgments. Second level students appear not to recognise that successful engagement with many third level courses is dependent upon mathematics knowledge and skills and that, without these competencies, progress may be severely hampered—irrespective of the grades achieved or the number of points obtained in the Leaving Certificate.

8 Equality issues

Issues of equality in mathematics education have tended to focus almost exclusively on perceived gender differences in relation to provision, uptake and achievement. This focus has persisted, despite research evidence that socio-economic status and educational disadvantage also represent significant factors in explaining differences in uptake and achievement between boys and girls across a range of subjects. In PISA 2003, for example, the difference in mean scores between boys and girls was about one-sixth of a standard deviation, whereas the difference in mean scores of students in Ireland with low and high economic, social and cultural status is around nine-tenths of a standard deviation (Cosgrove et al., 2004).

In relation to gender, girls' uptake of mathematics courses has improved in recent years, and so has achievement. Smith and Hannan (2002), while noting that the pattern of take-up for mathematics at senior cycle in Ireland contrasts with that in many other countries where it is an optional subject, found significant differences in the take-up of Higher-level mathematics. A greater proportion of girls than boys take Higher level mathematics in the Junior Certificate, but this is reversed for Leaving Certificate. Moreover, in schools which focus on preparation for higher education, a greater proportion of Leaving Certificate students study the Higher level mathematics course.

The age at which gaps in achievement appear to exist has been rising, but there are differences in the achievement of higher grades between boys and girls at the different course levels (Elwood and Carlisle, 2003). While Lyons et al (2003) found that girls tend to achieve overall better results than boys across all levels in Junior Certificate mathematics, a greater proportion of boys than girls obtain A grades at Higher level, with boys in single-sex and comprehensive schools obtaining the highest proportion of A grades. This disparity in top grades has decreased in recent years and more recent studies 'have suggested that gender may not be as important a variable in explaining performance differentials in schools as it once was, particularly in the field of mathematics' (Lyons et al. 2003, p.12). However, differences have persisted with regard to how total scores on the examinations are obtained. Stereotypically, girls were better at routine work and boys at problem-solving.

Another possible area of differentiation is the use or non-use of contexts. The received view in the past has appeared to be that boys could deal with 'real, hard' mathematics, whereas girls were better when work was embedded in 'soft' contexts … or that girls were able to deal with mathematics embedded in contexts whereas boys were not so well able.

Classroom style may also have a role to play. Again, conventional wisdom has been that the individual, competitive approach—easily associated with the 'typical' classrooms described earlier—would suit boys better than girls; girls might be better suited by co-operative group work, which does not appear as a regular feature in studies of Irish classrooms.

Our examinations have not greatly emphasised good communication (stereotypically a female skill); on the other hand, little use is made of multiple choice questions, which have been seen as favouring boys. The projected increased emphasis on good communication in the Junior Certificate papers may in fact favour girls. Research by Elwood and Carlisle on gender and achievement in mathematics examinations has supported the opinion that

> *'a narrow view of achievement in mathematics is promoted by the Junior Certificate and Leaving Certificate examinations, and it is one that does not sit comfortably with the aims and objectives outlined in the syllabuses on which the courses of mathematics in schools in Ireland are based.'* (Elwood and Carlisle, 2003).

9 Conclusion

This paper sets out the background and context for a review of mathematics in the post-primary curriculum. It identifies a range of issues that surround mathematics education in Ireland and, in conducting the review, a number of distinct areas that need to be considered:

- the purposes of mathematics education, including societal expectations
- the mathematics curriculum/syllabuses and changes that may be needed
- the assessment and certification of mathematics
- the requirements of third level institutions and how these may (or should) be accommodated
- the perceived problems with regard to mathematics education in schools
- the role of information and communications technology in mathematics education
- the teaching and learning 'culture' of mathematics
- the inservice education and training of mathematics teachers.

In addition, there is a need to consider how the pre-service education of teachers can take into account curriculum (and assessment) changes that may have taken place since their own time in school, and with which they may not be familiar. In the case of those whose main teaching subjects do not include mathematics, but in which mathematics has a significant role, there is a need to develop strategies by which a coherent approach can be taken to the teaching and learning of mathematical concepts and processes.

Historical accidents which have militated against a root-and-branch consideration of junior cycle mathematics in the past have also affected revisions at senior cycle because the latter built on foundations that had not been critiqued for some time. A major 'root-and-branch' review of mathematics education in Ireland has not taken place since the 1960s. Such a review would afford the opportunity of considering the purposes of mathematics education and could extend also to the syllabus for Leaving Certificate applied mathematics, which has not undergone revision for a considerable time. The various models that were examined at the time of the 1992 revision of Leaving Certificate mathematics, but which were rejected for various reasons, should be re-visited, given the changes in the intervening period.

The appropriateness of the examination papers across the three levels needs to be considered, particularly since the uptake aspired to at the three Leaving Certificate levels has not been realised. A comparison needs to be made with the intended levels of difficulty as exemplified in the initial sample papers developed when the syllabuses were introduced. In particular, this would involve ensuring that Higher level Leaving Certificate examination papers are pitched at a level that provides for the needs of a greater number of students.

The non-acceptance of A and B grades at Foundation level by institutions of higher education and by the CAO (for points) has been a significant factor in the uptake at Foundation and Ordinary levels. Discussion around the acceptability of grades/levels in the Leaving Certificate, and the standards of achievement associated with them, should encompass the issue of whether, when a root-and-branch review of mathematics is complete, two levels may meet the needs of the student cohort and the education system as a whole.

The current problems at Ordinary level in the Leaving Certificate have deep-seated roots in cultural expectations about schooling and beliefs about mathematics. They start much further down the school than the Leaving Certificate and are about much more than the content of the courses and the associated pedagogy and assessment mechanisms. However, some of the issues are being addressed by the introduction of the revised Primary School Curriculum and the current in-career development being undertaken at junior cycle level. It is important that these moves are reinforced rather than disrupted by any 'quick fix'. (It may be worth noting here that the reforms in the Netherlands took some 25 years, starting in infant classes, and involved a concerted approach by teacher education institutions.) In the coming years, students coming into post-primary school will have experienced the full impact of the revised primary school mathematics curriculum. It will be important to have established an appropriate and cohesive programme of mathematics education at second level that builds on their previous learning so that they can maximise their potential in mathematics and related fields of study.

There is a need to consider both pre-service and inservice education and training for teachers of mathematics, and the extent to which real change in teaching and learning can take place. If a genuine re-appraisal of mathematics education is to lead to significant change, attention must be paid to the need for teachers to move away from the traditional approach, which may have been their own experience as students and/or which may have served them well as teachers up to now, and to embrace a new philosophy and associated methodology that will best serve future generations of students. Furthermore, teachers of other subjects in which concepts and processes of mathematics arise will similarly need to embrace change in practice.

The 'problem' with mathematics in our schools is not solely related to the issue of improving numeracy skills, although these undoubtedly need to be addressed, but goes far beyond this. The learning of mathematics also transforms our ability to conceptualise and structure relationships, to model our world and thus be able to both control and change it. Young people need to develop the ability to build connections across knowledge, to identify and explore patterns, to estimate and predict, to interpret and analyse numerical and statistical data, to communicate increasingly complex information, and to apply all of this in their daily lives and work.

As pointed out by Hoyles and Noss (2000), the argument is not that the world can only be understood through mathematics; it is that mathematics should be an essential tool for understanding it. This applies in particular to the digital technologies that rely for their development on the types of skills that mathematics education provides. Thus, mathematics education is a two-way street; it is inter-related with technology and can both support and be supported by modern developments in technology. An important feature of the review of post-primary mathematics must be the role that information and communications technology can play in facilitating greater engagement with mathematical concepts, in developing mathematical knowledge and skills and thereby, almost in cyclic fashion, contribute to future developments in this self-same technology.

There is a need to engage in discussion about the culture of mathematics in schools and in society, and to promote a 'can-do' approach to mathematics. In the words of Hoyles and Noss:

> 'We do need to find ways to connect mathematics with the broader culture. We need to find ways to break down the barriers between mathematics and art, music, humanities and the social sciences, as well as, of course, science and technology. We need to find entry points into the preoccupation and aspirations of children in ways that respect the integrity of their interests rather than patronizing and inevitably disappointing them. Most importantly, we need to introduce ways to make these connections by

ensuring that the solutions of problems given to our students need the use of mathematics, that children do have a chance to make choices of strategy for themselves and learn to reflect upon and debug them. We would like to encourage an appreciation that solving problems in and with mathematics is not a matter of routine or factual recall – although these might play a part.' (2000, p.155)

References

Beaton, A., Mullis, I., Martin, M., Gonzalez, E., Kelly, D.L. and Smith, T. (1996) *Mathematics Achievement in the Middle School Years.* Boston, MA: Center for the Study of Testing, Evaluation and Educational Policy, Boston College.

Brown , C., & Borko, H. (1992) Becoming a Mathematics Teacher. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 209-239). New York: Macmillan.

Committee of Inquiry into the Teaching of Mathematics in Schools (1982) *Mathematics Counts* [The Cockcroft Report]. London: Her Majexty's Stationery Office.

Cosgrove, J., Shiel, G., Sofroniou, N., Zastrutzki, S. and Shortt, F. (2004). *Education for Life: The Achievements of 15-year-olds in Ireland in the Second Cycle of PISA.* Dublin: Educational Research Centre.

Department of Education and Science / National Council for Curriculum and Assessment (2002) *Mathematics Junior Certificate: Guidelines for Teachers.* Dublin: Stationery Office.

Elwood, J., and Carlisle, K. (2003) *Examining Gender: Gender and Achievement in the Junior and Leaving Certificate Examinations 2000/2001.* NCCA Research Report No. 1. Dublin: National Council for Curriculum and Assessment.

English, J. and Oldham, E. (2004) 'Continuing Professional Development for Mathematics Teachers; Meeting Needs in the Republic of Ireland'. Paper presented to the 29th annual conference of the Educational Studies Association of Ireland, April 2004

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., and Human, P. (1997) *Making sense: teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Hoyles, C., and Noss, R. (2000) Facts and Fantasies: What Mathematics Should Our Children Learn. In C. Tickly and A. Wolf (eds) *The Maths We Need Now: Demands, deficits and remedies.* London. Bedford Way Papers/Institute of Education.

Hoyles, C., Wolf, A., Molyneux-Hodgson, S., and Kent, P. (2002) *Mathematical Skills in the Workplace.* London, Institute of Education and Science, Technology and Mathematics Council.

ICSTI (Irish Council for Science, Technology and Innovation) (1999) *Benchmarking School Science, Technology and Mathematics Education in Ireland Against International Good Practice.* Dublin, ICSTI

Lapointe, A. E., Mead, N. A. and Phillips, G. W. (1989) *A World of Differences: an International Assessment of Mathematics and Science.* Princeton: Educational Testing Service.

Lapointe, A. E., Mead, N. A. and Askew, J. M. (1992) *Learning Mathematics.* Princeton: Educational Testing Service.

Lyons, M., Lynch, K., Close, S., Sheeran, E., and Boland, P. (2003) *Inside Classrooms: a Study of Teaching and Learning.* Dublin: Institute of Public Administration.

Ma, Liping (1999) *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

Morgan, M., Flanagan, R., and Kellaghan, T. (2000). *A study of non-completion in Institute of Technology courses.* Dublin: Educational Research Centre.

McLeod, D. (1992) Research on Affect in Mathematics Education: A Reconceptualization. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 575-596). New York: Macmillan.

National Council of Teachers of Mathematics (1989) *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000) *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Nickson, M. (2000) *Teaching and Learning Mathematics: a Teacher's Guide to Recent Research and its Applications.* London: Cassell.

O'Donoghue, J., (2002) 'Mathematics: transition from second level to university'. Fr. Ingram memorial Lecture to the Irish Mathematics Teachers Association, November 2002.

OECD (Organisation for Economic Cooperation and Development) (1999). *Measuring student knowledge and skills: A new framework for assessment.* Paris, OECD.

OECD (Organisation for Economic Cooperation and Development) (2004). *First results from PISA 2003; Executive summary.* Paris, OECD.

Robitaille, D.F., and Garden, R. A. (1989) *The IEA Study of Mathematics II: Contexts and Outcomes of School Mathematics.* Oxford: Pergamon Press.

Shiel, G., Cosgrove, J., Sofroniou, N., and Kelly, A. (2001). *Ready for Life? The Literacy Achievements of Irish 15-year-olds with Comparative International Data.* Dublin: Educational Research Centre.

Shulman, L. (1986) Those who Understand: Knowledge Growth in Teaching. *Educational Researcher,* 15 (2), 4-14.

Smyth, E., McCoy, S., and Darmody, M. (2004). Moving Up: *The Experiences of First-Year Students in Post-Primary Education.* Dublin: Liffey Press/ESRI.

Suggate, J., Davis, A., and Goulding, M. (1999) *Mathematical Knowledge for Primary Teachers.* London: David Fulton Publishers.

Sutherland, R. (2000) Disjunctions between School and University: The Case of Mathematics. In C. Tickly and A. Wolf (eds) *The Maths We Need Now: Demands, deficits and remedies.* London. Bedford Way Papers/Institute of Education.

Thompson, A. (1992) Teachers' Beliefs and Conceptions: A Synthesis of Research. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 209-239). New York: Macmillan.

Tickly, C. and Wolf, A. (2000) The State of Mathematics Education. In C. Tickly and A. Wolf (eds) *The Maths We Need Now: Demands, deficits and remedies.* London. Bedford Way Papers/Institute of Education.

Travers, K. and Westbury, I. (1990) The IEA Study of Mathematics I: Analysis of Mathematics Curricula. Oxford: Pergamon Press.

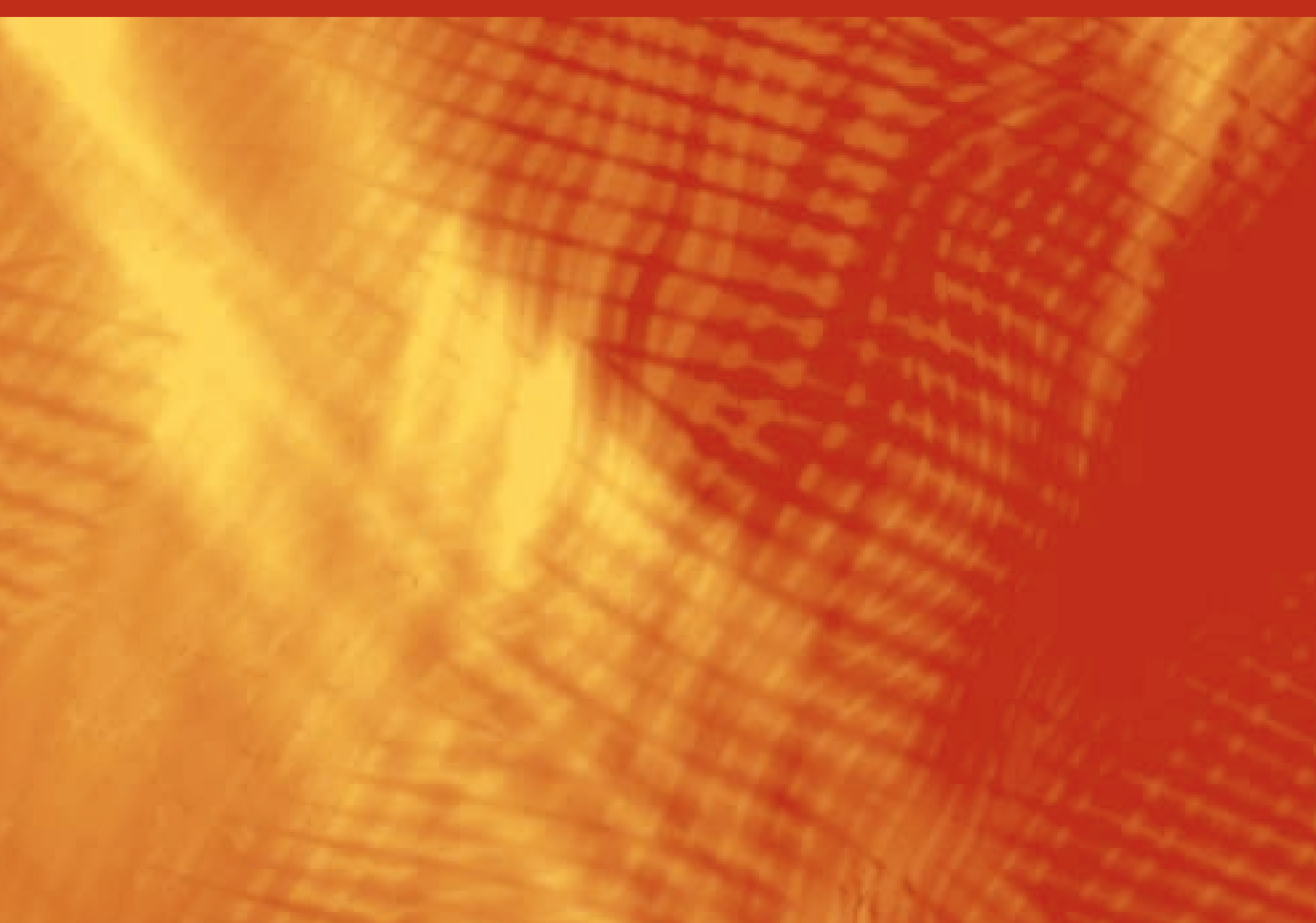
NCCA
National Council for Curriculum and Assessment
An Chomhairle Náisiúnta Curaclaim agus Measúnachta

Research Report No.5

International Trends in Post-Primary Mathematics Education:

Perspectives on Learning, Teaching and Assessment

Paul F. Conway and Finbarr C. Sloane



International Trends in Post-Primary Mathematics Education: *Perspectives on Learning, Teaching and Assessment*

Research report commissioned by the
National Council for Curriculum and Assessment

October 2005

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ISSN 1649-3362

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Acknowledgements

We are grateful to many people and organisations without whose timely and generous assistance and/or comments this report would not have been possible.

Individuals

Prof. Linda Allal, Prof. Sigrid Blömeke, Dr Caroline Brew, Paul Brady, Dr Stephen Buckley, Prof. David Clarke, Dr Sean Close, Dr Leland Cogan, Dr Judith Cosgrove, Simon Coury, Lorraine Crossan, Tom Daly, Claire Dooley, Dr Carol Gibbons, Prof. Jim Greeno, Tadelle Hagos, Hannah Joyce, Prof. Anthony E. Kelly, Kathrin Krammer, Dr David Leigh-Lancaster, John MacGabhann, Doreen McMorris, Barry McSweeney, Dr Kieran Mulchrone, Dr Tom Mullins, Marie Nash, Veronica O'Brien, Eileen O'Carroll, Elizabeth Oldham, Prof. Kurt Reusser, Dr Susan Sclafani, Prof. Denis O'Sullivan, Prof. Jack Schwille, Dr Ciaran Sugrue, Prof. Harm Tillema, Prof. Maria Teresa Tatto, Peter Tiernan. Prof. Yong Zhao.

We are especially grateful to Dr Sean Close and Elizabeth Oldham who provided assistance in a variety of ways, particularly in relation to the historical context of mathematics education in Ireland and the background to the PISA mathematical literacy framework.

We would like to thank members of the NCCA's Senior Cycle Review committee who provided thoughtful feedback on a presentation of this report's preliminary findings.

A special thank you is due to Simon Coury whose careful and timely proofreading of the draft document were important in completing this report.

Organisations

Boole Library, UCC; Department of Education and Science; Education Department, St. Patrick's College of Education; Library, Michigan State University; National Council for Curriculum and Assessment (NCCA); Office of the Chief Science Advisor to the Government; Printing Office, UCC; Research Office, UCC; Office of Marketing and Communications, UCC; Victorian Curriculum and Assessment Authority, Australia.

We would like to thank our colleagues in the Education Department at University College Cork (UCC), the Department of Counseling, Educational Psychology and Special Education, Michigan State University (MSU), the US National Science Foundation (NSF), and the College of Education, Arizona State University (ASU). Finally, we are grateful to the Executive of the NCCA: Dr Anne Looney, Chief Executive, John Hammond, Deputy Chief Executive, and Bill Lynch, Director, Curriculum and Assessment, whose support as well as detailed comments on a draft document were important in completing this research report.

Paul F. Conway and Finbarr C. Sloane

22nd October 2005

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Chapter 1

Mathematics education in an age of globalisation: *'we are all comparativists now'*

1.1 Introduction

An ideal education in which students have democratic access to powerful mathematical ideas can result in students having the mathematical skills, knowledge and understanding to become educated citizens who use their political rights to shape government and their personal futures. They see the power of mathematics and understand that they can use mathematical power to address ills in our society.

C. E. Malloy, Democratic Access to Mathematics through Democratic Education, 2002

Many important new developments have a mathematical basis. Documents and data sent electronically need to be securely and accurately transmitted. Cryptography is the science of the secret transmission of data, used nowadays on the Internet, by financial institutions and many others. Coding theory is used for the accurate transmission of data - error correcting codes are used, for example, in CDs. Formal methods provide a rigorous mathematical basis to software development and must be used for safety critical systems - needed in the aviation, military, medical and other fields. Storage, compression and recovery of large amounts of data are achieved using mathematical transforms - wavelet transforms are used to store fingerprints on FBI computers. Statistics and probability are an important part in the study of networking. The graphical images in Microsoft's Encarta encyclopaedia are stored in highly compressed form on the CD by means of techniques developed by mathematicians working in fractal geometry. Newer developments in the financial services sector need highly numerate and computer literate graduates. Neural networks are mathematical/computer models with a learning process that try to mimic the workings of the human brain. These have found extensive applications in many areas, as in the prediction and modelling of markets, signature analysis, selection of investments. Quote from a recent article: "Neural networking is the buzz word in the insurance industry."

Ted Hurley, Prof. of Mathematics, UCG, 'Computers need Maths', Irish Times, 4 February 1997

The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest…Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it…The reason for this failure of the science to live up to its great reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception (p. 1-2).

Alfred North Whitehead, An Introduction to Mathematics (1911)

…it has been widely recognised that mathematics occupies a rather special position. It is a major intellectual discipline in its own right, as well as providing the underpinning language for the rest of science and engineering and, increasingly, for other disciplines in the social and medical sciences. It underpins major sectors of modern business and industry, in particular, financial services and ICT. It also provides the individual citizen with empowering skills for the conduct of private and social life and with key skills required at virtually all levels of employment.

Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education, Roberts Report, United Kingdom (2004)

As articulated in Hurley's *Irish Times* article, an understanding of how mathematics is so crucial, and often central, to contemporary scientific developments underpins much of the ongoing drive to enhance mathematics education in the current wave of globalisation. This point of view could be characterised as 'mathematics for scientific advancement'. This 'mathematics for scientific advancement' position, together with the idea of 'mathematics for all', a democratic citizenship argument (see Malloy above), together represent two key forces that today drive the formulation of mathematics education policies. Consequently, few meetings about educational reform in any part of the world conclude without an endorsement of the central role that mathematics can and ought to play in education; the fruits of high-quality mathematics education, it is asserted, ensure a rich harvest for the economy and society. As such, mathematics is perceived as a high-yield school subject, especially so now in the current wave of globalisation with its attendant pressures on many economies to move up the value chain through the development of a 'high skill society' (Brown, Green, and Lauder, 2001). Despite what seems to be the unassailable position of those advocating enhanced mathematics education, either from the 'mathematics for all' or the scientific advancement positions mathematics is often perceived, as Whitehead so clearly articulates, as an impenetrable and abstract subject, and furthermore the teaching of it sometimes fosters such views. Thus, mathematics is seen simultaneously as increasingly important yet also as an especially difficult subject, despite its wondrous applications and essential role in today's world. Indeed, understanding our world today without the conceptual tools provided by mathematics would be unimaginable.

Mathematics, or more properly mathematical sciences, have played, it is argued, an important role as a powerful symbolic system leading and contributing to world-changing innovations in society, such as the development of new technologies, and a means of conveying in a powerful and compelling form significant applied and theoretical insights across a range of disciplines and professional fields.

1.2 Overview of research report

In the context of the review by the National Council for Curriculum and Assessment (NCCA) of mathematics education at post-primary level in Ireland, this study focuses on a number of key international trends in mathematics education. While it is cognisant of developments in Ireland, it does not focus on Irish developments per se. However, in the final chapter we note potential areas of overlap, areas where there are significant divergences and possible lines of development in mathematics education at post-primary level in Ireland.

The NCCA's companion document provides a review of developments in the Irish context (NCCA, April, 2005). In reviewing international trends in terms of their implications for both curriculum and assessment, we highlight selected countries and specific initiatives, and concentrate on a few key ideas: *new insights on mathematics teaching* from recent video studies; *new perspectives on learning and their implications for designing powerful learning environments;* and *conditions for high quality teacher education in mathematics* (including initial, early career and continuing professional development). Thus, we have chosen to focus on a few big ideas rather than adopting the encyclopaedic approach of numerous handbooks on mathematics published over the last decade (Grouws, 1992; English, 2002). In summary, the purpose of this report is to highlight the most significant trends in mathematics education internationally that might inform the current review of mathematics at post-primary level in Ireland.

Drawing on research in the learning sciences (Collins, Greeno and Resnick, 1996; Bransford, Brown, and Cocking, 2000), the study looks at developments in mathematics education within dominant approaches to learning as they pertain to current debates on the development of 'powerful learning environments' (De Corte *et al.*, 2003). The study is organised into five chapters. Chapter 1 will frame the study in terms of key concerns, images, and provision in relation to mathematics internationally. Chapter 1 provides an outline of the proposed study and reflects the outcome of consultation between the researchers and the NCCA with regard to the focus and content of this report, discusses the significance of mathematics education as an increasingly important curricular priority internationally, and describes key developments in relation to provision, curriculum and assessment in selected countries. As the current review of mathematics education in Ireland is focused on post-primary education, we look at the specific and changing educational goals and societal context of this phase of education. One of the main points we note in this first chapter is the manner in which countries are engaging in various cross-national efforts to enhance mathematics education.

Chapters 2 and 3 will address frameworks and philosophies that underpin mathematics education internationally, such as the realistic mathematics education movement and the concept of 'mathematical literacy' for all. Chapters 2 and 3 provide a review and analysis of two key trends in mathematics education internationally: (i) the cross-national efforts to understand the dynamics of mathematics teaching, and (ii) developments in the learning sciences, including their actual and potential impact on mathematics education. For the purposes of this report, Chapter 2 focuses on developments in understanding the teaching of mathematics, drawing primarily on compelling findings from the TIMSS 1995 and TIMSS-R 1999 video studies. As one of the chief insights of both video studies was the nature of culture-specific 'lesson signatures', the second section of Chapter 2 provides a review of developments in Japan and elsewhere in relation to lesson study as a form of professional development for teachers and what it suggests about curricular reform.

Chapter 3 addresses the impact of insights from research into the learning sciences on mathematics education, focusing on the current interest in brain-based research, the 'social' turn in learning theories, and the increasing prominence of self-regulated learning as a policy and research priority. The first section of the chapter provides an overview of learning sciences research and its role in understanding, designing and evaluating powerful learning environments in mathematics education. In examining behavioural, cognitive and socio-cultural/situated perspectives on learning, we note their impact, current status and potential to inform mathematics education. This section also describes the significant contribution of Han Freudenthal's views on the teaching and learning of mathematics and their influence on and relevance to contemporary debates on mathematics education. The subsequent section of this chapter outlines key developments in cognitive neuroscience and debates about the utility of such knowledge for cognitive psychology as a conceptual lens on the teaching and learning of mathematics.

Cognitive neuroscience has been the subject of considerable debate in education over the last decade; the OECD and US National Research Council among others have been examining its potential implications for education, especially literacy and numeracy. The final section of Chapter 3 reviews recent work on self-regulated learning (SRL), and considers its appeal to policy makers, researchers and reformers involved with mathematics education internationally, in a context where lifelong learning is becoming a priority. Building on Chapters 2 and 3, Chapter 4 will address key initiatives in mathematics education internationally to the extent that they illuminate issues of relevance to Irish post-primary mathematics education.

Chapter 5, the final chapter, notes key issues arising from previous chapters with a focus on issues that might form the basis for some of the discussion in the current review of post-primary mathematics education. This concluding section is framed within current curriculum developments in Irish post-primary education and discussions in Ireland of the 2003 PISA study of mathematics literacy (see Cosgrove *et al*., 2005), a video study of junior cycle mathematics education (Lyons *et al*., 2003) and recent research presented at the first Mathematics Education in Ireland conference (Close, Corcoran and Dooley, 2005).

1.3 What is mathematics?

> *A widespread public image of mathematics is that it is a difficult, cold, abstract, theoretical, ultra-rational but important and largely masculine. It also has an image of being remote and inaccessible to all but a few super-intelligent beings with 'mathematical minds'.*
>
> **(Ernest, 2004, p. 11)**

> *To most people, mathematics means working with numbers….this definition has been out of date for nearly 2,500 years. Mathematicians now see their work as the study of patterns real or imagined, visual or mental, arising from the natural world or from within the human mind.*
>
> **(Devlin, 1997, Mathematics: The Science of Patterns)**

In a classic text on the history of mathematics, Struik (1948) recognises and acknowledges reciprocal relationships between developments in mathematics and its historical and cultural contexts. In coming to an understanding of its origins, scope and influences, Struik reminds us that 'mathematics has been influenced by agriculture, commerce, and manufacture, by warfare, engineering, and philosophy, by physics, and astronomy' (p. 1). Furthermore, he notes that 'The influence of hydrodynamics on function theory, of Kantianism and surveying on geometry, of electromagnetism on differential equations; of Cartesianism on mechanics, and of scholasticism on calculus' (p. 1) all point to both the culturally-influenced nature of mathematics and the scope of mathematical ways of knowing. Struik's history of mathematics ends in 1945 because, he claims, 'the mathematics of the last decades of the twentieth century has so many aspects that it is impossible… to do justice even to the main trends' (p. 1). Tracing the origins of contemporary mathematics, over the last fifteen millennia, through the contributions of and affinity between Egyptian, Babylonian, Chinese, Indian, Arabian, Greek and other mathematical traditions (e.g. the less influential mathematical traditions of the Minoan-Mycenaeans, the Mayas and the Incas), Struik portrays mathematics as a dynamic and increasingly diverse field of knowledge influenced by but also influencing society in far-reaching ways. For example, he notes that 15,000 year-old cave paintings in France and Spain 'reveal a remarkable understanding of form; mathematically speaking, they reveal understanding of two-dimensional mapping of objects in space' (p. 9). These cave paintings were undertaken, as well as the more recent physical constructions in places such as Stonehenge in England, the pyramids in Egypt, or Newgrange in Ireland, point to sophisticated and relatively independent astronomical and mathematical knowledge traditions existing in different parts of the world. This brief reference to the history of mathematics illustrates one key position we adopt in this report - namely, the changing, dynamic and diverse nature of mathematical ways of knowing. As such, it reminds us that rather than being viewed as a timeless edifice, mathematics might more accurately be viewed as a complex growing tree. Indeed, the image of a tree has appealed to some mathematics educators (Romberg and Kaput, 1999). For example, Thurston's banyan tree metaphor has gained some adherents:

> *Mathematics isn't a palm tree, with a single long straight trunk covered with scratchy formulas. It's a banyan tree with many interconnected trunks and branches - a banyan tree that has grown to the size of a forest, inviting us to climb and explore. (p. 7)*

However, even adopting a historical perspective on the development of mathematics may not convey the current dynamic nature of evolving mathematical knowledge, in that it may be tempting to see historical aspects of mathematics as interesting and curious *ways of knowing* before the advent of a robust and relatively unchanging domain of mathematical knowledge in the more recent past.

The banyan tree image seems, at one level, overly abstract and, at another level, vivid and compelling. It clearly conveys a broad-ranging and expansive view of mathematics but in conveying a such a view, it could be argued that it might draw education and educators away from a focus on 'core' mathematical competences needed for living and learning in the 21st century. Indeed, over the last fifteen years there have been debates raging both nationally and internationally about the state and scope of mathematics education (and other curricular areas) in view of the perceived 'high skill' challenges of the 21st century. Debates about the state of mathematics education tend to focus on students' overall poor understanding of key mathematical concepts and procedures, typically reflected in routine rather than flexible use of mathematical ideas (APEC, 2004; Bramall and White, 2000; De Corte *et al.*, 1996;

Gardner, 1991). This has led to a world-wide drive to improve students' mathematical understanding. While the goal of teaching for mathematical understanding has been a goal for at least a hundred years (Romberg and Kaput, 1999), it has taken on a new meaning and urgency in current debates on the quality of mathematics education. These debates on the quality of mathematics education are occurring within the context of a wider global debate on the quality of educational outcomes (UNESCO, 2005; OECD, 2005). For example, in the European Union, the *Work Programme for 2010* has created a new tempo for European educational policies to focus on educational quality in order to meet the goals of Lisbon 2000 (Novoa and DeJong-Lambert, 2003; Conway, 2005a).

Debates about the scope of mathematics have been focusing on the extent to which school mathematics curricula are disconnected from real-world contexts and applications (De Corte *et al,* 1996), and the degree to which mathematics curricula ought to emphasise numeracy vis-á-vis the broader expanse of the mathematics curriculum; New Zealand's Numeracy Project[1] at post-primary level, for example, has been grappling with this issue. The current discussion, occurring in the context of the somewhat different foci of the Third International Mathematics and Science Study (TIMSS) and the OECD's PISA international comparative studies, captures this concern about the extent to which school mathematics is sufficiently connected to real-world contexts. In the IEA tradition, TIMSS and TIMSS-R tests were based on careful mapping of curricula across participating countries, whereas the PISA mathematical literacy studies are based on an agreed understanding of the type of *in and out-of-school mathematics* that students need to know for life and work in the 21st century. This has resulted in different focuses in the

[1] "The focus of the Numeracy Development Project is improving student performance in mathematics through improving the professional capability of teachers. It is intended that by 2005, almost every New Zealand teacher of year 1 to 3 children and many teachers of year 4 to 8 children will have had the opportunity to participate". http://www.nzmaths.co.nz/Numeracy/Intro.aspx.

TIMSS and PISA mathematics tests. The issues raised about both the content and format of mathematics tasks, surfacing in the different emphases of major and influential comparative studies, reflect a critical debate at the heart of mathematics education about the nature and scope of mathematics itself, and also of what counts as worthwhile mathematics for inclusion in school curricula and syllabi world-wide. For example, during 2005 in China, there have been 'math wars' a decade after the USA National Council of Teachers of Mathematics (NCTM)-modelled reform of mathematics curricula. As Zhao notes:

> *At the annual session of the Chinese National People's Congress (NPC), a group of members of the Chinese legislative body introduced a proposal calling for an immediate overhaul of the New Mathematics Curriculum Standards for elementary and secondary schools… The New Math Curriculum has been criticised for betraying the excellent education tradition, sacrificing mathematical thinking and reasoning for experiential learning, giving up disciplinary coherence in the name of inquiry learning, lowering expectations in the name of reducing student burden, and causing confusion among teachers and students. (2005, p. 1).*

According to Zhao, similar views have been voiced by leading Ministry of Education officials, parents, mathematicians, scientists and textbook publishers (see also S. J. Wang, 2005).

The debate in China over the last year is a good example of the internationalisation of mathematics education that has occurred in the field over the last forty years. As de Corte *et al*. note:

> *A further important development is the importance that the international dimension in mathematics education has come to assume in the last 20 years. Among the manifestations of this trend may be listed the growth of*

contact, collaboration, and international institutions; the carrying out of international comparisons of children's mathematical performance; and a more balanced recognition of the mathematical achievements of all cultures. (1996, p. 456)

The International Congress for Mathematics Education (ICME) has provided a forum for mathematics educators and other educational researchers working in mathematics to share ideas, and has ensured a flow of information and key ideas in mathematics education across national borders and across otherwise relatively impermeable political boundaries. For example, Hans Freudenthal was invited to China in the late 1980s to give a series of lectures on Realistic Mathematics Education (RME) in order to inform mathematics education reform proposals.

Mathematics as a 'basic' skill: three views

The focus on the 'real world' credentials of school mathematics overlaps in part with a debate about mathematics as a 'basic' skill in the sense that if maths is real it provides a basic, well-grounded or fundamental preparation for the future. Policy-wise and logically a strong and plausible case can be made for viewing mathematics as a basic skill. The policy and logic argument might run as follows: mathematics is a powerful symbolic system in itself and forms an essential foundation for a wide range of other disciplines and professional fields essential in fostering a 'high skill' society (see, for example, the opening quotations in this report). As such, it is 'basic' or foundational in that in its absence the economy and society would suffer in not having mathematically skilled workers and citizens ready to deploy and advance on these basic skills. This, however, is only one of the ways in which the term 'basic' is used to portray mathematics as important in a foundational sense (see the two collections of edited essays *Why learn maths?* [2000, Bramall and White] and *The maths we need now* [Tikly and Wolf, 2000] which discuss the role of mathematics and mathematics education in contemporary society).

A second and more problematic use of the term 'basic' in mathematics education comes from an epistemological and/or learning perspective. The argument runs as follows: mathematics - and especially core operations, concepts and procedures - can be ordered sequentially, and lines of development for learners can be specified reflecting a move from simple to more complex mathematics over time, as well as over the cycle of students' school careers. On the face of it, this argument seems compelling. Clearly, it is argued, some mathematics tasks are easier than others (complexity); clearly, it might be useful pedagogically to order operations, concepts and procedures from simple to complex (pedagogy); and clearly, mathematics, more than any other subject, reflects agreed-upon or high-consensus content (epistemology). We do not address all of these issues here, save to note that research into learning has demonstrated how even so-called 'basic' skills such as reading, writing and mathematics, even at their most elementary levels, involve complex and multi-dimensional cognitive and cultural factors (Means, Chelemer and Knapp, 1991). Means and Knapp note that cognitive science has provided ample evidence to question the logic that students should be given certain 'basic' skills in reading, writing and mathematics long before being exposed to more 'advanced' skills such as reading comprehension, writing composition and mathematical reasoning[2].

2 The argument we make here about reframing the relationship between what are termed basic and higher order skills in mathematics is very similar to the case made by some in science education that basic skills be taught in the context of investigations. (See for example Bennett, Lubben and Hogarth's [2003], review of context-based and Science-Technology-Society (STS) approaches in the teaching of secondary science.)

They note that:

> *By discarding assumptions about skill hierarchies and attempting to understand children's competencies as constructed and evolving within and outside of school, researchers are developing models of intervention that start with what children know and expose them to what has traditionally been thought of as higher-order thinking. (Means, Chelemer and Knapp, 1991, p. 181)*

Consequently, this second use of the term 'basic' to characterise mathematics can be problematic in optimally positioning mathematics within learning because it results in inappropriate skill hierarchies informing approaches to teaching; in curriculum because it misconstrues the relationships between different curriculum components; and in education policy debates because it can be used to argue for a *basics-first-and-advanced-skill-later* approach to guide the teaching of less able students (Means, Chelemer, and Knapp, 1991; Oakes and Lipton, 1998; Conway, 2002).

Finally, there is also a third sense in which 'basic' is often used, that is, to characterise the maths that is 'just enough' for weaker students, for students with special educational needs and for adults and returning learners. This use of the term basic emphasises the basic, functional and adaptive aspects of mathematics. As debates on mathematics unfold, it is important to distinguish between the various uses of 'basic' and the underlying assumptions of each.

1.4 Concerns about mathematics education

Internationally there are many concerns about the state of mathematics education. These concerns typically involve two sets of factors: that is, both *push* (poor levels of understanding and achievement gaps) and *pull* (the need for 21st century skills) factors.

In the case of *push* factors the focus tends to be on perceived poor levels of student understanding, as well as achievement gaps between boys and girls, between different ethnic groups, and between one country and another. Concerns about poor levels of understanding revolve around post-primary graduates' lack of capacity to apply mathematics in practical 'real world' contexts. Students' poor levels of mathematical understanding are typified by concerns about schools' focus on procedural, routine, inflexible, abstract, and inert knowledge rather than fostering students' capacity in conceptual, problem-focused, practical and flexible use of mathematical knowledge (de Corte, *et al.*, 1996; J. Wang, 2005). Summarising twenty years of research in the learning sciences, Gardner in his book *The Unschooled Mind* (1991) draws the attention of educational policy makers, educators and the wider public to the phenomenon whereby many degree holders do not have the capacity to apply their hard-earned knowledge appropriately in new contexts. He notes that there is now an extensive body of research examining this phenomenon in different disciplines and contexts and it clearly documents how those with advanced training are often unable to apply knowledge appropriately in new contexts or ill-structured problem domains. As Gardner notes,

> *Whether one looks at the physical sciences, the natural sciences, the human sciences, mathematics, history, or the arts, the same picture emerges: most students prove unable to master disciplinary content sufficiently so that they can apply it appropriately in new contexts. (p. 247)*

In an era of globalisation, with its increasingly rapid social, cultural and economic change, enhancing a system's as well as individuals' capacity to apply knowledge in new contexts is one of the key challenges education systems must address both within and across subject areas. In the context of mathematics, similar concerns have arisen based on observations of how students in school or after completion of school attempt to solve problems in routine fashion disregarding crucial information, especially contextual information (Verschaffel, Greer, and de Corte, 2000; Cooper and Harries, 2002).

A second long-standing push factor revolves around concerns about gender and ethnicity-related gaps in mathematical achievement. These concerns have been reflected in public debate, research and policy questions over many years. Historically, at least, concerns about gender differences in mathematical achievement have typically pointed out the consistent manner in which boys outperformed girls. More recently, the perceived underachievement of boys has become a source of concern across many countries, with girls now outperforming boys at the end of compulsory schooling in mathematics in many jurisdictions (see Elwood, 2005 for a detailed discussion of the role of examinations in the relationship between gender and achievement. Elwood's analysis focus on three aspects of examining: "styles of examinations and how they define achievement; coursework and the role it plays in contributing to gender differences in performance; and tiered entry systems in examinations and how they provide unequal opportunities for boys and girls to be successful", p.373).

Explanations about how to explain why boys outperformed girls historically revolve around two sets of factors: gender differences as deficiencies or hurdles (i.e. the spatial-visualization, mathematics anxiety, mathematics achievement, or the attribution patterns of girls) and social influences on mathematics participation (i.e. mathematics as a male domain, classroom culture, curriculum, and the manner in which teachers treated boys and girls differently). For example, the question as to whether there is a mathematics gene or not, and if boys were more likely to have it than girls, was the focus of a number of articles in the journal *Science* in the early 1980s (Benbow and Stanley, 1980; Kolata, 1981; Schafer and Gray, 1981). In line with the keen research and public interest in gender, genes and mathematics at that time, Kolata, for example, asked whether girls were born with less ability. The historical reality of girls' underachievement in mathematics has been and continues to be an issue in mathematics education in many countries. However, the nature of these concerns, and the related debates, are undergoing considerable change at present for reasons which include the increased opportunities for girls in mathematics and the more recent moral panic (mid-1990s onwards; see Skelton, 2001; Mac an Ghaill, Hanafin and Conway, 2004) about the perceived underperformance and low participation of some boys in mathematics - typically boys from poorer communities and/or low-status minority ethnic groups (Jacobs, Becker, Gilmer, 2001; Jencks and Phillips, 1998; Leder, 1992). Regardless of whether the concerns are about boys' or girls' perceived underachievement, they typically focus on the impact of stereotypical vocational choices and restricted labour market options on their future life-chances at personal, social and economic levels (Skelton, 2001).

In relation to ethnicity, the persistence of achievement gaps between ethnic groups has been a feature of debates in more ethnically diverse societies such as the USA, Canada, Australia and Germany. In the case of the USA, the perceived underperformance of Hispanic and African-American students in mathematics, compared to higher-performing Asian-American students, has led to extensive research on the cultural dimensions of mathematics achievement (Portes, 1996; Jencks and Phillips, 1998). Concerns about the underperformance of students from particular ethnic groups are typically part of wider debates on educational inequality, where achievement scores are one of the four primary indices of inequality, with years of education completed, grade retention and drop out or non-completion constituting the other three indices (Portes, 1996).

Ethnicity-related explanations of underachievement and inequality in mathematics and other subjects typically adopt on one of three positions: the cultural deficit, cultural difference, or post-cultural difference approaches (Portes, 1996). The cultural deficit approach 'draws attention to the adaptiveness in ethnic socialization practices and values relative to the norm, particularly in education' (Portes, 1996, p. 337). Cultural difference approaches focus on discontinuities between home and school cultures as reflected in different language practices, concepts and skills. Post-cultural difference models focus on one of two explanations: (i) double stratification of caste-like minorities and folk theory of success, or (ii) school-structured failure. The double stratification explanation focuses differences between the status of voluntary and involuntary minorities and subsequent school experiences of students from these distinctly different minorities (Ogbu, 1992). In general, voluntary minorities fare much better in the education system. For example, in the USA the very high performance of many students from East Asian immigrant voluntary minorities is contrasted with the underachievement in maths (and other subjects) of most African-American students who, as an involuntary minority, are subject to caste-like status rooted in a history of cultural and educational oppression. The second post-cultural difference explanation focuses on structural (e.g. curriculum, school organisation, textbooks) and/or functional (e.g. teaching practices and processes, assessments, time on task) that impede students from various ethnic groups in reaching their educational potential.

One of the distinctive features of recent analyses and commentaries on international comparative test results has been an intense media and research interest in understanding the reasons underlying the high performance of many East Asian countries. However, in their recent study of the role and conceptualisation of teacher knowledge in explaining the, more often than not, superior performance of China over the USA on international mathematics tests, Wang and Lin (2005) are very critical of the overly general categorisations of students from different East Asian cultural groups. Commenting on both the research studies and debates attempting to explain the cultural dimension of mathematics achievement, they note that:

> *…many are based on ambiguous cross-national categorization of East Asian students from Japan, China, Korea, and other East Asian regions and countries with little differentiation among them. Such ambiguous categorization is problematic considering that conceptual, institutional, and practical differences exist among those countries (Tobin, Wu, and Davidson, 1989). Similarly, U.S. students are often categorized as a homogeneous group without consideration of the similarities and differences among racial groups. For instance, findings on the performance of Asian Americans rarely distinguish among those whose ancestors are Japanese, Chinese, Korean, Vietnamese, Cambodian, Thai, Hmong, Laotian, Filipino, and so forth. Such broad categorizations may mask underlying ethnic and cultural differences and thus prevent adequate interpretation of differences related to student performance. (p. 4)*

Wang and Lin's comments and research point to the complexity involved in understanding the cultural dimension of mathematics achievement in an increasingly diverse world.

Cross-national studies in mathematics achievement tell a complicated story[3], raise many questions and tend to generate sporadic rather than

[3] For example, see O'Leary (2001) for a discussion of factors affecting the relative standing of countries in TIMSS.

sustained interest in educational matters. However, in many countries, the educational, public and media interest in the results of the IEA's TIMSS (1995), TIMSS-R (1999) and the OECD's Program for International Student Assessment (PISA) 2000 (where mathematics was a minor domain) and PISA 2003 (where mathematics was a major domain), have helped create and sustain a context in which national education systems are being held accountable to international standards in priority curricular areas. To cite a dramatic case, Germany's 'PISA-shock' in response to the mathematics literacy results in PISA 2003 is a good example of how international test results may have a wide ripple effect on public and educational opinion. In response to the perceived deficits in mathematics literacy in Germany, a new television programme on Saturday night, broadcast in a prime-time slot, presents PISA-like items in order to educate the public. In the USA, the *Christian Science Monitor*, in response to PISA 2003 results, had a headline 'Math + Test = Trouble for U.S. Economy'. The article noted that the performance of students on PISA's 'real-life' mathematics skill tests was a 'sobering' indicator of students' deficits on 'the kind of life-skills that employers care about.' Furthermore, comments by Ina Mullis, TIMSS co-director, about the USA's performance in TIMSS (2003), are indicative of a broad-based concern in the USA about its performance and ranking on international tests over the last decade: 'The United States has moved up a little, but there is still a huge gap between what our students can do and what students do in the highperforming Asian countries… Singapore has 44 percent of their students at the advanced level, while the United States has 7 percent' ('Math and Science Tests Find 4th and 8th Graders in U.S. Still Lag Many Peers', New York Times, 15 December 2004).

Moving beyond the performance of individual countries, international surveys in mathematics have complicated the interpretation of the gender dimension of test scores by illustrating how boys and girls differ in the manner in which they typically achieve their scores. Drawing on the results of IEA studies, Murphy (2001), for example, observed that, 'Girls' superior performance is associated with computation, whole number and estimation and approximation as well as algebra. Boys' superior performance is associated with geometry, particularly three-dimensional diagrams, measurement and proportional thinking' (p. 113). Situated within wider debates about the urgency of preparing students for knowledge societies in this era of globalisation, all of these analyses and commentaries point to the increasingly prominent international comparative context fuelling concerns about mathematics education.

With ever-increasing globalisation, educational policy-makers are identifying the new skills that students will need in order to prepare them to live in this century's knowledge society: the *pull* factor. Gardner (2004) classifies the knowledge, skills and dispositions needed in pre-collegiate education in an era of globalisation. They are:

- understanding the global system
- the ability to think analytically and creatively within disciplines
- the ability to tackle problem issues that do not respect disciplinary boundaries
- knowledge of other cultures and traditions as an end in itself and as a means to interact with people from other cultures
- knowledge of and respect for one's own cultural traditions
- fostering of hybrid or blended identities
- fostering of tolerance and appreciation across racial, linguistic, national and cultural boundaries.

In the context of mathematics education, some items on Gardner's list seem particularly relevant, such as the capacity for and importance of analytical and creative thinking, and problem solving within and across disciplinary boundaries.

1.5 Why is maths taught the way it is? Curricular cultures, textbooks and examinations/testing traditions

A wide variety of factors interact to shape the quality and content focus of the mathematics that is taught in schools. These include system-level features, wider community and social-cultural factors, approaches to teaching, and student characteristics such as motivation and ability. In this section, we focus on three system level features which profoundly shape the nature of students', teachers' and families' experiences of mathematics education: disciplinary/curricular culture in mathematics education, mathematics textbooks, and examination/testing traditions.

Curricular cultures in mathematics education: 'new/modern' maths and 'real world' maths

Two curricular cultures have been influential in mathematics education over the last forty years: a formal and abstraction-focused approach known as 'new' or 'modern' mathematics, and a more context-based, real-world and problem-focused mathematics education emanating from different sources including Piagetian constructivism, realistic mathematics education (RME) and situated cognition.

'New/modern' mathematics education

The 'new/modern' mathematics culture, especially influential in post-primary mathematics curricula of many countries since the early

1960s, has come under increasing scrutiny over the last decade or more. In Europe, it was given considerable support and impetus as an approach to mathematics education due to two main factors: an OECD-sponsored conference at Royaumont in 1959 (Goffree, 1993) and the ideas of the French Bourbaki school of mathematicians, whose abstract and logical views of mathematics had been percolating through the culture of university mathematics departments since the 1930s. In the USA, both a major conference at Woods Hole in 1959 and efforts to create a renewed focus on high-level mathematics in school curricula, in response to Russian's Sputnik achievements in space in 1958, culminated in a series of mathematics education reforms being identified, prominent among these the potential role of 'new math' (English and Halford, 1995). As a curricular culture, 'new' or 'modern' mathematics elevated abstraction as the most important value in mathematics education. This resulted in a formal, highly structured approach to teaching, with a focus on systematising mathematical ideas using mathematical symbols. As Goffree (1993) notes

> *Modern Mathematics is, proportionally speaking, a slight amount of knowledge with which you can do proportionally much more. Or, in other words, the more abstract mathematics is, the more applicable it can be. Abstract means: loose from context. (p. 29).*

English and Halford identify the key characteristics of the new math approach as a heavy emphasis on logical aspects of mathematics, explicit teaching of set theory, the laws of arithmetic, and Euclidean geometry. They note that these emphases were reflected in post-primary mathematics education textbooks' foci on set theory, systems of numeration and number theory. According to English and Halford (1995), 'The explicit teaching of the notation and algebra of sets and the general laws of arithmetic were by their very nature abstract' (pp. 7-8). As we noted above, much of the initial impetus for the development of new mathematics came from the universities through the impact of the Bourbaki school of mathematicians' who had been seeking to formulate a complete and integrated approach to mathematics. Their views had a profound impact on the philosophy of mathematics adopted by university mathematics departments in many countries from the 1930s right up to the present day, with far-reaching implications on the structure of the discipline as well as preferred approaches to teaching.

Mathematics in context: real-world problem-focused mathematics education

A second curricular culture, an approach to mathematics focused on real-world problem solving, has come to increasing prominence over the last two decades in mathematics education at second level in many countries. Its origins are less clear than the new mathematics movement and can be traced back to the influence of at least three major strands in mathematics education: Piagetian constructivism, realistic mathematics education (RME) and situated cognition. Two of these, Piagetian constructivism and situated cognition, have their roots in cognitive, educational and developmental psychology. RME grew out of the work of Hans Freudenthal, a Dutch-based mathematician turned mathematics educator, whose ideas have had a profound impact on mathematics education internationally. The importance of real-world contexts, as both a source of learning and site in which mathematical ideas can be applied, is perhaps the key idea uniting these different influences on mathematics education. Writing in the RME tradition, one of Freudenthal's colleagues notes that in RME '…reality does not only serve as an application area but also as the source for learning' (Goffree, 1993, p. 89). This position is in marked contrast to the new mathematics stance, whereby context is downgraded in favour of abstraction, and application is at best merely an add-on to abstraction.

Situated cognition and Piagetian constructivism share a number of features: a view of learners and learning as active; a view of knowledge as something constructed by learners in social and material contexts; and learners' sense-making as an important basis for the design of teaching and learning environments. Together with RME, they have now come to the forefront of thinking on mathematics education internationally, as evidenced by the OECD's adoption of situated cognition and realistic mathematics education as the bases for their preferred orientation to learning in the design of the PISA mathematics literacy study. In contrast with new/modern mathematics, the real world, context-focused mathematics education approach puts a premium on rich, socially-relevant contexts for learning, problem solving and the relevance of learners' prior experiences in planning and engaging with learners in classrooms or in multimedia learning environments (MMLEs). This move toward a more socially embedded view of mathematics reflects a wider change in the relationship between school and society as curricular knowledge is renegotiated in an era of globalisation.

"It's in the book": textbooks' role in shaping mathematics education

> *…a book is designed to give an authoritative pedagogic version of an area of knowledge.*
>
> **(Stray, 1994, p. 2)**

> *Textbooks are artefacts. They are a part of schooling that many stakeholders have the chance to examine and understand (or*

> *misunderstand). In most classrooms they are the physical tools most intimately connected to teaching and learning. Textbooks are designed to translate the abstractions of curriculum policy into operations that teachers and students can carry out. They are intended as mediators between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms. Their precise mediating role may vary according to the specifics of different nations, educational systems and classrooms. Their great importance is constant.*
>
> **(Valverde, Bianchi, Wolfe, Schmidt and Houang, 2002, p. 2)**

Textbooks play a vitally important role in shaping teachers', students' and families' views of various school subjects (Ravitch, 2003; Valverde *et al.*, 2002). Textbooks themselves reflect particular views of a disciplinary/curricular culture, even if this curricular culture is not necessarily made explicit in the textbooks themselves. Thus, textbooks written in the new or modern mathematics education tradition are likely to differ very significantly in both form and content from textbooks inspired by alternative views of mathematics education, such as Realistic Mathematics Education.

In this section, we draw on the results of the largest ever cross-national study of textbooks, which was undertaken as one of three components in the Third International Mathematics and Science Study (Valverde *et al.*, 2002). The TIMSS curriculum analysis examined mathematics and science textbooks from 48 countries and encompassed a detailed page-by-page inventory of the maths and science content, pedagogy, and other characteristics of hundreds of textbooks in the participating countries for three different age groups. Based on their findings, the authors discuss the rhetorical and pedagogical features in order to understand how textbooks create or constrain students' opportunities to learn complex, problem-solving oriented mathematics or science. In particular, they note the key role that textbooks can play in curriculum-driven educational reform 'in light of their role as promoters of qualitatively distinct educational opportunities' (Valverde *et al.*, 2002). One crucial question Valverde *et al.* (2002) examined was the extent to which textbooks' variability across countries could be linked to cross-national differences in achievement. For the purposes of this report, we concentrate mainly on their findings in relation to mathematics textbooks, focusing on their findings in relation to textbook characteristics, the relationship between textbooks and achievement, and the role of textbooks in translating policy into practice.

Adopting a tri-partite model of curriculum, that is, the intended, implemented and attained curriculum - Valverde *et al.* regard textbooks as mediators of the intended curriculum in that they 'reflect content in their own way - a way that is meant to be suggestive of the enactment of intention. This is done through paradigmatic lessons. Across most TIMSS nations, textbooks are made up of lessons. These lessons are units of instruction written to model valid ways to accomplish the substance of the book' (p. 12).

The TIMSS textbook analysis adopted five measures of textbook characteristics:

- the nature of the pedagogical situation posed by the textbook (e.g. a piece of narration to be read; a set of exercises to be worked on; or an example of how to solve a problem)

- the nature of the subject matter in the textbooks - not in terms of mathematics but rather in terms of topics included, whether they were concrete or abstract, and the degree of complexity of the topics

- sequencing of topics; that is, the number of times attention shifts from one to another
- the physical characteristics of the textbooks; that is, their size, length, weight, etc.
- the complexity of student behaviour 'textbook segments are intended to elicit' (p. 14).

A preliminary analysis, using multivariate analysis of variance (MANOVA), on four hundred textbooks revealed that 'textbooks exhibited statistically significant differences from each other on… characteristics, and that those differences were related to the subject matter of the book; the grade level of the student for which the books were intended; the region of the world in which the textbook was used; and the particular country within each region' (p. 16). Valverde *et al*. strongly emphasise how the variability in textbooks in relation to 'substantial differences in presenting and structuring pedagogical situations' (p. 17) debunks the notion that mathematics has the same connotations in different cultures.

A detailed exposition of the results of textbook characteristics analyses is beyond the scope of this report, but we note key findings. First 'textbooks around the world differ greatly in size, length, and other structural features. They also differ greatly in the types of units they contain and the ways they are laid out' (p. 21). Perhaps the most interesting and thought-provoking finding in relation to textbook size was that it was the sole area in which the USA was number one in the world (Valverde, 1999). Of particular significance was that high-scoring countries on TIMSS (1995) often had compact, focused, light textbooks with in-depth rather than brief focus on topics. In the light of the USA's low rankings in the 1995 TIMSS, the overall findings of the textbook study led US researchers to characterise its textbooks as 'a mile wide and an inch deep' in curricular terms, thereby providing very poor learning opportunities for students. Consequently, US mathematics textbooks were seen as providing a 'splintered vision' of mathematics with serious consequences for how teachers think about mathematics as subject matter, plan and teach lessons, as well as how they think about what it means to know and understand mathematics. Both phrases ('a mile wide and an inch deep' and 'a splintered vision') have come to take on rhetorical power in helping to reshape the form and content of mathematics textbooks and teaching in the intervening years (Cogan, 2005). Internationally, the TIMSS textbook study has provided a context for more careful analysis of how textbooks shape opportunities to learn in mathematics and science. This concern has been given particular impetus since the TIMSS textbook study revealed that, despite bold and ambitious curricular aims of promoting problem-solving and mathematics more focused on the real world, the textbooks analysed in TIMSS (Irish mathematics textbooks were included in two of the three populations) did not live up to these lofty goals. As Valverde *et al*. note:

> *Unfortunately, we have seen here that the textbooks from TIMSS countries are often poor exemplars of such a vision. New subject matter may enter textbooks swiftly but new ways of exploring subject matter in classrooms were largely absent. It is no doubt to be expected that current policies for educational change start out with documenting bold new visions of school mathematics and science. However, it is disquieting that the primary relationship between system-wide policies and the achievement of children is through textbooks. Clearly textbook embodiments of bold new visions in the TIMSS study were very rare. (2002, pp. 170-71)*

The message from the TIMSS textbook study is loud and clear: there is a mismatch in many countries between reform goals in mathematics and the actual mathematics embodied in textbooks. This observation provides a very real challenge for those interested in changing the practice of mathematics education in schools. There have been no studies of post-primary mathematics textbooks in Ireland[4]. However, we note that Looney (2003), in research with teachers working in the support services for post-primary, found that they believed the textbook was more influential than the curriculum in making decisions about classroom teaching. In conclusion, the findings of TIMSS textbook study highlight the manner in which mathematics textbooks and other organised resource materials, function as a potentially implemented curriculum, and thereby help us understand how textbooks act as mediators between intention and implementation. We now turn to tests and examinations which provide another, perhaps more powerful, means of putting in place a vision of mathematics.

Testing and examinations

> *Testing and assessment have been both the focus of controversy and the darling of policymakers…testing and assessment can be externally mandated. It is far easier to mandate testing and assessment requirements at the state or district level than it is to take actions that involve actual change in what happens inside the classroom.*
>
> **(Linn, 2000)**

The use of tests and examinations to select the most suitable for specific positions in society has a long history, going back to tests used to determine entry into the Chinese civil service over four thousand years ago (Zwick, 2002). In the modern era, tests and examinations were given new impetus with the Enlightenment and its faith in

4 Irish mathematics textbooks from 3rd/4th class (primary) and 1st and 2nd year (post-primary) were included in the sample of textbooks in the TIMSS Mathematics Textbook Study.

scientific thinking, rational systems and forms of organisation that could create tests that would more accurately and efficiently determine human abilities and achievements (Broadfoot, 2001). Various testing and measurement technologies, with complex statistical apparatus, have been developed over the last two centuries, and used to identify competence and make fair and just selections in order to allocate people for further training where there were scarce societal resources. Given the perceived and actual close links between test/examination results and future earnings, learners and their families have been acutely tuned to the types of knowledge required to do well in tests and examinations. The post-Enlightenment move away from craft or practical knowledge - highly valued for centuries in trades and crafts - toward the capacity to retain and reproduce knowledge in response to paper-and-pencil tests, has created a situation where retained cognitive knowledge has been elevated over more practical modes of knowledge display (Wells and Claxton, 2002).

One of the major ways in which exam and testing traditions determine curriculum is by shaping what is deemed valuable knowledge. As one Leaving Certificate student, interviewed by a national newspaper after securing high points, said, 'There's no point in knowing about stuff that's not going to come up in the exams.' Of course, if the tests are good in terms of what knowledge they seek to examine and the modes they use to do this, one might argue that this student's strategy is laudable. However, examinations have a powerful backwash effect on curriculum, shaping both what is taught and how it is taught, and often narrow the frame in terms of what counts as worthwhile knowledge. On the other hand, good tests may actually broaden and deepen the quality of what is taught. As such, the actual social and academic effects of tests and examinations may be productive or counterproductive (Mehrens, 1989; Elwood and Carlisle, 2003).

How do the combined effects of curricular cultures, textbooks and exam/test traditions shape how mathematics is taught in schools? In the case of mathematics education at second level in many countries, the impact of the often dominant new mathematics curricular culture and textbooks congruent with this tradition subtly but very powerfully keep in place a view of mathematics that prizes abstraction over concrete experience, formal knowledge over informal knowledge, logical thinking over intuition, reproduction over creative thinking, and routine procedures over problem-solving and problem posing.

1.6 Mathematics education as policy priority

One the major educational features of the current wave of globalisation is a restructuring of the school-society relationship in terms of attention to subject matter content, approaches to teaching, and accountability regarding the outcomes of schooling at school and systemic levels. In the context of setting reform agendas in both primary and post-primary education, globalisation has led to laser-like focus on the knowledge-fields of mathematics and science as they are perceived as providing a greater dividend for, and being more connected to, the marketplace than some other school subjects (Stromquist and Monkman, 2000). Furthermore, Stromquist and Monkman argue that the current wave of globalisation has led to increased attention to pedagogies oriented toward problem-solving and a heightened emphasis on issues of efficiency; note, for example, the growing importance accorded to performance in mathematics and reading tests (Tatto, in press). Across OECD countries, the intense interest in the results of the OECD PISA international comparative assessments of reading, mathematical and scientific literacies is reflected in the burgeoning academic and media publications focused on reporting results, reflecting on the policy implications for education systems, and reframing reform agendas. In the following section, we note the growth of an unprecedented international comparative movement in mathematics education during the last decade, and we highlight developments in post-primary mathematics education within APEC (Asia-Pacific Economic Cooperation) countries (Andrews, 2001a; Andrews, 2001b; Jaworski and Phillips, 1999; Brown, 1999). We then review developments in a number of selected countries: Australia, Japan, Singapore, the United Kingdom and the USA. We adopt a regional or trans-national focus in reviewing policy developments in mathematics education because increasingly countries are banding together at a regional level, in the face of globalisation, in order to calibrate and guide reforms across a wide range of areas, including education, frequently through a peer-review and policy-advisory process. For the purposes of this report, we focus on mathematics education in selected APEC countries for a number of reasons. Firstly, many countries scoring highest on mathematics achievement in TIMSS 1995, TIMSS 1999 and in mathematics literacy in PISA 2000 and PISA 2003 come from this region. Of particular interest in the TIMSS 1995 results was that Japan not only had high aggregate (mean) scores but also had a low level of variation around the mean score. This indicates that overall students, regardless of ability, were taught to a high level. From a policy perspective, Japan's results suggest it had addressed both excellence (high aggregate scores) and equity (little variation about the mean) issues, which is a perennial tension in educational policy and practice (see Table 1). Secondly, significant differences in how high-scoring countries attain high standards in mathematics point to the futility of trying to identify and borrow one or two key factors as reform levers in order to bolster mathematics attainment in any curricular reform efforts. Thirdly, on-going discussion in APEC about educational reform is focusing on how eastern and western education systems might learn

Table 1: TIMSS 1999 video study: participating countries and their average score on TIMSS 1995 and TIMSS 1999 mathematics assessments

	TIMSS 1995 mathematics score		TIMSS 1999 mathematics score	
Country	Average	Standard Error	Average	Standard Error
Australia	519	3.8	525	4.8
Czech Republic	546	4.5	520	4.2
Hong Kong SAR	569	6.1	582	4.3
Japan	581	1.6	579	1.7
Netherlands	529	6.1	540	7.1
Switzerland	534	2.7	-	-
United States	492	4.7	502	4.0
International average	-	-	487	0.7

from each other. In these efforts, policy makers have tried to be sensitive to the varying relationships between national societal priorities and educational practices. This focus within APEC on the need to identify country-specific reform policy instruments, taking account of both a country's existing traditions and its capacity for change, is, we think, valuable and reflects an informed self-review (Le Metais, 2002) approach to educational reform, rather then mere policy borrowing. Finally, given the prominent role the USA plays in setting educational research agendas in mathematics education, reflected in extensive funding over the last decade directed at helping US educators understand how they might learn from the higher-performing East Asian countries, we note some of the issues raised about substantial cross-national differences in 'curriculum policies and materials (such as content coverage, instructional requirements, and structures) but also in how policies and materials are developed in the United States and high performing East Asian countries' (J. Wang and Lin, 2005 p. 3).

1.7 Trans-national alliances in mathematics education policy

'We are all comparativists now'

(EU politician/educator, cited in Phillips, 2003)

One of the distinctive features of contemporary education reform is the focus on international comparative approaches in the curriculum. The development of national education systems has been a long-time focus of international collaboration. For example, a number of key high-level meetings in the late 1950s and early 1960s provided a context for significant increase in investment in education systems in many developed countries, based on a human capital view of education where expenditure came to be seen more as an investment than as a cost for national governments (Husén, Tuijnman and Halls, 1992). More recently, during the last decade, moving beyond what Brown *et al*. (2002) call an older 'walled' economies' approach to economic and educational development, there has been a significant shift toward international collaboration in understanding and improving educational practice in priority curricular areas. Whereas the pioneering work of the IEA, in both the First International Maths Study (FIMS) and Second International Maths Study (SIMS), focused solely on educational achievement, that is, a 'cognitive Olympics' approach, both the Third International Mathematics and Science Study (TIMSS, 1995) and its follow up study TIMSS-R (1999), adopted a much broader remit, focusing also on curriculum textbooks (TIMSS, 1995), a three-country video study (1995), and a seven-country video study (1999). The combined effect of these IEA studies, and the more recent OECD PISA studies (2000 and 2003), has been to unmistakably shift national debates on priority curricular areas into an international arena. As such, over the last decade many countries, either directly or indirectly, have been debating curricular approaches to the teaching and learning of mathematics informed by results of four international comparative studies of mathematics (TIMSS, 1995; TIMSS-R, 1999; PISA, 2000; and PISA, 2003) as well as the compelling insights from both TIMSS 1995 and TIMSS-R 1999 video studies. As many East Asian countries have performed at or near the top of these various international mathematics studies and the 'lesson signature' of Japanese mathematics lessons in the 1995 video study inspired a follow-up video study, we review some of the recent policy trends within the APEC region.

At the third APEC Education Ministerial Meeting in 2004, which focused on 'Skills for the Coming Age', priorities which had been identified over the previous years came together in four thematic strands that reflected the emerging educational needs of the APEC region. 'Stimulation of learning in mathematics and science to promote the technical skills necessary for the 21st century' was one of these four themes. Providing high quality exposure to mathematics and science is viewed as a policy priority within the context of their integration, with overarching educational goals directed at improving students' communication skills, facility with ICTs, and capacity for self-directed learning (APEC, 2004, p. 62). Among the key findings was the manner in which the wider educational governance and societal contexts and values affected approaches to mathematics and science education, as well as attempted curricular reforms in these subjects.

> *Eastern APEC economies tend to do well at promoting and aligning core content knowledge in standards, curricula, assessment and teacher training, but often in inflexible ways that shortchange the needs of individual students and teachers. In Western economies, on the other hand, stress is*

placed on individuality and on real world applications of mathematics and science, but core content is not always well taught or understood by all groups of students. (p. 62)

Both key research findings and reform trends across the region were presented under three headings: (i) curriculum/standards, (ii) assessment and (iii) teachers (see APEC, 2004; Park, 2004).

National initiatives in mathematics education policy

In this section, we discuss post-primary mathematics education in selected countries, focusing on: current policy concerns, current trends in mathematics curriculum, and assessment reform initiatives. In a 16-country study looking at the organisation of mathematics curriculum from ages 4-14 (compulsory schooling), Ruddock (1998) divided countries into two groups based on their educational-political governance structures:

- group A - centralised government: England, France, Hungary, Italy, Japan, Korea, the Netherlands, New Zealand, Singapore, Spain, Sweden
- group B - federal government: Australia, Canada, Germany, Switzerland, USA.

Ruddock noted that, 'There are considerable differences between the two groups in their organisation of mathematics education for ages five to 14, but regional flexibility and differences in local implementation are not unique to federal states.' (1998, p. 2). Ruddock contrasts the emphasis on local implementation in Hungary, Italy, and Spain with Singapore's highly specific guidelines curriculum, Japan's somewhat specific approach through national guidelines, and Sweden's specification of minimum attainment targets. Noting the governance context of Group B countries, she says that 'a national mathematics curriculum as such can only exist provided the regions agree to it both in principle and in practice' (p. 2). In Ireland, there is a national rather than regional approach to curriculum at primary level and syllabus development at post-primary level. Highly decentralised educational governance systems make curriculum reform efforts difficult, especially when it comes to formulating an 'instructional guidance system' to ensure coherence in initiatives intended to improve teaching and learning (Cohen and Spillane, 1992). In this report we can group countries we focus upon into:

- group A – *centralised government:* England, Japan, Singapore,
- group B – *federal government:* Australia, USA

Ruddock notes that the 'Division of the mathematics curriculum into content areas is fairly consistent in the examples available for study, although there are variations in terminology, and not every system includes each content area for a particular age group' (p. 2). She noted the existence of 'basic content building blocks across the curricula studied': number, algebra, geometry, measures, probability, and statistics. She observed that probability was 'absent from several curricula'. In an effort to tabulate the common trends across contexts, she notes that only one element did not fit easily and that was the 'Attitudes and Appreciations section of the Australian National Statement, which is also present in the Singapore framework' (p. 3). Given the importance of promoting an appreciation of mathematics this is a surprising finding. Ruddock emphasises national differences in 'how mathematical process is viewed, sometimes being regarded as a content area, but otherwise seen as cutting across content areas. This aspect of mathematics is absent from the curricula for Korea and Italy… In Singapore, this aspect of mathematics is central to the curriculum framework, but is not used as a content heading in the later primary years' (1998, p. 6). In addition to the specification of curricular content, Ruddock notes that a number of countries also specify performance standards and in some cases include annotated examples of student work at different levels of competence to assist teachers in understanding and implementing the intended curriculum. Ruddock also outlined other efforts being made to assist teachers in understanding reform initiatives. She notes that Singapore, due to its relatively small size relies on meetings with all teachers 'to introduce and implement changes in curriculum, and places particular stress on clarifying the aims of reforms for teachers' (p. 9). Furthermore, the use of assessment is evident in a number of jurisdictions as a means of 'clarifying the meaning of standards [as] is apparent in several systems, for example in Sweden [which has] detailed scoring instructions for tests'. She also drew attention to how national tests in England have shaped teachers' perceptions and 'understanding of the curriculum and their own classroom assessment against performance standards' (p. 10). Ruddock observed that it was difficult to determine the extent to which topics were repeated across years in the various curricula. Given the scale of Ruddock's study, this is not surprising, and it is only with larger and more intensive studies like the TIMSS textbook analyses that the depth and recurrence of particular topics can be traced across time. Furthermore, while Ruddock's study provides a useful set of findings it was not detailed in its analyses of the impact the intended curriculum has on learning, given its focus on curriculum policy documents. As such, it is more distant from the actual enacted curriculum than the TIMSS cross-national textbook study. Nevertheless, Ruddock's study is helpful in providing a curricular context for mathematics education. In the next section, we focus on a selected number of countries, highlighting notable developments in mathematics education.

Mathematical literacy, computing and algebra in Australia

The phrase 'mathematical literacy' is now being used to describe students' capacity to use their mathematical knowledge for informed citizenship. PISA, the new international assessment of 15-year-old students which was conducted under the auspices of the OECD (2001), defines mathematical literacy as an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to engage in mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD PISA, 2001).

The PISA study draws its understanding of mathematical literacy from socio-cultural theories of the structure and use of language. Gee (1996), for example, uses the term *literacy* to refer to the human use of language, the primary function of which is to scaffold the performance of social activities. A literate person knows the resources of the language and can use these for different social functions. In the same way a mathematically literate person knows the resources that mathematics offers and can use these for a variety of purposes in the course of everyday or professional life.

The resources that mathematics offers to a problem-solver include facts, concepts and procedures. Concepts provide the way in which a situation is understood and mathematised so that a problem can be crystallised in mathematical terms, and the procedures are needed to solve the mathematical problem. To harness the power of mathematics, students need to know facts, concepts and skills, the structure of ideas in the domain and how a situation can be mathematised. New technologies are altering the essential understandings of all of these.

Recent work in Australia, and most particularly Victoria, highlights Australia's recent focus on mathematical literacy, especially as it relates to the learning of algebra in the school system with the support of modern technologies. Australia's specific focus has been on the value added by the use of computed algebra systems (CAS): recent work in Australia, and most particularly Victoria, highlights Australia's recent focus on mathematical literacy, especially as it relates to the learning of algebra in the school system with the support of modern technologies. Australia's specific focus has been on the value added by the use of computed algebra systems (CAS) (Leigh-Lancaster, 2002; Leigh-Lancaster, et al, 2003; Stacey, et al, 2000). Their work (sponsored by the government) has focused on changes in the algebra curriculum, in supporting pedagogy and in assessment (classroom and state-wide assessment), with the following goals for mathematical literacy in mind:

- to make students better users of mathematics, by providing the possibility of working on more realistic problems and releasing curriculum time from learning procedures to reallocate to problem solving and modeling
- to increase congruence between mathematics done at school and in the world of work, by using a modern technology
- to achieve deeper learning by students, by using investigations and demonstrations not previously possible

- to promote a less procedural view of mathematics, by shifting curriculum emphasis from routine procedures to solving problems and investigating concepts
- to introduce new topics into the curriculum, using time freed by removing selected routine procedures which are well performed by CAS and have few other pedagogical benefits
- to increase access to mathematics by students with inadequate algebraic skills.

Interestingly, the final goal, increasing access to mathematics by students with inadequate algebraic skills, was one not anticipated by the government, or the researchers, but was by mathematics teachers. These goals are discussed by Stacey, Asp and McCrae (2000), along with the mechanisms by which CAS may help achieve these goals. The development of CAS in Victoria, Australia reflects long-standing collaborative research and policy-making initiatives between academics (e.g. Ball and Stacey, 2005; Stacey, et al, 2000) and the Victorian Curriculum and Assessment Authority (VCAA) (Leigh-Lancaster, et al, 2003). Furthermore, the VCAA also worked closely with other systems and jurisdictions that are using CAS technology in secondary mathematics (e.g. The College Board in the USA, Danish Baccalaureate, French Baccalaureat, IBO) (Leigh-Lancaster, et al, 2003). According to Leigh-Lancaster (personal communication) "the developments in Victoria could only take place within a positive policy orientation towards the use of technology in mathematics curriculum, pedagogy and assessment, including examinations. It is in this regard that the Authority and Board has provided clear and definite leadership and support – the assessment aspect is the hard part – many systems/researchers etc. have been positive about the curriculum and pedagogical inclinations. It takes a robust and progressive conceptualization and resolve to move on the assessment agenda and related issues". The VCAA views the CAS as meeting a wide range of ambitious mathematics teaching goals as well as addressing teachers' pragmatic concerns about the actual teaching of mathematics. According to the VCAA's Mathematics Manager (Leigh-Lancaster, personal communication) these include:

- the possibility for improved teaching of traditional mathematical topics
- opportunities for new selection and organisation of mathematical topics
- access to important mathematical ideas that have previously been too difficult to teach effectively
- as a vehicle for mathematical discovery
- extending the range of examples that can be studied
- as a programming environment ideally suited to mathematics
- emphasising the inter-relationships between different mathematical representations (the technology allows students to explore mathematics using different representations simultaneously)
- long and complex calculations can be carried out by the technology, enabling students to concentrate on the conceptual aspects of mathematics
- the technology provides immediate feedback so that students can independently monitor and verify their ideas

- the need to express mathematical ideas in a form understood by the technology helps students to clarify their mathematical thinking
- situations and problems can be modelled in more complex and realistic ways.

In recent years, "the VCAA has acknowledged and responded to the various reservations and concerns expressed about the use of technology, including by those with a generally positive inclinations, as well as those who are reserved or negative" (Leigh-Lancaster, personal communication). After an extensive consultation on examination models in December 2003 (VCAA, 2003), the VCAA opted for an examination structure that included separate technology free and technology assumed access examinations for 2006 to 2009 (VCAA, 2003).

Post-primary mathematics education in Japan

In Japan, Munbusho, the Japanese Ministry of Education, sets the number of class periods for the year, the length of the class periods, the subjects that must be taught, and the content of each subject for every grade from Kindergarten to 12th grade (K-12). For this reason, changes in the Japanese educational system are usually introduced more cautiously than in, for example, the United States, and possible curriculum revisions are evaluated more carefully before being put into effect. Technology-based courses of the type that one often sees in US classrooms are not as popular in Japan, and Japanese educators generally seem to prefer a more traditional, theoretical, and problem-solving based course. Even though the current curriculum standards encourage the use of calculators beyond the fifth grade, calculators are still not allowed in many Japanese classrooms, since university entrance exams do not permit their use. Computers seem to be more prevalent in the Japanese classroom than hand-held technology.

The elementary school curriculum is specified in Japan for grades 1-6[5]. The objectives of mathematics education at the elementary school level are to develop in children fundamental knowledge and skills with numbers and calculations, quantities and measurements, and basic geometric figures. The emphasis on geometry is far greater than in any of the other OECD countries.

Lower post-primary school in Japan consists of grades 7-9. Preparation to get into the best high schools and universities begins at this time. There is tremendous pressure on students to perform well. Students are asked to learn a tremendous amount of material in grades 7-12, which is perhaps one of the major reasons why university and postprimary school classrooms are often subdued. In

5 In grades 1-3, children learn about the concept of numbers and how to represent them, the basic concepts of measurement, how to observe shapes of concrete objects and how to construct them, and how to arrange data and use mathematical expressions and graphs to express the sizes of quantities and investigate their mathematical relationships. They acquire an understanding of addition, subtraction and multiplication, learn how to do basic calculations up to the multiplication and division of whole numbers, and learn how to apply these calculations. Children also become acquainted with decimal and common fractions during this time. The soroban or abacus is introduced in grade 3. Children learn basic concepts of measurement such as reading a clock, comparing quantities of length, area and volume, and comparing sizes in terms of numbers. They are also taught the concepts of weight and time and shown how to measure fundamental quantities such as length.

By the end of grade 4, children are expected to have mastered the four basic operations with whole numbers and how to electively apply them. They should also be able to do addition and subtraction of decimals and common fractions. In grades 5 and 6, children learn how to multiply and divide decimals and fractions. They are taught to understand the concept of area and how to measure the area of simple geometric figures and the size of an angle, as well as to understand plane and solid geometric figures, symmetry, congruence, and how to measure volumes. Children learn about the metric system during this time. Teachers show how to arrange data and use mathematical expressions and graphs to help children to become able to express the sizes of quantities. Letters such as x and a are introduced. Children also begin to learn about statistical data by using percentages and circle graphs (pie charts). It is recommended that calculators be introduced into the classroom in grade 5 to ease the computational burden.

contrast, elementary classrooms tend to be lively, with a great deal of interaction between students and teachers. In either case, classrooms are teacher-directed. The student-directed group learning that is found in some US classrooms is virtually non-existent in Japan.

In grade 7, students learn about positive and negative numbers, the meaning of equations, letters as symbols, and algebraic expressions. By the end of grade 8, they are able to compute and transform algebraic expressions using letter symbols and to solve linear equalities and simultaneous equations; they have also been introduced to linear functions, simple polynomials, linear inequalities, plane geometry, and scientific notation. In grade 9, students learn how to solve quadratic equations (those with real solutions) and are taught the properties of right triangles and circles, functions, and probability. In grade 7 and beyond it is recommended that calculators and computers should be efficiently used as the occasion demands.

In high school (grades 10-12), six mathematics courses are offered: Mathematics I, II III and Mathematics A, B and C. Although only Mathematics I is required of all students, those students intending to enter a university will usually take all six courses. In fact, Japanese high school students who take all of the courses offered will know more mathematics than many US students do when they graduate from college. In Mathematics I, students are taught quadratic functions, trigonometric ratios, sequences, permutations and combinations, and probability. Mathematics II covers exponential functions, trigonometric functions, analytic geometry (equations of lines and circles), as well as the ideas of limits, derivatives, and the definite integral. Calculus is taught in Mathematics III, including functions and limits, sequences and geometric series, differential and integral calculus. More advanced topics such as Taylor series are not usually taught in Mathematics III. Mathematics A deals with numbers and algebraic expressions, equalities and inequalities, plane geometry, sequences, mathematical induction, and the binomial theorem. Computation and how to use the computer are also taught in this course. In Mathematics B, students learn about vectors in the plane and three-dimensional space, complex numbers and the complex number plane, probability distributions, and algorithms. Mathematics C consists of a variety of topics, including matrix arithmetic (up to 3x3 matrices), systems of linear equations and their representation and solution using matrices, conic sections, parametric representation and polar coordinates, numerical computation including the approximate solution of equations and numerical integration, and some calculus-based statistics.

It is believed that students can attack this rigorous curriculum because of the problem-solving work conducted in the lower grades via the mechanism of 'lesson study' (a programme of teacher-led research on teaching described later in this document). However, the ministry is concerned that the post-primary school system is not producing enough students who complete the six-year mathematics sequence in high school. There is concern in some quarters (again, the ministry) that the post-primary school curriculum is too difficult and stunts student performance over time, especially the creativity developed via lesson study at the elementary school level. This concern has led to very recent meetings between the Ministry of Education in Japan and representatives of the National Science Foundation in the US, with reciprocal goals. The US are interested in understanding lesson study and their Japanese counterparts in how the US system fosters creativity and innovation in the learning of mathematics. We will see below that this search for creativity and the capacity for innovation is also at the forefront of educators' minds in Singapore.

Creativity in mathematics problem solving in Singapore

Being a tiny city-state of four million, Singapore is focused on nurturing every ounce of talent of every single citizen. That is why, although its fourth and eighth class students (roughly equivalent to fourth class and second year students in Ireland) already score at the top of the TIMSS international maths and science tests, Singapore has been introducing more innovations into schools. Its government understands that in a flattening world, where more and more jobs can go anywhere, it's not enough to just stay ahead of its neighbours. It has to stay ahead of everyone.

As Low-Sim Ay Nar, principal of Xinmin Post-primary School, told researchers, Singapore has got rote-learning down cold (Sloane, 2003). No one is going to outdrill her students. What Singapore is now focusing on is how to develop more of America's strength: getting Singaporean students and teachers to be more innovative and creative. 'Numerical skills are very important,' she said, but 'I am now also encouraging my students to be creative - and empowering my teachers… We have been loosening up and allowing people to grow their own ideas.' She added, 'We have shifted the emphasis from content alone to making use of the content,' on the principle that 'knowledge can be created in the classroom and doesn't just have to come from the teacher' (Sloane, 2003).

To that end, some Singapore schools have adopted a maths teaching programme called HeyMath, which was started four years ago in Chennai, India, by two young Indian bankers, Nirmala Sankaran and Harsh Rajan, in partnership with the Millennium Mathematics Project at Cambridge University.

With a team of Indian, British and Chinese mathematics and education specialists, the HeyMath group basically came to the following insight: if you were a parent anywhere in the world and you noticed that Singapore kids, or Indian kids or Chinese kids were doing really well in maths, wouldn't you like to see the best textbooks, teaching and assessment tools, or the lesson plans that they were using to teach fractions to fourth graders or quadratic equations to 10th graders? And wouldn't it be nice if one company then put all these best practices together with animation tools, and delivered them through the internet so any teacher in the world could adopt or adapt them to his or her classroom? That's HeyMath.

'No matter what kind of school their kids go to, parents all over the world are worried that their kids might be missing something,' Mrs Sankaran of HeyMath said in a recent conversation. 'For some it is the right rigour, for some it is creativity. There is no perfect system… What we have tried to do is create a platform for the continuous sharing of the best practices for teaching mathematics concepts. So a teacher might say: "I have a problem teaching congruence to 14-year-olds. What is the method they use in India or Shanghai?".'

According to Mrs Sankaran, Singaporean maths textbooks are very good. They are static and not illustrated or animated. 'HeyMath lessons contain animated visuals that remove the abstraction underlying the concept, provide interactivity for students to understand concepts in a "hands on" manner and make connections to real-life contexts so that learning becomes relevant,' Mrs Sankaran said.

HeyMath's mission is to be the maths Google – to establish a web-based platform that enables every student and teacher to learn from the 'best teacher in the world' for every maths concept and to also be able to benchmark themselves against their peers globally.

The HeyMath platform also includes an online repository of questions, indexed by concept and grade, so teachers can save time in devising homework and tests. Because HeyMath material is accompanied by animated lessons that students can do on their own online, it provides for a lot of self-learning. Indeed, HeyMath, which has been adopted by 35 of Singapore's 165 schools, also provides an online tutor, based in India, to answer questions from students stuck on homework. While Sankaran's comments have the feel and flavour of public relations and sales, it is likely that HeyMath (now with more than a 20% market share) will further catch on in Singapore because of the highly competitive nature of the schooling environment there.

High stakes testing in the US: driving the bull by the tail

In the 1990s, all 50 states in the USA embarked on education initiatives related to high standards and challenging content in mathematics (and science). A central focus of these policy efforts was to establish a common set of academic standards for all, not just elite students. Other components of these standards-based reforms included assessments that measure student performance and accountability systems that are at least partially focused on student outcomes. Bear in mind that graduation from post-primary school has been based solely on classroom teacher-made tests and student performance without comparison across schools at a state or national level. Although assessment has always been a critical component of the education system (Glaser and Silver, 1994; Linn, 2000), the growing focus on standards (for example the NCTM standards documents, 2000) and accountability has dramatically changed the role of tests in the lives of students, their teachers and their schools. Teachers continue to use the results of classroom and other types of tests to plan instruction, guide student learning, calculate grades, and place students in special programmes. However, with the passing of President George Bush's No Child Left Behind (NCLB) Act of 2001, policy makers are turning to data from large-scale state-wide assessments to make certification decisions about individual students, and to hold teachers accountable for their performance and progress of their students.

Provisions in the federal government's Title I programme have reinforced the role of assessment in standards-based reform. Title I of the Improving America's Schools Act (IASA) of 1994 required states to develop high-quality tests aligned with state standards in reading and mathematics in one grade per grade-span (elementary, middle and high school), and to use these data to track student performance and identify low-performing schools (somewhat akin to the league tables in use in Great Britain). The most recent amendments to Title I, contained in the NCLB Act of 2001, give even greater prominence to state assessment. The law expanded state testing requirements to include every child in grades 3 to 8 in reading and mathematics by the 2005-06 school year, and in science by 2007-08. These assessments must be aligned with each state's standards and allow for student achievement to be comparable from year to year. The results of the tests will be the primary measure of student progress toward achievement of state standards. States will hold schools and districts accountable for 'adequate yearly progress' toward the goal of having all students meet their state-defined 'proficient' levels by the end of the school year 2013-14. Students attending Title I schools that fail to make adequate progress are given the option of transferring to other public schools or receiving supplemental educational services outside of school. Title I schools that fail to improve over time can be restructured, converted to charter schools, or taken over by their district or state.

United Kingdom: making maths count and counting maths teachers

One of the big concerns in England, as against the more favourable situation in Scotland, is the lack of suitably qualified teachers to teach mathematics at second level. Furthermore as maths appears to count more and more in economic and scientific progress, counting (and reversing) the declining numbers of mathematics teachers is a key policy focus. These concerns are reflected in a recent report, *Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education* (2004). The genesis of the report was partly due to an earlier report on science, engineering and technology (the Roberts Report: SET for Success: the supply of people with science, technology, engineering and mathematical skills published in 2002) which according to Smith (2004) noted that, although

> *…relative to many other countries, the UK has a large and growing number of young people studying science and engineering, this overall growth has masked a decline in the numbers studying the physical sciences, engineering and mathematics. For example, the report drew attention to the drop during the 1990s of nearly 10 per cent in the numbers taking A-level mathematics in England. At the same time, the report also noted that the demand for graduates and postgraduates in these strongly mathematically oriented subjects has grown significantly over the*

past decade, not only in science and engineering areas, but also in the financial services and ICT sectors. In addition to the supply problem, the report identified concerns expressed by employers about the mismatch between skills acquired during formal education and those required in the workplace.' (2004, p. 1, Section 0.3)

Smith went on to note that the mismatch problem identified in the Roberts Report had potentially serious consequences for the economy because it 'is leading to skills shortages that will adversely affect the government's productivity and innovation strategy. These shortages will become increasingly serious unless remedial action is taken' (2004, p. 1, Section 0.3). The terms of reference for the Smith Report, commissioned by the UK government were as follows: to make recommendations on changes to the curriculum, qualifications and pedagogy for those aged 14 and over in schools, colleges and higher education institutions to enable those students to acquire the mathematical knowledge and skills necessary to meet the requirements of employers, and of further and higher education. (2004, p. 2, Section 0.8). Indicative of the ongoing concern about mathematics education in the United Kingdom is the appointment of Celia Hoyles as mathematics 'czar' for the UK government. This appointment reflects a rising concern about spiralling standards in mathematics education. A recent press release from the Department of Education and Skills typifies the politically sensitive nature of mathematics education:

No government has done more to get the basics right in our schools….Standards in maths are rising with last summer's test and exam results showing good progress. In 2004 maths had the the highest entry rate of any GCSE and the third highest entry rate at A-level…. Vacancies for maths teachers have declined every year since 2001 and to boost recruitment further, graduates from 2006 will be offered a £9,000 *bursary and £5,000 'golden hello' to train as maths teachers. (28th June 2005, BBC News, www.bbc.co.uk)*

Finally, the UK government is about to establish a national centre for excellence in the teaching of mathematics.

Ireland: slow emergence of concern about mathematics education

The PISA shock that led to national concern in Germany about mathematics or the 'splintered vision' concerns about mathematics education in the USA have not been mirrored in Ireland, despite the fact that on international tests Irish students perform at a moderate level (mid-ranking in both PISA 2000 and 2003 in mathematics literacy). The less dramatic reaction in Ireland may be wise or unwise depending on your perspective. Given that curriculum reforms, if undertaken, are best done in a considered rather than reactionary manner, less immediate concern may be justified. On the other hand, bearing in mind the ambitious social and economic goals being set in the fast-growing economy, it could be argued that a more concerted effort to address perceived weaknesses in mathematics education is merited. A series of studies over the last three years have drawn attention to the state of mathematics education in Ireland. These include the landmark video study *Inside Classrooms* (Lyons *et al.* 2003) which provided a rich portrait of mathematics education at junior cycle (lower post-primary). Recent national reports on PISA 2000 (published in 2003) and PISA 2003 (published in 2005), as well as two curriculum mapping exercises (one mapping current mathematics syllabi onto PISA and the other mapping PISA onto examination items), are providing a variety of very useful data in promoting debate about the state and vision of mathematics education in Irish post-primary classrooms. Among the insights emerging from these recent research studies are the following:

- post-primary mathematics education in Ireland is taught in a highly didactic and procedural fashion with relatively little emphasis on problem solving (Lyons, *et al.*, 2003)

- the 'new' mathematics curricular culture, with its elevation of abstraction as its core principle, has dominated post-primary mathematics teaching for the last forty years (Oldham, 2001)

- in both PISA 2000 and 2003, Ireland is ranked in the middle of OECD countries in mathematical literacy (Cosgrove, *et al.*, 2005).

1.8 Conclusion

This chapter has provided an overview of key issues in mathematics education internationally. We address many of the issues raised in Chapter 1 in greater detail in the rest of this report. Among the key issues raised in this chapter were

- the existence of multiple goals in debates on mathematics education. For example, the 'mathematics for scientific advancement' and the 'mathematics for all' goals

- the mathematical sciences as changing and dynamic areas of knowledge

- the changing definition of mathematics, that is, from 'working with numbers' to the 'science of patterns' (see Table 4, Chapter 3)

- the high level of policy priority accorded to mathematics by many countries, since it is seen as a high-yield subject in preparing workers and citizens for the knowledge society

- the combination of the push factor (that is, poor levels of understanding and achievement gaps related to gender, SES, ethnicity), and the pull factor (that is, the demands of 21st century economy and society), which are fuelling concern about mathematics education internationally
- mathematics education as a contested terrain: on the one hand, the formal and abstract view of maths in new or modern mathematics which has dominated since the 1960s; on the other, a more 'real world', problem-focused mathematics underpinned by cognitive science and realistic mathematics education
- the differing meanings of 'basic' in debates about mathematics
- three factors that help us understand why mathematics is taught the way it is: curricular cultures, textbooks, and testing/ examination traditions
- the mismatch between ambitious problem-solving oriented curriculum goals in mathematics education compared to the procedural approach and content of mathematics textbooks
- the important role played by educational governance structures in framing curriculum reform initiatives
- the research finding that post-primary mathematics education in Ireland is typically taught in a highly didactic and procedural fashion, reflecting the impact of the 'new' or 'modern' mathematics movement teaching over the last forty years
- recent developments that have helped to inform new visions of mathematics education: video and other technologies to understand teaching practices; new images of learning; new goals for education/schooling.

As video studies of post-primary mathematics teaching have had a major impact internationally on promoting an understanding of classroom practices in mathematics education, the next chapter focuses on key findings from video studies undertaken in the last decade.

CHAPTER 2:

Understanding and improving teaching: video studies and lesson study

2.1 Introduction

This chapter focuses on developments in understanding the teaching of mathematics, drawing primarily on compelling findings from the TIMSS 1995 and TIMSS-R 1999 video studies. As one of the key insights of both video studies was the nature of culturespecific 'lesson signatures', the second section of the chapter provides a review of developments in Japan and elsewhere in relation to *Lesson Study* as a form of teacher professional development and curricular reform. We think the focus on lessons learned both from these landmark video studies and Japanese Lesson Study reflects what Ball *et al.*, writing in the fourth *Handbook of Research on Teaching* (2001), identifed as a move *from teachers to teaching* as the focus of research on teaching. This shift in focus towards actual classroom practice is also found in research on mathematics education. For example, because of the impact of the cognitive revolution, research on teaching since the 1970s had focused on teacher thinking (Clark and Peterson, 1986) and knowledge: how do teachers think and what do they need to know to teach high-quality mathematics curriculum (Munby, Russell and Martin, 2001). This research has been complemented over the last decade by a shift towards carefully examining the actual practice of mathematics teaching through structured observation, case studies (sometimes supported by video data), video surveys, or video-ethnographies.

2.2 Understanding classroom practice: The critical role of video

Why use video as a form of data in the study of classroom instruction? Traditionally, attempts to measure classroom teaching on a large scale have used teacher surveys (i.e. questionnaires). In general, questionnaires are economical and relatively easy to administer to a large number of respondents, and responses usually can be transformed readily into data files ready for statistical analysis. However, using questionnaires to study classroom practices can be problematic because it is difficult for teachers to remember classroom events and interactions that occur quickly. Moreover, different questions can mean different things to different teachers, thereby creating validity issues (Stigler, 1999).

Direct observation of classrooms by coders overcomes some of the limitations of questionnaires, but important ones remain. Significant training problems arise when used across large samples, especially in cross-cultural comparisons. A great deal of effort is required to ensure that different observers are recording behaviours in comparable ways. In addition, as with questionnaires, the features of the teaching to be investigated have to be decided ahead of time. While new categories might occur to the observers as the study progresses, the earlier classes cannot be re-observed.

Video offers a promising alternative for studying teaching (Stigler, Gallimore and Hiebert, 2000), with special advantages. For example, by preserving classroom activity, video enables detailed examination of complex activities from different points of view. However, video is not without its disadvantages (see Jacobs *et al.*, in press); these are discussed later in this monograph. For the moment we focus on a component part of video that is rarely discussed, that of sampling.

2.3 Many types of video studies: From video surveys to video cases

Rapid advances in video technology, more user-friendly video technology, and declining costs have made it possible for relatively small-scale research projects to make effective use of video as a data-collection tool in research on classroom practices (Reusser, Pauli, and Hugener, 2005). Following Stigler *et al*. (1999), Roth (2004), and Reusser *et al*. (2005), we can summarise the advantages and affordances of video as follows:

- It allows research questions and analytic categories to emerge from the data: 'A "record" of actual events that can be observed repeatedly' (Reusser, Pauli and Hugener, 2005).
- It provides raw uninterpreted data.
- It is less theory bound than other methods of data collection.
- It creates the possibility of interdisciplinary and intercultural meaning negotiation.
- It enables repeated analyses of the same data from different theoretical angles at a later time.
- It provides 'concrete referents to find a shared language about teaching processes' (Reusser *et al*, 2005).
- It facilitates the integration of qualitative and quantitative analyses, and thus helps to transcend the dichotomy between qualitative and quantitative research.
- It creates the possibility 'for more nuanced analyses of complex processes that are tied to observable evidence' (Reusser *et al*., 2005).

Over the last decade a variety of video studies have been undertaken in mathematics education (see Table 2). While there are numerous issues of interest among the different foci, settings and findings of

Table 2: Video studies in mathematics education

Name + study type	Sample	Lessons: number and content focus	Key insight or question
TIMSS 1995 Video survey	3-country study: nationally representative sample of 8th grade maths lessons in USA, Germany and Japan.	231 lessons; one camera per classroom.	Is there such a phenomenon as 'cultural script' to describe lessons at national level?
TIMSS-R 1999 Video survey	7-country study: nationally representative sample of 8th grade maths lessons in Australia, the Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland and the United States.	638 lessons (including 1995 Japanese lessons); one camera per classroom.	High quality mathematics teaching takes many different forms.
Learner's Perspective Study (Clarke, 2002) Video survey	9-country study at 8th grade: Australia, Germany, Hong Kong, Israel, Japan, the Philippines, South Africa, Sweden and the USA.	Focus on sequence of 10 or more 'well-taught' lessons; three cameras per lesson: whole class, teacher and student.	Video-stimulated reconstructive recall of lessons is valuable in understanding student and teacher intentions.
Ms Ball (Ball, 1993) Video self-study	Self-study over one year at 4th grade, East Lansing, MI, USA.	Most lessons over the course of one school year were video-taped.	Student sense making provides many rich opportunities to foster important mathematical ideas.
Ms Smith (Cobb et al., 1997) Video case	Case study of one 1st grade teacher over a year in Nashville, TN, USA.	106 lessons; two cameras.	An example of integrating social and psychological frameworks in research on maths teaching.
Inside Classrooms Lyons et al., 2001) Video case	Junior cycle mathematics in Ireland at 9th grade.	20 lessons from 10 schools one camera per classroom.	Striking homogeneity across lessons/schools.

these studies, the somewhat different sampling of lessons is an important difference between them. The TIMSS studies focused on one lesson per teacher, whereas the Learner's Perspective Study, assuming that teaching involves thoughtful sequencing of content congruent with learners' developing understandings, focused on sampling a sequence of 'well taught' lessons (Clarke, 2002). For the purposes of this report, we focus on the TIMSS video studies as they provide a range of important insights into the rhythms of mathematics teaching and conditions for changing classroom practices.

TIMSS video studies

The broad purpose of the 1998-2000 Third International Mathematics and Science Study Video Study (TIMSS 1999 Video Study) was to investigate and describe teaching practices in Year 8 (i.e. second/third year mathematics in Ireland) mathematics and science in a variety of countries. It is a supplement to the TIMSS 1999 student assessment, and a successor to the TIMSS 1995. The TIMSS 1999 video study expanded on the earlier 1994-95 video study (Stigler *et al.*, 1999) by investigating teaching in science as well as mathematics, and sampling lessons from more countries than the original study.

TIMSS 1995 Video Study included only one country, Japan, with a relatively high score in Year 8 mathematics as measured by the TIMSS assessment instruments. It was tempting for some audiences to prematurely conclude that high mathematics achievement is possible only by adopting teaching practices like those observed in Japan. In contrast, the TIMSS 1999 Video Study addressed these concerns by sampling Year 8 mathematics lessons in more countries - both Asian and non-Asian, where students performed well on the TIMSS 1995 mathematics assessment. Countries participating in the TIMSS 1999

Video Study were Australia, the Czech Republic, Hong Kong SAR[6], Japan, the Netherlands, Switzerland, and the United States.

The aim of the TIMSS 1999 Video Study

The video survey methodology used in the TIMSS 1999 Video Study enabled very detailed snapshots of mathematics teaching to be collected. Internationally, a general aim was to use these snapshots to describe patterns of teaching practice in the participating countries. More specific aims included

- development of objective, observational measures of classroom instruction to serve as quantitative indicators of teaching practices
- comparison of teaching practices to identify similar or different features across countries
- development of methods for reporting results of the study, including preparation of video cases for both research and professional development purposes.

Scope of the study

The mathematics component of the TIMSS 1999 Video Study comprised 638 Year 8 lessons collected from all seven participating countries. The 50 lessons collected in Japan in the 1995 video study are included in this tally. The required sample size in 1999 was 100 lessons per country. One lesson per school was randomly selected from each of the approximately 100 randomly selected schools per country.[7]

6 It is this document Hong King SAR is referred to as a country. Technically, Hong Kong is a special administrative region (SAR) of the People's Republic of China.

7 The weighted response rate reached the desired 85 per cent or more in all countries except the United States, where it was 76 per cent.

In each school the selected teacher was filmed for one complete Year 8 mathematics lesson, and in each country (except Japan) videotapes were collected throughout the school year to try to capture the range of topics and activities that can occur across a whole school year. To obtain reliable comparisons among countries the data were appropriately weighted according to the sampling design.

Processing the data was a long, complex and labour-intensive undertaking. Several specialist teams were needed to decide what should be coded, what kinds of codes to use, and how reliably the codes could be applied. Many revisions were made to the codes before a satisfactorily reliable and common set were put in place. All coding was completed under the guidance of James Stigler at LessonLab on the campus of UCLA. Country representatives were encouraged to reside in Los Angeles for the period of coding.

Major findings

In general terms, the TIMSS 1999 Video Study of Year 8 mathematics teaching showed that there is no one way to undertake successful teaching of mathematics, bearing in mind that the countries studied were chosen because, other than the US, their students had performed well in the TIMSS assessments and they were willing to participate (see Table 3 below).

Table 3: Average scores on TIMSS 1995 and TIMSS 1999 mathematics assessments

Country	1995	1999
Australia	519	525
Czech Republic	546	520
Hong Kong SAR	569	582
Japan	581	579
Netherlands	529	540
Switzerland	534	-
United States	492	502

The results showed that teachers in these high performing countries used a variety of teaching methods and combined them in different ways. All countries shared come common features, while most countries were found to have some distinctive features.

Common features included

- At least 95 per cent of lesson time, on average, was spent on mathematical work (excluding time taken to organise students in relation to these tasks).

- At least 80 per cent of lesson time, on average, was spent solving mathematical problems, regardless of whether the main purpose of the lesson was review of previously explored content or the presentation or practice of new content.

- Lessons generally included some review of previous content as well as some attention to new content.

- Most of the time, lessons included some public, whole-class work and some private, individual or small group work - during the private time, students mostly worked individually rather than in pairs or larger groups.

- At least 90 per cent of lessons made use of a textbook or worksheet of some kind.

- Teachers talked much more than students, both in terms of numbers of words uttered and in terms of the lengths of such utterances. The ratio of teacher to student words was 8:1. Most teacher utterances were at least five words long while most student utterances consisted of fewer than five words.

Distinctive features

These features were found in relation to the introduction of new content, the emphasis on review of previous content, the use of various strategies to make lessons more coherent, the mathematical topics covered, the procedural complexity of the problems discussed and assigned, and the classroom practices regarding the use of individual work time and the use of class time for homework. Findings on these and other variables can be found in Hiebert *et al., Teaching Mathematics in Seven Countries* (2003).

2.4 Understanding classroom practice: Japanese lesson study

In this section we describe what has in the last 20 years become known as 'lesson study', a central component of Japan's major effort at teacher professional development (*kounaikenshuu*) in elementary and middle school teaching (infant through second year teaching in Irish schools). We begin by outlining the processes associated with lesson study and how they are developed and utilised. Next we describe the meta-context for education and schooling in Japan. We note that, given the number of students attending evening cram schools in Japan, Japan's international performance in mathematics draws on the combined effects of lesson study and *juku* (Japan's cram school system). We then highlight a finding from the US's National Academy of Science report (see Snow, Burns, and Griffin, 1998) on the current debate in reading circles about phonics versus whole-language learning. The Academy notes that reading with competence and comprehension is probably due to both strategies. We suggest that this is likely to be the case for the learning of mathematics. However, this should not detract from the fact that the quality of instruction found in Japanese classrooms is something that should be emulated where possible world-wide. In an effort to better link research and practice we ask if the process of lesson study could be improved by drawing on current research knowledge (in the first iterate of the lesson study process).

In their now classic book, *The Teaching Gap* (1999), Stigler and Hiebert contrast mathematics instruction in three countries: Germany, Japan and the United States. The authors draw on randomly sampled lessons from the TIMSS 1995 video study of eight grade mathematics instruction. The countries were picked for a number of reasons: professional, financial and political. Japan was chosen because of its high performance in SIMS (the Second International Maths Study) and expected high performance in TIMSS, while Germany was selected because of anticipated changes in performance due to the merging of East with West Germany in the period between SIMS and TIMMS. The US wanted to participate because it was felt that a lot could be learned through the contrasting of the three systems and because the US was underwriting a large proportion of the study.

Stigler and Hiebert (1999) use their video analysis to highlight instructional differences across the participating countries, asking why teachers in each country teach differently and what it is about the educational systems of each country that supports the different types of instruction that can be seen from nation to nation. Additionally, they outline the policy issues raised by their work for the teaching and learning of mathematics in American schools. It is in this regard that they provide insight into the processes underlying lesson study in Japan.

In lesson study, groups of teachers meet regularly over long periods of time (ranging from several months to a school year) to work on the design, implementation, testing and improvement of one or more 'research lessons'. Lesson study is a deceptively simple idea. If you want to improve instruction, what could be more obvious than getting together with your peers to collaborate on the development and implementation of lessons? While a simple idea, lesson study is quite complex in practice, with many component parts. Stigler and Hiebert (1999) focus our attention on what they believe to be the eight common parts of lesson study. These include: defining the problem; planning the lesson; teaching the research lesson; evaluating the lesson and reflecting on its effect; revising the lesson; teaching the revised lesson; evaluating and reflecting again; and sharing the results. Lewis (2002), on the other hand emphasises the metastructure in what she labels the lesson study cycle. She attends to four critical structures: goal setting and planning; researching the lesson; lesson discussion; and consolidation of learning. In the discussion that follows we combine the two renderings as appropriate. While we present the general features of lesson study we include, where appropriate, what teachers are likely to do. For more detail see Lewis (2002) and Fernandez (2004). Lewis provides rich examples of lesson study for the learning of primary school science, whereas Fernandez explicitly attends to the learning of mathematics. A discussion of lesson study in the Irish context is available in Kelly and Sloane's recent article in *Irish Educational Studies* (2002).

Goal setting and planning

1. Defining the problem. At its essence lesson study is the act of problem-solving (problem-solving by teachers about teaching and instruction). Consequently, the first step in the process is to define the problem to be resolved. What is the central problem that motivates the work of the lesson study group? Here the teachers identify the goals for student learning and for long-term development. These goals can be general or specific in nature,

focusing on student motivation or on the learning of a specific mathematical topic. Normally, teachers pick a topic that arises from their daily practice. Occasionally, topics come from the Ministry of Education. When this arises they are of the form 'How can we help students learn x?' The Ministry then invites a sample of schools throughout the country to look at this question using lesson study as a vehicle for generating this knowledge. The schools are then expected to report back on their findings. Additionally, the Ministry will issue recommendations in a top-down fashion. The interplay between the bottom-up and top-down mechanisms for change is unique to Japan. This feature of educational policy allows teachers direct input into national policy in a manner not available in other countries. This is not a common policy feature in Ireland.

2. Planning the lesson. When the learning goal has been set teachers meet regularly to plan the lesson. The aim here is twofold: to produce an effective lesson and also to understand why and how the lesson promotes (or doesn't promote) improved mathematical understanding on the part of students. Often teachers begin the planning process by looking at books and articles produced by other teachers who have studied a somewhat similar problem. This initial planning process can occur over weekly meetings for a period of months. In general, the lesson begins with the statement of a mathematical problem. Consequently, teachers engage in detailing the following:

   a. the mathematical problem, its exact wording, and the numbers to be used

   b. the materials students would need to be given to address and solve the problem

c. how the materials are likely to be used by the students

d. the possible solutions that students are likely to generate as they struggle with the problem; the probable misconceptions

e. the questions the teacher will use to promote student mathematical thinking during the lesson

f. the types of guidance that teachers could provide to students who showed one or another misconceptions in their thinking

g. the use of the blackboard by the teacher

h. the allocation of the finite amount of time devoted to the lesson and its component parts (the introduction of the problem, student work, etc.); how to handle individual differences in the level of mathematical preparedness of the students in the class

j. how to close the lesson. Lesson endings are considered crucial and involve teachable moments to advance student understanding.

The research lesson

3. Teaching the lesson. The team (that is, all teachers) prepare the lesson together with role-playing. One teacher teaches the lesson. The others attend the lesson with specific tasks in mind. When the students are asked to work at their desks, the other team members collect data on student thinking, what is being learned, student engagement, and student behaviour. They do this by observing and taking careful notes. Occasionally, the lesson is video-taped for further discussion.

Lesson discussion

Here the teachers share and analyse the data they have collected at the research lesson. They carefully examine evidence as to whether the goals of the lesson for student learning and development were fostered (or met). Finally they consider what improvements to the lesson specifically, and to instruction more generally, are necessary.

4. Evaluating the lesson and reflecting on its effect. On the day the lesson is taught the teachers will meet to discuss what they observed. In general the teacher who taught the lesson speaks first generating their own personal evaluation. Then the other teachers contribute. Comments tend to be quite critical in nature, but focus on the group product, the lesson itself, and not on the teacher. All team members feel responsible for the final group product. This is an important activity because the team places attention on the possibility and opportunity for lesson improvement.

5. Revising the lesson. Next the teachers revise the lesson as a group. They do this based on their observational data and on shared reflections. They may change any number of things in the lesson, including the materials, the specific activities, the opening problems posed, the questions asked, or on occasion all of these components. The teachers base their changes on misunderstandings demonstrated by the students they have observed.

6. Teaching the revised lesson. When the revised lesson is formulated it is then taught to a new classroom of students. It may be taught by the first teacher but it is often, and more than likely, taught by another team member. It is the lesson that takes central focus, not

the teacher. During this iterate other teachers may be invited to observe. Often the full complement of teaching staff is invited to the second or third iterate of the lesson. The goal is simple; it is one of slow, deliberate, iterative revision in search of perfection.

7. Evaluating and reflecting. Again, the teacher who taught the lesson comments first. The teacher will describe the goals of the lesson, his or her own assessment of how the lesson went and what parts of the lesson the teacher feels need re-thinking. Observers then critique the lesson and suggest changes based on their observations. An outside expert may be invited to these deliberations. The lesson conversations vary in focus from the specific issues of student learning and understanding, but also with respect to more general, or meta-issues, raised by the initial hypotheses used to guide the design and implementation of the lesson.

8. Consolidation of learning. If desired, the teachers refine and re-teach the lesson and study it again. When the iterative process is complete they write a report that includes the lesson plan, the student data, and the reflections on what was learned by the students, and by the teachers about the process of learning. This description has focused on a single lesson, but Japan, like Ireland and unlike the US for example, has a well-articulated national curriculum. Consequently, what the teachers have learned has relevance for other teachers locally and nationally. Teachers (and the school system) need to embrace the opportunity to share results. This can be undertaken in a number of ways. We highlight the ways that learning is consolidated locally and more globally in Japanese schools. First, as noted above, teachers write a report. The report documents the lesson, the reason for the lesson, what is

learned by students and by teachers, what the original assumptions were, and how these assumptions changed through the process of teaching, observation and lesson revision. The report is bound and is available to all teachers in the local school. It is read by faculty (consider the power of this resource for the new or neophyte teacher); it is read by the principal; it is shared within the prefecture if considered interesting enough. If a university professor collaborated with the group the work may find an even broader audience. Another way of consolidating and sharing the lesson study results occurs when teachers from other schools are invited to a rendering of the final version of the lesson. Sometimes, 'lesson fairs' are conducted at schools and teachers from a neighbouring geographic region are invited to watch research lessons in many subjects produced by a school over an extended time period. These are considered festive occasions, and a critical part of ongoing teacher development.

Some reflections

Lesson study is of course culturally based, and we discuss the culture of Japanese schooling below. We believe that the process of lesson revision is a powerful one and one that can be legitimately adopted and adapted to fit the Irish school system. Lesson study will not bring quick rewards but it is a deliberate and concentrated effort to improve the process and products of teaching in mathematics.

Lesson study re-situated: the wider context of schooling in Japan

Understanding the Japanese people and culture requires understanding the factors that mould them. Particularly important are those components which influence them in their formative years, notably the education system. Given the large amount of time that Japanese students spend in schools, it is little wonder that the education system plays a tremendous role in determining the fabric of Japanese society. An examination of the 'typical' post-primary school experience illuminates the function of the education system in Japanese society. One of the lessons from recent research in educational anthropology and cultural psychology is the importance of understanding educational activities and academic performance in their wider social and cultural contexts. Thus, in order to provide a context for the interpretation of Japanese post-primary mathematics education, we provide a brief description of students' school, extra-curricular, cram school and entrance examination experiences.

At school

Japanese students spend 240 days a year at school, approximately 60 days more then their Irish counterparts. Although many of those days are spent preparing for annual school festivals and events such as culture day, sports day and school excursions, Japanese students still spend considerably more time in class than their Irish counterparts. Traditionally, Japanese students attended school for half a day on Saturdays; however, the number of required Saturdays each month is decreasing as the result of Japanese educational reforms. Course selection and textbooks are determined by the Japanese Ministry of Education. Schools have limited autonomy in their curriculum development. Students in academic high schools typically take three years of each of the following subjects: mathematics, social studies, Japanese, science and English. Other subjects include physical education, music, art and moral studies. All the students in one grade level study the same subjects. Given the number of required subjects, electives are few.

At the end of the academic day, all students participate in *o soji*, the cleaning of the school. They sweep the classrooms and the hallways, empty rubbish bins, clean bathrooms, clean chalkboards and chalk dusters, and pick up litter from the school grounds. After *o soji*, school is dismissed and most students disperse to different parts of the school for club meetings.

Extracurricular activities

Club activities take place after school every day. Teachers are assigned as sponsors, but often the students themselves determine the club's daily activities. Students can join only one club, and they rarely change clubs from year to year. In most schools, clubs can be divided into two types: sports clubs (baseball, soccer, *judo, kendo*, track, tennis, swimming, softball, volleyball, rugby) and culture clubs (English, broadcasting, calligraphy, science, mathematics, yearbook). New students are usually encouraged to select a club shortly after the school year begins in April. Clubs meet for two hours after school each day and many clubs continue to meet during school vacations. Club activities provide one of the primary opportunities for peer group socialisation.

Most college-bound students withdraw from club activities during their senior year to devote more time to preparation for university entrance examinations. Although visible in the general high school experience, it is in the clubs that the fundamental relationships of *senpai* (senior) and *kohai* (junior) are established most solidly. It is the responsibility of the *senpai* to teach, initiate and take care of the *kohai*. It is the duty of the *kohai* to serve and defer to the *senpai*. For example, *kohai* students in the tennis club might spend one year chasing tennis balls while the upperclassmen practise. Only after the upperclassmen have finished may the underclassmen use the courts.

The *kohai* are expected to serve their *senpai* and to learn from them by observing and modelling their behaviour. This fundamental relationship can be seen throughout Japanese society, in business, politics, and social dealings.

Juku: cram schools

An interesting component of Japanese education is the thriving industry of *juku* and *yobiko*, after-school 'cram schools', where a large proportion of Japanese high school students go for supplemental lessons. *Juku* may offer lessons in non-academic subjects such as art, swimming, abacus and calligraphy, especially for elementary school students, as well as the academic subjects that are important to preparation for entrance examinations at all levels. *Juku* for high school students must compete for enrolment with *yobiko*, which exist solely to prepare students for university entrance examinations. Some cram schools specialise in preparing students for the examination of a particular school or university. Although it would seem natural for students to dread the rigour of additional lessons that extend their school day well into the late evening hours and require additional homework, many students enjoy *juku* and *yobiko*, where teachers are often more animated and more interesting than some of the teachers in their regular schools. Remember that the role of the teacher as portrayed by Stigler and Hiebert (1999) is to draw the mathematics out from the student. This requires that the teacher listen to the student more than is typical in Irish classrooms.

Juku and *yobiko* are primarily private, profit-making schools that attract students from a wide geographical area. They often are located near train stations, enabling students to transport themselves easily to *juku* directly from school. *Juku* and *yobiko* thrive in Japan, where it is believed that all people possess the same innate intellectual capacity, and it is only the effort of individuals, or lack thereof, that determines their achievement above or below their fellows. Much like Ireland, in Japanese schools there is the tendency to pass students up to the next 'year' with their entering cohort. Therefore, without the supplemental *juku* lessons, some students could fall well behind their classmates. *Yobiko* also exist to serve *ronin*, 'masterless samurai', students who have failed an entrance examination, but who want to try again. It is possible for students to spend a year or two as *ronin* after graduating from high school, studying at *yobiko* until they can pass a university entrance examination or until they give up. Cram school tuition is expensive, but most parents are eager to pay in order to ensure acceptance into a selective junior high school, high school or university, and thus a good future for their children.

Entrance examinations

In addition to university admission, entrance to high school is also determined by examination, and the subjects tested are Japanese, mathematics, science, social studies and English. Private high schools create their own examinations, while those for public high schools are standardised within each prefecture. Students (and their parents) consider each school's college placement record when deciding which examinations to take. Success or failure in an entrance examination can influence a student's entire future, since the prospect of finding a good job depends on the school attended. Thus, students experience the pressure of this examination system at a relatively early age. The practice of tests at school and *juku* helps teachers to direct students toward institutions whose examinations they are most likely to pass.

What do lesson study and the context of learning in Japan mean for mathematics teaching and learning in Ireland?

Le Métais (2003) warns policy makers that they should be careful when they simply copy one system of education with the vain hopes that such a system will work in the context of another culture. To that end we raise a number of salient questions to help us make local policy for Ireland that draws on cross-national educational study. To summarise Le Métais (2003), sensible policy making results from cross-national study when it helps us to (1) build informed self-review, and (2) clarify the goals of mathematics education.

Japanese performance in international comparisons is based not only on high quality lesson study but also on a system of cram schools. This should alert us to the dangers of drawing simple conclusions from the Japanese experience. Firstly, it is difficult if not impossible to disentangle the possible confounding effects of this dual system of education. For example, there is the anomaly that no homework is assigned in primary and the early years of secondary schooling. Stigler and Hiebert (1999, p. 30) note that 'no homework is typical in Japan' in their study of 8th grade. This is sometimes regarded as a weakness. As recently as 25 July 2005, Minako Sato wrote in the Japan Times: 'Cram schools cash in on the failure of public schools.' Further, she notes that, 'according to a 2002 survey by the Ministry of Education, Culture, Sports, Science and Technology, 39 percent of public elementary school students, 75 percent of public middle school students, and 38 percent of public high school students attend juku.' While one part of the system emphasises conceptual understanding, the other part leverages skills and practice. This bears an interesting resemblance to the NRC finding regarding the interplay of phonics and whole-language approaches to children's development of reading comprehension (Snow, Burns and Griggin, 1998).

Secondly, teacher development (*kounaikenshuu*) tends to be idiosyncratic in Japanese secondary schools. Instruction in the later secondary school years is not centrally linked to the process of lesson study. This may be due, in part, to departmental specialisation in Japanese secondary schools somewhat parallel to the Irish setting. It is also probably due to the pressure imposed by university entrance examinations and the focus on examination preparation (Yoshida, 1999). This is again quite similar to the Irish setting.

Thirdly, some view lesson study as a way of decreasing primary school teacher autonomy (anonymous presenter at a meeting funded by the National Science Foundation, June 2003). This rendering of the use of lesson study is in stark contrast to the manner in which lesson study is presented by most, if not all, authors. In sumarry lesson study is culturally bound and needs to be understood in the context of the Japanese culture. Some might argue that before we can fully understand and use the lessons of lesson study we need to know what fundamental social values are reflected in the different education systems of Ireland and Japan. Further, we should also know what are the intrinsic and extrinsic incentives motivating Irish and Japanese students.

Additionally, it would be wise to acknowledge the different definitions of democracy as applied to education in Ireland and Japan. In the Republic of Ireland, recognition of different talents is consonant with democracy. In Japan, 'equal access' based on standardised scores on entrance examinations is the implied, culturally-held definition of democracy. So what does this mean for teaching and learning in Ireland? Nevertheless, we think there are some important continuing professional development lessons to be learned from lesson study. In particular, lesson study highlights the importance of teachers' deep knowledge of content and pedagogical content knowledge (knowledge of how to represent subject matter in support of student learning). Furthermore, lesson study is a vivid and powerful image of how teachers can create contexts for collegial discussions of pedagogical practices.

Finding answers to these questions is quite daunting and takes us away from a central tenet of this document. That is, we believe that Irish teachers can learn a lot from the engaging in the process of lesson study. When used appropriately it is a very strong professionalising activity that improves both teaching and learning. However, we do not believe that lesson study alone will move Irish education forward, nor should it.

In embracing lesson study we also believe that the iterative process can be shortened if the research literature of mathematics education paid more attention to issues of practice, and teachers in turn attended to this 'new' literature (see, for example, recent work in variation theory approaches to the learning of mathematics, or the work of Lesh and others on model-eliciting mathematics problems). Embracing lesson study provides a unique opportunity for improvements in mathematics instruction and in the profession of teaching in Ireland, while also building bridges between research and practice in Irish schools.

2.5 Conclusion

In this chapter we have reviewed current trends in understanding and enhancing mathematics teaching. The video and lesson study examples provide new and important ways of engaging teachers in the nature of their professional practice, as well as providing important insights for policy makers and researchers. We highlighted a number of key findings in relation to international trends in mathematics education:

- The increasingly widespread use of various types of video studies to understand mathematics teaching.

- The power of video in promoting understanding of the nature of mathematics teaching, with a specific focus on a key component of teaching, that is, the lesson which is a daily reality in all teachers' lives.

- The move away from elevating Japanese teaching of mathematics as the most desirable way of improving mathematics (TIMSS 1995 video study) to an understanding of how there are many different ways of promoting high-quality mathematics (TIMSS 1999 video study). Nevertheless, there are important lessons to be learned both from the video study of Japanese mathematics teaching and Japanese lesson study as a model of subject-specific professional development.

- The role of new technologies in providing video-generated archives of teaching that can be used for future research and/or teacher professional development.

- Lesson study as a powerful model of teacher professional development rooted in Japanese culture.

- The limitations of borrowing teaching and/or professional development initiatives from another culture without a thorough understanding of both their original contexts and the constraints and affordances of introducing these in another culture.

We finish the chapter with one important insight from the TIMSS 1999 video study. Education Week, a free US-based online magazine for educators, reporting the launch of the report of the TIMSS 1999 video study, drew attention to differences in teaching emphases between the relatively low-scoring USA and high-scoring countries.

> *The study found that American middle school teachers use teaching approaches similar to those of their counterparts in higher-achieving countries. But the U.S. teachers, the report says, omit one critical ingredient: the underlying mathematical ideas that help students understand how the skills they're learning are part of a logical and coherent intellectual discipline. "Higher-achieving countries focus on developing conceptual underpinnings of the problems," said James Hiebert, a professor of education at the University of Delaware, in Newark, and one of the researchers. (Education Week, 2 April 2003, www.edweek.org)*

The key insight from Hiebert's observation is that the nature of the links between procedural skills and conceptual knowledge in classroom practice is a critically important dimension of high-quality mathematics education. The relationship between these two important dimensions of teaching has been the focus of considerable research in the learning sciences. In the next chapter, we look at the various perspectives on learning, each with a somewhat different emphasis in its approach to the relationship between procedural skills and conceptual knowledge.

CHAPTER 3

Cultures of learning in mathematics education: rethinking teaching and assessment

3.1 Introduction

Some of what the schools have adopted from the research disciplines has impeded deep learning and widespread achievement. The belief system in schools is consistent with beliefs held in the larger culture. For example, only recently have people come to believe that there might be alternative ways to think about the conditions of learning apart from individual capabilities and differences. Research concerned with individual differences has been held captive by its own ideas and the ideas of the larger culture. Breaking out of the box to imagine new possibilities for thinking and learning is both difficult and necessary.

Greeno and Goldman,
Thinking Practices in Mathematics and Science Education, 1998

Mathematics educators are moving from a view of mathematics as a fixed and unchanging collection of facts and skills to an emphasis on the importance in mathematics learning of conjecturing, communicating, problem solving and logical reasoning.

Lester, Lambdin and Preston,
Alternative Assessment in the Mathematics Classroom, 1997

How do different theories of cognition and learning help us think about mathematics education? How have theories of learning shaped assessment in mathematics education? Is there a consensus on how people learn? Can learners achieve mastery of routine procedures without actually understanding mathematical tasks? Under what set of teaching conditions are information and communication technologies likely to enhance mathematics learning? Why do learners typically suspend sense making in interpreting word story problems in school? Why do students typically give up trying to solve a mathematics problem if they cannot solve it in five minutes? What is the best way to teach basic skills and higher order concepts –

skills first then concepts, or vice versa? What is the role of learners' intuitive sense of important school concepts in the promotion of in-school learning? What is the role, if any, of learners' out-of-school experience in the promotion of school mathematics? What makes a mathematics activity realistic for students? These are among the questions that have been at the heart of research into mathematics education.

In addressing these questions we focus on different learning frameworks and philosophies that underpin mathematics education. In particular, we focus on the Realistic Mathematics Education (RME) movement and situated cognition, both of which underpin PISA's mathematical literacy framework (OECD, 2003; Romberg, 2000). Over the last hundred years, the behaviourist, cognitive and socio-cultural approaches to learning have been influential. Of these, the behaviourist and cognitive have been the most researched, with socio-cultural now becoming a widely researched and influential approach. However, people have been trying to understand learning and the mind for centuries and many of the assumptions underlying each of these three perspectives have a long history in philosophy (Case 1996; Greeno, Collins and Resnick, 1996). For example, key ideas underpinning the behavioural approach can be traced back to ideas of John Locke and his associationist perspective on the human mind. Cognitive psychology, particularly Piagetian constructivism, draws on Kant's conceptions of a priori mental categories. The various theories under the socio-cultural umbrella (e.g. situated cognition) have roots in Marx and Hegel's socio-historical epistemologies that locate learning in the social and material history of cultures into which learners become enculturated to a greater or lesser degree. Despite the long debate on cognition and learning, there is no agreement about the exact workings of the mind, in the sense that one theory dominates the discourse. However, a certain consensus has emerged in the last two decades that social and cultural influences on cognition and learning have been neglected (for a discussion see Bruner, 1996). This has resulted in the realisation by educators and researchers of the overemphasis on the isolated learner and his/her capability rather than considering the learner embedded within formative and potentially supportive social and cultural settings in which ability can be developed in various ways (Conway, 2002). One could make a strong case that the emphasis on individual capabilities has been a particularly strong feature of mathematics education. As Greeno and Goldman (1998) have argued, the focus on learners' individual capabilities in mathematics and science education has resulted in an under-utilisation of learners' out-of-school knowledge, unnecessary lowering of expectations about what learning is possible in classrooms, underestimation of the role of peers in contributing to learning, and a reliance on teaching strategies which overly compartmentalise teaching/learning activities.

Assumptions about learning mathematics are deeply embedded in the culture of schooling and evident in textbooks (see chapter one), parents' and educators' everyday or folk theories of learning (Olson and Bruner, 1996), modes of assessment, and the daily rhythm of lesson planning and lesson enactment. Olson and Bruner (1998) argue that, 'the introduction of any innovation will necessarily involve changing the folk psychological and folk pedagogical theories of teachers - and to a surprising extent, of pupils as well' (p. 11). The recognition of the necessity of paying more attention to both our folk/everyday and academic theories of learning is evident in the high profile being accorded research on learning over the last decade, as agencies (e.g. OECD, APEC, UNESCO) and governments consider optimal ways to enhance every citizen's learning in a period of rapid social, cultural, economic and educational change.

The first section of this chapter provides an overview of approaches to learning that have influenced mathematics education over the last hundred years, particularly behaviourist, cognitive and socio-cultural approaches, focusing on how each addresses procedural skills and conceptual knowledge, the development of learners' ownership of learning, and assessment. We then discuss 'realistic mathematics education' (RME) whose origins are in mathematics education rather than the learning sciences. However, in discussing the appeal of RME we note its similarities with the 'social turn' in learning theory. The next section of the chapter reviews recent interest internationally in brain-based research and its potential to inform developments in mathematics education. The final section of the chapter addresses the increasingly important role accorded learning to learn in international discourse on educational goals. We discuss learning to learn, drawing on cognitive and socio-cultural research on self-regulated learning.

3.2 Different approaches to learning in mathematics education

This section of this chapter provides an overview of three approaches to learning and also focuses on the very significant contribution of the Realistic Mathematics Education (RME) movement (see Section 3.2) to contemporary debates on mathematics education. As we have noted earlier (see Chapter 1), RME has become very influential in mathematics education, given that and situated cognition underpin the PISA mathematical literacy framework. The adoption of RME and situated cognition represents what Romberg (2000) calls a distinct 'epistemological shift' in mathematics education. At the heart of this epistemological shift are (i) a reconceptualisation of the relationship between procedural skills and conceptual knowledge and how they are understood to work together in 'applying' maths, and (ii) a recognition of the neglected role of the social and cultural setting as both the source of learning and the arena within which it is applied.

Historically, according to De Corte *et al*. (1996), approaches to teaching and learning in mathematics have emanated from two sources: (i) mathematicians and/or mathematics educators like Freudenthal, Polya, Poincaré, etc., and (ii) research in the learning sciences – primarily, cognitive/educational/developmental psychology and cognitive anthropology, and more recently cognitive neuroscience. Before we outline the three perspectives, we address the question as to whether or not academic or folk/everyday theories of learning matter in relation to understanding mathematics education practices.

At one level, theories of learning appear far removed from classroom practice in the sense that it may not be at all clear about the particular approach to learning underpinning specific practices, as these practices may have been shaped by a variety of influences such as curricular cultures, textbooks and examination traditions. However, a more detailed analysis often reveals some important links between teachers' beliefs about learning, their practices and student learning. Morine-Dershimer and Corrigan (1997), for example, interviewed student teachers in California regarding how they thought about mathematics learning and teaching. Based both on the interviews and videos of the student teachers' classroom teaching practices, they identified two groups: the first group had beliefs focused on getting knowledge and ideas across to students in an efficient and timely fashion; the second group emphasised the importance of attending to students' prior knowledge and experience and how they might integrate this into their lessons. Perhaps what was most interesting was how these beliefs were evident in quite different sets of teaching practices by the two groups of students; that is, they taught in ways congruent with their stated beliefs about mathematics teaching and learning.

A more recent longitudinal German study, undertaken by researchers from the Max Planck Institute, provides further evidence that teachers' beliefs about learning make a difference to student achievement. Staub and Stern's 2003 study involved 496 students in 27 self-contained 2nd and 3rd grade class in Germany in order to assess the degree to which teachers' beliefs were consistent with behaviourist/direct transmission or cognitive-constructivist views of teaching and learning. They also observed classroom teaching using an observation protocol to ensure the consistency and reliability of observations, and tested students on both routine procedural arithmetic and word story problem-solving measures. As Staub and Stern's study adopted a longitudinal research design, utilised robust statistical analyses (i.e. multi-level modelling which allowed the researchers to assess the nested effects of teachers on students and student growth over time), and focused on student learning with links back to teachers' beliefs and practices, we note the main findings (Raudenbush and Bryk, 2002). First, as in the Dershimer and Corrigan study, there were clear links between teachers' beliefs and their practices. Second, teachers who held the constructivist orientation provided more frequent conceptually-oriented learning opportunities for students that resulted in higher scores on both procedural and problem-solving tasks than teachers working from a behavioural orientation. This result is important, as Staub and Stern note, in that 'contrary to our expectations, there was no negative impact of a cognitive constructivist orientation on arithmetic tasks' (p. 353). Staub and Stern's study supports similar findings from an earlier US-based study by Peterson, Carpenter and Fennema (1989). Discussing the implications of their study, Staub and Stern note that children's prior mathematical achievement and ability have a stronger impact on their achievement than teachers' beliefs; nevertheless they argue that the cumulative impact of teachers' learning-related beliefs/practices over many years of schooling represent a significant impact on student learning.

Three perspectives on learning and assessment in mathematics education

Behavioural, cognitive and socio-cultural theories are the three schools or perspectives on learning that have had a significant impact on mathematics education over the last hundred years (De Corte, *et al.*, 1996; Greeno, Collins and Resnick, 1996). In this section, we address how each has affected approaches to mathematics education with particular reference to how each addresses

- procedural skills and conceptual knowledge
- their relative emphasis on the social and cultural dimensions of learning
- developing learners' ownership and responsibility for learning (with particular focus on self-regulated learning)
- assessment.

At the outset, we want to point out that of these three perspectives, two of them, the behavioural and cognitive, have had a more marked impact to date than the sociocultural perspective. However, in the last two decades a very significant shift towards socio-cultural theories reflects a wider movement in social science towards a more social and cultural understanding of human conduct. This is evidenced, as we noted earlier, in the adoption of situated cognition by the OECD as its preferred perspective on learning.

Definitions of learning

While a detailed exposition of the differences between these three perspectives on learning is beyond the scope of this report, we note how each understands learning and assessment in the context of mathematics education. In the behaviourist tradition learning is change in behaviour; in the cognitive tradition learning is change in thinking; and in the socio-cultural tradition learning is change in participation. These widely diverging definitions of learning draw the attention of mathematics teachers and curriculum designers to different sets of questions regarding assessment.

Learning theories and assessment practices

The three perspectives have different views of assessment. In the behaviourist tradition assessment involves an appraisal of how well learners have mastered component parts of skills, progressing from low-level to high-level skills. In the cognitive tradition, because learning is defined as change in thinking, the focus of assessment is on understanding change in learners' conceptual understanding, learning strategies and learners' thinking over extended periods, often involving real-world contexts. In the socio-cultural tradition, because learning is defined as engaged participation with agency, assessment focuses on learners' engagement with real-world tasks/problems, typically over an extended period of time. The cognitive and socio-cultural perspectives overlap in their focus on the value of real-world contexts and both involve an interactive component in assessment. In mathematics examination and assessment systems, the combined effect of the behavioural focus on decontextualised skills and the 'new' mathematics focus on abstraction have led to examinations and assessments that often pit the isolated learner against quite abstract tasks bearing little relation to real-world challenges of a mathematical nature.

Behaviourism: direct teaching followed by controlled practice

> *Conceptions of learning as incremental (with errors to be avoided or immediately stamped out), of assessment as appropriately implemented by reference to atomistic behavioural objectives, of teaching as the reinforcement of behaviour, of motivation as directly mediated by rewards and punishments, and of mathematics as precise, unambiguous, and yielding uniquely correct answers through the application of specific procedures remain prevalent in folk psychology and, as such, represent the legacy of behaviourism. Moreover, the development of behaviourism led to persisting views of learning hierarchies within mathematics, with harmful effects, according to L. B. Resnick. (1987a, pp. 48-49) (De Corte et al., 1996, p. 493)*

Learning and teaching in the behaviourist tradition

De Corte *et al.*'s summary of the impact of behaviourist thinking on mathematics education describes how it shaped conceptions of teaching, learning and assessment, and lent support to folk theories of teaching, learning and assessment. As we noted earlier (see chapter one), the persistence of and over-emphasis on learning hierarchies has been a distinct feature of mathematics education, and this is especially so for students deemed less able (Means and Knapp, 1991; Conway, 2002). Behaviourism put a premium on three basic pedagogical strategies: breaking down tasks into small and manageable pieces, teaching the basics first, and incrementally reinforcing or rewarding observable progress. From this perspective, knowledge can be seen as a hierarchical assembly or collection of associations or behavioural units.

Perhaps the most widely recognised and intuitively appealing implications of the behavioural perspective are its recommendations for designing teaching. These are the simplification and sequencing of tasks into discrete hierarchical steps and reinforcing successful approximations of desired activity. In sumarry, the hallmarks of behaviourism are presenting learning in small steps, in the simplest possible form, sequencing tasks in a hierarchy from the simple to the complex, and rewarding successful observed behaviours. Two problems associated with this approach to teaching are the assumption of 'vertical transfer' and the decomposition of activities such as reasoning and problem solving, resulting in a lack of task wholeness and authenticity. Vertical transfer assumes that learners will assemble the various associations or connections lower down on the learning hierarchy, and integrate these in order to eventually engage in higher order tasks. This vertical transfer problem is interwoven with what critics view as the lack of task authenticity when teaching is designed from a behavioural perspective. Thus, rather than involving learners in the full authenticity of say mathematical problem solving, a behavioural perspective focuses on teaching the fundamental elements (in the case of maths, basic arithmetic algorithms) prior to the more complex and contextually framed elements such as story problems (Koedinger and Nathan, 2004).

As De Corte *et al.* (1996) argue, behaviourism has had a powerful influence on views of teachers' and curriculum designers' understanding of learning and how best to assess it. In relation to assessment, behaviourism focuses on breaking down content into its constituent parts and assessing each part based on the assumption that, once it is known that learners can demonstrate their skill on the parts, the more integrative higher-order skills will flow naturally from these sub-skills. In contemporary context, the behaviourist perspective is evident in Precision Teaching[8] (Lindsley, 1990; Lindsley, 1991), and much drill and practice or computer-assisted instruction (CAI) software in mathematics (e.g. Mathblaster). In the case of drill-oriented computer software it is, typically, premised on a mastery framework of learning where students are rewarded for correct answers (typically 4 out of 5), and may then proceed to the next level of difficulty in the prescribed learning hierarchy.

Relationship between skills and development of concepts

One of the key features of behaviourism is its emphasis on teaching and assessing the component parts of skills prior to teaching the aspects of skill further up the learning hierarchy. In this sense, from a behavioural perspective learning involves the development of many component skills.

Learners' social and cultural background

Learners' social and cultural background is largely irrelevant from a behavioural perspective since what matters in terms of capacity to learn is the strength of learner's prior stimulus-response pairings related to key learning objectives. In fact this asocial view of the learner was one appealing feature of behaviourism in that it heralded and promised predictable programmed learning as a possibility for all learners.

8 Precision Teaching, as form of Applied Behaviour Analysis (ABA), is based on operant conditioning principles and focuses on pacing learners based on observations of learner behaviour on incrementally structured learning tasks. It has been widely used in special education, mathematics education and has also been employed to design computer-aided instruction (CAI).

Development of self-regulation

The development of self-regulation focuses on self-instruction with attention to identifying reinforcements that will strengthen desired behaviour. For example, Belfiore and Hornyak (1998) describe a homework completion programme involving a routine checklist to assist learners in self-instruction:

- Did I turn in yesterday's homework?
- Did I write all homework assignments in my notebook?
- Is all homework in my homework folder?
- Are all my materials to complete my homework with me?
- etc.

Learners can be helped in identifying and using self-reinforcement when an agreed criterion or standard has been reached by the student, e.g. when a certain number of mathematics problems have been completed.

In summary, a behavioural approach is consistent with a direct or transmission-oriented approach to teaching, and in the context of mathematics could provide support for the direct expository type of teaching associated with the new mathematics (see chapter one) movement where the role of the teacher is to convey/transmit the logical, hierarchically structured nature of mathematics. Behaviourally-inspired teaching focuses on supporting individual learners' movement through a prescribed hierarchy, from simple to more complex learning skills. The exact way in which learners vertically transfer and integrate basic skills with more complex in order to solve problems in mathematics is an aspect of learning that is not addressed very well in the behavioural approach.

Cognitive: promoting active learning and problem solving

> *From a contemporary perspective, Piaget's main contribution to mathematics education was his demonstration of the complexity of children's thinking and the qualitative differences in thinking at various stages of development. He is acknowledged as a major inspiration of the radical shift to the conception of the child as an active constructor of knowledge.*
>
> **(De Corte *et al.*, 1996)**

Learning and teaching

The emergence of cognitive perspectives on teaching and learning in the 1960s marked a very significant break with the traditions of behaviourism (Gardner, 1985; Collins, Greeno and Resnick, 1996). In the context of current developments in mathematics education, cognitive perspectives are informative in four ways: (i) their conception of active learning, and assessment instruments congruent with such a view of learning; (ii) the notion of cognitive challenge and how various degrees of cognitive challenge are embedded in all teaching and assessment situations, (iii) a conception of expert problem-solving or competent problem-solving in mathematics; and (iv) the development of significant literature demonstrating the applicability and efficacy of teaching self-regulated learning using various strategies (e.g. cognitive strategy instruction, goal setting, self-assessment, peer assessment, and formative feedback from teachers) for children of all ages and abilities.

Firstly cognitive perspectives represent a distinct break with the epistemology underpinning behaviourism. Resnick conveys the shift in epistemology well, commenting that, 'Learning occurs not by recording information but by interpreting it' (1989, p. 2). Over time, the notion that learning takes place through the active construction of knowledge by the individual learners has gained considerable attention among teachers and curriculum designers, and is now reflected in mathematics curriculum documents internationally. Knowledge, rather than being 'out there' (the basic assumption from behaviourist stance), is constructed by learners' actions on the world. As such, knowledge is made as learners engage with and experience the world.

The cognitive perspective has provided many important insights with which to plan both classroom teaching and assessment. Among the most important of these are that learning is active, learning is about the construction of meaning, learning is both helped and hindered by our prior knowledge and experience, learning reorganises our minds, the mind develops in stages, and learning is more often than not unsettling. Based on these insights, a diverse range of strategies has been developed for classroom practice, many of which have been evident in various textbooks, teacher handbooks and curricular documents over the last thirty years. Much of the appeal of cognitive theories, in the international context, grew out of the desire to move away from didactic and transmission-oriented teaching. Many advocates of active learning would echo Dewey (1933, p. 201), who in his book *How We Think*, in opposition to the didactic nature of classroom teaching at that time, spoke out against 'the complete domination of instruction by rehearsing second-hand information, by memorizing for the sake of producing correct replies at the proper time'. Anticipating some of the arguments and claims made by cognitive and educational psychologists over the last forty years, Dewey argued for the importance of students' active involvement in the learning process and problem-solving as the context within which to learn information.

Secondly in terms of assessment, cognitive perspectives focus on more extended, problem-focused and authentic tasks consistent with learners' mental models of key concepts in mathematics. Cognitive perspectives on assessment have provided complex models of cognition and the types of inference that can be drawn from performance on assessment items. For example, one of the legacies of the cognitive perspective was to draw educators' attention to levels of cognitive challenge embedded in assessment items as evidenced in the development and widespread use of Bloom's Taxonomy with its six levels of thinking representing increasing cognitive demands on learners. Bloom's original *Taxonomy of the Cognitive Domain* (Bloom *et al.*, 1956) is possibly one of the most widely used frameworks in education around the world. As Anderson (2002) commented, '…numerous classroom teachers can still recite them: knowledge, comprehension, application, analysis, synthesis and evaluation. It is not unusual to walk into a classroom and see a poster of the taxonomic categories (hand drawn or commercially produced) on a wall or bulletin board.' Furthermore, the original *Taxonomy* has been translated into 22 languages since its first publication in 1956 (Anderson, 2002). The revised *Taxonomy* (Anderson *et al.*, 2001), like the original, has six levels in the cognitive domain: remember, understand, apply, analyse, evaluate and create. However, the cognitive processes in the revised *Taxonomy* underwent some notable changes: three processes were renamed, the order of two was changed, and the verb form of processes was used instead of the noun form (see *Classroom Assessment: Enhancing the Quality of Teacher Decision-Making*, Anderson, 2003 for discussion of the Revised Bloom's Taxonomy in the context of teacher decision-making and its potential to change curriculum planning and assessment; Conway, 2005d). Consistent with the general principle of assessing cognitive challenge or cognitive complexity, this is a central design feature in the PISA mathematics literacy items with three levels of challenge, or 'competency clusters', to use the PISA terminology. These are: (i) *reproduction*, that is, performing calculations, solving equations, reproducing memorised facts or 'solving' well-rehearsed routine problems; (ii) *connections*, that is, integrating information, making connections within and across mathematical domains, or solving problems using familiar procedures in contexts; and (iii) *reflection*, that is, recognising and extracting the mathematics in problem situations, and using that mathematics to solve problems, analysing and developing models and strategies or making mathematical arguments and generalisations (OECD, 2003, p. 49; Close and Oldham, 2005, p. 176; Cosgrove *et al.*, 2005, pp. 7-9).

Thirdly cognitive perspectives provide a model of competent problem-solving and strategies for teaching it in mathematics and other domains. As De Corte *et al.* (1996) note:

> *A major finding of the analysis of expertise is that expert problem solvers master a large, well organized, and flexibly accessible domain-specific knowledge base… Indeed, it has been shown that experts differ from novices in that their knowledge base is better and more dynamically structured, and as a consequence more flexibly accessible. (Chi et al., 1988, p. 504)*

One of the most fruitful and practical ideas emerging from a number of strands in cognitive research is the efficacy of teachers modelling and making explicit the strategies they adopt in understanding and solving problems via teacher 'think alouds' and/or providing 'think sheets' for students to assist them in monitoring and controlling their own thinking while solving problems. Considerable research suggests that teachers rarely use think alouds and other strategies that model and make explicit complex and expert problem-solving. This is especially troubling as such strategies have been demonstrated to be effective with lower-achieving students in both primary and post-primary settings.

Relationship between components: lower-order and higher-order thinking

While it might be overstating it to say that the jury is out within the cognitive perspective as to the relative merits of adopting a bottom-up or top-down stance on the relationship between lower-order and higher-order thinking, this issue has been contested in the cognitive tradition. As Greeno, Pearson and Schoenfeld (1997) note:

> *According to some analyses, the elementary aspects of achievement - routine skills, facts and concepts - are prerequisite to learning more complex 'higher order' aspects of achievement. However, this is a disputed view. An alternative is that strategic, meta-cognitive, and epistemic aspects of achievement are more fundamental than detailed procedures and routine to effective intellectual functioning. (p. 159)*

For example, if students have not had experiences where they have learned to think mathematically (a higher-order strategic resource), it is unlikely that they will deploy computational skills in a timely and appropriate fashion. While it has been agreed that both lower and higher-order skills are important, it is vital, from a teaching perspective, to have some clarity on where the emphasis should lie: teaching basic skills first followed by higher-order skills, or embedding basic skills within complex problem-oriented situations where learners grapple with multiple levels simultaneously.

Development of self-regulation

The development of strategic, purposeful self-regulated learning (SRL) is one of the hallmarks of the cognitive perspective. SRL has become an increasingly important policy focus in mathematics education internationally. Its focus is on learning to learn strategies through a number of lines of research, including cognitive strategy instruction, motivation (particularly goal theory and related goal-setting strategies), and formative feedback (teacher, peer and self-assessment, and self-regulation). Again, these ideas from learning sciences have filtered into policy-making and international assessments. This is clear from the focus on self-regulation as a core competency in APEC policy documents (APEC, 2004) and the inclusion of items in PISA on self-regulation, as well as the development within PISA of a theoretical framework for understanding self-regulated learning as a critically important cross-curricular competence (Baumert, *et al.* 1999).

While behaviourist and cognitive theories are based on very different assumptions about learning, knowing and intelligence, and have very different implications for classroom practice, they share one defining feature, namely their focus on the individual learner with little emphasis on the cultural and historical context of mathematics learning. As Conway (2002) notes:

> *Rather than viewing the learner as part of family, community and social group embedded in a particular time and place, both the behavioural and cognitive perspectives portray learning as primarily a solo undertaking. Thus, what is neglected, in this focus on the solo learner, is how the learner is situated amidst levels of guidance by more knowledgeable others, nurtured via social support, influenced by peer norms, and shapes and is shaped through engaging in communication with other humans and various media within evolving cultural and historical circumstances. (p. 76)*

In a similar vein, De Corte *et al.* (1996) claim that, 'while arguing that ideas about social construction are integral to Piaget's genetic epistemology, it is acknowledged that there are grounds for the caricature of a Piagetian, as Youniss and Damon (1992) note, as the "apocryphal child who discovers formal properties of things, such as number, while playing alone with pebbles on the beach"' (p. 268). These and other criticisms of cognitive perspectives on learning and mathematics led researchers to look toward more culturally embedded conceptions of learning. Socio-cultural perspectives offered such a view and are best seen as a cluster of related theories, of which situated cognition is one of a range of possible theories.

Socio-cultural perspectives: engaged participation

Socio-cultural theories are consistent with constructivist learning frameworks although they adopt a more socially embedded view of learners than Piagetian/individual constructivist models (Prawat and Floden, 1994) and are most closely associated with the writings of Dewey and Vygotsky in the early part of the twentieth century. More recently, various authors writing from a situated cognition perspective (Brown, Collins and Duguid, 1989) have drawn upon the cultural and social views of learning in the writings of Vygotsky (his Russian contemporaries Luria and Leontiev) and Dewey. Socio-cultural theories assert that the mind originates dialectically through the social and material history of a culture which a person inhabits (Vygotsky, 1978). The emergence of socio-cultural perspectives in mathematics education reflects a wider 'social' turn in understanding learning in education (Lerman, 2002). This position is in marked contrast to the view that the mind has its primary origin in the structures of the objective world (behaviourist position) or that it has its origin in the order-imposing structures of the mind (cognitive perspective) (Case, 1996).

Learning and teaching

What are the implications of these assumptions for teaching and assessment in mathematics? In terms of pedagogy, rather than focusing on individualized teaching, socio-cultural theories put a heavy emphasis on fostering communities of learners (Prawat, 1992), which provide not only opportunities for cognitive development but also the development of students' identities as numerate members of knowledge-building communities. Brown (1994) outlined a coherent set of principles underpinning the notion of a 'community of learners' as well key strategies for its implementation. These principles are

- academic learning as active, strategic, self-motivated and purposeful
- classrooms as settings for multiple zones of proximal development through structured support via teacher, peer and technology-aided assistance of learners
- legitimisation of individual differences
- developing communities of discourse and practice, and
- teaching deep conceptual content that is sensitive to the developmental nature of students' knowledge in particular subject areas.

The integrated implementation of these five principles forms the support for the emergence of communities of learners in classroom settings (Brown, 1997). Two of the key constructs in Fostering a Community of Learners (FCL) are classroom discourse patterns and participation structures.

Firstly, based upon the insight that much academic learning is active, strategic, self-motivated and purposeful, Brown emphasised how FCLs ought to focus on the development of students' capacity to think about thinking, that is to engage in metacognition. As such, a key feature of FCLs is the promotion of a culture of meta-cognition, directed towards the development of learning to learn strategies. In the case of mathematics, for example, teaching students self-monitoring strategies becomes an essential part of teaching (see Schoenfeld, 1985). Verschaffel, De Corte, Lasure *et al.* (1999) describe this in terms of modelling various heuristic strategies for students in the course of problem-solving.

- Build a mental representation of the problem.
  Draw a picture; make a list, scheme or table; distinguish relevant from irrelevant; use your real-world knowledge.

- Decide how to solve the problem.
  Make a flow-chart; guess and check; look for a pattern; simplify the numbers.

- Undertake the necessary calculations.

- Interpret the outcome and formulate an answer.

- Evaluate the solution in terms of the problem.

Secondly, drawing upon Vygotsky's Zone of Proximal Development (ZPD), that is, the difference between what a learner can do by themselves versus what they can do with the assistance of another person and/or tool, Brown emphasised teaching toward the upper rather than lower bounds of students' competence. Much contemporary pedagogical practice, she claims, focuses on matching teaching with students' existing levels of competence – that is, the lower bounds of competence. In many mathematics classrooms, with their focus on students' individual capabilities, there are few opportunities for students to learn from each other (see TIMSS video study findings in chapter two). Japanese 8th grade mathematics classrooms were a notable exception in that teachers often started class by posing a complex written problem on the blackboard, telling students to think and consult with one another, after which students shared potential solutions with the whole class. In marked contrast, USA and German mathematics teaching focused on providing explicit fast-paced, teacher-led instruction in order to prepare students for individual practice with algorithms on a series of almost identical mathematical problems (Stigler and Hiebert, 1999).

Thirdly, FCLs are intended to value and nurture students' diverse cultural perspectives, support multiple entry points into subject matter (via art, music, drama, technology, story, text, etc.), and foster diversity in the distribution of expert knowledge. The emphasis on entry points in Brown's FCL shares similarities with Gardner's emphasis on entry points (1999) and RME's focus on rich contexts for engaging students in horizontal mathematising. For example, Freudenthal (1991) stressed the importance of rooting mathematising in rich contexts such as location, story, projects, themes, and newspaper clippings (pp. 74-75).

Fourthly, FCLs are premised on the belief that higher-level thinking is an internalised dialogue. Based on this premise, classrooms ought to be discursive environments where students can engage in conjecture, hypothesis testing, and speculation centred around 'hot' mathematical ideas in much the way mathematicians engage with mathematics.

Fifthly, in FCL one of the challenges for teachers is to teach deep conceptual content that is sensitive to the developmental nature of students' mathematical knowledge. One of the major findings from cognitive research over the last twenty years is that children and adolescents 'have a strong intuitive understanding of concepts that can support instruction, which in turn can further advance their conceptual understanding' (Greeno, Pearson, and Schoenfeld, 1997; see also Gopnik, Meltzoff and Kuhl, 1999 for a very readable account of emergent conceptual understanding in children, and Mix, Levine, Huttenlocher, 2002 for an account of the emergence of quantitative reasoning in infancy and early childhood). This is in marked contrast to the over-generalisation of the Piagetian perspective, which often resulted in teachers underestimating students' thinking capabilities. As De Corte *et al.* state, 'we doubt that educational practice needs to be guided very strongly by ideas about the development of general schemata of logico-deductive operations in children's reasoning' (1996, p. 18).

Relationship between basic skills and higher-order skills: lessons from cognition in the wild

Whereas there is disagreement in the cognitive perspective as to whether bottom-up or top-down teaching strategies are optimal, the socio-cultural perspective puts a strong emphasis on teaching/learning basic skills within the context of authentic, real-world teaching situations. Among the sources of evidence cited in support of this strategy, which is completely at odds with the behavioural bottom-up, skills-first stance, are a number of studies of how people of all ages learn complex skills in out-of-school contexts. These have investigated the way in which children and adults learn complex quantitative reasoning and problem-solving when tasks are authentic and provide opportunities for different levels of support and slowly increasing levels of participation (Lave, 1988). For example, research by Nuñes, Schleiman and Carraher

(1993) showed how children in Brazil who were working as street vendors had developed complex algorithms to calculate price and quantity in a fast and accurate fashion. The same children when presented with similar tasks in symbolic form in school situations performed well below their on-street level: their 'school scores' were between a quarter and half of their 'street scores'. The researchers in this study drew a number of conclusions including the following: school did not utilise these children's out-of-school mathematical knowledge; the children had developed sophisticated algorithms, often supported by the active use of materials in specific patterns to develop fast and accurate calculations; and the children learned basic computation within more complex problem-solving activities.

Development of self-regulated learning

The cognitive perspective provides valuable evidence of the efficacy of strategic and purposeful learning, and that such strategic and purposeful engagement can be taught to children and adolescents of varying abilities. The socio-cultural perspective builds on these insights by emphasising the activity settings within which such self-regulatory skills can be developed (Engeström, 1999; Allal and Saada-Robert, 1992; Allal, 2005). Rooted in a socio-cultural perspective, Allal has (Allal, 2005) developed a multi-level regulation framework in which she argues that regulation of learning must be addressed at school, classroom and individual student levels. Thus, in order for students to learn effective self-regulation skills there must be congruence between all three levels.

Three views of assessment

According to their different definitions of learning, each of the three approaches projects a different message about what is important in assessment. From a behavioural perspective, the assessment is concerned with facts, procedures, routines and skills. They are paramount because they are viewed as prerequisites for more advanced skills. Consequently, the assessment emphasis is on components of skills prior to composite skills. Consistent with the focus on learning hierarchies from simple to more advanced skills, assessments are often characterised by hierarchies where the test items in general, or within multiple items on sub-sections of a test, progress from simple to more complex challenges. Thus the behavioural emphasis on skill decomposition is reflected in both teaching and assessment.

A cognitive perspective on assessment emphasises the appraisal of a wide range of knowledge, including conceptual understanding, strategies, meta-cognition (i.e. thinking about thinking), and beliefs. Given the wide-ranging nature of the target of assessment in the cognitive perspective, a similarly wide range of assessment formats is necessary to reflect the different cognitive processes. For example, conceptual understanding and mathematical problem-solving are difficult to assess in test items that only ask the student to produce an answer that can only be marked as correct or incorrect. Assessment of the more challenging thought processes typical of the cognitive perspective can often only be appraised through open-ended response items, extended response items and/or portfolios where students can demonstrate their thinking processes over extended periods of time. Greeno, Pearson and Schoenfeld (1997) characterise the shift in emphasis from behavioural to cognitive perspectives in terms of a move from 'focusing on how much knowledge someone has to providing adequate characterization of just what is the knowledge someone has' (p. 13). A description of the nature of expert learners' knowledge as integrated, flexible and dynamic (Bransford, Brown and Cocking, 2000) has been one of the most valuable contributions of the cognitive perspective to understanding human competence. Consequently, the assessment of expertise ought to provide opportunities to appraise such knowledge. From a cognitive perspective, test/examination items that merely require students, in mathematics or any other subject, to deploy well-rehearsed procedures in solving predictable problem types do little to assess the degree of learners' expert knowledge. Open-ended and extended response type assessments allow assessors to see the variety of ways in which students solve problems (e.g. some may do so more intuitively than others, who may rely on careful step-by-step analysis). The key point here is that expertise is characterised by flexible and generative thinking which will often remain invisible unless assessments are suitably authentic and open-ended to bring forth such thinking (Greeno, Collins and Resnick, 1996; Bransford, Brown and Cocking, 2000).

The socio-cultural perspective on assessment is radically different in its focus from almost all contemporary school assessments. Assessment from a socio-cultural standpoint focuses on learners' capacity to participate in particular activities in order to demonstrate competence in disciplinary ways of knowing. In the case of mathematics, the capacity to engage in problem posing, formulate mathematical models, test such models in symbolic form, and when necessary reflect on the meaning of solutions for the real world, characterises the work of mathematicians. Socio-cultural assessments seek to assess participation as it is embedded in meaningful activities since it is assumed that learners' knowledge is intimately linked with various tools and social supports within such activities. The socio-cultural preference for real-world performance assessment seems idealistic and impractical when held up against the dominant assessment formats used in almost all large-scale mathematics assessments. A sociocultural perspective does not attempt to assess sub-skills in isolation (as in the behavioural model), but assumes that performing complex skills which require the meaningful deployment of relevant routines and procedures, as resources, is a far better way of assessing learners' actual integration of such procedural and factual resources. Assessments of the type preferred from a socio-cultural perspective are rarely seen in mathematics. However, not utilising such assessments seems very much like trying to assess a good band by sitting them down to perform an on-demand paper-and-pencil test rather than asking them to play some music! Typically, large-scale mathematics testing/examinations rarely if ever ask students to perform the mathematical equivalent of playing in a band or even as soloists.

Changing views of assessment: a three-part learning-based model

In this section, we build on some of the aforementioned key developments in the learning/cognitive sciences and also include concurrent developments in the field of measurement, which together are beginning to change the range and feasibility of assessment options for schools and educational policy makers. One feature of educational change we have noted earlier is the persistence of old forms of assessment despite ambitious new definitions of learning (Broadfoot, 2001; Bransford, Brown, and Cocking, 2001). This lag remains a real challenge in improving teaching in key curricular areas.

Educational assessment seeks to determine how well students are learning and is an integral part of the quest for improved education. When used appropriately, it provides feedback to students, educators, parents, policy makers and the public about the effectiveness of educational services. With the movement over the past two decades towards setting challenging academic standards and measuring students' progress in meeting those standards, especially in mathematics (see NCTM, 2000), educational assessment is playing a greater role in decision-making than ever before. In turn, education stakeholders are questioning whether current large-scale assessment/testing practices are yielding the most useful kinds of information for informing and improving education. For example, classroom assessments, which have the largest potential to enhance instruction and learning, are not being used to their fullest potential (Black and Wiliam, 1998a; Black and Wiliam, 1998b).

The US National Research Council (NRC) committee argues cogently that advances in the learning and measurement sciences make this an opportune time to rethink the fundamental scientific principles and philosophical assumptions serving as the foundations for current approaches to assessment. Advances in the learning/cognitive sciences have broadened the conception of those aspects of learning that it is most important to assess, and advances in measurement have expanded the capability to interpret more complex forms of evidence derived from student performance.

In the report *Knowing What Students Know* (Pellegrino, *et al*, 2001) the committee explains that every assessment, regardless of its purpose, rests on three pillars: a model of how students represent knowledge and develop competence in the subject domain, tasks or situations that allow one to observe students' performance, and an interpretative method for drawing inferences from the performance evidence obtained through student interactions with the chosen domain tasks. These three elements comprise what the committee refers to as the 'assessment triangle' and underlie all assessments.

All three must be co-ordinated; that is, the three constituent parts of all assessments include ideas about cognition, about observation and about interpretation. Moreover, and most importantly, these three elements must be explicitly connected and designed as a co-ordinated whole. It is this connectedness that distinguishes this model from most offered in mathematics education.

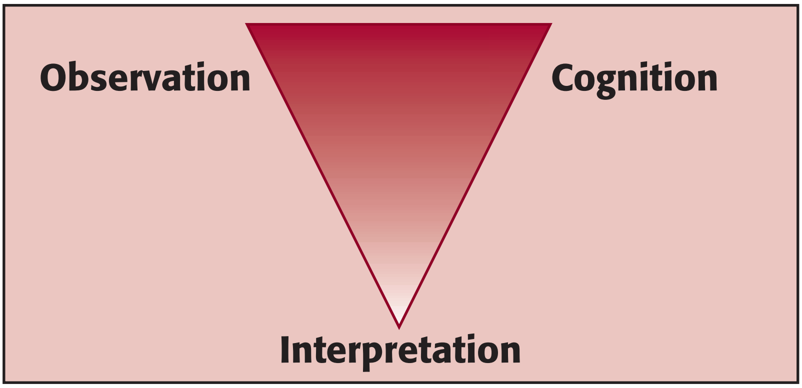


Figure 1: A model for the assessment of learning in mathematics

Traditionally, the mathematical task has been to the fore (see, for example, Kulm, 1990; Lesh and Lamon, 1992; and Mathematical Sciences Education Board, 1993), and discussion of cognition and interpretation has been most often ignored.

- Cognition
  Model of how students represent knowledge and develop competence in the domain.
- Observation
  Tasks or situations that allow one to observe students' performance.
- Interpretation
  Method for making sense of the data.

The committee further asserts that current assessment strategies are, generally speaking, based on highly restrictive models of learning. They note that the model of learning should serve as a unifying construct across the three elements. The model of learning should be based on current knowledge about human cognition, and should serve as the nucleus, or glue, that coheres curriculum, instruction, and assessment (see Gardner, 2006 for a detailed discussion of the increasingly central role accorded learning in framing assessment based on over fifteen years research undertaken by the A** in the England, Scotland Wales and Northern Ireland). Put simply, educational assessment must be aligned with curriculum and instruction if it is to support the learning of mathematics (or any other content). It is striking to again see the parallels here with Cognitively Guided Instruction and lesson study for example. The insights from the Realistic Mathematics Education movement can be seen as providing a very rich understanding of cognition, as well as opportunities for careful observation of students engaging with well-structured tasks - that is, ones likely to provide optimal inferences of what students know and understand.

Despite the plethora of evidence making a case for new approaches to assessment, and a considerable number of small-scale, well-researched projects presenting generally positive findings on the use of alternative assessments more closely linked to contemporary cognitive and socio-cultural conceptions of learning and mathematics education, the adoption of these assessment modes for large-scale assessment in mathematics has been slow, and so far the 'preponderance of assessment, in practice, remains unreformed' (Verschaffel, Greer and De Corte, 2000, p. 116). We provide three examples of initiatives in assessment which put learning more to the fore: (i) assessment within RME, (ii) a statewide reform of the upper post-primary mathematics assessment system in Victoria, Australia, and (iii) NCTM-inspired reforms in classroom assessment in the US.

Assessment within Realistic Mathematics Education (RME)

A detailed account of assessment in RME is available in Van den Heuvel-Panhuizen (1996). As a basic principle, RME (see section 3.2) seeks to design assessment and learning opportunities that are genuine problems - 'rich, non-mathematical contexts that are open to mathematization' (Van den Heuvel-Panhuizen, 1996, p. 19). Van den Heuvel-Panhuizen (1996) distinguishes RME assessment items from traditional word story problems which are often 'rather unappealing, dressed up problems in which context is merely window dressing for the mathematics put there' (p. 20). Thus RME textbooks include practical application problems rather than artificial word story problems. For example, a typical RME application problem reads as follows:

> *Mr Jansen lives in Utrecht. He must be at Zwolle at 9.00 Tuesday morning. Which train should he take? (Check the train schedule.)*

There are number of characteristics of this problem that mark it as different from the traditional word story problems in most mathematics textbooks. Van den Heuvel-Panhuizen (1996) explained the RME rationale for such assessment items as follows:

> *This problem is nearly unsolvable if one does not place oneself in the context. It is also a problem where the students need not marginalize their own experiences. At the same time, this example shows that true application problems have more than one solution and that, in addition to written information, one can also use drawings, tables, graphs, newspaper clippings and suchlike. Characteristic of this kind of problem is the fact that one cannot learn to do them by distinguishing certain types of*

problems and then applying fixed solution procedures. The object here is for the student to place him or herself in the context and then make certain assumptions (such as how far Mr. Jansen lives from the station and how important it is that he arrives at his destination on time). (p. 20-21)

Verschaffel, Greer and De Corte (2000) note the defining characteristics of RME assessments as follows: extensive use of visual elements; provision of various types of materials (e.g. train timetables); all the information may not be provided; there is a general rather than single answer; a focus on relevant and essential contexts; asking questions to which students might want to know the answer; and using questions that involve computations before formal techniques for those computations have been taught. This final RME strategy, using questions involving computations before formal techniques for those computations have been taught, is consistent with the sociocultural principle of presenting material above what children are able to do independently but providing sufficient support so that they can accomplish the task with others prior to independent performance - i.e. working in the zone of proximal development (Vygotsky, 1978; Bruner, 1996).

State-wide reform of an upper post-primary mathematics assessment system

In Victoria, Australia a study of the impact of alternative mathematics assessment in the state-level Victorian Certificate of Education (VCE) demonstrated that there was a positive backwash effect on teaching in Years 7-10 after changes were made in assessment practices in Years 11 and 12. Year 11 and 12 assessments comprise four components: (a) multiple-choice skills test, (b) an extended answer analytic test, (c) a 10-hour 'Challenging Problem', and (d) a 20-hour 'Investigative Project' (Clarke and Stephens, 1996). These assessment tasks are undertaken in the context of a curriculum in Victoria which involves three types of formal work requirements for all students (Barnes, Clarke and Stephens, 2000):

> ***problem-solving and modelling:*** *the creative application of mathematical knowledge and skills to solve problems in unfamiliar situations, including reallife situations*
>
> ***skills practice and standard applications:*** *the study of aspects of the existing body of mathematical knowledge through learning and practising mathematical algorithms, routines and techniques, and using them to find solutions to standard problems*
>
> ***projects:*** *extended, independent investigations involving the use of mathematics. (p. 630)*

Clarke and Stephens' study, based on document analysis and the administration of questionnaires and interviews to teachers, demonstrated a strong positive ripple effect in which changes in Year 11 and 12 assessment practices were reshaping teaching and learning lower down the school system. Finally, it should be noticed that the third and fourth assessment tasks, the 10-hour 'Challenging Problem' and the 20-hour 'Investigative Project', are consistent with socio-cultural and cognitive perspectives on learning given their emphasis on extended performance assessments rooted in realworld contexts.

The Australian experience is informative in a number of ways: (i) the ripple effect on teaching and assessment practice; (ii) the pressure to scale back the 1990s assessment system due to workload and verification problems; and (iii) the continued use of problem-solving and focus on non-routine in the scaled back assessment since 2000. Firstly the 'ripple effect'; that is, how making a change in assessments can affect curriculum in terms of teachers' classroom practice. A four-year study comparing assessment practices in upper post-primary mathematics in Victoria and New South Wales (Barnes, Clarke, and Stephens, 2000) provides strong evidence that changes in assessment leveraged change in curriculum. Barnes *et al*. (2000) note that they 'sought to examine empirically the prevailing assumption that changing assessment can leverage curricular reform' (p. 623). They demonstrate that there was indeed 'congruence between mandated assessment and school wide instructional practice' and that this was found in the case of 'two states whose high-stakes assessment embodied quite contrasting values' (p. 623). Barnes *et al*. note that the support provided to teachers in terms of professional development and guidance in relation to assessment was an important factor in influencing the degree to which changing the assessments in Victoria in the early 1990s had a 'ripple effect' on classroom practices:

> *…these assessments exercise a significant leverage on teaching and forms of assessment especially when the assessment tasks are set by the examining body with extensive guidance for teachers in applying criterion-based assessment. (p. 645)*

Secondly the four-part assessment system outlined above was scaled back in 2000 and Victoria now uses a two-part system. The reasons for the change were that there were serious concerns about the authentication and verification of student work, and teacher workloads. As Brew, Tobias and Leigh-Lancaster (2001) note:

> *Following the 1997 review of the Victorian Certificate of Education (VCE) the school-based, but centrally set and externally reviewed Common Assessment Task (CAT) was discontinued in the revised VCE 2000. This action was taken in response to real and perceived problems associated with excessive student and teacher workloads and authentication of student work. (p. 98)*

Thirdly despite the scaled-back assessment system, students and teachers are still engaging in problem-solving, modelling approaches and addressing non-routine problems, partly as a legacy of familiarity with such tasks during the earlier 1990s VCE, and partly as a result of the emphasis on such tasks in the revised curriculum. As Brew, Tobias and Leigh-Lancaster note:

> *In Victoria the implementation of the new school coursework structure using application tasks, analysis tasks and tests, has been supported by the publication of considerable resources by the former Board of Studies and the Victorian Curriculum and Assessment Authority (VCAA) to encourage teachers to continue to include a variety of contexts for application tasks and different types of analysis tasks in the new school-based assessment. Within this current structure teachers are able to draw on ideas and approaches from previous extended investigative and problem solving CATs. Brew et al. (2000) provided evidence that in the first year of implementation of the revised VCE, investigations, problem-solving and modelling approaches continued to be an important component of the school-based assessment in many Victorian schools. In part, this arises from the nature of the outcomes for the revised VCE Mathematics courses that require students to apply mathematical processes in non-routine contexts. (p. 99)*

NCTM-inspired reforms in classroom assessment in the US

In the US, the National Council of Teachers of Mathematics (NCTM) has made a strong case over the last fifteen years for a range of assessment tools that will provide the kind of rich data needed to understand the ambitious teaching-for-understanding stance it has been espousing (NCTM, 1989; NCTM, 1995). NCTM has argued that assessment should be an integral part of the learning process, and that students should have opportunities to 'express mathematical ideas by speaking, writing, demonstrating, and depicting them visually' (NCTM, 1989, p. 14). The NCTM's position on assessment reflects a movement internationally in which 'mathematics educators are moving from a view of mathematics as a fixed and unchanging collection of facts and skills to an emphasis on the importance in mathematics learning of conjecturing, communicating, problem-solving and logical reasoning' (Lester, Lambdin and Preston, 1997, p. 292). Lester *et al.* (1997) identify three key shifts in thinking that have occurred in mathematics education, encompassing the nature of mathematics, the nature of mathematics learning, and the nature of mathematics teaching. These in turn have promoted changes in mathematics assessment in the US (see Table 4).

Table 4: Changes in views of mathematics, mathematics learning and mathematics teaching.

Assumptions	Traditional view	Modern view
The nature of mathematics	Mathematics is nothing more than a list of mechanistic condition/ action rules	Mathematics is a science of patterns
The nature of mathematics learning	Mathematics learning is a cumulative process of gradually adding, deleting, and debugging facts, rules and skills	Humans are model builders, theory builders; they construct their knowledge to describe, explain, create, modify, adapt, predict, and control complex systems in the real world (real or possible)
The nature of mathematics teaching	Teaching involves demonstrating, monitoring student activity, and correcting errors	Teaching is an act of enabling students to construct and explore complex systems

They also note that a new generation of ICTs such as 'handheld calculators and notebook computers with graphing and symbol manipulation capabilities enables students to think differently, not just faster' (p. 293), and argue that, in this new landscape, teachers have to determine the optimal uses of technology to enhance learning and then decide on how best to assess the types of mathematical reasoning made possible by ICTs. Approaches to alternative assessments in mathematics education include the use of portfolios, assessments of cooperative group work, use of more structured observations of students, interviews with students, open-ended and extended tasks (see Victoria, Australia), concept maps, revision of student work (this was the strategy used in the Connected Mathematics Project), and student journals. Scoring of these various alternative assessments has adopted one of three strategies: general impression, analytic or holistic scoring. In using general impression scoring, teachers apply their past experience to gauge student work and award a score/mark rather than relying on specified criteria. In analytic scoring, the most time-consuming, teachers focus on the components of, for example, a problem-solving task and rate students' response on pre-specified criteria for each component of the problem. In the case of holistic scoring, a pre-specified rubric or set of criteria are used to provide an appraisal of students' whole response to a unit of work or problem-solving task.

In addition to creating a context for new approaches to assessment, the NCTM identified six standards for judging the quality of mathematics assessments: mathematics, learning, equity, openness, inferences and coherence.

- Standard 1 focuses on mathematics in terms of the value of choosing worthwhile mathematical ideas and by implication raises questions about the atomistic content of many traditional standardised mathematics tests with their focus on isolated bits of knowledge.

- Standard 2 focuses on learning in order to ensure that assessments are embedded in the curriculum and inform teaching.
- Standard 3 focuses on equity and opportunity and emphasises that assessments should give every student an opportunity to demonstrate mathematical competence.
- Standard 4 focuses on openness and notes that traditionally testing and examinations have been a very secretive process where test questions, sample answers and criteria for assessing responses have been closely guarded by unidentified authorities. NCTM argues for a more open approach where criteria and scoring procedures in particular are made public.
- Standard 5 focuses on inferences and redefining the traditional psychometric testing principles of reliability and validity given new curriculum-embedded uses of assessment. In the case of reliability, it makes little sense to think of new assessments being reliable across test instances (i.e. test-retest reliability, one of the traditional psychometric conceptions of reliability) when the goal is to use these assessments to assess change in learners' understanding. In the case of validity, NCTM makes a case for moving away from thinking of it in terms of an inherent feature of the test itself towards a view of validity as the quality of inferences made from the test/assessment.
- Standard 6 focuses on coherence and stresses the 'goodness of fit' between each type of assessment instrument in terms of the purposes for which it is used.

Many authors make cogent arguments for alternative assessment in mathematics education (e.g. Lester *et al.*, 1997). Despite the many advantages of alternative assessments, Lester *et al.* (1997) also identify problems with their implementation including time constraints, monetary costs, teachers' limited knowledge of alternative assessments, and difficulties in creating authentic tasks. We might add another problem: the challenge of creating alternative assessments which are fair, as some research demonstrates, according to Elwood and Carlisle (2003), that there are gender differences in how students respond to more realistic-oriented assessment items. Furthermore, Cooper and Dunne (2000) have argued that there are significant differences in how students from different socio-economic groups respond to realistic-oriented assessments, with lower SES students less likely to draw upon relevant knowledge. In the USA, the NCTM view of mathematics reform was influenced by developments in cognitive and learning sciences, rather than a domain-specific view of mathematics as in the highly influential Realistic Mathematics Education movement which originated in the Netherlands.

3.3 Realistic Mathematics Education (RME) and learning

This section of the chapter examines the work of Hans Freudenthal and RME, focusing on four aspects: the roots of RME, big ideas in RME, the legacy of RME, and challenges in living out RME in practice.

RME's roots

> *During his professional life, Hans Freudenthal's views contradicted almost every contemporary approach to educational reform: the 'new' mathematics, operationalized objectives, rigid forms of assessment, standardized quantitative empirical research, a strict division of labour between*

> *curriculum research and development, or between development and implementation. (Gravemeijer and Terwel, 2000, p. 777)*

RME has its roots in the work of Hans Freudenthal, a mathematician turned mathematics educator, highly critical of mainstream mathematics education from the 1950s given its 'instructional design' emphasis and related hierarchical assumptions, its basis in Bloom's *Taxonomy*, and its measurement focus (see Freudenthal's book *Weeding and Sowing*, 1980). Freudenthal died in the early 1990s but his work and ideas are being developed through research undertaken in the Freudenthal Institute in the Netherlands, as well as by an increasing number of researchers using RME to inform their own work (e.g. Cobb *et al.*, 1997; Romberg, 2000). Good primary source overviews of RME are available in Freudenthal's last book *Revisiting Mathematics Education: China Lectures* (1991) and a collection of articles published by his colleagues in 1993 and reprinted in 2001 titled *The Legacy of Hans Freudenthal* (Streefland, 2001).

The big ideas in RME

Freudenthal's view of maths education cannot be appreciated without understanding his objections and resistance to the formal and abstract ideas at the core of the new or mathematics movement. While new mathematics elevated abstraction as its highest value, Freudenthal saw this as its primary weakness, stating that, 'In an objective sense the most abstract mathematics is without doubt also the most flexible. But not subjectively, since it is wasted on individuals who are not able to avail themselves of this flexibility' (Freudenthal, 1968, p. 5). Rather, based on his own experiences as a mathematician, he viewed mathematics as a human activity deeply embedded in real situations. Furthermore, his understanding of the growth of mathematics, as a discipline or set of related areas of human inquiry, led him to stress its social and cultural embeddedness. For the purposes of this report, we focus on three ideas in RME: reality and related notions of rich pedagogical contexts; the horizontal and vertical mathematising cycle; and the four-level framework for classifying curricular emphases in mathematics education.

Firstly Freudenthal's conception of reality is central to his espoused pedagogy, as it is from a basis in some real situation that learning mathematics has its source and it is also the context for the application of formal mathematical ideas. This is a very important reconceptualisation of the role of reality in mathematics education, as reality is typically seen only as the site where mathematical models are applied, rather than as a rich source of mathematical ideas. While Freudenthal objected strongly to constructivism, viewing it no more than empty sloganising, his own conception of reality has a distinctly constructivist feel. Freudenthal's phenomenological conception of reality and its emphasis on the learner's perspective, is constructivist in the sense that he recognised that particular experiences have the potential to be more real - or to mean something different - from one learner to another. For example, he argues that: 'Real is not intended to be understood ontologically… but instead commensically… it is not bound to space-time world. It includes mental objects and mental activities. What I called "expanding reality" is accounted for on ever higher levels of common sense and witnessed by levels of everyday language or various technical languages' (p. 17).

Secondly the notion of mathematising or mathematical modelling is central to RME. Initially proposed by Treffers (1987), Freudenthal was reticent to endorse it but came to see its value in helping to formalise key ideas in RME (Freudenthal, 1991, pp. 41-44). After he agreed on the value of the distinction, Freudenthal described it as follows:

> *Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is a symbol world with regard to abstraction. To be sure, the frontiers of these worlds are vaguely marked. The worlds can expand and shrink also at one another' s expense. (1991, p. 71)*

Based on this definition of mathematising, new mathematics stresses - and indeed almost exclusively confines mathematising to - the vertical dimension. The vertical-horizontal distinction is evident in the PISA mathematical literacy framework which we discuss later in this chapter (see Section 3.5, Figure 2).

Thirdly Freudenthal provides a four-level framework for classifying curricular emphases in mathematics (1991, pp. 133-37). This is a valuable heuristic in appraising the policy direction to be taken, or not taken, in efforts to change mathematics education (Table 5).

Table 5: Curricular emphases on horizontal and vertical mathematising

	Horizontal	Vertical
Mechanistic	-	-
Empiricist	+	-
Structuralist	-	+
Realistic	+	+

Furthermore, this four-level framework is useful in considering the extent to which different levels of mathematics syllabi (e.g. Foundation, Ordinary and Higher) provide all learners with opportunities to experience and engage comprehensively with the full cycle of mathematising, including both horizontal and vertical aspects of mathematical modelling. Freudenthal's emphasis on the importance of mathematising reality as the basis for all mathematics education is consistent, as Gravemeijer and Terwel (2000) stress, with the principle of 'mathematics for all'. Freudenthal delineates four possible types of curricular stance in mathematics: mechanistic, empiricist, structuralist and realistic. Each is defined by the presence or absence of vertical and/or horizontal mathematising. Mechanistic mathematics employs neither horizontal nor vertical mathematising, but is focused on routine mathematical drills. The empiricist approach emphasises horizontal mathematising, working mathematically real life situations to symbols without delving into the world of symbolic manipulation. Structuralist mathematics, consistent with 'new' mathematics, focuses on symbolic manipulation, that is, vertical mathematising. Realistic mathematising combines both the horizontal and vertical. In doing so, the realistic approach can encompass the dual nature of mathematics, as described by De Corte *et al.* (1996):

> *On the one hand, mathematics is rooted in the perception and description of the ordering of events in time and the arrangement of objects in space, and so on ('common sense - only better organised', as Freudenthal (1991, p. 9) put it), and in the solution of practical problems. On the other hand, out of this activity emerge symbolically represented structures that can become objects of reflection and elaboration, independent of their own real-world roots. (p. 500)*

In the Irish situation, one of the dangers in reforming mathematics within a system where there are tiered mathematics syllabi (e.g. Foundation, Ordinary and Higher level) and/or different types of mathematics syllabi (e.g. pure versus applied mathematics) is that learners will be deemed suitable for only a portion of the full mathematising cycle, thus encompassing only one aspect of the dual nature of mathematics. For example, more advanced mathematics students might be seen as more suited to vertical mathematising and thereby miss out the value of moving between the vertical and horizontal. On the other hand, a more horizontally-oriented syllabus might be designed for less advanced students, thereby excluding them from the value of vertical mathematics. We want to emphasise that there is no case being made in Freudenthal's writings or in the wider RME research community that vertical and horizontal mathematising are more suited to particular learners, either in terms of age, developmental level or cultural background. Rather, the focus in RME is on mathematics for all and the value of providing all students with experiences of the full mathematising cycle in the context of mathematics as a human activity. Thus, from an RME perspective, all syllabi, whether at Foundation, Ordinary or Higher level, should provide many opportunities for students to experience horizontal and vertical mathematising and explore their inter-relationship.

The legacy

The Freudenthal Institute, the impact of RME on conceptions of maths education including textbook development, the Netherlands' high rankings in PISA, the ongoing research in RME tradition, and the impact of RME on mathematics education, most notably its adoption by PISA in the context of developing a framework for mathematics literacy, are the lasting legacy of Freudenthal. As such, the work of Freudenthal represents a huge contribution to mathematics education. Freudenthal's ideas were marginal when 'new' mathematics was dominant but the tide has turned, in part due to the emergence of new understandings of learning from research on situated cognition, as well as the widely perceived need for a more socially embedded mathematics education in order to prepare students to think mathematically and apply knowledge in new contexts. Furthermore, the adoption of horizontal (real world to symbol) and vertical (symbol to symbol) mathematising represents a powerful model for framing the scope of mathematics in planning and enacting curriculum, giving due regard to student experience, the complexity of learning from - and applying mathematics in - real world contexts, and the powerful conceptual traditions of various branches within mathematics.

The challenges: how realistic is real-world maths?

Perhaps the main challenge of implementing RME is coming to grips with the meanings of 'realistic' and 'context'. There are other challenges, including demands on teachers' time and understanding of maths, access to resources, and the manner in which historically dominant curricular cultures distort the meaning of RME through the subtle influence of alternative perspectives embedded in textbooks and tests/examinations. But the biggest challenge is defining what counts as a rich context for mathematising.

A move towards more applications-focused mathematics is one of ways in which mathematics educators have been attempting to make mathematics more realistic (Cooper and Harries, 2002). Textual representation of mathematics problems as word stories is the primary means of achieving this goal. But although presenting problems in the form of stories is the traditional way of making maths real, word story problems are problematic. As numerous studies internationally demonstrate, students appear to engage in widespread 'suspension of sense-making' when presented with word story problems, though it is a centuries-old method of making mathematics real for students. For example, research in various education systems (e.g. Belgium, Northern Ireland, Germany and Switzerland) reveals a similar pattern of response to word story problems, whereby students tend to approach them in such a way that they marginalise their out-of-school knowledge.

We illustrate the challenges of creating more authentic school mathematics experiences by drawing on the mathematical word story problem research of Verschaffel, Greer and De Corte (2000). For example, they posed the following three word story problems to primary and post-primary students in different countries over a number of years:

> *450 soldiers must be bussed to an army barracks. Each bus holds 36 soldiers. How many buses are needed? (Bus problem)*
>
> *Steve has bought four planks, each 2.5m. How many planks 1m long can he saw from these planks? (Plank problem)*
>
> *Joan's best 100m time is 17 seconds. How long will it take her to run 1km? (Running problem)*

As Cooper and Harries (2002) note, students may respond 'realistically' in two ways. First, knowing the predictable and solvable format of school-sourced mathematics story problems in which the story loosely disguises a set of numbers, students may discard any real-world considerations in developing a mathematical model. So, for example, in the running problem, they may ignore the question of whether Joan will be able to maintain the 17-second per 100m rate for a full kilometre. Such lack of realism in many school-sourced word story problems has accumulated in students' mathematics learning histories and results in a marginalisation of students' real-world knowledge. Second, students may draw upon their everyday out-of-school experience. In terms of the goals of promoting 'realistic' mathematics, in constructivist-and RME-inspired curricula, the hope is that they will.

How do students typically respond? Based on either written or oral responses, only 49% of students in the case of the bus problem, 13% in the case of the plank problem and 3% in the case of the running problem responded in ways hoped for from a 'realistic' mathematic perspective. Not surprisingly, students provide answers which indicate that they respond by accepting the logic implied in the genre of school word story problems. They typically assume textbook problems are solvable and make sense, that there is only one correct answer, that the answer is numerical and precise, that the final result involves 'clean' numbers (i.e. only whole numbers), and that the problem as written contains all the information needed to solve it (Verschaffel, Greer and De Corte, 2000, p. 59).

Verschaffel *et al*. took another version of the bus problem:

> *328 senior citizens are going on a trip. A bus can seat 40 people. How many buses are needed so that all the senior citizens can go on the trip?*

They presented this problem and a similar one in two ways to the same group of students in an effort to understand problem authenticity. They first presented the problem as above, and second they placed the students in a more 'reality-based situation'. In the 'reality-based situation', students were asked to make a telephone call using a tele-trainer from a local telemarketing company and were given the following information:

> ***Facts***
>
> *Date of party: Fri. April 15 Time: 4-6pm*
>
> *Place: Vinnie's Restaurant, Queen's*
>
> *Number of children attending party: 32*
>
> *Problem: We need to transport the 32 children to the restaurant so we need transportation. We have to order minivans. Board of Education minivans seat 5 children. These minivans have 5 seats with seatbelts and are prohibited by law to seat more than five children. How many minivans do we need? Once we know how many minivans are needed, call 998-2323 to place the order.*

In the first (restrictive) presentation, only 2 of the 20 students responded appropriately; the other 18 gave an incorrect interpretation of the remainder. In the 'reality-based situation' 16 of the 20 students gave an appropriate response; that is, they justified their answer for using the number of minivans (e.g. 6 minivans plus a car for the remaining 2 students).

Commenting on the 'unrealistic' nature of most school mathematics textbook problems, Freudenthal noted that:

> *In the textbook context each problem has one and only one solution: There is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms.*

> *Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why children's immunity against this mental deformation is so varied? (1991, p. 70)*

The work of Verschaffel *et al.* (2000), Cooper and Harries (2002) and others questioning how schools purport to present 'realistic' mathematics provides a challenge for mathematics educators and curriculum designers alike as they seek to create more 'realistic' mathematics education, whether they are motivated by constructivist, situated cognition or RME perspectives on mathematics.

3.4 Situated cognition in mathematics education

Another important voice in this discussion is represented by situated cognition. Much of the research conducted from a situated perspective has demonstrated how students utilise particular mathematical strategies, not solely because of some cognitive level of achievement, but in part because of the socio-cultural context: the nature of the activity, working practices, features of the tasks, etc. Many of the theoretical insights in situated cognition grew out of ethnographic studies of everyday mathematics in out-of-school contexts, and were undertaken by cognitive anthropologists and cultural psychologists (Lave, 1988; Walkerdine, 1988; Brown, Collins and Duguid, 1989; for reviews see de Abreu, 2002; Saxe, 1999; Saxe and Esmond, 2005). For example, in section 3.1 of this chapter, we noted research in this tradition undertaken by Nuñes, Schleimann and Carraher (1993) which demonstrated how street children in Brazil working as street vendors had developed complex algorithms to calculate price and quantity in a fast and accurate fashion. The same children when presented with similar tasks in symbolic form in school situations performed well below their on-street level.

Collectively such research has demonstrated that children and adults often have robust expertise in out-of-school settings.

As a further example, Schliemann, Goodrow and Lara-Roth (2001), in a study of third-grade children, used Vergnaud's (1983) scalar versus functional distinction to characterise the relative merits of different problem-solving strategies:

> *In our earlier work (Nuñes, Schliemann, and Carraher, 1993), we found that street sellers compute the price of a certain amount of items by performing successive additions of the price of one item, as many times as the number of items to be sold. The following solution by a coconut vendor to determine the price of 10 coconuts at 35 cruzeiros each exemplifies this:*
>
> *"Three will be one hundred and five; with three more, that will be two hundred and ten. [Pause]. I need four more. That is…[Pause] three hundred and fifteen…I think it is three hundred and fifty."*
> **(Nuñes, Schliemann, and Carraher, 1993, p. 19).**
>
> *The street sellers perform operations on measures of like nature, summing money with money, items with items, thus using a scalar approach (Vergnaud, 1983). In contrast, a functional approach relies upon relationships between variables and on how one variable changes as a function of the other. While work with scalar solutions can constitute a meaningful first step towards understanding number or quantity, a focus on scalar solutions does not allow for broader exploration of the relationships between two variables. Schools should therefore provide children with opportunities to explore functional relationships.*

In conceptual analyses or cognitive models, *functional* strategies are typically conceived as more sophisticated than *scalar* strategies.

However, in work and everyday situations, practitioners rely more heavily on scalar strategies and often employ them even when a functional strategy would be easier computationally. In part this is because the scalar strategies allow people to keep track of the measurable attributes of quantities in situations (e.g., dollars, pounds, feet), and thus preserve their ability to engage in sense-making in situations. That is, people can work with quantities and relationships and not just numbers stripped of contextual details when working with scalar strategies. Similarly, Säljö and Wyndhamn (1996) demonstrate how significantly students' maths strategies change when the same task is posed in a mathematics class as against a social studies class. In short, it is important for researchers creating cognitive models also to account for learners' goals, the activity setting, artefacts, socio-cultural practices, and phenomenological experiences. Thus, learning that appears to be hierarchically ordered and following a single path from a behaviourist or cognitive perspective may appear to be more multi-path in nature when that learning is viewed from a situated perspective.

We believe that one must consider conceptual change in mathematics as both a shared characteristic and as an individual psychological phenomenon. Students learn mathematics as they participate in communities of practice (Cobb and Yackel, 1996) and engage in both individual and social processes of learning. While one could consider social interaction simply as a catalyst for individual psychological development, other researchers have criticised this view for not acknowledging that students' interpretations of events as interpersonal conflict are in fact influenced by the classroom practices in which they participate (Salomon, 1993). Instead, we contend that cognition should be viewed as inherently social and as distributed across individuals as well as occurring within individuals.

A well-rounded study of mathematics learning should take into account not only interactions with materials and the cultural context of activity, but also the contribution of social interactions. Thus we consider the work of social theorists (Bowers, Cobb and McClain, 1999; Cobb, McClain and Gravemeijer, 2003; McClain and Cobb, 2001; Sfard and Kieran, 2001; Yackel, Cobb and Wood, 1999) to be critical in helping the mathematics education community to better understand the distributed, social nature of learning.

This expanded view of learning can explain some phenomena not accounted for within a purely psychological perspective. For example, it allows researchers to: (i) examine the ways in which cognition can be partly offloaded onto the external environment via interaction with tools, artefacts and other people (Brown, Collins and Duguid, 1989; Hutchins, 1995a; Hutchins 1995b); (ii) study the structuring resources for cognition that are distributed through the personal, social and historical settings in which people live and work (Salomon, 1993; Nuñes, Schliemann and Carraher, 1993); and (iii) capture interpersonal discourse as a tool for directly analysing cognitive events (Sfard and Kieran, 2001). However, attention to situated activity as a research focus could result in lack of attention to individual understanding and development. The study of internal schemes and mental operations are typically downplayed when a social perspective guides data collection and analysis (Lobato, 2005). Cobb and Yackel (1996) cautioned that a group analysis tends to downplay qualitative differences in individual students' mathematical interpretations, except to the extent to which they can be tied to their participation in communities of practice. While tracking changes only in individuals' conceptions might leave out important data about emerging social understanding and the reflexive relationship between the two, tracking group conceptions alone could similarly result in a half-told story. In summary, situated cognition challenges in very significant ways contemporary notions of the individual learner, emphasising the central role cultural and social settings as well as various technologies play in shaping mathematical ways of knowing in or out of school. The artificial nature of school word story problems is a good example of situated cognition in that over time learners come to see their artificial nature and respond accordingly, that is, by suspending sense making. A situated cognition perspective challenges educators to think beyond individual ability as the sole indicator of mathematical competence, and puts an emphasis on viewing mathematics within the context of activities, language, and social and educational expectations in or out of school. Finally, a situated cognition perspective challenges the education system to consider ways in which it can leverage social, cultural and technological resources to equip students with skills and knowledge to engage with mathematics both in and out of school.

3.5 The PISA mathematics literacy framework: situated cognition and RME

We now turn to how PISA addresses the challenges posed by situated cognition and RME. The PISA framework is based on the assumption that mathematics is a human activity, and reflects the strong influence of both RME and situated cognition. We can use an example to illustrate the PISA mathematical literacy framework (source: OECD, 2003, Mathematics Example 1, pp. 26-27).

Mathematics Example 1: ***Streetlight***

The Town Council has decided to construct a streetlight in a small triangular park so that it illuminates the whole park. Where should it be placed?

According to the PISA Framework (OECD, 2003), the above social problem 'can be solved by following the general strategy used by mathematicians, which the mathematics framework will refer to as mathematising. Mathematising can be characterised as having five aspects' (p. 26):

1. Starting with a problem situated in reality.
   Locating where a streetlight is to be placed in a park.

2. Organising it according to mathematical concepts.
   The park can be represented as a triangle, and illumination from a light as a circle with the street light as its centre.

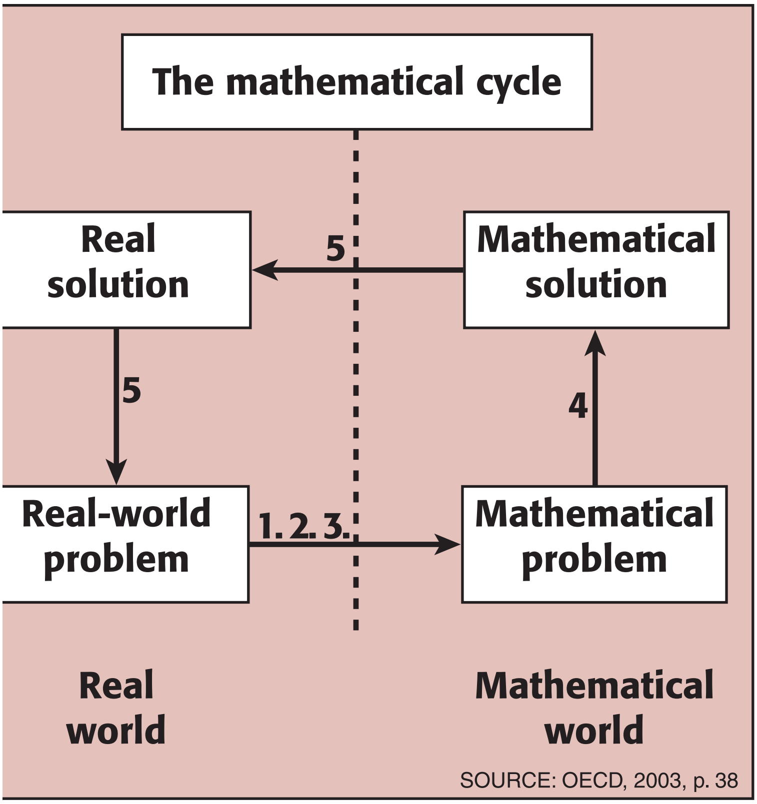
3. Gradually trimming away the reality through processes such as making assumptions about which features of the problem are important, generalising and formalising (which promote the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the situation).
   The problem is transformed into locating the centre of a circle that circumscribes the triangle.

4. Solving the mathematical problem.
   Using the fact that the centre of a circle that circumscribes a triangle lies at the point of intersection of the perpendicular bisectors of the triangle's sides, construct the perpendicular bisectors of two sides of the triangle. The point of intersection of the bisectors is the centre of the circle.

5. Making sense of the mathematical solution in terms of the real situation. *Relating this finding to the real park. Reflecting on this solution and recognising, for example, that if one of the three corners of the park were an obtuse angle, this solution would not be reasonable since*

the location of the light would be outside the park. Recognising that the location and size of trees in the park are other factors affecting the usefulness of the mathematical solution.

Figure 2: The mathematical cycle



The five phases in this mathematising cycle can be represented diagrammatically (see Figure 2) From a new/modern mathematics perspective, the mathematical 'action' is almost all on the right hand side of the cycle, that is, in the mathematical world. In the language of RME, it corresponds to a focus on vertical mathematics. However, providing students with opportunities to experience the full mathematising cycle, as a routine part of classroom mathematics culture, is vital in the promotion of mathematics in context.

1. Starting with a problem situation in reality.

2. Organising it according to mathematical concepts and identifying the relevant mathematics.

3. Gradually trimming away the reality through processes such as making assumptions, generalising and formalising, which promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation.

4. Solving the mathematical problem.

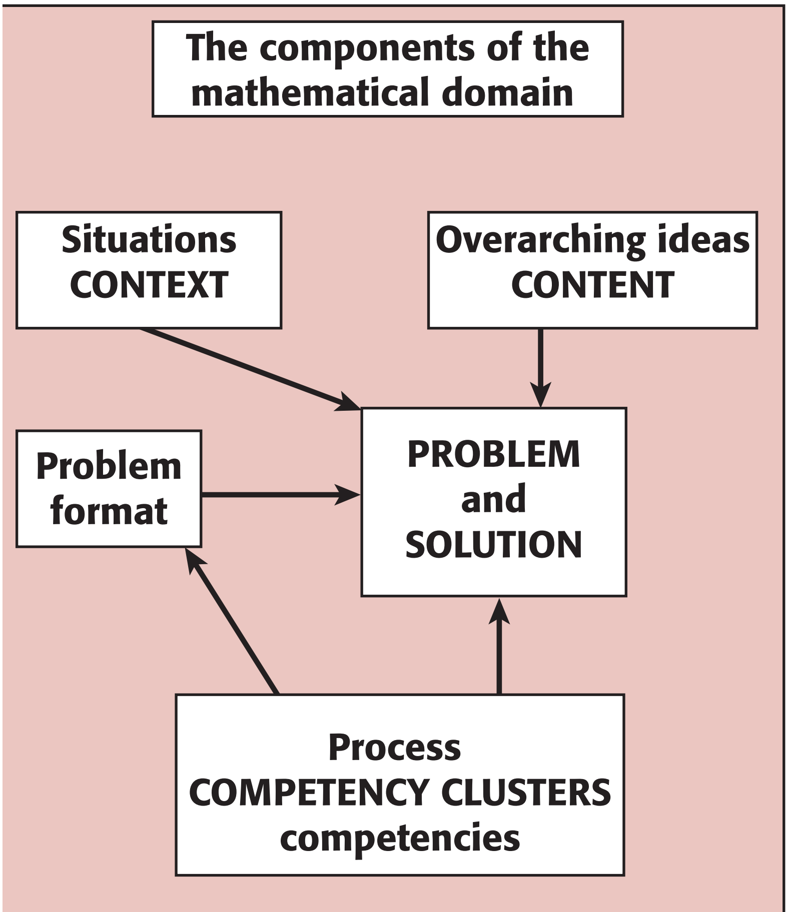
5. Making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution.

Typical of RME application problems, the PISA mathematical literacy items focus on application problems rather than traditional word story problems. The problem is rooted in the real world and students are expected to traverse both horizontal and vertical mathematising in coming to a solution. The vertical-horizontal boundary is one that students are expected to be able to move seamlessly across as they consider they validity of any chosen mathematical models and then reflect on any solutions in terms of the real world. Thus, if the real and mathematical worlds are opposite banks of a river, students are expected to cross and re-cross the river, viewing one side from the perspective of the other.

The components in the PISA mathematical domain

After starting with an example of a 'real-world' PISA item, we now turn to the components in the PISA mathematical domain made up of competency clusters, contexts (situations), and content (overarching ideas). Competency clusters reflect the different cognitive challenges intended in various types of problems. They are defined and discussed above in the section on the cognitive approach to learning (Section 3.1). In contrast with, for example, behaviourally-based assessment items, PISA assessment items reflect an assumption that various mathematical processes and strategies will be used concurrently: 'The process of mathematics as defined by general mathematical competencies… include the use of mathematical language, modelling and problem-solving skills.

Figure 3: The components of the mathematical domain



SOURCE: OECD, 2003, p. 30

Such skills, however, are not separated out in different test items, since it is assumed that a range of competencies will be needed to perform any given mathematical task' (OECD, 2003, p. 16). Based on the assumption, drawn from both RME and situated cognition, that mathematical literacy involves engaging with mathematics in different situations, PISA specifies four contexts in which mathematics can be presented to students: personal, educational/ occupational, public and scientific (OECD, 2003, p. 32). These situations are defined 'based on their distance to the students' (OECD, 2003, p. 16), that is students' experience of everyday life.

Finally, mathematical literacy content is conceptualised in terms of four overarching ideas:

- quantity
- space and shape
- change and relationships
- uncertainty.

The organisation according to overarching ideas contrasts with organisation by curricular strand typical of many curriculum documents, as explained in the PISA Framework document, 'the mathematical content [is] defined mainly in terms of four "overarching ideas" (quantity, space and shape, change and relationships, and uncertainty) and only secondarily in relation to "curricular strands" (such as number, algebra and geometry)' (p. 15).

3.6 Neuroscience as a basis for mathematics education: is it a bridge too far?

This section provides an overview of research into neuroscience and debates about the utility of such knowledge as a conceptual lens on the teaching and learning of mathematics. There is considerable interest internationally in the potential of brain-based research to inform mathematics education. The OECD's recent publication *Understanding the Brain: Towards a New Learning Science* (2002) details recent neuroscience research outlining its implications for, and the conditions for its use in, education. The OECD's interest in brain-based research is indicative of an international movement directed at understanding the potential of recent rapid advances in what is known about the brain, how it operates, and how it can be influenced, largely thanks to research based on sophisticated brain scanning technologies such as positron emission tomography (PET) and functional magnetic resonance imaging (fMRI). These technologies have shed light on normal neurobiological development as well as developmental disabilities (e.g. dyslexia). Brain-based research has attracted the attention of some educational policy makers. For example, one of Asia-Pacific Economic Cooperation's (APEC, 2004) five proposed priority action steps, in relation to APEC-wide collaboration to stimulate learning in mathematics and science, is a focus on brain-based research:

> *APEC should work to determine how brain research applies to the teaching of mathematics and science concepts, in determining appropriate sequencing of concepts, and in helping students of different ages to retain mathematical and science concepts. APEC should also consider information collected through UNESCO and OECD brain research projects. (p. 63)*

In this section, we examine some key insights from neuroscience research. We draw on two key reports, the US National Research Council (NRC) report *How People Learn* (2000) and the OECD's *Understanding the Brain: Towards a New Learning Science* (2002). We conclude the section by assessing whether brain-based research might guide development of policies and practices in the teaching and learning of mathematics. As such, we focus on whether current brain-based research knowledge can extend to form a bridge with the practice of mathematics education, or whether this is a bridge too far (Bruer, 1997).

The argument in favour of a brain-based approach to education, according to Bruer (1997), rests on 'three important and reasonably well established findings in developmental neurobiology' (p. 4). The first of these notes that from early infancy until middle childhood there is a period of rapid brain development characterised by the proliferation of synapses, then followed by a later period of pruning or elimination. Secondly there are experience-dependent critical periods in sensory and motor development. Thirdly considerable research over the last thirty-plus years has demonstrated the positive effects of experience-rich environments on rats' brain development, and by extrapolation there is a strong case for the occurrence of similar process in humans (Bruer, 1997). The NRC Report summarises its main points in relation to the current state of knowledge on learning from the field of neuroscience as follows:

- Learning changes the physical structure of the brain.
- These structural changes alter the functional organisation of the brain; in other words, learning organises and reorganises the brain.
- Different parts of the brain may be ready to learn at different times.

In order to outline the manner in which learning changes the physical structure of the brain, it is important to identify two components of the brain important in development: the neurons, or nerve cells, and synapses, the brain's information junctions. Two mechanisms account for changes in synapse development. Over the course of the first ten years of life, synapses are overproduced and pruned or selectively lost. A second mechanism involving the addition or growth of synapses is due to the nature of an individual's experiences. This second process is 'actually driven by experience' and forms the basis for memory (Bransford, Brown and Cocking, 2000). Studies comparing the brains of animals raised in complex environments with those of animals raised in environments lacking stimulation, provide convincing evidence that enriched experiences result in an 'orchestrated pattern of increased capacity in the brain that depends on experience' (Bransford, Brown and Cocking, 2000, p. 119). Such orchestrated changes occur in localised brain areas depending on the type and quality of experiences. Furthermore, synapse loss and growth occur at different rates in different parts of the brain depending both on learning experiences and internal development processes.

In what ways - and to what extent - might brain-based research inform mathematics education? Firstly brain research has demonstrated that different types of experiences have different effects on the brain. Secondly at present there is not sufficient evidence from a neuroscientific perspective to recommend certain activities as pedagogical strategies supported by brain-based research. As such, the exact implications for classroom practice are not at a level where specific practices can be recommended to ensure neural branching (Bransford, Brown and Cocking, 2000). Thirdly a critical approach to recommendations from various quarters, citing brain-based research for their educational recommendations, is merited, as these are often based on tenuous scientific evidence. Finally, it is important that teachers are aware of the insights and current limitations of brain-based research in informing pedagogical practice in mathematics.

3.7 Fostering ownership of learning: learning to learn

> *…the development of education and training systems in a lifelong learning and in a worldwide perspective has increasingly been acknowledged as a crucial factor for the future of Europe in the knowledge era.*
>
> **(Detailed work programme on the follow up of the objectives of education and training systems in Europe, European Union, 2000, p. 9)**

> *Teaching students self-regulatory skills in addition to classical subject-matter knowledge is currently viewed as one of the major goals of education. At the same time, self-regulated learning (SRL) is a vital prerequisite for the successful acquisition of knowledge in school and beyond, and is thus of particular importance with respect to lifelong learning.*
>
> **(Baumert *et al.*, 'Self-regulated learning as a cross-curricular competence', OECD-PISA Consortium, 1999, p. 1)**

'It was drilled into me' as an expression of a learner's relationship to learning represents the antithesis of the idea of promoting ownership of learning. Whereas the learner's sense that knowledge was drilled into him or her leaves the learner in a passive or receptive state, the promotion of a learning to learn capacity seeks to enlist the learner in his or her own education. For example, in mathematics problem-solving many learners assume that if after five minutes a problem has not been solved it is not possible to solve it, and therefore they typically give up (Greeno, Pearson and Schoenfeld, 1997). Furthermore Greeno, Pearson and Schoenfeld (1997) note that, in many fields, people think that 'you either know it or you don't' and therefore give up prematurely in the face of difficult problems.

A recurrent focus on the importance of *learning to learn* in the context of promoting *lifelong learning* is one of the most distinctive features of contemporary educational and economic policy-making at national and international levels. As such, the development of a learning to learn capacity has become a key educational policy priority around the world, animating discussions about the purpose of schooling as well as debates about how learning to learn can be developed in specific subject areas. As we have noted already, it has been emphasised as a cross-curricular goal in APEC countries. Self-directed learning, capacity for independent and collaborative problem-solving, developing ownership of learning, and self-regulated learning are terms used interchangeably in terms of highlighting learning to learn as an educational aim. In the case of mathematics, the promotion of skilled problem-solving has been a long-standing concern among mathematics educators (De Corte, Greer and Verschaffel, 1996). For example, early work by Polya (1945) on problem-solving heuristics (understand the problem, find the connection between the data and the unknown by possibly considering related problems, develop a plan, carry it out, and examine the solution) and more recent work by Schoenfeld (1985, 1987, 1992) attempted to teach mathematical problem-solving long before the contemporary focus on learning to learn and lifelong learning. Schoenfeld (1992), in particular, focused on delineating heuristic strategies as well as meta-cognitive control skills.

Why is learning to learn appealing and important to policy makers? Firstly there is an acknowledgement that with the accelerating pace of social change, economic development and most importantly knowledge production, it is crucial that learners have the capacity to continue engaging with new knowledge and ideas over the course of their personal and professional lives. Secondly the current wave of globalisation presents challenges, such as the explosion of information, environmental sustainability, pandemics (e.g. HIV/AIDS, SARS), terrorism, and national and global inequalities, that demand deep disciplinary knowledge, the capacity for interdisciplinary knowledge construction, and competence in dealing with non-routine problems (Gardner, 2001). As such, future challenges that today's students will have to address will demand competence in how they can manipulate, reframe, connect and apply knowledge to ill-structured problems (within and across and disciplines) and continue to do this throughout their lives. Thirdly in an era of lifelong learning, school graduates are expected to enter the workforce, higher education, or further education with the capacity for promoting their own learning.

In addition to APEC, the OECD's PISA framework also identifies self-regulated learning as a necessary stepping-stone in the promotion of lifelong learning, which has become a cradle-to-grave educational aim in many regions of the world. Baumert *et al.* (1999), in an article conceptualising self-regulated learning in the context of the OECD PISA studies, notes (i) the many definitions of SRL, (ii) the increasing move away from a sole focus on cognitive strategies to incorporate motivation and context, and (iii) the complexity of assessing students' SRL through questionnaires which focus on SRL as a domain-general capacity (i.e. non subject-specific competence) given that considerable research has demonstrated important subject-specific aspects of SRL (Chi, 1987; De Corte, Greer and Verschaffel, 1996; Resnick, 1987; Zimmerman, 2001).

While cognisant of Baumert's observation about the proliferation of SRL definitions, for the purposes of this report we use Pintrich's definition as, we think, it captures the current understanding of self-regulated learning as a multi-dimensional construct encompassing both self-regulatory processes (planning, monitoring, control and review) and areas for self-regulation (cognition, motivation, behaviour and context) (Schunk, 2005). According to Pintrich, self-regulated learning, or self-regulation, is 'an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and the contextual features in the environment' (2000, p. 453). Pintrich's definition of SRL provides a framework for guiding the design and evaluation of SRL interventions. Schunk (2005), citing Boekaerts *et al*. (2000) and Schunk and Zimmerman (1998), notes that the design and evaluation of SRL interventions in subject areas has demonstrated that self-regulation can be taught to students, and the outcome of training studies

> *…supports the idea that students' self-regulatory processes can be enhanced and that better self-regulation results in higher academic performance. Beneficial effects on self-regulation have been obtained from interventions designed to improve students' goal orientations, learning strategies, self-monitoring and self-evaluations. (Schunk, p. 2005)*

He continues by noting the need for further research to examine the efficacy of promoting SRL in specific subject areas.

Research on academic self-regulated learning began as an outgrowth of psychological investigations into self-control among adults and its development in children (Zimmerman, 2001). This early initial self-regulation research was focused on its application in therapeutic contexts. For example, researchers taught participants to recognize and control dysfunctional behaviours such as aggression, addictions, and various behavioural problems (Zimmerman, 2001). In school settings, we can think of SRL as providing students with opportunities to: orient action towards a learning goal, monitor progress towards that goal, provide feedback (reflexivity), confirm or re-orient the trajectory of action towards the goal and/or re-define the learning goal (Allal, 2005). As with our discussion of cognition, in which we emphasised the importance of taking both the social and psychological dimensions into account, we adopt a similar approach here.

Two perspectives on self-regulated learning

We note two somewhat different strands of work on SRL, both of which it is important to consider in the promotion of SRL in schools: one adopting an individual cognitive psychological orientation, and the other adopting a multilevel socio-cultural orientation.

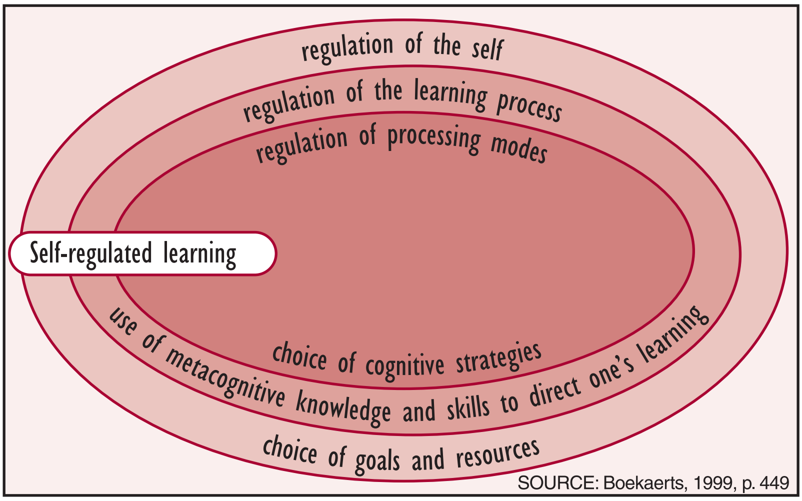
The first strand focuses on the individual learner (see figure 4, Boekarts, 1999). As illustrated by the model in figure 4, self-regulated learning comprises

- regulation of the self
- regulation of the learning process
- regulation of processing modes

- choice of cognitive strategies
- use of metacognitive knowledge and skills to direct one's learning
- choice of goals and resources.

The focus is very much on the individual learner, and reflects the psychological focus of much of the SRL literature. Allal (2005) presented a model of regulation (she writes about regulation rather than self-regulation) that stresses that more than the self is involved in regulation within school settings. Using a multi-level model, she argues that any effort to promote self-regulation must ensure that other aspects of the learners' environment must be congruent with such efforts.

Figure 4: A three-layered model of self-regulated learning



She identifies four levels as follows: overall teaching environment in the school; the teachers' approach to teaching; the quality of peer interaction; and students' own individual self-regulation strategies. Allal identifies various physical and psychological tools (e.g. teacher modelling strategies, multi-media learning environments, scripted peer feedback sheets, use of teacher think aloud to model mathematical problem-solving) that can be used as bridges between these levels.

3.8 Conclusion: rapid changes in approaches to learning

Each of the perspectives on mathematics teaching and learning discussed in this chapter could be portrayed in terms of the typical rhythms and patterns of teacher and student interaction. They each present an idealised image of learning. They also suggest that there is an emerging consensus in the learning sciences in relation to the design of powerful learning environments. Research is producing models that may have considerable import for the practice of teaching mathematics in schools. These new models of learning are more socially embedded than traditional ones. They include practical multi-level models of self-regulated learning and involve new understandings of the scope of assessment, including but extending beyond teacher assessment to include self and peer assessment. They suggest ways to integrate multi-media learning environments in mathematics education. How might some of these work in practice? What might future classroom learning for many or most students look like?

The Jasper Project is a well-researched initiative which has resulted in improvements in students' understanding of and skill in computation, enhanced problem-solving and improvements in transfer of learning to new situations (Roschelle, *et al*, 2000) (for a related learning and social interaction analysis of a technology-rich learning environment, see Järvelä, 1995). Early versions of Jasper used video to teach mathematics curricula focused on problem-solving, but researchers found that while students improved in their problem-solving skills and flexible use of knowledge the gains were not as great the designers had hoped. So the Jasper designers created a second wave of mathematics curricula. This second wave was called Jasper Challenge (the first wave was Jasper), and involved a multi-media learning environment (MMLE) called SMART that provided the following four types of support for learners:

- SMART Lab, providing comments on and summaries of student responses.
- Roving Reporter, involving videoclips of students grappling with the same problems in the learning community.
- Toolbox, comprising a toolbox for creating problem-related visual representations to aid mathematical representation and modelling.
- The Challenge: a new but related problem.

Jasper and Jasper Challenge provide a way to summarise key ideas in this chapter around four key themes: learner agency, reflection, collaboration and culture (Bruner, 1996; Brown, 1997). As we have noted, a move towards a more social and communal approach to learning is one of the defining characteristics of contemporary research on the design of powerful learning environments[9] (Roschelle, *et al*, 2000; De Corte *et al.*, 2003). The Jasper emphasis on Fostering a Community of Learners (Brown, 1997) is typical of efforts to understand and design *classroom cultures* where students share, negotiate and produce work that is presented to others. Learners' *reflection* on their own learning is a hallmark of effective learning and is seen in current efforts to promote self-regulated

9 Roschelle et al (2000), for example, identify several examples of computer-based applications to illustrate ways technology can enhance how children learn by supporting four fundamental characteristics of learning: (1) active engagement,(2) participation in groups, (3) frequent interaction and feedback, and (4) connections to real-world contexts.

learning. Jasper provides many opportunities for feedback in order to enhance students' SRL. The Roving Reporter and Toolbox in Jasper Challenge emphasise *collaboration* as a survival strategy; learners need each others' support and ideas to learn in the Jasper environment. This collaborative imperative draws attention to how knowledge is distributed in the classroom (Salomon, 1993). Finally, among Jasper learners, both teachers and students are seen as active meaning-making people with *agency*. As such, students are presented with real-world rather than artificial problems and supported in their efforts to create meaning.

In this chapter we have outlined very significant developments in the learning sciences and how these have influenced, sometimes in subtle ways (e.g. the behavioural underpinnings of much mathematics skill software), sometimes in very explicit ways (e.g. the impact of RME and situated cognition in PISA's mathematical literacy framework), conceptions of mathematics education. Increasingly there are calls for assessments to catch up with new perspectives on learning. The recent use of RME and situated cognition to frame PISA assessment items may be providing a context in which education systems, at least in some countries, especially those where new or modern mathematics is dominant, are beginning to grapple with whether and how they might begin to change their conceptions of mathematics curriculum and learning in order to measure up to new images of the competent mathematics student. Key issues raised in this chapter were as follows:

- New education and schooling goals demand new approaches to teaching and learning.

- Contemporary changes in mathematics education are being shaped by new visions of mathematics, mathematics learning and mathematics teaching.

- New visions of mathematics, mathematics learning and mathematics teaching have significant implications for assessment.
- A social turn has occurred in which the social and cultural dimension of learning and learners is becoming more central to policy and practice (e.g. the nature of PISA test items reflects a more socially embedded view of mathematics than most post-primary curricula internationally).
- Realistic Mathematics Education and situated cognition have emerged to play a key role in new understandings of the human mind which raise fundamental questions about the nature of mathematics education.
- Cognitive neuroscience has been the focus of increasing attention internationally (e.g. OECD, UNESCO) in debates about learning, although it will be a considerable while before this research translates into pedagogical policy and practice.
- Current research (e.g. NRC report on assessment and learning) is highlighting the more central role that learning ought to play in defining the nature of assessments.
- A variety of alternative assessments are being used in mathematics education including: portfolios, student journals, concept maps, assessment of cooperative group work, and revision of student work.
- Fostering students' responsibility for and ownership of learning is now a key education policy goal around the world. There is extensive research demonstrating that students of all ages, backgrounds and abilities can be taught self-regulation strategies.

A comprehensive approach to fostering ownership of learning is best viewed as a systemic issue involving school, classroom and student factors (Allal and Saada-Robert, 1992; Allal, and Pelgrims Ducrey, 2000; Allal, and Mottier Lopez, 2005; Allal, Mottier Lopez, Lehraus, and Forget, 2005).

CHAPTER 4

Five initiatives in mathematics education

Paul F. Conway, Finbarr C. Sloane, Anne Rath and Michael Delargey

4.1 Introduction

This chapter identifies five initiatives or important directions in mathematics education internationally and we discuss these using a common framework with the following headings: rationale, background, goals, key features, impact/outcomes, issues and implications for curriculum and assessment in an Irish post-primary context. The initiatives are:

- Mathematics in Context (MiC): RME-inspired curriculum materials
- 'Coaching' as a model for Continuing Professional Development in matehmatics education (West and Staub, 2004)[10]
- ICT (Information and Communication Technology) and mathematics education[11]
- Cognitively Guided Instruction (CGI)
- IEA First Teacher Education Study: The education of mathematics teachers (Schwille and Tatto, 2004; lead countries: USA and Australia).

A common thread through the five initiatives is their focus on the quality of student learning, one of the key areas in international policy discourse on education (UNESCO, 2004).

10 The case study on 'Coaching as model for CPD in mathematics education' was written by Dr. Anne Rath, Education Department, University College, Cork (UCC).

11 The case study on 'ICTs and mathematics education' was written by Mr. Michael Delargey, Education Department, University College, Cork (UCC).

4.2 Case 1: Mathematics in Context (MiC)

Rationale

The introduction to the MiC series describes its background as follows:

> *…Mathematics in Context is really a combination of three things: the NCTM Curriculum and Evaluation Standards, the research base on a problem-oriented approach to the teaching of mathematics, and the Dutch realistic mathematics education approach. (Education Development Center, 2001, p. 3)*

Mathematics in Context represents a comprehensive four-year mathematics curriculum for the middle grades (late primary and early post-primary) consistent with the content and pedagogy suggested by the NCTM *Curriculum and Evaluation Standards for School Mathematics and Professional Standards for Teaching Mathematics.* The development of the curricular units occurred between 1991 and 1998 through a collaboration between research and development teams at the Freudenthal Institute at the University of Utrecht, research teams at the University of Wisconsin, and a group of middle school teachers. MiC has been evaluated extensively and findings suggest that the curriculum is having a significant positive impact on student learning (see Romberg and Schafer, 2003).

The central tenet underlying the work is as follows. If mathematics is viewed as a language, then students must learn two interdependent knowledge sets: the design features of mathematics, and the social functions of mathematics. These include: the concepts and procedures of the language (the design features of mathematics) and solution of non-routine mathematics problems (a social function of mathematics). They must also develop the capacity to mathematise in a variety of situations (again, a social function). As such, the learning principles underpinning MiC are based on Realistic Mathematics Education and also reflect the principles of situated cognition, both of which, as we noted in chapter three, underpin PISA's mathematical literacy framework.

The MiC goals rest on an epistemological shift. The shift involves moving from assessing student learning in terms of *mastery* of concepts and procedures to making judgements about student *understanding* of the concepts and procedures and their ability to mathematise problem situations. In the past, too little instructional emphasis was placed on understanding, and the tests used to assess learning failed to provide adequate evidence about understanding or about a student's ability to solve non-routine problems.

Background

Because the philosophy underscoring the units is that of teaching mathematics for understanding, the curriculum has tangible benefits for both students and teachers (Romberg and Schafer, 2003). For students, mathematics is presented in opposition to the notion that it is a set of disjointed facts and rules. Students come to view mathematics as an interesting, powerful tool that enables them to better understand their world. All students should be able to reason mathematically; thus, each activity has multiple levels so that the able student can go into more depth while a student having trouble can still make sense out of the activity. For teachers, the reward of seeing students excited by mathematical inquiry, a redefined role as guide and facilitator of inquiry, and collaboration with other teachers, result in innovative approaches to instruction, increased enthusiasm for teaching, and a more positive image with students and society (note the commonalities with lesson study, Realistic Mathematics and Cognitively Guided Instruction).

Key features

A total of forty units have been developed for Grades 5 through 8. In terms of other available mathematics textbooks, the units are innovative in that they make extensive use of realistic contexts. From the activity of tiling a floor, for example, flows a wealth of mathematical applications, such as similarity, ratio and proportion, and scaling. Units emphasise the inter-relationships between mathematical domains, such as number, algebra, geometry and statistics. As the project title suggests, the purpose of each unit is to connect mathematical *content* both across mathematical domains and to the real world. Dutch researchers, responsible for initial drafts of the units, have twenty years of experience in the development of materials situated in the real world (see the commentary on RME in chapter three). These RME units were then modified by staff members at the University of Wisconsin in order to make them appropriate for US students and teachers.

Each of the units uses a theme that is based on a problem situation developed to capture student interest. The themes are the 'living contexts' from which negotiated meanings are developed and sense-making demonstrated. Over the course of the four year curriculum, students explore in depth the mathematical themes of number, common fractions, ratio, decimal fractions, integers, measurement, synthetic geometry, coordinate and transformation geometry, statistics, probability, algebra and patterns and functions. Although many units may focus on the principles within a particular mathematical domain, most involve ideas from several domains, emphasising the interconnectedness of mathematical ideas. The units are designed to be a set of materials that can be used flexibly by teachers, who tailor activities to fit the individual needs of their classes. One might think of the units as the questions (or questions sets) that a teacher can pose in the context of either lesson study or CGI. The design of textbooks is critical to MiC and reflects the RME philosophy:

> *In traditional mathematics curricula, the sequence of teaching often proceeds from a generalization to specific examples, and to applications in context. Mathematics in Context reverses this sequence; mathematics originates from real problems. (Education Development Center, 2001)*

Thus, the manner in which MiC 'reverses the sequence' in starting with real world problems as contexts for developing mathematical ideas, and possibly, but not necessarily, completing a unit with applications, draws attention to RME-inspired reframing of the role of real world contexts in mathematics education.

Over the course of the four-year MiC curriculum, students explore and connect the following mathematical strands:

- number (whole numbers, common fractions, ratio, decimal fractions, percents, and integers)
- algebra (creation of expressions, tables, graphs, and formulae from patterns and functions)
- geometry (measurement, spatial visualisation, synthetic geometry, and coordinate and transformational geometry)
- statistics and probability (data visualisation, chance, distribution and variability, and quantification of expectations).

Students work individually and in (flexible, not fixed) group situations, which include paired work and co-operative groups. The curriculum writers believe that the shared reality of doing mathematics in co-operation with others develops a richer set of experiences than students working in isolation. This focus on the valuable contribution that can be played by peers is consistent with the socio-cultural emphasis on the essential role of social support in fostering learning (see chapter three).

It is important to understand that the critical features are about the MiC curriculum itself (i.e., the content) and of components associated with the delivery process. We remind the reader that the delivery process (i.e., instruction) is not fully fleshed out in the original materials (or even in the detailed teacher's guides available through the commercial arm of *Encyclopedia Britannica*).

Impact and outcomes

A recent review in the USA of K-12 curricular evaluations indicated that there have been many evaluations of MiC as it has developed over the last decade (Mathematics and Science Board, 2004). These longitudinal and cross-sectional evaluation studies indicate that MiC improves students' achievement scores on standardised tests and their capacity to address non-routine mathematical problems (Romberg, 1997; Romberg and Schafer, 2003; Webb and Meyer, 2002). MiC writers assumed that when the curriculum is implemented well, students completing *Mathematics in Context* (MiC) will understand and be able to solve non-routine problems in nearly any mathematical situation they might encounter in their daily lives. In addition, they will have gained powerful heuristics, vis-á-vis the interconnectedness of mathematical ideas, that they can apply to most new problems that require multiple modes of representation, abstraction and communication. This knowledge base will serve as a springboard for students to continue in any endeavour they choose, whether it be further mathematical study in high school and college, technical training in some vocation, or the mere appreciation of mathematical patterns they encounter in their future lives.

While we agree with the general philosophy, the care and the quality of the materials, we also believe that without appropriate teacher development, similar to that outlined in either Cognitively Guided Instruction (see chapter four) or lesson study (see chapter two), the potential value of the MiC materials may be lost.

Issues and implications

Assuming that Irish post-primary textbooks are consistent with a 'new' mathematics approach, reviewing the impact of this approach on the sequencing of ideas and the underlying pedagogic vision seems essential in considering reform of post-primary mathematics education. Given textbooks' function as mediators between curricular intention and implementation, a reform of post-primary mathematics toward a more problem-solving orientation will, it could be argued, necessitate a radical overhaul of mathematics textbooks. As noted in the case of MiC, one very practical way in which textbooks have changed is the way in which the sequence of teaching unfolds. MiC textbooks proceed from real world problems to mathematical ideas rather than the traditional approach involving the generalisation of specific examples followed by real world applications.

4.3 Case 2: coaching as a case of subject-specific mentoring: a professional development model for teachers of mathematics

Rationale

The underlying principle of Content-Focused Coaching (CFC) for mathematics teachers is that authentic professional change in the teaching of mathematics involves teachers changing their fundamental knowledge base and beliefs about the teaching and learning of mathematics. This is coupled with the need for time and support to implement those beliefs in specific classroom contexts with a specific student body (Staub, 2004; West and Staub, 2003). It is assumed that practice change is difficult and requires ongoing coaching and support by an expert mentor well versed in the practice, content and principles of teaching mathematics. Another assumption is that change occurs over time and in bursts of uneven development, often requiring teachers to reframe their habitual teaching and learning strategies. Contexts indelibly shape how change occurs and specific contexts will demand different competencies, skills and strategies of teachers. Therefore, an on-site coach is available to help with adaptive teaching and learning strategies so as to better meet the needs of teachers and students in the learning of mathematics and to improve their conceptual understanding of mathematics.

The coaching context is structured around an expert mentor/coach working with one classroom teacher to collaboratively design, teach and reflect on teaching mathematics in a specific site. The coach provides a context for thoughtful and deliberate dialogues that result in improved teaching and learning - dialogues that reconnect a teacher to their own goals and passion for teaching, as well as connecting teachers to the body of empirical research that informs new ways of thinking about teaching and student learning in mathematics. A focus for these conversations is the content knowledge of mathematics, and through the use of a set of conceptual frameworks coming to a shared understanding of the fundamental concepts that underlie the discipline of mathematics, as well as a shared understanding of the principles of teaching and learning.

Background

Coaching is more familiar as a professional development model in business and sport than in an educational context. In business the role of the coach is to facilitate reflection and growth on the part of the client within a particular business context. Identifying specific problems and goals is the client's task. In sport the coach motivates, observes, models, gives ongoing specific feedback on progress and guides the development of the athlete as an individual or within a team.

In education, the efficacy of many professional development models has been questioned by educational reformers and researchers. They have challenged the dominant in-service 'one size fits all' model where new curricula and teaching methods are introduced to teachers in large workshop-style groups with little room for conversation, inquiry or on-site coaching (Fullan, 1995; Huberman, 1995). Research demonstrates that deep change rarely occurs in a teacher's thinking about teaching. Teachers are often treated as mere technicians and receivers of subject knowledge and pedagogical knowledge devised and generated by out-of-school 'experts' in contexts far removed from the busyness of classroom life (Clandinin and Connolly, 1996). Little attention has been paid to context-specific needs and how these contexts affect the implementation of new curricula and/or strategies. Nor has there been enough attention paid to the meanings that teachers themselves bring to the teaching/learning context; their motivation and passion for teaching has often been overlooked, thus failing to acknowledge the individuality of each teacher. Their skills and competencies in adapting their teaching to meet new curricula or standards have often been a subject of 'deprofessionalisation' (i.e. the sort of CPD that exclusively gives information to teachers rather than one that draws upon their professional knowledge as well as informing them) rather than professionalisation. Research demonstrates that efforts to introduce new teaching strategies are more successful if in-class coaching is part of the training (Joyce and Showers, 1995; Showers, Joyce and Bennett, 1987). This research shows that the on-site guidance and collaboration of a mentor who has expertise in the discipline content and the teaching of that content has been valuable.

It is within this context that CFC in the teaching of mathematics (West and Staub 2003) has been developed by a team at the University of Pittsburg's Institute for Learning. CFC is a professional development model designed to promote student learning and achievement by having a coach and a teacher working collaboratively in specific settings, guided by conceptual tools developed at the Institute (West and Staub, 2003, p. 2). These tools include a framework for lesson design and analysis and a set of core issues in mathematics lesson design that guide coach and teacher in deciding what to focus on in coaching conversations and reflection about mathematics thinking and teaching (see Guide to Core Issues in Mathematics Lesson Design below, adapted from West and Staub, 2003, p. 11)

- What are the goals and the overall plan of the lesson?
- What is the mathematics in this lesson?
- Where does this lesson fall in this unit and why?
- What are students' prior knowledge and difficulties?
- How does the lesson help students reach the goals?
- In what ways will students make their mathematical thinking and understanding public?
- What will students say or do that will demonstrated their learning?
- How will you ensure that students are talking with and listening to one another about important mathematics with mutual respect?
- How will you ensure that the ideas being grappled with will be highlighted and clarified?
- How do you plan to assist those students who you predict will have difficulties?
- What extensions or challenges will you provide for students who are ready for them?
- How much time do you predict will be needed for each part of the lesson?

The Institute was set up in 1995 as a partnership of school districts, committed to standards-based education and system-wide reform. Acknowledging the vast knowledge base that the past three decades has generated on teaching and learning, the Institute has been committed to translating this research into site-specific professional development tools for teachers. To this end it has also generated a set of nine principles of learning. These learning principles are condensed theoretical statements summarising decades of learning research (Resnick 1995a, 1995b; Resnick and Hall, 2001; see LRDC Learning Principles below):

- Organising for Effort
- Clear Expectations
- Fair and Credible Evaluations
- Recognition of Accomplishment
- Academic Rigour in a Thinking Curriculum
- Accountable Talk
- Socializing Intelligence
- Self-Management of Learning
- Learning as Apprenticeship.

The collaboration of Lucy West, a master instructional mathematics professional who is building a coaching system as a professional development model in a New York district of education, and Fritz Staub, a Swiss educator who came to the Institute as a postdoctoral fellow and was steeped in the tradition of didactics and deep subject-matter analysis, led to the evolution of content-focused coaching as a model for mathematics teaching. It is the bringing together of these two activities - the developmental processes of coaching, and a deep subject-matter focus and analysis – that offers such a comprehensive and promising model of staff development. This CFC model is also being adopted in other areas of the curriculum, including literacy and social studies education, by districts in the US who started by implementing it in mathematics.

Initiative's goals

Content-Focused Coaching provides structures for ongoing professional development that are underpinned by the conceptual tools above. According to West and Staub (2003, p. 3) coaching has the following goals:

- It helps teachers design and implement lessons from which students will learn.
- It is content specific. Teachers' plans, strategies and methods are discussed in terms of students learning a particular subject.
- It is based on a set of core issues of learning and teaching.
- It fosters professional habits of mind.
- It enriches and refines teachers' pedagogical content knowledge.
- It encourages teachers to communicate with each other about issue of teaching and learning in a focused and professional manner.

Key features

CFC takes place in schools. The teacher and coach are jointly accountable for initiating and assisting effective student learning. This feature ensures that the coach is intimately involved in all aspects of the lesson, including design, teaching and evaluation. The coach is a master teacher of mathematics and is well versed in the curriculum, standards, and principles of mathematics teaching and learning. The coach's responsibility is to set up and sustain a good working relationship between coach and teacher. This relationship is based on mutual respect of both roles. Key features that characterise this relationship include content-rich conversations, coaching conversations, site-specific interventions and observations, meticulous co-planning using a series of conceptual tools to guide design, and a commitment to work over a period of time to improve student learning.

The following structures are integral to this approach: the coach and teacher have a pre-lesson conference; they observe, teach, or co-teach the lesson; and they have a post-lesson conference. The coach comes to the teacher's classroom and together they talk and think about the mathematics to be focused on.

The pre-lesson conference

The first task of the coach is to get the teacher talking about how they view and feel about their teaching of mathematics and to gain their trust in having a coach as a mentor. This talk is focused on assessing the prior knowledge and experience of the teacher and the anticipated difficulties of the students. It is very important for the coach to ascertain how motivated and confident the teacher is in his/her own teaching and to understand the specific context that the teacher is working in. Then the coach and teacher collaboratively set goals and criteria for reaching these goals using the conceptual tools identified above. Establishing clear, explicit learning goals for students linked directly with mathematical content increases the possibility that important mathematical concepts will be included. Developing a shared view on the strategies, tasks, concepts and mathematical skills that the students are working towards, and also on a lesson design, paves the way for the focus on student learning which will be the subject of the post-lesson conference.

The lesson

The teacher and coach will have decided who will teach, observe and co-teach this lesson. Ordinarily the coach will have invited the teacher to choose a path that they are comfortable with and the teacher will have decided what they would like feedback on. However, the coach's role will vary a lot depending on the context and the teacher's needs. It might include modelling a specific strategy, co-teaching with the teacher, observing a class and focusing on a particular area that the teacher needs feedback on and so on. At all times, however, the coach's role is a collaborative one with the teacher. The goals and lesson design have been collaboratively reached and both teacher and coach are jointly responsible for student learning, which is where the focus lies.

Post-lesson conference

The post-lesson conference focuses on how successfully the lesson plan was implemented. Did students learn what they were supposed to? Examining student work is often a part of this conference - looking for evidence of learning. What problems arose? What strategies were successful and what strategies were less so? Were the learning goals appropriate to this group? How successfully were the mathematical concepts taught and linkages made to previous material? This conversation often leads into a pre-lesson conference for the next lesson where new goals are set. The coach supports and guides the teacher in thinking through these questions and also offers strategies, specific feedback and challenges the teacher when appropriate.

Impact and outcomes

One of the most important potential outcomes of CFC is the development of a learning community committed to both improving classroom practice (as in lesson study) and meeting high standards of teaching and learning. Talking becomes a main conduit for teachers to learn how to sustain and promote further learning. However, not all talk sustains learning. Teachers and coach become adept at developing a kind of talk that is called 'accountable talk'; that is, accountable to deep learning for all students, to accurate and appropriate knowledge in the domain of mathematics, and to rigorous thinking. Because both coach and teacher are working jointly together with a group of students they have a common context within which to think through problems of teaching. The talking that they do is accountable to developing a thinking curriculum together and is linked to a further research community and a shared understanding of the discipline of mathematics. Thinking and problem solving on the part of teachers and students is focused on, and both learn to become responsible for promoting and deepening further learning. Teachers become empowered and become adept at using powerful conceptual tools that further their learning: these tools can be used in their classrooms and with peers in thinking about problems of practice. Teachers become critical reflective practitioners in their field and become confident in adapting and developing pedagogical content knowledge (Shulman, 1987). More importantly, this model of professional development holds out the possibility of bridging the theory-to-practice gap that has characterised much of the history of teaching and learning (Schön, 1987).

Issues and implications

One of the most important issues that this initiative underlines is the need for ongoing support and professional development of mathematics teachers if students are to reach the necessary performance levels for a knowledge-based 21st century society. In Ireland, the present system that certifies teachers to teach mathematics with a degree in mathematics and a one-year Postgraduate Diploma in Education[12] (PDE) is insufficient. Teachers need to understand how to help students to think mathematically and to understand deeply the development process of learning mathematics. For teachers to teach in this way they need professional development opportunities so that they can engage in the very practices that a new teaching for thinking curriculum requires. They need to have space to think through the design, teaching and evaluation of their teaching with colleagues and expert mentors/coaches on a regular basis. Onsite work provides a model for giving teachers the help they need in very specific and concrete ways, and allows the coach to give feedback that is situation-specific and also content-specific. By the same token, there needs to be greater clarity as to the content of mathematics and the developmental process that students go through in understanding mathematical concepts. Students need to be given opportunities to generate their own knowledge, whilst at the same time they are in a context where the teacher is skilfully guiding and probing for ever more depth in understanding. The imperative of covering the curriculum needs to be replaced by a push to develop learning environments where students are set tasks that develop their problem-solving skills and self-regulating skills. In the same way teachers need such learning contexts for their ongoing development.

12 The Higher Diploma in Education (HDE), often just referred to as the 'HDip' or 'The Dip', was renamed the Postgraduate Diploma in Education (PDE) in 2005 in order to align itself with the National Qualifications Authority qualifications framework.

4.4 Case 3: teaching mathematics using ICTs

Rationale

Why use ICT to teach mathematics? One reason offered by the Joint Mathematical Council of the United Kingdom's review of algebra teaching, which was carried out for the Royal Society, is that:

> *The growth of IT has made it possible for students to manipulate many different types of external representations on the screen, involving symbolic, graphical and tabular forms. It is now possible to manipulate graphical representations in ways which were not possible on paper. Harnessing this new power within mathematics and school mathematics is the challenge for the 21st century.*
>
> **(Royal Society, 1995 cited in Oldknow and Taylor, 2000 pp. 90-91)**

According to Cockcroft (1982) once technology in general enters the school mathematics classroom and curriculum there are two ways in which technology such as computers and calculators can impact upon the school mathematics curriculum. Firstly they can assist and improve mathematics teaching. Secondly they can shape the mathematics curriculum itself. Tobin (1998) is of the opinion that new technological tools provide an impetus and opportunity for curriculum reform and teachers need to determine the most effective use of this new technology in the classroom. Technology helps students to learn mathematics and develops their knowledge of the subject, to such an extent that the US National Council of Teachers of Mathematics view it as an important component of a high-quality mathematics education programme: 'technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.' (NCTM, 2000, p. 24). Teachers need to apply ICT in ways that enhance the teaching and learning of the current established curriculum (NCTM, 2000; Oldknow and Taylor 2000) so that ICT can enable students to concentrate on more interesting and important aspects of content (Oldknow and Taylor, 2000) and that ICT can make the teaching of mathematics more efficient and effective (Butler, 2005).

Background

Historically, mathematics teachers were typically at the forefront of integrating computers into their teaching (Kelman, Bardige, Choate, Hanify, Richards, Roberts, Walters, Tornrose, 1983). Reflective of this early adoption of technology by mathematics teachers in some settings, Venezky and Davis (2002) found that ICT diffusion into a school curriculum in Luxembourg was due to innovation by a small group of male science and mathematics teachers motivated by an interest in the 1980s era of computer programming. Today, teachers' interest in technology is rarely the result of an interest in programming and early adopters of ICTs in teaching come from a variety of subject backgrounds.

An examination of ICT's place in international contemporary mathematics curricula provides some insight into current ICT use in school mathematics. One way of bringing technology into school mathematics practice is to make it a mandatory part of the syllabus. This has been the case in a number of countries such as Austria, Denmark, Luxembourg, Singapore, and New Zealand. According to the Danish senior cycle syllabus documents:

> *IT must be included as part of the instruction, e.g. by using application programs or programs for illustrating and teaching subject-specific concepts or methods. In addition, the instruction must include examples of how certain mathematical procedures can be expressed in algorithmic terms.*
>
> **(Danish Ministry of Education, 1999)**

According to Venezky and Davis (2002), ICT use in teaching and learning will be explicitly mentioned in the mandatory curriculum and ICT skills in mathematics will be part of the final examination for the school leaving certificate for post-primary schools in Luxembourg. The use of computers has been fixed in mathematical instruction in Austria since the last curricular reforms in 1993 and 2000 (Wurnig 2003). Leaving Europe and turning towards Asia, the focus of Singapore's revised syllabus is mathematical problem-solving. The integration of IT into mathematics teaching and learning will give leverage to the development of mathematical problem-solving (Singapore Ministry of Education, 2001 p. 8).

In Korea, the current 7th National curriculum, published in 1997, does not presuppose computer use (Hwang, 2004), although teachers can use them if they wish to do so (Hee-Chan Lew, 1999). Lew also writes that software development in Korea is moving from concentrating on low-level computational skills to higher-order software, such as games, tool-like applications, simulation and databases. Thus, Korea seems to be migrating from Sinclair and Jackiw's (2005) wave 1 type to wave 2 type (see below), which considers the context of learning. Hwang's comparative analysis of mathematics curricula in Korea and England (2004) notes general recommendation in the instructional guidelines section of the Korean curriculum that the 'active utilization of calculators and computer is recommended in order to improve the understanding of concepts, principles, and rules, and to enhance problem-solving abilities' (p. 9). He is impressed with the concrete presentation of content area for computer use in the English curriculum and he gives a few examples (Hwang 2004 p. 9):

- use systematic trial and improvement methods with ICT tools to find approximate solutions of equations where there is no simple analytical method, for example $x^3 + x = 100$ (the sub-area of 'numerical methods' at key-stage 3)
- plot graphs of: simple cubic functions… the reciprocal function… the exponential function… the circular function… using a spreadsheet or graph plotter as well as pencil and paper; recognise the characteristic shapes of all these functions (the subarea of 'other functions' at key-stage 4 higher).

The 1992 mathematics curriculum introduced into New Zealand schools assumes in the technology curriculum statement that both calculators and computers will be available and used in the teaching and learning of mathematics at all levels (New Zealand Ministry of Education, 1992). The curriculum statement views graphics calculators and computer software as 'tools which enable students to concentrate on mathematical ideas rather than on routine manipulation, which often intrudes on the real point of particular learning situations' (p. 14). Computer programs such as Logo facilitate mathematical experimentation and open-ended problem-solving. The curriculum document makes reference where appropriate to ideas concerning the students' use of technology, for example: 'students use a calculator or graphics package and the remainder theorem to identify factors or to locate intervals containing the roots to an equation' (p. 167).

Key features

There are four roles technology can play in the development of mathematical knowledge (Alagic, 2002). Firstly, technology empowers teachers and students to deal with multiple representations. So for example, Dugdale (2001) uses spreadsheets to investigate chaotic behaviour. She states that the students' investigation combined tabular and graphical representations. Understanding the relationships among several of these representations was essential to understanding the chaotic behaviour of the functions studied.

Secondly, technology enhances our ability to visualise. Dynamic geometry software packages such as *The Geometer's Sketchpad* (GSP) and *Cabri Geometry*, as well as graphing packages such as *Autograph or TI InterActive*, have made the teaching and learning of various geometrical and algebraic topics more interesting. For example, research carried out by Dixon (1997) concluded that 8th grade students in the United States who were taught about the concepts of reflection and rotation in a GSP environment significantly outperformed their traditionally taught peers on content measures of these concepts. Similarly they scored well on measures of 2-D visualisation.

Thirdly, technology increases the opportunities for development of conceptual understanding. Butler (2005) refers to the development of web-based *Java* and *Flash* applets and their transformation of the teaching of calculus. The concept underlying the very important chain rule in differentiation or that underlying integration (area under a curve) may be understood visually, thus reducing the extent of routine drill and practice. Almeqdadi's 2005 study concerning the effect of using GSP on Jordanian students' understanding of some geometric concepts concluded that there was a significant effect of using this software and consequently he recommends that more emphasis should be placed on computer use in mathematics and in education.

Finally, the opportunity for individualised learning is enhanced by technology. Sinclair and Jackiw (2005) identify three waves of ICT development, namely ICT for learners of mathematics, developing the context of learning and tomorrow's ICT. The relationship between the individual learner and the mathematics itself is the heart of first-wave technologies. They class *Logo* and the multiple-choice tests of the 1970s computer-aided instruction (CAI) as belonging to this technological wave. They have both achieved individual learning experiences at the cost of 'neglecting classroom practice, teacher habits and beliefs, as well as the influence of the curriculum, by imposing entirely new and perhaps inappropriate classroom practices' (Sinclair and Jackiw, 2005, p. 238). Individualised learning and independent student learning may also be facilitated by on-line technologies. Nicholas and Robertson (2002) give a brief account of the SCHOLAR[13] project developed at Heriot-Watt University in Scotland. The programme was used in a number of Scottish schools and as well as being used by teachers it is also designed for flexible learning by the independent student. Sinclair (2005) in her account of mathematics on the internet suggests that sites such as Ask Dr. Math[14] provide opportunities for curious students to investigate non-school-related mathematics independent of the school situation.

On the assessment side, ICT can be used in a number of ways. Firstly, it may be used as a mandatory component of the assessment process. Oldknow (2005) states that in the UK there is now compulsory data-handling coursework in mathematics which specifies the use of ICT. Singapore considers that the main purpose of mathematical assessment should be to improve the teaching and learning of mathematics and recommends that information technology be incorporated where appropriate (Singapore Ministry

13 SCHOLAR may be accessed at http://scholar.hw.ac.uk/heriotwatt/scholarlogin.asp

14 http://mathforum.org/dr.math/

of Education, 2001 p. 8). ICT skills in mathematics will be part of the final examination for the school-leaving certificate for post-primary schools in Luxembourg (Venezky and Davis, 2002). One of the assessment criteria for the internal assessment assignments in the International Baccalaureate is the use of technology (Brown 2002). Parramore (2001) informs us that the assessment of modelling is difficult to do under the conditions of a written examination. Coursework is the medium to do this. He proposes one approach at incorporating computer-based examining into A-level mathematics. He suggests that if one uses a computer as part of the modelling process then it should be reflected in ongoing school-based assessment. Computers may also be used to perform assessment tasks. Sangwin (2003) reports a recent development in mathematical computer-aided assessment (CAA) which employs computer algebra to evaluate students' work using the internet. He claims that these developments are of interest to all who teach mathematics, including class-based school teachers. He shows how technology may be used as a tool for uncovering students' misconceptions about simple topics. Uses of internet-driven technology may include the marking of existing problem sets and providing instant tailored feedback based on properties of a student's answer. The SCHOLAR project in Scotland reports positive and encouraging student and teacher experiences.

Impact and outcomes

Oldknow (2005) gives an account of an innovative project in the application of ICT in the mathematics classroom in 2000-01. The 'MathsAlive' project focussed on the use of ICT in twenty Year 7 (11-12 year-olds) mathematics classes in England. This project developed the hardware (PCs and interactive whiteboards) and software (*The Geometer's Sketchpad, TI InterActive!*) infrastructure necessary to integrate ICT into mathematics classrooms. Oldknow (2005) includes extracts from the evaluator's final report of September 2001, including overall indicators of success of this pilot project (pp. 182-83):

- teachers have been extremely positive about the values and usefulness of the resources throughout the project, and have wanted to continue with the project beyond the period of the pilot
- teachers have felt that the resources and the training offered have enabled them to implement the objectives and needs of the National Numeracy Strategy, using technology to support their teaching
- teachers have felt that the technology has added to their teaching strategies and approaches
- students have reported positively throughout the period of the project on the value of the resources and the impact it has had on their learning and on their positive attitudes towards mathematics
- the resources have been shown in practice to support both teaching and learning.

There were, however, some weaknesses in the project. There were too many new things for the teachers to take in, and a lack of time for training, discussion, revision and maturation. Some teachers found it hard to share control of the whiteboard with the students. Relatively few adapted the prepared teaching materials and even fewer designed their own. But, as Oldknow (2005) points out, the project was a success and the ICT did impact on both the students' and teachers' enjoyment and understanding of the underlying mathematics. Other authors, such as Passey (2001), Glover, Miller and Averis (2003) and Clark-Jeavons (2005), have written on the use of interactive whiteboards in teaching mathematics. Letting the user 'touch the mathematics' is how Passey (2001) describes the use of this technology. Interactive whiteboards offer learners the opportunity to get closer to mathematical systems and processes in an exploratory way (Clark-Jeavons 2005). In general, however, the integration of ICT into mathematics teaching in the United Kingdom has been poor, as evidenced by recent HMI Inspection Reports. One such Ofsted (2002a) report made the following comments on ICT practice in mathematics teaching within UK schools:

> *The use of ICT to support learning in mathematics is good in only one quarter of schools. It is unsatisfactory in three schools in ten. Typically there is some use of ICT with some classes, but it is not consistent across the department. Students' access to a range of mathematics software also varies greatly. Most departments have access to spreadsheets, graph-plotting software, LOGO and specific items of software to support skills learning. In general, however, very little use is currently made of the powerful dynamic geometry or algebra software available. Many mathematics teachers use ICT confidently outside the classroom in the preparation of teaching materials and in the management and analysis of students' achievement. Despite this, only a small proportion of departments have reached the point where they can evaluate critically their use of ICT and decide where it most benefits learning in mathematics. Too often, teachers' planning and schemes of work lack any reference to specific ICT applications, and students have difficult recalling when they have used ICT in mathematics. (pp. 8-9)*

A second report concerning the implementation of government ICT initiatives relating to post-primary mathematics published in 2002 states that overall good practice in relation to ICT use in mathematics teaching remains uncommon and that although some use is made of ICT in around two-thirds of mathematics departments, it is not an established part of the curriculum (Ofsted, 2002b). It seems that little has changed since Taverner (1996) wrote that the use of ICTs in mathematics teaching in schools is claimed to be as little as once per term on average.

In relation to using ICT in the assessment of mathematics, Ashton, Schofield and Woodger (2003), in their paper on piloting summative web assessment in post-primary education, state that the challenge for on-line assessment is not a technical but a pedagogical one: does on-line assessment measure the same learning outcomes as traditional paper based one? Beevers, Fiddes, McGuire and Youngson (1999) remind us that students consider it important that the computer issues feedback if it is to grade their work fairly and according to Bower (2005) providing students with their nonpreferred form of feedback has a significantly negative impact on their mathematics ability self-rating.

Issues and implications

ICTs have impacted upon school mathematics in a number of ways, from enhancing the teaching of the subject, to changing the curriculum content, to being used as an assessment tool. The role which ICT has assumed varies across the international arena. The current situation in Irish mathematics teaching is far from encouraging. Lyons, Lynch, Close, Sheerin, and Boland (2003) state that the use of educational technology in mathematics classrooms remains the exception rather than the norm.

Mulkeen (2004) confirms this by reporting that just 17% of post-primary schools used ICT in mathematics monthly or more in the year 2002. This same survey found that 67% of schools used it occasionally. The Impact of Schools IT2000 report (National Policy Advisory and Development Committee [NPADC], 2001) found that post-primary principals stated that mathematics software was been used in their schools, but the report did not elaborate on the type of software and manner in which it was being used by mathematics teachers. The NPADC report indicated that technology teachers were the most likely to be integrating ICTs into their teaching. Mariotti (2002) reminds us that computers have slowly entered schools and have been integrated into practice even more slowly. A radical change of objectives and activities is required if computer technologies are to be integrated into school practice. This review has referred to countries which have made ICT compulsory in both the teaching and assessing of mathematics, such as Denmark, New Zealand, Austria and the UK. Perhaps Ireland should follow suit and incorporate more explicit use of ICT in senior and junior cycle mathematics. The ways in which this may be achieved include:

- developing the ICT mathematics infrastructure in schools: e.g. provision of more technical support for schools in the ICT area
- incorporating an ICT coursework assignment as part of the assessment of junior and senior cycle mathematics similar to those in Luxembourg, Austria and the UK
- including explicit ICT curriculum statements in the preamble to curricular documents like those in Denmark, Singapore and New Zealand

- increasing teacher awareness of the potential of ICT in teaching and assessing mathematics through more training, as it happened in the SCHOLAR project in Scotland and the MathsAlive project in England.

However, the challenge for integrating ICT more fully into the Irish mathematics curriculum is to learn from international best practice and localise it to suit the Irish context.

4.5 Case 4: Cognitively Guided Instruction (CGI)

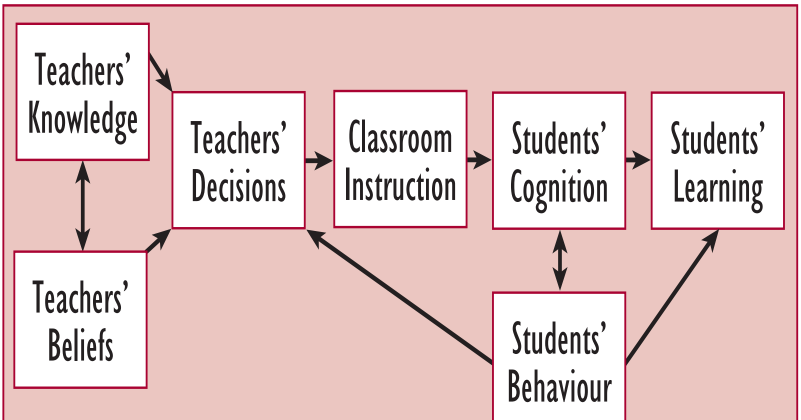
Rationale

The basic philosophy underlying Cognitively Guided Instruction (CGI) is that teachers need to make instructional decisions based on knowledge drawn from cognitive science about how students learn particular content. The critical viewpoint is that student understanding (or learning) involves linking new knowledge to existing knowledge (Vygotsky, 1978; Collins, Greeno and Resnick, 1996). Fennema, Carpenter and Petersen (1989) explain that teachers need to be cognisant of what knowledge their students have at various stages of the instructional process so they can provide appropriate instruction. Here we see a parallel with lesson study (LS) as Japanese teachers explicitly ask, in advance of teaching, just how they think students will respond to the mathematical content being taught. As we have seen, Japanese teachers carefully craft their strategies in anticipation of the most likely student responses.

Background

The CGI philosophy described above has, over the past twenty years, served as a guideline for research in which the 'major focus has been to study the effects of programs designed to teach teachers about learners' thinking and how to use that information to design and implement instruction' (Carpenter and Fennema, 1988, p. 11). The model illustrated below provides the framework for CGI research.

Figure 5: The Cognitively Guided Instruction research model



SOURCE: Fennema,et al, 1989, p. 204

The diagram illustrates that classroom instruction is the result of teacher decision-making in real time. These decisions are assumed to derive from a number of sources: teacher knowledge and beliefs, and teacher assessment of student knowledge. The latter are generated when teachers carefully observe (and interpret) their students' mathematical behaviours and the mathematics artefacts that result from those behaviours.

Key features

Fennema *et al*. argue that the main features or tenets of CGI are: '(1) instruction must be based on what each learner knows, (2) instruction should take into consideration how children's mathematical ideas develop naturally, and (3) children must be mentally active as they learn mathematics' (1989, p. 203).

The Wisconsin group (Fennena, Carpenter, Petersen, and their graduate students, including Franke [née Loef] and Chiang) have continued this line of CGI mathematics education research work over the past twenty years. Here we provide a short summary of an early study and note that more recent vignettes for teachers are available in the following publications: Jacobs, Carpenter, Franke, Levi and Battey, D. (under review); Franke, Kazemi and Battey (in preparation); Franke, Carpenter and Battey (in press); Franke, Kazemi, Shih, Biagetti and Battey (2005); Battey, Franke and Priselac, (2004); Carpenter, Franke and Levi (2003); Franke, Carpenter, Levi and Fennema (2001); Franke and Kazemi (2001); Carpenter, Fennema, Franke, Levi and Empson (1999).

Impact and outcomes: insight from the 1989 study

Carpenter, Fennema, Petersen, Chiang and Loef (1989) conducted a study to explore the three central components of CGI listed above. Their study involved 40 1st grade teachers (half of whom were randomly assigned to the treatment group). The treatment group attended a one-month summer workshop where participants were introduced to the research literature on the learning of addition and subtraction concepts. All forty teachers and their students were then observed throughout the autumn, winter and spring terms as they taught. Each teacher was observed on at least 16 separate occasions between November and April. The researchers collected data including preand post-test measures of student achievement (with various measures of mathematical problem-solving). They also collected measures of student beliefs and confidence, and conducted student interviews. Results showed that students in the treatment classes performed significantly better than control students on both the recall of number facts and the measures of problem-solving. The researchers showed that experimental teachers spent more time on word problems than the control teachers, who in turn spent more time on number fact problems. Experimental teachers focused more of their instructional attention on the processes that students used to solve problems, while control teachers attended to the answers that their students produced. Finally, experimental teachers allowed their students to respond with a wider range of strategies in order to solve mathematical problems (again, a central tenet of LS is to allow students the opportunity to generate and evaluate their own solutions to mathematical problems).

The following vignette excerpted from Carpenter, Fennema, Petersen, Chiang and Loef, (1989), provides parallel insights into the actions of the treatment teachers and their students.

> *A typical activity that was observed in CGI classes was for a teacher to pose a problem to a group of students. After providing some time for the students to solve the problem, the teacher would ask one student to describe how he or she solved the problem. The emphasis was on the process for solving the problem, rather than on the answer. After the student explained his or her problem-solving process, the teacher would ask whether anyone else solved the problem in a different way and give another student a chance to explain the new solution. The teacher would continue calling on students until no student would report a way of solving the problem that had not already been described… In contrast to the CGI teachers, control teachers less often (a) posed problems, (b) listened to students' strategies, and (c) encouraged the use of multiple strategies to solve problems. They spent more time reviewing material covered previously, such as drilling on number facts, and more time giving feedback to students' answers. (p. 528)*

The parallels between CGI and the problem-posing approach to teaching mathematics in Japanese classrooms are striking.

Issues and implications

CGI presents important insights for all levels of teacher education: pre-service, induction, early career, and continuing professional development. As we noted earlier, teachers' beliefs play a critically important role in practice and can act as a support for or impediment to reform in any subject area. In terms of pre-service education, CGI presents real challenges by pressing the case for a review of how much of an impact current teacher education is actually having on neophyte teachers' beliefs. From a CGI perspective, one could make a case for teaching practice tutors and mentors of beginning teachers to direct considerable energy into exploring beginning teachers' beliefs about learning and mathematics as an important feature both of the content of teacher education and a component in the appraisal of teacher competence.

4.6 Case 5: first IEA teacher education study: the education of mathematics teachers

Rationale

The Teacher Education Study in Mathematics, a cross-national study of elementary and secondary teacher preparation (TEDS-M 2008), is 'designed to inform and improve the policy and practice of how future teachers learn to teach a challenging mathematics curriculum in elementary and secondary school' (TEDS-M, 2005). As of 2005, the study had been under development for three years and was discussed and approved at successive IEA General Assembly meetings. The study is designed to help participating countries respond to a number of urgent concerns about mathematics education and will:

(a) help to strengthen the knowledge base to address participating countries' national priorities such as increasing the number of fully competent mathematics teachers; (b) gather empirical data on the experience of the participating countries to help resolve conflicts over the nature, benefits and costs of teacher education, in order to support improved policies for selection, preparation, induction, and professional development of mathematics teachers; (c) foster a more systematic and scientific approach to the study of teacher education and teacher learning in mathematics; (d) develop concepts, measurement strategies, indicators and instrumentation to strengthen the research in this field, and the knowledge base of teacher education cost-effectiveness (TEDS-M, 2005).

This study addresses the following questions:

- What is the level and depth of the mathematics and related teaching knowledge attained by prospective primary and lower secondary teachers that enables them to teach the kind of demanding mathematics curricula currently found across countries? How does this knowledge vary from country to country?

- What learning opportunities available to prospective primary and lower secondary mathematics teachers allow them to attain such knowledge? How are these structured? What is the content taught in teacher education programmes, and how is instruction organised?

- What are the intended and implemented policies that support primary and lower secondary teachers' achieved level and depth of mathematics and related teaching knowledge? How do teacher policies influence the structure of primary and lower secondary mathematics teachers' opportunities to learn mathematics at national and institutional levels?

Background

The IEA (International Association for the Evaluation of Educational Achievement) has undertaken major cross-national surveys of educational achievement since the early 1960s. These studies were designed to provide a rich empirical base from which both to understand factors influencing educational outcomes and inform educational policy makers. One of the original attractions of such cross-national studies was that cross-national variation found in the relationships between variables would help highlight the importance of particular variables and also draw attention to educational and wider cultural factors shaping relationships between relevant variables. In the case of IEA mathematics studies, Ireland participated in the curriculum analysis component of the Second International Mathematics Study (SIMS, 1980-82), and the achievement and textbook study in the Third International Mathematics and Science Study (TIMSS, 1995) (Cosgrove *et al.*, 2005). Ireland did not participate in the First International Mathematics Study (FIMS), the Third International Mathematics and Science Study (TIMSS-R, 1999) or in Trends in Mathematics and Science Study (TIMSS, 2003).

TEDS-M is the result of an increasing awareness of the importance of teacher education in both understanding existing classroom practice and changing classroom practice. Given the focus on mathematics education as a policy priority in many countries it is not surprising that the first IEA teacher education study chose to focus on mathematics (see chapter one). TEDS-M is being funded in

2006 (commencing September), 2007 and 2008 by the US National Science Foundation (NSF), as well as through a fee levied on participating countries. The development and piloting of data-gathering instruments/protocols have been undertaken through preliminary sub-study P-TEDS. Data-gathering protocols developed include survey instruments for mathematics lecturers in university departments, mathematics educators, future mathematics teachers, and two institution-focused surveys. The institution-focused surveys will gather data on the routes into teaching taken by teachers of mathematics (recognising that not all teachers of mathematics are trained mathematics teachers or have a degree in the area) and a survey of the teacher education institutions involved in the education of lower secondary mathematics teachers.

Issues and implications

The Report of the Review Group on Post-primary Teacher Education in Ireland (2002) observed that there is scope for more research on teacher education in Ireland. While there is a considerable amount of research by individuals or small-scale collaborative initiatives (e.g. the Standing Conference on Teacher Education North and South, SCoTENS), large scale programmatic cross-institutional studies of teacher education, with or without the international comparative dimension envisaged in TEDS-M, have not been undertaken to date. This is in our view problematic, both for teacher education in general and more specifically for implementing any reforms in post-primary mathematics education.

4.7 Conclusion

In this chapter, we have outlined a number of diverse initiatives of innovation in mathematics education (see Table 6).

Table 6: Five initiatives in mathematics education

Initiative (agency)	Rationale and goals	Issues/questions for maths education in Ireland
Mathematics in Context (MiC) (NSF)	Context-focused maths is neglected Goal: To provide RME-type curriculum materials for students in USA	How do Irish post-primary textbooks compare with MiC textbooks?
Coaching (LRDC)	Enhancing teacher knowledge and classroom practice Goal: To create mentoring pairs in the context of maths	How can evolving models of continuing professional development (e.g. induction) integrate a subject coaching perspective?
ICTs	New ICTs are changing conceptions of knowledge Goal: To transform learning using ICTs	Given the low usage of ICTs by Irish post-primary mathematics teachers, under what conditions might teachers use the increasing array of mathematics-related ICTs (including new handheld technologies)?
Cognitively Guided Instruction (UWM)	The lessons of cognitive science and their implications for teaching mathematics Goal: To develop teachers' capacity to use contructivist compatible teaching strategies	How can continuing professional development initiatives in mathematics provide opportunities for teachers to understand and reframe, where appropriate, their beliefs about teaching and learning mathematics?
IEA-Teacher Education Study (IEA, NSF, ACER)	Understanding links between teacher education, teacher knowledge and student learning Goal: To develop models of teacher education in order to enhance classroom maths teaching	A hypothetical question - What could be learned about the education of mathematics teachers if Ireland participated in the IEA's first Teacher Education study?

The Mathematics in Context (MiC) initiative provides a useful model of what a move towards more realistic mathematics education entails for the development of textbooks and supporting web or other ICT-based materials. The coaching initiative provides a model of the kind of subject-specific mentoring support that we think is essential in fostering a new era of co-operation and sharing of pedagogical practice. Coaching is a good example of what Hargreaves (2000) has identified as a move toward a new type of professionalism in teaching, defined by collegiality rather than the traditional image of the autonomous professional teacher. The coaching of early career and more seasoned teachers by thoughtful subject specialists extends notions of mentoring beyond a focus on general teaching skills characteristic of teacher induction, and highlights the important role of subject matter knowledge. Indeed, one of the strengths of the coaching case we described is its emphasis on the importance of mentor teachers' deep knowledge of subject matter and how it can be represented to enhance student learning. This is an important reminder of the value of the knowledge base residing among practising mathematics teachers, a vital feature and essential resource in any attempted reform of mathematics education.

The various examples of how ICTs can promote transformation in mathematics are a timely reminder of how access to advanced technologies which support representation of ideas in new ways is changing the work of mathematicians (Lei, Conway and Zhao in press; Roschelle et al, 2001). It presents a real challenge and opportunity for revising mathematics syllabi. In school settings, Cuban (2000) has characterised the current wave of technological innovation as one of overselling and underuse. Based on detailed analysis of ICT use in three classroom settings (kindergarten, high school and the medicine faculty in Stanford University) in Silicon

Valley (probably the world's most technology-rich area), Cuban presented convincing evidence that innovative ICTs are used more by teachers for preparation than for day-to-day teaching in the classroom. In the Irish context, a helpful start in increasing the use if ICTs can be made by ensuring they have a more central role in syllabus documents and related policies (e.g. NCCA documents identify ICT as a compulsory component of the new art syllabus and the four new technology syllabi awaiting implementation). Cognitively Guided Instruction (CGI) is an important case because it presents some of the challenges in moving toward more constructivist teaching. As we noted earlier, teachers' everyday or folk theories of learning have a profound impact on the structure and flow of teachers' daily lessons. Finally, the innovative IEA Teacher Education Study in Mathematics (TEDS-M 2008) is the first of its kind and is a vital opportunity for understanding the dynamics of the development of mathematics teachers' pedagogical and subject knowledge-base and how it impacts both their practice and student learning.

CHAPTER 5

Redefining and reforming mathematics education: issues and implications for Irish post-primary education

5.1 Introduction

> *Education systems tied to the formation of nation-state citizens and consumers bonded to local systems to the neglect of larger global forces are likely to become obsolete, while those that proactively engage globalization's new challenges are more likely to thrive.*
>
> **Suarez-Orozco, 2004**

> *There is little doubt that Kevin Myers's recent, thought-provoking article in the Irish Times, in which he questioned the emphasis placed on mathematics in our education system, struck a sympathetic chord in mathophobes whose experience of mathematics consisted of hours of blank incomprehension dealing with 'cosines and algebraic abstracts'. He states that for many pupils the 'useless hours spent on Euclid' would have been better spent on "understanding the most basic life-skills: mortgages, money management".*
>
> **John White, Irish Times, 5 June 2003**

> *What will mathematics education look like in 2000? The answer is simple. There will be no more mathematics education in 2000, it will have disappeared. There will be no subject called mathematics, no math programme, no math textbook to teach from… It is there to be lived and enjoyed, just as reading, writing, handicrafts, arts, music, breathing, in integrated education.*
>
> **Freudenthal, 1977, p. 294**

One hundred years ago access to post-primary education was only open to a small elite bound for a career in the civil service or professions. Today in Ireland, three to five years of post-primary education is a shared feature of almost every person's educational experience, with four out of five students who start school going on to complete the Leaving Certificate. In Ireland, mathematics has been and is a core aspect of students' post-primary school experience until the completion of schooling as, unlike some other jurisdictions, students cannot choose to opt out of mathematics at senior cycle. In Northern Ireland, for example, students can choose not to take mathematics as an A level subject, thus completing their post-primary mathematics education at GCSE level - the equivalent of the Junior Certificate. The Northern Ireland pattern is similar to that across the developed world (Le Metais, 2003) where mathematics is not a compulsory subject in upper post-primary education. The fact that mathematics is a part of every Irish student's experience throughout their post-primary school years presents a particular challenge in radically reforming, partially revising or tinkering with mathematics education in order to meet the diverse capabilities and interests of entire student cohorts. In discussing the nature of any proposed reforms in mathematics education, we draw attention to the importance of interlinking these with wider post-primary reform directions and strategies in order to increase the likelihood of economies of scale and synergistic forces than can operate in broadly based systemic reforms.

The current focus on redefining and reforming mathematics education in many countries constitutes a *conjuncture* (Goodson, 2002), that is, a powerful educational movement which sweeps around the world, knowing no respect for national boundaries. In light of these two significant challenges, of redefinition and reform, this final chapter focuses on their implications for post-primary mathematics education. Both from a national and international perspective, these challenges have emerged as urgent issues and are an illuminating barometer of the changing relationship between school and society in an era of globalisation, with its calls for a move toward a knowledge-based society. As we noted earlier, the cultural pressure to redefine mathematics emanates from a variety of sources, including disenchantment with the overly abstract focus of the now longstanding 'new' or 'modern' mathematics curricular culture, alarm among the business community (and some educators) at students' limited capacity to apply knowledge in new contexts, pressure from the learning sciences to revise our deeply held ideas and assumptions about both learning and mathematical understanding, the unprecedented elevation of international comparative test results onto government and cabinet tables, and deep concern about perceived and/or actual gender, socioeconomic status (SES) and ethnicity gaps on mathematics achievement tests. The cultural pressure to redefine mathematics education has led or is leading to significant educational reform agendas in a number of countries/regions (e.g. China; Victoria, Australia; Germany; Singapore; USA; UK).

This chapter addresses the issues raised in this report in terms of their potential to contribute to the current review of post-primary mathematics education in Ireland. The review is the first such opportunity in Ireland for over forty years, and is intended to be a root-and-branch initiative rather than a tinkering with the current model of mathematics at both junior and senior cycle (NCCA, 2005). In earlier chapters, we have sought to provide a wide range of examples from various countries around the world to illustrate how they are grappling with redefining mathematics education. In doing so, we have drawn on debates on mathematics education, mathematics education policy initiatives, new cross-national research on classroom practice using video technologies, recent developments in the learning sciences and mathematics education, and current understanding of educational systems as complex ecologies (Hoban, 2002). Considering the myriad challenges that face any attempted reform of mathematics education, we emphasise that it is highly unlikely that merely changing one feature of the system -the mathematics curriculum culture, textbooks or testing/examinations alone - will result in new forms of mathematics education in Irish post-primary classrooms. Even if all three were to be the focus of reform efforts, it is highly unlikely that actual changes in classroom practice or student scores in international tests would be evident on a wide scale for at least five to ten years. Nevertheless, in writing this chapter we assume the need for some moderate - if not significant - mathematics education reform given Ireland's ranking in international comparative studies vis-à-vis the country's ambitious economic, educational, social and research goals.

This does not necessarily mean that PISA results alone should be used, for example, as a reason for curriculum change in mathematics, but that a careful consideration of the relative merits of adopting a PISA-like approach to mathematics education may have considerable value. The merits of such an approach are not based solely on how a PISA-like reform of mathematics education might deliver a higher ranking in, for example, a PISA study a decade from now, but on how it may change the meaning of mathematical competence and create, in the long term, a more mathematically literate society ready to embrace the challenges of the 21st century. The necessary conditions for a radical review of mathematics education have far-reaching implications, encompassing primary and post-primary teacher education; the teaching of mathematics in tertiary institutions (especially those where future mathematics teachers enrol); the scope, rigour, practice-related and sustained nature of teacher professional development; the design of assessment and examinations; the nature of teaching resources (i.e. textbooks and ICTs); and the assumptions underpinning people's 'everyday' and the system's 'official' conception of mathematics education.

Ireland's current economic and social goals, whether expressed in terms of its desire to become a knowledge economy or for Irish universities to be ranked in the top quarter of OECD country universities, are ambitious ones that present definite challenges in light of existing human capital (of which mathematical literacy can be seen as a component part) and infrastructure. In relation to the knowledge society goal, some would argue that it has already been achieved, whereas others would claim that there is only a vision not the reality of a knowledge society. Whichever is the case, mathematics is typically seen as playing a very important role. For example, the former Chief Science Advisor (McSweeney, 2005) to the Irish government argued that mathematics may be even more central than the hard sciences such as physics (important as they are) in providing an underpinning for the knowledge society. Ernest (2000), reflecting on mathematics education in England, argues that 'the utility of school and academic mathematics is greatly overestimated, and the utilitarian argument provides poor justification for the universal teaching of the subject throughout the compulsory years of schooling' (p. 2). However, he extends his critique by making a case for a more expansive set of aims for school mathematics, encompassing (i) skills and enhancement of knowledge-based capability, (ii) the development of creative capabilities in mathematics, (iii) the development of empowering mathematical capabilities and a critical awareness of the social applications and uses of mathematics, and (iv) the development of an inner appreciation of mathematics, including its big ideas and nature.

However, in considering the role of mathematics in the wider curricular and societal context, neither education systems nor curricular areas derive their full rationale from the needs of the economy alone. It is a marked feature of curricula everywhere that they must balance multiple educational goals. The competing nature of those important goals presents real challenges at system, school and student levels. For example, debates about reforming mathematics education at senior cycle might differ significantly if only a small percentage of each senior-cycle cohort were studying the subject. The fact that mathematics is a subject taken by all students until the completion of their post-primary education means that curriculum review and reform must address both the two most significant mathematics education goals: 'mathematics for scientific advancement' and 'mathematics for all'[15].

The remainder of this chapter is divided into two parts: (i) a brief overview of the post-primary mathematics education context in Ireland; and (ii) a discussion of five key challenges which emerge from the preceding chapters in the context of the current review of mathematics education in Irish post-primary education.

5.2 Context of post-primary mathematics education in Ireland

In this section, we summarise key issues related to mathematics education in Ireland under two headings: (i) concerns about post-primary mathematics education; and (ii) recent research on post-primary mathematics education in Ireland.

Concerns about post-primary mathematics education

Concerns about mathematics education have been in the news in Ireland over the last few years. In particular, the publication of PISA mathematics literacy results (a preliminary report was published in

15 The NCCA's Discussion Paper (Review of Post-primary Mathematics Education, NCCA, October, 2005) on post-primary mathematics education refers to 'mathematics for scientific advancement' and 'mathematics for all' as 'specific' and 'general' mathematics education respectively.

December 2004 and full report in April 2005[16] by the Educational Research Centre) and the Leaving Certificate results in August 2005 drew attention to a number of concerns about mathematics education - some new, some not so new. A perusal of the related newspaper headlines presents a range of concerns reflecting the *push* and *pull* factors found in other countries (see chapter one, section 1.4). In August and September 2005, concerns about mathematics at post-primary level that were expressed in national newspapers partly repeated the annual discussion of exam results on the occasion of the publication of the Leaving Certificate results, but they also reflected a broader disquiet about the subject at post-primary level. Among the headlines were the following:

- **Poor maths results just don't add up**
  (Irish Times, 12 August 2005, p. 14)

- **Overhaul of Leaving Cert Maths urged by key group**
  (Bottom front page, Irish Times, 16 August 2005)

- **High rates of failure in Leaving Certificate maths and science**
  (Main headline on front page of Irish Times, 17 August 2005)

Typically, these concerns are linked, both by politicians and the business sector, to the country's economic development and/or the important role of mathematics in helping create a knowledge economy. Although taking a somewhat wider perspective than mathematics alone, a recent article on Ireland's low output of knowledge-economy ready students claimed that:

> *In recent years students (and their parents) have shown a declining interest in third-level courses that lead to careers in science and technology.*

16 Proceedings of the April 2005 national conference reporting on the PISA 2003 findings were published in September 2005. The proceedings include a summary of papers of key findings in Education for Life (Cosgrove, et al., 2005) followed a by a summary of the feedback from educationalists to the ERC and PISA steering committee. Available online at: www.erc.ie/pisa/

> *At second level, this is reflected in lower take-up of relevant school subjects, as well as lower performance in them. This trend has now reached crisis proportions. (O'Hare, Irish Times, 17 August 2005, p. 14).*

O'Hare's comments reflect a concern that applies to mathematics as well as other high-yield subjects in the development of a knowledge economy. In a similar vein, the Irish Business and Economic Federation's (IBEC) director of enterprise (Butler, *Irish Times*, 12 August 2005, p. 14) contrasted the improvement, between 1994 and 2004, in the percentage of honours students in nine other LC subjects with the decline in the percentage of students achieving a C or greater in LC honours maths: 84 percent got an honour in 1994 compared to 77 percent in 2004. He elaborated by arguing that the number of students not passing ordinary level mathematics (approximately 5,000 in 2005), combined with the decrease in interest in science subjects, 'should ring alarm bells around the cabinet table, not just in the Department of Education'. Furthermore, he warns that, given the Industrial Development Authority's (IDA) prediction that biotechnology, a mathematics-dependent sector, will be the next big driver of Ireland's knowledge-based economy, what is especially worrying is the mismatch between stated government aims of fostering a knowledge society and the fact that fewer students are 'leaving school with the skills and interest needed to support the projected growth of these sectors'. Stressing the vital role of mathematics, he claims that, 'virtually no quality career… will be available in high-tech sectors without a high-level knowledge of maths'. Three days later, in a front-page article in the *Irish Times*, IBEC again expressed concerns about mathematics, claiming that, 'The current curriculum has failed to give students a good understanding of the practical uses of maths outside of the classroom and must be addressed' (*Irish Times*, 17 August 2005, p. 1).

The above concerns reflect a mixture of *push* factors (that is, a perceived declining standards and poor capacity to apply mathematics in real world contexts) and *pull* factors (that is, the likely mathematical needs of the economy in newly emerging areas of science that will demand sophisticated knowledge and use of mathematics in problem-solving contexts). While the emphasis is clearly on 'mathematics for scientific advancement', the concerns being raised about foundation level mathematics reflect a wider concern about the mathematics capacity of more than just an elite group of high-flying high-tech-bound mathematics students. As such, there is some evidence of a 'mathematics for all' emphasis in some of the concerns currently being raised about mathematics in Ireland. However, the 'mathematics for all' argument, as used in mathematics education, typically focuses not only on mathematics for economic purposes, but also its role in educating a critically informed, numerate citizenry.

To the extent that media debates about mathematics focus on a restricted version of 'mathematics for all', it is likely that some important functions of mathematics in society may be overlooked. Scribner's three-metaphors approach to characterising literacy (1986) is helpful in this context: that is, literacy as adaptation, literacy as power, and literacy as a state of grace. While she was referring solely to reading literacy, she stressed the important role of each type of literacy. The *literacy* as adaptation metaphor focuses on how reading literacy or mathematics literacy allows the person to survive and adapt in society. Note, though this is more than just 'basic survival' and can also include high-level adaptation, the emphasis is nevertheless on the adaptive, survival function of literacy within existing economic and political structures. *Literacy as power* stresses the political empowerment that can accrue to people through access to literacy. In a similar fashion, one might ask in what way might access to mathematics enhance people's capacity as citizens to transform the world around them. Finally, Scribner writes about *literacy as a state grace*: that is, a form of individual or collective access to literacy's privileged forms of power, knowledge and aesthetic experience. With different but related emphases, the multi-dimensionality of literacy is evident in the OECD's expansive definition of mathematics literacy in PISA:

> *…an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage with mathematics in ways that meet the needs that of the individual's life as a constructive, concerned and reflective citizen (OECD, 2003, p. 24)*

The OECD's definition of mathematics is based on its literacies framework rooted, as we have noted earlier, in socio-cultural understandings of language and literacy.

Specifically, PISA drew on Gee's (1998) definition of language and the PISA studies extend Gee's understanding of language and its allied framework to view mathematics as a language in its own right, in terms of people's knowledge of both its design features (facts, signs, symbols and procedures) and the potential uses of these resources in a variety of non-routine situations for different social functions (see OECD, 2003, p. 26) (see also RME in chapter three and Mathematics in Context, that is, case one in chapter four). The adoption by the OECD of this socially embedded model of knowledge in PISA represents a very significant move toward acknowledging that understanding the use or application of knowledge is not merely a curricular afterthought, along the lines of, 'nice if we can manage to fit in some applied bit of maths but real maths is more abstract!'

It ought instead to be a constituent part of how teaching and learning are conceptualised, in order to support knowledge building by drawing upon a range of cultural and psychological tools and resources: classroom social interaction and support, technologies, symbols, and non-routine problems. The expansive definition of mathematical literacy in PISA, given its focus on the development of constructive, concerned and reflective mathematically literate citizens, presses for a vision of mathematics beyond one hitched only to economic imperatives.

Irish 15-year-olds' mathematical literacy scores: crisis or no crisis?

The process of making claims about the extent and nature of a crisis, if any, depends on appraising the meaning of a number of different types of test results in the light of educational and societal goals. Results of tests typically report two types of scores: measures of central tendency (known as the 'average' in our everyday use of that term but referred to as the 'mean') and measures of variation or distribution of scores. In the case of PISA, a country's mean score is used to create a ranking of where a country lies in relation to the OECD average and other countries - what some have called a 'cognitive Olympics' approach to education. Measures of dispersion provide an indication of how student scores differ from the mean.

The moderate scores of Irish 15-year-old students on PISA mathematics literacy results have drawn unprecedented attention to, not only the actual ranking of Irish students, but also to an emerging debate in the Irish mathematics education and wider education community about the nature of post-primary mathematics. Close and Oldham (2005), commenting on the 'marked contrast' between Irish students' high ranking in reading literacy (5th of 39 countries) and their mid-ranking in mathematical literacy (20th of 40 countries), warn that:

> *The mathematics performance is of some concern since mathematics is considered a key area of competency in moving towards a knowledge-based society and economy. In this context, the contrast between the type of questions in the PISA mathematics tests and those typical of the Irish Junior Certificate examinations has been the subject of discussion. (p. 174)*

However, given the different philosophies underpinning the Junior Certificate and PISA mathematics literacy, one could argue that the performance of Irish 15-year-olds was higher than expected: i.e. Irish students performed at about OECD average (Cosgrove, Oldham and Close, 2005). Overall, Ireland's scores in mathematical literacy resulted in a ranking of 17th of 29 OECD countries. In terms of the distribution of scores, Ireland had 'comparatively few very high achievers and very low achievers'. That is, 17% of Irish students achieved Level 1 or below Level 1 (the lowest two levels on the proficiency scale), compared to an OECD average of just over 21%, whereas approximately 11% of Irish students achieved Levels 5 and 6 (the highest levels), compared to an OECD average of 15% (Cosgrove *et al.*, 2005). The comparatively low level of very high achievers raises concerns about whether current post-primary mathematics education is preparing a sufficient number of students to meet the demands of a knowledge society which will need a pool of researchers and other professionals for a range of fields underpinned by mathematics, as well as mathematics graduates who may enter the teaching profession. One other indicator of the pattern of Irish scores is particularly informative. The average difference in

Ireland (220.9) between the 10th and 90th percentile scores was lower than the OECD average difference (259.3), providing evidence that there is a narrower band of achievement in Ireland than in many other countries (Finland's and Korea's were also small). This can be seen as a positive outcome in that the Irish education system provides, relative to most other OECD countries, a more equitable outcome in mathematical literacy. Cosgrove *et al*. (2005) note that the combination of high achievement and a narrow spread of achievement in Finland and Korea is important in demonstrating that high average achievement scores and low variation in achievement scores are not incompatible. In chapter two we noted a similar pattern in the case of Japan's TIMSS (1995) results and that this pattern was one of the main reasons for the attention accorded Japanese mathematics education over the last decade. In summary, the ranking based on the mean scores raises serious questions about Ireland's overall performance; the pattern of dispersion raises questions about the dearth of very high achieving students; and the narrow spread of achievement is a welcome finding.

Is there or is there not a crisis in mathematics education? On the one hand, there is the no-crisis argument that the existing mathematics education system has provided a sufficient pool of mathematically competent people to support the Celtic Tiger economy. On the other hand, Ireland's mid-ranking PISA scores may not be sufficient to meet the social and economic aspirations in today's Ireland. Overall, given the ambitious social, economic and research goals being set by the government and research agencies and institutions, we think that the mid-ranking scores and the comparatively low level of very high achieving students are a cause for considerable concern. To the extent, as Dewey (1938) has argued, that education is about providing experiences with which students can grow into more fruitful and expansive engagements with future experiences (even if the nature of these is as yet somewhat unclear), it is important, at the very least, to consider carefully the nature of post-primary mathematics education. The decision as to whether a reform of post-primary mathematics education might move toward a PISA-like approach to mathematics is inextricably linked with a vision of what the competent mathematical learner of 2010 and 2020 ought to look like. That in turn is based on our incomplete knowledge of how society will evolve in the next few decades and what that will mean for mathematical ways of knowing.

Recent research on mathematics in Ireland

There has been a considerable amount of research on mathematics education at primary and post-primary levels in Ireland over the last five years which can contribute in important ways to the current review of post-primary mathematics. In light of Lyons *et al.*'s (2003) observation that over the last thirty years there has been 'a limited amount of research undertaken on mathematics education in the Republic of Ireland' (p. 2), the recent publication of their landmark study and two other major publications (Cosgrove *et al.*, 2005; Close *et al.*, 2005) on mathematics education this year represents a significant development. In this section, we summarise major findings rather than review the research. Three recent publications bring together much of the relevant research: *Inside Classrooms: the Teaching and Learning of Mathematics in Social Context* (Lyons, Lynch, Close, Sheerin and Boland, 2003); *Education for Life: The Achievements of 15-year-olds in Ireland in the Second Cycle of PISA*[17] (Cosgrove, Shiel, Sofroniou, Zastrutzki and Shortt, 2005); and the *Proceedings of the First National Conference on Research in Mathematics Education* (Close, Dooley and Corcoran, 2005).

17 Education for Life (Cosgrove et al., 2005) can be downloaded at the following web address: www.erc.ie/pisa/

Inside Classrooms involved a video study of junior-cycle mathematics classes in coeducational and single-sex post-primary schools. It is a landmark study in Irish education in that it is the first video study of classroom teaching in Ireland, and provides insights into pedagogy and its impact on student attitudes to mathematics and their opportunities to learn. The *Inside Classrooms* study had a specific focus on questions of gender and its relationship to classroom learning opportunities in mathematics. In line with the video studies discussed in chapter two of this report, it reflects a move toward a contextual and up-close analysis of classroom teaching.

Education for Life (Cosgrove *et al.*, 2005) provides a detailed analysis of the PISA 2003 results, in which mathematical literacy was a major domain, for Irish 15-year-olds, as well as a detailed account of the PISA mathematical literacy framework (Cosgrove *et al.*, 2005, pp. 4-11).

In the context of the current review, the 300-page-plus *Proceedings of the First National Conference on Research in Mathematics Education* provides a timely and detailed set of twenty-three papers, many of which are directly relevant to post-primary mathematics education.

What shapes student experiences of maths? Curriculum culture, textbooks and examinations

In earlier chapters, we focused on three system-level features which profoundly shape students', teachers' and parents' experiences of mathematics education: curricular culture, textbooks, and testing/examination traditions. We now outline relevant research in Irish post-primary mathematics education pertaining to these features.

The curricular culture of Irish post-primary mathematics education

The *Inside Classrooms* video study provides convincing evidence[18] that Irish post-primary mathematics education at junior cycle is dominated by a culture of teaching and learning 'which presents the subject as static rather than dynamic, abstract, formal and remote rather than relevant and accessible' (p. 363). The video study documents the close relationship between teachers' stated beliefs about teaching and learning mathematics and their actual classroom practices (p. 366). When interviewed, teachers stressed the importance of the 'demonstration of procedures and monitoring of students' progress and the video analysis bore this out' (p. 366). This finding reflects well-established findings in research on teacher thinking which has demonstrated a considerable degree of consistency between teachers' stated views on teaching and their classroom practices (Clark and Peterson, 1986; Morine-Dershimer, 1997). Lyons *et al*. conclude that:

> *…all twenty lessons that we videotaped were taught in a traditional manner… most of the time was spent on exposition by the teacher, followed by a programme of drill and practice. Overall teacher-initiated interaction comprised 96% of all public interactions in the classes, and within this context a International Trends In Post-Primary Mathematics Education procedural rather than a conceptual and/or problem-solving approach to subject matter prevailed. (p. 367)*

Oldham (2002) anticipated these findings in her comments on the historical roots of Ireland's post-primary mathematics education culture:

> *At second level, Ireland adopted 'modern mathematics' in the 1960s to greater extent than many other countries (Oldham, 1989; Oldham, 2001); the syllabuses and, in particular, the examination papers,… still*

18 A single class group for two class periods was videoed in ten different schools. While the sample of lessons and teachers was small, and not nationally representative, we think the video study is convincing, given the relative homogeneity of the education system in Ireland.

reflect the focus on precise terminology and abstraction that is characteristic of the movement. The recent revision of the junior cycle syllabus was less profound than the revision at primary-school level; it was essentially a minor update to deal with areas of the course that were giving difficulty rather than a root-and-branch review. (p. 43)

The video study evidence and historical context both suggest that the mathematics education culture existing in Irish post-primary schools (at junior and senior cycles) is didactic and procedural, and as such consistent with the new mathematics education movement. Lyons *et al.* go as far as to say that, based on evidence of curricular cultures in other subjects at second level, 'the evidence from this study is that mathematics classes are especially didactic' (p. 367).

Textbooks in Irish post-primary education

Whereas the *Inside Classrooms* video study and the historical context of Ireland's adoption of new/modern mathematics is now well documented, there is a dearth of evidence on the nature of post-primary mathematics textbooks. What mathematics education curricular culture informs and is reflected in the textbooks? What is the role of examinations in shaping textbook content and format? What types of mathematical ways of knowing are privileged in post-primary mathematics textbooks? From the PISA competency clusters perspective, to what extent do textbooks focus on each of the three clusters, namely, reproduction, connections and reflection? To use PISA terminology, what is the range of 'contexts' provided in presenting mathematical ideas? These and other questions are important in considering the role of textbooks in any effort to redefine post-primary mathematics education. Some tentative findings in the Irish context may be gleaned from the TIMSS textbook analysis (see chapter one, section 1.5). Internationally, the mismatch between stated mathematics education reform aims and the marked dearth of textbook content reflecting these reform aims, identified in the TIMSS textbook analyses, in which Irish textbooks were included for two of the three populations, suggests that it is unlikely that Irish post-primary textbooks are any different. Indeed, scanning the textbooks it appears that they, like the Junior and Leaving Certificate examinations, rely on a similar pattern of exposition as evidenced in examination papers: that is, a focus on precise terminology, symbol manipulation and abstraction with little attention devoted to the provision of rich contexts to concretise mathematical ideas. From a Realistic Mathematics Education perspective, we think it is reasonable to argue that textbooks at post-primary level typically focus on vertical mathematising[19] rather than providing students, teachers and families with opportunities to experience the full horizontal and vertical mathematising cycle. In the context of the review, we note that one of the features of RME-influenced reforms in the Netherlands was a revision of textbooks in order to create contexts in which teachers and students could grapple with mathematical ideas in ways consistent with the horizontal and vertical mathematising cycle underpinning RME.

Testing/examination traditions in mathematics at post-primary level

> *They were only weeks away from the examination halls. So much work still had to be done, so much work had to be gone over again. The chance-throw of the exam would almost certainly determine the quality of the rest of their lives. Sheila had dreams of university. Much could be won, a great deal could be lost, and there was always England.*
>
> **(John McGahern, Amongst Women, pp. 72-73).**

19 Horizontal mathematising refers to the process of taking real-life situations and turning them into mathematical models. Vertical mathematising refers to the manipulation of symbols and advancement of mathematical modelling and/or translating this back to the horizontal level. The horizontal-vertical distinction has its origins in RME (see chapter three). A good example of failed translation of mathematical models back to the horizontal is how IMF and World Bank economists (see Stiglitz, 2003) are often accused, unfairly or not, of failing to understand the implications of their econometric model-based prescriptions on real-life economics in vastly diverse settings, where culture, or other important variables, cannot be reduced to a specific amount of variance in multiple regression models. Such arguments and criticisms draw attention to the hermeneutic dimension of translating mathematical models of social and economic dynamics to real-life situations.

The centrality of the examination tradition in Irish post-primary education provides a good opportunity to examine the type of knowledge valued in any subject area in Irish education. In the case of Irish post-primary mathematics, Close and Oldham (2005) have recently undertaken a comparative content analysis of certificate examinations and the PISA framework, in which they mapped the 2003 Junior Certificate and the 1974 Intermediate Certificate onto the three-dimensional (content, competencies and situations) PISA mathematical literacy framework. Their study is illuminating on two levels. Firstly it provides insights into the curricular culture of post-primary mathematics and reveals the consistency across time of a particular view of mathematics that is aligned with the 'new' mathematics movement. Secondly it reveals striking differences between the type of mathematical knowledge that was required for success on both the 2003 Junior Certificate and 1974 Intermediate and that in the PISA mathematical literacy framework. As Close and Oldham observe:

> *…major differences can be seen between the PISA tests and Irish papers. There are considerable discrepancies across specific Irish papers in terms of the percentages of items in two of the three competency clusters (Reproduction and Connections), again reflecting the levels of the examinations. (p. 185)*

They also identified a 'major difference' in the situations component between the PISA framework and examination papers. The Close and Oldham (2005) and Cosgrove *et al*. (2005) studies draw attention to the significant discrepancies between PISA and Irish post-primary mathematics. It is important to note that PISA mathematical literacy assessments were never intended to reflect or assess national mathematics syllabi; rather PISA mathematical literacy is based on a vision of mathematical competence that PISA views as necessary for students' life as productive and reflective workers and citizens in the 21st century. The Close and Oldham study, in particular, draws attention to how the examination tradition emphasises a certain type of mathematical competency (i.e. reproduction) and draws upon narrow range of mathematical situations (i.e. mainly intra-mathematical).

Table 7: Percentage of items in the 2003 Junior Certificate and 1974 Intermediate Certificate mathematics papers corresponding to the three clusters of the PISA mathematics framework

PISA framework dimension	Dimension category	PISA 2003 maths test (85 items) % of	2003 JC Higher level maths (77 items) % of	2003 JC Ordinary level maths (81 items) % of	2003 JC Foundation level maths (32 items) % of	1974 JC Higher Course maths (79 items) % of	1974 IC Lower Course maths (79 items) % of
Competency clusters	Reproduction	30.6	83.1	95.1	100	84.8	88.2
	Connections	47.1	16.9	5.0		15.2	11.8
	Reflection	22.4					

SOURCE: Close and Oldham, 2005, p. 183

Table 8: Percentage of items in the 2003 Junior Certificate and 1974 Intermediate Certificate mathematics papers corresponding to the context categories of the PISA mathematics framework

PISA framework dimension	Dimension category	PISA 2003 maths test (85 items) % of	2003 JC Higher level maths (77 items) % of	2003 JC Ordinary level maths (81 items) % of	2003 JC Foundation level maths (32 items) % of	1974 JC Higher Course maths (79 items) % of	1974 IC Lower Course maths (79 items) % of
Situations	Personal	21.2		12.3	21.9	1.2	13.2
	Educational Occupational	24.7	6.5	23.4	6.2	7.6	7.4
	Public	34.1	16.9	2.5	12.5	8.8	10.3
	Scientific	20.0	10.4				
	Intra-mathematical		66.2	61.7	59.4	82.2	69.1

SOURCE: Close and Oldham, 2005, p. 184

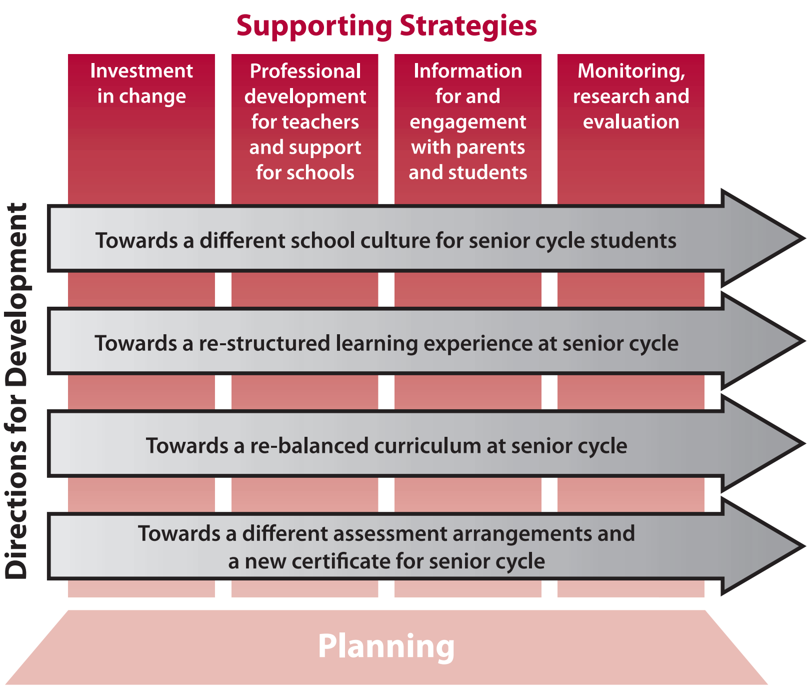
The intra-mathematical dominance of context (situations) illustrates how the abstract focus of new mathematics has filtered into the type of examination questions likely to be asked, and thus likely to guide teachers and students in their exam preparation (see table 8) (Elwood and Carlisle, 2003). Elwood and Carlisle (2003, pp. 72-74), in their examination of Leaving Certificate mathematics papers (they examined both content and question type) found that questions in the Leaving Certificate were abstract, focused on 'pure' mathematics, relied more heavily on traditional notations and symbols than comparable examination papers in other jurisdictions, and were notable also for the lack of real-life contexts in questions. In the next section we note key reform challenges in mathematics in the light of the wider reform context at junior and senior cycle in Irish post-primary education.

5.3 Five challenges within a wider post-primary reform context

> *…systems thinking is a discipline for seeing the 'structures' that underlie complex situations…systems thinking offers a language that begins by restructuring how we think.*
>
> **(Senge, 1990)**

Post-primary education in Ireland has been the focus of considerable attention over the last few years, due largely to the acknowledgement that while the curriculum and examination system has served society well it may not be providing students with the necessary learning experiences for the 21st century, especially in an emerging knowledge society. The National Council for Curriculum and Assessment proposals for senior cycle sets out possible reforms for upper post-primary education (NCCA, 2003; NCCA, 2005). The wider reforms outlined by the NCCA set a context for any proposals to reform post-primary mathematics education. Developing a different school culture, reforming students' experiences of teaching and learning, re-balancing curriculum at senior cycle, and developing different assessment arrangements and a new certification system for senior cycle, were the directions for development specified in the NCCA document (see Figure 6). The documents outline a series of support strategies that would be necessary to achieve these ambitious goals, encompassing investment for change, professional development for teachers and support for schools, provision of information to and engagement with stakeholders and parents, and monitoring, research and evaluation. Given the interlocking nature of the forces impacting teaching and learning, any effort to reform post-primary mathematics education will have to adopt this or another similarly comprehensive systemic approach.

Figure 6: Proposals for the Development of Senior Cycle Education Directions for Development and Supporting Strategies



The wider post-primary review and reform context is important in considering the potential for serious reform of mathematics education. Roschelle *et al.* (2000), for example, in a comprehensive review of ICTs and learning, observed that technological innovation in education needs a wider reform context for ICTs to have significant impact on teachers' classroom practice and students' learning (for a discussion ICT innovation and educational change in the Irish context see Conway, 2005c; Fitzpatrick and Conway, 2005). Similarly, we want to draw attention to how any effort to reform mathematics education will be supported and/or inhibited by the wider post-primary reform context. For example, should mathematics education adopt a more problem-solving, realistic mathematics education (RME) approach, its impact might be significantly constrained by students' experiences of mathematical ideas in other subjects. If mathematics is presented in subjects such as physics, economics, and geography only as the application of procedures, it might detract from their sense of how both horizontal and vertical mathematising provides a powerful means through which students can engage with the world. In essence, there has to be congruence between students' experiences of mathematical ways of thinking and practices across different subjects. Increasingly, over the last decade, ecological and systems perspectives of educational change have gained prominence over linear, additive and mechanistic change models (Hoban, 2002). Ecological and systems perspectives draw attention to the inter-relationship of a variety of forces and how they might impact educational change: school culture, teachers' lives and learning, assessment systems, teaching and other resources, everyday/'folk' conceptions of teaching and learning, politics, leadership and parental/societal expectations. From a systemic and ecological perspective, any effort to reform mathematics education without attending to the wider context is highly unlikely to bring about the desired changes.

The cumulative impact of the interlocking forces of a curriculum culture, textbooks and examinations mutually reinforcing each other cannot be underestimated in terms of the challenge it poses for any proposed reforms in post-primary mathematics education. The challenges that might be addressed can be outlined under five headings:

- defining a vision
- changing examinations and assessment practices
- the excellence/equity tension
- teacher education
- scaling up: the change challenge.

Defining a vision for mathematics education today

Any claims about maintaining the status quo, engaging in minor revisions, or making radical reforms of mathematics education reflect 'values, interests and even ideologies of certain individuals and groups' (Ernest, 2000, p. 5) Ernest provides an overview of interest groups and their aims for mathematics teaching in Great Britain (see table 9). There is considerable resonance between the aims and locations described by Ernest and aims and locations in the Irish context. Consequently, it is important to consider the value-laden nature of any claims made in relation to the state of mathematics, the interpretation of evidence in discussing the state of mathematics in schools, the discursive space accorded various publics (or stakeholders), and the scope, focus and appropriateness of current and potential aims for mathematics education. For example, to what extent do all students need maximal mathematical knowledge? Is basic numeracy sufficient for a large majority of students? Should appreciation of the big ideas of mathematics be a feature of any, all or just some students' mathematics education? To what extent ought mathematics education focus on a critical appreciation of the social applications and uses of mathematics?

Table 9 Interest groups and their aims for mathematics teaching

Interest group	Social location	Mathematical aims
Industrial trainers	Conservative politicians	Acquiring basic mathematical skills and numeracy
Technological pragmatists	Meritocratic industry-centred managers	Learning basic skills and learning to solve practical problems with mathematics and ICTs
Old humanist	Conservative mathematicians preserving rigour of proof and purity in mathematics	Understanding and capability in mathematics with some appreciation of mathematics (pure mathematics and/or 'new' mathematics-education centred)
Progressive educators	Professionals, liberal educators, welfare state supporters (child-centred progressivist)	Gaining confidence, creativity and self-expression through mathematics
Public educators	Democratic socialists and reformers concerned with social justice	Empowerment of learners as critical and mathematically literate citizens in society

SOURCE: Ernest, 2000, p. 6

Is the development of students' skill or capability in mathematics necessarily competing with the development of students' awareness of big ideas in mathematics? To what extent should some or all students experience horizontal and vertical mathematising (see chapter three)? Ought school mathematics focus first on teaching basic skills prior to teaching higher-order conceptual understanding in mathematics? It is probably important to address some or of all of these - and other - questions in redefining post-primary mathematics education, aware that answers to each of these questions inevitably reflect value positions, some more amenable to empirical investigation than others, but each important nonetheless. Different sectors in the education system (primary, post-primary, tertiary), government, industry[20], business and the wider society have legitimate interests in and claims on how mathematics is and could be taught in post-primary education, and there will be differences in how each might respond to these questions because of their different interests.

In this report, we have emphasised the powerful role of the mathematics curricular culture, textbooks, and examination/assessment traditions, each with identifiable assumptions and values in relation to mathematics education, particularly the nature of what counts as worthwhile mathematical knowledge. In many ways, these can be thought of as interlocking forces - each shaping and reinforcing the other over time. Of these three system-level features, curricular culture is probably the most difficult to change as it is threaded through other aspects of mathematics education, including textbooks, examination and assessment

20 For example, the Irish Council for Science, Technology and Innovation (ICSTI), identified effective primary and second level science, technology and mathematics (STM) education in Irish schools as a priority area for its consideration, and undertook a benchmarking study titled Benchmarking School Science, Technology and Mathematics Education in Ireland against international good practice (ICSTI, 1999). Available online at: http://www.forfas.ie/icsti/statements/benchmark/foreword.htm

traditions, and teachers' and students' mathematical beliefs and practices in the classroom, as well as parents' understandings of mathematics as they support their post-primary students' mathematics learning. Questioning epistemologies, examining philosophies of mathematics education, and reforming teaching and learning practices are central to developing any new vision of post-primary mathematics education. These aspects of reform have implications at system, school and classroom level; we address them in general terms, noting issues of relevance to teacher education (initial, induction and in-career) as teachers' opportunities to learn will be central in any proposed reform of mathematics. For example, if a culturally appropriate adaptation of lesson study was to be implemented it would have significant implications for investment in teacher education at all phases in the teaching life-cycle.

Questioning epistemologies

Underpinning any discussion of curricular culture lies a conception of knowledge in the particular domain. These conceptions are reflected in the personal epistemologies of all involved (teachers, students, parents, curriculum and test/examination designers) and encompass questions about certainty of knowledge in the domain, the basis for making knowledge claims, and the organisation of knowledge in the domain (Hofer, 2004). In this report, we address epistemology through our description of the differing epistemologies underpinning behavioural, cognitive and constructivist approaches to learning.

The realistic mathematics movement adopts a phenomenological epistemology: that is, it focuses on the lived mathematical experiences of learners as the basis for mathematising. On the other hand, new/modern mathematics education's epistemology regards mathematics as an 'objective, value-free, logical, consistent and powerful knowledge-based discipline which students must accept and understand and manipulate' (Burton, 1994, p. 207). The 'real world' and problem-focused approaches to mathematics, inspired by Piagetian constructivism, situated cognition and realistic mathematics education, have been more attentive to questions of epistemology. Although we note that Freudenthal (1991) was reluctant to get drawn into discussions of 'isms' in mathematics education and was deeply sceptical of constructivism, which he viewed as slogan ridden (Freudenthal, 1991, pp. 142-46; Goffree, 1993), he nevertheless adopted an epistemology with a family resemblance to constructivism, given RME's phenomenological view of the learner and learning.

Olson and Bruner (1996) argue that any innovation in education must engage actively with the everyday/folk theories of learning held by teachers and educators. In chapter three, we noted two studies which demonstrated the important role of teachers. In chapter three, we also outlined the role that epistemologies play in shaping teachers' classroom teaching practices (Morine-Dershimer and Corrigan, 1997; Staub and Stern, 2002).

In order to highlight the challenge of addressing educators' everyday theories of learning, we draw on Morine-Dershimer and Corrigan (1996) who, having examined twenty years' research on teacher thinking, provide important insights for those promoting reform of mathematics education:

> *The strength of traditional prior beliefs, reinforced by experiences as students and teachers, makes real change extremely difficult. Teachers implementing mandated changes interpret those mandates through the*

> *screen of their prior beliefs, modifying… desired reform strategies. New practices require new beliefs. Changing beliefs involves cognitive stress, discomfort and ambiguity. In changing beliefs, individuals must reconcile or realign other related beliefs to resolve conflicts and contradictions, and come to terms with what actions guided by previous beliefs meant. Such cognitive reorganization is not easily or quickly accomplished. (p. 308)*

They outline three strategies and four conditions for changing teacher beliefs, noting that sometimes teacher practices create changes in beliefs and other times vice versa. The strategies for changing beliefs (these could be thought of as strategies for initial teacher education and/or continuing professional development) are: changing images via exploration of teachers' images and metaphors for teaching; confronting contradictions; and addressing cases. The four conditions for change in teacher beliefs are: time, dialogue, practice and support. They pose an equally daunting challenge in the design and provision of high-quality teacher education. The necessary conditions for changing teacher beliefs share remarkable similarities with key dimensions of Japanese lesson study (see chapter two).

Examining philosophies of mathematics education

The different philosophies underpinning the current approach to post-primary mathematics education and the PISA mathematical literacy framework have helped to highlight the existing culture of Irish post-primary mathematics. Indeed, the high visibility of the PISA results draws attention to the quality of mathematics education in schools, even if only temporarily. In the context of the current review of post-primary education, it is fortunate that the focus on mathematical literacy as a major domain in PISA 2003 (rather than minor domain as was the case in PISA 2000 and as will be in PISA 2006) has fixed a degree of educational and media attention on mathematics.

The differing philosophies underpinning PISA[21] and the 'new' mathematics-influenced post-primary syllabi have inspired two curriculum mapping exercises in Ireland: (i) a test-curriculum rating involving a measurement of the expected curriculum familiarity students might have with PISA concepts, contexts and formats, based on an analysis of Junior Certificate papers at Higher, Foundation, and Ordinary levels (Cosgrove, Oldham and Close, 2005); and (ii) the mapping of the 2003 Junior Certificate and 1974 Intermediate Certificate against the PISA three-dimensional framework (Close and Oldham, 2005). Cosgrove *et al.* (2005) found that 'some key topic areas of sets, geometry, and trigonometry are not assessed at all by PISA items. There is also little coverage in PISA of algebra, and functions and graphs' (p. 203). Such discrepancies help to highlight the consequences of, for example, reorienting syllabi at post-primary level toward concepts that are the focus of PISA mathematical literacy. Debates focused only on whether one topic or another ought to be on future syllabi will be unlikely to provide the type of root-and-branch review of mathematics education currently under way. In order to clarify the curriculum components that might be addressed, the PISA mathematics components (see table 10) may be useful in ensuring broad-ranging discussion. A discussion of the relative strengths of the current post-primary mathematics syllabi using the PISA components (as has already been initiated by the curriculum mapping exercises noted above) provides a useful set of questions in redefining mathematics education at post-primary level.

21 The OECD's PISA tests are not without their critics. For example, in England there has been considerable debate about the technical adequacy of PISA, the implications sampling of an age group rather than a class group [i.e. PISA sampled 15-year olds rather than a class/year-group as in the case of IEA studies] and PISA designers' lack of attention to linking PISA tests with other international comparative tests in mathematics such as those devised by the IEA (see Prais, 2003; Adams, 2003; Prais, 2004).

Table 10: Mathematics components in PISA and post-primary mathematics

Mathematics component	PISA	'New' mathematics-inspired Irish post-primary mathematics	Reformed post-primary mathematics education
Contexts (Situations)	Diverse range of contexts including personal, educational/ occupational, public, scientific, and intramathematical	Dominated by intramathematical contexts: 60-80% + in Junior Cert (see Close and Oldham, 2005)	What range of contexts will the syllabus include and what will be the emphasis on each in curriculum and assessment?
Content (Overarching Ideas)	4 strands: Space and Shape Change and Relationships Quantity Uncertainty	Syllabi are constructed around curricular strands rather than focused on overarching ideas	What approach to defining content will the syllabus take?.. curricular strands... overarching ideas... other?
Competency Clusters	Reproduction Connections Reflection	Dominated by reproduction (80% + in Junior Cert) but with some focus on connections (see Close and Oldham, 2005)	What types of, competencies, if any, will be expected of students in the new syllabus?

We use the PISA mathematical literacy components not because PISA's version of these components is the only way ahead in curriculum reform, but rather because the components provide a useful set of questions that, we think, are worth addressing in reviewing the post-primary mathematics education. That is to say:

- CONTEXTS: What range of contexts will the syllabus include, and what will be the emphasis on each in curriculum and assessment?

- DEFINING CONTENT: What approach to defining content will the syllabus take:
  - curricular strands
  - over-arching ideas, or
  - other?
- DEFINING COMPETENCIES: What types of competencies, if any, will be expected of students in the new syllabus?

In this section we have noted key dimensions that might be addressed in the redefining of post-primary mathematics education, including discussions of underlying epistemologies, the important role of personal epistemologies for all those engaged in design, enactment and evaluation of mathematics education, and the value of adopting the components of mathematics in PISA as a way of addressing contexts, competences and content in the review process.

In examining post-primary mathematics education, developments at primary level in Ireland provide some useful resources as well as a challenge in terms of how the post-primary system might provide continuity with the more problem-solving and real world focus that has been a feature of primary mathematics - especially over the last six years, since the advent of the Revised Primary School Curriculum (1999) (Conway, 2005b; Coolahan, 2005), and to a lesser extent since the Deweyan- and Piagetian-inspired constructivist orientation embedded in 1971 Primary School Curriculum (see Department of Education, 1971; and for a review Gash, 1993). At present there is distinct discontinuity between philosophies underpinning primary and post-primary mathematics, although the revised junior cycle mathematics makes some moves toward a more problem-solving and constructivist orientation with its focus on fostering students' active learning in mathematics. However, the move towards a problem-solving orientation was reflected more in the Guidelines for Teachers and in-service support for teachers than in the syllabi themselves.

To what extent ought a review of mathematics concern itself with mathematics as used in other curricular areas? For example, the Science, Technology and Society (STS) movement in science education is a kindred spirit to the Realistic Mathematics Education movement in mathematics education. In both movements, there is a range of common assumptions and preferred teaching methodologies, including approaches to learning (both value social support and ICTs in the design of learning environments), the importance of teaching basic skills in the context of real and higher-order activities, and the dual role of reality as both a key source of ideas and an arena for application of models. Recognising the substantial overlap in curricular cultures is an important stance in an ecological approach (Hoban, 2002) to thinking about the future of schools and curriculum reforms. Is it important, in moving toward a knowledge society, that curricular cultures be aligned so that from the perspective of students' educational experience they get a coherent vision of mathematics and learning? So, for example, what are the short and long-term consequences for student learning, if in reforming post-primary mathematics it is poorly aligned with how science, geography or economics teachers teach and use mathematical ideas? For society and the student, the ultimate result may be a weaker grasp of the potential of mathematics and its role as a powerful way of thinking and an important human activity with relevance both in 'real life' and the symbolic world of abstract mathematical modelling. We pose the following thought experiment: what would we learn about one student's experiences of mathematical ways of knowing by shadowing the student for one week across all classes in which mathematics is used and/or taught? Will we learn that this amalgam of experiences leaves the student better off in the sense of being better equipped to think about mathematics? Will the student experience both the real world of mathematical modelling (horizontal mathematising) and symbol manipulation and modelling (vertical mathematising) in each mathematical experience across curricular areas? Will the student get mixed messages about what it means to think mathematically? If yes, what are the consequences, both for the student and for society?

Changing learning, assessment and examination practices

> *Assessment should be recognised, not as a neutral element in the mathematics curriculum, but as powerful mechanism for the social construction of mathematical competence*
>
> **(Clarke, 1996, p. 327)**

> *…the issue now seems to be whether policy makers have both the wisdom to demand that we apply what we already know to design a program that maximizes the benefits and minimizes the negative consequences and the patience it takes to get that job done. In designing such a program, policy makers must realize that, while the effects of tests are often portrayed as uniformly good or bad, tests affect different types of students quite differently and that any particular affect is often a two-sided coin*
>
> **(Madaus, 1991, p. 226)**

Over the last decade there has been a significant realignment of the relationship between learning and assessment (see chapter three). Largely due to very significant developments in both the learning sciences and measurement theory, there are now new ways of thinking about learning and assessment not only as loosely linked ideas, but as closely intertwined but distinct and powerful ways of understanding schooling. For example, developments in self-regulated learning have brought prominence to issues of self, peer and teacher-assessment and how each might support students' ownership of learning. Furthermore, the powerful impact of feedback to learners, alongside clearly articulated domain-specific performance criteria, has been strongly linked to enhanced student learning (Black and Wiliam, 1998a; Black and Wiliam, 1998b; Gardner, 2006). In drawing attention to the potential of utilising assessments in support of learning, we are distinguishing assessment from the more narrowly defined goals of selection and certification typical of high-stakes examination systems. The realignment of the relationship between learning and assessment is important in considering the redesign of examination and assessment systems. In that redesigning, two aspects of assessment and examinations must be addressed: (i) *how* examinations and assessments are organised and (ii) *what* they examine/assess. Regardless of how sophisticated the design and enactment of any examination/assessment system is, it will inevitably be, in Shulman's (1987) terms, a 'union of insufficiencies' in that our visions and practices of learning more often than not outstrip our mechanisms for its assessment.

How do we assess?

Firstly we address the ways in which examinations and assessments are undertaken. The question of new assessment and examination traditions arises, given the importance of the latter in enshrining as almost sacred particular forms of knowledge. One of the advantages of an 'exam tradition' in any educational culture is the very fact that it is reflected in some degree of shared understanding about what knowledge is valued, how all involved (students, families and teachers as well as examinations administrators, test/question designers, curriculum developers, media commentators) can prepare for the high-stakes examinations, how students go about the actual exam (typically, a sit-down paper-and-pencil mode of assessment), and most importantly there is typically a very significant degree of credibility attached to the results in terms of both their validity and fairness. Madaus (1992) catalogues the advantages of highstakes examinations as follows (p. 229):

- they are relatively objective and impartial means of distributing educational benefits
- they engender a degree of national homogeneity in educational standards and practice
- they give teachers a sense of purpose and provide tangible benefits for students
- they diminish the conflict between roles of teaching and assessment by providing an assessment procedure that is unaffected by personal relationships between teachers and students
- they are widely accepted by society
- they create some sense of social and educational standards among the young, while meeting some definition of comprehensiveness, equal access, and shared experience.

These examination advantages tend to create a powerful force for stability. In the present context, given the *push* and *pull* factors which are presenting schools and society with new visions of both subject matter and learning, some examination/assessment systems are being reformed by changing assessment modes congruent with these new visions of learning (e.g. the introduction of some school-based project work and ICT-supported testing in Victoria, Australia). Furthermore, taking the learning principles underpinning RME and situated cognition-inspired PISA mathematics literacy to their logical conclusion, suggests that assessments ought to involve students in actual real-world settings with real-world problems. As Cosgrove *et al*. (2005) observe, 'Students would work in groups, each one contributing ideas on how a problem might be solved, and what the results mean' (p. 5). They then note the constraints under which PISA operates which resulted in PISA relying on 'traditional' paper-and-pencil modes of assessment. This 'practical' fall-back option is a familiar response of education systems given the logistical, bureaucratic and financial burden of designing more authentic assessments (e.g. project work, portfolios, collaborative assessment tasks) on a large scale (Mehrens, 1992). However, in the long term, the 'opportunity cost' of not providing some form of more authentic or real-world assessment modes in post-primary mathematics seems significant and might be measured in terms of student alienation and disenchantment with the subject, the persistence of a narrow vision of what counts as mathematical knowledge and competence, and decreasing the likelihood that students will develop their capabilities to use mathematical knowledge in the context of new and non-routine real-world contexts. Thus the negative systemic effects of high-stakes examinations may be added to other, well documented drawbacks of the 'traditional' timed paper-and-pencil examination format. Among the disadvantages Madaus (1992) catalogues are the following (p. 30):

- High-stakes tests in upper grades can have an undesirable backwash or trickle-down effect on class work and on study in the lower grades.

- They tend to encourage undue attention to material that is covered in the examinations, thereby excluding from teaching and learning many worthwhile educational objectives and experiences.
- Scores on them come to be regarded by parents and students as the main, if not sole, objective of education.
- They are usually carried out under artificial conditions in a very limited time frame. They are not suitable for all students and can be extremely stressful for some. In addition, they can negatively affect such personality characteristics as self-esteem and self-concept.
- There is often a lack of congruence between course objectives and examination procedures (e.g. there may be no examinations for oral or practical objectives).
- Some kinds of teaching to the test enable students to perform well on examinations without engaging in higher levels of cognition.
- Preparation for high-stakes tests often overemphasises rote memorisation and cramming by students and drill and practice as a teaching method.
- High-stakes tests are inevitably limited in the range of characteristics that they can assess, relying heavily on verbal and logico-mathematical areas.

This catalogue of disadvantages may sound all too familiar in the Irish context, but it can be seen as a source of inspiration in designing more authentic assessments in mathematics. It is also a warning about how the high-stakes nature of any type of examination or assessment can be distorted by wider socio-political and economic forces. However, as we noted in chapter three, an Australian study of the impact of alternative mathematics assessment in the state-level Victorian Certificate of Education (VCE) demonstrated that there was a positive backwash effect on teaching in Years 7-10 after changes were made in assessment practices in Years 11 and 12. Year 11 and 12 assessments consist of four components: (a) a multiple-choice skills test, (b) an extended answer analytic test, (c) a 10-hour 'Challenging Problem', and (d) a 20-hour 'Investigative Project' (Clarke and Stephens, 1996; Verschaffel, Greer and de Corte, 2000). However, as we noted earlier (see chapter three) the VCE has been revised and it now consists of only two rather than four components; that is, an examination and a school-based assessment (Barnes, Clarke and Stephens, 2000).

What is assessed?

Whether in APEC or the OECD countries, the question of what needs to be assessed has presented a new challenge in terms of the appraisal of students' 'learning to learn' capacity. One of the features of globalisation, and its attendant calls for the development of a knowledge society, is a recognition that students cannot and will not be able to learn all they need to know in school (OECD, 2003). This has pressed education systems to focus, not just on subject matter, but on the promotion of learning to learn both within subject matter and as a cross-curricular competence (Baumert, 1999). In the case of the OECD/PISA, the assessment of problem-solving as a domain (in addition to reading literacy, mathematical literacy and scientific literacy) is indicative of this new focus on not just teaching problem-solving but assessing it in a formal manner. However, assessing problem-solving separately (as a proxy indicator for learning to learn competence) seems contradictory, given Resnick's (1987) observations that learning to think/learn as well as the teaching of thinking/learning skills has to occur within some subject/content area. The actual promotion of students' learning to learn competence crystallises the relationship between teaching/learning and assessment, and demands a greater focus on a number of different types of assessment, such as assessment for learning (AfL) as well as peer and self-assessment (see chapter three). In the context of the current review of post-primary mathematics education, reforming the examination system is likely to present the biggest challenge, given the stability-inducing effect of an examination tradition, as well the logistical, bureaucratic and financial implications of reforming mathematics education examinations system-wide in line with new visions of both *learning mathematics* and *learning to learn in the context of mathematics*. However, we note that examination system reforms can have a strong ripple effect on teaching, learning and assessment down through the post-primary system, as has been the case in Victoria, Australia over the last fifteen years (Barnes, Clark and Stephens, 2000). A change in the assessment system at post-primary level may be essential in overcoming what Elwood and Carlisle (2003) see as the 'narrow view of achievement in mathematics… promoted by these examinations, and it is one that does not sit comfortably with the aims and objectives outlined in the syllabi' (p. 111). Close (2005) makes a similar point in his comparison between the content of questions in the Junior Certificate examination and the syllabus aims and objectives, and notes that only four of the ten aims outlined in the syllabus are actually assessed in the Junior Certificate examination.

> *The main goals of the Junior Certificate mathematics Syllabus would seem to focus more on mathematics needed for continuing education and*

> *less on the mathematics needed for life and work. The 10 sub-goals (objectives) listed in the syllabus have some similarities with objectives of the PISA framework but 6 of these sub-goals are not currently assessed in any formal way, including objectives relating to mathematics in unfamiliar contexts, creativity in mathematics, motor skills, communicating, appreciation, and history of mathematics (2005, p. 7).*

In summary despite the broad view of mathematical competence in PISA, there appears to be a narrow focus in terms of both style and content in Irish post-primary mathematics examinations.

Equity and excellence as policy and practice challenge

Balancing equity and excellence

Balancing the demands of excellence and equity is a perennial challenge for education systems, and it is often exacerbated by public and business demands for excellence rather than equity in high priority areas such as mathematics education. At a time when the traditional measures of human capital, such as years of education, have begun to shift towards more teaching/learning focused measures, such as those described by the reading, mathematical and scientific literacies of OECD-PISA (OECD, 2003), the relationship between excellence and equity in the distribution of human capital in these priority educational areas is an important educational and public policy issue. As we noted in chapter one, one of the reasons for international interest in Japan's post-primary mathematics education was not only its high aggregate scores but also the low variation between high and low-scoring students in the 1995 Third International Mathematics and Science Study (TIMSS). In a nutshell, Japanese post-primary mathematics education managed to achieve a greater degree of excellence and equity than other countries. In the context of PISA 2003, the Irish report (Cosgrove *et al*., 2005) noted that the distribution of mathematical literacy scores for Irish 15-year-old students indicated that, compared to most other countries, there was a relatively low number of students scoring at the high end and a high number of students scoring at the lower end. This phenomenon presents a policy challenge: to ensure both excellence and equity within the education system. As Cosgrove *et al*. note:

> *The relatively low performance of higher-achieving students in mathematics in Ireland is noteworthy and suggests that any forthcoming review of mathematics education at post-primary education should consider this finding, with a view to identifying ways in which performance of high achievers can be enriched (p. xxiii)*

We might add here that any discussion of high achievers ought also to consider the wider public and educational policy implications of efforts to reform mathematics education in terms of its role in the distribution of mathematical literacy-based human capital.

Three capitals and mathematics education: human, social and identity capitals

The benefits of learning for people's life-chances have become a central policy focus of governments, as evidenced by the commitment to and interest in OCED/PISA, now that learning is becoming the new labour in the 21st century. High achievement in high priority areas such as mathematics is likely to contribute to significant enhancement in both individuals' and society's *human* capital (the knowledge and skills possessed by individuals). That has been the basis for massive investments in education since the 1960s (Husen, *et al*, 1992). Over the last decade, two other types of capital, social (networks of power and privilege which allow people to contribute to common goals) and *identity* (personal resources to define themselves and have others define them in a changing world), have come to the forefront in understanding the wider benefits of learning (Schuller, Preston, Hammond, Brasset-Grundy and Bynner, 2004). Noting the intertwined nature of the three capitals, Schuller *et al*. (2004) make a case for the interpretation of learning experiences and outcomes in terms of all three types of capital.

Reviewing post-primary mathematics education provides an opportunity to consider the wider benefits of mathematics education, incorporating not only human capital but also social and identity forms of capital. Inclusion of these other capitals in considering the wider benefits of mathematics might help address questions of how the access to power and privilege is related to achievement or lack of achievement in mathematics (social capital), and the role of mathematics in defining a sense of identity and personal capacity, given the manner in which numeracy is often popularly misused as a proxy measure of intelligence (identity capital), which in turn constrains or affords certain types of self-definition/classification and definition/ classification by others.

The teacher education challenge

> *…Consistent and increasing pressure on teachers, school leaders, administrators, policy-makers and researchers to construct new understandings, insights and practices to bring about transformations in the schools as organizations while simultaneously inventing more appropriate, efficient and effective approaches to teaching and learning consonant with individual needs, national aspirations and economic competitiveness . .[puts] increasing pressure on teachers to be accountable not only for the attainment and achievement of their students but also for the ways in*

which they teach . . the central message internationally is . . that business as usual for schools and teachers is no longer an adequate response to the rapidly changing landscape.

(Sugrue and Day, 2002, p. xv).

Sugrue and Day highlight the new context of teachers' work in an era of demands for higher standards and increased educational accountability. Calls for new teaching and learning approaches put pressure on teacher education at all levels in terms of how it can provide the type of experiences that will make it possible (Conway and Clark, 2003). Despite the calls for teachers to teach in new ways, initial and induction phases of teacher education and ongoing professional development opportunities often fail to live up to what is now a substantial knowledge-base concerning high-quality professional development. As Hiebert *et al*. (2002) note, 'There is a growing consensus that professional development yields the best results when it is long-term, school-based, collaborative, focused on students' learning, and linked to curricula' (p. 3). The high-quality professional development we have outlined in looking at lesson study (chapter two) and coaching of beginning, mid-career and veteran mathematics teachers (chapter four), are good examples of the type of continuing professional development necessary to begin any process of mathematics education reform. Hiebert *et al*. identify five distinguishing characteristics of quality professional development:

- elaborating the problem and developing a shared language for describing the problem
- analysing classroom practice in light of the problem
- envisioning alternatives, or hypothesising solutions to the problem

- testing alternatives in the classroom, and reflecting on their effects
- recording what is learned in a way that is shareable with other practitioners.

The criteria for high-quality teacher learning outlined by Hiebert *et al*. are consistent with a situated cognition perspective on teacher learning. Over the last fifty years, each of the three influential theories on learning - that is, the behaviourist, cognitive and sociocultural theories - have influenced conceptions of good teaching, visions of professional development and the evaluation of teachers and teaching (Conway and Artiles, 2005). As Putnam and Borko (2000) note, an influential and developing body of knowledge on cognition and learning suggests that cognition is situated, social and distributed. Thus, in terms of teacher development, teacher thinking cannot be isolated from the context of teaching, that is, the classroom and the school. Just as the situated cognition perspective reframes student learning in significant ways, so it does in the case of teacher professional development. A situated cognition view of teacher learning suggests the need to

- ground staff-development in teachers' learning experiences in their own practice by conducting it on-site at schools and in the classroom
- encourage teachers to bring experiences from their own classroom to staff development workshops that are extended over a number of weeks or months
- incorporate multiple contexts for teacher learning (both site-based, drawing on teachers' own practice, and those involving the perspectives of 'outsiders' such as in-service providers, inspectors, university lecturers, etc.).

In contrast to the high quality professional development outlined by Hiebert *et al* and supported by insights from situated cognition perspective on teacher learning, teacher professional development typically falls far short of these standards.

As Fullan has noted:

> *Professional development for teachers has a poor track record because it lacks a theoretical base and coherent focus. On the one hand, professional development is treated as a vague panacea - the teacher as continuous, lifelong learner. Stated as such, it has little practical meaning. On the other hand, professional development is defined too narrowly and becomes artificially detached from 'real-time' learning. It becomes the workshop, or possibly the ongoing series of professional development sessions. In either case, it fails to have a sustained cumulative impact. (Fullan, 1995, cited in Guskey and Huberman, 1995, p. 253)*

In light of Huberman's observations we draw on two studies to highlight key issues in relation to teacher professional development in the Irish context ([i] Sugrue, Morgan, Devine, and Raftery, 2001; Sugrue, 2002; and [ii] Delaney, 2005). Firstly Sugrue, Morgran, Devine and Raftery's (2001) Department of Education and Science-commissioned survey and interview study of primary and post-primary teachers about their professional learning provides a good overview of the state of teacher professional development. We highlight number of their main findings here. Key claims of relevance to this report are as follows:

- Teachers were generally positive about their continuing professional development (CPD) experiences.

- Teachers said they felt their CPD was making some impact on classroom practice. Sugrue (2002), in subsequent article drawing on the same data, notes that in the absence of evidence linking teachers' professional learning opportunities with their classroom practices and ultimately student learning, many questions about the impact of CPD remain unanswered.

- Teachers felt that 'professional learning provision has been more successful in communicating cognitive knowledge than impacting positively on competencies and skills' (p. 334).

- The approach to provision of CPD tended to favour 'talking at' teachers rather than more interactive approach where teachers have a chance to explore ideas with others. Sugrue (2002) notes that even when CPD was more interactive, there was little support for teachers once they returned to their schools in terms of addressing issues raised at professional learning sessions.

Thus, this first study portrays a professional learning landscape that is at one level satisfactory (teachers are generally positive about it), but at another appears to leave teachers poorly served in terms of the kind of professional learning needed to initiate and sustain significant changes in teachers' classroom practices.

Delaney (2005), in a study reviewing professional development in mathematics for primary teachers, noted that, based on the 1999 National Assessment of Mathematics Achievement, only just over a quarter of teachers (29%) had attended in-service training in mathematics, and of these just less than half (47%) were either dissatisfied or very dissatisfied with the courses they attended. Furthermore, he notes that the one-day and one-week professional development opportunities are unlikely to provide the time needed to develop a discourse with which to share and examine mathematics teaching practices in a collegial setting. In summary, any radical reform of mathematics education in Ireland presents a challenge to create the continuing professional development CPD structures and opportunities which are needed to support it.

Scaling up:[22] the change challenge

> *'…An ecological model of education goes far beyond schools in seeking to embrace people, things and institutions in a systemic, interrelated whole.'*
>
> **(Goodlad, 1975)**

Given the acceleration in social, economic and cultural change around the world, education systems have been asked to (and asked themselves to) meet the change challenge as it relates to almost every aspect of the educational environment within their control. This has placed both a heavy burden and an exciting range of possibilities at the feet of educators and all those involved in education. Mathematics education has been made a policy priority, and presents an especially urgent challenge in curriculum and assessment. Educational change researchers have not been very optimistic about the capacity of educational systems to change, noting the powerful forces for stability often very useful in preserving what is valuable in a culture and education system. The title of Sarason's classic text *The Predictable Failure of Educational Reform*, based on analyses of change initiatives in the USA over several decades, captures the insights of those sceptical of revolutionary change agendas in education. In Ireland today, one important source of support for change is the individual school's own capacity to change as a result of the rolling

22 See Coburn (2003) for an analysis of how scale has been conceptualised in the educational change literature. She notes that 'definitions of scale have traditionally restricted its scope, focusing on the expanding number of schools reached by a reform' (p. 3). Coburn argues for multidimensional conception of scale that addresses its interrelated dimensions: depth, sustainability, spread, and shift in ownership.

review of policies and practices brought into motion by School Development Planning (SDP) and Whole School Evaluation (WSE) processes.

However, reforming mathematics education will need to go beyond school level capacity for change, and adopt a system-level approach given the scale of the task involved. The National Council for Curriculum and Assessment proposals for senior cycle sets out possible reform framework for upper post-primary education (NCCA, 2003; NCCA 2005b). These wider reforms outlined by the NCCA set an essential context for any proposals to reform post-primary mathematics education. Developing a different school culture, reforming students' experiences of teaching and learning, re-balancing curriculum at senior cycle, and developing different assessment arrangements and a new certification system for senior cycle, were the directions for development specified in the NCCA document. These documents outlined a series of support strategies that would be necessary to achieve these ambitious goals, encompassing investment for change, professional development for teachers and support for schools, provision of information to and engagement with stakeholders and parents, and monitoring, research and evaluation. What might be the best place to start in terms of reform? Assuming that some significant reform agenda is adopted, the post-primary mathematics education in Victoria, Australia reform strategy[23] provides some evidence that reforming the assessment system can produce a significant ripple effect right through secondary education. It is examples such as the VCE reforms and the insights on the mismatch between mathematics textbooks and mathematics education reform goals that identify potential system-level levers for change.

[23] See VCAA, 2004 and 2005 for relevant resources.

5.4 Conclusion

Mathematics education has become the focus of considerable attention in Ireland and elsewhere since mathematical competence is seen as playing a critical role in the development of a knowledge society, as well as an essential skill for productive and reflective citizenship. Recent research on post-primary mathematics education in Ireland has begun to question the dominance of the 'new' mathematics education movement and its impact on approaches to teaching and examinations. In this report we have not attempted to outline a strategy for reforming mathematics education but have focused on key aspects that might inform such a strategy, such as the role of curriculum culture, textbooks and examinations. From our review of the trends, it is clear that there is no one template for reforming post-primary mathematics education. There are, however, trends, of which the move towards a more 'real-life' focus in mathematics curriculum and assessment is the most distinctive and significant shift in mathematics education in many countries.

REFERENCES

Abreu, G. de (2002). 'Mathematics learning in out-of-school contexts: a cultural psychology perspective.' In L. D. English (Ed.), *Handbook of International Research in Mathematics Education: Directions for the 21st Century* (pp. 323- 353). Lawrence Erlbaum.

Adams, R. J. (2003). Response to 'Cautions on OECD's recent educational survey (PISA)', Oxford Review of Education, 29, 3, 377 - 389

Alagic M. (2002). What Change is Technology Bringing to Conceptual Understandings of Mathematical Ideas? Paper presented at the IMECT3 - Third International Conference on Mathematics Enrichment with Communication Technology at Robinson College, University of Cambridge, Cambridge England.

Allal, L. (2005). 'Regulation of learning: Processes and contexts.' Keynote address at the European Association for Research on Learning and Instruction (EARLI) conference, University of Cyprus.

Allal, L. and Saada-Robert, M. (1992). La métacognition: Cadre conceptuel pour l'étude des régulations en situation scolaire. Archives de psychologie, 60, 265-296.

Allal, L., and Pelgrims Ducrey, G. (2000). Assessment of–or in–the zone of proximal development. *Learning and Instruction*, 10, 137-152.

Allal, L. and Mottier Lopez, L. (2005). Formative assessment of learning: A review of publications in French. In Formative Assessment - Improving Learning in Secondary Classrooms (pp. 241-264). Paris: OECD Publication.

Allal, L., Mottier Lopez, L., Lehraus, K. and Forget, A. (2005). Whole-class and peer interaction in an activity of writing and revision. In T. Kostouli (Ed.), Writing in context(s): Textual practices and learning processes in sociocultural settings (pp. 69-91). New York: Springer.

Almeqdadi F. (2005) 'The Effect of Using The Geometer's Sketchpad (GSP) on Jordanian Students' Understanding Some Geometrical Concepts'. *International Journal for Mathematics Teaching and Learning* [Online]. Available: http://www.ex.ac.uk/cimt/ijmtl/almeqdadi.pdf

Anderson, L.W. (2003). *Classroom Assessment: Enhancing the Quality of Teacher Decision-Making*. Mahwah, NJ: Lawrence Erlbaum Associates.

Andrews, P. (2001a) Comparing international practice in the teaching of mathematics, in P. Gates (Ed.) Issues in the teaching of mathematics, London: Routledge Falmer.

Andrews, P. (2001b) Mathematics education and comparative studies: what can we learn from abroad? *The Scottish Mathematics Council Journal* 30, 56-59.

APEC (2004). *3rd APEC Education Ministerial meeting: Skills for the coming challenges.* April 29-30, 2004, Santiago, Chile. Santiago: Ministry of Education.

Ashton H.S., Schofield D.K. and Woodger S.C. (2003). 'Piloting Summative Web Assessment in Secondary Education'. In *Proceedings of 7th International CAA Conference, Loughborough University 2003*. Available online at: http://sd.lboro.ac.uk/caaconference/conf2003/pdfs/ashton.pdf

Ball, D. L. (1993). 'With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics.' *Elementary School Journal,* 93, 4, 373-397.

Ball, D.L., Lubienski, S.T., and Mewborn, D. S. (2001). 'Mathematics.' In V. Richardson (Ed). *Handbook of Research on Teaching (4th edition)*. New York: Macmillan.

Ball, L. and Stacey, K. (2005). Teaching strategies for developing judicious technology use. In W. J. Masalski and P. C. Elliott (Eds.), *Technology-Supported Mathematics Learning Environments.* 2005 National Council of Teachers of Mathematics Yearbook, 2005 (pp. 3-15). Reston, VA: NCTM.

Barnes, M., Clarke, D.J. and Stephens, W.M. (2000). 'Assessment as the engine of systemic reform.' *Journal of Curriculum Studies* 32, 5, 623-650.

Battey, D., Franke, M., and Priselac, J. (2004). 'Scaling up professional development: Using cases to initiate change in the core of teaching mathematics.' Paper presented at the annual meeting of the American Educational Research Association: San Diego, CA.

Baumert, J., Klieme, E., Neubrand, M., Prenzel, M, Schiefele, U., Schneider, W. Tillmann, K. J. and Weib, M (1999). 'Self-regulated learning as cross-curricular competence.'
Available online at: www.mpib-berlin.mpg.de/en/Pisa/pdfs/CCCengl.pdf retrieved June 1st 2005.

Beevers C.E., Fiddes D.J., McGuire G.R. and Youngson M.A. (1999). 'The emerging philosophy behind computer-based assessment.' *Teaching Mathematics and its Applications* 18, 4, 147-149.

Belfiore, P. J., and Hornyak, R. S. (1998). 'Operant theory and application to selfmonitoring in adolescents.' In D. H. Schunk and B. J. Zimmerman (eds.), *Self-Regulated Learning: From reflection to self-reflective practice* (pp.184-202). New York: Guilford Press.

Benbow, C. P., and Stanley, J. C. (1980). 'Sex differences in mathematical ability: fact or artifact?' *Science* 210, 1262-1264.

Bennett, J., Lubben, F., Hogarth, S. (2003). A systematic review of the effects of context-based and Science Technology-Society (STS) approaches to the teaching of secondary science. In Research Evidence in Education Library. London: EPPI-Centre, Social Science Research Unit, Institute of Education, University of London. Available online at: http://eppi.ioe.ac.uk/reel/

Black, P., and Wiliam, D., (1998a). 'Assessment and classroom learning.' *Assessment in Education* 5, 1, 7-74.

Black, P., and Wiliam, D. (1998b). 'Inside the black box: Raising standards through classroom assessment.' *Phi Delta Kappan* 80, 2, 139-148. Available online at: http://www.pdkintl.org/kappan/kbla9810.htm

Boekaerts, M. (1999). 'Motivated learning: studying students situation transactional units.' In M. Boekaerts and P. Nenniger (Eds.), Advances in motivation from the European viewpoint (Special Issue). *European Journal of Psychology of Education* 14, 1, 41-55.

Bower, M. (2005). 'Online Assessment Feedback: Competitive, Individualistic, or…Preferred Form!' *Journal of Computers in Mathematics and Science Teaching* 24 (2), 121-147. Available online at: http://dl.aace.org/16980

Bowers, J. S., Cobb, P., and McClain, K. (1999). 'The evolution of mathematical practices: a case study.' *Cognition and Instruction*, 17, 25 - 64.

Bramall, S., and White, J. (2000). (Eds.), *Why learn maths?* London: Institute of Education (Bedford Way Papers).

Bransford, J. D., Brown, A., and Cocking, R. (Eds.), (2000). *How people learn: Mind, brain, experience and school.* Washington DC: National Academy Press.

Brew, C., Tobias, S., and Leigh-Lancaster, D. (2001). 'VCE mathematics revised 2000: the future impact on teaching and learning mathematics at the junior secondary level' In J. Bobis, B. Perry and M. Mitchelmore, (Eds.), *Proceedings of the Mathematics Educational Research Group of Australasia* (MERGA) 24, 98-105.

Broadfoot, P. (2001). 'Liberating the learner through assessment,' In J. Collins and D. Cook, (Eds.), *Understanding learning: Influences and outcomes* Milton Keynes: Open University Press.

Brown, A. L. (1994) 'The advancement of learning' *Educational Researcher* 23, 8, 4-12.

Brown, A. L. (1997a) 'The advancement of learning' In H. J. Walberg, and G. D. Haertel, (Eds.), *Psychology and Educational Practice* San Pablo, CA: McCutchan Publishing, pp. 52-76.

Brown, A. L. (1997b) 'Transforming schools into communities of thinking and learning about Serious Matters' *American Psychologist* 52, 4, 399-413.

Brown R. (2002) 'Mathematical modelling in the international baccalaureate, teacher beliefs and technology usage' *Teaching Mathematics and its Applications* 21, 2, 67-74.

Brown, M. (1999). Problems of interpreting international comparative data. In B. Jaworski, B and D. Phillips, D. (Eds.) Comparing standards internationally: research and practice in mathematics and beyond. Oxford: Symposium Books.

Brown, P., Green, A., and Lauder, H. (2001). *High skills: globalization, competitiveness and skill formation.* London: Cambridge University Press.

Brown, J. S., Collins, A., and Duguid, P. (1989). 'Situated cognition and the culture of learning.' *Educational Researcher* 18, 1, 32-42.

Brown, A. L., Ash, D., Rutherford, M., Nakagawa, K., Gordon, A., and Campione, J. (1993) 'Distributed expertise in the classroom' In G. Salomon (ed.) *Distributed Cognitions: Psychological and educational considerations* (pp.188-228). New York: Cambridge University Press.

Bruer, J. T. (1997). 'Education and the brain: A bridge too far?' *Educational Researcher.* 26, 8, 4-16.

Bruner, J. (1996). *The Culture of Education.* Cambridge: Harvard University Press.

Burton, L. (1994). *Who Counts?:Assessing Mathematics in Europe* Stoke-on-Trent: Trentham Books.

Butler, B. (2005). 'Poor maths results just don't add up.' *Irish Times*, (p. 14), 12th August.

Butler D. (2005). 'The school mathematics curriculum in a technological age' In JohnstonWilder S., and Pimm D. (Eds.), *Teaching Secondary Mathematics with ICT.* Maidenhead: Open University Press, pp. 123-141.

Callan, J. (1997). 'Active learning in the classroom: a challenge to existing educational values and practices. *Issues in Education.* Dublin: ASTI.

Case, R. (1996). 'Changing views of knowledge and their impact on educational research and practice.' In Olson, D. R., and Torrance, N., (Eds.), *The Handbook of Education and Human Development*. Oxford: Blackwell.

Carpenter, T.P., Fennema, E., Franke, M., Levi, L., and Empson, S. (1999). *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Carpenter, T., Franke, M., and Levi, L. (2003). *Thinking Mathematically*. Portsmouth, NH: Heinemann.

Carpenter, T. P., and Fennema, E. (1988). 'Research and cognitively guided instruction.' In E. Fennema, T. P. Carpenter, and S. J. Lamon (Eds.), *Integrating research on teaching and learning in mathematics* (pp.2-19). Madison, WI: University of Wisconsin, Wisconsin Center for Education Research.

Carpenter, T. P., Fennema, E., Petersen, P. L., Chiang, C., and Loef, M. (1989). 'Using children's mathematics thinking in classroom teaching: An experimental study.' *American Educational Research Journal* 26, 4, 499-531.

Case, R. (1996). 'Changing views of knowledge and their impact on educational research and practice.' In D. R. Olson, and N. Torrance, N., (Eds.), *The Handbook of Education and Human Development*. Oxford: Blackwell.

Chi, M. T. H., Glaser, R., and Farr, M. J. (Eds.), (1988). *The nature of expertise*. Hillsdale, NJ: Erlbaum.

Clandinin D. J., and Connolly, F. M. (1996). 'Teachers' professional knowledge landscapes.' *Educational Researcher* 25, 3,24-30.

Clark-Jeavons A. (2005). 'Interactive whiteboards: developing a pedagogy for mathematics classrooms.' In Johnston-Wilder S. and Pimm D. (Eds.), (pp. 159-173), *Teaching Secondary Mathematics with ICT* Maidenhead: Open University Press.

Clark, C. M., and Peterson, P. L. (1986). 'Teachers' thought processes.' In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.). (pp. 255-296). New York: Macmillan.

Clarke, D. (2002). 'The Learner's Perspective Study: Exploiting the Potential for Complementary Analyses.' Paper presented as part of the symposium 'Primary Research, Secondary Research and Research Synthesis: Theory, Value and Linkage' at the Annual Conference of the American Educational Research Association, New Orleans, April 2-5, 2002.

Clarke, D.J., and Stephens, M. (1996). 'The ripple effect: The instructional impact of the systemic introduction of performance assessment in mathematics.' In M. Birenbaum, and F.J.R.C. Dochy (Eds.). *Alternatives in assessment of achievements, learning processes and prior knowledge* (pp. 63-92). Boston, MA: Kluwer.

Close, S., Dooley, T., and Corcoran, D. (Eds.), (2005). Proceedings of the First National Conference on *Research in Mathematics Education*. Dublin: St. Patrick's College.

Close, S., and Oldham, E. (2005). 'Junior Cycle Second level mathematics examinations and the PISA mathematics framework.' In Close, S., Dooley, T. and Corcoran, D. (Eds.), *Proceedings of the First National Conference on Research in Mathematics Education* (pp. 174-192). Dublin: St. Patrick's College.

Cobb, P., McClain, K., and Gravemeijer, K. (2003). 'Learning about statistical covariation.' *Cognition and Instruction* 21, 1-78.

Cobb, P., and Yackel, E. (1996). 'Constructivist, emergent and sociocultural perspectives in the context of developmental research.' *Educational Psychologist* 31, 175-190.

Cobb, P., Gravemeijer, K., Yackel, E., McCalin, K., and Whitenack, J. (1997). In D. Kirshner and J. A. Whitson, *Situated Cognition: Social, semiotic and psychological perspectives*. Mahwah, NJ: Lawrence Erlbaum Associates.

Coburn, C. E. (2003). 'Rethinking scale: Moving beyond numbers to deep and lasting change.' *Educational Researcher* 32, 6, 3-12.

Cockcroft W. H. (1982). '*Mathematics Counts*.' London: HMSO.

Cohen, D. K., and Spillane, J. P. (1992). 'Policy and practice: The relations between governance and instruction.' *Review of Research in Education* 18, 3-49.

Conway, P. F. (2002). 'Learning in communities of practice: Rethinking teaching and learning in disadvantaged contexts.' *Irish Educational Studies* 21, 3, 61-92.

Conway, P. F. (2005a). 'Trends in secondary curriculum reform in Europe: Cubs and tigers swapping tricks.' Keynote address at US-China Centre for Educational Excellence conference on Secondary School Curriculum Reform, East Lansing, MI, USA.

Conway, P. F. (2005b). 'The 1999 Primary School Curriculum and your child as a learner.' In E. Gaire and O. Mahon (Eds.) *Primary Options: What Every Parent Needs to Know about Primary Schools in Ireland*. Dublin: Liffey Press.

Conway, P. F. (2005c). *Laptops Initiative for Students with Dyslexia or other reading and writing difficulties: Early Implementation Evaluation Report.* Dublin: National Centre for Technology Education (NCTE). Available on-line at: http://www.ncte.ie/SpecialNeedsICT/Projects/LaptopsInitiative PublicArea/

Conway, P. F. (2005d). Book Review of L. W. Anderson's (2003) Classroom Assessment: Enhancing the Quality of Teacher Decision-Making, Teaching and Teaching: Theory into Practice, 11, 1,

Conway, P. F. and Artiles, A. A. (2005). 'A socio-cultural perspective on teacher performance assessment.' In F. P. Peterman (Ed.) *Designing Performance Assessment Systems for Urban Teacher Preparation*. Mohawk, NJ: Lawrence Erlbaum.

Conway, P. F., and Clark, C. M. (2003) The journey inward and outward: A re-examination of Fuller's concern-based model of teacher development. *Teaching and Teacher Education*,19, 5, 465-482.

Coolahan, J. (2005). 'The Primary School Curriculum.' Paper presented at joint National Council for Curriculum and Assessment and DES symposium on the revised Primary School Curriculum. Available online at: www.ncca.ie

Cooper, J., and Dunne, M. (2000). 'Constructing the legitimate goal of a "realistic" maths item: a comparison of 10-11 and 13-14 year olds.' In A. Filer, *Assessment: Social Practice and Social product*. London: RoutledgeFalmer.

Cooper, J., and Harries, T. (2002). 'Children's responses to contrasting realistic problems: Just how realistic are children ready to be?' *Educational Studies in Mathematics* 29, 1-23.

Cosgrove, J., Shiel, G., Sofroniou, N., Zastrutzki, S. and Shortt, F. (2005). *Education for Life: The achievements of 15-year-olds in Ireland in the Second Cycle of PISA*. Dublin: Educational Research Centre.

Danish Ministry of Education (1999). *The Danish Gymnasium General Rules and the Subjects - Supplement 23 Mathematics.* Available online at: http://us.uvm.dk/gymnasie/almen/lov/bek/supplement23.html

De Corte, E., Greer, B., and Verschaffel, L. (1996). 'Mathematics teaching and learning.' In Berliner, D. C. and Calfee, R. C. (Eds.) *Handbook of Educational Psychology* New York: Macmillan, pp.15-47.

De Corte, E., Verschaffel, L., Entwistle, N., and van MerriÎnboer, J. (Eds.), (2003). *Powerful Learning Environments: Unravelling Basic Components and Dimensions*. Oxford: Pergamon.

Delaney, S. (2005). 'Mathematics professional development for primary teachers: how have we got to here? Where can we go? Where should we go?' In Close, S., Dooley, T., and Corcoran, D. (Eds.) *Proceedings of the First National Conference on Research in Mathematics Education* (pp. 235-249). Dublin: St. Patrick's College.

Department of Education (1971). *Curaclam na Bunscoile: Primary School Curriculum, Volumes 1 and 2*. Dublin: Government Publications.

Devlin, K. (1997). *Mathematics: The science of patterns*. New York: W. H. Freeman and Company.

Dewey, J. (1933/1993). *How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process*. Boston: D.C. Heath, and Co.

Dewey, J. (1938). *Experience and Education*. New York: Macmillan Company.

Dixon J. (1997). 'Computer Use and visualization in Students' Construction of Reflection and Rotation Concepts.' *School Science and Mathematics* 97, 7, 352-358.

Dolly, J.P. (1992). 'Juku and the performance of Japanese students: An American perspective.' Paper presented at the Annual Meeting of the Japan-United States teacher education consortium, Tokyo, June 1992.

Druker, S. and Roth, K. J. (2004). 'Implications of the TIMSS 1999 Science Video Study Analysis Tools for Science Education Research and Professional Development.' Paper presented at the American Educational Research Association Conference, San Diego.

Dugdale S. (2001). 'Order out of Chaos: A Spreadsheet Excursion Into a Mathematical Frontier.' *Journal of Computers in Mathematics and Science Teaching*, 20, 3, 323-341. Available online at:
http://dl.aace.org/16980 http://dl.aace.org/6488

Elwood, J. (2005). Gender and achievement: what have exams got to do with it? *Oxford Review of Education*, 31, 3, 373–393.

Elwood, J., and Carlisle, K. (2003). 'Examining gender.' *Gender and achievement in the Junior and Leaving Certificate Examinations 2000/2001.* Dublin: National Council for Curriculum and Assessment.

English, L. D. (Ed.) (2002). *Handbook of International Research in Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum.

English, L. D., and Halford, G. S. (1995). *Mathematics education: Models and processes*. Hillsdale, NJ: Lawrence Erlbaum.

Engeström, Y., Miettinen, R., and Punamäki, R. L. (Eds.), (1999). *Perspectives on activity theory. Learning in Doing: Social, Cognitive, and Computational Perspectives*. New York, NY: Cambridge University Press.

Ernest, P. (2000). 'Why teach mathematics?' In S. Bramall and J White, *Why learn mathematics?* London: Institute of Education (Bedford Papers Series).

Ernest, P. (2004). 'Images of mathematics, values and gender: a philosophical perspective.' In B. Allen and S. Johnston-Wilder (Eds.). *Mathematics education: exploring the culture of learning*. London: Routledge.

Fennema, E., Carpenter, T. P., and Petersen, P. L. (1989). 'Learning mathematics with understanding: Cognitively guided instruction.' In J. Brophy (Ed.) *Advances in research on teaching* (pp.195-221). Greenwich, CT: JAI Press.

Fitzpatrick, S., and Conway, P. F. (2005). 'On-line learning and differential participation in a democratic society: Ireland as a case study.' In A. Carr-Chellman, (Ed.), *Global perspectives on E-Learning: Rhetoric and Reality*. Thousand Oaks, CA: Sage Publications.

Franke, M. L., and Kazemi, E. (2001). 'Learning to teach mathematics: Focus on student thinking.' *Theory Into Practice* 40, 2, 102-109.

Franke, M. L., and Kazemi, E. (2001). 'Teaching as learning within a community of practice.' In T. Wood, B. Nelson, and J. Warfield (Eds.), *Beyond classical pedagogy in elementary mathematics: The nature of facilitative teaching.*

Franke, M. L., Kazemi, E., and Battey, D. (in preparation). 'Exploring classroom practices.' *Handbook of Research on Mathematics Teaching and Learning*. New York: National Council of Teachers in Mathematics, Macmillan.

Franke, M. L., Carpenter, T. P., and Battey, D. (in press). 'Content matters: The case of algebraic reasoning in teacher professional development.' In J. Kaput, D. Carraher, and M. Blanton, (Eds.), *Algebra in the Early Grades*. Hillside, NJ: Lawrence Erlbaum.

Franke, M. L., Carpenter, T.P., Levi, L., Fennema, E. (2001). 'Capturing teachers' generative growth: A follow-up study of professional development in mathematics.' *American Educational Research Journal* 38, 653-689.

Franke, M., Kazemi, E., Shih, J., Biagetti, S., and Battey, D. (2005). 'Changing teachers' professional work in mathematics: One school's journey.' In T. Romberg, T. Carpenter, and Dremock (Eds.), *Understanding Mathematics and Science Matters*. Hillside, NJ: Lawrence Erlbaum.

Freudenthal, H. (1968). 'Why to teach mathematics so as to be useful.' *Educational Studies in Mathematics* 1, 3-8.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1978). *Weeding and Sowing: Preface to a Science of Mathematical Education*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education: China lectures*. Dordrecht: Kluwer.

Fullan, M. G. (1995). 'The Limits and the Potential of Professional Development.' In T.R. Guskey and M. Huberman (Eds.) *Professional Development in Education*, (pp. 258-267). New York: Teachers College Press.

Gardner, H. (1985). *The Mind's New Science*. New York: Basic Books.

Gardner, H. (1991). *The unschooled mind: How children think and how schools should teach*. New York: Basic Books.

Gardner, H. (2004). 'How education changes: Considerations of history, science and values.' In M. Suarez-Orozco and D. Qin-Hilliard (Eds.), *Globalization: Culture and education in the new millennium*. Berkeley: University of California Press.

Gash, H. (1993). 'The status of Piaget today in mathematics education.' *Compass* 22, 65-75.

Gee, J. (1996). *Social linguistics and literacies: Ideology in discourses.* (Second Edition) London: Taylor and Francis.

Gee, J. (1998). *Preamble to a literacy program*. Madison, WI: Department of Curriculum and Instruction.

Glaser, R. and Silver, E. (1994). 'Assessment, testing and instruction. Retrospect and prospect.' In L. Darling-Hammond (Ed.), *Review of research in education* (pp. 393-419). Washington, DC: American Educational Research Association.

Glover D., Miller D., and Averis D. (2003). 'The impact of interactive whiteboards on classroom practice: examples drawn from the teaching of mathematics in secondary schools in England.' In *The Mathematics Education into the 21st Century Project*. Proceedings of the International Conference The Decidable and the Undecidable in Mathematics Education, Brno, Czech Republic, September 2003, Available online at: http://math.unipa.it/~grim/21_project/21_brno03_Miller-Averis.pdf

Goffree, F. (1993). 'Hans Freudenthal: Working on mathematics education.' *Educational Studies in Mathematics* 25, 21-50.

Goodlad, J. I. (1975). *The dynamics of educational change: Toward responsive schools*. New York: McGraw-Hill.

Goodson, I. (2002). 'Teachers' responses to new educational reforms: International perspectives.' Keynote address at the joint Educational Studies Association of Ireland (ESAI) and National Council for Curriculum and Assessment conference on Curriculum Contestation, Dublin Castle, January 31st and February 1st.

Gopnik, A., Meltzoff, A. N., and Kuhl, P. K. (1999). *The Scientist in the crib: What early learning tells us about the mind*. New York: Harper Collins.

Gravemeijer, K., and Terwel, J. (2000). 'Hans Freudenthal: a mathematician on didactics and curriculum theory.' *Journal Curriculum Studies* 32, 6, 777-796.

Greeno, J.G., Pearson, P.D., and Schoenfeld, A.H. (1997). 'Implications for NAEP of research on learning and cognition.' *Assessment in Transition: Monitoring the Nation's Educational Progress*. Washington: National Academy Press.

Greeno, J. G., Collins, A. M., and Resnick, L. B. (1996) 'Cognition and Learning.' In Berliner, D. C. and Calfee, R. C. (Eds.), *Handbook of Educational Psychology*. pp.15-47. New York: Macmillan.

Greeno, J. and Goldman, S.G. (Eds.), '*Thinking practices in mathematics and science learning*.' Mahwah, NJ: Lawrence Erlbaum and Associates.

Grouws, D. A. (1992). (Ed.) *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company.

Guskey, T. and Huberman, M. (1995). 'Introduction.' In T. Guskey and M. Huberman, (Eds.), *Professional development in education: New paradigm and practices* (pp. 1-6). New York: Teachers College, Columbia University.

Hiebert, J., Gallimore, R., Garnier, H., Bogard Givven, K., Hollingsworth, H., Jacobs, J., Miu-Ying Chui, A., Wearne, D., Smith, M. Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., and Stigler, J. (2003). *Teaching mathematics in seven countries: results from the TIMSS 1999 Video Study.* Washington D.C.: U.S. Department of Education, Institute of Education Sciences. National Center for Education Statistics.

Hoban, G. (2002). '*Teacher Learning for Educational Change*.' Buckingham, Philadelphia: Open University Press.

Hofer, B. K. (2004). 'Introduction: paradigmatic approaches to personal epistemolog.' *Educational Psychologist* 39, 1, 1-4.

Huberman, M. (1995). 'Networks that alter teaching: Conceptualisations, exchanges, and experiments.' *Teachers and Teaching: Theory and Practice* 1, 2, 193-211.

Husen, T., Tuijnman, A., and Halls, W. (1992). *Schooling in Modern European Society*. London: Pergamon.

Hutchins, E. (1995b). *Cognition in the wild*. Cambridge: MIT Press.

Hutchins, E. (1995). 'How a cockpit remembers its speeds.' *Cognitive Science* 19, 265-288.

Hwang H.J. (2004). 'A comparative analysis of mathematics curricula in Korea and England focusing on the content of the algebra domain.' *International Journal for Mathematics Teaching and Learning*. Available online at: http://www.ex.ac.uk/cimt/ijmtl/hwang.pdf

Jacobs, J. E., Becker, J. R., and Gilmer, G. F. (Eds.), (2001). *Changing the faces of mathematics: Perspectives on gender*. Reston, VA: The National Council of Teachers of Mathematics.

Jacobs, J., Garnier, H., Gallimore, R., Hollingsworth, H., Givvin, K.B., Rust, K., Kawanaka, T., Smith, M., Wearne, D., Manaster, A., Etterbeek, W., Hiebert, J., and Stigler (in press). *TIMSS 1999 Video Study Technical Report: Volume I: Mathematics Study*. U.S. Department of Education. Washington, D.C.: National Center for Educational Statistics.

Jacobs, V., Carpenter, T., Franke, M., Levi, L., and Battey, D. (under review). 'Algebraic thinking in elementary school'. Submitted to Journal for Research in Mathematics Education.

Järvelä, S. (1995). 'The cognitive apprenticeship model in a technologically rich learning environment: Interpreting the learning interaction.' *Learning and Instruction* 5, 3, 237-59.

Jaworski, B., and Phillips, D. (1999). *Comparing standards internationally: research and practice in mathematics and beyond*. Oxford: Symposium Books.

Jaworski, B., and Phillips, D. (1999). Looking abroad: international comparisons and the teaching of mathematics in Britain. In B. Jaworski, B and D. Phillips, (Eds.) Comparing standards internationally: research and practice in mathematics and beyond. Oxford: Symposium Books.

Jencks, C., and Phillips, M. (1998). 'The black-white test score gap: An introduction.' In C. Jencks and M. Phillips (Eds.), *The black-white test score gap* (pp. 1-51). Washington, DC: Brookings Institution.

Joyce, B., and Showers. B. (1995). 'Student Achievement through Staff Development.' *Fundamentals of School Renewal (2nd edition).* White Plains, NY: Longman.

Kelly, A. E., and Sloane, F. C. (2003). 'Educational research and the problems of practice.' *Irish Educational Studies*[24] 22, 1, 29-40.

Kelman P., Bardige A., Choate J., Hanify G., Richards J., Roberts N., Walters J., and Tornrose M.K. (1983). *Computers in Teaching Mathematics*. Reading: Addison-Wesley.

Koedinger, K. R., and Nathan, M. J. (2004). 'The real story behind story problems: Effects of representations on quantitative reasoning.' *Journal of the Learning Sciences* 13, 2, 129-164.

Kolata, G. B. (1981). 'Math and sex: Are girls born with less ability?' *Science* 2, 11.

Kulm, G. (Ed.) (1990). *Assessing higher order thinking in mathematics*. Washington, DC: American Association for the Advancement of Science.

Lave, J. (1988). *Cognition in practice*. Cambridge, UK: Cambridge University Press.

Lave, J., and Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. New York: Cambridge University Press.

24 Available online from 2005, 24, 1 with all previous volumes available online in 2006 i.e. Volumes 1-23. http://www.tandf.co.uk/journals/titles/03323315.asp

Le Métais, J. (2003). *International developments in upper secondary school education: Context, provision and issues*. Dublin: The National Council for Curriculum and Assessment.

Leder, G. (1992). 'Mathematics and gender: Changing Perspectives.' In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 597- 622). New York: Macmillan Publishing Company.

Lei, J., Conway, P.F., and Zhao, Y. (in press). *The digital Pencil: One-to-one computing for children*. Mahwah, NJ: Lawrence Erlbaum.

Leigh-Lancaster, D. (2002). The Victorian Curriculum and Assessment Authority (VCAA) Mathematical Methods (CAS) pilot study 2001 – 2005. *Australian Senior Mathematics Journal*, 16, 2, 20 – 28.

Leigh-Lancaster, D., and Stephens, M. (1997). Anticipating the impact of widely available access to Computer Algebra Systems on Teaching and Assessment in Mathematics: the next millennium. In K. Baldwin and J. Roberts (Ed.). Adelaide: Australian Association of Mathematics Teachers.

Leigh-Lancaster, D., Norton, P., and Evans, M. (2003). The Victorian Curriculum and Assessment Authority Mathematical Methods computer algebra pilot study and examinations. *The International Journal of Computer Algebra in Mathematics Education*, 2003, 10, 3.

Lerman, S. (2001). 'The 'social' turn in mathematics education research' In J. Boaler (Ed.) *Multiple perspectives on mathematics teaching and learning*. Connecticut: Ablex Publishing.

Lester, F. K., Lambdin, D.V., and Preston, R.V. (1997). 'A New Vision of the Nature and Purposes of Assessment in the Mathematics Classroom.' In G. D. Phye (Ed.), *Classroom Assessment. Learning, Achieving and Adjustment* (pp. 287 - 319). San Diego, CA: Academic Press.

Lew, H.C. (1999). 'New Goals and Directions for Mathematics Education in Korea.' In C. Hoyles. C. Morgan. and G. Woodhouse (Eds.) *Rethinking the Mathematics Curriculum* (pp. 218-227). London: Falmer Press.

Lewis, C. (2002). *Lesson Study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools, Inc.

Lesh, R., and Lamon, S. (1992). *Assessment of authentic performance in school mathematics*. Washington, DC: American Association for the Advancement of Science.

Lindsley, O. R. (1990) 'Precision Teaching: By teachers for children.' *Teaching Exceptional Children* 22, 3, 10-15.

Lindsley, O. R. (1991). 'Precision teaching's unique legacy from B.F. Skinner.' *Journal of Behavioral Education* 2, 253-266.

Linn, R. L. (2000) 'Assessments and Accountability.' *Educational Researcher* 29, 4-16.

Lobato, J. (2005). 'Accounting for phenomenology, artefacts, tasks and norms in cognitive models: The case of ration-as-measure.' Paper presented at the American Educational Research Association (AERA) annual conference, Montreal, Canada, April 9-14.

Looney, A. (2004). 'The Practice of Policy? A study of the role of the support services in the implementation of curriculum and assessment policy in the Republic of Ireland.' Unpublished Ed. D. thesis. Institute of Education: London.

Lyons, M., Lynch, K., Close, S., Sheerin, E., and Boland, P. (2003). *Inside classrooms: The teaching and learning of mathematics in social context*. Dublin: Institute of Public Administration.

Mac an Ghaill, M., Hanafin, J., and Conway, P. F. (2004). *Gender politics and Exploring Masculinities in Irish Education: Teachers, materials and the media*. Dublin: National Council for Curriculum and Assessment. Available online at: http://www.ncca.ie/j/index2.php?name=publ

McClain, K., and Cobb, P. (2001). 'The development of sociomathematical norms in one first-grade classroom.' *Journal for Research in Mathematics Education* 32, 234-266.

Malloy, C. (2002). 'Democratic access to mathematics through democratic education: An introduction.' In L. D. English, (Ed.), *Handbook of International Research in Mathematics Education*. Lawrence Erlbaum Associates.

Mariotti M.A. (2002). 'The influence of technological advances on students' mathematics learning.' In L. D. English (Ed.), *Handbook of International Research in Mathematics Education*. New Jersey: Lawrence Erlbaum Associates.

Matusov, E. (1999) 'How Does a Community of Learners maintain itself? Ecology of an innovative school.' *Anthropology and Education Quarterly* 30, 161-86.

Means, B., Chelemer, C., and Knapp, M. S. (Eds.), (1991) '*Teaching Advanced Skills to At-risk Students: Views from Research and Practice*.' San Francisco: Jossey-Bass Inc., Publishers.

Mehrens, W. A. (1989). 'Preparing students to take standardized achievement tests.' *Practical Assessment Research and Evaluation* 1, 11. Available online at: http://PAREonline.net/getvn.asp?v=1andn=11

Mix, K., Levine, S.C., and Huttenlocher, J. (2002). *Quantitative development in infancy and early childhood*. Oxford: Oxford University Press.

Morine-Dershimer, G., Corrigan, S. (1996). 'Teacher beliefs.' In H. J. Walberg and G. D. Haertel, (Eds.). *Psychology and educational practice*. San Pablo, CA: McCutchan Publishing.

MSEB (1993). *Measuring up: Prototypes for mathematics assessment*. Washington DC: MSEB, National Research Council. National Academy Press.

Mulkeen A. (2004). *Schools for the Digital Age: Information and Communication Technology in Irish Schools.* Dublin: National Centre for Technology in Education.

Munby, H., Russell, T., and Martin, A.K. (2001). 'Teachers' knowledge and how it develops.' In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 877904). Washington, DC: American Educational Research Association.

Murphy, P. (2001). 'Gendered learning and achievement.' In J. Collins and D. Cook (eds), *Understanding learning: Influences and outcomes* (pp. 109-121). London and Milton Keynes: Paul Chapman with Open University Press.

National Council for Curriculum and Assessment (2003). *Directions for Development of Senior Cycle Education*. Dublin: National Council for Curriculum and Assessment. Available online at: www.ncca.ie

National Council for Curriculum and Assessment (2005). *Proposals for the Future Development of Senior Cycle Education in Ireland*. Dublin: National Council for Curriculum and Assessment. Available online at: www.ncca.ie

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluations Standards for School Mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (1995). *Connecting mathematics across the curriculum*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, Virginia: National Council of Teachers of Mathematics.

National Policy Advisory and Development Committee (2001). *The Impact of Schools IT 2000*. Dublin: National Centre for Technology in Education.

New Zealand Ministry of Education (1992). *Mathematics in the New Zealand Curriculum*. Wellington: Learning Media.

Nicholas L., and Robertson G. (2002). Testing Times for Mathematics Education in Scotland, *Mathematics Today*, 38, 1, 9-12.

Novoa, A., and deJong-Lambert, W. (2003). 'Educating Europe: An analysis of EU educational policies.' In D. Phillips and H. Ertl (Eds.), *Implementing European Union Education and Training Policy*. Dordrecht: Kluwer Academic Publishers.

Nuñes, T., Schliemann, A. D., and Carraher, D. W. (1993). *Mathematics in the Streets and in Schools*. Cambridge, UK: Cambridge University Press.

O'Hare, D. (2005). 'Science left on the back burner.' *Irish Times*, (p. 14), 17th August.

Oakes, J., and Lipton, M. (1999) '*Teaching to Change the World*.' Boston: McGraw-Hill College.

OECD PISA (2001). Accessed June 20, 2005: http:www.pisa.oecd.org/pisa/math.htm

OECD (1991). *Ireland. Reviews of National Policies for Education*. Paris: OECD.

OECD (2003). *PISA Assessment Framework: Mathematics, reading, science and problem solving*. Paris: OECD.

OECD (2005). *Attracting, developing and retaining effective teachers*. Paris: OECD.

Ofsted (2002a). *Mathematics in Secondary Schools* (HMI 818). London: Ofsted.

Ofsted (2002b). *ICT in Schools: Implementation of Government Initiatives - Secondary Mathematics* (HMI 705). London: Ofsted.

Ogbu, J. U. (1992). 'Understanding cultural diversity and learning.' *Educational Researcher* 21, 8, 5-14.

Oldham, E. (1989). 'Is there an International Curriculum?' In Greer, B. and Mulhern, G., *New Directions in Mathematics Education* pp. 185-224. London: Routledge.

Oldham, E. (1991) 'Second Level Mathematics Curricula: The Republic of Ireland in International Perspective.' *Irish Educational Studies* 10, 122-138.

Oldham, E. (2001). 'The Culture of Mathematics Education in the Republic of Ireland: Keeping the Faith?' *Irish Educational Studies* 20, 266-277.

Oldknow A., and Taylor R. (2000). *Teaching Mathematics with ICT*, London: Continuum.

Oldknow A. (2005). "MathsAlive": lessons from twenty Year 7 classrooms. In Johnston-Wilder S. and Pimm D. (Eds.) *Teaching Secondary Mathematics with ICT* (pp. 174-190). Maidenhead: Open University Press.

O'Leary, M. (2001). 'Item format as a factor affecting the relative standing of countries in the Third International Mathematics and Science Study' (TIMSS), *Irish Educational Studies* 20,153-176.

Olson, D. R., and Bruner, J. (1996). 'Folk psychology and folk pedagogy.' In D. R. Olson, and N. Torrance (Eds.), *The Handbook of Education and Human Development* (pp. 927). Oxford: Blackwell.

Parramore K. (2001). 'Computer-based examining in GCE (A level) Decision and Discrete Mathematics.' *Teaching Mathematics and its Applications* 20, 2, 89-97.

Passey D. (2001). *MathsALIVE! Final Evaluation Report of the RM/DfEE Project to Develop Year 7 Mathematics On-line Course Materials*. Oxford: RMplc.

Pelligrino, J., Chudowsky, N., and Glaser, R. (Eds.). (2001). *Knowing What Students Know: The Science and Design of Educational Assessment.* Washington: National Academies Press.

Pettersen, L. (1993). 'Japan's cram schools.' *Educational Leadership* 50, 5, 56-8.

Phillips, D., and Ertl, H. (Eds.), (2003). *Implementing European Union Education and Training Policy: A Comparative Study of Issues in Four Member States*. Kluwer Academic Publishers.

Pintrich, P.R. (2000). 'The role of goal orientation in self-regulated learning.' In M. Boekaerts, P.R. Pintrich and M. Zeidner, (Eds.), *Handbook of self-regulation* (pp. 451-502). San Diego: Academic Press.

Portes, P. R. (1996). 'Ethnicity and culture in educational psychology.' In D. C. Berliner and R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 331-357). New York: Macmillan.

Postlethwaite, T. N. (1999). An overview of issues in international comparative achievement studies. In B. Jaworski, B and D. Phillips, D. (Eds.) '*Comparing standards internationally: research and practice in mathematics and beyond'*. Oxford: Symposium Books.

Prais, S. J. (2003). 'Cautions on OECD's recent educational survey (PISA)'. *Oxford Review of Education*, 29, 2, 139 - 163

Prais, S. J. (2004). 'Cautions on OECD's recent educational survey (PISA): rejoinder to OECD's response'. *Oxford Review of Education*, 30, 4, 569 - 573

Prawat, R. S. (1992). 'From individual differences to learning communities: Our changing focus.' *Educational Leadership* 49, 7, 9-13.

Prawat, R. S., and Floden, R. W. (1994). 'Philosophical perspectives on constructivist views of learning.' *Educational Psychologist* 29, 37-48.

Putnam, R. T., and Borko, H. (2000). 'What do new views of knowledge and thinking have to say about research on teacher learning?' *Educational Researcher* 29, 1, 4-15.

Raudenbush, S., and Bryk, A. 2002. *Hierarchical Linear Models (2nd edition).* Thousand Oaks: Sage.

Ravitch, D. (2003). *The language police: How pressure groups restrict what students learn.* New York: Alfred A. Knopf.

Resnick, L.B. (1987). *Education and learning to think.* Washington, DC: National Academy Press.

Resnick, L.B. (1995a). *From the bell curve to all children can learn.* (Video-based lecture.) Pittsburgh, PA: University of Pittsburgh, Institute for Learning.

Resnick, L.B. (1995b). 'From aptitude to effort: A new foundation for our schools.' *Daedulus* 124, 4, 55-62.

Resnick, L. B., and Hall, M. W. (2001). *The principles of learning: Study tools for educators* (CD-Rom, Version2.0). Pittsburgh, PA: University of Pittsburgh, Learning Research and Development Centre, Institute for Learning. Available online at: www.instituteforlearning.org

Reusser, K., Hugener, I., and Pauli, C. (2005). 'Video-based classroom studies: integration of different methods of video analysis for a differentiated perspective on teaching and learning.' Invited paper at European Association for research on Learning and Instruction (EARLI), University of Cyprus.

Rohlen, T.P. (1983). *Japan's high schools.* Berkeley: University of California Press.

Romberg, T. (2000). 'Changing the Teaching and Learning of Mathematics.' *The Australian Mathematics Teacher* 56, 4, pp. 6-9.

Romberg, T., and Kaput, J. (1999). 'Mathematics worth teaching, mathematics worth understanding.' In T. A. Romberg and J A. Kaput (Eds.), *Mathematics classrooms that promote understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Romberg, T. A., Schafer, M. C., and Webb, N. (in press). 'The design of the longitudinal/cross-sectional study'. In T. A. Romberg and M. C. Schafer (Eds.), *The impact of teaching mathematics using Mathematics In Context on student achievement*.

Roschelle, J. M., Pea, R. D., Hoadley, C. M., Gordin, D. N., and Means, B. M. (2000). 'Changing how and what children learn in school with computer-based technology.' *Children and Computer Technology* 10, 2, 76-101.

Ruddock, G. (1998). *Mathematics in the School Curriculum: an International Perspective*. Available online at: http://www.inca.org.uk

Saljo, R., and Wyndhamn. N. (1993). 'Solving everyday problems in a formal setting: An empirical study of the school as context for thought.' In S. Chaiklin and J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 327-342). New York: Cambridge University Press.

Salomon, G. (1993). Editor's introduction. In G. Salomon (Ed.), *Distributed cognitions. Psychological and educational considerations* (pp. xi-xxi). New York: Cambridge University Press.

Sangwin, C. (2003). 'Assessing mathematics automatically using computer algebra and the Internet.' *Teaching Mathematics and its Applications* 23, 1, 1-14.

Sarason, S. (1993). *The predictable failure of educational reform: Can we change course before it's too late?* San Francisco: Jossey-Bass.

Sato, M. (2005). 'Cram schools cash in on failure of public schools.' *The Japan Times*. Accessed via the web on August 8th, 2005.

Saxe, G. B. (1999). Cognition, development, and cultural practices. In E. Turiel (Ed.) *Culture and development: New directions in child development*. San Francisco, CA, US, Jossey-Bass Inc, Publishers, pp. 19-35.

Saxe, G. B., and Esmonde, I. (2005). Studying cognition in flux: A historical treatment of fu in the shifting structure of oksapmin mathematics. *Mind, Culture, and Activity*. 12, 3 & 4, 171-225.

Schafer, A. T., and Gray, M. W. (1981). 'Sex and mathematics.' *Science* 211.

Schafer, M., and Romberg, T. (1999). 'Assessment in classrooms that promote understanding.' In T. A. Romberg and J A. Kaput (Eds.), *Mathematics classrooms that promote understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Schliemann, A. D., Goodrow, A., and Lara-Roth, S. (2001). 'Functions and graphs in third grade.' Paper presented at National Council of Teachers of Mathematics (NCTM) conference, Orlando, Florida.

Schmidt, W. H. (2004). 'Curriculum and coherence: does the logic underlying the organization of the subject matter.' Paper presented at the Asia Pacific Economic Cooperation (APEC) Educational Reform Summit, Beijing, China, January 11-14.

Schoenfeld, A.H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A.H. (Ed.) (1987). *Cognitive science and mathematics education*. Hillsdale, NJ: Erlbaum.

Schoenfeld, A.H. (1992), 'Learning to think Mathematically: Problem Solving, Metacognition, and Sense-making in Mathematics.' In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* pp.334 370. New York, NY, Macmillan.

Schön, D. (1987). *Educating the Reflective Practitioner: Toward a New Design for Teaching and Learning in the Professions*. San Francisco: Jossey Bass.

Schuller, T., Preston, J., Hammond, C., Brassett-Grindy, A., and Bynner, J. (2004). *The Benefits of Learning: The Impact of Education on Health, Family Life and Social Capital*. London: RoutledgeFalmer.

Schunk, D. H. (2005). Commentary on self-regulation in school contexts, *Learning and Instruction* 15, 2, 173-177.

Senge, P. (1990). *The Fifth Discipline, The Art and Practice of the Learning Organization*. London: Random House.

Sfard, A., and Kieran, C. (2001). 'Preparing teachers for handling students' mathematical communication: Gathering knowledge and building tools.' In F. L. Lin and T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 187-205). Dordrecht, The Netherlands: Kluwer.

Showers, B., Joyce, B., and Bennett, B. (1987). 'Synthesis of research on staff development: A framework for future study and a state-of-the-art analysis.' *Educational Leadership* 45, 3, 77-87.

Shulman, L.S. 'A union of insufficiencies: strategies for teacher assessment in a period of educational reform.' *Educational Leadership* 46, 3, 36-41.

Shulman, L.S. (1987). 'Knowledge and teaching: foundations of the new reform.' *Harvard Education Review* 57, 1, 1-21.

Sinclair, N. (2005). 'Mathematics on the Internet.' In Johnston-Wilder S. and Pimm D. (Eds.), *Teaching Secondary Mathematics with ICT* (pp. 203-216). Maidenhead: Open University Press.

Sinclair, N., and Jackiw, N. (2005). 'Understanding and projecting ICT trends in mathematics education.' In S. Johnston-Wilder and D. Pimm (eds), *Teaching Secondary Mathematics with ICT* (pp. 235-251). Maidenhead: Open University Press.

Singapore Ministry of Education (2001). *Lower Secondary Mathematics Syllabus*. Singapore: Author.

Skelton, C. (2001). *Schooling and the boys: Masculinities and primary education*. Milton Keynes: Open University Press.

Sloane, F. (2005). Unpublished document: 'Interviews in Singapore.' National Science Foundation.

Sloane, F. C. and Hamilton, E. (2003). Internal document of the National Science Foundation. Arlington, Virginia. NSF.

Smith, A. (2004). *Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education*. London: The Stationery Office.

Snow, C., Burns, M. S., and Griffin, P. (1998). *Preventing reading difficulties in young children*. Washington: National Academy Press.

Stacey, K., Asp, G., and McCrae, B. (2000). 'Goals for a CAS-active senior mathematics curriculum.' In M.O.J. Thomas (Ed.), *Proceedings of TIME 2000 An International Conference on Technology in Mathematics Education December 244-252*). Auckland, New Zealand: The University of Auckland and Auckland University of Technology.

Stacey, K., McCrae, B., Chick, H., Asp, G., and Leigh-Lancaster, D. (2000). Research-led policy change for technologically-active senior mathematics assessment. In J. Bana and A. Chapman (Eds.), Mathematics Education Beyond 2000. Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 572-579. MERGA. Freemantle.

Stigler, J.W., and Hierbert, J. (1999). *The teaching gap*. New York: The Free Press.

Stigler, J. W., and Hiebert, J. (1999). Understanding and improving classroom mathematics instruction: an overview of the TIMSS video study. In B. Jaworski, B and D. Phillips, (Eds.), *Comparing standards internationally: research and practice in mathematics and beyond*. London: Symposium Books.

Stigler, J. W., Gallimore, R., and Hiebert, J. (2000). 'Using Video Surveys to Compare Classrooms and Teaching Across Cultures: Examples and Lessons from the TIMSS Video Studies.' *Educational Psychologist* 35, 2, 87-100.

Stiglitz, J. E. (2002). *Globalization and its discontents*. London: Penguin.

Stray, C. (1994). 'Paradigms regained: Towards a historical sociology of textbooks. *Journal of Curriculum Studies* 26,1, 1- 29.

Streefland, L. (Ed.). *The Legacy of Hans Freudenthal*. Dordrecht: Kluwer Academic Publishers.

Stromquist, N., and Monkman, K. (Eds.), (2000). *Globalisation and education: Integration and contestation across cultures*. Lanham, MD: Rowman and Littlefield.

Struik, D. J. (1948). *A history of mathematics*. London: Dover Publications.

Suarez-Orozco, M., and Qin-Hilliard, D. B. (2004) (Eds.). *Globalization: Culture and education in the new millennium*. Berkeley: University of California Press and Ross Institute.

Sugrue, C. (2002). 'Irish teachers' experiences of professional learning: implications for policy and practice' *Journal of In-service Education* 28, 2, 311-338.

Sugrue, C., and Day, C. W. (2002). Introduction. In C. W. Day and C. Sugrue, (Eds.), *Developing Teachers and Teaching Practice: International research perspectives*. London: RoutledgeFalmer.

Sugrue, C., Morgan, M., Devine, D., and Raftery. (2001). *Policy and Practice of Professional Development for Primary and Post -Primary Teachers in Ireland: A Critical Analysis*. Dublin: St Patrick's College Drumcondra and Education Department UCD.

Tatto, M. T. (Ed.), (in press). *Teacher education reform and accountability systems in a global time: A comparative analysis*. London: Symposium Books.

Taverner, S. (1996). 'Comparing boards in A-level mathematics.' *Mathematical Gazette* 488, 362-6.

Tikly, C., and Wolf, A. (Eds.), (2000). *The maths we need now: demands, deficits and remedies*. London: Institute of Education (Bedford Way Papers).

TEDS-M (2005). Report on TEDS-M 2008 for IEA General Assembly, September, 2005.

Tobin, P. (1998). 'New technology and curriculum change.' *Teaching Mathematics and its Applications* 17, 3, 97-105.

Tomlinson, T. (1992). *Hard Work and High Expectations: Motivating Students to Learn*. Issues in Education. Office of Educational Research and Improvement, Washington, D.C. Report. April 1992.

Treffers, A. (1987). *Three Dimensions: A Model of Goal and Theory Description in Mathematics: The Wiskobas Project*. Dordrecht, The Netherlands: Reidel.

Tsukada, M. (1988). 'Institutionalized supplementary education in Japan: The Yobiko and Ronin student adaptations'. *Comparative Education* 24, 3, 285-303.

UNESCO (2005). *The quality imperative: the EFA global monitoring report*. Paris: UNESCO.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., and Houang, R. T. (2002). *According to the book: using TIMSS to investigate the translation of policy into practice through the world of textbooks.* Dordrecht: Kluwer Academic Publishers.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht: Freudenthal Institute.

Venezky, R.L., and Davis, C. (2002). *Quo Vademus? The Transformation of Schooling in a Networked World*. Paris: OECD/CERI.

Vergnaud, G. (1983). 'Multiplicative structures.' In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.

Verschaffel, L., De Corte, E., Lasure, S., Van Vaerenbergh, G., Bogaerts, H., and Ratinckx, E. (1999). 'Design and evaluation of a learning environment for mathematical modeling and problem solving in upper elementary school children.' *Mathematical Thinking and Learning* 1, 195-229.

Verschaffel, L., Greer, B., and De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets and Zeitlinger.

Victorian Curriculum and Assessment Authority (2004). *Mathematics methods (CAS): Teacher Support Resources*. Melbourne: Author. [CD-ROM].

Victorian Curriculum and Assessment Authority (2005). *Mathematics: Victorian Certificate of Education Study Design*. Melbourne: Author.

Victorian Curriculum and Assessment Authority (2003). VCAA Bulletin, December, No. 9. Consultation paper for VCE examinations: 2006 and beyond. Available online at: http://www.vcaa.vic.edu.au/correspondence/bulletins/2003/December/03DECSU2.pdf

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.

Wang, J., and Lin, E. (2005). 'Comparative studies on US and Chinese mathematics learning and the implications for standards-based mathematics teaching reform.' *Educational Researcher* 34, 5, 3-13.

Webb, D. C., and Meyer, M. R. (2002). *Summary report of achievement data for Mathematics in Context: A Connected Curriculum for Grades 5–8.* Madison, WI: Wisconsin Center for Education Research.

Wells, G., and Claxton, G. (2002). 'Introduction: Sociocultural perspectives on the future of education.' In G. Wells and G. Claxton (Eds.), *Learning for life in the 21st century: Sociocultural perspectives on the future of education.* Oxford: Blackwell.

West, L., and Staub, F. (2003). *Content-Focused Coaching: Transforming Mathematics Lessons*. University of Pittsburgh: Heinneman.

White, M. (1993). *The material child: Coming of age in Japan and America.* New York: The Free Press.

Whitehead, A.N. (1911). *An introduction to mathematics*. London: Williams and Norgate.

Wurnig, O. (2003). 'Using PC and TI-92 in teaching Binomial and Normal Distribution in the final Grades of Austrian Grammar Schools.' In *The Mathematics Education into the 21st Century Project Proceedings of the International Conference The Decidable and the Undecidable in Mathematics Education*. Brno, Czech Republic, September 2003. Available online at: http://math.unipa.it/~grim/21_project/21_brno03_Wurnig.pdf

Yackel, E., Cobb, P., and Wood, T. (1999). 'The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example.' *Journal of Mathematical Behavior* 17, 469-488.

Yoshida, M. (1999). 'Lesson Study: An ethnographic investigation of school-based teacher development in Japan.' Unpublished doctoral dissertation, University of Chicago.

Youniss, J., and Damon, W. (1992). Social construction in Piaget's theory, In H. Berlin and B. Pufal (Eds.), *Piaget's theory: Prospects and possibilities*, (pp. 267-286). Hillsdale, NJ: Erlbaum.

Zhao, Y. (2005). Increasing Math and Science Achievement: Sharing best (worst) practices from East and West. Paper presented at US-China Centre for Educational Excellence conference on Secondary School Curriculum Reform, East Lansing, MI, USA.

Zimmerman, B.J. (2001). Theories of self-regulated learning and academic achievement: An overview and analysis, In B.J. Zimmerman and D.H. Schunk (Eds.), *Self-regulated learning and academic achievement: Theoretical perspectives* (2nd ed., pp. 1-37). Mahwah, NJ: Erlbaum.

Zwick, R. (2002). *Fair Game? The Use of Standardized Admissions Tests in Higher Education*. London: Falmer.

ISSN 1649-3362

National Council for Curriculum and Assessment
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An Chomhairle Náisiúnta Curaclaim agus Measúnachta

Review of Mathematics in Post-Primary Education

Report on the Consultation

April 2006

Contents

Introduction

The review of post-primary mathematics education arose in the context of concerns about the low level of mathematical skills displayed by students emerging from post-primary schools. Also, a number of studies had highlighted issues in relation to curriculum provision and uptake, the teaching and learning of mathematics in our schools, the appropriateness and effectiveness of the assessment arrangements, and the performance of candidates in the Junior Certificate and Leaving Certificate mathematics examinations. Mathematical knowledge and skills are held in high esteem in Ireland and are seen as having a significant role to play in the development of the knowledge society to which Ireland aspires.

In October 2005 the NCCA published a discussion paper which presented an overview of the issues surrounding mathematics education at post-primary level in Ireland. The paper presented data on the uptake of mathematics at the three different syllabus levels and considered the performance of candidates in the state examinations and in international tests of achievement. It identified particular areas of concern that needed to be addressed as part of the review, with a view to stimulating discussion on the nature and purpose of post-primary mathematics education as well as on the particular difficulties facing it at this point in time.

A companion paper on international trends in mathematics education was commissioned by the NCCA to further inform the debate on the issues identified in the discussion paper, and to give an insight into how other countries are addressing similar concerns about mathematics education.

This report describes the consultation process and the responses and submissions received. It revisits the issues raised in the discussion paper in light of the consultation responses, and summarises the findings that have emerged. Finally, it draws together a set of conclusions and makes recommendations on progressing the review so that post-primary mathematics education can be re-shaped to fulfil its many roles and functions in the lives of learners.

2. The consultation process

Consultation documents and feedback mechanisms

Following the publication of the NCCA discussion paper and its companion paper on international trends in mathematics education, a consultation process was engaged in during October and November to allow those with an interest in the issues raised to respond to these and to raise any other concerns which they considered should be addressed under the review.

To facilitate feedback, a consultation questionnaire was developed that was based on the main issues identified in the discussion paper. An online version of this questionnaire was made available on the NCCA website. Submissions by post and email were also invited and, to encourage as wide an audience as possible to participate in the consultation, a free text messaging service (SMS) was established. A 'flyer' was designed to draw attention to the response channels available.

Consultation documentation was circulated widely, including to all post-primary schools and to the education partners. In particular, third-level education departments and the science/engineering departments of Universities and Institutes of Technology were invited to respond. Consultation documents were also sent to individuals who had expressed an interest in the review. The initial period of consultation was extended to mid-December to facilitate those groups and organisations who wished to engage their members in discussing particular aspects of the review, and who needed time to facilitate the collation of their views.

Focus discussions

As part of the consultation, the NCCA held discussions with a number of groups to engage in focused consideration of the issues as they perceived or experienced them. These included two parents' representative groups, three branches and the Council of the Irish Mathematics Teachers' Association, and the Junior Certificate Mathematics Support Service team. Discussions also took place with a representative of the Union of Students in Ireland and informally with individual mathematics teachers who were attending inservice courses in the period of the consultation. A combined meeting was convened of three NCCA course committees in the mathematics area (Junior Certificate and Leaving Certificate mathematics, and applied mathematics at Leaving Certificate). In collaboration with the School of Mathematical Sciences, University College Cork, a well-attended meeting was held in January 2006 at which teachers of mathematics and lecturers (from both University College Cork and the Cork Institute of Technology) engaged in wide-ranging discussion of aspects of mathematics education that were of common interest.

Collecting the data

Towards the end of the consultation period, questionnaires received in hard copy were uploaded to the online facility, and email and other submissions were collated under headings similar to those used in the questionnaire. This enabled the responses and submissions to be considered in a structured manner. According as further responses and submissions were received these were included in a similar fashion.

The structure of the questionnaire was such that those responding could be easily grouped into pre-determined categories (2^{nd} or 3^{rd} level students, teachers or lecturers, parents, school principals, employers, etc.). The survey facility that was used also enabled the identification of 'skipped' items and the tabulation of items in a multi-part section that required response on a three-part Likert scale.

The SMS provider facilitated the collection of the text messages received in the form of a spreadsheet, which allowed for easy handling and collation of the data. This also enabled the identification of repeat text messages, so that the number of individual SMS respondents could be gauged accurately.

3. Responses and submissions

The total number of responses to the consultation is indicated below in Table 1. Two-thirds of the responses were in the form of returned questionnaires (by post, as an email attachment or using the online facility). When repeat or trivial responses (mainly text messages) were taken into account this number was reduced to 340. Of these, 21 contained limited comment on just some of the issues, while about half of the questionnaires contained one or two skipped sections. It should be noted that about 10% of those who made a response did so on behalf of a group or organisation and that, in some cases, this was the result of focused internal discussion within the group or organisation. In addition to the formal submissions, notes were also taken at the discussion meetings mentioned previously.

Table 1. Number of responses or submissions by type

Response format	**No.**	**Comment**
Questionnaire (postal or email attachment)	150	15 of these had less than three of the twelve sections completed
Questionnaire (online)	114	7 of these were blank except for a name, while a further 6 had less than three of the twelve sections completed
Text message	98	36 of these were repeat or trivial messages
Other submissions	22	Including email comments and feedback obtained at group meetings
Total	384	

Table 1 shows the total number of submissions by type while Table 2 opposite provides a further breakdown of the returned questionnaires by category of respondent, where this was indicated in the response. Sixty-five text messages of a non-trivial nature were received from individuals. In a small number of cases written submissions were received that did not follow the format of the questionnaire; these tended to focus on specific aspects of mathematics education, such as syllabus content, syllabus levels, examinations, or the low level of mathematical knowledge and skills of students emerging from post-primary schools. Just over half of the non-questionnaire/SMS submissions dealt in great detail with all of the issues raised in the discussion paper.

Table 2. Categories of questionnaire respondents

Category	Number	% of total
2nd level student	24	9.3
Teacher (mathematics)	117	45.5
Teacher (other subject)	7	2.7
3rd level student	22	8.6
Lecturer (mathematics)	26	10.1
Lecturer (other subject)	12	4.7
Parent	11	4.3
School Principal	14	5.4
Employer	0	0
Other	16	6.2
Not indicated	8	3.1
Total	257	

As can be seen, almost half of all questionnaires came from teachers, the majority of whom were teachers of mathematics. No individual business or employer completed a questionnaire, despite the concerns that this sector has expressed in recent years about standards and achievement in mathematics. The ‘other’ category includes a small number of professional individuals or groups, including ICT Ireland which represents the high technology/ICT sector within IBEC. A submission was also made on behalf of the Office of the Financial Regulator.

The remainder of this section of the report describes and summarises the submissions received using the section headings provided in the consultation questionnaire, even where this structure was not used by the respondents. These are not evaluated or responded to in this section; comment is provided in sections four and five in relation to what has emerged in the consultation submissions.

Where it was deemed relevant in the case of particular points of view expressed in a submission, indication is given in the commentary as to whether these were from an individual or from a group. In a small number of cases, the type of group is identified (teachers, representative organisation, parents, etc.).

3.1 Role and purpose of mathematics education

All respondents commented on the importance of mathematics education to the individual and to society. They pointed to its significance in the development of logical thinking and problem-solving skills, as well as its importance as a foundation for other subjects, especially the science and technology subjects. At senior cycle, mathematics was noted as having an added importance, given its requirement for admission to third-level courses.

Many respondents commented on the need for post-primary mathematics to be more in tune with real-life contexts so that students can see its relevance to themselves and to their current and future lives. Some also commented on the need for students to appreciate the beauty of mathematics, as well as the need to overcome the sometimes negative attitudes towards the subject. In one group discussion, a parallel was drawn with English, which students also use in other subjects. Indeed, the learning of mathematics as a series of formulae to be learnt, or procedures and techniques to be practised, was likened to learning English through the study of grammar, but with no exposure to literature or poetry.

Some respondents also commented on the need to refer to applications of mathematics, including its use in what could be termed non-mathematical fields or situations. They see it as important that students at post-primary level should know how and where they will use mathematics in their lives and work, and how it underpins many of the developments that affect our daily lives.

(Mathematics) provides an essential foundation and language for science and technological subjects.

Numeracy is an essential life skill…poor mathematical skills impede progress.

Learners … (should) acquire a certain amount of competence in applying the mathematics they learn to meaningful problems drawn from both the personal and scientific domains.

3.2 Concerns about mathematics education

The discussion paper identified specific areas of concern in relation to post-primary mathematics education:

- the emphasis on procedural skills rather than on understanding
- poor application of mathematics in real-world contexts
- low uptake of Higher level mathematics, especially in the Leaving Certificate
- low grades achieved at Ordinary level, especially in the Leaving Certificate
- gender differences in uptake and achievement
- difficulties in mathematics experienced by some students in third-level courses.

A common theme across many submissions was the emphasis that is placed in classrooms on 'rote learning' of mathematical knowledge and on 'procedural skills' which students can then demonstrate in examinations. This approach means that teaching for understanding does not receive much consideration. The revised syllabus in Junior Certificate mathematics is seen as going some way towards bringing about a change, but more needs to be done.

There is a real need to explain the ideas behind … procedures.

> *The emphasis in maths education needs to change to one where understanding and an ability to apply knowledge are the main goals.*
>
> *More emphasis on experimentation, investigations and project work in the classroom would have a beneficial effect.*

However, some respondents pointed to the benefits of developing a reasonable level of procedural skills, as distinct from rote learning, so that students can call on these skills when solving problems. The application of mathematics in real-world contexts entails a measure of practised skills in solving problems.

Many respondents commented on the fact that mathematics is perceived as a difficult subject, in which only those with a high level of mathematical ability can expect to achieve success.

> *I feel (at least from my own school days, which aren't so far away) that a lot if emphasis is put on 'strange' mathematical questions... I never really understood these types of questions...and still fail to understand their full importance in after school life* (3rd level student)

This perception is seen as contributing to the fall-off in uptake of Higher-level mathematics at Leaving Certificate where, currently, the uptake levels are 11% at Foundation level, 71% at Ordinary level, and 18% at Higher level. The lack of recognition of Foundation level mathematics for entry to many courses at third level is viewed as contributing to the 'problem' of the excessive numbers taking Ordinary level mathematics at Leaving Certificate.

Many teachers commented on the lack of sufficient class time for mathematics, especially in the junior cycle, and the need to 'cover the course' so that students can do well in the Junior Certificate examinations. The reduction of time for mathematics is attributed to an overloaded curriculum as a result of the introduction of additional subjects or modules.

As a result, they argue, it is difficult to find time to explain the concepts involved. In order to prepare students for the examinations, the focus is mainly on student practice of the technical application of procedures.

> *A lot of the time with maths I just follow the method and do not understand why I should do so. After a while I get some understanding of the reasoning.* (2nd level student)

Some teachers commented on the declining commitment on the part of students to make the effort that is required to achieve a good level of understanding, but acknowledge that this is not confined to mathematics. The growing tendency for post-primary students, particularly in the senior cycle, to engage in part-time work is also noted; this is seen as contributing to lower attendance rates and less time being devoted to homework. Other factors identified as contributing to difficulty in mathematics at Leaving Certificate level are the 'gap' in standards between the junior cycle and senior cycle mathematics syllabuses (especially since the revision at junior cycle), a lack of basic numeracy skills, which some attribute to over-reliance on—or inappropriate use of—calculators.

> *The fact that Junior Cert maths was reduced in content and the topics dropped are still on the LC means that the gap is too wide for many LC students.* (mathematics teacher)

> *The recent changes at both primary level and Junior Cert. level are steps in the right direction. The main emphasis is on teaching and learning for understanding. These ideas must be carried on to Leaving Cert. The problem arises at second level in that these changes are not continuous. There is a huge void between Junior Cert. and Leaving Cert. at present.* (mathematics teacher)

> *The intuition associated with mental arithmetic (tables) is gone as a result of the introduction of calculators. Revision of maths in primary and junior cert with the associated revision in senior cycle (is required).* (group of mathematics teachers)

While some respondents see the lack of choice in the Junior Certificate mathematics examination as problematic, leading to lower achievement levels, others see this as of benefit in ensuring that students have fully covered the course.

Only a small number of respondents commented on gender differences, most of these making the point that the number of girls taking Higher level mathematics needs to increase. It was noted that girls generally perform better than boys in the examinations.

Mathematics lecturers also cited evidence of low levels of understanding on the part of students, even those who had studied Higher level mathematics at Leaving Certificate. They claimed that many students lack a basic understanding of concepts in some areas of mathematics, including arithmetic, algebra and geometry. These difficulties may go back to junior cycle, or even to primary school. Some commented on the narrow interpretation of the syllabus in textbooks and in the examinations.

> *At present one might describe the teaching of mathematics as a preoccupation with the 'how' of the subject to the almost total neglect of the 'why' of the subject. It is vital that the mathematical material presented in future texts and examinations cater more to applications so that students can acquire a greater understanding and appreciation of mathematics as a problem-solving discipline.* (mathematics lecturer)

Concern was also expressed that teachers of mathematics, especially in the junior cycle, may not themselves have an in-depth understanding of the subject, or have received pre-service training in the teaching of mathematics, thus leading to a narrow focus or 'comfort zone' in their teaching. Some respondents noted that not all teachers of mathematics attended the inservice courses that were provided to support the implementation of the revised Junior Certificate mathematics syllabuses. In discussions with the Junior Certificate Mathematics Support Service, it was pointed out that longer-term support for teachers is needed if they are to make the kinds of changes in practice that are called for in the move towards teaching for and learning with understanding, particularly if further syllabus and assessment changes are introduced.

The Irish Mathematics Teachers' Association and some mathematics lecturers pointed out that a move to greater emphasis on applying or relating mathematics to the world in which we live is a complex task, one which is likely to prove challenging for teachers, particularly for those who are not specialist mathematicians. The continuing professional development of teachers of mathematics will need to be supported through the preparation of appropriate resources to facilitate this change of approach in the teaching and learning of mathematics.

3.3 Recent developments in mathematics education

Many respondents indicated that the 'teaching for understanding' approach in Junior Certificate mathematics has not been fully embraced by teachers. Some see this being due to the 'traditional' style of the revised syllabuses and examnations, while others point to the lack of time required to do this well. There is a general view that teachers need further support in changing their classroom methodology.

Individual teachers have commented that students find it difficult to make the transition from primary school mathematics to the kind of mathematics they meet at second level. Teachers at second level may not be aware of the changes that have taken place in primary school mathematics and vice versa.

> *How much communication is there between Primary and Post-Primary teachers with regard to the knowledge / implementation of these curricula? Virtually none, I would say.* (mathematics teacher)

Some respondents (parents, teachers) raised concerns about the early use of the calculator and its effect on numeracy skills, some arguing for a separate, calculator-free assessment of these skills. The lack of connectedness to real-life contexts in the way mathematics is taught was also commented on. There is a need to ensure a seamless progression in the kind of mathematics being taught from primary school to Leaving Certificate. There was no follow-through at Leaving Certificate following the changes in Junior Certificate mathematics and 'gaps' exist as a result.

While a small number of respondents commented on the lack of choice in the Junior Certificate mathematics examination papers, there were differing views on this issue. One second-level student expressed the view that the lack of choice ensures full course coverage and avoids situations where students can omit some sections of the (Junior Certificate) course:

> *For years, topics such as trigonometry were left out by candidates – showing their lack of understanding, only to realise they needed to understand it more fully for Leaving Cert.*

Mathematics lecturers were generally very much in favour of the 'no choice' situation in the Junior Certificate mathematics examination, some even suggesting that this could also be beneficial at Leaving Certificate. On the other hand, some teachers expressed the view that the lack of choice served only to reinforce the perception that mathematics is difficult. It was suggested that, in making curriculum changes, the views of a wide-ranging group of experts should be sought; this should include primary school teachers and mathematics lecturers. One lecturer suggested that third-level scientists could provide suitable insights and examples for topics that might be included in mathematics syllabuses at second level.

Respondents also referred to the influence of the examinations on what is taught and how it is taught in mathematics classrooms. Teaching and learning continue to be exam driven.

> *There is strong consensus that while the examination remains the same, classroom practice will be to resist change.* (lecturer in mathematics education)

In expressing concern about the competence levels of students, especially in the junior cycle but also at senior cycle, some respondents suggested that there should be a separate assessment of basic mathematical skills, which all students should be required to achieve. If a high 'pass' mark was required, this would give assurances to the system regarding the ability of students in basic mathematics. Students should be allowed to undertake this assessment as many times as required to achieve a 'pass'.

Some respondents (students and teachers) identified algebra as an area of particular concern, which needs to be addressed by different teaching methods. Geometry is another area seen to pose difficulties for students.

Finally, a number of respondents commented on the reluctance of (second-level) teachers to change their methods, since their students are very successful in the examinations. One school principal expressed the view that mathematics teachers would feel very insecure in an environment that promotes and assesses an explorative, innovative and investigative assignment type of mathematics. Another school principal noted that

> *project work and/or continual assessment through different methodologies and skills should give time and space to enjoy maths and its essence, thereby breaking the emphasis on procedural skills, etc and relying more on understanding of maths and its principles.*

3.4 Current trends in mathematics education

This section had the highest level of non-response in the questionnaire. Where it was addressed in detail, respondents tended in the main to be lecturers in mathematics. Some second-level teachers indicated a lack of familiarity with current trends.

The comment of a mathematics lecturer captures the points expressed in a number of the responses about ‘modern mathematics’:

> *… its nature and emphases rendered it susceptible to the type of reductionism to which it fell victim: where abstract techniques were taught and examined while the applications that should have accompanied them were neglected because of the greater demands they place on both teacher and learners.*

This respondent went on to say that adopting a genuine ‘modern mathematics’ approach would require careful work to see it implemented successfully, including a change in textbook and examination treatment of many syllabus topics and a change in the mode of teaching to address student motivation and to include the application or evident purpose of these mathematics topics.

Some respondents commented on the need for a balance between the various approaches to mathematics education, noting that, while students need to learn mathematics in context, skills of logical thinking, rigorous argument and abstraction must not be neglected. The various approaches to mathematics education are not independent or mutually exclusive, they argue; the focus should not be on one to the virtual exclusion of the other.

A group of teachers pointed out that opportunities arise, especially in the Transition Year, to take alternative approaches to topics in mathematics, allowing students to look at these topics differently, but that time constraints are a definite hindrance at other levels.

One group (a professional body), summarised this need for balance as follows:

> *It is likely that the RME approach will do more to address the problems we encounter. However, a balance must be found here; students should also appreciate the elegance of mathematical proof and the rigour involved. Beyond that, there is no need for us to 'prescribe' minutiae – if students enter third level with an appropriate*

disposition towards mathematics, we should be capable of providing them with the rest, whether it be in guiding them into the world of 'pure' mathematics, or facilitating their acquisition of the additional 'knowledge' required for data analysis, 'modelling' etc.

There seemed to be general agreement that more concrete experiences of mathematics in real-world contexts are needed for primary and junior cycle students, with gradual movement towards abstraction as they become more comfortable with the underlying concepts and structures of mathematics. It should be possible to integrate some applications of mathematics as well as problem-solving approaches within the current syllabuses, but the examinations would need to change in order for this to be taken seriously.

Many respondents referred to the fact that students will need mathematics in many aspects of their daily lives, so these should be used as the contexts when they learn mathematics: interest calculations, discounts, tax, statistics, currency conversion, calculation of area, volume, etc. While some of these contexts may not have immediate relevance to junior cycle students, reference to them at this level is important so that the students can appreciate how and where mathematics affects their lives.

3.5 Mathematics in relation to other subjects

This section focused on the dual nature of mathematics—geared to applications but also worthy of study in its own right—and on its relationship to other subjects, and was responded to in sixty percent of submissions. Many respondents pointed to the importance of a good foundation in mathematics so that students could engage productively in a wide range of subjects, not just the science and technology areas (specific mention was made of business subjects, music, geography, P.E., home economics, history, CSPE, ICT). These subjects can also contribute to mathematics. While commenting on this overlap, it was noted (mainly by teachers, but also by some students) that the manner in which topics are treated in the different subjects can vary significantly.

Teachers teach in isolation and they are happy to do so. How does the business teacher teach percentages? How does the science teacher (unless they are one and the same) teach measure, scientific notation, area and volume? How does the geography teacher teach scale, data handling and data presentation? (mathematics teacher)

Some of this variation is attributed to the lack of communication or discussion between teachers of these subjects (and, perhaps, a lack of opportunities for this) and/or the lack of in-depth mathematical knowledge on the part of teachers of subjects other than mathematics. The establishment of good in-school planning structures can help this situation. Linkages with other subjects will also enable students to see the connectedness of mathematics and its applications in other disciplines. This could be facilitated if there was coursework in mathematics.

Some respondents pointed out that it is important not to treat mathematics solely as a 'service' subject; it has a beauty in itself. As with many other subjects, 'usefulness' is not the only criterion by which mathematics should be judged or justified.

Mathematics as a universal language and as a subject to enrich our students in their problem solving and creative abilities rather than as a subject of routine procedures to service other disciplines is a must for the education of our future generations. (mathematics teacher)

Third-level students commented on the need for second-level students to be aware of the importance of mathematics for many courses at third level and how understanding of mathematics is critical for success in such courses.

3.6 Provision and uptake of mathematics

In this section respondents were asked to comment on the adequacy of the current mathematics courses in meeting the needs of all students at both junior cycle and senior cycle. Views were also invited on the relatively low uptake in Higher level mathematics. Just over forty percent of respondents commented on these points.

The responses were very wide-ranging, reflecting the diversity of opinion that exists regarding the current courses and their assessment. Many of the points made in responding to this section were repeated elsewhere in submissions.

A small number of respondents made initial remarks concerning the lack of continuity between mathematics in the primary school curriculum and mathematics syllabuses at junior cycle and senior cycle. This is seen as creating difficulties for students, who may not make the connections themselves. A lack of awareness and understanding among teachers of the mathematics curriculum/syllabuses at each level means that there is not a coherent treatment of topics. When added to the different methodologies that are used to address the same concept, students are confused; it is little wonder that they consider mathematics difficult.

> *In Ireland there is a serious discontinuity between the mathematics curriculum for senior primary school and that for junior post-primary school. … This discontinuity has been highlighted in past reports but little has been done to address the situation. By the time students reach senior post-primary level, many of them have become alienated from mathematics and will find the easiest option available in terms of further study of mathematics.* (lecturers in mathematic education)

On the other hand, teachers generally (and some mathematics lecturers) were of the view that the revised mathematics syllabuses at Junior Certificate did not require much alteration, but that the approaches used in the teaching and learning of the topics were critical. Many of the teachers pointed out that there is potential to introduce topics in context and to adopt a problem-based approach to these, while ensuring that students develop the requisite skills that will enable them to progress to more advanced topics and problems. However, as noted already, this is not a simple step and will require

> *the development of a different mindset and much greater pedagogical content knowledge among teachers – at least among many of those who teach mathematics as a second subject – … many good teachers already model the approaches one would like to see; the current in-service initiative taps the resource that they can provide.* (lecturer in mathematics education)

A common theme among respondents was the need to alter the style of the examination paper if assessment is to reflect a problem-solving emphasis and a context-based approach in teaching and learning.

Most respondents recognised the need to encourage more students to study Higher level mathematics in the Junior Certificate; some reference was made to timetabling and in-school structures that could facilitate this. It was recognised that a concerted effort is required to change the attitude that mathematics is difficult, and this message must be communicated to students, teachers, parents, and the public generally.

Some respondents recommended a focus on numeracy and algebra in the junior cycle, so that students grasp the basics concepts and skills. Teachers, especially those for whom mathematics may not be their main subject, should be given support to ensure that they cover these aspects well.

Reference was made once more to the gap between Junior Certificate and Leaving Certificate mathematics courses. A number of mathematics lecturers favoured a reduction in the breadth of the Leaving Certificate courses so that a deeper understanding of some topics could be achieved by students at all levels. As one lecturer pointed out, the current choices mean that students can opt out of some topics:

> *Many students can completely disregard whole sections of the course, e.g. trigonometry, and still enter a third level college where this can be a major component of their required knowledge.*

The issue of non-recognition of Leaving Certificate Foundation level mathematics for third level courses continues to give rise to difficult choices for students, some of whom find themselves in classes with others who are capable of talking Higher level mathematics, but who choose to take the Ordinary level course and focus their efforts to gain points in other subjects. Some students opt out of Higher level through fear of failure in the examination; they need to 'pass' mathematics to get into college.

In a discussion with a parents' group, one parent suggested that the timetable for the exams should be changed so that students following the Higher level course could sit the Ordinary level examination in order to secure the required 'pass', and then also take the Higher level examination without the fear of 'failing'.

As an alternative to this, it was suggested, where a school offers a three-year senior cycle along the lines of the NCCA senior cycle proposals, students could take the Ordinary level examination in mathematics at the end of their fifth year and then proceed to study the Higher level course for sixth year. This would require some re-alignment across the courses, with perhaps a common course for one year followed by a measure of specialisation in the final year. As noted by some respondents elsewhere in the questionnaire, a common examination paper, focusing on basic mathematical knowledge and skills (perhaps a calculator-free one), could form one element of assessment. The second paper could offer a range of choices, depending on what courses or careers students were aiming for.

A different approach was suggested by some respondents. This advocates the merging of Foundation and Ordinary level mathematics syllabuses into a common course, accepted by third level, which should be aimed at those who do not intend to study pure mathematics at third level. The common course should be designed to ensure that students will acquire a good grasp of fundamental mathematics concepts and will develop an adequate level of mathematical skill to enable them to use mathematics in their chosen courses at third level.

A re-structuring of Higher level mathematics was also suggested, with a reduction on the ‘options’ and giving relevance to the topics retained. By way of illustrating how real-life context could be used in the treatment of a topic (difference equations) one lecturer in mathematics offered the following suggestion.

> …*the recurrence relation governing the amortisation of loans should be treated in depth as this is perhaps one of the most useful applications of mathematics lying outside of the scientific situation that bears on a student's (future) personal life and which can be undertaken at the level of senior cycle.*

The same respondent, having commented on the ‘technical exercise’ nature of many examination questions went on to state:

> *The manner by which a difference equation is arrived at is more often than not a greater practical skill than the routine by which the solution of the difference equation is obtained*.

Many of the respondents, but particularly students, commented on the fall-off in the numbers taking Higher level mathematics at Leaving Certificate. Students pointed out that, while they may begin studying mathematics at Higher level, they drop back to Ordinary level when they find the time demand too great and believe that they can get points in a much easier fashion in other (Higher level) subjects.

Although it is not possible to determine the categories of respondents who submitted text messages, a large proportion of comments related to maths as being a ‘hard’ subject to study/learn, with difficult topics to be covered especially at Higher level, making success (as measured in points) beyond the reach of all but the most able students.

3.7 Influence of the examination papers – ‘teaching to the test’

In this section, respondents expressed a variety of views on assessment of mathematics. The influence of the examinations on student choice was mentioned frequently in responses. Teachers acknowledged that they tend to focus on ‘getting their students through’ the exams successfully. The exam focus was also noted by students, parents and lecturers.

> *Maths is taught almost exclusively for the purpose of getting students the best possible results in the exams. Our present syllabus can be ruined by bad exam papers. Our present exam tests predominately a student’s ability to memorise – and to substitute into a formula*. (2nd level student)

> *As long as teachers are judged on their results teachers will `teach to the test'. Note the interest in recent times on who goes to college.* (mathematics teacher)

> *Assessment is the driving force for change and the current system will not change unless exam questions demand thinking skills from our students. The repetitive nature of examination questions forces rote learning rather than understanding.* (mathematics teacher)

Once again, there was extensive comment on the difficulty that students experience with Level Certificate Higher level mathematics, and the wide gap in standard between this and the Ordinary level course. While recognising that sole reliance on a terminal examination paper puts pressure on students, caution is advised by some respondents as regards the introduction of coursework. Students of average or below average ability in mathematics, it is argued, might not be well served by coursework assessment, depending on how it is conducted.

> *In the UK there seems to be evidence that investigations are becoming procedural. Assessment of coursework also raises social issues as students from privileged backgrounds often have advantages.* (mathematics teacher)

However, in general, respondents seem to be favourably disposed to the introduction of coursework, 'practical work', or some other form of second assessment component, in mathematics as an aid to deepening students' understanding of topics. This would also facilitate closer alignment between the syllabus aims and objectives and its assessment.

A number of respondents stressed the need for extensive professional development for teachers in relation to any second assessment component that might be developed, so that they are well prepared and the system can be assured that this is not a 'dumbing down' of the mathematics course.

Following on from earlier views on the need to move away from rote learning, many respondents in this section stressed the need for the examinations to adopt a different style of question, which is designed to assess the candidate's mathematical understanding as well as problem solving skills rather than practised procedural skills in predictable question types. This could also include

> *unseen applied questions … which test the ability of students to apply familiar techniques in unfamiliar situations.* (mathematics lecturer)

Some respondents referred to the need for continuous assessment, both as a formative tool and to enable students to obtain some credit/marks in advance of the final examination.

> *I think continuous assessment would be a brilliant idea in all subjects - particularly maths! You spend two years studying a subject and then it all comes down to what you can remember or write on a page in the space of 5 hours….* (2nd level student)

> *Sílim gur cheart measúnú leanunach a dhéanamh le linn na mbliana agus go mbeadh bfhéidir 20% nó 30% den pháipéar déanta acu roimh an Teastas Sóisearach agus Ard Teist – ní bheadh an méid céanna brú orthu… sílim go mbeadh torthaí níos airde.* (mathematics teacher)

The potential for using ICT in assessment was mentioned by a number of respondents, with some commenting on the need for greater use of ICT by teachers in the mathematics classroom before students could be expected to use it as both a formative and a summative diagnostic/assessment tool. Steps would need to be taken to ensure that its use is monitored and that external and/or moderation procedures are established. A number of submissions advocated a piloting of possible alternative assessment arrangements, perhaps in Transition Year.

If there is to be a specific assessment of basic mathematical knowledge and skills (especially in arithmetic and algebra), then ICT has a role to play in this. Students could take (and re-take, if necessary) such a test to obtain a 'satisfactory' grade, which could then be incorporated into their final result in mathematics.

Many respondents pointed to the 'fairness' of the current system and the trust that it enjoys. Any changes would need to be well researched and also well resourced, and their introduction preceded by extensive information and education of all concerned, students, parents, teachers, examiners, third level institutions, employers, etc.

3.8 Syllabus levels and range of courses

Less than half of the questionnaire responses commented on the issues raised in this section. Some referred back to earlier comments, especially in relation to provision and uptake, while others focused almost exclusively on the issue of non-recognition of Leaving Certificate Foundation level mathematics for course entry at third level. A few submissions made the point that students who are capable of studying Foundation level mathematics only are probably not suited to third level courses where there is any mathematical content.

As already noted, the lack of follow-on changes at Leaving Certificate after the revisions in mathematics syllabuses at Junior Certificate is seen as a major difficulty for both teachers and students. This needs to be addressed as a matter of urgency. Other changes can be developed and introduced more gradually. While some respondents advocated a reduction to two course levels, others saw benefit in retaining the three levels, provided a re-balancing was undertaken to bring about a more evenly spread progression across the levels. If there was to be a re-structuring of mathematics syllabuses, this could accommodate both general and specialist mathematics, with the former taking a more context-based and applications approach while the latter could have greater emphasis on ‘pure’ mathematics.

In some responses and group discussions, the status of applied mathematics at Leaving Certificate came in for consideration. The present course is seen as offering students with an interest in mathematics the opportunity to solve problems based on realistic applications (albeit mostly restricted to mechanics). If appropriate aspects of this course were incorporated into the main mathematics syllabuses, this could provide an opportunity to extend the coverage of (advanced) applications to other interesting and more recently developed areas of mathematics. ICT is seen as offering great potential in this regard.

As an alternative, it is suggested, if Higher level mathematics were made accessible to a greater number of students, following an amalgamation of the Foundation and Ordinary level syllabuses, an additional, more advanced mathematics course might be considered that would provide a challenge to more able students who were interested in further study of mathematics.

3.9 Student achievement in mathematics

This section sought respondents' views on the effectiveness of identified measures for improving student performance in mathematics examinations. Approximately two-thirds of respondents completed this section of the questionnaire. Table 3 shows the levels of agreement with each of the three effectiveness ratings for the suggested measures (respondents were invited to suggest other measures).

Table 3. Effectiveness of measures for improving the performance of students in mathematics examinations (% of responses; rounded).

Suggested measure	Very effective	Effective	Not effective
allocation of more class time to mathematics	51	35	14
better pre-service and inservice education for teachers of mathematics	64	27	9
improved mathematics textbooks and other learning resources	51	42	7
provision of learning support for students who are experiencing difficulties with the subject	73	26	1
provision of 'general' as well as 'specialist' mathematics courses	52	34	14
increased emphasis in examination questions on the application of mathematics to real-world problems	53	37	10
the introduction of additional forms of assessment, such as coursework	46	29	25
improving the perception of mathematics among parents and the general public	43	37	20

Over sixty percent of respondents completed this section. As shown in the table above, the provision of learning support for students who experience difficulty is almost universally seen as very effective or effective in improving their performance in examinations, whereas the introduction of additional forms of assessment is judged to be an ineffective measure by over a quarter of respondents.

Over ninety percent of those who responded consider that better pre-service and inservice for teachers, as well as improved mathematics textbooks and other resources, are likely to be effective or very effective in this regard, although comments elsewhere in responses point out that the 'performance' should not be equated with the technical execution of practised routines in answering questions.

The allocation of additional class time for mathematics was seen as a very effective measure by most of the respondents who had previously drawn attention to the reduction of class time for mathematics as a result of an expanded junior cycle curriculum. In some cases, teachers had advocated a minimum of one class period per day for mathematics, which they felt would allow them to develop teaching strategies that did not rely almost exclusively on rote learning of techniques and procedures.

Among the other measures suggested by respondents were

- use of ICT and other innovative teaching methods
- continuous assessment linked to overall results
- topics on the history, evolution and applications of mathematics
- greater emphasis on (and reward for) accuracy in mathematics
- reduction in class size to allow for group work and discussion
- better information on the mathematics required in third level courses
- teachers of mathematics to be qualified in mathematics
- increase in the points allocated to maths for some courses
- improving the connectivity between mathematics and other subjects
- mathematics courses for parents.

Finally, a number of respondents (in questionnaires and text messages) commented on the need to make mathematics more 'real' and to develop additional resources for students, such as web-based tutorials.

3.10 Teaching and learning in mathematics

Almost sixty percent of respondents expressed views in this section. Many focused on the need to change the way mathematics is taught; they see both syllabus revision and changes in assessment as the drivers for this. There was some repetition of points made earlier in responses regarding the need for teachers of mathematics to have a qualification in mathematics and for the availability of resources to facilitate and encourage changes in classroom practice. However, respondents did comment on the good work being done by inspiring teachers and recommended that best practice needs to be disseminated widely so that all teachers can gain insights into the varied approaches that could be taken in the mathematics classroom.

Reference was made to developments in our understanding of how students learn and the need for teachers to adapt their approach so that all students are catered for. Active teaching methodologies help to engage students more and to encourage enquiry-based, constructivist learning. A move to teach for skills development was advocated, allied to changes in assessment so that the emphasis is on students applying their mathematical knowledge and skills in answering examination questions.

> *A more hands on approach is needed involving active methodologies leading to a greater understanding of topics and an ability to apply knowledge to realistic problems. This would require a reduction in content of the courses with topics taught to a greater depth… (and) a commitment to CPD for maths teachers involving both content and pedagogy.* (mathematics teacher).

The issue of continuous assessment was also raised in the context of changing teaching and learning practice. Student confidence can be improved and learning can be reinforced if students are required to demonstrate skills in problem solving over a period of time. This can lead to the assessment of their application of such skills to unfamiliar contexts in a terminal examination paper.

> *There is definitely a place for the teaching of procedural skills and teaching for answering exam-type questions as repetition of certain mathematical skills does have merit. However, I feel there would need to be a dramatic cultural change among teachers and parents to accommodate the highly desirable shift of emphasis towards different, unfamiliar contexts. (school principal)*

The valuable support provided to Junior Certificate mathematics teachers in recent years is acknowledged by teacher respondents, who want to see this support continued. Teachers should have available to them expert tutors or guides who can assist them in changing their teaching methods, but this will take time.

There is generally good support for establishing a better balance between the 'traditional' approach and emphasis on understanding of concepts. Many students come to understand the concepts when they have achieved a measure of confidence in their own ability to follow the procedure for answering examination-type questions.

3.11 Attitudes to and beliefs about mathematics

A little over half of the respondents offered opinions on the issues surrounding perceptions, attitudes and beliefs about mathematics. There was general acknowledgment in the submission that mathematics is perceived as a difficult subject, with many students and adults lacking confidence in dealing with mathematical issues and processes. The symbolic language that is used in mathematics is off-putting for some and the fact that there is little understanding of the concepts means that, for many, any mathematical skills they have are the result of rote learning.

Respondents see a need to make school mathematics more interesting and engaging for students, through activities and learning approaches that relate the mathematics being learned more closely with students' everyday lives and experiences. Teachers should make connections between and across the often isolated topics as they are presented in textbooks. This requires that teachers be confident and have a 'big picture' understanding of the subject.

> *… students need their best maths teachers at a young age. Teachers who really know what they are doing and really understand the simplicity of what they are doing. Once confidence is in place at a young age, I think the other issues… will right themselves*. (mathematics lecturer)

Many third-level respondents commented on the need to give second-level students insights into the beauty of mathematics and to provide them with appropriate challenges so that they can experience the satisfaction that comes from solving a difficult problem or discovering an interesting aspect of a problem or solution. The quality of teaching, especially in the early years, is very significant. Some students also commented on 'good' and 'bad' mathematics teachers that they have had in school.

> *A teacher who loves mathematics will communicate this to students. Many successful mathematicians will attribute their success to the enthusiasm of teachers who inspired them to work at the subject.* (mathematics lecturer)

In discussions with parents, the point was made that many parents do not themselves have a sufficient grasp of mathematics (or the methodologies now being used in class) to enable them to help their children. Mathematics is one of two subjects (Irish is the other) where parents feel inadequate and regularly seek outside help for their children when they are struggling with homework. This begins as early as first year at second level, and reinforces the perception of mathematics as a difficult subject.

> *… provide support (to be defined) to parents who can ultimately influence student attitudes to maths and who can supervise homework assignments.* (text message)

A number of students (and some second-level mathematics teachers) commented that it is not worth the effort, in terms of getting points, to spend a lot of time at mathematics when it is easier to get high grades in other subjects. There is a large discrepancy between the effort required to achieve a grade C at Higher level and a grade A at Ordinary level in mathematics. Students who do persist with Higher level mathematics tend to be those who are very good at it or who are targeting courses or careers that rely heavily on mathematics. This contributes to the ‘elitist’ perception of the subject. A text message stated that large class size means teachers do not have the chance to give individual attention to students who are struggling with mathematics, and so the students just ‘give up’.

Some respondents suggested that greater use of ICT (computers and calculators) in mathematics classes could help students to overcome the drudgery or boredom of long computational problems or repetitive routines and allow them to gain insights into patterns and concepts so that mathematics is made more interesting and enjoyable. For the majority of students, it is argued, a focus on real-life applications will help them to relate mathematics to aspects of their own lives in ways that will hold their attention. The media have a role to play also, particularly in not perpetuating negative attitudes towards the subject.

The view was expressed that the ‘fear’ of mathematics is associated with getting the ‘wrong’ answer, and this may begin at primary school. It was advocated that students need to focus more on the processes involved in solving problems of a mathematical nature and realise that their accuracy will improve once they have confidence in the methods used. Teachers can help by pointing out that a variety of approaches may often be possible, all of which are valid and will give the correct result. This requires teachers themselves not to have an absolutist view of the subject. In this regard, one respondent quoted the Russian mathematician Kolmogorov:

> *"Mathematics can be taught really well only by a person who himself is fascinated by it and perceives it as a living, evolving science."*

One third-level college introduced a ‘primer’ mathematics course for its Higher Certificate Electronic Engineering course as an intervention aimed at bringing students back into contact with mathematics in a manner that will boost their confidence, as well as being a foundation for the main course. A lecturer in the college noted that, in particular, many mature students returning to education have a negative ‘mathematical self-image’. Material for the primer course is chosen carefully so that students are engaged initially in concrete concepts related to number manipulation, then progressing to simple symbol manipulation and finally to realistic applications of the concepts and processes. The course is structured in such a way that students can learn at their own pace and in an individually supported manner.

Not all students need to undertake this primer course; a diagnostic test, which has no bearing on student grades, has been developed to determine students’ needs. Other third-level colleges also report successful operation of mathematics learning centres and/or tutorial programmes in mathematics for students who find it difficult to make the transition from the rote-learned mathematics that they experienced at second level. In some of these colleges diagnostic testing is also carried out on the first-year intake.

Respondents pointed to the need for students at second level to be given opportunities in class to build confidence in their own ability at mathematics, building on what they know rather than presenting them with a set of procedures to practise when answering questions. It was suggested that second-level mathematics teachers might benefit from training in diagnostic testing.

3.12 Other influences affecting mathematics

Many of the responses under this heading referred back to previous comments they had made regarding attitudes and beliefs, the pressure to get ‘points’ at Leaving Certificate, and the need to make greater use of ICT in the teaching and learning of mathematics. Some respondents favoured the return of ‘bonus points’ so that more students would be encouraged to study mathematics at Higher level. The status of Foundation level mathematics at leaving Certificate also surfaced again in responses.

The lack of coherence and appropriate progression between primary and post-primary mathematics, and between the different syllabus levels in post-primary mathematics were again pointed to as being problematic. There was a repeated call for immediate action to be taken at Leaving Certificate in light of the changes that have been introduced at Junior Certificate, and for professional development and support for teachers to be maintained. It was recommended that opportunities for teachers to support their peers should be facilitated and encouraged. Mention was made of the wealth of web-based material that is suitable for use in mathematics classes, but teachers need to be made aware of it and be supported in using it appropriately. Some applications can ease the workload of the teacher by providing students with opportunities to practise (and assess) their technical skills in problem solving and to be facilitated in some measure of self-directed learning.

There was very little comment on some other issues that had been identified in this section of the Discussion Paper: cultural, equality, gender (uptake and achievement), socio-economic factors, educational disadvantage, and students with disabilities or special educational needs.

Additional comments

Almost one-third of questionnaire respondents provided additional comments, many of these using this section to summarise or re-state their main concerns and recommendations. A number of respondents used this opportunity to look forward to what they would like to see happening in mathematics education in relation to syllabus revision or development, assessment arrangements, accreditation, and teacher professional development.

> *If we wish to see the vast majority of our 15-18 year olds continuing to study mathematics then understanding and an ability to apply*

> *knowledge rather than an ability to follow procedures should be the benchmark. Syllabus review by itself will not achieve this. The culture in the maths classroom needs to be changed. This needs to be supported by time for interaction and reflection for maths teachers and a proper support service to foster and inform this change.* (mathematics teacher)
>
> *What is needed is a slow but steady change in the way mathematics is taught in Ireland today. Most of the relevant issues are adequately covered in the discussion paper, however now the real challenge lies in addressing these issues. This will require a long look at best practices in other countries, supply of adequate resources to teachers especially in the form of training and materials to aid education. Making maths relevant to the students is one area that needs urgent attention at all levels from primary to third level.* (mathematics lecturer)

By way of contrast with their earlier comments, which stressed the urgency of addressing the issues in mathematics education, in this section respondents acknowledged that it will take time and will require a range of actions to be undertaken. All respondents were in agreement on one point – change is required; there was not the same measure of agreement on the extent or nature of that change.

Responses in this section also contained a number of specific suggestions for addressing identified issues. These included

- combining Foundation level and Ordinary level syllabuses, with a greater emphasis on ‘practical’ mathematics and reflect this in the assessment arrangements
- introducing an element of coursework that would involve students researching topics outside the narrow confines of the syllabus, thus enabling them to see how/when mathematics is applied in real-world contexts

- making examination questions less predictable; including ‘unseen’ questions/problems so that students are required to apply familiar techniques in unfamiliar situations
- introducing some form of continuous assessment (perhaps piloting this in schools)
- providing the resources and training for teachers to make greater use of ICT in mathematics; computer-based assessment for some aspects of mathematics should be investigated
- supporting mathematics teachers with short courses, developed by the relevant authorities and interest groups and delivered by personnel with in-depth knowledge and experience of teaching the subject, so that they can renew their enthusiasm and reflect on their understanding of mathematics and how they teach it in their classes
- re-visiting the issue of the mathematics requirement for entry to courses at third level that do not have particular mathematics content; if necessary, including at OL and FL topics that students need to have covered before going to third level
- researching best practice elsewhere and adopting whatever would improve mathematics education here
- commissioning the development of resources (CDs, web, handbooks, etc.) for teachers so that they can adopt different teaching methods
- encouraging more mathematics graduates to take up teaching
- finding out the real-life contexts for mathematics that students are interested in
- making more use of an assessment for learning approach in class
- considering the re-introduction of bonus points for Higher level mathematics in respect of some third-level courses.

4. Reflecting on the issues

The consultation set out to stimulate discussion of a range of issues affecting post-primary mathematics education; in this it has certainly succeeded. The responses received reflect a wide variety of perspectives on mathematics and mathematics education, ranging from mathematics in the primary school, through second and third level, encompassing its past and present developments as an area of human interest, and touching on its current and potential applications in the world in which we live.

Many individuals and groups have given serious thought to the issues raised in the consultation, and communicated their views in a manner which is both an honest recognition of the challenge and an expression of hope that serious consideration will be given to this important area of learning. The concerns surrounding mathematics education arise not only because of its importance in the development of human knowledge and skills, but also because of the high esteem in which it is held within and outside of education in Ireland and the potential contribution that it can make as Ireland aspires to become a knowledge-based society.

The review has provided the first opportunity for almost forty years to take an in-depth look at mathematics education and there is broad welcome for this. Many respondents point to the discontinuity that now exists between mathematics education in the primary school and that at second level. The philosophy underpinning post-primary mathematics, which has its origins in the 'modern mathematics movement' of the sixties is at odds with the realistic mathematics education (RME) philosophy which underpins the revised curriculum for mathematics in primary schools. The review affords the opportunity to take a 'bigger picture' look at mathematics education so that that students will experience a continuum of mathematics learning over their years in formal education.

The remainder of this section presents a summary and analysis of the issues in light of the responses received during the consultation.

There is a general recognition – and expectation – that change is required in the mathematics education which students experience in post-primary school. The discussion documents and the consultation responses present the rationale for change in respect of four broad aspects of mathematics education which are inter-related: the mathematics curriculum, teaching for and learning with understanding, assessment, and the culture that surrounds mathematics. Other issues identified in the NCCA discussion paper which elicited a range of responses as presented in the previous section of this report, can be considered under these four headings.

4.1 Looking at curriculum

Although not a compulsory subject to Leaving Certificate, mathematics is studied by the vast majority of post-primary students, most of whom remain in school until the end of the senior cycle. Internationally, the proportion of students who study mathematics in upper-secondary education is comparatively lower than is the case in Ireland. Students' performance in Leaving Certificate mathematics plays a significant role in progression to third-level education as well as in preparing them for their future personal, social and working lives. This has given added weight to the concerns that have arisen over the past number of years in relation to their mathematical knowledge and competence.

A traditional mathematics curriculum allied to a relatively narrow emphasis on the development of de-contextualised procedural skills and the backwash effect of examinations on teaching and learning have meant that many students leave school with only a superficial understanding of the subject and little or no conceptual knowledge. While this may serve them well in the short term, that is in terms of examination results, its longer-term effect is one of limited value in respect of their future mathematical needs.

The Junior Certificate mathematics syllabuses were revised in 2000, but this was more a minor adjustment of content than a genuine revision and was followed (rather than informed) by a move to focus on teaching for understanding. Although arising just after the introduction of the revised primary school curriculum in 1999, the revised syllabuses for Junior Certificate mathematics did not adopt the emphasis placed in primary school mathematics on developing mathematical ideas in real-life contexts and their application to real-life problems. They were still very much a content-based presentation of mathematics that was almost totally devoid of any meaningful context. Leaving Certificate mathematics syllabuses are of a similar style, not having been revised since the early nineties.

Many of the submissions point to the need for a complete overhaul of the post-primary mathematics syllabuses due to a number of factors:

- discontinuity with the mathematics curriculum in the primary school and the 'gaps' that have arisen at Leaving Certificate as a result of the syllabus changes at Junior Certificate
- over-emphasis on technical, procedural skills at the expense of conceptual understanding
- the lack of context for much of the syllabus content and the virtual absence of reference to applications of mathematics in real life
- the perception that mathematics is a difficult subject in which only the intellectually talented students can expect to succeed
- questions arising with regard to the appropriateness of the three syllabus levels and the mismatch between the aims and objectives of the current syllabuses and their assessment
- the identified mathematical deficiencies of students emerging from post-primary school.

A large number of submissions indicate topics in mathematics that should be considered for inclusion in revised syllabuses, and some that could be considered for reduced treatment or removal altogether. Some of the respondents have suggested that two different types of mathematics courses are required to accommodate the needs and abilities of all students, including their future mathematical needs in terms of their further studies and careers.

4.2 Looking at teaching and learning

The teacher is the critical agent of change. Syllabus changes can indicate a change in content, emphasis and approach to mathematics education and can set out appropriate strategies and structures for its assessment, but the context and conduct of the teaching and learning situation is critical to effecting a change in the way students develop their knowledge, understanding, skills and attitudes in mathematics.

International trends suggest that an approach which develops from the concrete to the abstract and which presents mathematics in a context that relates it to real-life situations is likely to engage students' interest and enable them to develop their knowledge and skills to an appropriate level. Many of the respondents to the consultation pointed to the need to make mathematics more related to the lives of students, to let them see how it applies in real-world contexts and how it enables them to develop their thinking and problem-solving skills. They need to realise that mathematics is not an unconnected series of procedures whose meaning and logic are impenetrable to all but a small minority, but that it has an integrity and beauty in itself as well as myriad applications in their daily lives.

There is an acknowledgement that this requires a fundamental change in the way in which many teachers teach mathematics, but such a major change is required if students are to learn with understanding. Some respondents (teachers and lecturers), referring to the findings from research on the impact of change on teaching practice, argue that this change will take time, that it needs to be well resourced, and that sustained teacher professional development and support will be required for it to become embedded in practice.

A major concern in the responses is the level of pedagogical content knowledge that some mathematics teachers have, particularly those for whom mathematics is not their principal subject and who may not have any third-level qualification in mathematics. This concern also extends to teachers in primary school, where some of the students' perceptions of and attitudes to mathematics are formed. It has been suggested that serious consideration needs to be given to both pre-service and inservice aspects of mathematics teaching. Lecturers in mathematics education who responded to the consultation commented on the difficulty that teachers experience in teaching mathematics in a manner that is qualitatively different from the way they themselves learned mathematics.

The use of ICT in mathematics can dramatically change the students' engagement with the subject. The use of computers and calculators acts as a motivator and increases productivity, overcoming the tedium and boredom of repetitive procedures and computation and allowing for a level of intuitive exploration that was impossible heretofore. The potential of this technology needs to be recognised when syllabuses are being developed.

The backwash effect of examinations on teaching and learning was noted in many of the comments from students, parents, teachers and lecturers. In recognising this, there needs to be a closer alignment between learning and assessment. Assessment for learning has a significant role to play in this regard but, once again, the need for teacher professional development was highlighted.

The following were among the suggestions made by respondents in relation to the teaching and learning of mathematics.

- Undertake an audit of the mathematics qualifications of teachers of mathematics in post-primary schools; identify the needs of teachers in terms of professional support and put measures in place that will address these comprehensively.
- Develop teaching resources that will facilitate teachers in changing their teaching methods where necessary and support teachers in making this change.
- Prepare examples of well-structured contexts and applications that relate mathematics to the lives of students and to real-world situations (these need to be genuine, not pseudo-applied) and that will stimulate discussion and exploration by the students.

4.3 Looking at assessment

Throughout the various sections of the questionnaire responses and in the many discursive submissions that were received, the centrality of the examinations and their dominant influence on teaching and learning was evident. This was also recognised in the discussion groups, where some suggestions were made for steps that could be taken to alleviate the situation. However, there was no agreement on whether such suggestions would gain the necessary support. In the absence of alternative forms of assessment, students and teachers view preparation for the terminal examination as the focus of much of the learning in mathematics. The predictable nature of the examination papers, in many cases at the level of the individual question, reinforces—some would argue rewards—this focus.

In meetings with parents and, to a lesser extent, with teacher groups, the possibility was discussed of a common examination that would be aimed at assessing basic mathematical competency. This examination, which students could undertake as often as necessary to obtain a satisfactory rating, could be operated independently of the final examination (Junior or Leaving Certificate). It could provide a student with a certificate of competency, a sort of 'mathematics driver's licence', which would give assurances to the system that those who obtained this certificate had demonstrated a minimum level of knowledge, skills and application in particular aspects of mathematics. Much of this learning and assessment could be technology based, allowing students to progress at their own pace and to build up their competency as well as confidence in their ability to 'do' the mathematics. Students who experience difficulty in making progress could be identified easily and given individual support to overcome the particular difficulty. In this way, individual barriers to progress could be surmounted.

The focus on examinations that prevailed in most of the responses is in itself indicative of the need to broaden the debate about assessment and the different roles that it can and should play in teaching and learning. The benefits of some form of continuous assessment were mentioned in many submissions and responses, although it was not always clear what the respondent meant as ‘continuous assessment’. In some cases, this appeared to refer to the undertaking of coursework over the two or three years of the course, which would be assessed towards the end of the cycle and the result combined with that obtained on the examination paper(s). In other cases, it seemed to mean a number of assessment events spread over the period of the course, each contributing a part mark towards the final grade. Some respondents expressed reservations about the former, indicating disadvantages that accompanied it, while others were not in favour of the latter due to its lack of flexibility for accommodating different rates or stages of intellectual development.

Very few teachers indicated that they used forms of assessment other than traditional tests to provide their students with formative feedback on their knowledge and understanding of mathematical concepts. This is an area which needs to be addressed. Assessment for learning initiatives currently being undertaken by the NCCA may provide insights into ways in which teachers can be assisted in making greater use of this approach in mathematics.

As mentioned above, and also in conjunction with teaching and learning, there is great potential for the use of ICT in assessing specific aspects of mathematical knowledge, understanding and skills. At present, initiatives of this kind are mainly confined to calculator and/or computer-based tests that focus on routine computational and procedural ‘drill and practice’ learning in mathematics. However, computer software is available that assesses logical reasoning and problem-solving skills in a mathematics context, which might easily be adapted to encompass a range of syllabus topics.

4.4 Looking at the culture of mathematics education in Ireland

Despite the high regard in which mathematics is held in this country—or perhaps because of it—the perception of mathematics as a difficult subject persists. The majority of respondents acknowledge this and agree that this perception needs to be challenged and changed if mathematics education is to achieve its potential in contributing significantly to the development of both the individual and society.

A few respondents noted that there are some indications that the changed approach to more active methodologies in mathematics classrooms in primary schools is having an effect on pupil's attitudes to the subject. This is supported by the findings of the Primary Curriculum Review, which reported very positive attitudes to mathematics among children who participated in the case study schools. As primary school pupils progress to the post-primary junior cycle, it will be important that these attitudes are reinforced. More needs to be done to ensure greater coherence between the approaches in mathematics education at both of these levels. As mentioned elsewhere in this report, the mismatch in the underpinning philosophies as well as in the classroom experiences are barriers that must be overcome.

Parents indicated that their own perceptions of mathematics as being difficult are picked up on by their children, especially as they progress through second level. There is a role here for the wider community to adopt a more positive attitude and to be conscious of how negative messages regarding mathematics can reinforce already poor attitudes.

Teachers need to be encouraged to adopt more imaginative approaches in their teaching and to encourage their students to 'make sense' of the mathematics they learn. This in turn will require them to have confidence in their own mathematical ability as well as in their teaching of the subject. Once the student's confidence is established at a young age, they will not be daunted by the more challenging aspects of the subject as they progress through school. A number of respondents recommended that the 'best' mathematics teachers in a school should teach at first year so that a solid foundation in mathematical knowledge and skills is laid at that stage. Some lecturers point to the success of mathematics tutorial or assistive initiatives in overcoming the negative attitudes of third-level students to the subject, and cite examples of students who emerge from second level with relatively low grades in Leaving Certificate mathematics, yet who manage to graduate with the highest honours in courses that have a significant mathematical content.

Mathematics needs to be seen as a subject which, although it may present challenges, provides the means for overcoming these challenges. Many respondents cautioned against falling into the trap of ‘dumbing down’ the subject through the removal of essential mathematical concepts from syllabuses or by a lowering of the standards required in the examinations (particularly the Leaving Certificate). Rather, they suggest, efforts should be made to support students in their learning of mathematics by identifying the problems they experience and developing methods of overcoming them. In this way, their confidence and self-esteem will be enhanced.

5. Towards effecting change

The most significant finding in the consultation is the broad agreement that teaching and learning practices have the greatest influence on students' understanding of mathematics. Teachers, and their own attitudes to mathematics, have a major role to play in student attitudes to and perceptions of the subject.

Syllabus change will not of itself bring about a transformation of student attitudes and perceptions or in their mathematical knowledge, skills and understanding, or result in improved examination results. While it is recognised that syllabus change is needed, as is a change in the assessment arrangements—including the examinations—it is in teaching and learning the 'why' and 'what if' of mathematics as well as the 'how' that deeper understanding of mathematical concepts can be achieved. This will, in turn, bring about a change in the student perception of, and attitude towards, mathematics.

A dominant theme in the consultation feedback is the need to make mathematics more related to the lives of students. Two main approaches are advocated for this:

(i) introducing mathematics concepts in real-life contexts so that it leads from the concrete to an appropriate level of abstraction

(ii) highlighting the many ways in which mathematics is applied in the real world.

It has been pointed out that this is not a straight-forward or simple process, not can it be expected to succeed in a short time-frame. Appropriate contexts and applications need to be selected carefully; their ability to underpin the mathematical ideas and processes being taught and learned must be assured. Teachers must be confident in their own ability to use contexts and applications in the class situation and in the effectiveness of such an approach to facilitate student learning and understanding.

The development of a range of teaching resources that can be drawn on in planning classroom activities is critical to its success. A collaborative effort between experienced teachers (primary and post-primary) and third-level lecturers (of mathematics and mathematics education) may provide a means of selecting or developing appropriate exemplars. A representative sample of students should be consulted to ensure that the contexts used are indeed of interest or relevance to them.

ICT is seen as having a significant role in enabling greater exploration and investigation in mathematics, both within and beyond the classroom. Its potential as a tool for teaching and learning remains to be exploited fully, although some progress is being made in this area as a result of the teacher support programmes put in place in recent years. Teachers are gradually availing of this technology (advanced calculators and computers) to enhance students' learning of mathematics. The ICT framework being developed by the NCCA can provide a means for teachers to both explore and develop exemplars of ICT use in mathematics classes.

Most of the responses in the consultation identified the rote learning of mathematics, particularly of procedural techniques specifically aimed at answering predictable types of examination questions, as one of the major problems that must be overcome. Textbooks were mentioned as narrowing down mathematics to this feature. Some respondents, mainly mathematics lecturers and lecturers in mathematics education, stressed the need to shift the emphasis in procedures from drill-and-practice routines to a deeper understanding of the processes involved in solving problems and the varied strategies and approaches that can be adopted to achieve a solution.

Students need to be given insights into the reasoning behind the kinds of decisions that are taken in devising solutions to problems, and to see how 'real' and 'live' mathematics can be. Teachers can also facilitate this by encouraging class discussion, by brainstorming ideas, and by considering alternative approaches to the solution of similar problems.

As noted by many of the respondents in the consultation, change is also required in the manner in which learning in mathematics is assessed. There needs to be debate on how the development of basic mathematical knowledge and skills can be supported and assessed. The potential of alternative forms of assessment, and assessment events, should be explored and the role that ICT might play in these should be investigated.

Potential avenues for change need to be explored, which base change on the findings of research. They should take an inclusive approach so that all involved, students, teachers, parents, and the wider community and education interests can work together to bring about much-needed improvement in how students engage with mathematics and in their understanding of it.

The remainder of this section sets out a number of short-term and longer term steps that can be taken to re-shape the teaching, learning and assessment of mathematics. These are grouped under the four main headings identified previously.

5.1 Changing curriculum

The opportunity presented by the current re-balancing exercise for Junior Certificate syllabuses (in which mathematics is included) should be availed of to reflect, especially in the learning outcomes, the change in approach to one of teaching for and learning with understanding. This emphasis had not been a driver of the syllabus change in mathematics that took place in 2000, but was subsequently the focus in much of the implementation support for teachers that accompanied the introduction of the revised syllabuses. Added value can be obtained if this is combined with the context and applications approach suggested earlier, and accompanied by focused support for teachers.

The types and range of mathematical skills that students are expected to develop through their study of mathematics need to be identified. These could be categorised (computational, graphical, communication, procedural, analytical, logical reasoning, problem-solving, decision-making, etc.) and syllabus topics and learning outcomes could be selected that will support and emphasise the development of these skills. Resources should be developed that teachers can use in the classroom and teachers should be supported in adopting a changed approach, where necessary. On an ongoing basis, syllabus revision that is deemed necessary can be informed by developments along these lines, and can move the focus away from content, facilitating greater emphasis on contexts, applications and skills.

Another aspect of mathematics that featured prominently in submissions was the need for students to realise and appreciate the many ways in mathematics is applied in the real word. Although applications of mathematics are myriad, it is important that those chosen for reference in class should be appropriate to the students' level of knowledge and understanding and should underpin the particular topic(s) being considered. As an initial step, links should be identified between mathematical concepts and processes as they arise in the mathematics class and the same topics (or their applications) as they arise in other subjects.

The integration of applications of mathematics into the mainstream subject will mean that the position and nature of the syllabus for applied mathematics at Leaving Certificate must be considered. This should take into consideration the potential for short courses in one or more specialised areas of mathematics and its applications. The current Leaving Certificate applied mathematics syllabus, which has not changed for over thirty years and is studied by a small proportion of Leaving Certificate students, is almost exclusively focused on mechanics topics from mathematical physics. Many other applications of mathematics have emerged in recent times, facilitated in no small way by computer technology which itself is based on mathematical principles.

Post-primary syllabus change in mathematics, especially in the junior cycle, must build on the changes that have taken place in the primary school curriculum so that there is greater coherence between the two levels. Students and teachers must be able to see a progression in the mathematics that is taught and learned. The ‘gap’ between junior cycle and senior cycle syllabuses also needs to be addressed. Of course, syllabus change *per se* will not transform the mathematics experiences of students. Their engagement with mathematics in the classroom is much more than learning what is set down in the syllabus.

Much attention in responses to the consultation was focused on the impact that lack of recognition for Foundation level Leaving Certificate mathematics has on student uptake at Ordinary level and, consequently, on examination results. If the suggestions outlined in this section of the report are implemented and evaluated, any subsequent revision of syllabuses can take into account whether the continued existence of three syllabus levels is either desirable or necessary.

5.2 Changing teaching and learning

If change is to occur in the mathematical experiences of students, then teachers will need to consider ways in which the approaches they use in class can become more effective in providing the kinds of experiences that will engage students. Bringing about a change in teaching practice is a challenging process that needs to be managed carefully. Over a period of time, teachers have built up and honed their repertoire of resources and teaching strategies, which have proven successful with successive cohorts of students. It is important to appreciate that changing practices will involve extending the teachers’ comfort zones to embrace new approaches, and that they will need to be supported in doing this.

Teaching involves building on students’ existing knowledge, understanding and skill. A problem that has constantly faced teachers in first year at second level is that of knowing what it is that students have learned at primary school and how well they have done so. Using standardised tests, all primary schoolchildren will be tested twice on their mathematical achievement. The NCCA is working on the development of report card templates and on supporting the process of transfer of information on student progress from primary to post-primary schools. Post-primary teachers will thus know more about the mathematical achievement of primary schoolchildren as they enter second level. However, they will need support in using this information effectively. Opportunities for primary and post-primary teachers to engage in discussions about mathematics and the teaching methodology employed at each level should be both encouraged and facilitated.

To bring about a change in the teaching of post-primary mathematics, two specific approaches have already been identified: a move to more context-based mathematics (including its applications) and an emphasis on skills development. To gauge the suitability and effectiveness of a context-based approach to teaching and learning, a number of mathematics topics in the current syllabuses could be selected and some exemplar lessons developed for teachers to use in class. The support services, the subject association (IMTA) and the emerging teacher professional networks operating through Education Centres could be involved in providing support for teachers in adapting and using these resources in their teaching of mathematics. This initiative could focus on junior cycle mathematics for a defined initial period.

In their review of the literature on international trends in mathematics education, which the NCCA commissioned, Conway and Sloane described in detail a lesson study initiative which is proving very successful among teachers in Japan. This involves groups of teachers collaborating in the development of specific lessons which are then implemented in class. The group reviews the lessons, refines them in light of practice, and puts them through a repeat cycle of implementation and evaluation. Further iterations are undertaken until the group is satisfied with the quality of the lessons. This method of professional collaboration could be tried on a pilot basis in conjunction with the proposals outlined in 5.1 above. The development of appropriate assessment procedures and mechanisms for classroom use would enable teachers to gauge the extent to which students have developed their mathematical understanding and skills in these topics, and at the same time allow them to evaluate the effectiveness of their own teaching using this approach.

By limiting the number and variety of topics involved, teachers could gain confidence in the new approach and, perhaps, begin to devise ways of using it in other mathematics topics. Such an initiative could also inform, and be informed by, a process of ongoing syllabus review, in an alternative model for syllabus revision that involves teachers as action researchers, contributing as professionals to subject development. Teachers should be encouraged to see themselves as learners, adopting and adapting a variety of strategies in their teaching. Where students and teachers engage in open discussion of mathematics, this can be to the benefit of both. Students can gain insights that they might not otherwise obtain, and teachers can gain a better understanding of the students' thinking.

Students will not necessarily make the connections within and between subjects, so it is important that the teacher does this with them. As mentioned already in connection with a move to include relevant applications of mathematics, this will require collaboration between teachers at the school level. Some respondents suggested the development of support materials and/or short courses for teachers so that they can improve their familiarity with such applications of mathematics, thus building their confidence in referencing these in class. Here again, the Japanese lesson study approach referred to earlier comes to mind.

5.3 Changing assessment

Much has already been said regarding the need to reform the assessment of mathematics, including the development of complementary assessment components, not least because of the backwash effect of the examinations on teaching and learning. The benefits of assessment for learning (AfL) in building students' confidence in their own learning and in the early detection and solution of difficulties which students experience must become more widely appreciated and accepted. The current NCCA initiative in AfL includes mathematics. If this is extended in conjunction with a context and applications based approach at Junior Certificate, as suggested above, all teachers can engage in formative assessment as part of a developmental process. Feedback provided through AfL can also help to reassure students that appropriate standards are being attained or to identify areas of knowledge or skills that require attention if the student is to achieve a particular target.

A greater focus on skills and applications of mathematics in both the curriculum and classroom practice must be supported and reinforced in the examinations. Many respondents pointed to the backwash effect of the examinations, particularly due to the predictable nature of the questions, resulting in classroom emphasis on practising routine procedures that can be reproduced in the examination. They advocate the inclusion of questions that require students to apply their knowledge and skills in unfamiliar contexts (through 'unseen' questions). However, this should not be the only way in which a changed teaching and learning approach is recognised in the examination papers. There needs to be a move away from a focus on demonstration of routine procedures in other questions also.

Syllabus changes will need to be reflected in changes to the assessment procedures, including the examination papers. Closer liaison with further and higher education in syllabus revision and in teaching and learning initiatives would ensure confidence in the revised arrangements from the point of view of progression to courses at third level.

5.4 Changing the culture surrounding mathematics

The review of mathematics undertaken by the NCCA has generated much discussion and debate, but this has mainly been within the education community. There is a need to widen the debate and to challenge the traditional, negative views of mathematics. Students need to be given both the opportunity and the means to achieve their potential in mathematics, so that they can have a better self-image and confidence that they can 'do' mathematics. They need to see the subject as a set of challenges that they can meet and overcome rather than ones they should avoid.

The rationale for and nature of the changes to be implemented will need to be publicised. Parents, especially, will need to be informed of the changes and assisted in understanding their implications. They will need to be facilitated in helping their children to develop their mathematical ability. Changing attitudes and perceptions will involve a long-term campaign. It took time for the present situation to develop; it will take time to reverse that development.

Children in primary school are experiencing a different kind of mathematical education from that experienced by their parents. The Primary Curriculum Review indicates that children are enjoying the active engagement with mathematics and the methodologies being employed in class. This must be continued and reinforced at second level and teachers of mathematics at both levels must be supported in adopting practices that will ensure its continuation. The suggested approaches outlined above, if implemented fully, will go a long way to ensuring that for students their years in post-primary education and, more specifically, their experiences in mathematics will also prove to be enjoyable.

In the past, significant numbers of students have emerged from post-primary education with negative memories of mathematics as a result of ‘failure’. These can be overcome, as evidenced by successful interventions at third level, but the problem must be tackled where it arises – in primary and post-primary mathematics classrooms.

Looking ahead

The next phase of work under the review is about effecting change. This includes not only curriculum and assessment change but also engaging with students, teachers and schools, and those in further and higher education, in developing and implementing new approaches in mathematics education that will impact positively on the experiences of post-primary students.

Ireland has a long tradition of student engagement in mathematics education throughout the post-primary years, unlike in many other countries where a much smaller proportion of the student cohort in upper secondary education studies mathematics. The concerns that exist about the level of knowledge and skills among school leavers can be harnessed into support for the steps needed to bring about improvement, for the benefit of the individual learner, the learning community and society generally.

Appendix 1: Mailing list for the consultation

Groups or Organisations

All Post-primary Schools

An Chomhairle um Oideachas Gaeltachta agus Gaelscolaíochta

Association of Community and Comprehensive Schools

Association of Primary Teaching Sisters

Association of Secondary Teachers, Ireland

Association of Teachers' Centres in Ireland

CDVEC Curriculum Development Unit

Centre for Early Childhood Development and Education

Chambers of Commerce of Ireland

Chief Executive Officers of Vocational Education Committees

Church of Ireland Board of Education

Co-operation of Minority Religions and Protestant Parents' Association (COMPASS)

Curriculum Councils for England, Scotland, Wales and Northern Ireland

Department of Education and Science

Directors of Education Centres

Economic and Social Research Institute

Educate Together

Education Departments in Third-Level Colleges

Educational Research Centre

Engineers Ireland

Federation of Catholic Brothers and other Catholic Schools Parent Councils (FED CBS)

Forfás

Heads of Mathematics, Science and Engineering Departments in Universities and Institutes of Technology

Heads of Universities and Institutes of Technology

Industrial Development Authority

Institute of Public Administration

Irish Business and Employers Confederation

Irish Mathematics Teachers' Association

Irish National Teachers' Organisation

Irish Primary Principals' Network

Irish School Heads' Association

Irish Small and Medium Enterprises Association

Irish Universities Association

Joint Managerial Body

Junior Certificate Mathematics Support Service

Mater Dei Institute of Education

National Association of Principals and Deputy Principals

National College of Art and Design

National Education Office for Travellers

National Educational Psychological Service

National Parents Association of Vocational Schools and Community Colleges (NPAVSCC)

National Parents' Council (Post-primary)

National Qualifications Authority of Ireland

NCCA Council Members

NCCA Senior Cycle Committee

NCCA Junior Cycle Committee

NCCA Committee for Junior Certificate mathematics

NCCA Committee for Leaving Certificate mathematics

NCCA Committee for Leaving Certificate applied mathematics

Office of the Chief Science Advisor

Parents Association of Community and Comprehensive Schools (PACCS)

Primary Curriculum Support Programme

Queen's University Belfast

Royal Irish Academy

Second Level Support Service

Shannon Curriculum Development centre

St. Mary's University College, Belfast

State Examinations Commission

Teachers' Union of Ireland

Union of Students in Ireland

University Libraries

University of Ulster

Individuals

Aidan Savage	National Co-ordinator, School Completion Programme
Aideen Cassidy	National Co-ordinator, Junior Certificate School Programme
Alan Gilbert	COMPASS
Alan Mulligan	COMPASS
Anna Walshe	National Co-ordinator, Junior Certificate Science Support Service
Anne Marie Ryan	National Co-ordinator, Leaving Certificate Home Economics
Anne O'Sullivan	Castlebar, Co. Mayo
Brendan Duane	National Co-ordinator, Leaving Certificate Chemistry
Christy Tyrrell	NCCA Education Officer, Accounting
Ciarán O'Sullivan	Lecturer, Institute of Technology, Tallaght
Connie Carolan	PACCS
Daithí Mac Sithig	Union of Students in Ireland
Diane Birnie	Lucan, Co. Dublin
Eamonn Sheppard	Thurles, Co. Tipperary
Eileen Flynn	National Co-ordinator, School Development Planning
Eleanor Petrie	COMPASS
Elizabeth Oldham	Trinity College, Dublin
Frances Holohan	National Co-ordinator, Leaving Certificate Vocational Programme
Geraldine Horgan	NCCA Education Officer, Junior Certificate Science
Geraldine Mooney Simmie	University of Limerick
Gerard O'Reilly	FED CBS
Geraldine Perkins	FED CBS
Gerry Nolan	Intel Ireland
Humphrey O'Riordan	PACCS
Jerry Shiel	Educational Research Centre
Jim Jackman	PACCS
Joan Crowley-	National Co-ordinator, Special Education Support

O'Sullivan	Service
Joe Kennedy	NPAVSCC
John Mulcahy	NCCA Education Officer, Geography
John O'Donoghue	University of Limerick
Joyce Ryder	COMPASS
Louise Holden	The Irish Times
Margaret Donohue	PACCS
Marion Palmer	Institute of Art, Design and Technology, Dún Laoghaire
Maureen Connolly	NPAVSCC
Michael O'Leary	National Co-ordinator, Transition Year Support Service
Paddy Flood	National Co-ordinator, Leadership Development for Schools
Pádraig Ó Siochrú	Daingean Uí Chúis, Co. Chiarraí
Pat Murphy	PACCS
Pat O'Connor	Coachford, Co. Cork
Pat Younger	NCCA Education Officer, Economics
Patricia Forde-Brennan	NPAVSCC
Patricia O'Malley	COMPASS
Paul McElwee	St. Catherine's College, Sion Hill
PJ Garvan	NPAVSCC
Rose Tully	NPAVSCC
Sean Close	St. Patrick's College, Drumcondra
Sheila O'Driscoll	National Co-ordinator, Leaving Certificate Applied
Sinéad Breathnach	National Co-ordinator, School Development Planning Initiative
Tim Regan	National Co-ordinator, Leaving Certificate Physics
Tom Geary	University of Limerick
Tom Mullins	University College, Cork
Valerie O'Dowd	Assistant National Co-ordinator, PCSP
Vincent Brett	FED CBS

Appendix 2: List of respondents

Aileen Clancy	Teacher (other subject)
Alex Hogarty	3rd level student
Alfie O'Doherty	Lecturer (mathematics)
Amanda Fennell	Teacher (mathematics)
Andrew Wood	Teacher (mathematics)
Ann Allen	Teacher (mathematics)
Ann Vereker	Lecturer (mathematics) WIT
Anne Brosnan	Teacher/Researcher NUIM
Anne McNamara	Teacher (mathematics)
Aoife Maher	2nd level student
Association of Community and Comprehensive Schools	
Association of Secondary Teachers, Ireland	
Ayla Tuohy	3rd level student
B McDonnell	Teacher (mathematics)
Bernadette McLean	Teacher (mathematics)
Bernie McMahon	Teacher (mathematics)
Bernie O'Callaghan	Teacher (mathematics)
Bertie Keely	Teacher (mathematics)
Br Thomas Hickey	Teacher (mathematics)
Breda Collins	Teacher (mathematics)
Breda Doherty	Teacher (mathematics)
Breda Fallon	Teacher (mathematics)
Breda Morrissey	Teacher (mathematics)
Brendan Kelly	Teacher (mathematics)
Brendan McGill	Teacher (mathematics)
Brendan O Sullivan	Teacher (mathematics)
Bríd Galligan	Teacher (mathematics)
Brien Nolan	Lecturer (mathematics)
Cammie Gallagher	Teacher (mathematics)

Carmel McGee	Teacher (other subject)
Cathal Jordon	Teacher (mathematics)
Catherine O Donnell	Parent
Catherine Roddy	Teacher (mathematics)
Catherine Walsh	Teacher (mathematics)
Christine Dunne	Teacher (mathematics)
Christy Maginn	Teacher (mathematics)
Ciaran O'Sulllivan	Lecturer (mathematics) IT Tallaght
Claire Thomas	Teacher (mathematics)
Claire O Neill	2nd level student
Coláiste Íde	School Principal
Colm Doyle	3rd level student
Colm McGuinness	Lecturer (mathematics)
Colm O Connor	School Principal
Conall Kelly	Other
Diarmuid Lalor	Teacher (mathematics)
Daithí Ó Máirtín	Múinteoir (matamaitic)
Damian Cooke	SEC
David Doyle	Lecturer (mathematics)
David Flannery	Lecturer (mathematics)
David Hobson	Teacher (mathematics)
Declan Casey	Teacher (mathematics)
Declan Dunne	Teacher (mathematics)
Declan McConnell	Other
Deirdre Barry	Teacher (mathematics)
Deirdre Gardiner	Teacher (mathematics)
Deirdre O Halloran	2nd level student
Denis Dunne	Parent
Dominic Guinan	Teacher (mathematics)
Donal Hurley	Lecturer (mathematics) UCC

Donna McGowan	Teacher (mathematics)
Donncha Ó hÉallaithe	Lecturer (mathematics) GMIT
Dorothy Hughes	
Dr. Aidan Seery	Lecturer (other subject) TCD
Dr Ann O'Shea, Dr David Wraith	Lecturers (mathematics) NUIM
Dr Brien Nolan + others	Lecturers DCU
Dr. Diarmuid O Sé	Lecturer (mathematics) IT Carlow
Dr James Grannell	Dept. of Mathematical Sciences, UCC
Dr. John Corr	Lecturer (mathematics) IT Tralee
Dr Joseph Manning	Lecturer (other subject)
Dr Leo Creedon	Lecturer (mathematics) IT Sligo
Dr. Michael Brennan	Lecturer (mathematics)
Dr Noel Colleran	Tipperary North VEC
Dr Paul Robinson	Lecturer (mathematics) IT Tallaght
Dr S M McMurray	Lecturer (mathematics) TCD
E Kernan	Teacher (mathematics)
Eamonn Grennan	Lecturer (mathematics) IT Sligo
Eamon McNulty	Teacher (mathematics)
Eileen Gildea Moran	Teacher (mathematics)
Eileen McCrory	Teacher (mathematics)
Elizabeth Oldham	Lecturer (maths; maths ed.) TCD
Emer O Neill	3rd level student
Emma Keenan	2nd level student
Ena Boyle	Teacher (mathematics)
Engineers Ireland	
Eugene Hickey	Lecturer (other subject)
Eugene Kernan	Lecturer (mathematics)
Fiona Desmond	Teacher (mathematics)
Flora-Louise Carey	3rd level student
Frances Weymes	Teacher (mathematics)

Francis Kavanagh	Other (school psychologist)
Frank Ryan	Teacher (mathematics)
Gary Hammond	Other
Gemma O’Dwyer	Teacher (mathematics)
Geraldine Browne	Teacher (mathematics)
Gerard Brennan	Teacher (mathematics)
Gerry Kelly	Lecturer (other subject) IT Letterkenny
Gina Hyland	Teacher (mathematics)
Grace Poole	3rd level student
Grainne Ni Mhuiri	Teacher (mathematics)
Helen O'Mahony	Teacher (mathematics)
Helena McLoughlin, Mary Waters-Wynne	Teachers (mathematics)
Higher Education and Training Awards Council	
Ian McCulloch	Teacher (mathematics)
Ian O Donnell	3rd level student
ICT Ireland	High-tech sector within IBEC
Imelda Butler	2nd level student
Irish Mathematics Teachers’ Association	
Irish National Teachers’ Organisation	
Jacqui Lehane	Teacher (mathematics)
James Reilly	Lecturer (mathematics) IT Tallaght
Jessica Stack	2nd level student
Joan Cleary	Lecturer (mathematics) IT Tralee
Joe Bridges	Teacher (mathematics)
John	Teacher (other subject)
John A Desmond	Teacher (mathematics)
John Colleran	Teacher (mathematics)
John Fanning	Teacher (mathematics)
John King	Teacher (mathematics)
John Mac Sweeny	Lecturer (mathematics)

John McGuinness	Teacher (mathematics)
John Moore	Teacher (mathematics)
John S Davin	School Principal
John Scannell	Teacher (mathematics)
Joseph McCarthy	Lecturer (mathematics)
Karan Murphy	Teacher (mathematics)
Karina Plunkett	Parent
Kathleen Cotter	Teacher (mathematics)
Kevin J Kelly	Dept. of Mathematics, CIT
Kevin Lynch	Lecturer (other subject) IT Tralee
Kevin Swords	Teacher (mathematics)
Lauranne Kelly	Teacher (mathematics)
Leo Hogan	School Principal
Liam Ó Callanáin	Teacher (mathematics)
Lorraine Doherty	Teacher (mathematics)
Lynda	3rd level student
Mairéad Hourigan	Other
Mairéad Ó Shaughnessy	Teacher (mathematics)
Majella Healy	Teacher (mathematics)
Marc van Dongen	Lecturer (other subject)
Margaret Convey	Teacher (mathematics)
Margaret McKeon	
Maria O'Brien	Parent
Marie Griffen	School Principal
Marie Reilly	Teacher (mathematics)
Marion Palmer	Lecturer (other subject) DLIADT
Martha Burton	
Martina Plunkett	Teacher (mathematics)
Mary Clancy	Teacher (mathematics)
Mary Clayton	Teacher (other subject)

Mary Cronin	Teacher (mathematics)
Mary Doohan	Teacher (mathematics)
Mary Fahey	Teacher (mathematics)
Mary Friel	School Principal
Mary Gallagher	Teachers (mathematics)
Mary Geary	Teacher (mathematics)
Mary Irving	Parent
Mary Lafferty	Teacher
Mary McHugh + Maths teachers	Teachers (mathematics)
Mary Morrissey	3rd level student
Mary O Donnell	Teacher (mathematics)
Mary O'Malley	Other
Mary Smith	3rd level student
Mary Storey	2nd level student
Mary T Reany	Teacher (mathematics)
Maths Dept, Bishopstown	Teachers (mathematics)
Maths Dept, Boherbue	Teacher (mathematics)
Maura Carroll	Teacher (mathematics)
Maura Farrell	Teacher (mathematics)
Maureen Spain	Teacher (mathematics)
Melissa Finn	3rd level student
Michael Cannon	Teacher (mathematics)
Michael FitzGerald	Lecturer (mathematics) GMIT
Michael Kelly	Teacher (mathematics)
Michael McCann	School Principal
Michael Stacey	School Principal
Michael White	Teacher (mathematics)
Mícheál D.O Muineacháin	Teacher (mathematics)
Michel Schellekens	Lecturer (other subject)
Michelle Conroy	Teacher (mathematics)

Monica McKenna	3rd level student
Nadia Duffy	Parent
Neil Hallinan	Teacher (mathematics)
Nicholas Sweetman	School Principal
No Name	Teacher (mathematics)
No Name	3rd level student
No Name x 11	2nd level student
Noreen Noonan	Teacher (mathematics)
Nuala Ní Cheallabhui	Teacher (mathematics)
Office of The Financial Regulator	
Oliver McCormack	Teacher (mathematics)
Oliver Murphy	Teacher (other subject)
Orna Lucey	Parent
Owen McConway	Teacher (Mathematics)
Owen O'Mahony	2nd level student
Paddy Ward	Teacher
Pamela Reidy	Teacher (mathematics)
Patricia Bridges	Parent
Patricia Mc Hugh and Geraldine O Shea	Teacher (mathematics)
Patricia Nunan	Teacher (mathematics)
Patrick Flood	Teacher (mathematics)
Patrick McVicar	School Principal
Patrick Plunkett	Teacher (mathematics)
Paul Allen	3rd level student
Paul Brady	Parent
Paul Curran	Lecturer (other subject) UCD
Paul Doyle	Teacher (mathematics)
Pauline Ryan	Parent
Peadar Hanratty	Teacher (mathematics)
Peter McLoughlin, Angela Donoghue	Teachers (mathematics)

Rachel Maguire	2nd level student
Rasmus Noeske	3rd level student
Rebecca Quirke and Yvonne Hannafin	2nd level students
Regina Casey	
RIA Mathematical Sciences Committee	Royal Irish Academy
Richard O'Donnell	3rd level student
Rita Murphy	Teacher (mathematics)
Rita O Donoghue	Teacher (mathematics)
Roísín Scally	Teacher (mathematics)
Roisin O’Sullivan	Teacher (mathematics)
Rynagh McNally	3rd level student
S Moloney	Teacher (mathematics)
Seamus O Neill	School Principal
Seán Close, Dolores Corcoran, Therese Dooley, St. Patrick’s College Drumcondra	
Seán Dowling	Teacher (mathematics)
Sean Kealy	Teacher (mathematics)
Shane Dowdall	Lecturer (mathematics)
Sheila Carroll	Teacher (mathematics)
Sinéad Fitzsimons	Teacher (mathematics)
Sr Berchmans Whelan	School Principal
Students’ Union IT Carlow	3rd level students
Stuart Barry	
Syed Ahmed	3rd level student
Teresa Cushen	Teacher (mathematics)
Teresa Mc Namara	Teacher (mathematics)
Teresa Mulhall	3rd level student
Teresa Nolan	Teacher (mathematics)
Thomas Doyle	2nd level student
Thomas K Murphy	Teacher (mathematics)
Tim Brophy	Teacher (mathematics)

Tom O Connor	Teacher (mathematics)
Tomás Mac Eochagáin	
Una Healy	2nd level student
Union of Students in Ireland	USI
Union of Secondary Students in Ireland	USSI
Valerie O'Keeffe	Teacher (mathematics)
Veronica Kerin	Teacher (mathematics)
Victoria Clarke	3rd level student
Winifred O'Toole	Teacher (mathematics)
Yvonne Hassett	3rd level student
Yvonne O Neil	Teacher (mathematics)

National Council for Curriculum and Assessment

REVIEW OF MATHEMATICS IN POST-PRIMARY EDUCATION

NCCA
National Council for Curriculum and Assessment
An Chomhairle Náisiúnta Curaclaim agus Measúnachta

Consultation Questionnaire

As part of its review of mathematics in post-primary education, the National Council for Curriculum and Assessment has published a discussion paper, and has commissioned a companion paper on international trends in mathematics education. A consultation on the issues identified in the papers is being conducted in October/November 2005.

You are invited to participate in the consultation by responding to this questionnaire, which summarises the issues identified in the discussion paper. Feel free to skip any item that is not particularly relevant or significant for you, or to use additional sheets where the space provided is not enough.

Name:

Address:

Responding on behalf of:
(if applicable)

Date:

I'm responding as a:
(please tick one)

❑ 2nd level student ❑ Teacher (mathematics) ❑ Teacher (other subject)

❑ 3rd level student ❑ Lecturer (mathematics) ❑ Lecturer (other subject)

❑ Parent ❑ School Principal ❑ Employer

❑ Other (please specify)____________________________________

For additional copies of the questionnaire or discussion paper contact the NCCA, tel. 01-6617177, or email mathsreview@ncca.ie. The questionnaire and both papers may be downloaded from the Maths Review section of the NCCA website (**www.ncca.ie**) where there is also a link to an online version of the questionnaire.

Enquiries should be addressed to: **Bill Lynch**, Director, Curriculum and Assessment
Tel: 01-661 7177 or email **mathsreview@ncca.ie**

Completed questionnaires should be returned by Wednesday, 30th November, 2005 to

Maths Review, NCCA, 24 Merrion Square, Dublin 2.

1. Role and purpose of mathematics education

Mathematics has traditionally played an important role in the education of young people in Ireland. It is valued as a component of general education (both as a subject in its own right and as a support for other subjects), as preparation for employment, and as a foundation for further or higher education.

Please comment on the level of importance you would attach to post-primary mathematics education and your views regarding its role and purpose(s).

2. Concerns regarding mathematics

Internationally, there are concerns about the type and quality of the mathematics education that students experience in schools. Many of these concerns are echoed in Ireland. The discussion paper identifies concerns about

- the emphasis on procedural skills rather than on the understanding of mathematics
- the poor application of mathematics in real-world contexts
- the low uptake of Higher level mathematics, especially in the Leaving Certificate
- the low grades achieved at Ordinary level, especially in the Leaving Certificate
- gender differences in uptake and achievement in mathematics
- difficulties in mathematics experienced by some students in third-level courses.

We would welcome your views on these issues or other concerns that you may wish to raise.

3. Recent developments in mathematics education in Ireland

In the past five years, a revised mathematics curriculum has been implemented in primary schools and there has also been syllabus revision in Junior Certificate Mathematics.

Please comment on the impact of these changes and whether they go far enough to address the problems in mathematics that have been identified.

4. Current trends in mathematics education

The discussion paper describes some of the approaches that are used in mathematics education, including

- 'modern mathematics' with its emphasis on abstraction, logical structure, rigorous argument, set theory, number theory, etc.
- real-world or context-based mathematics, also referred to as 'realistic mathematics education' (RME).

Please comment on the relative merits of such approaches for Junior Certificate and Leaving Certificate mathematics courses.

5. Mathematics in relation to other subjects

The discussion paper notes the dual nature of mathematics. It is geared to applications but is also worthy of study in its own right. It plays an important role in other subjects, especially the science and technology subjects.

Please comment on this dual nature and on the relationship between mathematics and other subjects, including the contribution that mathematics can make to other subjects and their contribution to mathematics.

6. Provision and uptake of mathematics

Internationally, the proportion of students who study mathematics in senior cycle or upper second level education is lower than in Ireland. In some countries, where a high proportion of students remain in school, students can choose between 'general' and 'specialist' mathematics courses. In Ireland, practically all students in post-primary schools study mathematics. However, the proportion of students taking the Higher level syllabus is lower than had been expected when the three syllabus levels were introduced.

Please comment on the adequacy of the current mathematics courses in meeting the needs of all students in (i) the junior cycle and (ii) the senior cycle of post-primary education. We would also welcome your views on the relatively low uptake in Higher level mathematics and any suggestions you might have for increasing this uptake.

7. Influence of the examination papers

Research has shown that teaching and learning in mathematics is strongly influenced by the examination papers, with firm evidence of 'teaching to the test' (this is true for other subjects also). The absence of other forms of assessment, such as coursework, is noted as contributing to this dominant influence.

Please give us your views on the assessment of mathematics.

8. Syllabus levels and range of courses

The discussion paper (pages 5, 13, 33) points to some issues that have arisen in relation to the existence of three syllabus levels in mathematics (Irish is the only other subject with a Foundation level). Views vary greatly on issues such as non-recognition of Foundation level mathematics for entry to many third level courses, or the challenge of meeting demands for both 'general' and 'specialist' mathematics courses in the one subject (the uptake of Leaving Certificate Applied Mathematics is very low).

Please comment on these issues and on how they might be addressed within the current review.

9. Student achievement in mathematics

Section 5 of the discussion paper considers the results of the Junior Certificate and Leaving Certificate mathematics examinations in recent years and the Chief Examiners' reports, as well as evidence from cross-national studies. It is noted that higher-performing Irish students do less well than their counterparts in countries which record comparable overall levels of achievement in international mathematics tests.

How effective, in your view, would each of the following measures be in improving the performance of students in mathematics examinations?

		very effective	effective	not effective
(i)	allocation of more class time to mathematics	❑	❑	❑
(ii)	better pre-service and inservice education for teachers of mathematics	❑	❑	❑
(iii)	improved mathematics textbooks and other learning resources	❑	❑	❑
(iv)	provision of learning support for students who are experiencing difficulties with the subject	❑	❑	❑
(v)	provision of 'general' as well as 'specialist' mathematics courses	❑	❑	❑
(vi)	increased emphasis in examination questions on the application of mathematics to real-world problems	❑	❑	❑
(vii)	the introduction of additional forms of assessment, such as coursework	❑	❑	❑
(viii)	improving the perception of mathematics among parents and the general public	❑	❑	❑

(ix) other (please specify)____

10. Teaching and learning in mathematics

Research in Irish classrooms indicates that mathematics is taught and learned in a 'traditional' manner, mainly involving teacher exposition or demonstration of procedural skills and techniques for answering examination-type questions, followed by student practice of these techniques (in class or as homework) using similar questions. There appears to be little or no emphasis on students understanding the mathematics involved, or on its application in different or unfamiliar contexts.

Please comment on the strengths and weaknesses of this approach. We would also welcome your views on the degree to which syllabus change, assessment change, teacher professional development and support would contribute to bringing about changes in teaching and learning.

11. Attitudes to and beliefs about mathematics

The discussion paper – and the companion paper on international trends in mathematics education – raises, on a number of occasions, issues surrounding the perceptions, attitudes and beliefs that exist in relation to mathematics, such as

- the view that mathematics is a difficult subject
- negative attitudes towards mathematics including, for some, a 'fear' of the subject
- the perception and advocacy of mathematics, particularly Higher level mathematics, as an elite subject for only the 'best' students
- research findings that suggest a connection between teachers' views of mathematics and their approach to teaching it.

We would welcome your views on these or other issues associated with mathematics.

12. Other influences

The discussion paper draws attention to a range of other cross-cutting themes or issues that affect mathematics education in schools:

- cultural issues related to the value of education in general and mathematics education in particular
- equality issues (gender, uptake and achievement; socio-economic factors; educational disadvantage; students with disabilities or special educational needs)
- the 'points' system, including its influence on the uptake of the different syllabus levels in mathematics
- recent developments in, and availability of, information and communications technology (ICT) in schools.

Please comment on any of these issues, or on other factors that impact on mathematics education in schools.

Conclusion

The purpose of this review is to map out the direction that must be taken in planning curriculum and assessment provision for post-primary mathematics education in the years ahead.

Please use the space below to make any additional comments on current issues in post-primary mathematics education or to give us your views regarding its future.

Thank you for taking the time to complete this questionnaire. Please return the completed questionnaire by Wednesday, November 30th to: Maths Review, NCCA, 24 Merrion Square, Dublin 2.

Log on to the NCCA website www.ncca.ie for updates on the review, or to participate in a weekly poll on a topical question in relation to mathematics.

NCE-MSTL

National Centre for Excellence
in Mathematics and Science Teaching and Learning

A Review of School Textbooks for Project Maths

Lisa O'Keeffe and John O'Donoghue

Published by the National Centre for Excellence in
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The NCE-MSTL is a distributed centre established under the aegis of the Shannon Consortium and funded by the HEA through SIF Cycle 2. The University of Limerick is the lead institution and is supported by existing centres of expertise in the Shannon Consortium partners.

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National Centre for Excellence in Mathematics and Science Teaching and Learning

A Review of School Textbooks for Project Maths

Lisa O'Keeffe and John O'Donoghue

July 2011

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Preface

We are charged under our brief at the National Centre to engage in evidence-based world class research in Science and Mathematics Teaching and Learning and to bring our findings to bear, *inter alia*, on advice offered to stakeholders in Irish education. This is the third report in the series following the NCE's ground-breaking report, *Out-of-field teaching in Post-Primary Mathematics Education: An analysis of the Irish context* (2009) and it presents a review of school textbooks published commercially for Project Maths.

Historically, mathematics teaching and classrooms in Ireland have been strongly influenced by commercially produced school textbooks and mathematics education has come to reflect the view of mathematics teaching and learning portrayed in these textbooks. This situation has not also worked in the best interests of mathematics education.

This report is timely as teachers, students, parents and publishers are working to come to grips with the new reality of Project Maths in our schools. A significant number of new textbooks are now available for Project Maths but anecdotal evidence suggests that they are not a good match for Project Maths. Given the central role of textbooks in curriculum development and change, it is prudent that care is taken at policy level to ensure new second level mathematics textbooks are aligned with the reform vision of the mathematics curriculum embodied in Project Maths, and are 'fit for purpose'. This report offers an objective evaluation of a selection of new textbooks available for Project Maths.

In this study the authors look to the Third International Mathematics and Science Study (1995) (TIMSS) for theoretical underpinnings and methodology. The TIMSS mathematics curriculum framework as it evolved is adapted by the authors (LO'K) and further refined for use in this report as TIMSS+. This present report, the first of its kind in Ireland, produced a wealth of interesting data recorded. These are recorded in the appendices and support the conclusion that this selection of textbooks is not well aligned with the intended Project Maths curriculum and expectations. Further, a careful reading of the report and data show where mismatches occur and potential avenues for improvement.

The Directors are pleased to discharge their brief to advise on matters related to Science and Mathematics teaching in this way and commend this report to all who have a stake in Irish education and particularly to those front-line agencies involved in improving matters in Mathematics and Science teaching at all levels.

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Summary

The Department of Education and Skills asked the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL) to review the current mathematics textbooks available for schools for Project Maths, and evaluate them using a version of the TIMSS framework already in use, adapted for the purpose of this study.

This study is part funded by the NCCA and is undertaken by the National Centre as part of its brief to advise on matters related to Science and Mathematics teaching and learning.

A very tight reporting deadline and resource constraints precluded a more extensive study, however, the study as implemented and reported aspires to best practice standards within its frame.

Methodology

The study summarised here is based on analysis of data obtained by the application of TIMSS+ (a modified TIMSS instrument for mathematics textbook analysis) to a selection of published textbooks (Appendix A) that were available during the school year 2010/11. The TIMSS mathematics curriculum framework is applied to the Project Maths Syllabus documents and is used as a referent for analysis and comparison purposes.

Key Findings

- All textbooks included in the study fall short of the standard needed to support Project Maths (intended curriculum) effectively,
- These textbooks display a genuine attempt to match the intentions of Project Maths but no one textbook meets all the needs of Project Maths,
- The most significant overall finding is the mismatch between textbook expectations and Project Maths expectations,
- It is noteworthy that there are topic omissions in the reviewed textbooks when the Project Maths syllabus treats all topics as compulsory,
- A key topic omission is the integration of ICT throughout all textbooks,
- Structure and content analysis uncovers disparities between the textbooks in their approaches to teaching for understanding and problem solving.

Conclusions and Recommendations

If textbooks are to contribute to the success of Project Maths then more needs to be done. The obvious lack of attention to key Project Maths expectations needs careful consideration. While the developmental nature of Project Maths is on a strand by strand basis, this militates against topic integration, and, when the roll out is complete, a more integrated approach should feature in textbooks.

The report recommends that:

- An exemplar textbook series for Project Maths be produced by a specially selected and constituted writing team appointed and funded by the DES,
- All commercially produced textbooks for Project Maths be reviewed against this exemplar textbook series
- Such a review procedure leads to an approved list of mathematics textbooks for Project Maths

1. Introduction

1.1 The influence of mathematics textbooks on teaching and learning

It is accepted worldwide that mathematics textbooks have a major influence on classroom practice (Valverde et al., 2002). Textbooks are important vehicles for the promotion of specific types of mathematics curricula. They are organised in a purposeful way, and consequently their content and structure are very important for the promotion of a specific vision of mathematics curriculum such as Project Maths. Given the central role of textbooks in curriculum development and change it is prudent that care is taken at policy level to ensure new second level mathematics textbooks are aligned with the reform vision of mathematics curriculum embodied in Project Maths, and are 'fit for purpose'.

Historically, mathematics teaching and classrooms in Ireland have been strongly influenced by commercially produced school textbooks that have promoted a view of mathematics concerned mainly with skills and instrumental learning (NCCA, 2005). This view of mathematics curriculum is not compatible with Project Maths, and if these emphases dominate through the new generation of mathematics textbooks then the success of Project Maths is likely to be severely compromised.

The aim of the report is to inform decision making at policy level through evidence-based research regarding school mathematics textbooks at second level.

1.2 Outline structure of report

The report is presented in eight sections. The first section of the report describes the context and the influence of mathematics textbooks on teaching and learning. Section 2 discusses the theoretical underpinnings of the report and the origins of the textbook analysis instrument used in the study. Section 3 is devoted exclusively to methodology and related issues. Data analysis and findings are developed in Section 4. Section 5 contains a summary of the report and the main conclusions. A short list of references is included in section 6. Section 7 contains a glossary of important terms and definitions used throughout the study. The appendices (section 8) contain all the tables, diagrams and figures developed from the primary data.

2. Theoretical framework for this study

2.1 The TIMSS curriculum frameworks

This study looks to the Third International Mathematics and Science Study (1995) (TIMSS) for its theoretical underpinnings. Curriculum is a central variable in TIMSS and is used to compare national systems of education. The conceptual framework for TIMSS is based on the now well known tripartite model of curriculum (Robitaille et al., 1997):

- intended curriculum
- implemented curriculum
- attained curriculum.

TIMSS devised a common framework to compare systems of education through analyses of curricula, related documents and artefacts. They are known as *curriculum frameworks*. Each framework is characterised by the same three elements that are further sub-divided (Robitaille et al., 1997):

- subject matter content
- performance expectations
- perspectives or context.

These frameworks are applied to the curriculum or any piece of the curriculum that is seen as promoting the intended, implemented or attained curriculum and includes artefacts such as textbooks, curriculum guides, standards documents etc. TIMSS employs two separate frameworks viz. the curriculum framework for mathematics, and curriculum framework for science. The TIMSS model was formulated to deal with evolving curricula and is appropriate for use with Project Maths. This model does not deal with language analysis or readability of mathematics textbooks which would require a separate study.

2.2 The mathematics framework as a tool for textbook analysis

The mathematics framework is a tool for studying curriculum or any piece of curriculum or artefact. Indeed the view supported by TIMSS is that 'A textbook is a surrogate curriculum…' (Robitaille et al., 1997: 50). In this study the mathematics framework is adapted for use as a tool for mathematics textbook analysis.

Thus we start with the TIMSS mathematics framework as a tool for textbook analysis. It has three dimensions:

- Structure
- Performance expectations
- Perspectives.

Subsequently, the mathematics framework was adapted and refined for use in TIMSS as an instrument for mathematics textbook analysis per se.

3. Methodology

3.1 TIMSS instrument for mathematics textbook analysis

In this study the TIMSS mathematics curriculum framework as it evolved is adapted and further refined as outlined below. The 'perspectives' dimension captures student data and is not used in this study. In any case it was not envisaged that all three dimensions would be applied to every piece of curriculum. The structure dimension encompasses issues concerning content and the structure of knowledge and information in the textbook and the make-up of the textbook. This line of reasoning led to an analytical tool with two dimensions and three elements as follows:

- Structure
  - Structure
  - Content
- Expectation.

3.2 The TIMSS+ instrument

Further refinements were added to this TIMSS instrument by O'Keeffe (2011) in order to allow for a finer-grained analysis. Refinements based on the work of River's (1990) and Mikk (2000) that reinforce and add to the TIMSS model around content and expectation and structure analysis respectively, are included here. The evolved model is identified as the TIMSS+ instrument (Figure 3.1).

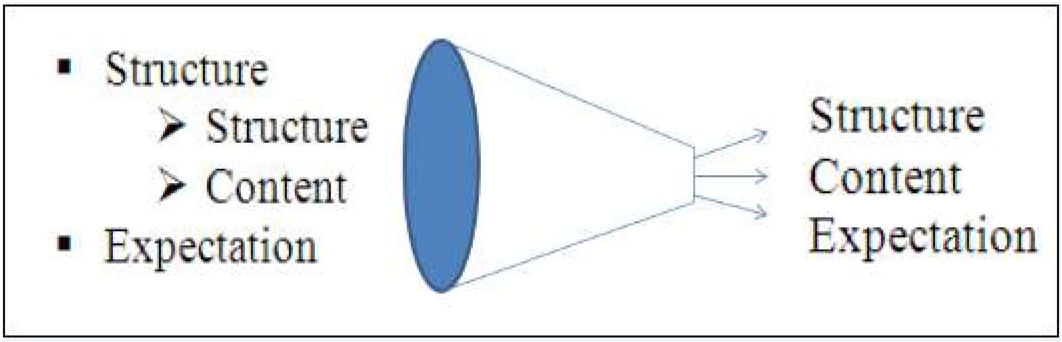

Figure 3.1: Development of the TIMSS+ Instrument

3.3 Exploiting the link between curriculum and textbooks for methodological purposes

TIMSS posits and develops a powerful link between curriculum and textbooks. Indeed the textbook is described as a 'surrogate curriculum'. This point is expanded by Vanezky (cited in Robitaille et al., 1997: 50) when he points out that a single set of curriculum guidelines can spawn a myriad of textbook representations. In this context the bi-directional link between the textbook and curriculum is a powerful insight. This line of reasoning is exploited to advance this study. The alignment of this study with TIMSS theory and methodology makes available the TIMSS superstructure as needed.

Each textbook analysed for the study is treated as a representation of the intended Project Maths curriculum, and may be treated as individual stand-alone representations or compared to the others in order to advance our understanding. A full list of textbooks and disaggregated textbooks is given in Appendix A.

3.4 Specially constructed curricula (SCC)

For the purposes of this study a further refinement is necessary. The TIMSS+ instrument is systematically applied to a number of mathematics textbooks or series of textbooks by disaggregating textbooks into chunks of curricula or strands identified as units called *specially constructed curricula*. This step is necessary because there are no complete textbooks for Project Maths (Strands 1-5) covering the entire curriculum from year 1 to year 5 at Higher and Ordinary levels; several of the textbooks comprise curriculum strands only, and consequently render analysis and comparison extremely difficult or impossible without some such device. A selection of several recently published mathematics textbooks are analysed and discussed in the context of their associated SCC and Project Maths.

Specially constructed curricula (SCC) are represented by a book or books or selected chapters from a book according to the information supplied by the publisher and are keyed to the Common Introductory Course (CIC), Junior Cycle and Senior Cycle as appropriate. In all, this study compares data from 10 selected textbooks and 6 specially constructed curricula as follows:

1 Common Introductory Course (CIC)
2 Junior cycle curricula
3 Senior cycle curricula.

Details of the SCC and their associated textbooks are given in Tables 3.1 and 3.2 together with their respective codes. For the purposes of this study a workbook is treated as part of the associated textbook.

Table 3.1: Junior Cycle Constructed Curricula

Constructed Curriculum:	Level:	Curriculum Material:
1. Strand 1 - 5	Ordinary Level	Textbook 1 Syllabus
2. CIC	All Levels	Textbook 2 Textbook 3 Syllabus
3. Strands 1 & 2	Ordinary Level	Textbook 4 Textbook 5 Textbook 6 Syllabus

Table 3.2: Senior Cycle Constructed Curricula

Constructed Curriculum:	Level:	Curriculum Material:
4. Strands 1 & 2	Ordinary Level	Textbook 7 Textbook 8 Textbook 9 Syllabus
5. Strands 1 & 2	Higher Level	Textbook 10 Textbook 11 Syllabus
6. Strand 2	Higher Level	Textbook 12 Textbook 13 Textbook 14 Syllabus

Also for the purposes of analysis the Project Maths Syllabus documents are disaggregated into constructed syllabi matching the 6 SCC in the above list. The TIMSS mathematics curriculum framework is applied to these constructed syllabi for use as a reference for comparative purposes. This gives an objective benchmark for the Project Maths 'intended curriculum'.

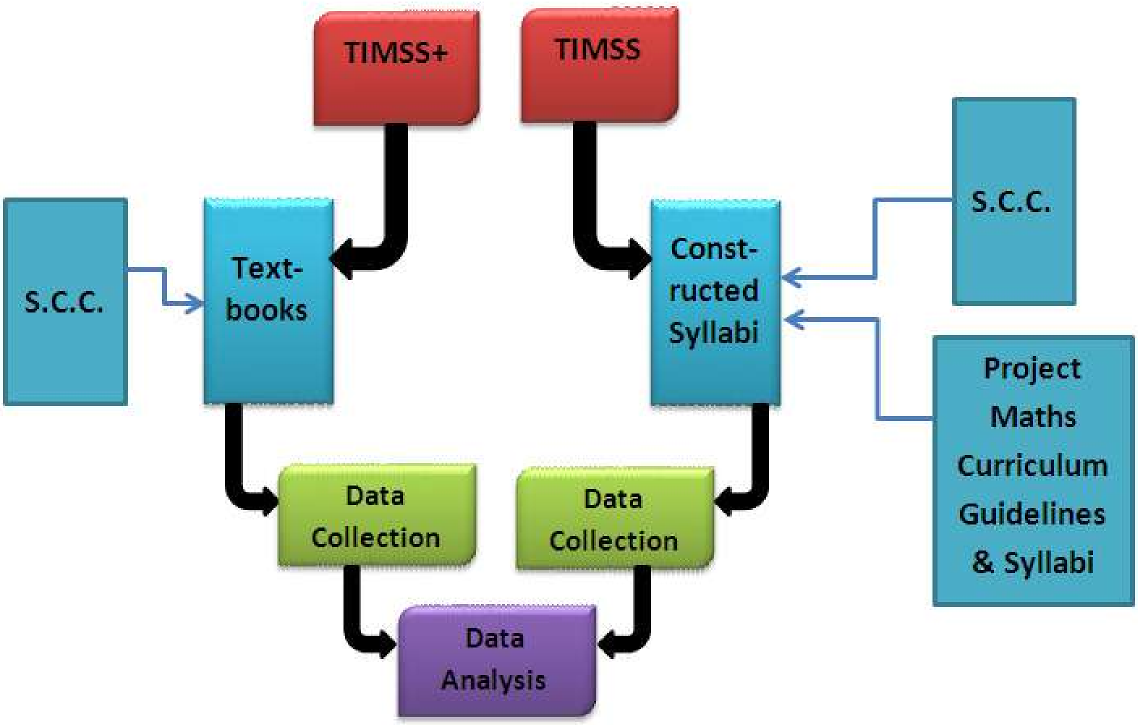

Figure 3.2: Overview of methodology

3.5 The process of analysing textbooks

The textbook analysis proceeds by dividing every textbook into blocks of 'text' and applying the three lenses from the TIMSS+ instrument to each block. This process involves every page and every chapter of each textbook. 'Text' may include ordinary literary text, exam items, graphs or charts, etc. An ordinary literary paragraph in a curriculum document may be taken as a block for the purposes of this analysis.

Generally, the data from each block is recorded as counts that in turn contribute to the production of various graphical displays. The TIMSS template for data collection was used systematically throughout (Appendix B). The data collected in this manner were entered into an Excel file and subsequently used for computer analysis.

4. Data analysis and findings

4.1 Structure Analysis

Table 4.1 Structure Analysis

Specially Constructed Curriculum and Associated Textbooks	**Structure Analysis:**
Constructed Curriculum 2: **JC CIC** **Associated textbooks:** **TB 2;** **TB 3**	Table D.1 and Figure D.1 show that textbook 2 displays lower levels of narration but higher levels of related narration in comparison to textbook 2. Also textbook 3 has a much higher rate of instructional narration throughout its text. Textbook 2 has three times as many definitions, seven times as many theorems and four times as many axioms highlighted from the text than textbook 3. Graphics are plentiful across both textbooks (Figure D.3) with textbook 2 using 171 of its graphics for problem solving and textbook 3 using 196. Textbook 2 has a much higher presence of photo-like graphics, 126 compared to 12. Textbook 3 boasts a greater number of exercises (1843 compared to 782), and problems (532 compared to 349) (Table D.4). Of the total number of problems and exercises present in these two textbooks, 32% of these in textbook 2 are devoted to problems while 22% are devoted to problems in textbook 3 (Table D.4a). Further analysis of problem solving showed that 20 of the 349 problems present in textbook 2 are non- routine problems and the ratio in textbook 3 is 89/532. Further classification of the routine problem types (Figure D.7) identifies that Real and Realistic problems are the most commonly found in textbook 2, whereas Realistic and Purely Mathematical problems are the most commonly found in textbook 3. Of the 20 non-routine problems in textbook 2, 16 of these are realistic type problems and in textbook 3 of the 89 non-routine problems 29 are realistic and 44 are purely mathematical (Figure D.8). Textbook 2 contains 162 worked examples, 56 of which represent real or realistic problems, giving a total of 1 worked example for every 7 exercises/problems. The corresponding ratio for textbook 3 is 21/108 giving a total of 1 worked example for every 22 exercises/problems.
Constructed Curriculum 3: **JC S1&2/O**	As can be seen from Table D.1 and Figure D.1, textbook 4 exhibits the lowest levels of narration but highest levels of related narration. Textbook 6 exhibits the highest rate of instructional narration throughout its text. Textbook 4 also has the greatest number of definitions, theorems and axioms highlighted from its text (Figure D.2). Graphics are plentiful across all three textbooks (Table & Figure D.3) with textbook 4 exhibiting the greatest number of graphics in total, the highest number of

Associated textbooks: **TB 4;** **TB 5;** **TB 6**	graphics used for problem solving and the greatest frequency of photo-like graphics. Textbook 4 has the highest frequency of exercises and problems present (Table & Figure D.4). 31% of the total number of problems and exercises in textbook 4 are devoted to problems; the corresponding figures for textbooks 5 and 6 are 32% and 21% respectively (Table D.4a). Further analysis of problem solving identified that the ratio of non-routine problems to problems is 16/280, 6/225 and 2/150 for textbooks 4, 5 and 6 respectively. A further classification of the routine and non-routine problem types (Figure D.7 & D.8) identifies that realistic problems are the most commonly found in all textbooks. Textbook 4 contains 110 worked examples, 45 of which represent real or realistic problems, giving a total of 1 worked example for every 8 exercises/problems. The corresponding data for textbooks 5 and 6 respectively are: 58 worked examples, 16 of which represent real or realistic problems, giving a total of 1 worked example for every 12 exercises/problems; 67 worked examples, 23 of which represent real or realistic problems, giving a total of 1 worked example for every 11 exercises/problems.
Constructed Curriculum 4: **LC S1&2/O** **Associated textbooks:** **TB 7;** **TB 8;** **TB 9**	As can be seen from Table E.1 and Figure E.1, textbook 8 exhibits the highest levels of narration while textbook 7 exhibits the highest presence of related narration. Textbook 7 also has the greatest number of definitions, theorems and axioms highlighted from its text (Table & Figure E.2). Graphics are plentiful across all three textbooks (Table & Figure E.3) with textbook 7 containing the greatest number of graphics and real life diagrams and photo-like graphics. However the highest number of graphics used for problem solving is found in textbook 8. On the other hand textbook 9 exhibits the highest frequency of exercises and textbook 7 has the highest frequency of problems present (Table E.4). The percentage of the total number of problems and exercises present in these three textbooks devoted to problems is 35 % , 49%, and 33% respectively for textbooks 7, 8 and 9 (Table E.4a). Further analysis of problems shows that 46 of 389 problems in textbook 7 are non- routine problems. This ratio for textbooks 8 and 9 is 51/380, and 118/288. A further classification of the routine and non-routine problem types (Tables E.10a and E.10b and Figures E.7 & E.8) identifies that realistic and purely mathematical problems are the most commonly found across all three textbooks. Textbook 7 contains 189 worked examples, 86 of which represent real or realistic problems, giving a total of 1 worked example for every 7 exercises/problems. The corresponding data for textbooks 8 and 9 are: 114 worked examples, 35 of which represent real or realistic problems, giving a total of 1 worked example for every 8 exercises/problems; 146 worked examples, 48 of which represent real or realistic problems, giving a total of 1 worked example for every 8 exercises/problems.
Constructed Curriculum 5: **LC S1&2/H** **Associated textbooks:**	Table F.1 and Figure F.1 show that textbook 10 displays higher levels of both narration and related narration in comparison to textbook 11. Textbook 10 has more than four times as many definitions and almost twice as many theorems. Textbook 11 fails to highlight any axioms or corollaries from the narration (Table and Figure F.2). Graphics are plentiful across both textbooks (Table & Figure F.3) with textbook 10 using 330 of its graphics for problem solving and textbook 11 using 290. Textbook 10 has a much higher incidence of photo-like graphics, 58 compared to 0. The difference in the number of exercises in both textbooks is

TB 10; **TB 11**	marginal (604 in textbook 10 compared with 615 in textbook 11) and textbook 11 exhibits a greater number of problems (689 compared to 656) (Table F.4 and Figure F.4), while textbook 10 boasts a higher presence of activities (29 compared with 3). Of the total number of problems and exercises present in each of these textbooks just over 52% (in each textbooks, see Table F.4a) are devoted to problems. Further analysis of problem solving demonstrated that 183 of the 656 problems present in textbook 10 are non- routine problems and the ratio in textbook 11 is 226/689. Further classification of the non-routine problem types (Figures F7 & F8) identifies that realistic and purely mathematical problems are the most commonly found in both textbooks. Textbook 10 contains 188 worked examples, 59 of which represent real or realistic problems, giving a total of 1 worked example for every 7 exercises/problems. The corresponding ratio for textbook 11 is 68/177 giving a total of 1 worked example for every 7 exercises/problems.
Constructed Curriculum 6: **LC S2/H** **Associated textbooks:** **TB 12** **TB 13;** **TB 14**	Textbook 13 displays higher levels of narration (Table F.1 and Figure F.1) while textbook 12 has the highest levels of related narration. Textbook 12 has the greatest number definitions (94), theorems (41) and axioms (14) (Table F.2 and Figure F.2). Textbook 12 has considerably more graphics (943) than textbook 13 (525) or textbook 14 (494) (Table F.3). The distributions of graphics used for problem solving across all three textbooks are similar with 222 in textbook 12, 195 in textbook 13 and 212 in textbook 14. Textbook 12 has a higher incidence of real life diagrams and photo-like graphics, 108 and 25, compared to 32 and 0 in textbook 13 and 36 and 0 in textbook 14 (Table F.4 and Figure F.4). Textbook 13 has the lowest number of exercises (217) but the highest number of problems (442) compared with 420/351 in textbook 12 and 466/428 in textbook 14 (Table F.5). Of the total number of problems and exercises present in textbook 12, 46% are devoted to problems compared with 67% in textbook 13 and 48% in textbook 14 (Table F.4a) Further analysis of problem solving showed that 138 of the 351 problems present in textbook 12 are non- routine problems. The ratio in textbook 13 is 147/442 and in textbook 14 it is 188/428. Further classification of the non-routine problem types (Figures F7 & F8) identifies that realistic and purely mathematical problems are the most commonly found in all three textbooks. Textbook 12 contains 117 worked examples, 4 of which represent real or realistic problems, giving a total of 1 worked example for every 7 exercises/problems. The corresponding ratio for textbook 13 is 15/122 giving a total of 1 worked example for every 5 exercises/problems and for textbook 14 it is 1/96 with 1 worked example for every 9 exercises/problems.

4.2 Content Analysis

Table 4.2 Content Analysis

Specially Constructed Curriculum and Associated Textbooks	**Content Analysis:**
Constructed Curriculum 1: **JC S1-5/O** **Associated textbook:** **TB 1**	Content analysis identified two content topics that are outlined in the syllabus (Tables D.6a and D.6b) but omitted from this textbook. These topics are 'Domain and Range' and 'use of Computer Software'. While the River's Matrix (Table D.9) identified 12 references to 'Computer Software', all of these references are found in one chapter which forms part of Strand 1. The Project Maths syllabus also makes direct reference to use of the internet particularly for strand 3, and this textbook makes two references to the internet in two separate chapters, one of which is in Strand 1 and the other is in Strand 3. The River's Matrix also identifies the presence of motivational factors (Table D.7); this textbook contains 25 historical notes, 5 biographical facts and 9 notes of career information, while also including 6 instances of direct humour or quotes. There is consistent use of colour, with background colours placing text in context, for example all hints are in a purple box and all keywords are in a green box etc. (see Table D.8).
Constructed Curriculum 2: **JC CIC** **Associated textbooks:** **TB 2;** **TB 3**	Analysis of the content grids identified that 'Estimating Computations', a significant element of strand 1 is omitted from textbook 3 (Tables D.6c to D.6e). The River's Matrix (Table D.7) shows that textbook 2 has a greater number of historical notes, biographies, career information and humour/quotes. Both textbooks demonstrate consistent use of colour. While textbook 3 demonstrates consistent use of colour, the range of colours used is more limited (see Table D.8). The analysis of Technical Aids (Table D.9) identifies 12 references to 'Computer Software' and one reference to the 'Internet' in textbook 2 (all of these references are found in one chapter which forms part of Strand 1). Textbook 3 has no references to either type of technical aid.
Constructed Curriculum 3: **JC S1&2/O** **Associated textbooks:** **TB 4;** **TB 5;** **TB 6**	Content analysis of the three textbooks in this section identified one content topic that is outlined in the syllabus but omitted from the textbooks (Tables D.6f to D.6i). This topic is 'Computer Software'. While the River's Matrix (Table D.9) identified 12 references to 'Computer Software' and one reference to the 'Internet' in textbook 4 all of these references are found in one chapter that forms part of Strand 1. There are other omissions; 'Counting Principles' is omitted from textbook 5 and '2D Geometry – The Circle' is omitted from textbook 6. The River's Matrix (Table D.7) shows that textbook 4 has a greater presence of historical notes, biographies, career information and humour/quotes. Consistent use of colour is evident in textbook 4 as is the case for the other Junior Cycle textbooks from this publisher. While textbooks 5 and 6 also demonstrate consistent use of colour, the range of colours used across both is more limited, with textbook 5 making use of more colour and colour backgrounds from context (see Table D.8).

Constructed Curriculum 4: **LC S1&2/O** **Associated textbooks:** **TB 7;** **TB 8;** **TB 9**	Four content topics that are outlined in the syllabus (Tables E.6a to E.6d) are omitted from all three textbooks. These topics are 'Negative Numbers & their properties', 'Set Properties', 'Set Operations' and 'Linear Functions'. Also all textbooks omit the topic content 'Computer Software' and the 'Internet' (with the exception of Strand 1 in textbook 7, see Table E.9). Content analysis also shows that 'Percentages' and 'Exponents, roots and radicals' are not given a strong weighting in textbooks 8 and 9. The Rivers Matrix (Table E.7) indicates that textbook 7 has a greater number of historical notes, biographies, career information and humour/quotes than the others in the set. Textbook 7 demonstrates consistent use of colour in line with the same publisher's textbooks for Junior Cycle. While textbooks 8 and 9 also demonstrate consistent use of colour the range of colours used across both is more limited, with textbook 8 making use of more colour and colour backgrounds from context (see Table E.8).
Constructed Curriculum 5: **LC S1&2/H** **Associated textbooks:** **TB 10;** **TB 11**	Content analysis of textbooks 10 and 11 identified four content topics that are outlined in the Project Maths syllabus but omitted from both textbooks (Tables F.6a. to F.6c). These topic are 'Negative Numbers, Integers are their Properties', 'Set Properties', 'Set Operations'(all omitted from strand 1), and 'Linear Functions' (omitted from strand 2). The content analysis of textbook 10 also indicates that 'Proportionality Problems' are omitted from strand 2. Textbook 11 also exhibits a lack of emphasis to 'Randomisation' and 'Defining Probability' in strand 1. The River's Matrix (Table F.7) shows that textbook 10 has a marginally greater number of historical notes and a significantly greater number of references to career information. Both textbooks demonstrate consistent use of colour. While textbook 11 demonstrates consistent use of colour, the range of colours used is more limited (see Table F.8). The analysis of Technical Aids (Table F.9) identifies 14 references to 'Computer Software' and 12 references to the 'Internet' in textbook 10 in comparison to 17 and 2 references in textbook 11.
Constructed Curriculum 6: **LC S2/H** **Associated textbooks:** **TB 12** **TB 13;** **TB 14**	The content analysis of textbook 12 indicates that 'Proportionality Problems' are omitted, while in textbook 13 an emphasis on 'Fractions' is omitted. The River's Matrix (Table F.7) shows that textbook 12 has a marginally greater number of historical notes and biographical information and a significantly greater incidence of career information. All three textbooks demonstrate consistent use of colour. While textbook 14 demonstrates consistent use of colour, the range of colours used is more limited (see Table F.8). The analysis of Technical Aids (Table F.9) identifies 1 reference to 'Computer Software' in textbook 12 and none in textbook 13 or 14. None of these textbooks make reference to the 'Internet'.

4.3 Expectation Analysis

Table 4.3 Expectation Analysis

Specially Constructed Curriculum and Associated Textbooks	**Expectation Analysis:**
Constructed Curriculum 1: **JC S1-5/O** **Associated textbook:** **TB 1**	The expectation analysis identified that two expectations are omitted from textbook 1, one of which the syllabus suggests should be evident in every strand (see Tables D.11a and D.11b). These expectations are 'Developing Algorithms' which the Project Maths syllabus suggest should be present in strand 3 and 'Generalising' which it suggests should be evident in all five strands. Other expectations which the syllabus suggests should have greater coverage across all strands are 'Use of Vocabulary & Notation', Relating Representations', 'Critiquing', 'Inter subject Connections' and 'Across Subject Connections'. The S1-5 syllabus also places weight on the presence of 'Instrumental and Relational Learning', 'Fostering Positive Attitudes', 'Inquiry Based Learning', 'Applications' and making 'Connections between Solutions and Questions', problem solving in context and the use of graphics to assist with problem solving.
Constructed Curriculum 2: **JC CIC** **Associated textbooks:** **TB 2;** **TB 3**	The expectation 'Generalising' which the syllabus suggests should be evident in strands 3 and 4 is omitted from both textbooks (see Tables D.11c to D.11e). Textbook 2 has a number of expectations missing in various strands in comparison to the syllabus. 'Performing more Complex Procedures', 'Formulating & Clarifying Problems' and Developing Strategies' are omitted from strands 2 and 4. 'Predicting' is omitted from strand 4 and 'Developing Algorithms' is omitted from strand 3. A number of expectations are also omitted from textbook 3: 'Recognising Equivalents' is omitted from strand 2; 'Developing Strategies' is omitted from strands 2, 3 and 4 while 'Predicting' is omitted from strands 1 and 4. Both textbooks are missing a number of expectations that are outlined in the syllabus for strand 4, these are 'Conjecturing', 'Justifying & Proving', 'Axiomatising', 'Using Vocabulary & Notation', and 'Describing/Discussing'. The syllabus also makes direct references to 'Inquiry Based Learning', 'problem solving in context and the use of graphics to assist with problem solving.'
Constructed Curriculum 3: **JC S1&2/O** **Associated textbooks:** **TB 4;** **TB 5;** **TB 6**	A number of expectations are omitted from all three textbooks, (see Tables D.11f to D.11i). The first of these expectations is 'Generalising' which the syllabus suggests should be evident in all strands. The elements 'Developing Strategies' and 'Across Subject Connections' are omitted from strand 2 of each textbook while 'Relating Representations', 'Inter Subject Connections' and 'Across Subject Connections' are also omitted from strand 1 of each textbook. 'Critiquing' is also absent from strand 2 of textbook 4. Textbook 5 and 6 are both missing 'Predicting' from strand 1 and 'Axiomatising' from strand 2. 'Using more Complex Procedures' and 'Conjecturing', are also omitted from strand 2 of textbook 5 while textbook 6 is also missing 'Using more Complex Procedures' and 'Critiquing' from strand 1. The Project Maths S1&2 syllabus also places weight on the presence of 'Instrumental and Relational Learning', 'Fostering Positive Attitudes', 'Inquiry Based Learning' and 'Applications'.

Constructed Curriculum 4: LC S1&2/O **Associated textbooks:** **TB 7;** **TB 8;** **TB 9**	The expectation analysis identified that a number expectations are omitted from all three textbooks, (see Tables E.11a to E.11d). The first of these expectations is 'Generalising' which the syllabus suggests should be evident in all strands. 'Developing Strategies' is omitted from strand 2 of each textbook. 'Relating Representations' is omitted from strand 1 in textbooks 7, 8 and 9 and in strand 2 for textbooks 8 and 9. 'Inter Subject Connections' is absent from both strands in textbooks 7, 8 and 9. ''Critiquing' is absent from strand 2 in both textbooks 7 and 8. The S1&2 Project Maths syllabus also places weight on the presence of 'Instrumental and Relational Learning', 'Fostering Positive Attitudes', 'Inquiry Based Learning', 'Applications' and making 'Connections between Solutions and Answers'.
Constructed Curriculum 5: LC S1&2/H **Associated textbooks:** **TB 10;** **TB 11**	The syllabus suggests that the expectation 'Generalising' should be evident in strand 1. This expectation is omitted from both textbooks (see Tables F.11a to F.11c). Both textbooks are also missing three further expectations from strand 2; 'Developing Strategies', 'Developing Algorithms', and 'Critiquing'. There are two further expectations omitted from textbook 10, 'Critiquing' is omitted from strand 1 and 'Across Subject Connections' is omitted from strand 2. For textbook 11 there are three expectations absent from strand 1, 'Using Equipment', 'Inter Subject Connections' and 'Across Subject Connections'. The Project Maths syllabus also makes direct references to a number of further expectations; 'Instrumental and Relational Learning', 'Fostering Positive Attitudes', 'Inquiry Based Learning', 'Applications' and making 'Connections between Solutions and Questions'.
Constructed Curriculum 6: LC S2/H **Associated textbooks:** **TB 12** **TB 13;** **TB 14**	The expectation analysis of all three textbooks identified two expectations that are outlined in the Project Maths syllabus but omitted from all textbooks (Tables F.11d. to F.11g). These expectations are 'Developing Strategies' and 'Critiquing'. Textbook 12 is also missing 'Developing Algorithms' and 'Across Subject Connections'. 'Generalising' and 'Across Subject Connections' are omitted from textbook 13 while textbook 14 is missing Developing Algorithms' and 'Generalising'. The Project Maths syllabus also makes direct references to a number of further expectations 'Instrumental and Relational Learning', 'Fostering Positive Attitudes', 'Inquiry Based Learning' and 'Applications'.

5. Conclusion

5.1 Discussion and conclusions

Project Maths places particular emphasis on teaching for understanding, problem solving and real life applications promoted through active teaching methodologies including ICT and appropriate assessment strategies. Textbooks will be considered successful to the extent that they facilitate the needs of Project Maths. The data analysis and findings provide a perspective on the alignment between the textbooks and Project Maths.

Table: 5.1 Discussion and conclusion for each textbook

Textbook	
1	This Junior Cycle textbook has a high level of narration and a low level of instructional narration. This indicates a genuine concern for explanations and a move away from procedural-type materials. The definitions, axioms, theorems, etc are accentuated by printing devices such boxes and colour thus adding learning impact. The large number of graphics in this textbook adds little to the educational value of the text because a very small proportion of these graphics are related to problems as Project Maths stipulates. The ratio of exercises to problems is 3:1 and this balance is inconsistent with Project Maths particularly in the context of non-routine problems being 2% of the overall total of exercises and problems. An unusual feature in presentation occurs in strand 4 (Algebra) where problems are not integrated throughout the associated chapters (8). Content analysis shows gaps in textbook content compared to the corresponding Project Maths syllabus. One of these omissions viz. 'Use of Computer Software' has particular significance in light of the emphasis that Project Maths places on use of ICT. Consideration for motivational factors is evident and this is captured in the data on related narration. The gaps that are evident from the expectation analysis indicate a lack of focus on mathematical thinking e.g. 'Generalising', 'Inter and Across Subject Connections' and 'Relating Representations'. Problem solving in context is underrepresented despite being highlighted in Project Maths syllabus. These findings run counter to the general expectation of Project Maths.
TB 2	This textbook for the Common Introductory Course is a subset of textbook 1 (chapters 1-14; chapters 26, 27) previously discussed. The only concern here is whether this book is aligned with Project Maths in terms of content and expectation since it inherits the other characteristics. Despite some minor omissions from strand 3 and other omissions as identified for the previous

	textbook this book is reasonably well aligned with the CIC syllabus in content terms. In terms of expectations strand 4 (Algebra) seems to be far removed from Project Maths with key expectations omitted such as ‘Conjecturing’, ‘Justifying and Proving’, ‘Using Vocabulary and Notation’ and ‘Describing/ Discussing’. Apart from strand 4 the main issues in relation expectations are in the area of problem solving, particularly in strands 2 and 4 that relate ‘Performing more Complex Procedures, ‘Formulating and Clarifying Problems’, and ‘Developing Strategies’.
TB 3	This Junior Cycle textbook exhibits high levels of narration and high levels of instructional narration compared to other CIC textbook. While a high level of narration is reflective of concern for explanations the level of instructional narration is not so high as to cause concern. An exceptionally high number of exercises in this book contributes to a poor ratio of exercises to problems (3.5:1). This ratio masks a good incidence of problems throughout the textbook. Of the two CIC textbooks this book has a much higher incidence of non- routine problems. However, this textbook has a special extra section for the problem solver. The only noteworthy content omission is ‘Estimating Computations’ from strand 1. Motivational factors such as historical references, biographies and career information are given little consideration which is reflective of the low levels of related narration found in the structure analysis. Again strand 4 (Algebra) seems to be far removed from Project Maths with key expectations omissions as identified previously. Other individual strands fail to target expectations such as ‘Generalising’, ‘Recognising Equivalents’, ‘Developing Strategies’ and ‘Predicting’.
TB 4	Once again this textbook is a subset of another textbook, TB 1 (chapters 3, 7, 14, 11, 12, 13, 18, 24-28). As previously, the only concern here is with syllabus alignment as regards content and expectation. The only content gap evident is ‘Use of Computer Software’ which was addressed previously. Issues similar to those in textbook TB 1 related to mathematical thinking were identified in the expectation analysis.
TB 5	This Junior Cycle textbook exhibits a high level of narration and instructional narration combined with low levels of related narration. The levels of instructional narration are not so high as to cause concern. As previously outlined high narration identifies a concern for explanations. The number of graphics present in this textbook is plentiful and 20% are concerned with problems which is a step towards the Project Maths expectation of graphic use. A surprising finding is the reduction in the number of exercises in comparison to its counterpart CIC textbook. This reduction improves the proportion of exercises and problems that are problems but there is no objective measure to judge whether this

	improves the efficacy of the text for the learner. The content of this textbook aligns well with the Project Maths syllabus apart from the 'Use of Computer Software' content topic. The only minor omission is 'Counting Principles'. In conjunction with the ICT deficiency noted in the content analysis, further analysis identified that references to computer software and the internet are minimal. While colour is consistent across this textbook the use of other motivational factors is minimal. The gaps that are evident from the expectation analysis indicate a lack of general focus on mathematical thinking e.g. 'Generalising' and strand-specific omissions, 'Inter and Across Subject Connections' and 'Relating Representations'. Problem solving is an issue for strand 2 where 'Developing Strategies' 'Using more Complex Procedures' and 'Conjecturing' do not feature. These findings run counter to the general expectation of Project Maths.
TB 6	High narration, instructional narration and low related narration are features of this textbook and these features are similar to previous textbook discussed. Similar findings for graphics are evident in both textbooks with 19% of graphics used for problem solving. However, this textbook exhibits a much lower incidence of problems in relation to the comparable textbooks for this syllabus. An important content gap evident is 'Use of Computer Software' which was addressed for previous textbooks. In conjunction with this ICT deficit noted in the content analysis, further analysis identified that references to computer software and the internet are absent. The use of colour is more limited in this textbook than the previous books and also the incidence of motivational factors is almost non-existent with just one reference to career information. Issues similar to those identified for the previous textbook related to mathematical thinking and problem solving were again evident in the expectation analysis.
TB 7	This Senior Cycle Ordinary level textbook is characterised by high levels of narration and related narration. There are a large number of definitions, theorems and axioms highlighted in comparison to comparable textbooks. The learning impact of definitions, axioms, theorems, etc is accentuated by printing devices such boxes and colour. The large number of graphics in this textbook adds little to the educational value of the text because only a small proportion of these graphics are related to problems counter to what Project Maths stipulates. The ratio of exercises to problems (2:1) is a better alignment with Project Maths expectations. The content topic 'Use of Computer Software' is an issue of concern for this textbook particularly in strand 2. However, strand 1 includes 6 references to computer software and 6 references to the internet. This book exhibits the highest incidence of motivational factors compared to the other Senior Cycle Ordinary level textbooks which is consistent with a high level of related

	narration. There is also a wide variety and consistent use of colour which contributes to student comprehension. The gaps that are evident from the expectation analysis indicate a lack of focus on mathematical thinking e.g. 'Generalising' and 'Inter and Across Subject Connections'. 'Developing Strategies' is omitted from strand 2.
TB 8	This Senior Cycle Ordinary level textbook exhibits a high level of narration and instructional narration combined with low levels of related narration. As previously outlined high narration identifies a concern for explanations, the levels of instructional narration are not so high as to cause concern. The number of graphics present in this textbook is plentiful and almost 40% are concerned with problems which is a step towards the Project Maths expectation of graphic use. The ratio of exercises to problems is almost 1:1 and is again a good alignment with the Project Maths expectations. As outlined in previous textbooks an important content gap evident is 'Use of Computer Software', further analysis also noted an omission of 'Exponents, Roots and Radicals' from strand 2. The use of colour is consistent throughout the textbook, however, there is a very low incidence of other motivational factors. The gaps evident from the expectation analysis indicate a lack of focus on mathematical thinking e.g. 'Generalising' and 'Across Subject Connections'. 'Developing Strategies' and 'Critiquing are also omitted from strand 2.
TB 9	High narration, high instructional narration and low related narration are features of this textbook and these features are similar to previous textbooks discussed. Similar findings for graphics are evident in both textbooks with 35% of graphics used for problem solving here. This textbook also makes use of printing devices to improve the learning impact of definitions e.g. colour and boxes. While the ratio of exercises to problems is almost 3:1, this textbook does present the highest number of non-routine problems (118) compared with the comparable textbooks for this syllabus. Similar content findings are evident for this textbook and the one previously discussed. The main areas of content concern are the omission of 'Use of Computer Software' from both strands and the absence of an emphasis on the use of 'Exponents, Roots and Radicals' from strand 2. Further analysis with regard to ICT identified that this textbook makes only 2 references to the internet; minor reference to motivational factors, and a limited use of colour. The gaps evident from the expectation analysis again indicate a lack of focus on mathematical thinking e.g. 'Generalising' and 'Across Subject Connections'. 'Inter Subject Connections' is omitted from strand 1 while 'Developing Strategies' is also omitted from strand 2.

TB 10	This Senior Cycle Higher level textbook contains high levels of narration and related narration. However, it is important to note that the levels of instructional narration in this textbook are much higher than in the previous textbooks from this series. This textbook also demonstrates a concern for explanations with the large number of definitions, theorems and axioms highlighted in comparison to comparable textbooks for this syllabus. It has a much higher number of graphics than comparable textbooks, however, only a small proportion of this number is related to problem solving. The balance between problems and exercise is improved from the ordinary level counterpart textbook by a ratio of 0.9:1 exercises to problems, and by the presence of 183 non-routine problems and 29 activities. This is a much better reflection of the Project Maths intentions. Again, lack of emphasis on 'Use of Computer Software' is evident from the content analysis of this textbook. There is one other omission; 'Proportionality Problems' which is absent from strand 2. Further analysis of the content shows that this textbook includes 14 references to computer software and 12 internet references. These findings in conjunction with the higher incidence of motivational factors, and variety and consistent use of colour suggest a concern for the Project Maths intentions. Once again gaps evident from the expectation analysis indicate a lack of focus on mathematical thinking e.g. 'Generalising' omitted from strand 1, and 'Developing Algorithms', 'Developing Strategies' & 'Across Subject Connections' from strand 2. 'Critiquing is also absent from both strands.
TB 11	This Senior Cycle Higher level textbook exhibits a high level of narration and instructional narration combined with low levels of related narration. As previously outlined, high narration identifies a concern for explanations, the level of instructional narration is not so high as to cause concern. The number of graphics present in this textbook is generous and almost 40% are concerned with problems which is a step towards the Project Maths expectation of graphic use. The ratio of exercises to problems is 0.9:1, and combined with the high number of non-routine problems (226) this is a good alignment with the Project Maths expectations. Again the main area of concern with regard content is the omission of 'Use of Computer Software'; further analysis identified 17 references to computer software and 2 internet references. A weak attempt is made to include motivational factors in this textbook while the use of colour is also limited. The gaps evident from strand 1 of the expectation analysis indicate a lack of focus on mathematical thinking e.g. 'Generalising' and 'Inter & Across Subject Connections'. Problem solving is a lesser issue for strand 2 where 'Developing Strategies' and 'Critiquing' do not feature.

TB 12	This Senior Cycle Higher level textbook is a subset of textbook TB 10 (chapters 4-6 and 8-11). As previously, the only concern here is with syllabus alignment as regards content and expectation. The only content gap evident is ‘Use of Computer Software’ which was addressed previously. The main expectation concerns are evident in the omission of ‘Developing Algorithms’, ‘Developing Strategies’ & ‘Across Subject Connections’ and ‘Critiquing’ from strands 2. These expectations are of concern for mathematical thinking and problem solving as espoused by Project Maths.
TB 13	This Senior Cycle Higher level textbook exhibits a high level of narration and combined with low levels of instructional narration and related narration. The number of graphics present in this textbook is plentiful and 37% are concerned with problems. Again, this is in line with Project Maths expectations of graphic use. The ratio of exercises to problems (0.5:1) combined with the high number of non-routine problems (147) signals a good alignment with the Project Maths expectations in this regard. The content topic ‘Use of Computer Software’ is again omitted. Further analysis identified no references to computer software or the internet, and few incidences of motivational factors. The gaps evident from the expectation analysis indicate a lack of focus on mathematical thinking e.g. ‘Generalising’, ‘Developing Algorithms’ and ‘Critiquing’.
TB 14	This Higher level textbook is a subset of textbook TB 11 (chapters 3-6) which was previously discussed. As previously, the structural concerns include a low level of related narration. The main issues with this textbook are with syllabus alignment as regards content and expectation. The only content gap evident is ‘Use of Computer Software’ which was addressed previously and expectation gaps are again evident in terms of mathematical thinking; ‘Developing Algorithms’, ‘Developing Strategies’ & ‘Critiquing’ and ‘Across Subject Connections’.

5.2 Summary

All textbooks included in the study fall short of the standard needed to support Project Maths (intended curriculum) effectively, as envisaged in the Project Maths Syllabus documents for Junior Cycle, including the Common Introductory Course (CIC), and Senior Cycle. However, some of the new textbooks are better aligned to Project Maths expectations than others. The individual profiles of textbooks as developed in this study are not always consistent with each other across their associated curricula.

A number of issues stand out when these profiles are considered together. It is noteworthy that there should be any topic omissions when the Project Maths syllabus treats all topics as compulsory. For example, Project Maths stresses the integration of ICT into the curriculum and despite this emphasis 'Use of Computer Software' repeatedly emerged as a content omission. Project Maths expectations in relation to teaching for understanding, problem solving and using real life applications, and integration of ICT are addressed to variable degrees within the new textbooks. All these textbooks display a genuine attempt to match Project Maths expectations but no one textbook meets all the Project Maths expectations as identified above.

The textbook analysis captures elements that contribute to all the expectations. Structure and content analysis uncovers disparities between the textbooks in their approaches to teaching for understanding and problem solving. This is evident in features such as the distribution of narration, narration types, and abundance of non-routine problems and problems in context. Some textbooks show greater consideration for teaching for understanding in terms of narration and related narration, and the use of printing devices to aid comprehension. The treatment of problem solving is variable across all textbook series and no one textbook series deals with this aspect satisfactorily. However, some individual Senior Cycle textbooks are better in this regard. Another contribution to teaching for understanding is the inclusion of motivational factors that help to place mathematics in context.

The most significant overall finding is the mismatch between textbook expectations and Project Maths expectations. The lack of obvious attention to key Project Maths expectations must be attended to successfully if textbooks are to fulfil their role in the success of Project Maths. While the developmental nature of Project Maths is on a strand by strand basis, this militates against topic integration, and, when the roll out is complete, a more integrated approach should feature in textbooks.

5.3 Recommendations

These considerations lead to a single integrated recommendation:

- An exemplar textbook series for Project Maths should be produced by a specially selected and constituted writing team appointed and funded by the DES,
- All commercially produced textbooks for Project Maths should then be reviewed against this exemplar textbook series
- Such a review procedure should lead to an approved list of mathematics textbooks for Project Maths.

6. References

Conway, P and F. Sloane, (2005). *International Trends in Post Primary Mathematics Education: Perspectives on Learning, Teaching and Assessment*, NCCA, Dublin: Government Publications.

Mikk, J. (2000). Textbook Research and Writing. Oxford: Lang

O'Keeffe, L. (2011). *An investigation into the nature of mathematics textbooks at Junior Cycle and their role in mathematics education.* Unpublished PhD thesis, University of Limerick.

NCCA. *Project Maths Syllabus Documents*, undated [obtained from NCCA June 2011].

River's, J. (1990). Contextual analysis of problems in Algebra 1 textbooks. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, MA, April

Robitaille, D.F., Schmidt, W.H., Raizen, S.A., McKnight, C.C., Britton, E.D., and Nicol, C. (1993). *Curriculum Frameworks for Mathematics and Science, Vol 1.* Vancouver, Canada: Pacific Educational Press.

Valverde, G.A., Bianchi, L.J., Wolfe, R.G., Schmidt, W.H., and Houang, R.T. (2002). *According to the Book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

7. Glossary of terms

Narration:

this refers to all text and information presented in the textbook outside of exercises and examples. Narration was counted in blocks, sentence by sentence.

Related Narration:

this refers to all text and information devoted to providing additional information such as historical significance, biographical and career information.

Instructional Narration:

this refers to all commands and instruction which are outside the exercise and example context. High levels of instructional narration indicate an emphasis on the procedural nature of mathematics.

Definitions:

this refers to all definitions which are bolded and highlighted from the narration. Embedded definitions were not counted.

Routine Problem:

this refers to all problems which are 'dressed up exercises'.

Non- Routine Problem:

this refers to all problems which cannot be answered by a routine procedure or problems in which it is not immediately obvious what one must do.

Real Context Problem:

this refers to the type of problems which are created in a real environment and which involve or engage the student.

Realistic Context Problem:

this refers to the type of problems which have the possibility of being reproduced, involves a simulation of reality.

Fantasy Context Problem:

this refers to the type of problems which are not based on reality and are the product of imagination.

Purely Mathematical Context Problem:

this refers to the type of problems which are exclusively mathematical and relate only to mathematical objects.

Project Math Syllabus Strands:

Strand 1 (S1): Probability and Statistics
Strand 2 (S2): Geometry and Trigonometry
Strand 3 (S3): Number
Strand 4 (S4): Algebra
Strand 5 (S5): Functions

8. Appendices

Appendix A

Table A.1: List of Junior Cycle reviewed textbooks

Constructed Curriculum:	Publisher	Level:	Code	Textbooks
1. Strand 1 - 5	A	Ordinary Level	JC TB 1 S1-5/O	Textbook 1
			JC SY S1-5/O	Syllabus
2. CIC	A	All Levels	JC TB 1 CIC	Textbook 2
	B		JC TB 2 CIC	Textbook 3
			JC SY CIC	Syllabus
3. Strands 1 & 2	A	Ordinary Level	JC TB 1 S1&2/O	Textbook 4
	B		JC TB 2 S1&2/O	Textbook 5
	C		JC TB 3 S1&2/O	Textbook 6
			JC SY S1&2/O	Syllabus

Table A.2: List of Senior Cycle Ordinary level reviewed textbooks

Constructed Curriculum:	Publisher	Level:	Code	Textbooks
4. Strands 1 & 2	A	Ordinary Level	LC TB 1 S1&2/O	Textbook 7
	B		LC TB 2 S1&2/O	Textbook 8
	C		LC TB 3 S1&2/O	Textbook 9
			LC SY S1&2/O	Syllabus

Table A.3: List of Senior Cycle Higher level reviewed textbooks

Constructed Curriculum:	Publisher	Level:	Code	Textbooks
5. Strands 1 & 2	A	Higher Level	LC TB 1 S1&2/H	Textbook 10
	C	Higher Level	LC TB 2 S1&2/H	Textbook 11
			LC SY S1&2/H	Syllabus
6. Strand 2	A	Higher Level	LC TB 1 S2/H	Textbook 12
	B	Higher Level	LC TB 2 S2/H	Textbook 13
	C	Higher Level	LC TB 3 S2/H	Textbook 13
			LC SY S2/H	Syllabus

Appendix B

Document Analysis Form

Date: ____________ Document ID Code: ____________ Unit ID Number: ____________ No. ____ of ____

Page No.												
Block ID												
Structure (Block Type)												
Content												
Expectation												
Notes:												

Appendix C

A further refinement of the curriculum material analysis tool in TIMSS identified a number of textbook content topics. This topic list which is intended for content analysis was further refined for this study. The TIMSS content list originally had ten key elements:

1.1 Numbers
1.2 Measurement
1.3 Geometry: position, visualisation and shape
1.4 Geometry: symmetry, congruence and similarity
1.5 Proportionality
1.6 Functions, relations and equations
1.7 Data representation, probability and statistics
1.8 Elementary analysis
1.9 Validation and structure
1.10 Other

TIMSS then further categorised each of these ten categories into subdivisions of content topics. For the purpose of this analysis six additional explicit categories were included. These are:

- The use of Computer Software
- Measurement (as a content topic as opposed to a section as outlined in TIMSS)
- Factorisation
- Trigonometry (as a separate section from Geometry)
- Definition of Function
- Irrational Numbers
- Use of the Internet

The above topics and the TIMSS content topics combine to give the following list of content topics:

Content Data List:

1. Meaning
2. Operations
3. Properties of Operations
4. Common Fractions
5. Decimals
6. Relationships with common & decimal
7. Percentages
8. Properties of common & Decimal
9. Negative Numbers & their properties
10. Rational Numbers and their properties
11. Real numbers, their subsets & properties
12. Binary Arithmetic
13. Exponents, roots and radicals
14. Complex numbers and their properties
15. Number Theory
16. Counting
17. Estimation & Number Sense
18. Estimating quantity & size
19. Rounding & significant figures
20. Estimating computations
21. Exponents & orders of magnitude
22. Set properties
23. Set operations
24. Venn Diagrams
25. Units
26. Perimeter, Area & Volume
27. Estimation & Errors
74. Measure
28. 2D Geometry: Coordinate Geometry –
29. 2D Geometry: Coordinate Geometry –
30. 2D Geometry: Basics – angles & lines
31. 2D Geometry: Basics – shapes &
32. 2D Geometry: Circles
33. 3D Geometry/problems
34. Transformations
35. Congruence & Similarity
36. Constructions using straightedge &
37. Proof & Theorems
38. Proportionality Concepts
39. Proportionality Problems
40. Linear interpolation & extrapolation
76. Trig
41. Area
42. Trig Ratios
43. Patterns, Relations & Analysis
44. Equations & formulae
77. Definition of Function
45. Domain & range
46. Function classification
47. Linear Functions
48. Quadratics Functions
49. Cubic Functions
50. Exponential Functions
51. Logarithmic Functions
52. Trigonometric Functions
53. Inequalities
75. Factorisation
54. Approximating Values
55.Maximum & Minimum Values
56. Data Representation & Analysis
57. Classification of Data
58. Classification of Studies (including
59. Summary Statistics
60. Randomisation (including Bias)
61. Inferential Statistics
62. Counting principles
63. Permutations & Combinations
64. Defining Probability
65. Measuring Probability
66. Laws of Probability
67. Probability Experiments
68. Infinite Processes
69. Change
70. Motion
71. Validation & Justification
72. Structuring & Abstracting
73. Software
78. Irrational Numbers
79. Internet

Appendix D

Junior Cycle Data

TIMSS+ Analysis – Structure, Content & Expectation

Appendix D1

Narration

Table D.1: Distribution of Narration and Narration type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Narration	Related Narration	Instructional Narration
1 (JC S1-5 O)	1	1107	99	25
	SY	-	-	-
2 (JC CIC)	2	631	76	10
	3	884	14	45
	SY	-	-	-
3 (JC S1&2 0)	4	505	74	10
	5	703	16	32
	6	579	7	47
	SY	-	-	-

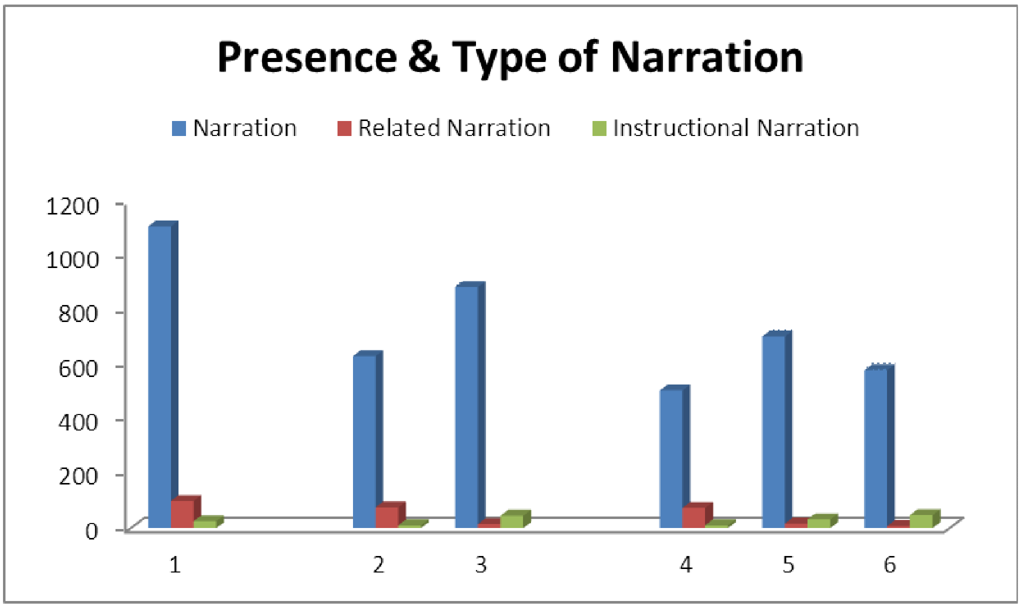

Figure D.1: Distribution of Narration and Narration type throughout the Junior Cycle Mathematics Textbooks

Appendix D2

Definitions

Table D.2: Distribution of Definitions, Theorems & Axioms throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Definitions	Theorems	Axioms & Corollaries
1 (JC S1-5 O)	1	152	11	7
	SY	-	-	-
2 (JC CIC)	2	99	7	4
	3	33	1	1
	SY	-	-	-
3 (JC S1&2 0)	4	74	11	7
	5	19	2	1
	6	32	1	0
	SY	-	-	-

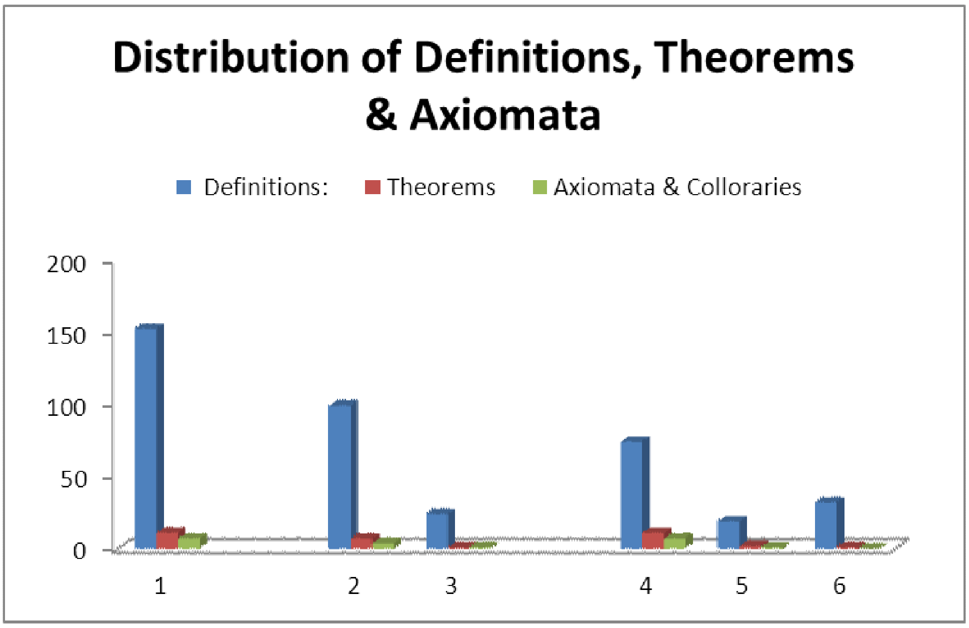

Figure D.2: Distribution of Definitions, Theorems & Axioms throughout the Junior Cycle Mathematics Textbooks

Appendix D3

Graphics

Table D.3: Distribution and Purpose of Graphics throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Graphics	Graphics for Problem Solving	Realistic Diagrams	Photos
1 (JC S1-5 O)	1	1616	334	193	204
	SY	-	-	-	
2 (JC CIC)	2	853	171	103	126
	3	949	196	150	12
	SY	-	-	-	
3 (JC S1&2 0)	4	1096	211	164	125
	5	822	177	112	10
	6	720	139	83	0
	SY	-	-	-	

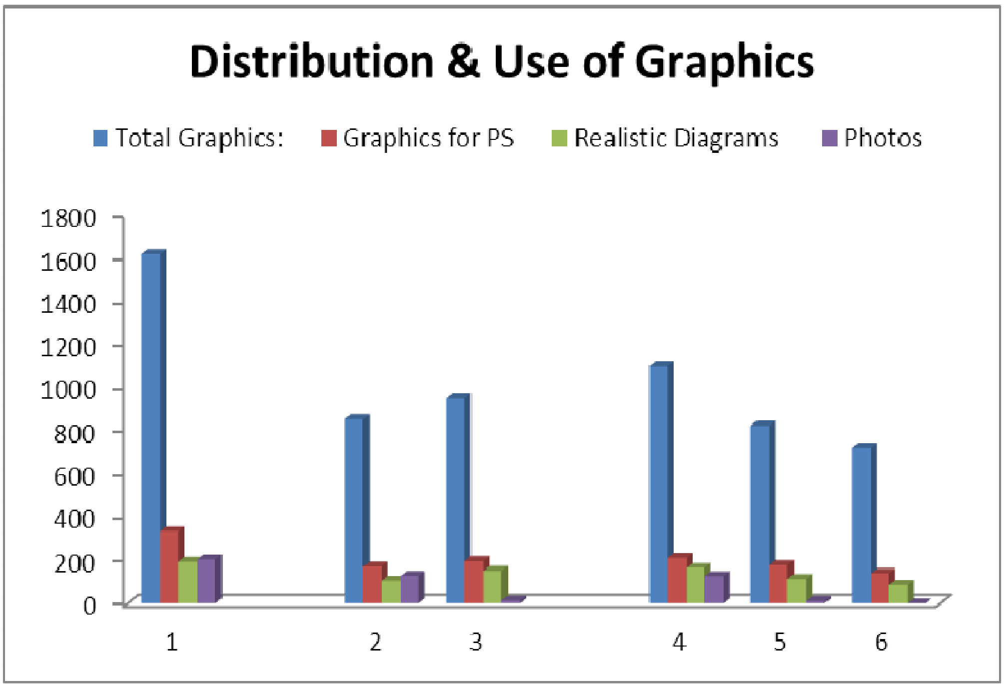

Figure D.3: Distribution and Purpose of Graphics throughout the Junior Cycle Mathematics Textbooks

Appendix D4

Exercises

Table D.4: Distribution of Exercises, Problems and Activities throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Exercise Blocks	Problem Blocks	Exercises	Problems	Activities
1 (JC S1-5 O)	1	343	100	2025	690	16
	SY	-	-	-		
2 (JC CIC)	2	189	55	782	349	13
	3	150	89	1843	532	8
	SY	-	-	-		
3 (JC S1&2 0)	4	176	45	612	280	16
	5	58	35	482	225	5
	6	46	27	556	152	1
	SY	-	-	-		

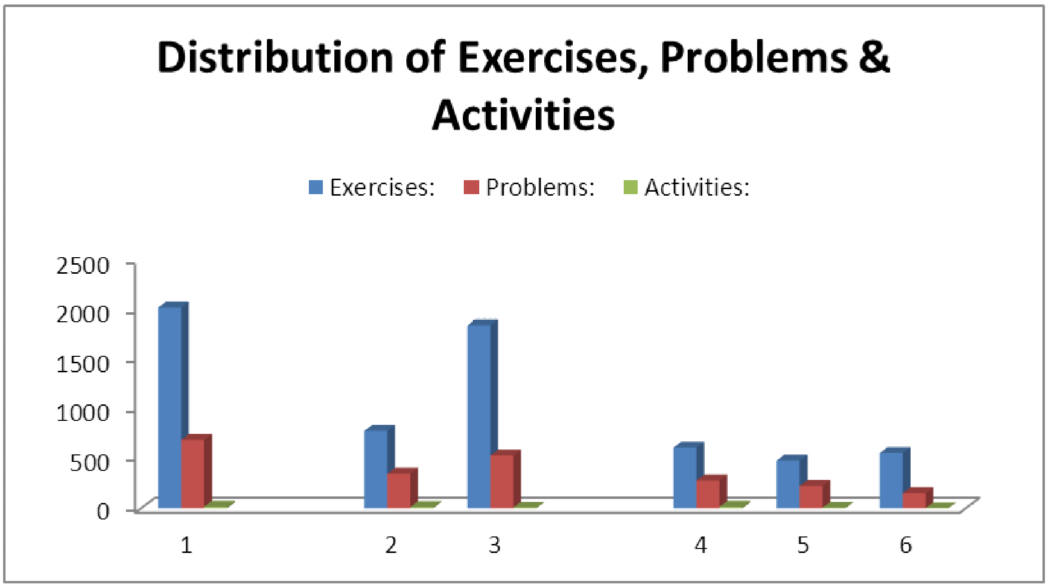

Figure D.4: Distribution of Exercises, Problems and Activities throughout the Junior Cycle Mathematics Textbooks

Table D.4a: Percentage Breakdown of Exercises and Problems throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Total Exercises + Problems	% of which are Problems
1 (JC S1-5 O)	1	2715	25.41%
	SY	-	-
2 (JC CIC)	2	1081	32.28%
	3	2375	22.4%
	SY	-	-
3 (JC S1&2 0)	4	892	31.39%
	5	707	31.83%
	6	708	21.47%
	SY	-	-

Appendix D5

Examples

Table D.5: Distribution of Worked Examples throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Total No. of Worked Examples	Real/Realistic	Mathematical
1 (JC S1-5 O)	1	309	102	207
	SY	-	-	-
2 (JC CIC)	2	162	56	106
	3	108	21	87
	SY	-	-	-
3 (JC S1&2 0)	4	110	45	65
	5	58	16	42
	6	67	23	44
	SY	-	-	-

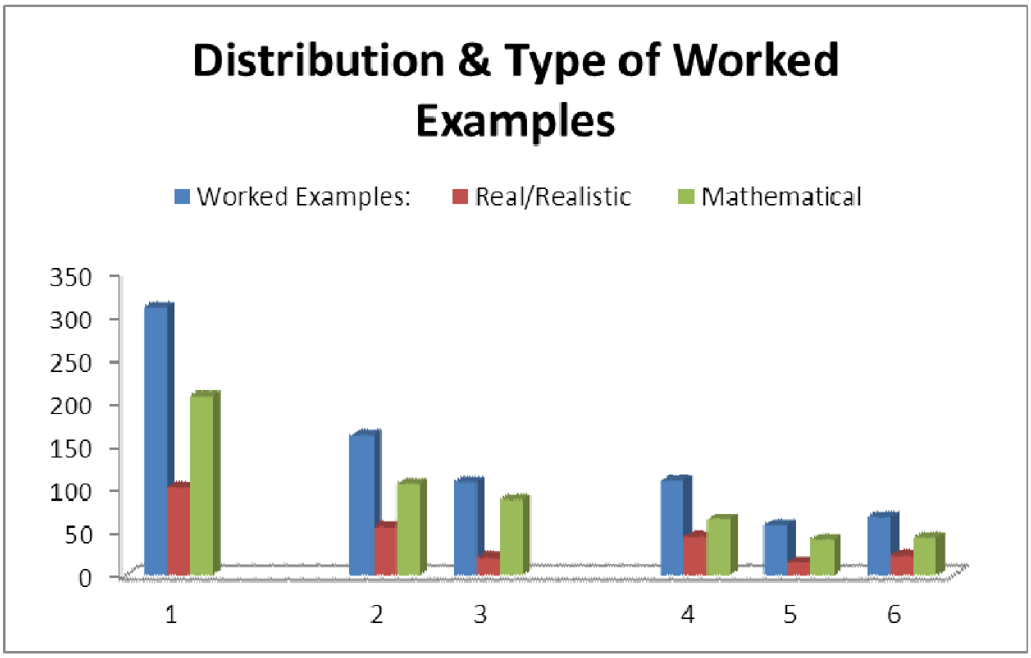

Figure D.5: Distribution of Worked Examples throughout the Junior Cycle Mathematics Textbooks

Table D.5a: Ratio of Worked Examples to Exercises, Problems & Activities throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Total Worked Examples	Ratio Examples: Exercises
1 (JC S1-5 O)	1	309	1:08.79
	SY	-	-
2 (JC CIC)	2	162	1:06.67
	3	108	1:21.99
	SY	-	-
3 (JC S1&2 0)	4	110	1:.08.11
	5	58	1:12.19
	6	67	1:10.57
	SY	-	-

Appendix D6

Junior Cycle Mathematics Textbook Content Analysis

The following tables (Table D.6a to D.6i) represent the content data from the Junior Cycle Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project Maths Curriculum and each row represents a specific content topic as derived from TIMSS. A list of 79 content topics was originally devised (see Appendix C), non-applicable content topics were removed after data collection hence the numbering is not continuous.

Content Data List:

1. Meaning
2. Operations
3. Properties of Operations
4. Common Fractions
5. Decimals
6. Relationships with common & decimal
7. Percentages
8. Properties of common & Decimal
9. Negative Numbers & their properties
10. Rational Numbers and their properties
11. Real numbers, their subsets & properties
13. Exponents, roots and radicals
16. Counting
17. Estimation & Number Sense
18. Estimating quantity & size
19. Rounding & significant figures
20. Estimating computations
21. Exponents & orders of magnitude
22. Set properties
23. Set operations
24. Venn Diagrams
25. Units
26. Perimeter, Area & Volume
27. Estimation & Errors
74. Measure
28. 2D Geometry: Coordinate Geometry –
30. 2D Geometry: Basics – angles & lines
31. 2D Geometry: Basics – shapes &
32. 2D Geometry: Circles
33. 3D Geometry/problems
34. Transformations
35. Congruence & Similarity
36. Constructions using straightedge &
37. Proof & Theorems
38. Proportionality Concepts
39. Proportionality Problems
76. Trig
41. Area
42. Trig Ratios
43. Patterns, Relations & Analysis
44. Equations & formulae
77. Definition of Function
45. Domain & range
47. Linear Functions
48. Quadratics Functions
49. Cubic Functions
50. Exponential Functions
53. Inequalities
75. Factorisation
54. Approximating Values
56. Data Representation & Analysis
57. Classification of Data
58. Classification of Studies (including
59. Summary Statistics
60. Randomisation (including Bias)
62. Counting principles
63. Permutations & Combinations
64. Defining Probability
65. Measuring Probability
66. Laws of Probability
67. Probability Experiments
69. Change
70. Motion
73. Use of Computer Software
79. Internet

Table D.6a: TB 1

S. 1	S. 2	S. 3	S. 4	S. 5	Content
		■			1
■	■	■	■	■	2
		■	■		3
■	■	■	■		4
	■	■			5
		■			6
		■			7
		■			8
		■	■		9
		■	■		10
			■		11
		■			13
■		■	■		16
	■	■			17
					18
■	■	■		■	19
■	■			■	20
		■			21
		■			22
		■			23
		■			24
	■				25
	■	■			26
	■				27
	■				74
	■			■	28
	■				30
	■				31
	■				32
	■				33
	■				34
	■				35
	■				36
	■				**37**
		■		■	38
		■			39
	■				76
	■				41
	■				42
		■	■	■	43
	■	■	■		44
				■	77
					45
		■	■	■	47
		■	■	■	48
			■		49
		■		■	50
			■	■	53
			■		75
				■	54
■		■			56
■					57
■					58
■					59
■					60
■					62
■					63
■					64
■					65
■					66
■					67
				■	69
		■		■	70
					73

Table D.6b: JC SY S1-5

S. 1	S. 2	S. 3	S. 4	S. 5	Content
		■			1
		■	■		2
		■	■		3
	■	■			4
		■			5
		■			6
		■			7
					8
		■			9
		■			10
		■			11
		■			13
					16
					17
					18
					19
■					20
					21
		■			22
		■			23
		■			24
					25
	■	■			26
					27
					74
	■				28
	■	■			30
	■				31
	■				32
		■			33
	■				34
					35
	■				36
	■	■			**37**
		■	■		38
		■	■		39
	■				76
					41
					42
			■		43
	■		■		44
					77
				■	45
			■	■	47
			■	■	48
					49
				■	50
					53
			■		75
				■	54
■			■	■	56
■					57
■					58
■					59
					60
■					62
■					63
■					64
■					65
■					66
■					67
			■		69
			■		70
■	■		■	■	**73**
		■			**79**

Table D.6c: TB 2

S. 1	S. 2	S. 3	S. 4	Cont.
		■		1
■	■	■	■	2
		■	■	3
■		■		4
		■		5
		■		6
		■		7
		■		8
		■	■	9
		■		10
		■		11
				13
■		■		16
		■		17
				18
■		■		19
■	■			20
				21
		■		22
		■		23
		■		24
				25
	■	■		26
	■			27
	■			74
	■			28
■	■			30
	■			31
				32
				33
	■			34
				35
	■			36
	■			**37**
		■		38
		■		39
				76
				41
				42
			■	43
	■		■	44
				77
				45
			■	47
			■	48
			■	49
				50
				53
				75
				54
				53
■		■		56
■				57
■				58
■				59
■				60
■				62
■				63
■				64
■				65
■				66
■				67
				69
				70
				73

Table D.6d: TB 3

S. 1	S. 2	S. 3	S. 4	Cont.
		■		1
■	■	■	■	2
		■	■	3
■		■		4
		■		5
		■		6
■		■		7
		■		8
		■		9
				10
				11
		■	■	13
				16
		■		17
		■		18
		■		19
		■		20
				21
		■		22
		■		23
		■		24
	■			25
	■		■	26
				27
	■	■		74
	■			28
	■		■	30
	■			31
				32
				33
	■			34
				35
	■			36
	■			**37**
		■		38
		■		39
				76
				41
				42
■	■	■	■	43
			■	44
				77
				45
				47
				48
				49
				50
		■		53
				75
		■		54
■				56
■				57
■				58
				59
				60
■				62
■				63
■				64
■				65
■				66
■				67
				69
		■		70
				73

Table D.6e: JC SY CIC

S. 1	S. 2	S. 3	S. 4	Cont.
		■		1
				2
		■		3
		■		4
		■		5
		■		6
		■		7
				8
		■		9
		■		10
		■		11
				13
				16
				17
				18
				19
■				20
				21
		■		22
		■		23
		■		24
				25
	■			26
				27
				74
	■			28
	■			30
				31
				32
				33
	■			34
				35
	■			36
	■	■		**37**
		■		38
				39
				76
				41
				42
			■	43
				44
				77
				45
				47
				48
				49
				50
				53
				75
				54
			■	56
■				57
■				58
				59
■				60
■				62
■				63
■				64
■				65
■				66
■				67
				69
				70
				73

Table D.6f:
TB 4

S. 1	S. 2	Cont.
		1
■	■	2
		3
■	■	4
	■	5
		6
		7
		8
		9
		10
		11
		13
■		16
	■	17
		18
■	■	19
■	■	20
		21
		22
		23
		24
	■	25
	■	26
	■	27
	■	74
	■	28
■	■	30
	■	31
	■	32
	■	33
	■	34
	■	35
	■	36
	■	**37**
		38
		39
	■	76
	■	41
	■	42
		43
	■	44
		77
		45
		47
		48
		49
		50
		53
		75
		54
■		56
■		57
■		58
■		59
■		60
■		62
■		63
■		64
■		65
■		66
■		67
		69
		70
		73

Table D.6g:
TB 5

S. 1	S. 2	Cont.
		1
■		2
		3
■	■	4
■	■	5
		6
■		7
		8
		9
		10
		11
		13
		16
		17
		18
		19
■		20
		21
		22
		23
		24
	■	25
	■	26
		27
	■	74
	■	28
■	■	30
	■	31
	■	32
		33
	■	34
		35
	■	36
	■	**37**
		38
		39
	■	76
	■	41
	■	42
		43
	■	44
		77
		45
		47
		48
		49
		50
		53
		75
		54
■		56
■		57
■		58
■		59
■		60
		62
■		63
■		64
■		65
■		66
■		67
		69
		70
		73

Table D.6h:
TB 6

S. 1	S. 2	Cont
		1
■	■	2
		3
■	■	4
■	■	5
		6
■	■	7
		8
		9
		10
		11
	■	13
		16
		17
		18
		19
■	■	20
		21
		22
		23
		24
	■	25
	■	26
		27
	■	74
	■	28
	■	30
	■	31
		32
		33
	■	34
		35
	■	36
	■	**37**
		38
		39
	■	76
		41
	■	42
		43
	■	44
		77
		45
		47
		48
		49
		50
		53
		75
		54
■		56
■		57
■		58
■		59
		60
■		62
■		63
■		64
■		65
■		66
■		67
		69
	■	70
		73

Table D.6i:
JC SY S1&2/O

S. 1	S. 2	Cont.
		1
		2
		3
	■	4
		5
		6
		7
		8
		9
		10
		11
		13
		16
		17
		18
		19
■		20
		21
		22
		23
		24
		25
	■	26
		27
		74
	■	28
	■	30
	■	31
	■	32
		33
	■	34
		35
	■	36
	■	**37**
		38
		39
	■	76
		41
		42
		43
	■	44
		77
		45
		47
		48
		49
		50
		53
		75
		54
■		56
■		57
■		58
■		59
		60
■		62
■		63
■		64
■		65
■		66
■		67
		69
		70
■	■	**73**

Appendix D7

Motivational factors (Rivers Matrix)

Table D.7: Distribution of Motivational Factors throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Historical Notes	Bio- graphies	Career Information	Problem Solving	Photos	Humour/ Quotes
1 (JC S1-5 O)	1	25	5	9	690	204	6
	SY	-	-	-	-	-	-
2 (JC CIC)	2	21	4	4	349	126	5
	3	6	0	0	532	12	0
	SY	-	-	-	-	-	-
3 (JC S1&2 0)	4	15	5	7	280	125	3
	5	4	1	1	225	10	0
	6	0	0	1	150	0	0
	SY	-	-	-	-	-	-

Appendix D8

Comprehension Cues (Rivers Matrix)

Table D.8: Summary of colour use throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Page Background Colour	Font Colour	Graph-line Colour
1 (JC S1-5 O)	1	Hints: Purple Formula: Blue Exercises: blue Examples: green Definitions: Green Keywords: green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green, Pink	Red, Blue, Black, Green, Purple, Navy
	SY	-	-	-
2 (JC CIC)	2	Hints: Purple Formula: Blue Exercises: blue Examples: green Definitions: Green Keywords: green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green	Red, Blue, Black,
	3	Hints/Def/Formulae: Yellow Keywords: Blue Examples: Green	Red, Black, Blue, Green	Black, Red
	SY	-	-	-
3 (JC S1&2 0)	4	Hints: Purple Formula: Blue Exercises: Blue Examples: Green Definitions: Green Keywords: Green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green, Pink	Red, Blue, Black, Green, Purple, Navy

	5	Hints/Definitions/Formulae: Yellow Keywords, Examples & Definitions: Interchange dependant on chapter colour (Purple (pink), Blue, Green, Red (orange))	Black & dependant on chapter colour. For example chapter 1 heading is Purple and writing is black & purple	Blue, Black, Red, Green
	6	Hints/Definitions/Formulae: Blue Examples & exercises: Yellow	Red, Black, Blue,	Blue, Red, Black
	SY	-	-	-
[redacted]				

Appendix D9

Technical Aids (Rivers Matrix)

Table D.9: Distribution of Technical Aids throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Computer Software	Calculator	Internet
1 (JC S1-5 O)	1	12	56 (22)	2
	SY	-	-	-
2 (JC CIC)	2	12	14 (16)	1
	3	0	19 (12)	0
	SY	-	-	-
3 (JC S1&2 0)	4	12	22	1
	5	0	11	0
	6	0	7	0
	SY	-	-	-

*The figure in brackets represents instances where the calculator is referred to in the context of 'Do not use your calculator'

Appendix D10

Problem Solving

Table D.10: Distribution of Routine & Non- Routine Problems throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Routine Problems	Non-Routine Problems
1 (JC S1-5 O)	1	645	45
	SY	-	-
2 (JC CIC)	2	329	20
	3	443	89
	SY	-	-
3 (JC S1&2 0)	4	264	16
	5	219	6
	6	150	2
	SY	-	-

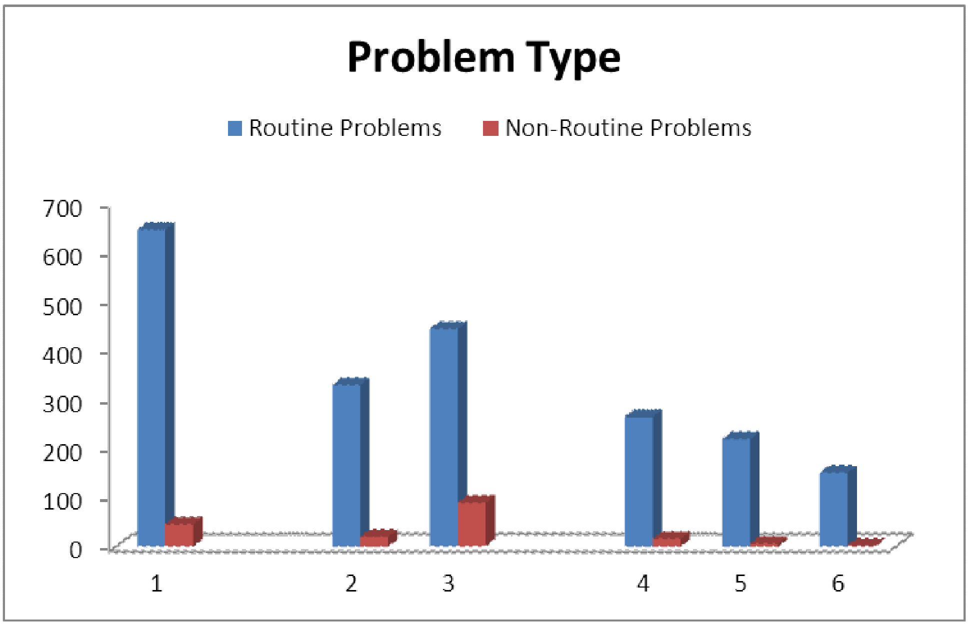

Figure D.6: Distribution of Routine & Non- Routine Problems throughout the Junior Cycle Mathematics Textbooks

Table D.10a: Breakdown of Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
1 (JC S1-5 O)	1	90	447	29	79
	SY	-	-		
2 (JC CIC)	2	65	235	4	25
	3	49	239	12	60
	SY	-	-		
3 (JC S1&2 0)	4	43	190	20	11
	5	29	163	5	22
	6	19	97	3	29
	SY	-	-		

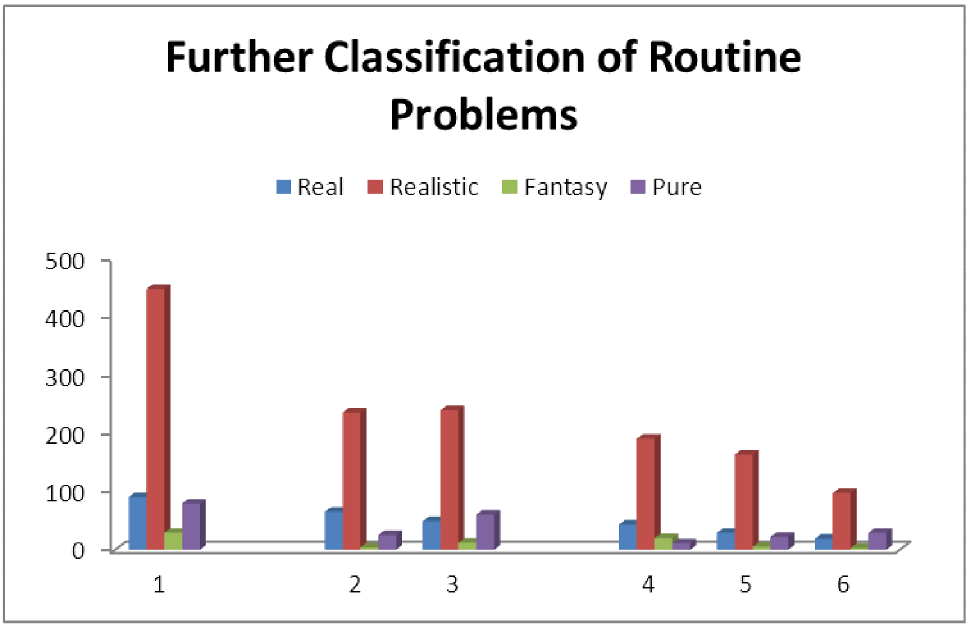

Figure D.7: Breakdown of Routine Problems throughout the Junior Cycle Mathematics Textbooks

Table D.10b: Breakdown of Non-Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
1 (JC S1-5 O)	1	4	32	2	7
	SY	-	-		
2 (JC CIC)	2	3	16	0	1
	3	0	29	16	44
	SY	-	-		
3 (JC S1&2 0)	4	0	12	0	4
	5	0	6	0	0
	6	0	2	0	0
	SY	-	-		

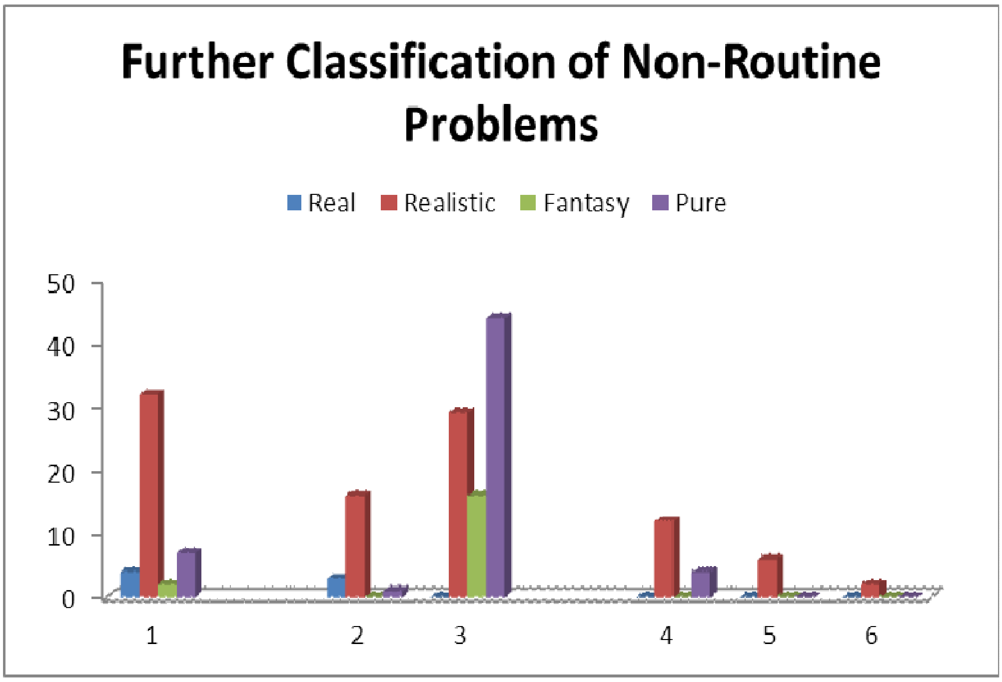

Figure D.8: Breakdown of Non-Routine Problems throughout the Junior Cycle Mathematics Textbooks

Appendix D11

Junior Cycle Mathematics Textbook Expectation Analysis

The following tables (Table D.11a to D.11i) represent the expectation data from the Junior Cycle Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project Maths Curriculum and each row represents a specific expectation as derived from TIMSS. A list of 24 expectations was originally devised, an additional 6 project Maths specific expectations are noted in the syllabi analysis.

Expectation Data List:

Knowing

1. Representing
2. Recognising Equivalents
3. Recalling Mathematical Objects & Properties

Using Routine Procedures

4.Using Equipment
5.Performing Routine Procedures
6.Using More Complex Procedures

Investigating & Problem Solving

7.Formulating & Clarifying Problems &
8.Developing strategies (Designing)
9.Solving
10.Predicting
11.Verifying

Mathematical Reasoning

12. Developing Notation & Vocabulary
13. Developing algorithms
14. Generalising
15.Conjecturing
16. Justifying & proving
17. Axiomatising

Communicating

18.Using Vocabulary & Notation
19.Relating Representations
20.Describing/Discussing
21.Critiquing

Making Communications

22.Inter Subject Connections
23.Across Subject Connections
24.Give Real Life Examples
25.Instrumental Learning
26.Relational Learning
27.Fostering Positive Attitudes
28.Inquiry Based Learning
29. Applications
30.Connecting Solutions & Questions

Table D.11a: TB 1

S. 1	S. 2	S. 3	S. 4	S. 5	Exp.
■	■	■	■	■	1
	■	■	■		2
■	■	■	■	■	3
■	■	■			4
■	■	■	■	■	5
■	■	■	■	■	6
■	■	■	■	■	7
■		■			8
■	■	■	■	■	9
■		■			10
	■	■	■	■	11
■					12
					13
					14
■	■	■	■	■	15
■	■	■		■	16
	■	■			17
■	■	■	■		18
					19
■	■	■	■	■	20
■		■			21
	■				22
					23
					24

Table D.11b: JC SY S1-5

S. 1	S. 2	S. 3	S. 4	S. 5	Exp.
■	■	■	■	■	1
	■	■			2
■	■	■	■	■	3
			■	■	4
■	■	■	■	■	5
■	■	■	■	■	6
■	■	■	■	■	7
■	■	■	■	■	8
■	■	*	*	*	9
■			■		10
		■			11
					12
		■			13
■	■	■	■	■	14
■	■	■	■	■	15
■	■	■	■	■	16
	■				17
■	■	■	■	■	18
■		■	■	■	19
■	■	■	■	■	20
■	■	■	■	■	21
■	■	■	■	■	22
■	■	■	■	■	23
					24
■	■	■	■	■	25
■	■	■	■	■	26
■	■	■	■	■	27
■	■	■	■	■	28
■	■	■	■	■	29
					30

*The red box identifies the expectation that problem solving should be in context and should make use of graphics.

Table D.11c: TB 2

S. 1	S. 2	S. 3	S. 4	Exp.
■	■	■	■	1
	■	■	■	2
■	■	■	■	3
■	■			4
■	■	■	■	5
■		■		6
■		■		7
■		■		8
■	■	■	■	9
■		■		10
	■	■		11
■				12
				13
				14
■	■	■		15
■	■	■		16
	■	■		17
■	■	■		18
				19
■	■	■		20
■		■		21
	■			22
				23
				24

Table D.11d: TB 3

S. 1	S. 2	S. 3	S. 4	Exp.
■	■	■		1
		■	■	2
■	■	■	■	3
	■	■		4
■	■	■	■	5
■	■	■	■	6
■	■	■	■	7
■				8
■	■	■	■	9
				10
	■			11
				12
	■			13
				14
■		■		15
■	■	■		16
				17
■	■	■		18
				19
■	■	■		20
				21
■		■		22
				23
				24

Table D.11e: JC SY CIC

S. 1	S. 2	S. 3	S. 4	Exp.
■		■	■	1
	■	■		2
■	■	■	■	3
				4
■	■	■	■	5
■	■	■	■	6
■	■	■	■	7
■	■	■	■	8
■	■	*	■	9
■			■	10
		■		11
				12
		■		13
		■	■	14
■	■	■	■	15
■	■	■	■	16
	■	■	■	17
■	■	■	■	18
				19
■	■	■	■	20
				21
				22
				23
				24
				25
				26
				27
	■	■		28
				29
				30

*The red box identifies the expectation that problem solving should be in context and should make use of graphics.

Table D.11f:
TB 4

S. 1	S. 2	Exp.
■	■	1
	■	2
■	■	3
■	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
	■	11
■		12
		13
		14
■	■	15
■	■	16
	■	17
■	■	18
		19
■	■	20
■		21
	■	22
		23
		24

Table D.11g:
TB 5

S. 1	S. 2	Exp.
■	■	1
	■	2
■	■	3
	■	4
■	■	5
■		6
■	■	7
		8
■	■	9
		10
	■	11
		12
	■	13
		14
■		15
■	■	16
		17
■	■	18
		19
■	■	20
■		21
		22
		23
		24

Table D.11h:
TB 6

S. 1	S. 2	Exp.
■	■	1
■	■	2
■	■	3
	■	4
■	■	5
	■	6
■	■	7
■		8
■	■	9
		10
	■	11
		12
		13
		14
■	■	15
■	■	16
		17
■	■	18
		19
■	■	20
	■	21
	■	22
		23
		24

Table D.11i:
JC SY S1&2/O

S. 1	S. 2	Exp.
■	■	1
	■	2
■	■	3
		4
■	■	5
■	■	6
■	■	7
■	■	8
■	■	9
■		10
		11
		12
		13
■	■	14
■	■	15
■	■	16
	■	17
■	■	18
■		19
■	■	20
■	■	21
■	■	22
■	■	23
		24
■	■	25
■	■	26
■	■	27
■	■	28
■	■	29
		30

Appendix E

Senior Cycle Ordinary Level Data

TIMSS+ Analysis – Structure, Content & Expectation

Appendix E1

Narration

Table E.1: Distribution of Narration and Narration type throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Narration	Related Narration	Instructional Narration
4. (LC S1&2/O)	7	689	75	15
	8	891	5	39
	9	665	6	67
	SY	-	-	-
[redacted]				

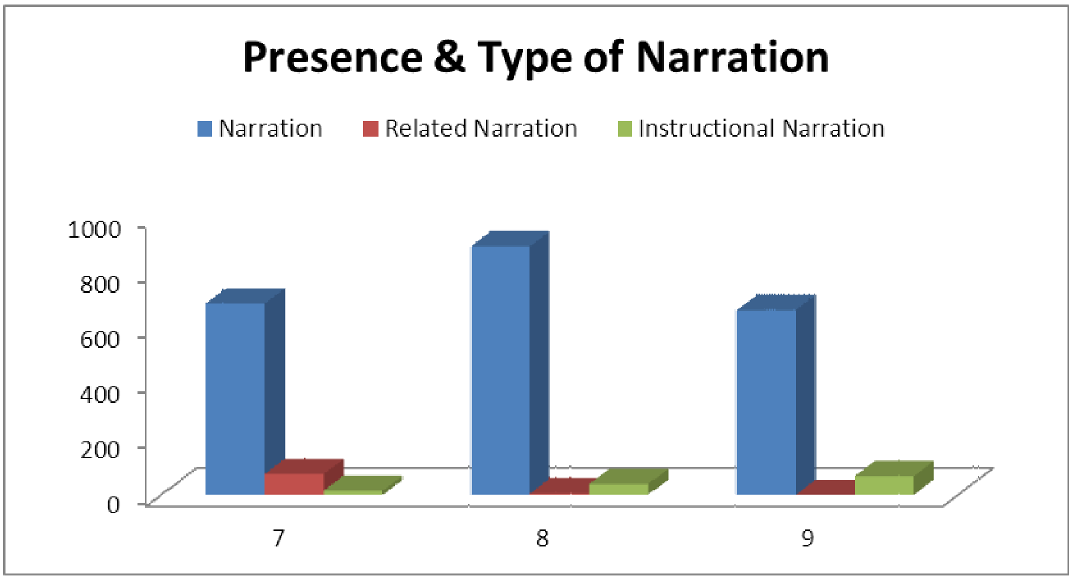

Figure E.1: Distribution of Narration and Narration type throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Appendix E2

Definitions

Table E.2: Distribution of Definitions, Theorems & Axioms throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Definitions	Theorems	Axioms & Corollaries
4. (LC S1&2/O)	7	119	22	9
	8	44	13	1
	9	71	1	0
	SY	-	-	-
[redacted]				

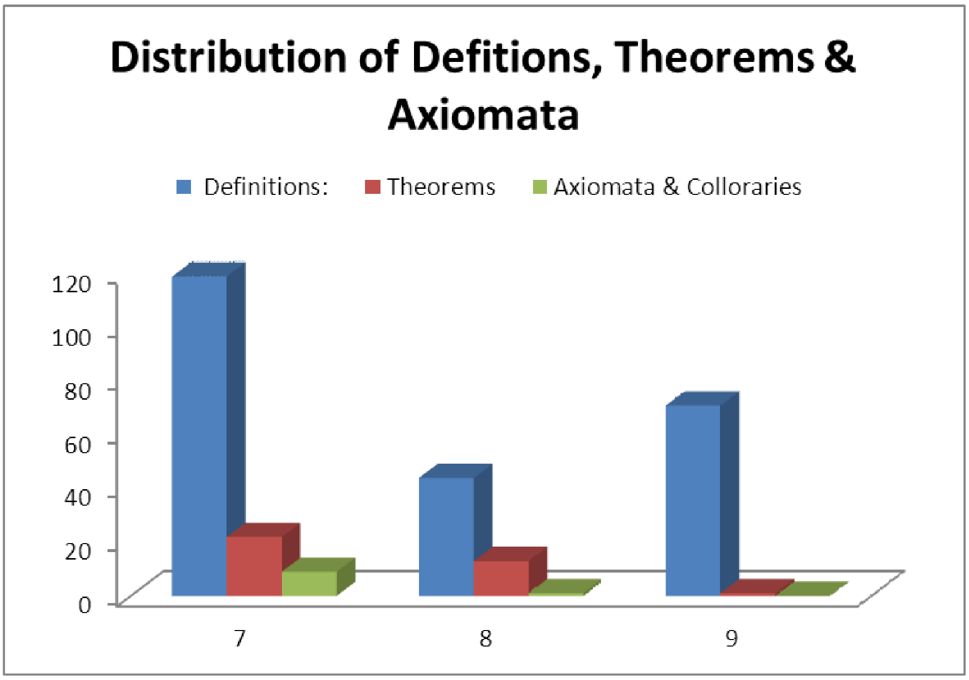

Figure E.2: Distribution of Definitions, Theorems & Axioms throughout the Senior Cycle Ordinary Level Mathematics Textbooks

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Appendix E3

Graphics

Table E.3: Distribution and Purpose of Graphics throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Graphics	Graphics for Problem Solving	Realistic Diagrams	Photos
4. (LC S1&2/O)	7	1218	230	151	96
	8	736	290	78	5
	9	641	227	78	2
	SY	-	-	-	
[redacted]					

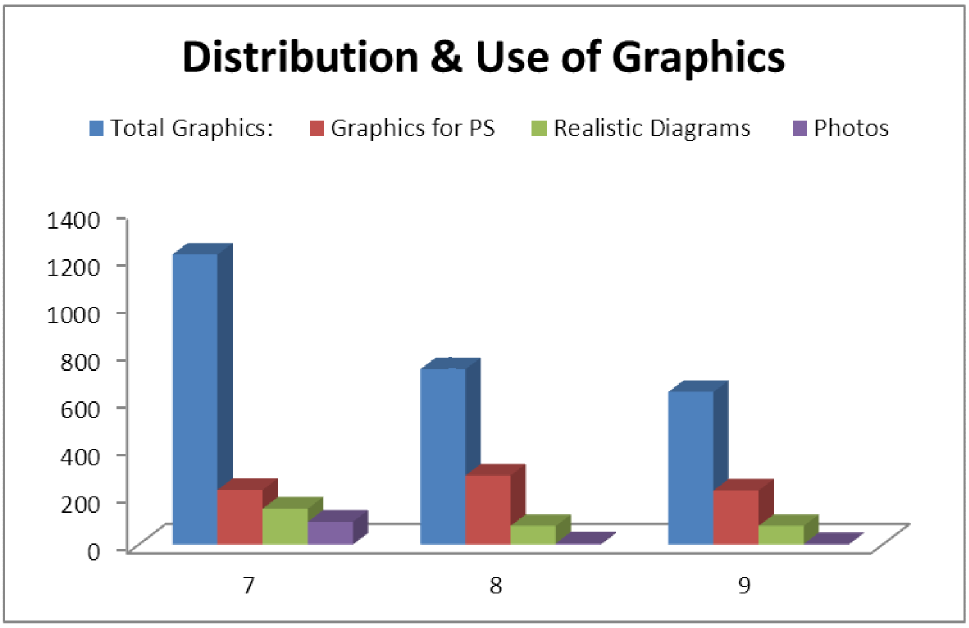

Figure E.3: Distribution and Purpose of Graphics throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Appendix E4

Exercises

Table E.4: Distribution of Exercises, Problems and Activities throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Exercise Blocks	Problem Blocks	Exercises	Problems	Activities
4. (LC S1&2/O)	7	207	81	806	435	15
	8	54	54	448	431	1
	9	67	61	828	404	9
	SY	-	-	-		
[redacted]						

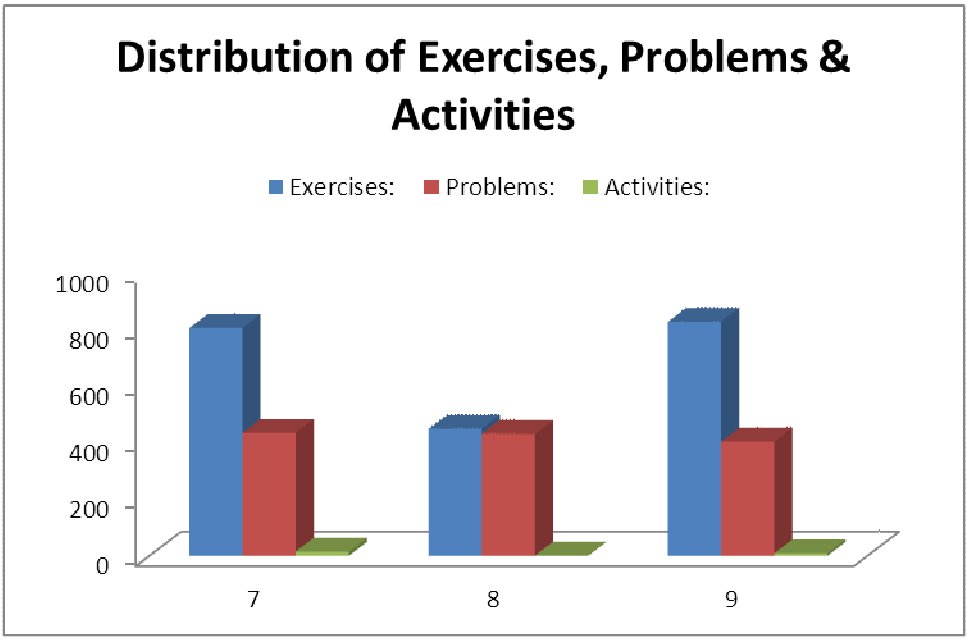

Figure E.4: Distribution of Exercises, Problems and Activities throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Table E.4a: Percentage Breakdown of Exercises and Problems throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Total Exercises + Problems	% of which are Problems
4. (LC S1&2/O)	7	1241	35.05%
	8	879	49.03%
	9	1232	32.79%
	SY	-	-
[redacted]			

Appendix E5

Examples

Table E.5: Distribution of Worked Examples throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Total No. of Worked Examples	Real/Realistic	Mathematical
4. (LC S1&2/O)	7	189	86	103
	8	114	35	79
	9	146	48	98
	SY	-	-	-
[redacted]				

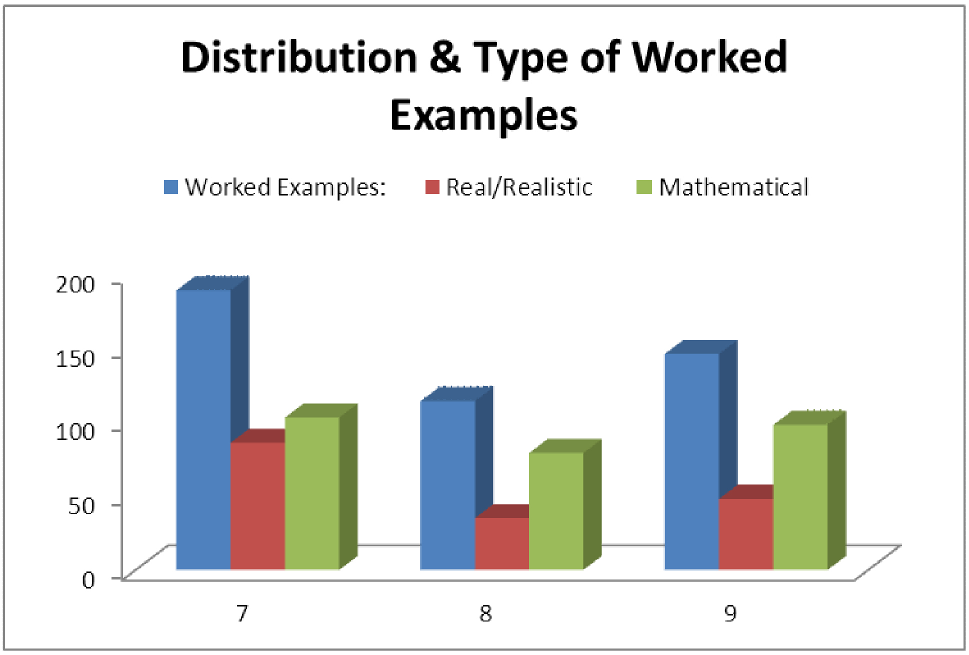

Figure E.5: Distribution of Worked Examples throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Table E.5a: Ratio of Worked Examples to Exercises, Problems & Activities throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Total Worked Examples	Ratio Examples: Exercises
4. (LC S1&2/O)	7	189	1:6.57
	8	114	1:7.71
	9	146	1:8.44
	SY	-	-
[redacted]			

Appendix E6

Senior Cycle Mathematics Textbook Content Analysis

The following tables (Table E.6a to E.6d) represent the content data from the Senior Cycle Ordinary Level Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project maths Curriculum and each row represents a specific content topic as derived from TIMSS. A list of 79 content topics was originally devised, irrelevant content topics were removed after data collection hence the numbering is not continuous.

Content Data List:

4. Common Fractions
5. Decimals
7. Percentages
9. Negative Numbers & their properties
13. Exponents, roots and radicals
17. Estimation & Number Sense
19. Rounding & significant figures
20. Estimating computations
22. Set Properties
23. Set Operations
24. Venn Diagrams
26. Perimeter, Area & Volume
74. Measure
28. 2D Geometry: Coordinate Geometry – Line
29. 2D Geometry: Coordinate Geometry - Circle
30. 2D Geometry: Basics – angles
31. 2D Geometry: Basics – shapes
32. 2D Geometry: Polygons & Circles
34. Transformations
35. Congruence & Similarity
36. Constructions using straightedge & Compass
37. Proof & Theorems
38. Proportionality Concepts
39. Proportionality Problems
76. Trig
41. Area
42. Trig ratios
43. Patterns, Relations & Analysis
44. Equations & formulae
47. Linear Functions
53. Inequalities
54. Approximating Values
56. Data Representation & Analysis
57. Classification of Data
58. Classification of Studies (including limitations,
59. Summary Statistics
60. Randomisation (including Bias)
62. Counting principles
63. Permutations & Combinations
64. Defining Probability
65. Measuring Probability
66. Laws of Probability
67. Probability Experiments
73. Software
79. Internet

Table E.6a:
TB 7

S.1	S. 2	Cont.
■	■	4
	■	5
■		7
		9
	■	13
		17
	■	19
■	■	20
■		22
		23
■		24
	■	26
	■	74
	■	28
	■	29
	■	30
	■	31
	■	32
	■	34
	■	35
	■	36
	■	**37**
	■	38
	■	39
	■	76
	■	41
	■	42
		43
	■	44
		47
	■	53
	■	54
■		56
■		57
■		58
■		59
■		60
■		62
■		63
■		64
■		65
■		66
■		67
■		**73**

Table E.6b:
TB 8

S.1	S. 2	Cont.
■	■	4
■	■	5
		7
		9
		13
	■	**17**
	■	19
■	■	20
■		22
		23
■		24
	■	26
	■	74
	■	28
	■	29
	■	30
	■	31
	■	32
	■	34
	■	35
	■	36
	■	**37**
	■	38
	■	39
	■	76
	■	41
	■	42
	■	43
	■	44
		47
		53
	■	54
■		56
■		57
■		58
■		59
■		60
		62
■		63
■		64
■		65
		66
■		67
		73

Table E.6c:
TB 9

S.1	S. 2	Cont.
	■	4
		5
		7
		9
		13
		17
	■	19
■		20
■		22
		23
■		24
	■	26
	■	74
	■	28
	■	29
	■	30
	■	31
	■	32
	■	34
	■	35
	■	36
	■	**37**
	■	38
	■	39
	■	76
	■	41
	■	42
	■	43
	■	44
		47
		53
	■	54
■	■	56
■		57
■		58
■		59
■		60
■		62
■		63
■		64
■		65
■		66
■		67
		73

Table E.6d:
LC SY S1&2/O

S.1	S. 2	Cont.
	■	4
		5
■		7
■		9
	■	13
		17
		19
■		20
■		22
		23
■		24
	■	26
		74
	■	28
	■	29
		30
	■	31
	■	32
	■	34
	■	35
	■	36
	■	**37**
		38
		39
	■	76
	■	41
	■	42
		43
		44
		47
		53
		54
■		56
■		57
■		58
■		59
■		60
■		62
■		63
■		64
■		65
■		66
■		67
■	■	**73**
■		**79**

Appendix E7

Motivational factors (Rivers Matrix)

Table E.7: Distribution of Motivational Factors throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Historical Notes	Bio- graphies	Career Information	Problem Solving	Photos	Humour/ Quotes
4. (LC S1&2/O)	7	19	6	15	435	96	2
	8	4	2	3	431	5	0
	9	1	0	3	404	2	0
	SY	-	-	-	-	-	-
[redacted]							

Appendix E8

Comprehension Cues (Rivers Matrix)

Table E.8: Summary of the use of colour throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Page Background Colour	Font Colour	Graph-line Colour
4. (LC S1&2/O)	7	Hints: Purple Formula: Blue Exercises: blue Examples: green Definitions: Green Keywords: green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green, Pink	Red, Blue, Black, Green, Purple, Brown, Orange
	8	Hints/Formulae: Yellow Keywords, Examples & Definitions: Interchange dependant on chapter colour (Purple (pink), Blue, Green, Red (orange))	Black & dependant on chapter colour. For example chapter 1 is Purple and writing is black & purple	Blue, Black, Red,
	9	Hints/Definitions/Formulae: Blue Examples & exercises: Yellow	Red, Black, Blue, White	Blue, Red, Black
	SY	-	-	-
[redacted]				

Appendix E9

Technical Aids (Rivers Matrix)

Table E.9: Distribution of Technical Aids throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Computer Software	Calculator	Internet
4. (LC S1&2/O)	7	6	9	6
	8	0	48	0
	9	0	25(5)*	2
	SY	-	-	-
[redacted]				

*The figure in brackets represents instances where the calculator is referred to in the context of 'Do not use your calculator'

Appendix E10

Problem Solving

Table E.10: Distribution of Routine & Non- Routine Problems throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Curriculum:	Text - book:	Routine Problems	Non-Routine Problems
4. (LC S1&2/O)	7	389	46
	8	380	51
	9	286	118
	SY	-	-
[redacted]			

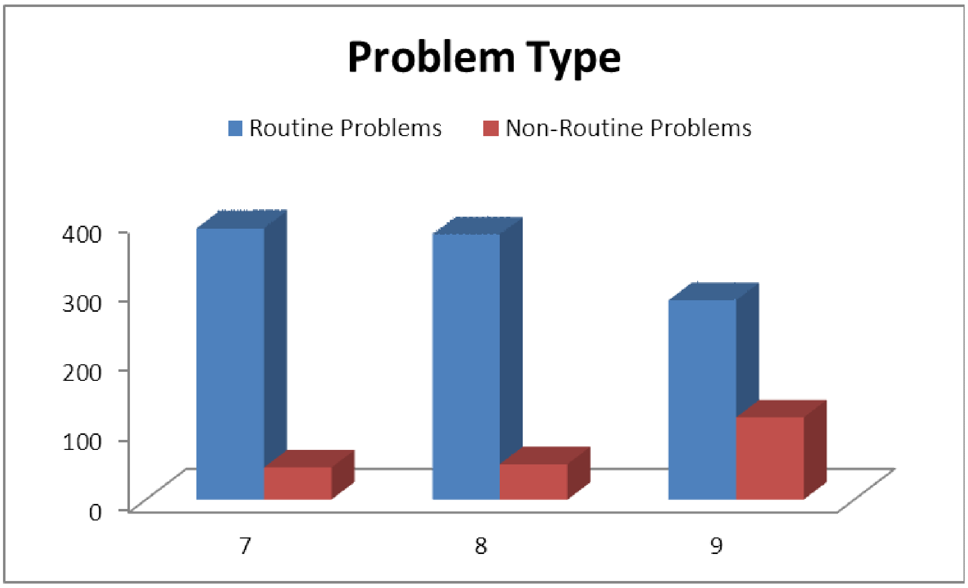

Figure E.6: Distribution of Routine & Non- Routine Problems throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Table E.10a: Breakdown of Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
4 (LC S1&2 0)	7	45	243	1	100
	8	31	180	1	168
	9	19	131	1	135
	SY	-	-		
[redacted]					

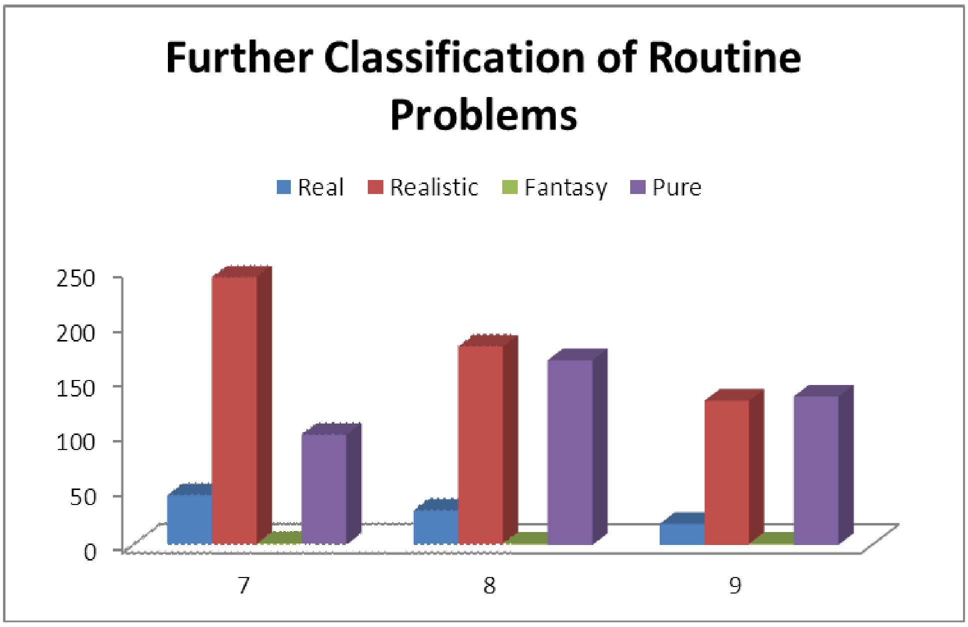

Figure E.7: Breakdown of Routine Problems throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Table E.10b: Breakdown of Non-Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
4 (LC S1&2 0)	7	7	21	0	18
	8	2	18	0	31
	9	2	60	0	54
	SY	-	-		
[redacted]					

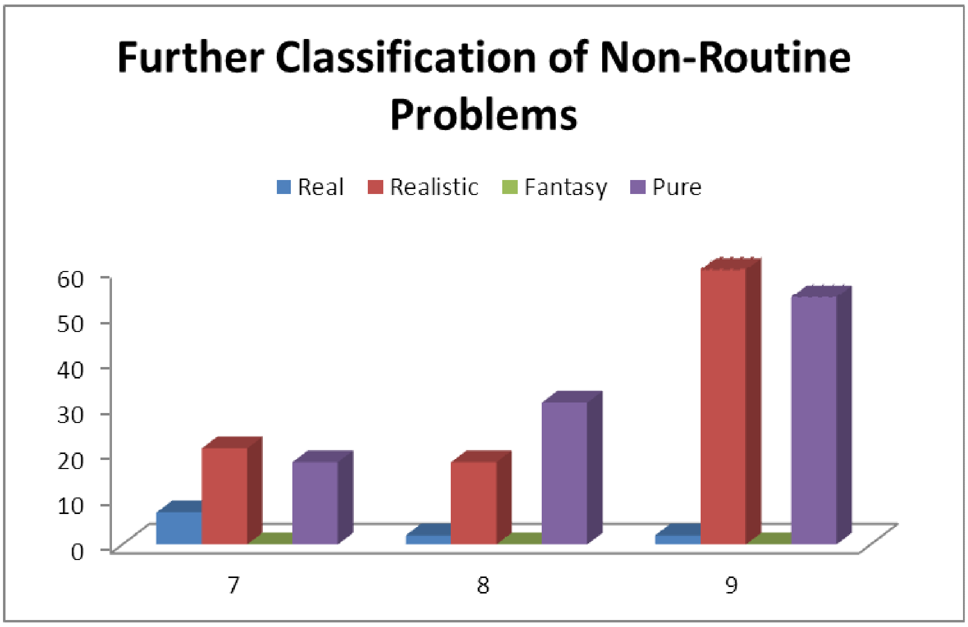

Figure E.8: Breakdown of Non-Routine Problems throughout the Senior Cycle Ordinary Level Mathematics Textbooks

Appendix E11

Senior Cycle Mathematics Textbook Expectation Analysis

The following tables (Table E.11a to E.11d) represent the expectation data from the Senior Cycle Ordinary Level Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project Maths Curriculum and each row represents a specific expectation as derived from TIMSS. A list of 24 expectations was originally devised, an additional 6 Project Maths specific expectations are noted in the syllabi analysis.

Expectation Data List:

Knowing

1. Representing
2. Recognising Equivalents
3. Recalling Mathematical Objects & Properties

Using Routine Procedures

4.Using Equipment
5.Performing Routine Procedures
6.Using More Complex Procedures

Investigating & Problem Solving

7.Formulating & Clarifying Problems &
8.Developing strategies (Designing)
9.Solving
10.Predicting
11.Verifying

Mathematical Reasoning

12. Developing Notation & Vocabulary
13. Developing algorithms
14. Generalising
15.Conjecturing
16. Justifying & proving
17. Axiomatising

Communicating

18.Using Vocabulary & Notation
19.Relating Representations
20.Describing/Discussing
21.Critiquing

Making Communications

22.Inter Subject Connections
23.Across Subject Connections
24.Give Real Life Examples
25.Instrumental Learning
26.Relational Learning
27.Fostering Positive Attitudes
28.Inquiry Based Learning
29. Applications
30.Connecting Solutions & Questions

Table E.11a:
TB 7

S.1	S. 2	Exp.
■	■	1
	■	2
■	■	3
■	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
■	■	11
		12
		13
		14
■	■	15
■	■	16
	■	17
■	■	18
■		19
■	■	20
■		21
■	■	22
		23
		24

Table E.11b:
TB 8

S.1	S. 2	Exp.
■	■	1
		2
■	■	3
■	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
■	■	11
		12
	■	13
		14
■	■	15
■	■	16
		17
■	■	18
■		19
■	■	20
■		21
■	■	22
		23
■		24

Table E.11c:
TB 9

S.1	S. 2	Exp.
■	■	1
		2
■	■	3
	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
■	■	11
		12
		13
		14
■	■	15
■	■	16
		17
■	■	18
■		19
■	■	20
■	■	21
	■	22
		23
		24

Table E.11d:
LC SY S1&2/O

S.1	S. 2	Exp.
■		1
		2
■	■	3
■		4
■	■	5
		6
■	■	7
■	■	8
■	■	9
		10
		11
		12
		13
■	■	14
■	■	15
■	■	16
		17
■	■	18
■		19
■	■	20
■	■	21
■	■	22
■	■	23
		24
■	■	25
■	■	26
■	■	27
■	■	28
■	■	29
■		30

Appendix F

Senior Cycle Higher Level Data

TIMSS+ Analysis – Structure, Content & Expectation

Appendix F1

Narration

Table F.1: Distribution of Narration and Narration type throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Narration	Related Narration	Instructional Narration
5. (LC S1&2/H)	10	973	55	32
	11	938	4	87
	SY	-	-	-
6. (LC HS2)	12	434	44	17
	13	529	7	20
	14	315	3	69
	SY	-	-	-

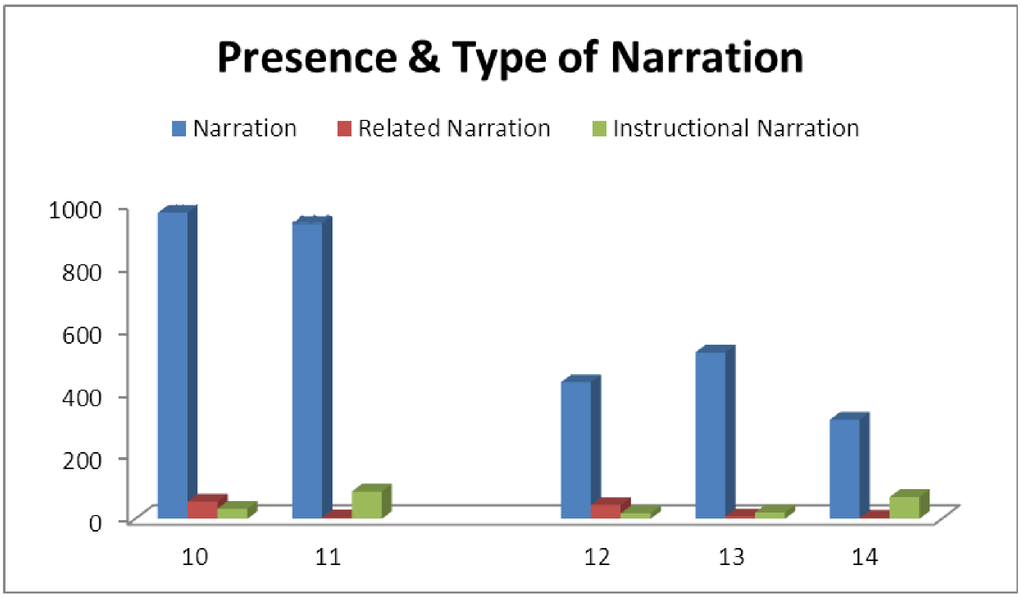

Figure F.1: Distribution of Narration and Narration type throughout the Senior Cycle Higher Level Mathematics Textbooks

Appendix F2

Definitions

Table F.2: Distribution of Definitions, Theorems & Axioms throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Definitions	Theorems	Axioms & Corollaries
5. (LC S1&2/H)	10	136	41	14
	11	32	21	0
	SY	-	-	-
6. (LC S2/H)	12	94	41	14
	13	36	12	3
	14	17	21	0
	SY	-	-	-

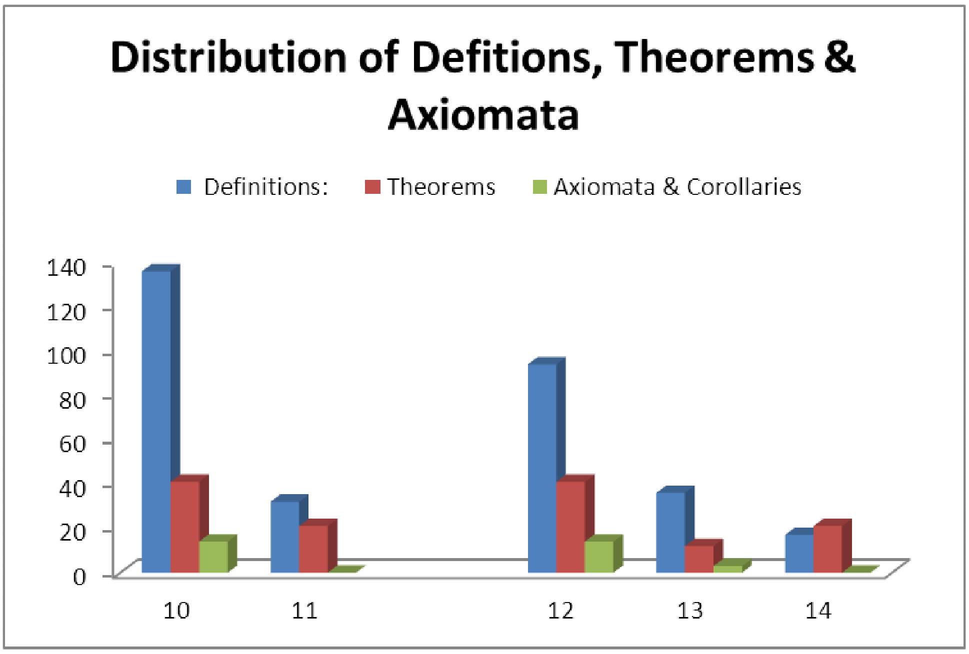

Figure F.2: Distribution of Definitions, Theorems & Axioms throughout the Senior Cycle Higher Level Mathematics Textbooks

Appendix F3

Graphics

Table F.3: Distribution and Purpose of Graphics throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Graphics	Graphics for Problem Solving	Realistic Diagrams	Photos
5. (LC S1&2/H)	10	1305	330	122	58
	11	744	290	56	0
	SY	-	-	-	
6. (LC S2/H)	12	943	222	108	25
	13	525	195	32	0
	14	494	212	36	0
	SY	-	-	-	

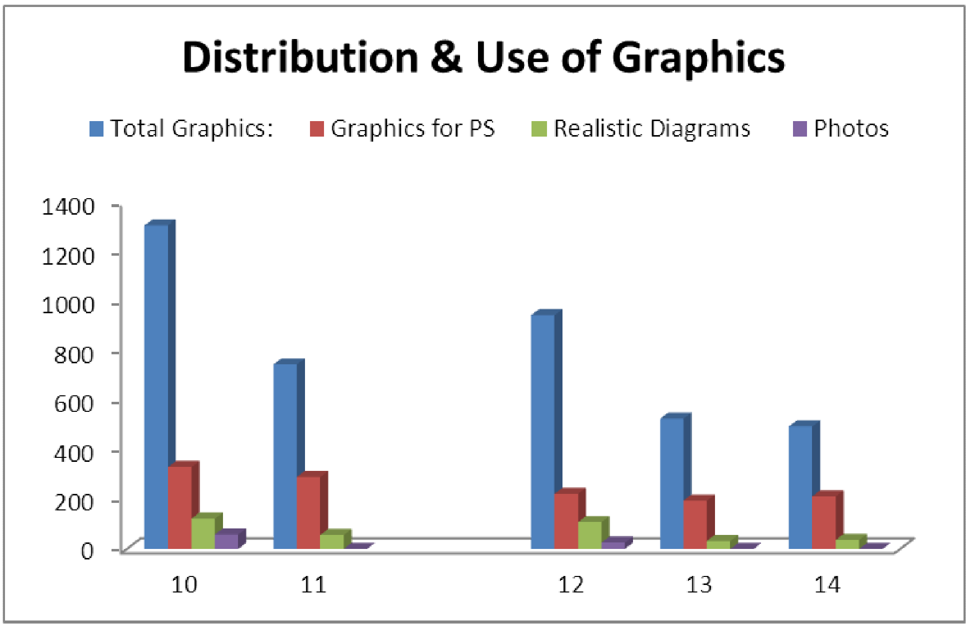

Figure F.3: Distribution and Purpose of Graphics throughout the Senior Cycle Higher Level Mathematics Textbooks

Appendix F4

Exercises

Table F.4: Distribution of Exercises, Problems and Activities throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Exercise Blocks	Problem Blocks	Exercises	Problems	Activities
5. (LC S1&2/H)	10	155	111	604	656	29
	11	50	61	615	689	3
	SY	-	-	-	-	-
6. (LC S2/H)	12	107	68	420	351	2
	13	35	39	217	442	0
	14	29	35	466	428	0
	SY	-	-	-	-	-

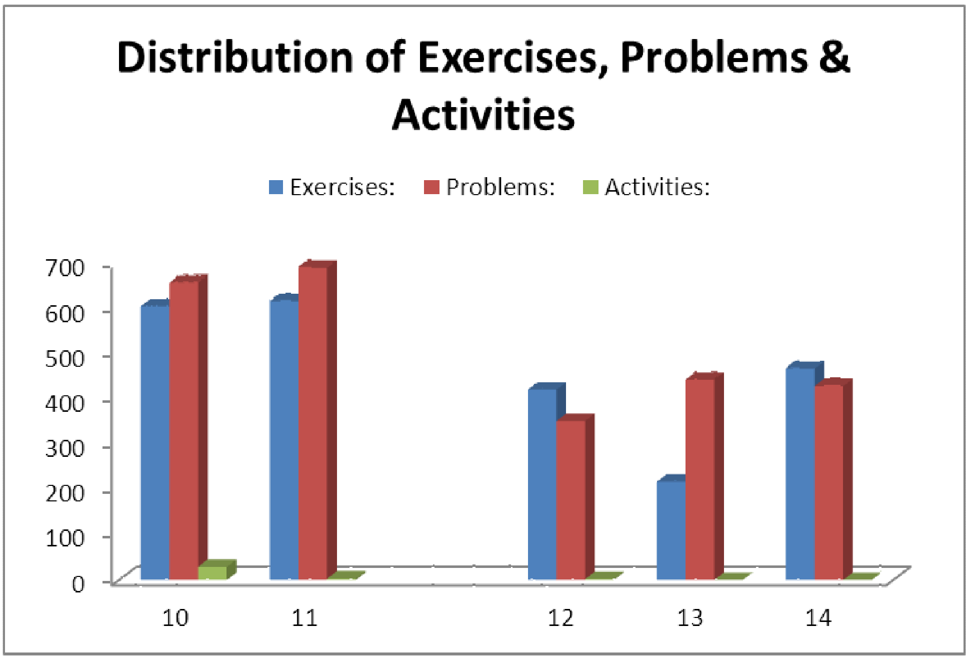

Figure F.4: Distribution of Exercises, Problems and Activities throughout the Senior Cycle Higher Level Mathematics Textbooks

Table F.4a: Percentage Breakdown of Exercises and Problems throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Total Exercises + Problems	% of which are Problems
5. (LC S1&2/H)	10	1260	52.06%
	11	1304	52.84%
	SY	-	-
6. (LC S2/H)	12	771	45.53%
	13	659	67.07%
	14	894	47.87%
	SY	-	-

Appendix F5

Examples

Table F.5: Distribution of Worked Examples throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Total No. of Worked Examples	Real/Realistic	Mathematical
5. (LC S1&2/H)	10	188	59	129
	11	177	68	109
	SY	-	-	-
6. (LC S2/H)	12	117	4	113
	13	122	15	107
	14	96	1	95
	SY	-	-	-

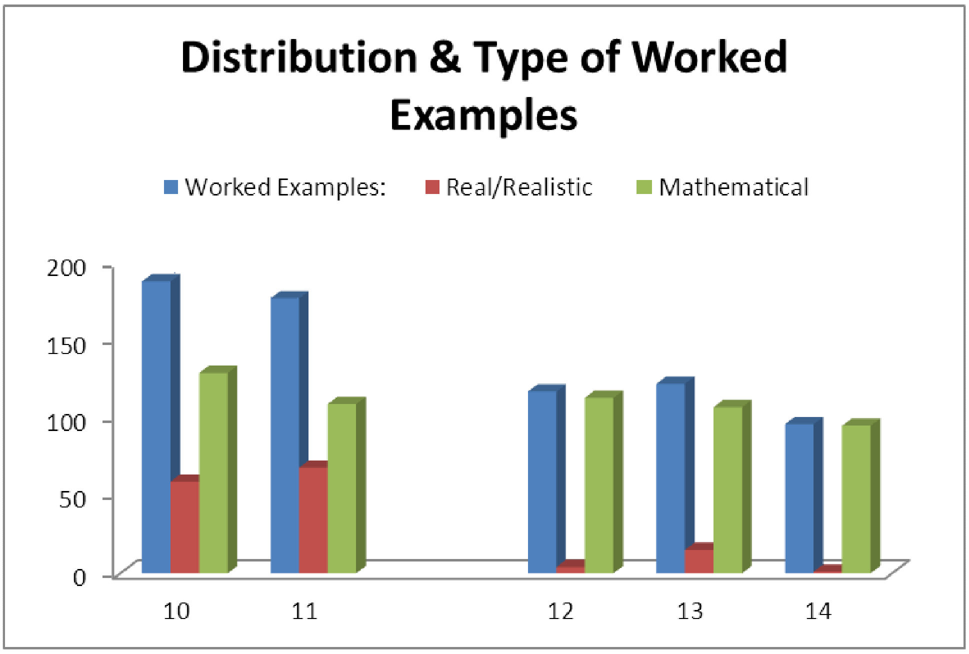

Figure F.5: Distribution of Worked Examples throughout the Senior Cycle Higher Level Mathematics Textbooks

Table F.5a: Ratio of Worked Examples to Exercises, Problems & Activities throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Total Worked Examples	Ratio Examples: Exercises
5. (LC S1&2/H)	10	188	1:06.70
	11	177	1:07.37
	SY	-	-
6. (LC S2/H)	12	117	1:06.59
	13	122	1:05.40
	14	96	1:09.31
	SY	-	-

Appendix F6

Senior Cycle Mathematics Textbook Content Analysis

The following tables (Table F.6a to F.6g) represent the content data from the Senior Cycle Higher Level Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project Maths Curriculum and each row represents a specific content topic as derived from TIMSS. A list of 79 content topics was originally devised, irrelevant content topics were removed after data collection hence the numbering is not continuous.

Content Data List:

4. Common Fractions
7. Percentages
9. Negative Numbers & their properties
13. Exponents, roots and radicals
19. Rounding & significant figures
20. Estimating computations
22. Set Properties
23. Set Operations
24. Venn Diagrams
25.Units
26. Perimeter, Area & Volume
74. Measure
28. 2D Geometry: Coordinate Geometry – Line
29. 2D Geometry: Coordinate Geometry - Circle
30. 2D Geometry: Basics – angles
31. 2D Geometry: Basics – shapes
32. 2D Geometry: Polygons & Circles
33. 3D Geometry problems
34. Transformations
35. Congruence & Similarity
36. Constructions using straightedge & Compass
37. Proof & Theorems
38. Proportionality Concepts

39. Proportionality Problems
76. Trig
41. Area
42. Trig ratios
43. Patterns, Relations & Analysis
44. Equations & formulae
47. Linear Functions
53. Inequalities
54. Approximating Values
56. Data Representation & Analysis
57. Classification of Data
58. Classification of Studies (including limitations,
59. Summary Statistics
60. Randomisation (including Bias)
61. inferential Statistics
62. Counting principles
63. Permutations & Combinations
64. Defining Probability
65. Measuring Probability
66. Laws of Probability
67. Probability Experiments
73. Software
79. Internet

Table F.6a:
TB 10

S.1	S. 2	Cont.
	■	4
■		7
		9
	■	13
	■	19
■		20
		22
		23
■		24
	■	25
	■	26
■	■	74
	■	28
	■	29
	■	30
	■	31
	■	32
	■	33
	■	34
	■	35
	■	36
	■	**37**
	■	38
		39
	■	76
	■	41
	■	42
	■	44
		47
	■	53
	■	54
■	■	56
■		57
■		58
■		59
■		60
■		61
■		62
■		63
■		64
■		65
■		66
■		67
		73

Table F.6b:
TB 11

S.1	S. 2	Cont.
	■	4
■		7
		9
	■	13
	■	19
■	■	20
		22
		23
		24
	■	25
■	■	26
	■	74
	■	28
	■	29
■	■	30
	■	31
	■	32
	■	33
	■	34
	■	35
	■	36
	■	**37**
	■	38
	■	39
	■	76
	■	41
	■	42
■	■	44
		47
	■	53
	■	54
■	■	56
■		57
■		58
■		59
		60
■		61
■		62
■		63
		64
■		65
■		66
■		67
		73

Table F.6c:
LC SY S1&2/H

S.1	S. 2	Cont.
	■	4
■		7
■		9
	■	13
		19
■		20
■		22
		23
■		24
	■	25
	■	26
		74
	■	28
	■	29
■	■	30
	■	31
	■	32
	■	33
	■	34
	■	35
	■	36
	■	**37**
	■	38
	■	39
	■	76
	■	41
	■	42
	■	44
	■	47
		53
		54
■	■	56
■		57
■		58
■		59
■		60
■		61
■		62
■		63
■		64
■		65
■		66
■		67
■	■	**73**

Table F.6d:
TB 12

S. 2	Cont.
■	4
	7
	9
■	13
■	19
	20
	22
	23
	24
■	25
■	26
■	74
■	28
■	29
■	30
■	31
■	32
■	33
■	34
■	35
■	36
■	**37**
■	38
	39
■	76
■	41
■	42
■	44
	47
■	53
■	54
■	56
	57
	58
	59
	60
	61
	62
	63
	64
	65
	66
	67
	73

Table F.6e:
TB 13

S. 2	Cont.
	4
	7
	9
■	13
■	19
	20
	22
	23
	24
■	25
■	26
	74
■	28
■	29
■	30
■	31
■	32
■	33
■	34
■	35
■	36
■	37
■	38
■	39
■	76
■	41
■	42
■	44
	47
	53
■	54
■	56
	57
	58
	59
	60
	61
	62
	63
	64
	65
	66
	67
	73

Table F.6f:
TB 14

S. 2	Cont.
■	4
	7
	9
■	13
■	19
■	20
	22
	23
	24
■	25
■	26
■	74
■	28
■	29
■	30
■	31
■	32
■	33
■	34
■	35
■	36
■	**37**
■	38
■	39
■	76
■	41
■	42
■	44
	47
■	53
■	54
■	56
	57
	58
	59
	60
	61
	62
	63
	64
	65
	66
	67
	73

Table F.6g:
LC SY S2/H

S. 2	Cont.
■	4
	7
	9
■	13
	19
	20
	22
	23
	24
■	25
■	26
	74
■	28
■	29
■	30
■	31
■	32
■	33
■	34
■	35
■	36
■	**37**
■	38
■	39
■	76
■	41
■	42
■	44
	47
	53
	54
■	56
	57
	58
	59
	60
	61
	62
	63
	64
	65
	66
	67
■	**73**

Appendix F7

Motivational factors (Rivers Matrix)

Table F.7: Distribution of Motivational Factors throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Historical Notes	Bio- graphies	Career Information	Problem Solving	Photos	Humour/ Quotes
5. (LC S1&2/H)	10	9	0	23	656	58	0
	11	2	0	5	689	0	0
	SY	-	-	-	-	-	-
6. (LC S2/H)	12	6	2	16	351	25	0
	13	1	1	2	442	0	0
	14	1	0	3	428	0	0
	SY	-	-	-	-	-	-

Appendix F8

Comprehension Cues (Rivers Matrix)

Table F.8: Summary of the use of colour throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Page Background Colour	Font Colour	Graph-line Colour
5. (LC S1&2/H)	10	Hints: Purple Formula: Blue Exercises: blue Examples: green Definitions: Green Keywords: green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green, Pink	Red, Blue, Black, Green, Purple, Brown,
	11	Hints/Definitions/Formulae: Blue Examples & exercises: Yellow	Red, Black, Blue, White	Blue, Red, Black
	SY	-	-	-
6. (LC S2/H)	12	Hints: Purple Formula: Blue Exercises: blue Examples: green Definitions: Green Keywords: green Theorems/Axioms: Orange Reminders: Change per chapter (Blue, Pink, Green, Orange)	White, Black, Blue, Green, Pink	Red, Blue, Black, Green, Purple, Brown,
	13	Hints/Def/Formulae: Yellow Keywords, Examples & Definitions: Interchange dependant on chapter colour (Purple (pink), Blue, Green, Red (orange))	Black & dependant on chapter colour. For example chapter 1 is Purple and writing is black & purple	Blue, Black, Red, Green, Purple
	14	Hints/Def/Formulae: Blue Examples & exercises: Yellow	Red, Black, Blue, White	Blue, Red, Black
	SY	-	-	-

Appendix F9

Technical Aids (Rivers Matrix)

Table F.9: Distribution of Technical Aids throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Computer Software	Calculator	Internet
5. (LC S1&2/H)	10	14	59(12)*	12
	11	17	17	2
	SY	-	-	-
6. (LC S2/H)	12	1	25(12)*	0
	13	0	19(11)*	0
	14	0	4	0
	SY	-	-	-

*The figure in brackets represents instances where the calculator is referred to in the context of 'Do not use your calculator'

Appendix F10

Problem Solving

Table F.10: Distribution of Routine & Non- Routine Problems throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Routine Problems	Non-Routine Problems
5. (LC S1&2/H)	10	473	183
	11	463	226
	SY	-	-
6. (LC S2/H)	12	213	138
	13	295	147
	14	240	188
	SY	-	-

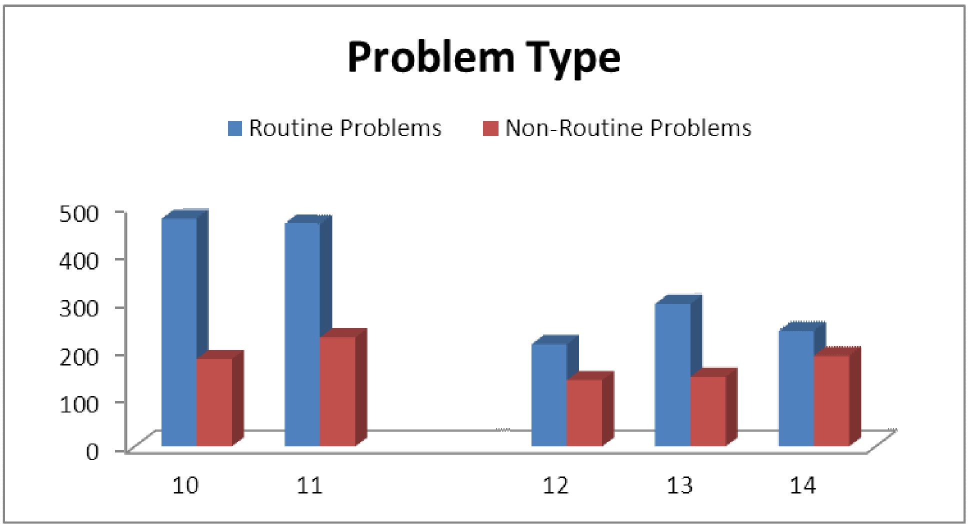

Figure F.6: Distribution of Routine & Non- Routine Problems throughout the Senior Cycle Higher Level Mathematics Textbooks

Table F.10a: Breakdown of Routine Problem Type throughout the Senior Cycle Higher Level Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
5. (LC S1&2/H)	10	41	212	1	223
	11	6	215	1	240
		-	-	-	-
6. (LC S2/H)	12	2	26	0	185
	13	1	24	1	214
	14	1	19	0	277
	SY	-	-	-	-

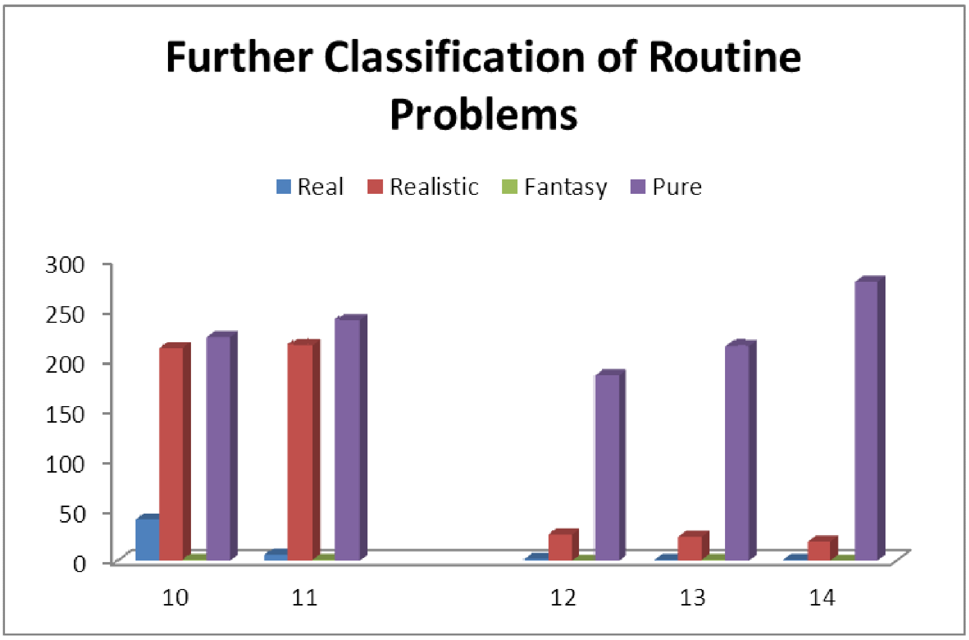

Figure F.7: Breakdown of Problem Type throughout the Senior Cycle Higher Level Mathematics Textbooks

Table F.10b: Breakdown of Non- Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Curriculum:	Text - book:	Real	Realistic	Fantasy	Purely Mathematical
5. (LC S1&2/H)	10	20	42	0	117
	11	8	90	0	129
		-	-	-	-
6. (LC S2/H)	12	1	21	0	116
	13	2	63	0	123
	14	8	90	0	129
	SY	-	-	-	-

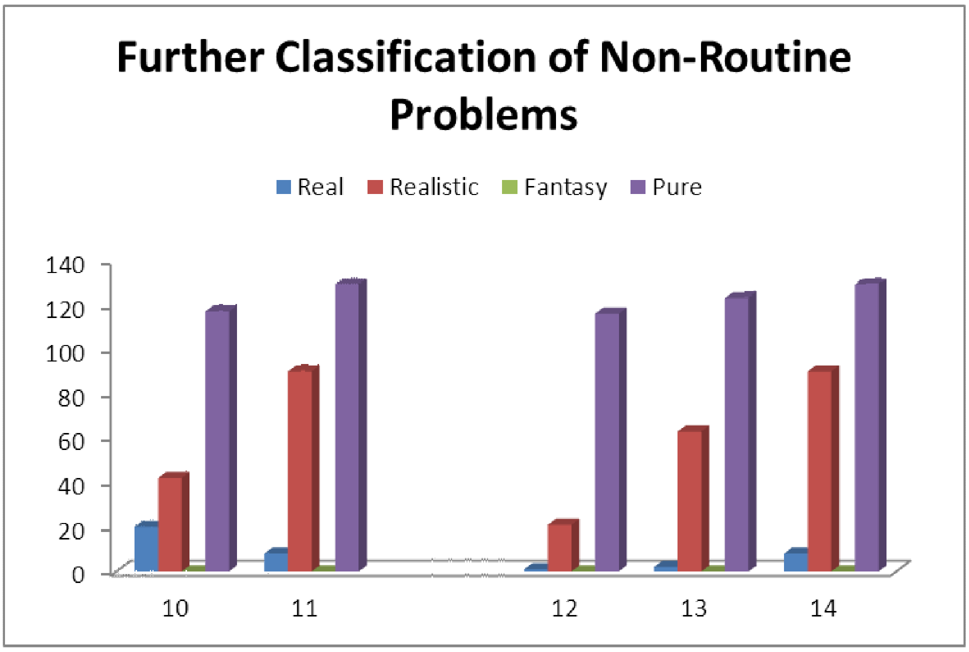

Figure F.8: Breakdown of Non- Routine Problem Type throughout the Junior Cycle Mathematics Textbooks

Appendix F11

Senior Cycle Mathematics Textbook Expectation Analysis

The following tables (Table F.11a to F.11g) represent the expectation data from the Senior Cycle Higher Level Mathematics Textbooks. Each column in the following grids represents a strand of content from the Project Maths Curriculum and each row represents a specific expectation as derived from TIMSS. A list of 24 expectations was originally devised, an additional 6 Project Maths specific expectations are noted in the syllabi analysis.

Expectation Data List:

Knowing
1. Representing
2. Recognising Equivalents
3. Recalling Mathematical Objects & Properties

Using Routine Procedures
4.Using Equipment
5.Performing Routine Procedures
6.Using More Complex Procedures

Investigating & Problem Solving
7.Formulating & Clarifying Problems &
8.Developing strategies (Designing)
9.Solving
10.Predicting
11.Verifying

Mathematical Reasoning
12. Developing Notation & Vocabulary
13. Developing algorithms
14. Generalising
15.Conjecturing
16. Justifying & proving
17. Axiomatising

Communicating
18.Using Vocabulary & Notation
19.Relating Representations
20.Describing/Discussing
21.Critiquing

Making Communications
22.Inter Subject Connections
23.Across Subject Connections
24.Give Real Life Examples
25.Instrumental Learning
26.Relational Learning
27.Fostering Positive Attitudes
28.Inquiry Based Learning
29. Applications
30.Connecting Solutions & Questions

Table F.11a:
TB 10

S. 1	S.2	Exp.
■	■	1
	■	2
■	■	3
■	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
■	■	11
		12
		13
	■	14
■	■	15
■	■	16
	■	17
■	■	18
■	■	19
■	■	20
		21
■	■	22
■		23
■		24

Table F.11b:
TB 11

S. 1	S.2	Exp.
■	■	1
		2
■	■	3
	■	4
■	■	5
■	■	6
■	■	7
■		8
■	■	9
■		10
■	■	11
		12
	■	13
		14
■	■	15
■	■	16
		17
■	■	18
■		19
■	■	20
■		21
	■	22
	■	23
		24

Table F.11c:
LC SY S1&2/H

S. 1	S.2	Exp.
■		1
		2
■	■	3
■		4
■	■	5
■		6
■	■	7
■	■	8
■	■	9
■		10
		11
		12
	■	13
■		14
■	■	15
■	■	16
		17
■	■	18
■		19
■	■	20
■	■	21
■	■	22
■	■	23
		24
■	■	25
■	■	26
■	■	27
■	■	28
■	■	29
■		30

Table F.11d:
TB 12

S. 2	Exp.
■	1
■	2
■	3
■	4
■	5
■	6
■	7
	8
■	9
	10
■	11
	12
	13
■	14
■	15
■	16
■	17
■	18
■	19
■	20
	21
■	22
	23
	24

Table F.11e:
TB 13

S. 2	Exp.
■	1
	2
■	3
■	4
■	5
■	6
■	7
	8
■	9
	10
■	11
	12
■	13
	14
■	15
■	16
	17
■	18
■	19
■	20
	21
■	22
	23
	24

Table F.11f:
TB 14

S. 2	Exp.
■	1
	2
■	3
■	4
■	5
■	6
■	7
	8
■	9
	10
■	11
	12
	13
	14
■	15
■	16
	17
■	18
	19
■	20
	21
■	22
■	23
	24

Table F.11g:
LC SY S2/H

S. 2	Exp.
	1
	2
■	3
	4
■	5
	6
■	7
■	8
■	9
	10
	11
	12
■	13
■	14
■	15
■	16
	17
■	18
	19
■	20
■	21
■	22
■	23
	24
■	25
■	26
■	27
■	28
■	29
	30

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HEA
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An Roinn Oideachais agus Eolaíochta
Department of Education and Science

NDP
National Development Plan 2007 - 2013

NCE-MSTL
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Evidence for Excellence in Education

Final report

Research into the impact of Project Maths on student achievement, learning and motivation

Jennifer Jeffes
Emily Jones
Máirín Wilson
Emily Lamont
Suzanne Straw
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Published in November 2013
by the National Foundation for Educational Research,
The Mere, Upton Park, Slough, Berkshire SL1 2DQ
www.nfer.ac.uk

How to cite this publication:
Jeffes, J., Jones, E., Wilson, M., Lamont, E., Straw, S., Wheater, R. and Dawson, A. (2013). *Research into the impact of Project Maths on student achievement, learning and motivation: final report.* Slough: NFER.

Contents

Acknowledgements

The authors would like to thank the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), for the opportunity to undertake this work, and particularly to Rachel Linney and Anne Looney for their helpful guidance and feedback. We would like to extend our gratitude to the schools who have participated in this research, especially the teachers and students who have given their time to share their views and experiences with us. We would also like to acknowledge the valuable support of the Church of Ireland College of Education and the research team at the National Foundation for Educational Research (NFER), in particular Marian Sainsbury, the Centre for Statistics and Research Operations colleagues.

Executive summary

About the research

The National Foundation for Educational Research (NFER) has been commissioned by the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), to undertake research into the impact of Project Maths on student achievement, learning and motivation.

Project Maths is a major national reform of post-primary mathematics curriculum and assessment in the Republic of Ireland for both junior and senior cycles. The syllabuses are divided into five strands, as follows:

- Strand 1: Statistics and Probability
- Strand 2: Geometry and Trigonometry
- Strand 3: Number
- Strand 4: Algebra
- Strand 5: Functions.

Aims of the research

The overarching aim of the research is to explore the impact of Project Maths on students' achievement, learning and motivation in mathematics in:

- the initial schools (phase one schools), which introduced the revised mathematics syllabuses in September 2008
- all other post-primary schools (non-phase one schools), which introduced the revised mathematics syllabuses in September 2010.

Methodology

The methodology for this research comprises four parts:

- **assessment of student achievement** in all strands of the revised mathematics syllabuses in spring and autumn 2012, with two separate cohorts of Junior Certificate and Leaving Certificate students
- **attitude surveys** exploring students' experiences of the revised mathematics syllabuses, administered to the same cohorts of students above
- **case studies** in eight phase one schools, and eight non-phase one schools, exploring in more depth students' and teachers' experiences of the revised mathematics syllabuses
- **qualitative analysis of students' work**, exploring trends in the processes being promoted in the revised mathematics syllabuses.

Students' experiences of the revised mathematics syllabuses

Students' experiences of learning mathematics

Students report that they are frequently undertaking activities commonly associated with the revised syllabuses (for example, making connections between mathematics topics, and applying mathematics to real-life situations). However, more traditional approaches (for example, using textbooks and copying from the board) also continue to be widespread.

Students' perspectives on their progression in mathematics

While Leaving Certificate students appreciate the value of gaining a rich understanding of mathematics, they have found the change in learning approach between Junior Certificate and Leaving Certificate challenging.

In general, Junior Certificate students are more positive about their transition from primary school than Leaving Certificate students are about their transition from Junior Certificate. In part, this may be because Junior Certificate students have experienced greater continuity of learning styles across their transition from primary school.

Students' overall learning and motivation

Students are largely positive about their experience and confidence in learning mathematics, tending to agree that they enjoy it, do well in it and learn quickly in it. Junior Certificate students are more positive and more confident in their ability than Leaving Certificate students.

International comparison suggests that students following the revised syllabuses are slightly *less positive about mathematics, but more confident in their mathematics ability* than those who participated in TIMSS 2007. This implies that, in this research, enjoyment of, and confidence in, mathematics do not necessarily go hand in hand.

Evidence of the extent to which students are using the approaches promoted in the revised mathematics syllabuses

Based on the small sample of students' work included in this study there is emerging evidence that the revised syllabuses are impacting on students' learning in the key process areas. However, the processes promoted through the revised syllabuses are not yet embedded in the written output from mathematics lessons (although they may be evident in other aspects of lessons). This may be expected due to the early stage of the implementation of the revised syllabuses.

Whilst some processes of the revised mathematics syllabuses are visible in some of the student material reviewed, there does not appear to have been a substantial shift in what teachers are asking students to do, and few differences between the phase one and non-phase one students.

It is possible that teachers are currently emphasising the content of the revised syllabuses rather than the processes promoted within it. This reflects earlier findings from the interim report[1] that traditional approaches to mathematics teaching and learning continue to be widespread.

The evidence strongly suggests that students have a good mastery of

[1] Jeffes, J., Jones, E., Cunningham, R., Dawson, A., Cooper, L., Straw, S., Sturman, L. and O'Kane, M. (2012). *Research into the impact of Project Maths on student achievement, learning and motivation: interim report.* Slough: NFER.

mathematical procedures and, to a slightly lesser extent, problem solving and making mathematical representations. There is very little evidence in the work reviewed that students are demonstrating reasoning and proof and communication, or making connections between mathematics topics.

The findings suggest that students need to be regularly given high quality tasks that require them to engage with the processes promoted by the revised syllabuses, including: problem solving; drawing out connections between mathematics topics; communicating more effectively in written form; and justifying and providing evidence for their answers.

Students' achievement and attitudes towards mathematics

The variables that affect students' attitudes and achievement

Students' confidence and achievement in mathematics is significantly associated with examination entry level and gender. Of note, girls are less confident about mathematics than boys and perform less well at Junior Certificate.

Overall, following a greater number of strands, or schools having greater experience of teaching the revised syllabuses, does not appear to be associated with any improvement in students' achievement and confidence.

Students' achievement and confidence across the five strands

Whilst achievement is highest in Strand 1 (Statistics and Probability) and lowest in Strand 5 (Functions), confidence is actually highest in both of these strands and lowest in Strand 3 (Number) and Strand 4 (Algebra).

The relationship between confidence and achievement

In this research study, confidence in mathematics does not always correspond to achievement. Although students who are further through their studies perform better than those who are at an earlier stage, higher levels of confidence are not associated with students who have almost completed their studies.

Furthermore, results show that students are confident in relation to Strand 5 (Functions), but do not perform highly when assessed in this area. While girls are less confident in their mathematical ability across all strands explored, at Leaving Certificate level, they perform as highly as boys.

Students' aspirations for further study and careers involving mathematics

Students' perceptions of the wider relevance and application of mathematics

Students tend to recognise the broader application of mathematics, particularly in helping them to secure a place at the university of their choice and in their daily life. Reflecting the broader difference in the attitudes of Junior and Leaving Certificate students, Junior Certificate students are generally more positive about the broader application of their mathematics study than students studying for the Leaving Certificate.

Students' aspirations to further study of mathematics

Almost all Leaving Certificate students plan to go on to further study when they finish their Leaving Certificate, and around half of all students intend to pursue further study involving mathematics.

Almost all Junior Certificate students plan to stay on at school after their Junior Certificate, and the majority plan to take the Higher Level Leaving Certificate examination. The aspirations of students for Higher Level examination in phase one schools are higher than students from non-phase one schools. This may be a result of the revised syllabuses beginning to embed in phase one schools, and therefore instilling a greater enjoyment of, and confidence in mathematics amongst their students.

Students' appreciation of careers involving mathematics

Around two-thirds of Leaving Certificate students stated that they do not intend to go into a job that involves mathematics.

It appears that students are developing a general awareness of the importance of mathematics in further study and of its broader application, but in some cases, the specifics of this, such as a sound understanding of what careers will draw on their mathematical skills and knowledge, appears to be lacking.

Concluding comments

The research highlights that considerable progress has been made in implementing the revised mathematics syllabuses since the inception of the Project Maths initiative in 2008. There are numerous examples of promising practice in the way that mathematics is being delivered in the classroom, and emerging evidence of positive impacts on students' experiences of, and attitudes towards, mathematics.

However, there is also evidence that more traditional approaches to teaching mathematics remain widespread and, in many cases, the approaches described by teachers and students are not yet being evidenced in students' written work. Moreover, at this stage of the curriculum's implementation, the revised mathematics syllabuses taken as a whole do not appear to be associated with any overall deterioration or improvements in students' achievement.

1. Introduction

The National Foundation for Educational Research (NFER) has been commissioned by the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), to undertake research into the impact of Project Maths on student achievement, learning and motivation.

Project Maths is a major national reform of post-primary mathematics curriculum and assessment in the Republic of Ireland for both junior and senior cycles. Introduced in 24 phase one schools in September 2008, and rolled out to all post-primary schools in September 2010, Project Maths was designed to change not just *what* students learn about mathematics, but *how* they learn and are assessed. Project Maths represents a philosophical shift in Irish post-primary education towards an investigative, problem-focused approach to learning mathematics, emphasising its application in real-life settings. Both the Junior Certificate (students aged 12 to 15 years) and Leaving Certificate (students aged 15 to 18 years) syllabuses are divided into five strands, as follows:

- Strand 1: Statistics and Probability
- Strand 2: Geometry and Trigonometry
- Strand 3: Number
- Strand 4: Algebra
- Strand 5: Functions.

Students may follow a syllabus at a number of levels. Junior Certificate students can follow a syllabus at Ordinary or Higher Level, while Leaving Certificate students can follow a syllabus at Foundation, Ordinary or Higher Level. Both Junior and Leaving Certificate students may sit their examinations at Foundation, Ordinary or Higher Level. Across all strands and levels, students are encouraged to test out and apply their knowledge to meaningful contexts, and to take responsibility for their own learning through, for example, setting goals and developing action plans. The revised syllabuses are designed to offer a continuous learning experience for students throughout junior and senior cycles, building upon the foundations of mathematical knowledge acquired at primary school.

The research explores students' achievement, learning and motivation in:

- the initial schools (phase one schools), which introduced the revised mathematics syllabuses in September 2008
- all other post-primary schools (non-phase one schools), which introduced the revised mathematics syllabuses in September 2010.

Further background and contextual information about Project Maths can be found on the NCCA website at www.ncca.ie/projectmaths. In addition, earlier findings of the evaluation can be found in NFER's interim report to DES and NCCA[2].

The report is divided into the following sections:

- about the evaluation, covering the research aims and methodological considerations (section 2)
- students' experiences of the revised mathematics syllabuses (section 3)
- the extent to which the processes promoted in the revised mathematics syllabuses are evidenced in students' work (section 4)
- impacts of the revised mathematics syllabuses (section 5)
- students' aspirations for further study and careers involving mathematics (section 6)
- concluding comments (section 7).

[2] Jeffes, J., Jones, E., Cunningham, R., Dawson, A., Cooper, L., Straw, S., Sturman, L. and O'Kane, M. (2012). *Research into the impact of Project Maths on student achievement, learning and motivation: interim report.* Slough: NFER.

2. About the evaluation

This section sets out the approach taken to this research into the impact of Project Maths on student achievement, learning and motivation including:

- the overarching aims and key research questions
- the research activities undertaken and methodological considerations.

2.1 Aims of the research

The overarching aim of the research is to explore the impact of Project Maths on students' achievement, learning and motivation in mathematics, in both phase one and non-phase one schools. Table 2.1 sets out the key research themes for this study, mapped against each element of the research (outlined in more detail in section 2.2).

Table 2.1: Key research themes

Research theme	Assessment of student performance	Student attitude surveys	Analysis of students' work	Case studies
Students' achievement in mathematics, across each individual strand of the revised mathematics syllabuses	✓			
Comparison of students' performance in mathematics with international standards	✓			
Students' motivations and attitudes to mathematics, in general and in relation to the revised mathematics syllabuses		✓		✓
Students' perceptions of the effectiveness of different strands and approaches used in the revised mathematics syllabuses		✓		✓
Students' perceptions of their knowledge, understanding, confidence and achievement in mathematics		✓		✓
Students' aspirations for further study of mathematics		✓		✓
Students' views on the relevance and application of mathematics more generally		✓		✓
Students' understanding of the processes being promoted in the revised mathematics syllabuses			✓	✓
The impact of the revised mathematics syllabuses on individual students' progress and standards			✓	
Trends in students' approaches to, and performance in, the revised mathematics syllabuses	✓		✓	

2.2 Methodology

The methodology for this research comprises four parts, each described in detail later in this section:

- **assessment of student achievement in all strands of the revised mathematics syllabuses**. Data was collected at two time points, each time from four groups of students. In Spring 2012, this involved students in the Junior and Leaving Certificate classes of 2012 in both phase one and non-phase one schools. In Autumn 2012, this involved different Junior and Leaving Certificate students in the classes of 2013, again in phase one and non-phase one schools
- **attitude surveys exploring students' experiences** of the revised mathematics syllabuses, administered to the same year groups of students as described above
- **data-rich case studies** in eight phase one schools, and eight non-phase one schools, exploring in more depth students' and teachers' experiences of the revised mathematics syllabuses
- **qualitative analysis of students' work** exploring trends in the processes being promoted in the revised mathematics syllabuses.

Note that examination classes of 2012 and 2013 refer to the group of students reaching the end of their studies during each of these years. Throughout this report, the term examination class will be replaced simply with class (for example, class of 2013).

2.2.1 About the survey and assessment samples

The assessment of student achievement and attitude surveys involved students at both Junior Certificate and Leaving Certificate, in the classes of 2012 and 2013. The strands of the revised mathematics syllabuses followed by these year groups of students are set out in Table 2.2.

Table 2.2: Strands studied by students participating in the assessment of performance and attitude survey

	Year group	Years of study	Strands studied by phase one students	Strands studied by non-phase one students
Junior Certificate	**Students who completed the survey and tests in Spring 2012**	**2009-12**	Strands 1-4	No strands
	Students who completed the survey and tests in Autumn 2012	**2010-13**	Strands 1-5	Strands 1-2
Leaving Certificate	**Students who completed the survey and tests in Spring 2012**	**2010-12**	Strands 1-5	Strands 1-2
	Students who completed the survey and tests in Autumn 2012	**2011-13**	Strands 1-5	Strands 1-4

All phase one schools were invited to participate in the research, and a sample of non-phase one schools was drawn to be representative of school type and size, as well as those included in the Delivering Equality of Opportunity in Schools (DEIS) programme, which aims to address educational disadvantage.

The sample was also drawn to be representative of gender of students and provide coverage of geographical location: all 26 counties in the Republic of Ireland were included. Students were selected so that, across the sample, the distribution of predicted examination levels was broadly based on previous State Examination Commission (SEC) entry patterns.

Timing differences in data collection within the classes of 2012 and 2013

Surveys and assessments for the class of 2012 took place in Spring 2012, as students were reaching the end of their studies. By contrast, these took place in Autumn 2012 for class of 2013, as students were entering their final year of study. As a result, the class of 2013 had experienced fewer months of teaching than those in the class of 2012 at the time they participated in the research. This approach was governed by the research period and maximised data collection in the time available. Where appropriate, variations in students' attitudes and performance as a result of these timing differences are addressed.

Table 2.3 provides a breakdown of the number of schools and students who participated in these parts of the research, by phase and level of study.

Table 2.3: Number of students who participated in the assessment and survey phases of the research

			Phase one				Non-phase one			
			Assessment		Attitude survey		Assessment		Attitude survey	
	Assess-ment date	Exam year	Number of students	Number of schools[3]	Number of students	Number of schools	Number of students	Number of schools	Number of students	Number of schools
Junior Certificate	Spring 2012	2012	303	19	375	19	910	52	2,375	124
	Autumn 2012	2013	421	17	417	17	795	43	2,248	128
Leaving Certificate	Spring 2012	2012	370	19	299	19	722	52	2,004	124
	Autumn 2012	2013	413	17	413	17	788	43	2,161	128

[3] Some phase one schools were unable to participate in the research due to other commitments (e.g. participation in PISA 2012, other school priorities).

The use of comparative data, to measure the impact of the revised mathematics syllabuses relative to the previous ones, is central to the research design. However, as this research commenced in January 2012, the revised syllabuses had been rolled out nationally to most cohorts of students. Therefore, involvement of non-phase one Junior Certificate students in the class of 2012 represents the only comparison group included in this research. Multi-level modelling has enabled further associations to be investigated between extent of involvement in the revised syllabuses and achievement, learning and motivation in mathematics. Multi-level modelling is a development of a statistical technique called 'regression analysis'. This is a technique for finding relationships between variables (for further detail, see Appendix A).

2.2.2 Assessment of student performance

Assessment of student achievement aims to gather quantitative data, focusing on:

- students' achievement in mathematics, across each individual strand of the revised mathematics syllabuses
- trends in students' approaches to, and performance in, the revised mathematics syllabuses.

Assessment of student performance at Junior Certificate level

Assessment booklets were created in order to assess students' performance in individual items pertaining to each strand of the revised mathematics syllabuses, thereby giving an indication of their overall performance across each strand. The phase one classes of 2012 and 2013 were both assessed in Strands 1-4. Additionally, students in the class of 2013 were assessed in two items relating to Strand 5 (Functions). This reflects the strands studied by these year groups[4].

For reference, the booklets are labelled according to the strands covered. Table 2.4 shows the syllabus strands covered by each booklet, and the number of items in each booklet[5]. Two Junior Certificate booklets were produced, which were randomly allocated to students at each school so that phase one students each completed one booklet. Students from non-phase one schools in the class of 2013 only completed questions relating to Strands 1 and 2. They were not assessed on Strands 3-5, which they had not yet been taught.

[4] Assessment booklets were made available in the Irish medium. However, participating Irish language schools included in the sample opted to administer the assessment in English.

[5] The Autumn 2012 booklet JC1/2/5 contains all the items presented in the Spring 2012 booklet JC1/2. The Autumn booklet LC3/4/5 contains all the items presented in the Spring 2012 booklet JC3/4.

Table 2.4: Booklets for the Junior Certificate

Booklet	Syllabus strand	Syllabus area	Number of items	Number of students completing booklet in the class of 2013[6]
JC1/2/5	Statistics and Probability	• concepts of probability • outcomes of random processes • statistical reasoning with an aim to becoming a statistically aware consumer • representing data graphically and numerically • analysing, interpreting and drawing conclusions from data	11	1006
	Geometry and Trigonometry	• synthetic geometry • transformation geometry • co-ordinate geometry • trigonometry	10	
	Functions	• graphing functions	2	
JC3/4/5	Number	• number systems • indices • applied arithmetic • applied measure	11	210
	Algebra	• representing situations with tables, diagrams and graphs • finding formulae • examining algebraic relationships • relations without formulae • expressions • equations and inequalities	10	
	Functions	• graphing functions	2	

[6] Class of 2012 figures are provided in the first interim report (November 2012).

The booklets were made up of items from two international surveys: ‘released’ items[7] from TIMSS 2007 (Trends in International Mathematics and Science Study, a survey of 13-14 year olds) and sample items[8] from PISA 2000, 2003 and 2006 (Programme for International Student Assessment, a survey of 15 year olds). The use of the TIMSS items allowed a direct comparison to be made between international results and the results of the phase one (and comparison group) students. Details of the origin of each item in the booklets, along with further explanation of the suitability of the international surveys for this research, are given in the interim report (November 2012).[9]

Assessment of student performance at Leaving Certificate level

Again, assessment booklets were created in order to assess students’ performance in individual items pertaining to each strand of the revised mathematics syllabuses, thereby giving an indication of their overall performance across each strand[10]. Table 2.5 shows the syllabus strands covered by each booklet, and the number of items in each booklet.

Both phase one and non-phase one students completed one of two booklets, which were randomly allocated to students at each school. However, students from non-phase one schools did not complete the last set of items in the booklets assessing Strand 5 (Functions). Due to the phased introduction of the revised syllabuses, this strand had not yet been taught to the non-phase one students[11].

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[7] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS.
[8] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no comparative data available.
[9] It should be noted that the number of students, at both Junior Certificate and Leaving Certificate, completing each of the booklets varies. Facilities based on relatively small numbers of students taking each item are not estimated to a high level of precision so should be treated with a degree of caution. To estimate facility with a reasonable degree of precision we would usually need to sample around 400 students in each group to be reported.
[10] Booklet LC1/2/5 contains all the items previously presented in SPLC1, GTLC2 and FLC5. Likewise booklet LC3/4/5 contains all the items previously presented in NLC3, ALC4 and FLC5. All items were unchanged and new items were not added.
[11] Assessment booklets were made available in the Irish language. However, participating Irish language schools opted to administer the assessment in English.

Table 2.5: Booklets for the Leaving Certificate

Booklet	Syllabus strand	Syllabus area	Number of items	Number of students completing booklet in the class of 2013[12]
LC1/2/5	Statistics and Probability	• concepts of probability • outcomes of random processes • statistical reasoning with an aim to becoming a statistically aware consumer • representing data graphically and numerically	9	596
	Geometry and Trigonometry	• synthetic geometry • co-ordinate geometry • trigonometry	10	
	Functions	• functions • calculus	9	
LC3/4/5	Number	• number systems • length, area and volume • problem solving and synthesis skills	10	605
	Algebra	• expressions • solving equations • inequalities • complex numbers	7	
	Functions	• functions • calculus	9	

The booklets were made up of items from three international surveys: released items[13] from the Trends in International Mathematics and Science Study (TIMSS 2007, 8th grade) and

[12] Class of 2012 figures are provided in the first interim report (November 2012).

[13] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS.

TIMSS Advanced (2008), and sample items[14] from the Programme for International Student Assessment (PISA) surveys of 2000, 2003, and 2006. Leaving Certificate items were specifically selected to match the revised mathematics syllabus and to assess a variety of mathematical skills. Some items are common across both Junior and Leaving Certificate booklets allowing for some comparison to be made across years. Again, further details are given in the interim report (November 2012).

Administration and marking of the assessment booklets

The booklets were administered to students by their teachers and returned to NFER before being marked using the NFER's own on-line system. Multiple-choice items were double marked by the NFER's data capture staff. The remainder of the items were marked by the NFER team. Teachers in the participating schools were not involved in marking the booklets. All participating schools were offered individual online feedback on their students' results. Fifty-one schools (out of a possible 80) downloaded their assessment results, and 115 (out of a possible 176) downloaded the survey results.

Versions of the revised mathematics syllabus referenced in this report

All references to the syllabus in the assessment sections of this report have been taken from the Junior Certificate Mathematics syllabus for examination in June 2015 and the Leaving Certificate Mathematics syllabus for examination in 2014 only. Where referring to a sub-section of a strand, numbering from the syllabus is provided in brackets. For example, 'representing data graphically and numerically' is labelled as sub-section 1.6 of Strand 1 (Statistics and Probability).

2.2.3 Student attitude surveys

The student attitude survey gathered quantitative data, focusing on:

- students' motivations and attitudes to mathematics, in general and in relation to the revised mathematics syllabus
- students' opinions on the revised mathematics syllabus, including the effectiveness of different strands and approaches
- students' perceptions of their knowledge, understanding, confidence and achievement in mathematics
- students' aspirations for further study of mathematics
- students' views of the relevance and application of mathematics more generally.

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[14] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no comparative data available.

Surveys were administered to two separate year groups of students by their teachers. Students in the class of 2012 completed surveys in Spring 2012, and students in the class of 2013 completed surveys in Autumn 2012[15]. This report primarily focuses on the findings for the class of 2013, drawing comparisons to the class of 2012 as appropriate[16]. Further detailed findings relating to the class of 2012 are provided in the interim report[17].

2.2.4 Case studies

To explore in further depth students' attitudes and experiences of the revised mathematics syllabuses, 16 school case studies were conducted. These included:

- eight phase one schools, which were selected to include schools from different geographical areas and with varying characteristics (for example, school type, level of deprivation), in consultation with NCCA
- eight non-phase one schools, again including schools in different geographical areas and with varying characteristics, chosen from a self-selecting sample of 29 schools who had volunteered to take part in the first survey and assessment phase, and indicated an interest in participating as a case-study school.

The visits took place in Autumn 2012, following early telephone consultations with mathematics coordinators in several schools during the spring and summer terms. Focus groups were conducted with Junior and Leaving Certificate students, as well as face-to-face and telephone interviews with teachers and mathematics coordinators to understand how the revised syllabuses are being implemented. Senior leaders were also invited to participate in an optional interview. The number of participants involved in the case-study phase is presented in Table 2.6.

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[15] Attitude surveys were made available in the Irish language. However, participating Irish language schools opted to administer the surveys in English.
[16] Significant differences in the descriptive statistics between phase are presented where appropriate. Comparisons between cohorts are also provided, based on an indicative exploration of the raw data rather than statistical analysis. As this is a cross-sectional study, caution should be applied when interpreting these results as they do not control for any factors which might differ between the two time points. Detailed findings taking account of these factors are explored in the multi-level modelling and presented in section 5.
[17] Jeffes *et al.*, 2012

Table 2.6: Number of participants involved in the case studies

Role	Number of participants	
	Phase one	Non-phase one
Principal/Senior leader	8	4
Mathematics coordinator	8	7
Teacher	15	10
Junior Certificate student	46	43
Leaving Certificate student	46	40

2.2.5 Analysis of students' work

This element of the research focuses on analysing students' written work to identify common features of students' mathematical approaches, as well as information about the mathematical skills that characterise particular groups of students. The sample material was produced by students during their mathematics lessons whilst following the revised syllabuses. Material has been collected from both phase one and non-phase one schools, some of whom were also involved in the case studies. The data collected falls into two categories:

- samples of student work that is the product of a mathematics lesson
- information provided by teachers detailing: the context of the lesson; the learning outcomes expected; and the teaching and learning approaches used.18

As the focus of this part of the study is solely on the analysis of students' written work, contextual details of the lesson have not been included in the analysis. It is important to note that this element of the research focuses on the processes promoted through the revised syllabuses and their presence in students' written work, and it is not intended as an analysis of student performance.

A total of 154 samples of students' written work (37 samples from phase one schools and 117 from non-phase one schools) were collected from 58 lessons (17 lessons from phase one schools and 41 from non-phase one schools). These are in several formats, including copy books, worksheets and class tests. Details of strands covered in these work samples, across the range of levels within the education system (e.g. Foundation, Ordinary and Higher Level), are set out in Appendix B.

[18] At the time of reporting not all teachers had provided this information, though some provided additional commentary on the success of the lesson and the next steps proposed.

In interpreting the data it is important to note that this is a small-scale exercise and the samples collected provide only an indicative picture of how the processes promoted through the revised syllabuses are being evidenced in students' work. Also, written student materials submitted for analysis are not the only product of the mathematics lessons where they were produced (for example, in some cases, students also engaged in discussions, worked in groups and participated in hands-on activity learning). Such approaches generate ephemeral evidence, which would require audio or video to record and was not conducted as part of this research.

3. Students' experiences of the revised mathematics syllabuses

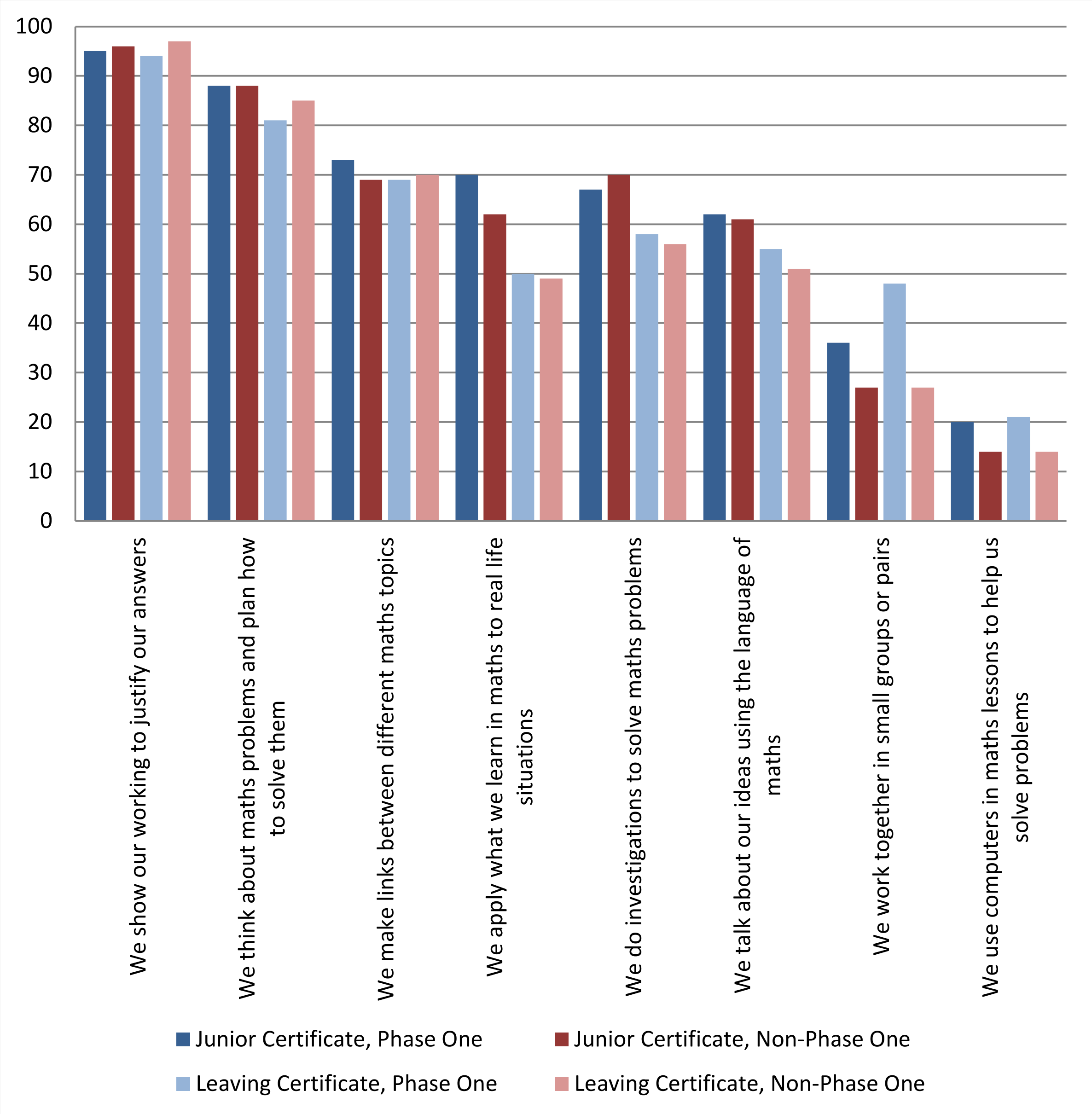
Key findings
Students' experiences of learning mathematics
• Students report that they are frequently undertaking activities commonly associated with the revised syllabuses (for example, making connections between mathematics topics, and applying mathematics to real-life situations). However, more traditional approaches (for example, using textbooks and copying from the board) also continue to be widespread.
Students' perspectives on their progression in mathematics
• While Leaving Certificate students appreciate the value of gaining a rich understanding of mathematics, they have found the change in learning approach between Junior Certificate and Leaving Certificate challenging.
• In general, Junior Certificate students are more positive about their transition from primary school than Leaving Certificate students are about their transition from Junior Certificate. In part, this may be because Junior Certificate students have experienced greater continuity of learning styles in their transition from primary school.
Students' overall learning and motivation
• Students are largely positive about their experience and confidence in learning mathematics, tending to agree that they enjoy it, do well in it and learn quickly in it. Junior Certificate students are more positive and more confident in their ability than Leaving Certificate students.
• International comparison suggests that students following the revised syllabuses are slightly less positive about mathematics, but more confident in their mathematics ability than those who participated in TIMSS 2007. This implies that, in this research, enjoyment of, and confidence in, mathematics do not necessarily go hand in hand.

This section explores students' experiences of learning mathematics and their perceptions about mathematics teaching. It draws on survey and case-study data collected from Junior Certificate and Leaving Certificate students in both phase one and non-phase one schools. Survey findings presented here primarily focus on the class of 2013, making comparison to the class of 2012 as appropriate. Further detailed findings from the class of 2012 are presented in the interim report (November 2012).

3.1 Students' experiences of the revised syllabuses

Students were asked about the frequency with which they participate in a range of activities which feature in the revised mathematics syllabuses. An overview of the answers given by students in the class of 2013 is presented in Figure 3.1 (for full data tables, see Appendix C, Tables 1-8).

Figure 3.1: Percentage of students who report that they 'often' or 'sometimes' participate in each of the following activities in mathematics lessons



Source: NFER student survey, Autumn 2012

Figure 3.1 shows that many students in the class of 2013 'often' or 'sometimes' undertake activities commonly associated with the revised syllabuses. Students most commonly report that they participate in activities to develop their understanding of the processes underpinning mathematics (for example, showing their working to justify their answers, and thinking about mathematics problems and planning how to solve them). By contrast, students do not commonly report that they regularly work together in small groups or pairs, or use computers in class to solve mathematics problems. This mirrors the findings of the analysis of students' work set out in section 4.

Further analysis reveals that there are statistically significant differences between the views of phase one students - who are in schools where teachers have been engaged in Project Maths for longer - and non-phase one students. Phase one students report that they 'often' or 'sometimes' participate in the following activities more frequently than their non-phase one peers:

- we make links between different maths topics
- we apply what we learn in maths to real-life situations
- we work together in small groups or pairs
- we use computers in maths lessons to help us solve problems.

This suggests that the approaches promoted through the revised syllabuses are becoming more widespread in the classroom as schools' familiarity and depth of involvement increases. Although use of computers in lessons, and working in small groups and pairs, appear to be the least frequent approaches used in lessons, observed comparison of the classes of 2012 and 2013 suggests that these are more frequent in the later year group. By contrast, application of learning to real-life situations appears to have become less common.

3.1.1 Findings from the case studies

Teachers' increased attention on raising students' depth of understanding of mathematical processes and the connections between mathematics topics

Teachers explain that they have increased the attention they give to **developing students' knowledge of the processes underpinning mathematics**, placing a greater emphasis on teaching for understanding, and introducing students to a range of different methods to solve mathematical problems. Many teachers report that they spend more time when introducing students to new topics, to ensure their basic understanding before moving on to more advanced concepts.[19]

> *We spend more time explaining* [mathematical concepts] *at the beginning of a topic rather than moving straight onto calculations, which is necessary to help their understanding as pupils are out of their comfort zone…*
>
> Mathematics teacher, phase one school

[19] While teachers involved in this research do not report that time spent introducing topics is particularly problematic, it is interesting to note that teacher feedback collated by the Irish Maths Teachers Association (IMTA) suggests that the amount of time required to cover many of the topics within the revised syllabuses is too great. (Irish Maths Teachers Association (2012). *Project Maths and the Irish Maths Teachers Association*. Cork: IMTA.)

Many students (including those studying at Higher Level) express difficulties with providing written explanations for their solutions to mathematical problems. Some report that they find the use of word-based problems challenging and difficult to interpret. Teachers and students alike report that students (particularly younger students) tend to be uncomfortable if there is room for interpretation and not just one ‘correct’ answer. There is a general consensus among students and teachers that across examination entry levels these are particularly challenging aspects of the revised syllabuses.

> *It’s hard to be specific... is my interpretation different from yours? That never happened in maths before. You know you had your solution and you got it right or wrong.*
>
> Mathematics subject coordinator, non-phase one school

Teachers feel that it is beneficial for students to engage in group and pair work in mathematics lessons, and feel that this helps students learn to communicate using mathematics terminology and by contributing knowledge from different strands. The majority of students report that they are **starting to make connections between different mathematics topics**. For example, students and teachers commonly describe making links between geometry and algebra, and between business studies and number systems (‘financial mathematics’). It is interesting to note, however, that such connections are not yet apparent in students’ written work (see section 4).

> *Your mind has to be very various in its abilities for Project Maths, to bring together all the different strands.*
>
> Leaving Certificate student, phase one school

Applying mathematics to real-life situations

The majority of students report that teachers use real-life contexts to explain mathematical concepts, for example measuring missile trajectories and activities based around modern technology. Students report that they enjoy applying mathematics to real-life contexts and find this beneficial for their learning.

> *Real-life applications are interesting, even if it* [the context] *is outside your subject. We’ve used lots of contexts from modern life, financial life and personal life.*
>
> Leaving Certificate student, non-phase one school

While students find learning through real-life contexts to be stimulating and motivating, they also describe the challenges of applying their learning to unfamiliar contexts. Phase one students comment on these challenges more frequently than non-phase one students, reflecting their broader experience of the style of questions used in the revised syllabuses. Many find the unpredictability of questions challenging, even at Higher Level.

In general, students prefer to apply their learning to actual real-life situations (for example, going outdoors to measure angles in nature) than contrived real-life situations (for example, measuring the volume of a grain silo). Junior Certificate students appear to have a broader experience of conducting investigations and undertaking practical activities than Leaving Certificate students and express a view that a more hands-on approach makes mathematical concepts more 'fun' and easier to understand.

Teachers observe that the deeper understanding of mathematics acquired by students as a result of the revised syllabuses will yield longer-term benefits for students. For example, one non-phase one teacher gave the example of two bridges, one built by an engineer who studied the previous mathematics curriculum and one built by an engineer who had studied the revised syllabus.

> [Given the choice], *I would rather cross the bridge that the Project Maths engineer built.*
> Mathematics teacher, non-phase one school

Use of ICT in the classroom

Use of information and communication technologies (ICT) in the classroom appears to be highly variable amongst the case-study schools, although some teachers report having introduced a range of approaches including, for example, digital media and mathematics software such as GeoGebra. This suggests that, in general, schools recognise the benefit of using computers to support mathematical learning. However, it should be reiterated that, as shown in Figure 3.1, students participating the attitude survey appear to use computers in their mathematics lessons relatively infrequently. Additionally, samples of students' work analysed as part of this study also provide limited evidence of the use of ICT (see section 4 for further details). Although all post-primary schools in Ireland were in receipt of an ICT infrastructure grant in 2012, some case-study schools remain concerned that they have insufficient resources to fully exploit this aspect of the curriculum and that this will disadvantage their students.

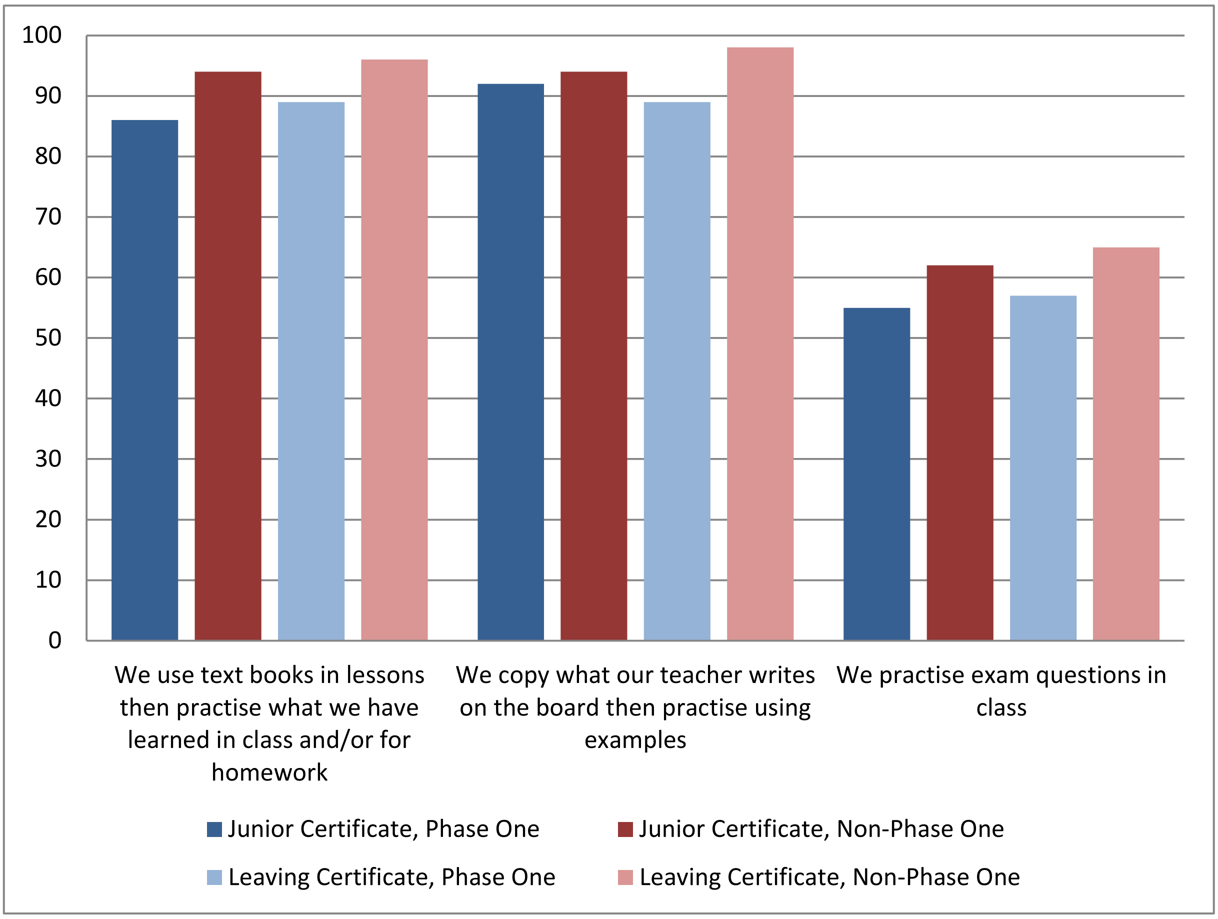
3.2 Students' experiences of more traditional approaches in the revised curriculum

Students were also asked about the frequency with which they participate in certain activities more typically associated with a traditional approach to teaching and learning mathematics. They were asked how often they:

- use textbooks in lessons then later practise what they have learned
- copy what their teacher writes on the board then practise using examples
- practise examination questions in class.

Although worthwhile as part of a balanced teaching of mathematics, it is hoped that, as the revised syllabuses become more embedded, students will report engaging with these activities less regularly as approaches promoted through the revised syllabuses take up a greater proportion of the available teaching time. Again, an overview of students' responses in each of these areas is presented in Figure 3.2 (see Appendix C, Tables 9-11 for more detail).

Figure 3.2: Percentage of students who report that they 'often' or 'sometimes' undertake each of the following activities in mathematics lessons



Source: NFER student survey, Autumn 2012

Figure 3.2 shows that more traditional approaches to mathematics teaching and learning continue to be widespread. This is also evident in students' written work (see section 4). All three approaches appear to be more widely used in non-phase one schools, which is expected given that they are at an earlier stage of implementing the revised syllabuses. The proportion of students who report that they 'sometimes' or 'often' use textbooks in lessons, and copy what their teacher writes on the board, suggests that these approaches become less common with increased exposure to the revised syllabuses.

3.2.1 Findings from the case studies

Traditional approaches to learning mathematics are evident and appear to be in use regularly, particularly in non-phase one schools. In non-phase one schools especially, there is some confusion about the availability of resources to support the teaching and learning of mathematics. For example, teachers report that they have adapted materials from existing textbooks and workbooks, some of which do not completely match the revised curriculum. By contrast, phase one teachers appear to have been more proactive in developing their own resources and materials for use in the classroom, perhaps because they have received a greater level of support as a result of their early involvement in delivering the revised syllabuses. In general, phase one students appear to have greater enjoyment of, and confidence, in mathematics than their non-phase one peers. One conclusion is that this is due to phase one teachers' own increased confidence in implementing the revised syllabuses.

Of note, case-study participants identify considerable concerns about 'exam readiness'. It is common for students to be using grinds[20] to support revision for their mathematics examinations. Students in over half of schools involved in the case studies, report that this is the case, which they feel is indicative of generally low confidence levels and concerns over examination preparation. This is experienced by both Junior Certificate and Leaving Certificate students in phase one and non-phase one schools alike. However, it should be noted that the use of grinds has historically been more common in mathematics than other subjects[21].

> *I know the whole reason behind Project Maths is to get rid of the whole structure* [of the examination]… *but you just felt that bit more confident* [with the previous curriculum] *because there was a structure to it. We have no idea what our exam is going to be like in June.*
>
> Leaving Certificate student, non-phase one school

3.3 Perceptions of mathematics teaching within the revised syllabus

To gain a deeper understanding of students' experiences of the revised mathematics syllabuses, teachers and students were asked about their perceptions of mathematics teaching. Students were asked to what extent they agreed with a range of statements about the mathematics teaching they had experienced, as presented in Figure 3.3 (see Appendix C, Tables 12-19 for more detail).

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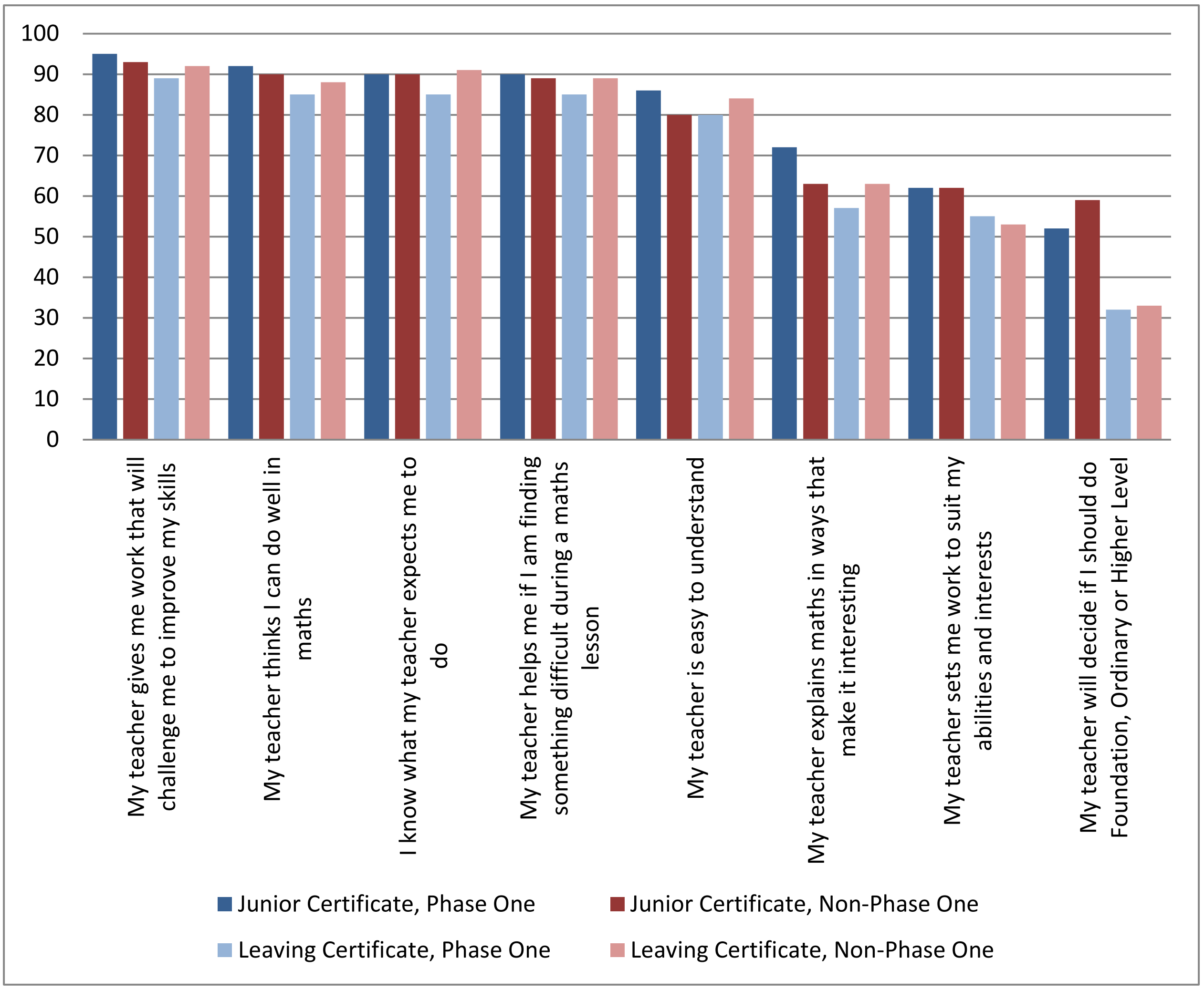
[20] Private tuition.

[21] NCCA. (2007). *ESRI research into the experiences of students in the third year of junior cycle and in transition to senior cycle: summary and commentary*. Available online: http://www.ncca.ie/uploadedfiles/publications/ESRI_3rdYr.pdf [Accessed 30th July 2013]

Figure 3.3: Percentage of students who report that they 'strongly agree' or 'agree' with the following statements about their mathematics teaching and learning



Source: NFER student survey, Autumn 2012

Figure 3.3 shows that Junior and Leaving Certificate students in both phase one and non-phase one schools have highly positive perceptions of their mathematics teaching. Statistically significant differences occur between phase one and non-phase one schools in relation to the following statements:

- my teacher helps me to understand if I am finding something difficult during a mathematics lesson
- my teacher is easy to understand
- my teacher explains mathematics in ways that make it interesting
- my teacher will decide if I should do Foundation Level, Ordinary Level or Higher Level.

Broadly, phase one students are more positive than their non-phase one peers in all of these areas. This suggests that students have increasingly positive perceptions as the revised syllabuses become more embedded in school. Additional analysis reveals that, of these four areas, there are statistically significant differences between Junior Certificate and Leaving Certificate students in relation to ‘my teacher explains mathematics in ways that make it interesting’ and ‘my teacher will decide if I should do Foundation Level, Ordinary Level or Higher Level’, with Junior Certificate students agreeing with this more strongly in both cases.

3.4 Students’ perspectives on their progression from primary and Junior Certificate mathematics

Students involved in the attitude surveys and case studies were asked to comment on their experiences of progression, in the case of Junior Certificate students, from primary level mathematics and, in the case of Leaving Certificate students, from Junior Certificate level. Students were asked to draw out key differences in their experiences of learning mathematics at different stages to ascertain how this had impacted upon their progress.

In general, Junior Certificate students are more positive about their transition from primary school than Leaving Certificate students are about their transition from Junior Certificate. Whilst Leaving Certificate students appreciate the value of gaining a rich understanding of mathematics, they have found the change in learning approach between Junior Certificate and Leaving Certificate challenging. Most Junior Certificate students report that the revised curriculum is ‘just maths’, and have experienced a more natural transition from primary to post-primary level.

A note on attitudes at Junior versus Leaving Certificate levels

As set out above, Junior Certificate students are more positive about their transition from primary school than Leaving Certificate students are about their transition from Junior Certificate. This finding reflects a general trend presented in Sections 3 to 6, as Junior Certificate students are generally more positive about mathematics, more confident in their ability and recognise the broader application of mathematics more than their older, Leaving Certificate counterparts.

As described above, Junior Certificate students have used approaches similar to those promoted in the revised syllabuses through their experience of mathematics at primary school. By comparison, some Leaving Certificate students experienced a shift in teaching and learning approaches in the transition from the previous curriculum at Junior Level. This may explain some of the difference in attitude.

Leaving Certificate students may be seeking additional university tariff points by studying mathematics at Higher Level. In some cases, highlighted through the case-study visits, students who would otherwise have followed the Ordinary Level syllabus are now working at Higher Level. As a result, some teachers and students feel that this affects the learning of the whole class.

3.5 Students' overall confidence, motivation and attitudes towards mathematics

While impacts on students' confidence are explored in depth in section 5, it is interesting to compare students' perceptions of their abilities and levels of engagement with mathematics with an international sample of students who participated in TIMSS 2007 (aged, on average, 14 years old). Comparison suggests that the students involved in this study are slightly less positive about mathematics, but slightly more confident in their mathematics ability than the average of those involved in the TIMSS study.

4. Evidence of the extent to which students are using the approaches promoted in the revised mathematics syllabuses

Key findings

- Based on the small sample of students' work included in this study there is emerging evidence that the revised syllabuses are impacting on students' learning in the key process areas. However, the processes promoted through the revised syllabuses are not yet embedded in the written output from mathematics lessons (although they may be evident in other aspects of lessons). This may be expected due to the early stage of the implementation of the revised syllabuses.
- Whilst some processes of the revised mathematics syllabuses are visible in some of the student material reviewed, there does not appear to have been a substantial shift in what teachers are asking students to do, and few differences between the phase one and non-phase one students.
- It is possible that teachers are currently emphasising the content of the revised syllabuses rather than the processes promoted within it. This reflects the earlier finding that traditional approaches to mathematics teaching and learning continue to be widespread.
- The evidence strongly suggests that students have a good mastery of mathematical procedures and, to a slightly lesser extent, problem solving and making mathematical representations. There is very little evidence in the work reviewed that students are demonstrating reasoning and proof and communication, or making connections between mathematics topics.
- The findings suggest that students need to be regularly given high quality tasks that require them to engage with the processes promoted by the revised syllabuses, including: problem solving; drawing out connections between mathematics topics; communicating more effectively in written form; and justifying and providing evidence for their answers.

This section sets out the findings from an analysis of a sample of students' written work. It aims to identify evidence of the processes introduced in the revised mathematics syllabuses in sample of students' written work, as well as information about the mathematical skills that characterise particular groups of students.

4.1 About the samples of students' work

4.1.1 Number of lesson samples

As set out in Table 4.1, a total of 154 samples of students' written work (37 samples from phase one schools and 117 from non-phase one schools) have been collected from 58 lessons (17 lessons from phase one schools and 41 from non-phase one schools). Further detail on the methodology for this research activity is provided in section 2.2.5.

Table 4.1: Number of lesson samples included in the analysis of students' work, by strand

		Leaving Certificate			Junior Certificate			Total
		Higher Level	Ordinary Level	Mixed	Higher Level	Ordinary Level	Mixed	
Strand 1	Phase one	0	1	0	1	0	0	**2**
	Non-phase one	0	4	0	5	1	3	**13**
	Total	0	5	0	6	1	3	**15**
Strand 2	Phase one	1	0	0	0	0	0	**1**
	Non-phase one	3	0	1	5	4	1	**14**
	Total	4	0	1	5	4	1	**15**
Strand 3	Phase one	1	1	0	0	0	1	**3**
	Non-phase one	1	3	0	0	1	0	**5**
	Total	2	4	0	0	1	1	**8**
Strand 4	Phase one	1	1	2	2	2	2	**10**
	Non-phase one	2	2	0	1	1	0	**6**
	Total	3	3	2	3	3	2	**16**
Strand 5	Phase one	0	0	1	0	0	0	**1**
	Non-phase one	2	0	1	0	0	0	**3**
	Total	2	0	2	0	0	0	**4**
Total	Phase one	3	3	3	3	2	3	**17**
	Non-phase one	8	9	2	11	7	4	**41**
	Total	**11**	**12**	**5**	**14**	**9**	**7**	**58**

4.1.2 Format of the samples

Table 4.2 outlines the formats for written work that were evident in the student material gathered. Students indicated the use of textbooks by reference to pages, exercises and questions.

Table 4.2: Number of lesson samples of student material by format

Format of students' work	Total number of samples received
Copy	32
Worksheets	
Tests	6
Mind map	3
Copy/worksheets	3
Total	**58 lessons**

A further breakdown of sample material format is set out in Table 4.3 where the specific types of worksheets used are identified and the copy-work based on textbook activity has been extracted.

Table 4.3: Details of the lesson samples of student material by format

Format of students' work	Total number of samples received
Copy	13
Copy and textbook	19
Test/sample examination paper	6
Worksheet only	5
Copy/Worksheet	3
Activity Maths worksheets	7
Project Maths worksheets	2
Mind maps	3
Total	**58 lessons**

Three mind maps were submitted as sample lesson materials[22]. These were of varying standards but indicated in every instance an attempt at synthesis of information, making connections and seeing patterns in the subject matter explored.

4.2 Interrogating students' written work

The samples collected were analysed to explore evidence of the following mathematical processes, which are promoted through the revised mathematics syllabuses, demonstrated in students' work:

- **problem solving**, including analytical consideration of different approaches and evidence of adjusting and self-correcting where appropriate
- **mastery of mathematical procedure**, including ability to solve a problem using accurate techniques to reach the correct solution

[22] Due to the complexity of some of these mind maps it is unlikely that they were the product of a single lesson.

- **reasoning and proof**, including the use of arguments, evidence and proof to support an answer to a mathematical problem
- **communication**, including effective explanation of arguments and appropriate use of mathematical language
- **connections between mathematics topics**, including observation of relationships between subjects and themes to solve mathematical problems
- **mathematics representations**, including construction of models and tools to analyse and interpret data.

A detailed description of these categories, and the criteria used to assess the type and extent of evidence present in students' written work, is presented in Table 4.4, adapted from process standards produced by the National Council of Teachers of Mathematics (NCTM)[23] and adapted by Exemplars[24].

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[23] National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM [online]. Available: http://www.nctm.org/standards/content.asp?id=322 [21 March, 2013].

[24] Exemplars (2013). *Exemplars: Standard Maths Rubric.* Underhill, VT: Exemplars [online]. Available: http://www.exemplars.com/resources/rubrics/assessment-rubrics [21 March, 2013].

Table 4.4: Framework for analysis of students' work

Process	No evidence	Novice level	Practitioner level	Expert level
Problem solving[25]	All examples show the same solution strategy. Where a single example is presented it is based solely on a worked example.	There is no strategy or the strategy chosen does not lead to a solution. Little or no engagement with the task presented.	A correct strategy is chosen based on the mathematical situation in the task. Planning or monitoring of the strategy is evident. Evidence is present of solidifying prior knowledge and applying it to the problem solving situation. Correct, or partially correct, answer is obtained.	An efficient strategy is chosen and progress towards a solution evaluated. If necessary, self-correction and adjustments are made along the way. Evidence is present of analysing the situation in mathematical terms and extending prior knowledge. Correct answer is obtained.
Mastery of mathematical procedures	No evidence of following any basic mathematical procedure.	Evidence of some familiarity with a basic mathematical procedure. Does not always lead to the correct answer.	Evidence of the application of a basic mathematical procedure. Some insecurity evident. Correct answer is not always obtained.	Mathematical procedures are accurately followed. Correct answer is obtained.
Reasoning and proof	No reason or proof provided.	Arguments made have no mathematical basis.	Arguments are constructed with adequate mathematical basis.	Deductive arguments are used to justify decisions and resulted in formal proofs.

---

[25] In general, it is challenging to effectively analyse problem solving within students' work. This is because sometimes the most able students simply select the simplest method and apply it in a straightforward way. By contrast, the less able may show more evidence of modifying their approach.

	The task/activity does not require a reason/proof.	No correct reasoning or justification for reasoning is present.	A systematic approach and/or justification of correct reasoning is present leading to the exploration of mathematical phenomena, noting pattern, structure and regularities.	Evidence is used to justify and support decisions made and conclusions reached. This may be indicated in self- corrections, the testing and rejection of hypotheses, explanation of phenomenon or generalising and extending the solution to other cases.
Communication	No communication evident. Task/activity does not require students to communicate in any way.	Everyday familiar language is used to communicate ideas.	Communication of an approach is evident through a methodical, organised, coherent, sequenced and labelled response. Formal mathematical language is used throughout the solution to clarify and express ideas.	Communication of argument is supported by mathematical properties. Precise mathematical language and symbolic notation are used to consolidate mathematical thinking and to communicate mathematical ideas precisely.
Connections between mathematics topics	No connections are made between maths topics.	Some attempt is made to relate the task to other maths topics.	Mathematical connections and observations are recognised and evident.	Mathematical connections and observations are used to extend the solution.
Mathematical representations	No attempt made to construct mathematical representations	An attempt is made to construct mathematical representation to record and communicate problem solving. Not always accurate.	Appropriate and accurate mathematical representations are constructed and refined to solve problems and portray solutions	Abstract or symbolic mathematical representations are constructed to order, record and analyse relationships, to extend thinking and clarify or interpret phenomena.

The analysis sought to explore the extent to which the processes emphasised in the revised syllabuses are evident in the student material. The analysis also explored the extent to which the evidence suggests that students are engaging with the approaches promoted through the revised syllabuses on a continuum of 'no evidence' to 'expert', as well as the proportion of examples which are based on students using structured worksheets or copies and text books (characteristic of the previous version of the curriculum), as an indicator of their prevalence in the classroom. At each level, the extent and nature of differences between the work of students in phase one and non-phase one students was explored.

Illustrations of students' work at both Junior and Leaving Certificate are provided in Figures 4.1 and 4.2. The students' work in Figure 4.1 demonstrates an **expert level mastery of the mathematical procedures** required to answer the problem posed. Mathematical procedures have been accurately followed and the correct answer has been obtained. The student also demonstrates **expert level use of mathematical representations**, evidenced in their illustration of their thought processes and as a means of providing proof and reasoning for their answer. The student demonstrates **practitioner level evidence of communication**, as whilst communication is logical, there is evidence of the student using language from earlier parts of the question to formulate their responses. There is no evidence of connections between mathematics topics evident in the student's work.

The students' work in Figure 4.2 demonstrates a **novice level evidence of making mathematical representations** due to the poor level of accuracy in the graph. The student's work demonstrates **novice level evidence of mastery of mathematical processes**, as many of the calculations in the table are inaccurate, even though the correct points are subsequently graphed. There is no evidence of reasoning and proof, communication, or making connections between mathematics topics.

Figure 4.1: Extract from a Junior Certificate student's work, studying towards Higher Level examination

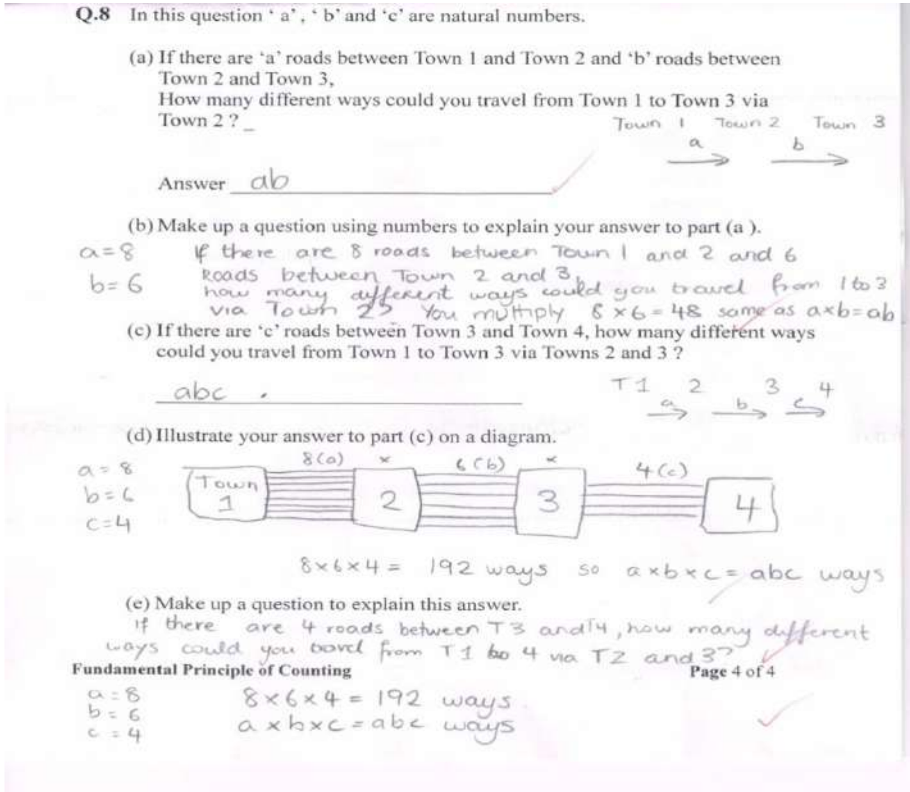
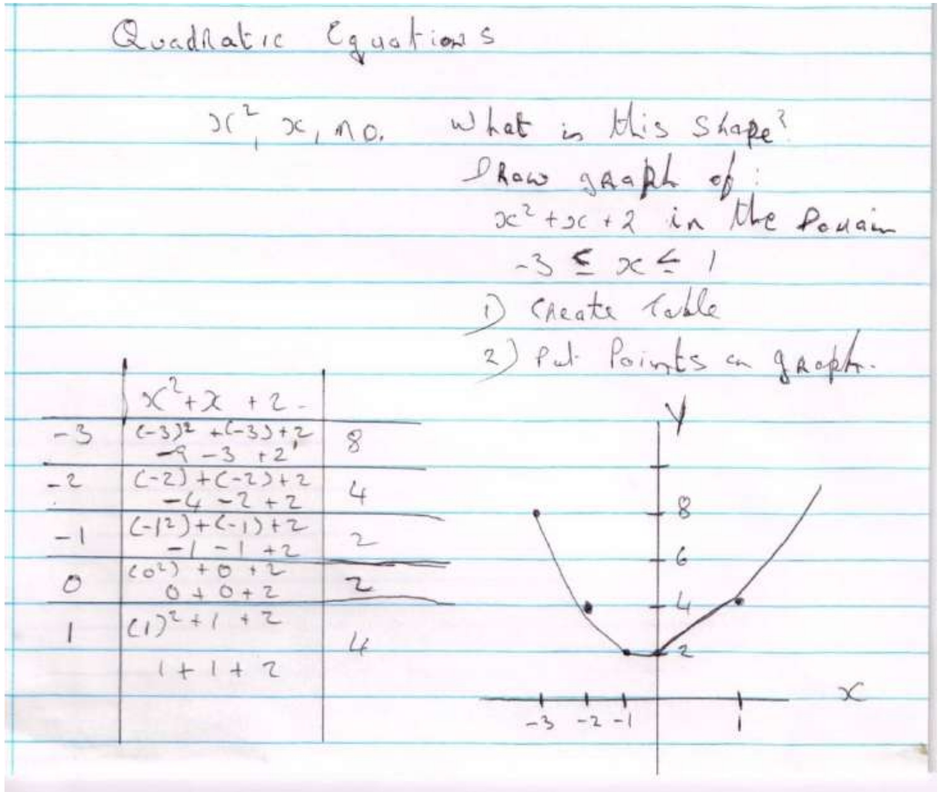


Figure 4.2: Extract from a Leaving Certificate student's work, studying towards Ordinary Level examination



4.3 Evidence of the mathematical processes present in students' work

There is variation in the extent to which the approaches promoted through the revised syllabuses are evidenced in students' work, which is as might be expected at this stage of the curriculum's implementation.

Evidence of mastery of mathematical procedures is visible in many examples of students' work from both phase one and non-phase one schools. To a slightly lesser degree, there is also evidence of problem-solving, communication and representations. There is very limited evidence of students demonstrating reasoning and proof, or making connections between mathematics topics, in their written work.

The following sections provide an overview and commentary on the analysis of students' work in relation to each of the mathematical processes described above. Further technical details are provided in Appendix B.

4.3.1 Problem solving

Six pieces of 6th year phase one students' work meet the 'expert' criteria. Of these, four are from a lesson where the standard of work was 'expert' overall. Of the eight students in 6th year whose work was at 'practitioner' level, six were the product of lessons by the same teacher.

Table 4.5: Evidence of problem solving in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	1	0	2	2
	3rd year	5	1	2	1
	5th year	6	0	1	0
	6th year	2	0	8	6
Non-phase one	1st Year	1	0	0	0
	2nd Year	0	3	3	0
	3rd year	15	5	9	1
	5th Year	3	9	4	1
	6th Year	0	3	25	0

In non-phase one schools, 'expert' level work in problem solving was not in evidence among the 1st and 2nd year samples. Most notably, among the sample material presented by 3rd year students there was often no evidence (19 cases out of 48) that an opportunity to engage in problem solving activity had been provided. The 'practitioner' level problem solving work was predominantly evidenced in 6th year material: this was the modal standard of problem solving at this level and was spread across a number of teachers, schools and strands.

4.3.2 Mastery of mathematical procedure

In the phase one schools, over half of students appear to be engaging in traditional mathematical procedures at 'expert' level. This standard of work is not always present in the other processes explored, suggesting that students are still engaging in processes more closely associated with the previous version of the curriculum, such as drill and practice. In some lessons, mastery of mathematical procedure was the only process evidenced in the sample material for these lessons. This suggests that these students are not required to engage in other mathematical processes and have not been given opportunities to do so; instead they present material that is based on worked examples.

Table 4.6: Evidence of mastery of mathematical procedure in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	0	0	3	2
	3rd year	0	1	4	4
	5th year	0	2	1	4
	6th year	0	2	5	9
Non-phase one	1st year	1	1	5	0
	2nd year	1	3	2	0
	3rd year	0	4	19	25
	5th year	0	3	10	4
	6th year	0	8	13	18

Similar to the phase one schools, high standards of mastery of mathematical procedures are evident in almost all sample material from the non-phase one schools. The material also indicates that students are being given opportunities to engage in mathematical procedure, often quite complex in nature, without engaging in the other processes that are integral to the revised mathematics syllabuses. A total of 128 samples of students' work (in both phase one and non-phase one schools) indicate that students are operating at 'practitioner' or 'expert' level when mastering the application of a method and practising a particular algorithm.

Where material is at 'novice' level, most of the students are deemed by their teacher to be 'struggling', at Foundation or Ordinary Level. At the standard of 'novice', fluency is sometimes insecure, mathematical engagement is shallow and many errors are evident.

4.3.3 Reasoning and proof

The sample material from phase one schools' lessons suggests that students are not engaging in any depth in reasoning and proof as a mathematical process within attempts at problem solving. Most sample material has no evidence of students giving any reasons or proofs.

Table 4.7: Evidence of reasoning and proof in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	2	2	1	0
	3rd year	4	3	1	1
	5th year	6	0	1	0
	6th year	2	7	3	4
Non-phase one	1st year	7	0	0	0
	2nd year	6	0	0	0
	3rd year	32	10	5	1
	5th year	4	11	1	1
	6th year	10	16	8	5

Similarly, in non-phase one schools, it could be conjectured that in most samples, students are not required to give reasons and proofs, and are therefore not doing so. At 'novice' level, some students who provide a proof presented incorrect, ineffective reasoning, or make arguments that have no mathematical basis. At the 'practitioner' and 'expert' level, students present adaptive, correct, effective reasoning when a proof is provided.

4.3.4 Communication

Although communication in mathematics has both oral and written elements[26]; this analysis considers only the written element. Where samples include an attempt at communication they range from familiar student language to more precise mathematical language and symbolic notation to consolidate mathematical thinking, to explain and communicate ideas.

In phase one schools, 26 samples have either no evidence or are at 'novice' level. Most students appear not have given explanations of their answers or to communicate their thinking in writing. Typically most 'novice' standard work makes use of everyday familiar language, although often not focussed and sometimes inaccurate. There is very little evidence of students exploring patterns or noting where patterns are evident in their work.

[26] Some teachers included comments on their use of oral communication/discussion in the lesson context information provided

Table 4.8: Evidence of communication in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	4	0	1	0
	3rd year	6	2	0	1
	5th year	3	3	1	0
	6th year	1	7	3	5
Non-phase one	1st year	6	1	0	0
	2nd year	5	1	0	0
	3rd year	21	15	9	3
	5th year	7	5	5	0
	6th year	13	20	2	4

Once again the pattern is replicated in the non-phase one schools, with the majority of samples showing no evidence or 'novice' standards of communication. In some material based on worksheets and class test papers, where clearly students are asked to explain their answers and have been given the opportunity to communicate their thinking, many have left this part of the question blank or provided incorrect, unclear statements.

In the four 'expert' level samples from 3rd years there are three excellent mind-maps where communication of ideas and synthesis of relevant information are present. These samples could only be evaluated under a limited number of process standards and it should be noted that due to their complexity, they are unlikely to be the outcome of a single lesson.

4.3.5 Connections between mathematics topics

There appears to be a tendency to see the various strands and topics in the revised syllabuses as discrete subject areas, and mathematics is not treated in the sample material as an integrated field of study. The poor use of connections across mathematical topics and subject areas is evident in a high proportion of material where there is no evidence of students making any connections outside of the narrow topic that they are studying: 31 out of 37 phase one lessons and 95 out of 117 non-phase one lessons fall into this category.

Table 4.9: Evidence of making connections between mathematics topics in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	5	0	0	0
	3rd year	9	0	0	0
	5th year	6	1	0	0
	6th year	11	0	1	4
Non-phase one	1st year	7	0	0	0
	2nd year	6	0	0	0
	3rd year	36	8	4	0
	5th year	15	2	0	0
	6th year	31	0	8	0

4.3.6 Representations

There is little evidence of students using the range of representations available across all of the strands of the revised syllabuses to solve problems, illustrate or extend their thinking.

Table 4.10: Evidence of representations in phase one and non-phase one schools

	Year group	No evidence	Novice	Practitioner	Expert
Phase one	2nd year	0	3	2	0
	3rd year	4	1	4	0
	5th year	6	1	0	0
	6th year	3	3	5	5
Non-phase one	1st year	2	5	0	0
	2nd year	2	4	0	0
	3rd year	19	14	13	2
	5th year	5	8	4	0
	6th year	11	18	8	2

The most common representations came from Strand 1 (Statistics and Probability): the use of stem and leaf plots; the number line; graphs; area models; Venn diagrams; bar charts; pie charts; and tables. A lack of accuracy means that many of the samples fall within 'novice' standard. Overall, the students in the non-phase one schools are less likely to use representations, or they produce ones that are at 'novice' standard. Only four students produced work at 'expert' level.

4.3.7 Discussion

There is emerging evidence that the revised mathematics syllabuses are impacting on students' learning in relation to each of the mathematical processes described in the preceding section. This is as might be expected at this stage of the curriculum's implementation. While analysis of students' work reveals that elements of the revised mathematics syllabuses are visible in almost all of the material reviewed, the findings suggest that the processes promoted through the revised syllabuses are not yet fully embedded in the written output from mathematics lessons. Evidence from the samples of students' work presented suggests that students are being presented with tasks that do not require them to engage widely with the mathematical processes promoted through the revised syllabuses. This is surprising given that in other parts of this research, teachers suggest that such approaches are being widely implemented.

There is evidence of some engagement with real-life contexts in students' written work. It is interesting to note that students' written work often includes visual models and tools (for example, an area model for solving quadratic equations and factorising in algebra). Reflecting the findings of the attitude survey, there is very limited evidence of students working collaboratively to solve mathematical problems and only one sample of students' work appears to include the use of ICT. Again, this is of interest as both teachers and students involved in the case studies provide examples of ways in which ICT has been introduced into mathematics lessons.

While it is likely that it will take time for students to consolidate new ways of learning in their written outputs, this also suggests that teachers may benefit from further support to translate their teaching approaches (for example, investigations and practical activities) into students' written work. This may include more opportunities for students to engage in problem-solving processes, draw out connections between mathematics topics and communicate more effectively in written form. It may be beneficial for students to receive further encouragement to explain, justify and provide evidence for their answers; and to understand more deeply the relationships and connections between mathematics topics within and across all strands of the revised mathematics syllabuses.

5. Students' achievement and attitudes towards mathematics

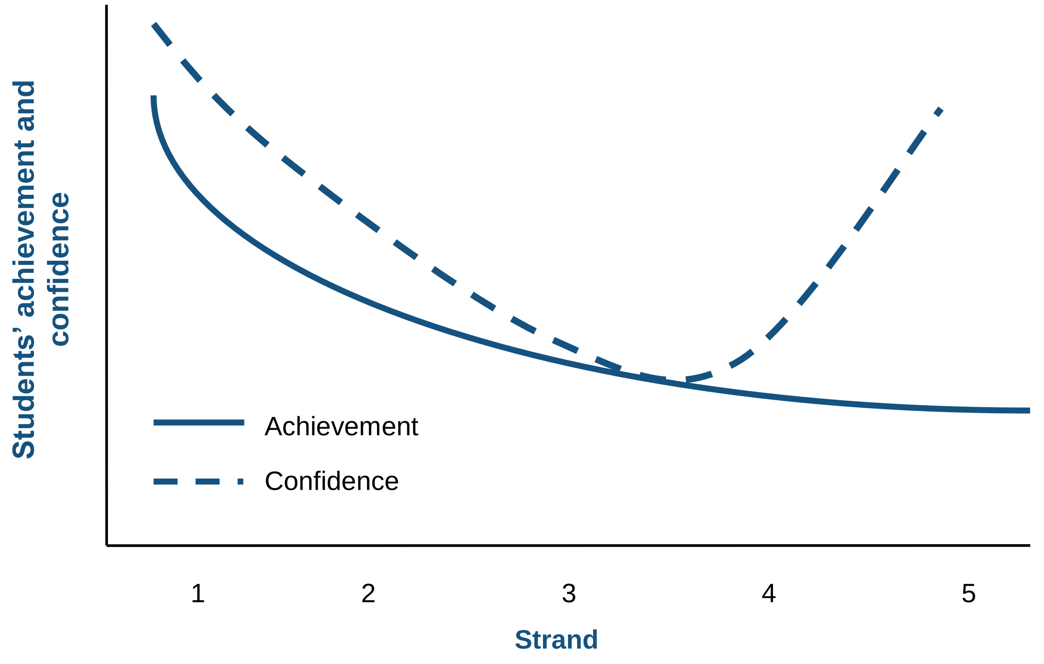
Key findings

The variables that affect students' attitudes and achievement

- Overall, following a greater number of strands, or schools having greater experience of teaching the revised syllabuses, does not appear to be associated with any improvement in students' achievement and confidence.
- Students' confidence and achievement in mathematics is significantly associated with examination entry level and gender. Of note, girls are less confident about mathematics than boys and perform less well at Junior Certificate.

Students' achievement and confidence across the five strands

Whilst achievement is highest in Strand 1 (Statistics and Probability) and lowest in Strand 5 (Functions), confidence is actually highest in both of these strands and lowest in Strand 3 (Number) and Strand 4 (Algebra). Note, this diagram is illustrative only, and not to scale.



The relationship between confidence and achievement

- In this research study, confidence in mathematics does not always correspond to achievement. Although students who are further through their studies perform better than those who are at an earlier stage, higher levels of confidence are not associated with students who have almost completed their studies.
- Furthermore, results show that students are confident in relation to Strand 5 (Functions), but do not perform highly when assessed in this area. While girls are less confident in their mathematical ability across all strands explored, at Leaving Certificate level, they perform as highly as boys.

This section explores the impacts of the revised mathematics syllabuses on students' achievement, learning and motivation. The section draws on evidence collected via the assessments and the surveys. Section 5.1 presents the findings of a multi-level modelling exercise to investigate the effect of a range of variables on students' achievement and confidence in mathematics. Sections 5.2 and 5.3 set out in more detail comparisons between phase one and non-phase one students' achievement in the classes of 2012 and 2013, including additional comparison with international TIMSS data where appropriate. Further details of students' performance in, and attitudes towards, individual strands of the revised syllabuses are provided in Appendix D.

5.1 Students' overall achievement and attitudes towards mathematics

Multi-level modelling[27] allows the impact on all students to be explored, taking account of a range of student characteristics (for example, phase of study, gender (boys or girls) and examination entry level). Multi-level modelling also enables us to take into account timing differences between the data collection exercises conducted with each year group.

This is important because students in the class of 2012 participated in the research as they were reaching the end of their studies, whereas those in the class of 2013 were just beginning their final year of study. As a result, the class of 2013 had received less mathematics tuition than the previous year group at the time of the study. This difference in schooling may lead to the expectation that the class of 2012 would perform better and achieve higher results than the class of 2013.

A number of background variables have been explored to investigate their effect on students' achievement and confidence in mathematics. These include:

- phase (phase one or non-phase one)
- survey date (Spring or Autumn 2012)
- gender (girls or boys)
- examination entry level (Foundation, Ordinary or Higher Level)
- school type (vocational, community and comprehensive or secondary school).

It is important to note that only Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry) are factored into the multi-level modelling analysis, as these are the only strands in which all students involved in the study have been assessed. Further consideration of the students' achievement and confidence across all five strands is provided later in this section.

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[27] Further technical details on the multi-level modelling analysis are provided in Appendix A

5.1.1 Variables impacting on students' achievement and confidence in mathematics

Multi-level modelling reveals that the following variables significantly impact on students:

- **assessment date:** the timing of the assessment is a significant predictor of achievement among Junior Certificate students, and among Leaving Certificate students in relation to Strand 2 (Geometry and Trigonometry). As might be expected, students in the class of 2013 perform less well than the class of 2012
- **examination entry level:** again, as might be expected, Foundation Level and Ordinary Level students generally have less confidence in mathematics and lower achievement than Higher Level students
- **gender:** gender is a significant predictor of confidence in mathematics at both Junior and Leaving Certificate. It is also a significant predictor of achievement. Girls are less confident about mathematics than boys and have lower achievement at Junior Certificate. At Leaving Certificate, girls have lower achievement than boys in relation to Strand 1 (Statistics and Probability)
- **some school types**[28]**:** Junior Certificate students in vocational schools have lower achievement than those in secondary schools.

5.1.2 Variables that are not impacting on students' achievement and confidence in mathematics

Multi-level modelling also reveals that the following background characteristics are **not** impacting on students' confidence and achievement in mathematics:

- **phase**: whether a student is in a phase one or non-phase one school does not significantly impact on their confidence or achievement in mathematics at either Junior and Leaving Certificate
- **survey date:** the timing of the attitude survey is not a significant predictor of students' confidence in mathematics at either Junior or Leaving Certificate
- **some school types:** again, whether the school is community and comprehensive or secondary does not significantly impact on students' confidence or achievement in mathematics at either Junior or Leaving Certificate.

5.1.3 Interpretation of the multi-level model analysis

School phase (phase one and non-phase one) is not a significant predictor of students' confidence or performance in the revised mathematics syllabuses. The findings suggest that

[28] The post-primary school types included in the analysis are: secondary schools (managed by boards of management, religious communities/trustees or individuals); vocational schools (managed by local vocational education committees (VECs)); and community and comprehensive schools (managed by management boards supporting local interests). For further information, see http://www.citizensinformation.ie/en/education/

background characteristics, such as gender rather than phase, are more closely associated with achievement and confidence in mathematics. This tells us that, overall, there does not appear to be any improvement in students' achievement and confidence as a result of following a greater number of strands, or as a result of schools having greater experience of teaching the revised syllabuses. However, delving deeper suggests some apparent differences between phase one and non-phase one students within individual strands of the revised syllabuses. Further details are provided in section 5.2.

The timing of the testing periods is a significant predictor of students' mathematics achievement[29], telling us that students' performance in mathematics continues to develop throughout their studies. The exception to this case is the similar performance of Leaving Certificate students in the classes of 2012 and 2013 in relation to Strand 1 (Statistics and Probability). It is likely that this is because students in the class of 2013 are able to draw on previously taught knowledge and practised skills in relation to this strand. By contrast, students are less likely to be able to use previous knowledge to answer questions in relation to Strand 2 (Geometry and Trigonometry) as success in this strand requires knowledge of specific theorems, which students may not have covered at the time of data collection.

Interestingly, confidence in mathematics is not affected by the survey date. This suggests that confidence remains consistent throughout the academic year, or is acquired early in students' studies and remains at this level, as well as that confidence and achievement in mathematics are not always closely associated.

Perhaps unsurprisingly, the findings also show that Foundation and Ordinary Level students have lower achievement and lower confidence in mathematics than their Higher Level peers[30]. It is interesting to note that there are larger differences between Foundation, Ordinary and Higher Level students in relation to Strand 1 (Statistics and Probability), than Strand 2 (Geometry and Trigonometry) at both Junior and Leaving Certificate. This indicates that students of all examination entry levels find Strand 2 difficult, whereas difficulties are more acutely experienced by Foundation and Ordinary Level students in relation to Strand 1.

In general, girls appear to have less confidence and have lower achievement than boys in the revised mathematics syllabuses. Additionally, students attending vocational schools appear to have lower levels of confidence and achievement than those in secondary or in community and comprehensive schools (which may have a different intake). This may point to differences between the learning preferences of different groups of students and resulting impact on their responses to the revised mathematics syllabuses, rather than being related to the characteristics of the new syllabuses.

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[29] Students who participated in the research in Autumn 2012 were at the beginning of the academic year, whereas those who participated in Spring 2012 were approaching the end of their studies. Therefore, students in the Autumn 2012 cohort had experienced less teaching time than the Spring 2012 cohort.

[30] Examination level was identified by students when completing the survey and assessment materials.

5.2 Comparison of the performance of phase one and non-phase one students in the classes of 2012 and 2013

This section considers in more depth student achievement in each individual strand of the revised syllabuses, and makes comparison between the performance of students in the classes of 2012 and 2013. To recap, an overview of the strands studied by each group of students is outlined in Table 5.1.

Table 5.1: Strands studied by students participating in the assessment

	Year group	Years of study	Strands studied by phase one students	Strands studied by non-phase one students
Junior Certificate	**Students who completed the survey and tests in Spring 2012**	**2009-12**	Strands 1-4	No strands
	Students who completed the survey and tests in Autumn 2012	**2010-13**	Strands 1-5	Strands 1-2
Leaving Certificate	**Students who completed the survey and tests in Spring 2012**	**2010-12**	Strands 1-5	Strands 1-2
	Students who completed the survey and tests in Autumn 2012	**2011-13**	Strands 1-5	Strands 1-4

In interpreting the results it is important to note that, overall, phase (phase one or non-phase one) is not associated with improvements in students' achievement. Therefore, this section takes a closer look at the patterns emerging across strands for illustrative purposes. Also, as set out in section 5.1, timing differences between the two data collection exercises is an important consideration. Discussion of the findings is therefore based on the expectation that the class of 2013 would perform **less well** than the class of 2012 as a result of receiving fewer months of teaching at the time of participating in the research.

5.2.1 Overview of students' performance across strands

Figure 5.1 provides an illustrative overview of both Junior and Leaving Certificate students' performance across each of the five strands of the revised mathematics syllabuses. This is true of students in the classes of both 2012 and 2013.

Figure 5.1: An illustrative overview of students' achievement in mathematics (not to scale)

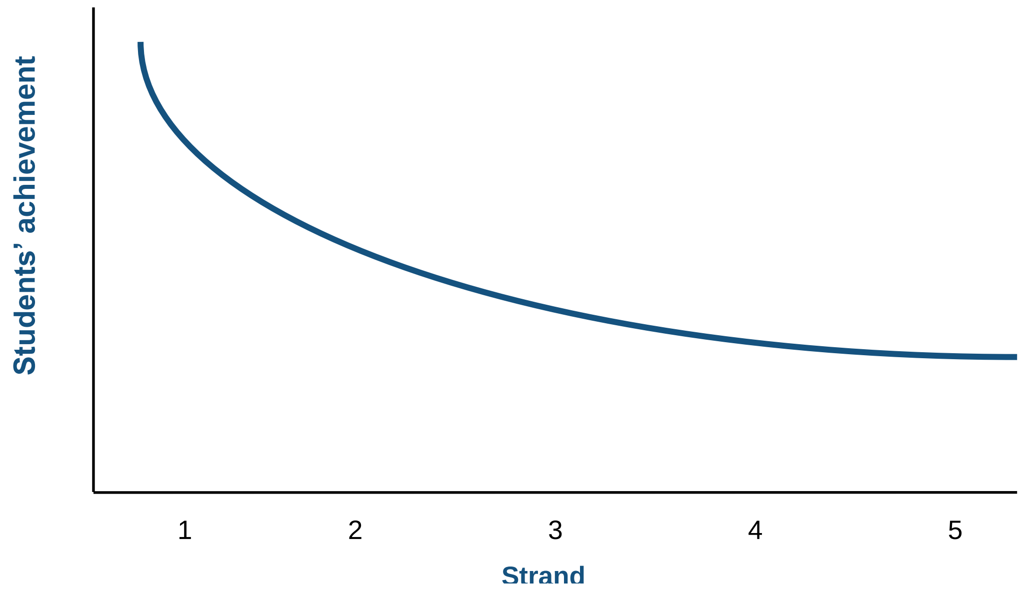

Figure 5.1 shows that, overall, both Junior and Leaving Certificate students have highest achievement in Strand 1 (Statistics and Probability). Students also perform well in items relating to Strand 2 (Geometry and Trigonometry) and Strand 3 (Number). Students perform least well in items relating to Strand 4 (Algebra) and Strand 5 (Functions). Broadly, this corresponds with students' confidence in each strand (see section 5.3). The exception is Strand 5 (Functions), where students generally report high levels of confidence. For greater detail of students' performance in each of the individual strands, including by examination entry level where appropriate, see Appendix D.

5.2.2 Comparison of phase one and non-phase one Junior Certificate students' performance

In all strands of the revised mathematics syllabuses, the performance of phase one students in the classes of 2012 and 2013 are broadly comparable. By contrast, non-phase one students in the class of 2013 do less well than the class of 2012. As set out above, we would expect the performance of the class of 2013 to be lower than the class of 2012 in both phase one and non-phase one schools as they had received less schooling at the time of participating in the research. As the performance of phase one students has not followed this trend, this suggests that, in some strands, phase one students are performing better than their non-phase one peers.

Comparison of Junior Certificate students' performance in the classes of 2012 and 2013 in a booklet covering Strands 1, 2 and 5[31]

This booklet measures students' performance in three strands of the revised syllabuses: Strand 1 (Statistics and Probability); Strand 2 (Geometry and Trigonometry) and Strand 5 (Functions). The booklet was completed by both phase one and non-phase one students in the classes of 2012 and 2013.

Phase one: Between the classes of 2012 and 2013, there was an increase in student achievement by over five per cent in five questions and a decrease in student achievement in only three questions. The proportion of questions omitted has remained broadly constant and at a lower level than among non-phase one students.

Non-phase one: Between the classes of 2012 and 2013, there has been a decrease in student achievement by over five per cent in 14 questions. The effect is most noticeable on the questions assessing Strand 2 (Geometry and Trigonometry).[32] The proportion of questions omitted for three of the items has increased by 6-10 per cent.

Comparison of the performance of Junior Certificate students in the classes of 2012 and 2013 in a booklet covering Strands 3, 4 and 5[33]

This booklet measures students' performance in three strands of the revised syllabuses: Strand 3 (Number); Strand 4 (Algebra) and Strand 5 (Functions). The booklet was completed by only phase one students in Spring and Autumn 2012.

Amongst phase one schools, there is a fairly consistent level of achievement with students' achievement in one question decreasing by 6-10 percentage points and increasing by more than five percentage points in three questions between the classes of 2012 and 2013. Two questions, which were omitted by the highest proportion of students in the class of 2012, have been completed by over ten per cent more students in the class of 2013.

[31] Booklet reference JC1/2/5. Please note that students were only assessed on Strand 5 (Functions) in Autumn 2012. Further detail of students' performance in all strands is provided in Appendix D.
[32] The facilities of all but one of the Strand 2 (Geometry and Trigonometry) items have decreased by more than five percentage points.
[33] Booklet reference JC3/4/5. Please note that students were only assessed on Strand 5 (Functions) in Autumn 2012. Further detail of students' performance in all strands is provided in Appendix D.

The key findings show that in Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry), the performance of phase one students in the classes of 2012 and 2013 are broadly comparable. By contrast, the performance of non-phase one students has fallen between the two year groups. As outlined above, we would expect students in the class of 2013 to perform less well than those in the class of 2012 as a result of receiving less schooling at the time of the research. This indicates that in relation to these strands, phase one students are performing better than would be expected.

The strong performance of students in phase one schools compared to their non-phase one peers suggests that they may have developed a wide range of mathematical skills as a result of studying a greater number of strands (for example, problem solving and an ability to make connections between mathematics topics), making them better able to apply their knowledge to unfamiliar topics. Given the timing differences in the assessment periods, the strong performance of students in the phase one class of 2013 relative to the class of 2012 (who studied fewer strands of the revised syllabuses) further supports the argument that students' ability to draw on their wider mathematical knowledge and apply this to new contexts appears to increase with the number of strands studied.

The contrast between the performance of phase one and non-phase one students is particularly notable in Strand 2 (Geometry and Trigonometry). While the performance of phase one students is broadly comparable between the classes of 2012 and 2013, in non-phase one schools the class of 2013 students do less well than the class of 2012. It is likely that there are topic areas within this strand (for example, knowledge of mathematical theorems) which the class of 2013 had not yet been taught at the time of the assessment. Without such knowledge, the students would have found these items difficult to complete. The performance of phase one students in this instance again suggests that they are better able, and more confident, to draw on their mathematical skills in other areas.

The performance of phase one and non-phase one students cannot be compared in relation to Strand 3 (Number), Strand 4 (Algebra) and Strand 5 (Functions) as only phase one students were assessed in these strands. However, again the performance of phase one students is comparable between the classes of 2012 and 2013.

Comparison of Junior Certificate students' performance with international standards

Students in the classes of 2012 and 2013 perform similarly to international students who participated in TIMSS 2007. Phase one students have shown a strong performance on items assessing Strand 1 (Statistics and Probability). However, phase one students appear to find Strand 4 (Algebra) especially difficult when compared to international standards. Overall, their knowledge of subject areas relating to Strand 2 (Geometry and Trigonometry) and Strand 3 (Number) appear to be similar to that internationally.

5.2.3 Comparison of phase one and non-phase one Leaving Certificate students' performance

This section explores the performance of the phase one and non-phase Leaving Certificate students in the class of 2013, and makes comparison to the performance of the class of 2012. Again, key findings are presented in the box below and further discussed in the interpretation which follows.

Comparison of the performance of Leaving Certificate students in the classes of 2012 and 2013 in a booklet covering Strands 1, 2 and 5

This booklet measures students' performance in three strands of the revised syllabuses: Strand 1 (Statistics and Probability); Strand 2 (Geometry and Trigonometry) and Strand 5 (Functions). The booklet was completed by phase one and non-phase one students in the classes of 2012 and 2013.

Phase one: Generally, students in the classes of 2012 and 2013 perform similarly. In Strand 5 (Functions), however, student achievement has decreased between the classes of 2012 and 2013 by more than ten per cent in three questions, but has increased in none of the questions. There is also a clear difference between the two year groups in the number of students omitting questions, with all items being missed out by a greater percentage of students (generally more than 10 per cent) in the class of 2013.

Non-phase one: The performance of non-phase one students in the classes of 2012 and 2013 are comparable. However, student achievement has decreased by more than five per cent in eight of the 13 questions assessing Strand 2 (Geometry and Trigonometry). Ten of the 13 items assessing Strand 2 were omitted by over five per cent more students in the class of 2013 than the class of 2012.

The key findings show that students studying at phase one schools have shown very little change in performance across the two year groups for Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry). Additional analysis reveals that this is also the case for Strand 3 (Number) and Strand 4 (Algebra)[34].

It is clear, however, that the phase one Leaving Certificate students in the class of 2013 are finding Strand 5 (Functions) more difficult than the class of 2012, and appear to have less confidence in attempting an answer. It is possible that as a more difficult topic within the syllabus, this strand is left until students' final year of study before it is taught, making the timing difference between the two testing periods more noticeable than for the other strands.

Non-phase one students in the classes of 2012 and 2013 have, like their phase one peers, performed similarly in Strand 1 (Statistics and Probability). This supports the findings of the multi-level modelling analysis, which shows that, at Leaving Certificate, the timing of the assessment is not a significant predictor of achievement in Strand 1 (Statistics and Probability). As mentioned in section 5.1, this suggests that students in the class of 2013 are able to draw on previously taught knowledge and practised skills in relation to this strand.

Students' performance in Strand 2 (Geometry and Trigonometry) reveals a different pattern. Here, phase one students outperform their non-phase one peers. Again, this is indicative that phase one students are better able to draw on their wider mathematical skills and knowledge and suggests that students' abilities within an individual strand may increase with the number of strands studied overall.

Non-phase one students did not complete items assessing Strand 3 (Number), Strand 4 (Algebra) or Strand 5 (Functions) so no further comparisons can be made between the year groups for this group of students.

[34] Comparisons across cohorts cannot be made for Strand 5 (Functions) as students were only assessed in these items in Autumn 2012. For further detail of students' performance in all strands, see Appendix D.

Comparison of Leaving Certificate students' performance with international standards

Phase one students in both the classes of 2012 and 2013 performed much better than the average score of international students on many of the items relating to Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry).

It is important to note that this may be expected, as Leaving Certificate students are older than those who participated in the international studies (the only ones available which explore students' achievement in, and attitudes towards, mathematics). However, the international studies are designed for a wide range of students, with some designed to challenge more able students and others to be more widely accessible. This makes them relevant for this research.

5.2.4 Types of questions that students find more difficult

Across all strands of the revised syllabuses there is variation in the achievement of students. Students appear to find particular types of questions more difficult than others. Analysis of the students' achievement patterns identifies three broad question types that appear most challenging for students, as set out below. These patterns are the same for both Junior and Leaving Certificate students in phase one and non-phase one schools.

Reading and interpreting large amounts of information

While students often perform well in this type of question, they appear to lack confidence when asked to draw conclusions from a considerable amount of written information. This is particularly notable for Leaving Certificate students in Strand 1 (Statistics and Probability), given their high performance in other questions types relating to this topic. Several factors may make this type of question more difficult, including that students often have to:

- interpret and classify a large number of statements
- understand the 'big picture' as well as individual data
- appreciate that some information is not relevant or is inconclusive in order to gain full credit.

Showing working and justifying answers

Students appear to find open-ended questions requiring them to show their working, and use reasoning to justify their answers, particularly difficult. Often, this is despite having arrived at the correct answer. This reflects findings based on the analysis of a small sample of students' work set out in section 4, which suggests that students are not being given these types of tasks in a classroom setting. In Strand 2 (Geometry and Trigonometry), for example, Junior Certificate students appear to have used mental or estimation methods rather than written justification to arrive at their answers. Therefore, they were often unable to receive full marks for this type of question.

Likewise, in both Strand 3 (Number) and Strand 4 (Algebra), Junior and Leaving Certificate students appear to find broader, open ended questions more challenging than multiple choice questions or those demanding a specific calculation to be performed. The challenge posed by this type of question is particularly pronounced when students are asked questions requiring them to apply their knowledge in unfamiliar ways. For example, in Stand 4 (Algebra) Leaving Certificate students were asked to form a quadratic function from its graph. The graph shows the points at which the function cuts both axes. In general, students struggled with this question. While all students following the revised syllabus study quadratic equations, it may be that they are more familiar with solving equations to find the roots, rather than working backwards as this item requires.

Multi-step questions

Students appear to find multi-step items more complex than others. For example, in Strand 2 (Geometry and Trigonometry), Leaving Certificate students were asked to calculate the width of a flat window in a semi-circular room. To answer correctly, students must recognise the need to bisect a sector of the circle to form two right-angled triangles. They must then apply the ratio for the sine of an angle to calculate the relevant length. Therefore, this item requires thorough knowledge of trigonometric ratios and the geometric properties of triangles. It also benefits from the ability to construct an accurate diagram. In general, students struggled with this type of question.

5.3 Comparison of phase one and non-phase one student attitudes between the classes of 2012 and 2013

This section considers in more depth students' attitudes towards the individual strands of the revised syllabuses. Results are taken from the surveys and expanded on using detail from the case studies. Again, comparison between phase one and non-phase one students is useful to draw out distinctions within individual strands. Overall, however, it is important to note that phase of study is not associated with improvements in students' attitudes.

5.3.1 Overview of students' attitudes across strands

In order to gauge students' confidence in mathematics, they were asked how they would feel about approaching a range of different mathematical problems which may arise during their lessons[35] (see Appendix C, Tables 30-37). Overall, students appear generally confident in many of the areas explored, spanning all five strands of the revised syllabuses. Figure 5.2 provides an illustrative overview of both Junior and Leaving Certificate students' confidence across each of the five strands of the revised mathematics syllabuses.

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[35] Please note that students were asked about their confidence in relation to particular aspects of each syllabus strand rather than a strand as a whole.

Figure 5.2: An illustrative overview of students' confidence in mathematics (not to scale)

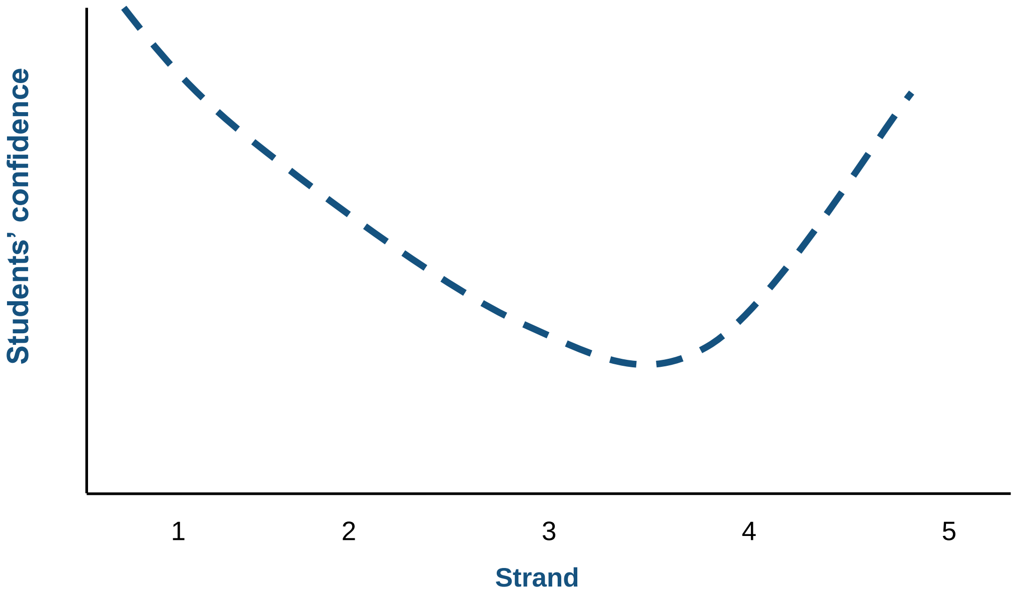


Figure 5.2 shows that Junior and Leaving Certificate students in both phase one and non-phase schools tend to feel most confident about Strand 1 (Statistics and Probability) and Strand 5 (Functions), and least confident in relation to Strand 3 (Number). Again, this corresponds with patterns of students' achievement in each of these strands (see section 5.2), with the exception of Strand 5 (Functions), where students generally demonstrate considerably lower levels of achievement than some other strands. Some key differences in students' attitudes within individual strands are detailed below.

5.3.2 Strand 1 (Statistics and Probability)

There is a statistically significant difference between the response of phase one and non-phase one students in the class of 2013, with phase one students appearing to be more confident in items relating to this strand than their non-phase one peers. Further comparison reveals that only amongst non-phase one Junior Certificate students is there a notable difference between year groups (with 85 per cent of the class of 2013 reporting they would find this 'very easy' or 'easy', compared to 72 per cent of the class of 2012).

Strand 1: Findings from the case studies

Students in both phase one and non-phase one schools are, in the main, confident in Strand 1 (Statistics and Probability) and find it enjoyable and interesting. Students tend to find it the most straightforward of all of the five strands, sometimes commenting that they are building on concepts learned at primary school or, in phase one schools, at Junior Certificate.

While Leaving Certificate students in phase one schools feel they are building on knowledge from Junior Certificate, those from non-phase one schools have experienced some difficulties with the unfamiliarity of the content and associated terminology.

In phase one schools, views of Strand 1 (Statistics and Probability) are similar between Junior Certificate and Leaving Certificate. However, in non-phase one schools, Leaving Certificate students are less positive than Junior Certificate students.

Some teachers feel this may be because older students are less comfortable with the learning approaches promoted, having studied a previous version of the syllabus at Junior Certificate.

5.3.3 Strand 2 (Geometry and Trigonometry)

Just over half of all students report relatively high levels of confidence in relation to Strand 2 (Geometry and Trigonometry). Students appear particularly confident in their ability to make different shapes. No significant differences in response emerge by phase or level of study.

Strand 2: Findings from the case studies

Generally, students from all strands and phases are positive, if slightly cautious, about Strand 2 (Geometry and Trigonometry). Students usually enjoy the introduction of real-life contexts into their learning (for example, how a professional rugby player chooses the angle at which to kick a rugby ball). However, Junior Certificate students at some non-phase one schools feel that these make questions difficult to access.

Leaving Certificate students at half of the non-phase one schools describe this strand as very difficult. Aside from difficulties with learning theorems and formulae, these students find it difficult to understand and develop a strategy to solve problems when they have to construct their own diagrams or visualise a situation. In contrast, this is mentioned as an issue by students at only one phase one school. This suggests that, in general, students' response to this type of learning improves with increased familiarity with classroom engagement in problem-solving techniques.

Junior Certificate students are generally more positive about this strand than Leaving Certificate students, particularly enjoying visual activities such as drawing and constructing shapes. Despite this, teachers generally feel that Leaving Certificate students would be well prepared for examination in these topics.

5.3.4 Strand 3 (Number)

Students appear generally positive about this strand, with around a half of all students reporting that they would find it 'easy' or 'very easy' to solve mathematical problems in this strand. Although a statistically significant difference emerges between the responses of phase one and non-phase one students, the percentages are very close and do not seem to support any meaningful interpretation of difference.

Strand 3: Findings from the case studies

Students' views about Strand 3 (Number) are quite mixed. They do not perceive it to be the most difficult strand, nor the most interesting. Students feel it is more similar to the previous version of the curriculum, and therefore a more familiar approach to mathematics, than other strands. Whilst some aspects of this strand appear to be challenging for students, this overall familiarity makes it seem easier which gives them more confidence in their abilities.

Minimal differences are evident between views of students in phase one and non-phase one schools. However, in some non-phase one schools, Junior Certificate students feel that this strand does not provide them with same number of engaging contexts and activities as other strands. There is little difference between the response of Junior Certificate and Leaving Certificate students to this strand. Notably, however, Leaving Certificate students perceive that it is more similar to their previous experience of mathematics compared to other strands. They are also aware of how they are applying skills and concepts learned in this strand to other areas of mathematics.

5.3.5 Strand 4 (Algebra)

Students responding to the survey report broadly positive levels of confidence for solving problems using algebra, with around two-thirds of all students finding it ‘easy’ or ‘very easy’. Leaving Certificate students are more confident than their Junior Certificate peers. No statistically significant difference emerged between the responses of phase one and non-phase one students and there are no notable differences between the responses of students in the classes of 2012 and 2013.

Strand 4: Findings from the case studies

In contrast to the survey findings, attitudes to Strand 4 (Algebra) amongst case-study schools are very mixed and, in general, there are no clear distinctions in students’ views by either phase or age group.

Students feel more motivated to learn algebra when it is taught in a way that makes it seem more relevant to everyday life, for instance using algebra to describe the growth pattern of sunflowers. Students’ motivation and interest is also higher when they can see that it interlinks with other mathematics topics (the corollary of this is that if they find algebra difficult, students feel concerned that they will struggle with other mathematics that builds on its foundations).

In some schools, algebra is described as ‘being taught the old way’ via textbooks and exercises on the board, which does not necessarily help to engage students’ interest. Teachers, particularly in non-phase one schools, comment that they need more time to develop more integrated practical approaches into their teaching to this strand, and would welcome further examples of real-life contexts to use in lessons.

5.3.6 Strand 5 (Functions)

Students report high levels of confidence in their ability to represent relationships graphically. Phase one students are significantly more confident than their non-phase one peers in this ability, however, there is an increase in the performance of non-phase one between the classes of 2012 and 2013 more so than their phase one peers. Leaving Certificate students are significantly more likely than Junior Certificate to state that they find items relating to this strand ‘a little difficult’ or ‘difficult’.

Strand 5: Findings from the case studies

Most teachers in both phase one and non-phase one schools, and at both Junior Certificate and Leaving Certificate, feel that Strand 5 (Functions) is fairly 'uncontroversial', straightforward to teach and does not pose as many challenges to students as other strands. Of note, teachers report that students cope reasonably well with the topic. However, the assessment conducted as part of this research suggests that students do not always achieve highly in this strand.

Students at non-phase one schools are, in general, quite accepting of this strand and do not have strong feelings about it either way: they often report that it is 'easy enough' and are particularly confident about drawing graphs. In comparison, students at phase one schools are more divided in their opinions, describing it variously as: easy; difficult; confusing; interesting; and boring, and students' attitudes seem to be down to individual preference. Given that students expressed considerably stronger views about Strand 4 (Algebra), their relative indifference to this strand suggests that they may not fully recognise the connections between algebra and functions topics.

Leaving Certificate students (in phase one schools) who like the topic tend to enjoy the visual and practical aspects of drawing and interpreting graphs, and feel these are useful skills to learn. They find learning the concepts to be initially challenging but that, with practice, they can consolidate their understanding.

6. Students' aspirations for further study and careers involving mathematics

Key findings

Students' perceptions of the wider relevance and application of mathematics

- Students tend to recognise the broader application of mathematics, particularly in helping them to secure a place at the university of their choice and in their daily life. Reflecting the broader difference in the attitudes of Junior and Leaving Certificate students, Junior Certificate students are generally more positive about the broader application of their mathematics study than students studying for the Leaving Certificate.

Students' aspirations to further study of mathematics

- Almost all Leaving Certificate students plan to go on to further study when they finish their Leaving Certificate, and around half of all students intend to pursue further study involving mathematics.
- Almost all Junior Certificate students plan to stay on at school after their Junior Certificate, and the majority plan to take the Higher Level Leaving Certificate examination. The aspirations of students for Higher Level examination in phase one schools are higher than students from non-phase one schools. This may be a result of the revised syllabuses beginning to embed in phase one schools, and therefore instilling a greater enjoyment of, and confidence in mathematics amongst their students.

Students' appreciation of careers involving mathematics

- Around two-thirds of Leaving Certificate students stated that they do not intend to go into a job that involves mathematics.
- It appears that students are developing a general awareness of the importance of mathematics in further study and of its broader application, but in some cases, the specifics of this, such as a sound understanding of what careers will draw on their mathematical skills and knowledge, appears to be lacking.

This section sets out the impact of the revised syllabuses on students' aspirations for further study and careers involving mathematics. It begins by highlighting perceptions of the wider relevance and application of mathematics, before exploring students' plans for further study or careers. It draws on findings from the attitude survey conducted with the class of 2013 and case studies, and makes international comparison with TIMSS 2007 results as well as comparison with the survey completed by the class of 2012, where appropriate.

6.1 Students' perceptions of the wider relevance and application of mathematics

Students' views on the broader application of mathematics are firstly considered, followed by their understanding of jobs and career pathways involving mathematics.

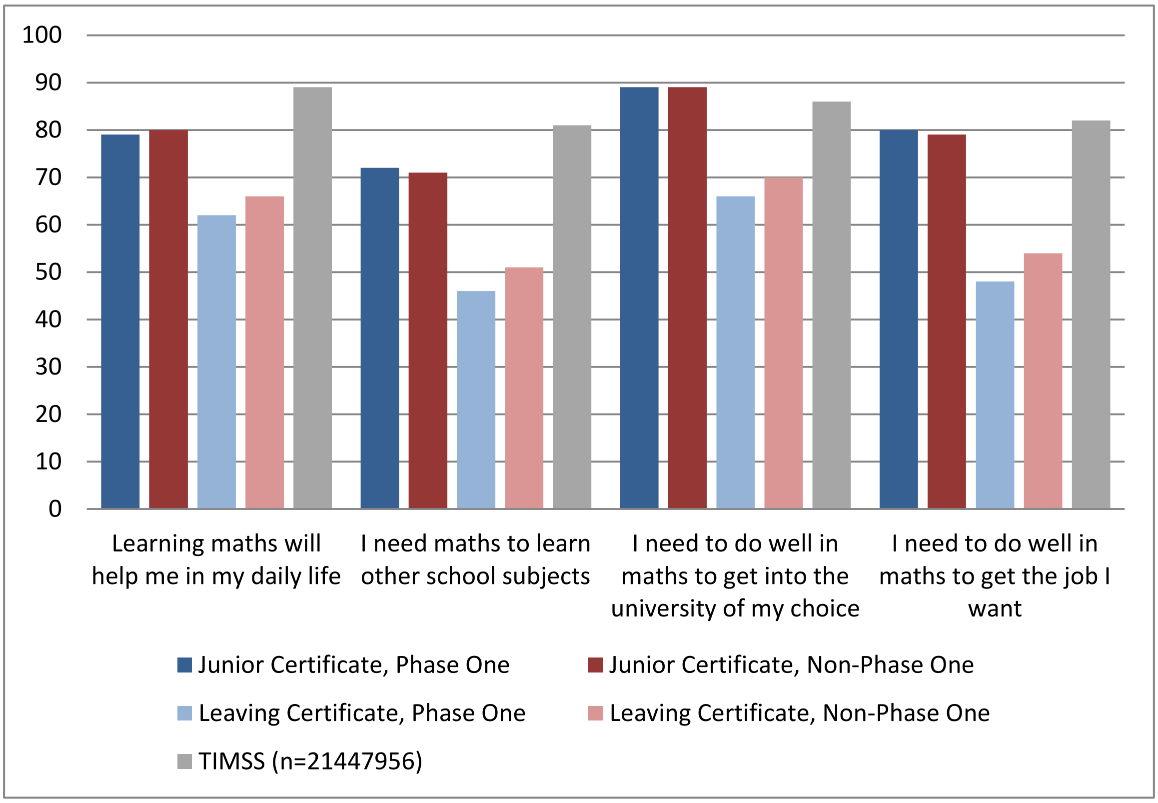
6.1.1 Students' views on the broader application of mathematics

To ascertain students' views on the broader application of mathematics, they were asked to comment on the extent to which they perceive it to be useful in the following ways:

- to help in daily life
- to aid learning in other school subjects
- to enable them to get into the university of their choice
- to enable them to get the job of their choice.

Students' responses to these questions are set out in Figure 6.1 (see Appendix C, Tables 41-44 for more detail). To provide international comparison, Figure 6.1 also contains the responses of students from TIMSS 2007 (aged, on average, 14 years old) who were asked the same question, to provide international comparison and context. Again, please note, that the TIMSS data is an international, rather than an Irish, average so direct comparisons cannot be made.

Figure 6.1: Percentage of students who agree 'a lot' or 'a little' with statements on the wider application of mathematics



Source: NFER student survey, Autumn 2012

As detailed in Figure 6.1, students tend to recognise the broader application of mathematics, particularly in helping them to secure a place at the university of their choice and in their daily life, but there are generally lower levels of agreement that they need mathematics to learn other school subjects. Of particular note, students studying for the Junior Certificate are significantly more likely than their Leaving Certificate counterparts to respond favourably to each of the statements, suggesting that they are more positive about the broader application of their mathematics study. This may reflect, or be a contributing factor, in their generally more positive attitudes towards mathematics. No significant differences were observed between the responses of students from phase one and non-phase one schools and further analysis reveals no notable differences between year groups.

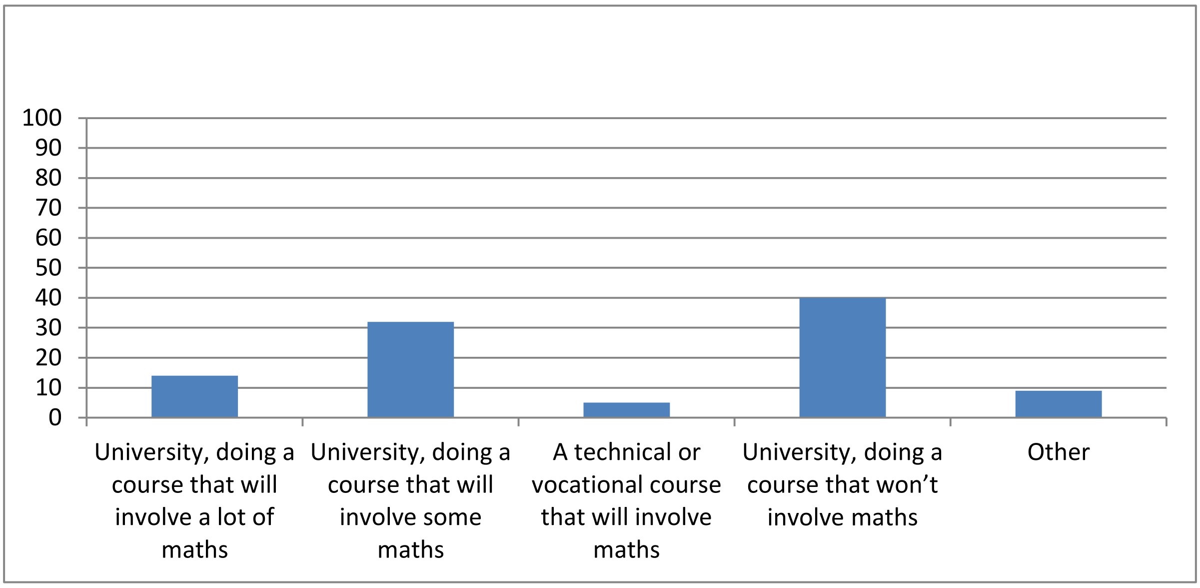
Figure 6.1 also shows that students in the TIMSS study responded more favourably to all of the statements, implying that they consider mathematics to be more relevant to their current and future study/careers, as well as to life in general. The views of Junior Certificate students are, however, often closer to the TIMSS data than their Leaving Certificate peers. As the Junior Certificate students are closer in age to the TIMSS average, it is possible that this may account for any difference.

6.2 Students' aspirations for further study of mathematics

6.2.1 Leaving Certificate students

Ninety-four per cent of Leaving Certificate students plan to go on to further study when they finish their Leaving Certificate. There are no significant differences in responses of students from phase one and non-phase one schools. Figure 6.2 sets out these students' plans. The views of students from phase one and non-phase one schools are not significantly different, and are therefore presented together in Figure 6.2 (see Appendix C, Table 45).

Figure 6.2: Percentage of Leaving Certificate students planning to go on to different types of further study



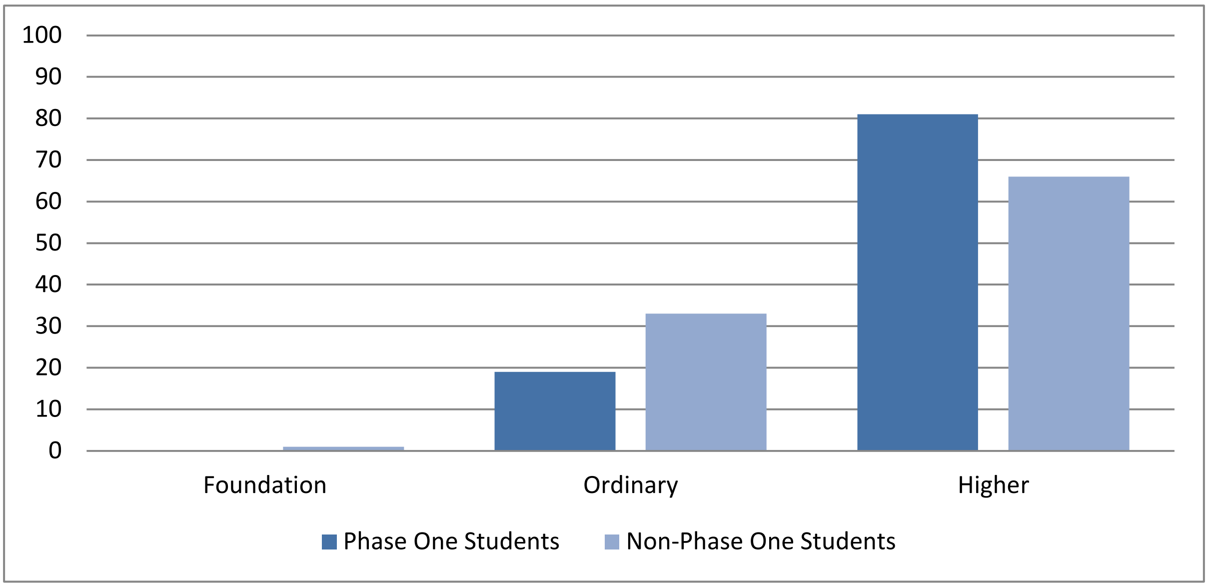
Source: NFER student survey, Autumn 2012

Figure 6.2 shows that around half (51 per cent) of all Leaving Certificate students in the class of 2013 who plan to go on to further study intend to go on to use mathematics to some extent. Thirty-two per cent intend to study a university course with 'some' mathematics involved, 14 per cent plan to study a course that will involve 'a lot' of mathematics and five per cent intend to take on a technical or vocational course that will involve mathematics (around the same as the class of 2012). This suggests that around half of students intend to study mathematics at a higher level, and appreciate its prominence in the further study that they intend to pursue.

6.2.2 Junior Certificate students

Almost all (98 per cent) of Junior Certificate students plan to stay on at school after their Junior Certificate. Figure 6.3 sets out these students' plans (see Appendix C, Tables 46-47).

Figure 6.3: Percentage of Junior Certificate students intending to take their Leaving Certificate at Foundation, Ordinary or Higher Level



Source: NFER student survey, Autumn 2012

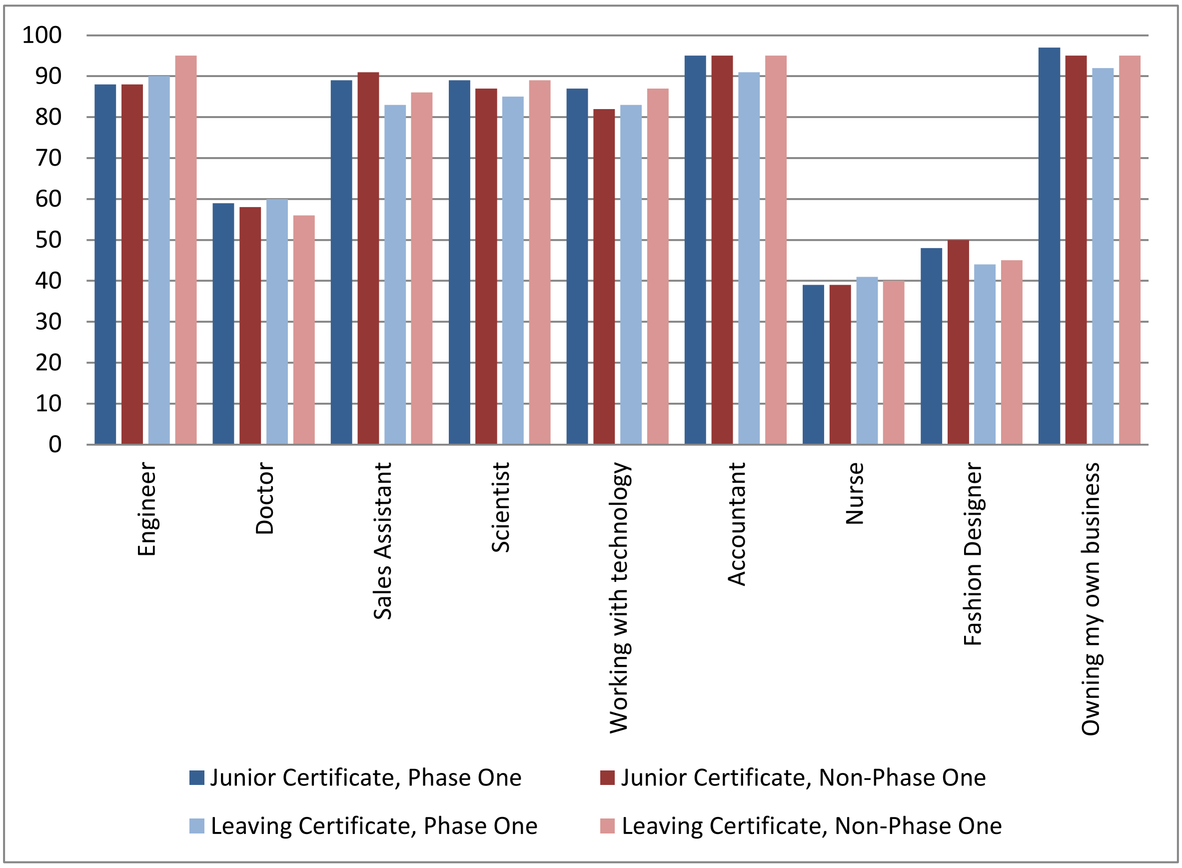
As set out in Figure 6.3, the majority of students from both phase one and non-phase one schools in the class of 2013 (81 and 66 per cent, respectively) who intend to stay on at school after their Junior Certificate plan to take the Higher Level Leaving Certificate examination (compared to 62 per cent of phase one students and 57 per cent of non-phase one students in the class of 2012). Thirty-three per cent of students from non-phase one schools plan to take their examination at Ordinary Level, compared to 19 per cent of students from phase one schools. The differences in responses between phase one and non-phase one students in the class of 2013 are statistically significant, suggesting that the aspirations of students for their Higher Level examination in phase one schools are higher than students from non-phase one schools. This may be a result of the revised syllabuses beginning to embed in phase one schools, and therefore instilling a greater enjoyment of, and confidence in mathematics amongst their students.

6.3 Students' appreciation of careers involving mathematics

6.3.1 Students' understanding of jobs and career pathways involving mathematics

To explore students' understanding of jobs and career pathways involving mathematics, they were provided with a list of ten different professions and asked to select which of these involved using mathematics. All of the jobs involved using mathematics to some extent. Students' responses are set out in Figure 6.4 (see Appendix C, Tables 48-56).

Figure 6.4: Percentage of students who think that mathematics is involved in different jobs/careers



Source: NFER student survey, Autumn 2012

There is a generally high level of awareness of the application of mathematics in jobs such as engineering, being a sales assistant, a scientist, working with technology, accountancy and owning your own business, and less awareness of the role of mathematics in being a doctor, nurse and fashion designer. Junior Certificate students are significantly more likely than Leaving Certificate students to think that being a sales assistant and being a fashion designer involve mathematics, and significantly less likely to recognise the role that mathematics plays in working with technology. Only one significant difference emerged between students in phase one and non-phase one schools: significantly more students from phase one schools recognise the role of mathematics in owning a business than those from non-phase one schools. There were no notable differences in the responses of students between the two year groups.

These results suggest that students appear to be generally aware of the role of mathematics in the majority of the jobs/careers listed. However, there is still a lack of awareness of the mathematical application of some jobs/careers. Perhaps further work is needed to demonstrate the application of mathematics across a broader spectrum of career choices. Section 6.3.2, below, explores this issue further.

6.3.2 Students' aspirations to pursue careers involving mathematics

Around two-thirds of Leaving Certificate students stated that they do not intend to go into a job that involve mathematics (see Appendix C, Table 57). This is an interesting finding given that around one-half of these students are intending to go on to further study that involves some mathematical elements. It is possible that students are aware of the broader application of mathematics and its importance in further study, but do not necessarily recognise its role in jobs or careers.

Those who do intend to go on to a job with a mathematical component were asked to specify what roles they were considering. The following were most frequently cited:

- engineering
- ICT/computer science/software development
- teaching
- business/management type roles
- finance/accountancy
- science-related
- medicine/health sciences.

Data collected from the case-study interviews suggests that some students lack an awareness of the role of mathematics in their chosen career or study pathway. For example, although they might state that mathematics will not feature in their plans, they go on to explain that they will be studying subjects such as chemistry, which has a substantial mathematical component.

The case-study interviews also asked students whether they felt they had a broad understanding of career routes that involve mathematics. Students' responses varied by school, suggesting that their teachers' focus on highlighting the relevance of mathematics to different career routes was central to their levels of awareness. Students do not always equate their awareness of different careers involving mathematics with their mathematics lessons. Rather, they default to considering their careers advice or careers lessons. In some cases, students talked about how their awareness of the importance of mathematics to careers or jobs generally had developed, but appeared to lack specific knowledge of which jobs contained mathematical elements. In sum, it appears that students are developing a general awareness of the importance of mathematics in further study and of its broader application, but in some cases, the specifics of this, such as a sound understanding of what careers will draw on their mathematical skills and knowledge, appear to be lacking.

7. Concluding comments

The methodology for this research has enabled patterns to be explored in: students' experiences of, and attitudes towards, mathematics (the attitude survey and case studies); their achievement (the assessment of students' performance); and the mathematics processes evident in students' work (analysis of students' written outputs from lessons). The research highlights that considerable progress has been made in implementing the revised mathematics syllabuses since the inception of the Project Maths initiative in 2008.

Although there is evidence that more traditional approaches to teaching mathematics remain widespread, across both phase one and non-phase one schools, there are numerous examples of promising practice in transforming the way that mathematics is delivered in the classroom. However, it appears that, in many cases, the approaches described by teachers and students are not yet being evidenced in students' written work. Whilst this is perhaps to be expected given the early stage of the curriculum's implementation, it may be useful to further support teachers to provide opportunities for students to engage more widely with the written processes promoted through the mathematics syllabuses.

At this stage of the curriculum's implementation, the revised mathematics syllabuses *taken as a whole* do not appear to be associated with any overall deterioration or improvements in students' achievement. Performance across each of the strands of the revised syllabuses varies. Overall, students perform most highly in Strand 1 (Statistics and Probability) and least well in Strand 4 (Algebra) and Strand 5 (Functions). It may, therefore, be beneficial to consider ways in which the delivery of these strands can be enhanced to improve outcomes for students. This could, for example, include teaching a greater proportion of the content of these strands at the beginning of the course to allow students more time to consolidate their learning, or to integrate the strands more closely.

There is emerging evidence of positive impacts on students' experiences of, and attitudes towards, mathematics. Furthermore, emerging impacts on students' achievement at individual strand level are apparent, and in some instances students appear to be successfully drawing together their knowledge across different mathematics topics. This suggests that students are beginning to acquire a deeper understanding of mathematics and how it can be applied.

It is interesting to note that within this study, students' confidence and achievement do not always appear to be linked. This issue is of particular importance for girls: whilst overall they appear to have lower confidence than boys, this not always associated with lower achievement. Conversely, in relation to Strand 5 (Functions), students report high levels of confidence despite having lower achievement than in other strands. It may be valuable to explore ways in which girls' confidence in mathematics can be increased, as well as to capitalise on students' apparent enthusiasm for particular topics more generally, with a view to improving achievement.

Generally, Junior Certificate students hold more positive views about mathematics than their Leaving Certificate peers. In part, this may be because they have experienced greater continuity of learning styles in their transition from primary school, compared to Leaving Certificate students who have experienced a more pronounced change to the previous version of the curriculum followed at Junior Certificate. This implies that continuity is central to the successful implementation of the revised syllabuses, and is a positive indication of students' direction of travel in mathematics. The difference in attitudes may also be a result of the added pressure experienced by Leaving Certificate students, whose performance in mathematics can impact on their assessment in all of their Leaving Certificate subjects, and ultimately impact on progression to further study and careers.

International comparison shows that students following the revised syllabuses are slightly less positive about mathematics, but more confident in their mathematics ability. Also, they do not appear to recognise the relevance of mathematics to their future study and careers to the same degrees as students in the international sample. For example, the majority of Leaving Certificate students state that they do not intend to pursue careers in mathematics, but seem not to recognise the prevalence of mathematics in many of their chosen careers. Therefore, it may beneficial to focus resources on widening students' awareness of the broader application of mathematics and its value in their academic and future careers.

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Teaching and Learning in Project Maths:

Insights from Teachers who Participated in PISA 2012

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http://www.erc.ie

Cataloguing-in-publication data

Cosgrove, Jude

Mathematics teaching and learning in Project Maths: Insights from teachers who participated in PISA 2012 /

Jude Cosgrove… [et al.].

Dublin: Educational Research Centre.

viii, 79p., 30cm

ISBN: 978 0 900440 38 0

1. Programme for International Student Assessment (PISA)
2. Project Maths
3. Teacher Surveys - Ireland
4. Mathematics Education - Ireland

2012

371.262-dc/23

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Preface

Since its launch in 2008, Project Maths has been the subject of considerable discussion and debate amongst the mathematics education community and the general public. The initiative, which is being implemented on a phased basis, involves the complete revision of the mathematics curriculum at junior and senior cycles at post-primary level, with all five revised syllabus strands scheduled to be examined in 2014 for the Leaving Certificate, and 2015 for the Junior Certificate.

Project Maths began in 24 post-primary schools in 2008, and was rolled out across all post-primary schools in the country beginning in the autumn of 2010. The initiative has necessitated considerable inservice training and support from the Project Maths Development Team, a gradual complete overhaul of the examination papers and marking schemes, and the development of new textbooks and other instructional materials. A Common Introductory Course has been devised for the beginning of junior cycle to help to ensure that all students have the opportunity to engage with the same set of core mathematical concepts and content areas. A Bridging Framework aims to promote continuity in mathematics education between the senior classes at primary level and junior cycle at post-primary level.

The scale of the initiative, its timeframe, and its phased implementation represent significant challenges to mathematics teachers, students and school principals. However, if Project Maths is successful, it is envisaged that it will result in a deeper engagement with and understanding of mathematics on the part of students, and increased uptake of Higher level mathematics for both the Junior and Leaving Certificates.

This report describes the findings of a survey of mathematics teachers and mathematics school co-ordinators, implemented as part of PISA 2012 in Ireland. It examines teachers' views on mathematics teaching and learning in general, and on the implementation of Project Maths more specifically. Since PISA 2012 is based on a nationally representative sample of schools, we are provided with an opportunity to gain insights into Project Maths that are generalisable to national level.

In December 2013, when the mathematics achievement data of students in the PISA 2012 schools become available, we will be able to contextualise achievement outcomes with data from the teacher survey. These 'second-stage' analyses will provide empirical results on the effects of the implementation of Project Maths, though it must be borne in mind that it will be 2017 before the first full cohort of students will have experienced Project Maths all the way through post-primary education, from First through to Sixth Year.

This report is aimed primarily at teachers of mathematics and those involved in mathematics education and policymaking. It is also likely to be of interest to the international research community. The report is published at around the same time as a second one drawing on data from PISA 2012 which concerns mathematics in Transition Year. Both are available at www.erc.ie/pisa.

This report is divided into seven chapters. Chapter 1 provides an overview of PISA, while Chapter 2 describes Project Maths and existing research and commentary on the initiative. Chapter 3 describes the survey design, content of questionnaires, and survey respondents. Chapter 4 provides a description of the characteristics of mathematics teachers and the teaching of mathematics, while Chapter 5 discusses teachers' views on the teaching and learning of mathematics. Chapter 6, the main focus of this report, describes teachers' views on Project Maths at junior cycle. Chapter 7 provides a set of conclusions and recommendations, which are made at school level and at the broader level of the education system.

Acknowledgements

PISA is a large and complex exercise, and its implementation would not be possible without advice and support from many. Thanks, first and foremost, to the students, teachers and principals in the 183 schools that participated in PISA 2012. Thanks also to the Inspectors from the Department of Education and Skills who, working in collaboration with staff in schools, helped to ensure that PISA was administered in line with rigorous international standards.

In Ireland, PISA is overseen by the Educational Research Centre with the support of the Department of Education and Skills. The PISA National Advisory Committee advises on all aspects of PISA, from the content of the survey, to analysis and reporting. We are indebted to the Committee for their work on PISA, including their review of this report. Members of the PISA 2012 National Advisory Committee, along with ERC staff, are:

- Pádraig MacFhlannchadha (DES, Chair, from February 2012)
- Éamonn Murtagh (DES, Chair, to February 2012)
- Declan Cahalane (DES, joined 2012)
- Conor Galvin (UCD)
- Séamus Knox (DES, joined 2012)
- Rachel Linney (NCCA, joined 2012)
- Bill Lynch (NCCA, joined 2012, previously a member)
- Hugh McManus (SEC)
- Philip Matthews (TCD)
- Brian Murphy (UCC)
- Maurice OReilly (St Patrick’s College, Drumcondra, joined 2012)
- Elizabeth Oldham (TCD)
- George Porter (DES, to February 2012).

Other ERC staff members involved in PISA 2012 are Peter Archer (Director), Gráinne Moran, Paula Chute, John Coyle, and Mary Rohan. We would also like to thank Seán Close for his review of an earlier draft of this report. Thanks to Jill Fannin, Breda Naughton and Anne O’Mahony in the Department of Education and Skills for their review and comments on the report.

Finally, our thanks to the OECD and to the international PISA 2012 consortium (led by ACER in Melbourne) for their work in overseeing PISA’s successful implementation at international level.

Acronyms and Abbreviations Used

ACER	Australian Council for Educational Research
CPD	Continuing Professional Development
DEIS	Delivering Equality of Opportunity In Schools
DES	Department of Education and Skills
ERC	Educational Research Centre
ICTs	Information and Communication Technologies
NCCA	National Council for Curriculum and Assessment
NCE-MSTL	National Centre for Excellence in Mathematics and Science Teaching and Learning
NCTE	National Centre for Technology in Education
OECD	Organisation for Economic Co-operation and Development
PISA	Programme for International Student Assessment
PDST	Professional Development Support Team
PMDT	Project Maths Development Team
RDO	Regional Development Officer
SD	Standard Deviation
SE	Standard Error
SEC	State Examinations Commission
SSP	School Support Programme
TALIS	Teaching and Learning International Survey

1. Introduction

1.1. PISA 2012: An Overview

The OECD's Programme for International Student Assessment (PISA) assesses the skills and knowledge of 15-year-old students in mathematics, reading and science. PISA runs in three-yearly cycles, beginning in 2000, with one subject area becoming the main focus, or 'major domain' of the assessment in each cycle.

In 2012, the fifth cycle of PISA, mathematics became the major focus of the assessment for the first time since 2003. A new element to PISA in 2012 is the computer-based assessments of mathematics and problem solving. Ireland also participated in the digital reading assessment that was introduced in PISA 2009. Sixty-seven countries/economies, including all 34 OECD members and 33 'partner' countries/economies participated in PISA 2012 (Table 1.1)[1].

Table 1.1. Countries/economies participating in PISA 2012

Albania	Estonia	*Latvia*	*Serbia*
Argentina	Finland	*Liechtenstein*	*Singapore*
Australia	France	*Lithuania*	Slovak Republic
Austria	*Georgia*	Luxembourg	Slovenia
Belgium	Germany	*Macao-China*	Spain
Brazil	Greece	*Malaysia*	Sweden
Bulgaria	*Hong Kong-China*	Mexico	Switzerland
Canada	Hungary	*Montenegro*	*Thailand*
Chile	Iceland	Netherlands	*Trinidad and Tobago*
China (Shanghai)	*Indonesia*	New Zealand	*Tunisia*
Chinese Taipei	Ireland	Norway	Turkey
Colombia	Israel	*Peru*	*United Arab Emirates*
Costa Rica	Italy	Poland	United Kingdom
Croatia	Japan	Portugal	United States
Cyprus	*Jordan*	*Qatar*	*Uruguay*
Czech Republic	*Kazakhstan*	*Romania*	*Vietnam*
Denmark	Republic of Korea	*Russian Federation*	

Note. Partner countries are in italics.

1.2. PISA in Ireland

In Ireland, around 5,000 students in 183 schools participated in PISA in March 2012. These students took paper-based tests of mathematics, science and reading, and completed a student questionnaire. The sample included students in each of the 23 initial Project Maths schools (referred to as 'initial schools' in this report)[2]. A sub-sample of these students, just under 2,400 also took part in the computer-based assessments of mathematics, problem solving and reading. It should be noted that, depending on the school and year level that students were in, they may or may not have

[1] Of these 67 countries, over 40 participated in the computer-based assessments of reading, mathematics, and/or problem solving.

[2] One of the original 24 Project Maths initial schools amalgamated with another school and therefore was not included as a Project Maths school in the sample for PISA 2012.

been studying some of the new Project Maths syllabus (see Chapter 2). Principals in participating schools were asked to complete a questionnaire about school resources and school organisation. In Ireland, teachers of mathematics were invited to complete a national teacher questionnaire. Mathematics school co-ordinators[3] were also invited to complete a short questionnaire. The survey sample and content of the mathematics teacher and mathematics co-ordinator questionnaires are described in more detail in Chapter 3 of this report.

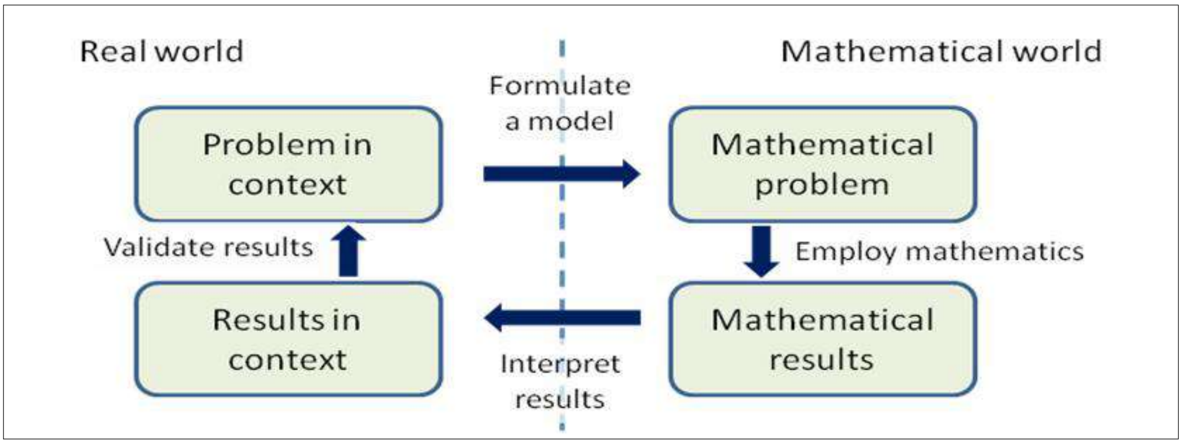
1.3. The Assessment of Mathematics in PISA

The PISA mathematics assessment focuses on active engagement in mathematics in real-world contexts that are meaningful to 15-year-olds. In PISA 2012, mathematical literacy (mathematics) is defined as

…an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens (OECD, in press).

Central to the PISA mathematics framework is the notion of mathematical modelling (Figure 1.1). This starts with a problem in a real-world context. The problem is then transformed from a 'problem in context' into a 'mathematical problem' by identifying the relevant mathematics and reorganising the problem according to the concepts and relationships identified. The problem is then solved using mathematical concepts, procedures, facts and tools. The final step is to interpret the mathematical solution in terms of the original 'real-world' context.

Figure 1.1. Mathematical modelling process in the PISA 2012 assessment framework



Source: OECD (in press).

The PISA mathematics framework is described in terms of three interrelated aspects: (i) the mathematical content that is used in the assessment items; (ii) the mathematical processes involved; and (iii) the contexts in which the assessment items are located.

PISA measures student performance in four content areas of mathematics: *Change and Relationships; Space and Shape; Quantity* and *Uncertainty*. The PISA 2012 survey will, for the first time, report results according to the mathematical processes involved (see Stacey, 2012). PISA

[3] A mathematics school co-ordinator is the staff member in each school who has overall responsibility for mathematics education – he or she is sometimes referred to as the head of the mathematics department or subject head.

mathematics items examine three mathematical processes: *formulating* situations mathematically; *employing* mathematical concepts, facts, procedures, and reasoning; and *interpreting*, applying and evaluating mathematical outcomes. PISA also identifies seven fundamental mathematical capabilities that underpin each of these reported processes. These are *communicating; mathematising; representing; reasoning and argumentation; devising strategies; using symbolic, formal, and technical language and operations;* and *using mathematical tools.*

An important aspect of mathematical literacy is the ability to use and do mathematics in a variety of contexts or situations and the choice of appropriate mathematics strategies is often dependent on the context in which the problem arises. Four categories of mathematical problem situations or contexts are defined: *personal, occupational, societal* and *scientific*. In total, 85 mathematics items, drawing on all four situations, were included in the PISA 2012 assessment, though individual students were asked to complete a subset of these items.

1.4. PISA Mathematics and the Mathematics Curriculum in Ireland

While a comparison of the PISA mathematics framework to the current junior cycle (Project Maths) curriculum has not yet been conducted, a comparison between PISA mathematics and the previous junior cycle curriculum can be found in the PISA 2003 national main report (Cosgrove, Shiel, Sofroniou, Zastrutzki & Shortt, 2005)[4]. This review found substantial differences between the content of the Irish junior cycle mathematics syllabi and the content of the PISA 2003 assessment. The concepts underlying PISA mathematics items were deemed to be unfamiliar to between a third to a half of junior cycle students, depending on syllabus level studied, and the majority of the contexts and item formats were also judged to be unfamiliar to most junior cycle students. In particular, none of the PISA items were deemed to fall into the junior cycle areas of geometry and trigonometry, and just 5% were located in the algebra strand. It may be noted that the PISA 2012 mathematics assessment now includes a higher proportion of items assessing algebra, trigonometry and geometry, in response to criticisms from some countries that the 2003 version had not included a sufficient emphasis on formal mathematics (OECD, in press).

Considerable differences were also found between the PISA assessment and the Junior Certificate mathematics examination (Cosgrove et al., 2005). While the majority of PISA 2003 items assessed Connections and Reflections competency clusters, the majority of items from Junior Certificate examination were classified as assessing skills associated with the Reproduction cluster. In other words, most of the questions on the Junior Certificate assessed routine mathematics skills in abstract contexts, rather than non-routine skills embedded in real-life situations. Also, the PISA assessments use a variety of item formats, such as multiple choice, short response and constructed response items, while the Junior Certificate examination mostly included short response items. A full comparison of the PISA assessments and the Junior Certificate examinations can be found in Close (2006).

1.5. Mathematics Achievement in Previous Cycles of PISA

The first three cycles of PISA indicate that mathematics performance of students in Ireland is at or just below the OECD average. In 2003, when mathematics was last a major focus in PISA, Ireland achieved a mean mathematics score of 502.8, which was not significantly different from the average

[4] The comparison focused on junior cycle mathematics rather than mathematics at senior cycle, since the majority of PISA students – about two-thirds – are in junior cycle.

across OECD countries[5]. However, there was variation in Irish performance across the different mathematical content areas assessed in PISA: students in Ireland performed significantly above the OECD average on the Change and Relationships and Uncertainty content subscales, while they performed significantly lower than the OECD average on the Space and Shape subscale and not significantly differently to the OECD average on the Quantity subscale (Table 1.2).

Table 1.2. Mean scores and standard deviations on the PISA 2003 mathematics content subscales: Ireland and OECD average

	Space & Shape		Change & Relationships		Quantity		Uncertainty	
	Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*	*Mean*	*SD*
Ireland	476.2**	94.5	506.0*	87.5	501.7	88.2	517.2*	88.8
OECD	496.3	110.1	498.8	109.3	500.7	102.3	502.0	98.6

*Significantly above OECD average.
**Significantly below OECD average.

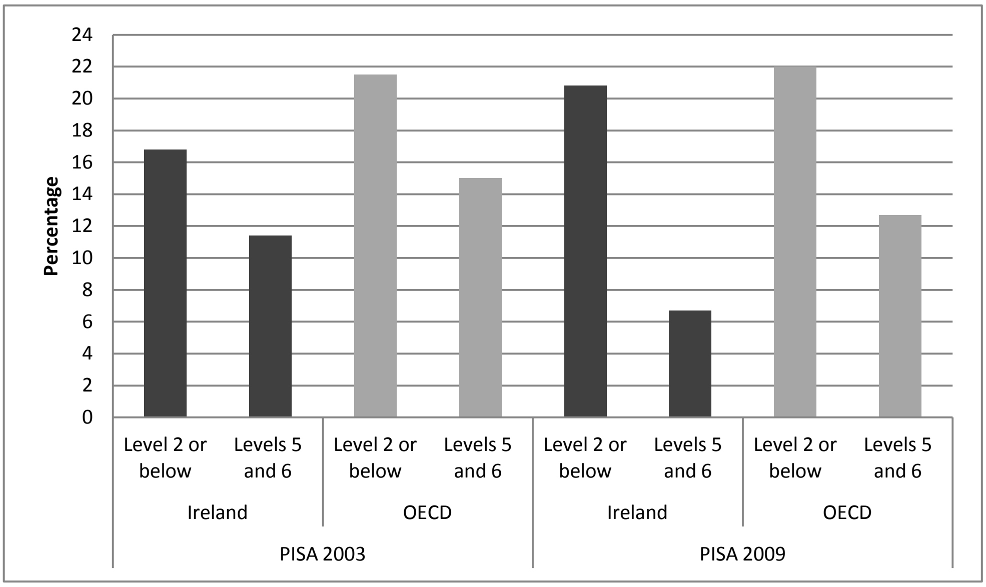
Ireland recorded a significant decline, of 16 points (about one-sixth of a standard deviation), in mathematics performance between 2003 and 2009[6] (at a time when the pre-Project Maths curriculum was in place). This was the second largest drop of all countries that participated in both cycles of PISA. The majority of this decline occurred between 2006 and 2009, when Ireland's mean score changed from 501.5 to 487.1. Ireland's position relative to the OECD average also changed, from being at the OECD average in 2003 and 2006, to being significantly below it in 2009. As mentioned previously, results for PISA 2012 will be available in December 2013.

As well as a drop in average mathematics achievement, there have been changes in the proportions of high and low achieving students in Ireland. In 2003, Ireland had significantly fewer low achieving students (i.e. students performing below proficiency Level 2) (16.8%) than on average across OECD countries (21.5%). In 2009 the percentage in Ireland increased to 20.8%, which did not differ significantly from the OECD average (22.0%). On the other hand, Ireland has seen a decline in the proportion of higher achieving students (i.e. students performing at Level 5 or above) in mathematics, from 11.4% in 2003 to 6.7% in 2009, which is below the corresponding OECD average (12.7%) (Figure 1.2). This indicates that, aside from an overall decline in mathematics achievement in Ireland, there has been a drop in the achievement of students which has been more marked at the higher end of the achievement distribution.

Males significantly outperformed females in Ireland in 2003 and 2006; however, in 2009 the gender difference was not significant. The performance of both male and female students dropped significantly from 2003 to 2009 (from 510.2 to 490.9 for males and from 495.4 to 483.3 for females), with most of the decline occurring between 2006 and 2009. In 2009, both male and female students in Ireland performed on average significantly lower than their OECD counterparts. Ireland saw an increase in the proportion of low-achieving males (from 15.0% to 20.6%) and females (from 18.7% to 21.0%) between 2003 and 2009, with the increase greater among male students. There has also been a marked decrease in the percentage of high-achieving males (from 13.7% to 8.1%) and females (from 9.0% to 5.1%) between 2003 and 2009.

[5] The OECD average for mathematics, set in 2003, is 500 points, and the standard deviation is 100.
[6] Comparisons of PISA results over different cycles assume that the scales are reliably consistent over time, which has not yet been conclusively demonstrated.

Figure 1.2. Percentages of students at or below Level 2, and at Levels 5 and 6 on PISA mathematics in 2003 and 2009: Ireland and OECD average



1.6. PISA 2012 Reporting

This report is published at around the same time as a second report that also draws on information collected in the national teacher and mathematics school co-ordinator questionnaires. The second one concerns Transition Year mathematics *(Transition Year Mathematics: The Views of Teachers from PISA 2012)*. These two reports are the first national publications on PISA 2012.

The first international results from PISA 2012 will be published by the OECD in December 2013. Results will be reported in four volumes:

- *Volume 1:* Performance in mathematics, reading and science
- *Volume 2:* Quality and equity
- *Volume 3:* Engagement and attitudes
- *Volume 4:* School and system-level policies and characteristics.

Two additional reports/volumes will be published by the OECD in the spring and summer of 2014. These are:

- *Volume 5:* Performance on computer-based problem-solving
- *Volume 6:* Performance on financial literacy (an optional assessment in which Ireland did not participate).

The ERC will release a national report on PISA 2012 in December 2013 which will complement the OECD's reporting. Additional reporting designed to provide a fuller understanding of PISA 2012 outcomes will also be published by the ERC in 2014.

All national PISA publications are at www.erc.ie/pisa, while the OECD's reports are at www.pisa.oecd.org.

1.7. Conclusions

It is reasonable to conclude that the performance of students in Ireland on PISA mathematics has, to date, been somewhat disappointing, although, as discussed in Chapter 2, there are a number of developments underway which aim to improve mathematics standards, along with changes to our education system more generally. The decline in mathematics achievement between 2003 and 2009 is nonetheless a cause for concern. Further consideration of the possible reasons for this decline, which highlight the complexity of the issue, are discussed in Cartwright (2011), Cosgrove, Shiel, Archer and Perkins (2010), LaRoche and Cartwright (2010), and Shiel, Moran, Cosgrove and Perkins (2010). We will not know how students fared on the PISA 2012 paper-based and computer-based assessments of mathematics until December 2013. As well as overall achievement in mathematics in PISA 2012, we will need to examine the performance of students at the high and low ends of the achievement distribution, since the PISA 2009 results suggest a dip in the performance of high-achieving students in particular.

Previous analyses that compare the junior cycle mathematics syllabus and examinations with PISA mathematics indicate that the syllabus in Ireland that was in place prior to the introduction of Project Maths tended to emphasise the application of familiar concepts and routines in abstract (purely mathematical) contexts. These points underline the importance of the Project Maths initiative, which is considered in Chapter 2.

As of yet, there has not been a comparison of the revised (Project Maths) syllabus and examinations on one hand, and the PISA 2012 assessment framework for mathematics and the PISA mathematics test on the other, and there would be merit in making this comparison as Project Maths becomes more established in schools.

2. Project Maths: An Overview

2.1. What is Project Maths?

Project Maths is a national curriculum and assessment initiative. The project, which involves changes in the syllabi, their assessment, and the teaching and learning of mathematics in post-primary schools, arose from detailed consideration of the issues and problems that had been identified over several years. These have been highlighted in a number of sources: research in Irish classrooms (Lyons, Lynch, Close, Sheerin & Boland, 2003), Chief Examiner's reports (for the Junior Certificate in 2003 and 2006, and for the Leaving Certificate in 2000, 2001, and 2005; see www.examinations.ie), the results of diagnostic testing of third-level undergraduate intake (Faulkner, Hannigan, & Gill, 2010), trends in international mathematics education (Conway & Sloane, 2006), and results of international assessments such as PISA (Cosgrove et al., 2005). Broadly speaking, these revealed major deficiencies in students' understanding of some of the basic concepts in mathematics, and significant difficulties in applying mathematical knowledge and skills in other than routine or well-practised contexts. For this reason, there was an identified need to provide significant support for teachers in adopting changed practices that were sustainable (NCCA, 2005). The mathematics syllabi that were in place prior to Project Maths attempted to incorporate some of the current changes, but 'because of the amount of change that had taken, and was taking, place in the junior cycle in other subject areas [at the time of introducing the previous syllabi, in 2000], *it was specified that the outcomes of the [NCCA's] review would build on current syllabus provision and examination approaches rather than leading to a root and branch change of either*' (NCCA/DES, 2002, p. 6, italics in original).

Project Maths focuses on developing students' understanding of mathematical concepts, the development of mathematical skills, and the application of knowledge and skills to solving both familiar and unfamiliar problems, using examples from everyday life which are meaningful to students (NCCA/DES, 2011a, 2011b). These aims are similar to those outlined in the PISA 2012 mathematics assessment framework, which is intended to represent the most up-to-date international views on mathematical knowledge and skills in adolescents (see Chapter 1), and although PISA is certainly not a key driver of the Project Maths initiative, it is one source of influence. One of the key elements of Project Maths is a greater emphasis on an investigative approach, meaning that students become active participants in developing their mathematical knowledge and skills. This implies not only changes in the content of the syllabi, but also, and more fundamentally, perhaps, changes to teaching and learning approaches.

Project Maths also aims to provide better continuity between primary school mathematics and junior cycle mathematics. To this end, a Bridging Framework has been developed, which maps the content of fifth and sixth class mathematics onto junior cycle mathematics[7]. A Common Introductory Course in mathematics[8] is now completed by all students in the first year of the junior cycle, meaning students do not study a specific syllabus level until a later stage. Also, in the revised syllabi, there is no separate Foundation Level syllabus. However, a Foundation Level examination will continue to be provided.

[7] http://action.ncca.ie/en/mathematics-bridging-framework
[8] http://www.projectmaths.ie/documents/handbooks_2012/handbooks_revised_feb_2012/first_yr_HB_2012.pdf

It is an objective of Project Maths to increase the uptake of Higher level mathematics at Leaving Certificate to 30%, and to 60% at Junior Certificate. To incentivise this, 25 bonus points[9] are now awarded to students who take Higher level mathematics for the Leaving Certificate and who are awarded a grade D3 or higher (www.cao.ie).

Learning outcomes are set out under five strands:

1. Statistics and Probability
2. Geometry and Trigonometry
3. Number
4. Algebra
5. Functions.

A comparison of the old and revised syllabi has not been published, partly to encourage a flexible interpretation of the revised syllabi[10]. However, an inspection of the old and revised syllabus documents indicates that some topics have been de-emphasised to allow for the development of a deeper understanding by students of the material that *is* covered. For example, there is a rebalancing of calculus at Leaving Certificate level[11], and vectors and matrices are not on the Leaving Certificate syllabus. An area which now receives more emphasis in the revised syllabi is statistics and probability.

Since Project Maths is as much about changing teaching and learning practices as it is about changing content, it was considered desirable to introduce the changes simultaneously at junior and senior cycles. This was intended to allow teachers to embed the changed teaching approaches at both junior and senior cycles at the same time. Furthermore, it was felt that teachers could focus on specific strands of mathematics regardless of the level at which these were being taught, and that support could be targeted at all mathematics teachers at the same time, although this approach meant that students commencing Fifth Year at the start of the implementation of Project Maths would not have had exposure to changes at junior cycle.

A phased approach to the changes in the syllabus was adopted. The combinations of strands to be changed in the first phase (Strands 1 and 2) was selected on the basis that these strands affected only one of the two examination papers; they also contained both familiar (Strand 2) and unfamiliar (some of Strand 1) material. By retaining some elements of the old syllabus, it was thought that teachers could concentrate on incorporating changes in the revised strands only.

Project Maths represents a new model of curriculum development in Ireland in that it involved implementing and testing out a draft curriculum from 'ground-level' upwards. It was introduced in an initial group of 24 schools in September 2008. These 24 schools have been referred to as both 'pilot' and 'initial' schools. In this report, we refer to them as initial schools, since Project Maths is not a pilot programme in the formal sense of the term.

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[9] In Ireland, students gain entry to post-secondary education through a 'points scheme' that is operated through the CAO (Central Applications Office). The provision of bonus points was not initiated as part of Project Maths.

[10] This stands in strong contrast to the syllabi previously in place, where a detailed topic-by-topic comparison between the 2000 syllabi and the previous ones was published (NCCA/DES, 2002, Appendix 1).

[11] That is, there is a reduction in the range of functions that students are expected to integrate, along with an increase in the range and types of applications that are expected, and a greater level of understanding of fundamental technical aspects of calculus.

The initial schools were selected (by the ERC) from 225 volunteer schools in such a way as to ensure that they were broadly representative of the national population of schools. This sample comprised four community/comprehensive schools, seven vocational schools, and 13 secondary schools, four of which were mixed sex. Roll-out of Project Maths to all schools began in September 2010, with the final strand being introduced into all schools in September 2012 (see Table 2.1).

Table 2.1. Timeline for Project Maths

	Junior Cycle			Senior Cycle		
Timeline	**Strands 1 and 2**	**Strands 3 and 4**	**Strand 5**	**Strands 1 and 2**	**Strands 3 and 4**	**Strand 5**
Sep-2010	Changes to Paper 2			Changes to Paper 2		
Jun-2011						
Sep-2011		Changes to Paper 1, New Paper 2			Changes to Paper 1, New Paper 2	
Jun-2012 **(PISA – Mar-2012)**						
Sep-2012			New Paper 1, New Paper 2			New Paper 1, New Paper 2
Jun-2013						
Sep-2013						
Jun-2014						
Sep-2014						
Jun-2015						

Strands 1 and 2 of the revised syllabi were first examined in all schools in 2012 at Leaving Certificate level. The Junior Certificate Examination will include these two strands in 2013, and the first examination of all Strands (1-5) takes place in 2014 at Leaving Certificate level and 2015 at Junior Certificate level. In 2017, a first cohort of students will have experienced all five strands of Project Maths right through post-primary, from First to Sixth Year.

The timeframe for the implementation of Project Maths should be borne in mind with respect to the time at which the PISA 2012 survey was conducted (i.e. spring 2012) in that the results in this report come at an early, and transitional, stage of implementation; a majority of PISA 2012 students would not have experienced the revised mathematics syllabus.

Teachers in initial schools participated in summer courses[12] that focused on the syllabus strands. Their work was also supported by school visits from a Regional Development Officer (RDO). In a general sense, the work of initial schools was supported by the RDOs through meetings, seminars, and online resources (Kelly, Linney, & Lynch, 2012). To support these changes across all schools, a programme of professional development consisting of workshops that focus on changing classroom practice, and evening courses that emphasise mathematics content are being delivered by the Project Maths Development Team (PMDT), and the National Centre for Technology in Education (NCTE)/Professional Development Support Team (PDST) is delivering courses on ICTs.

An additional support is the new Professional Diploma in Mathematics for Teaching, which is aimed at 'out-of-field' teachers of mathematics over the next three years. There are 390 places on the course, which began this autumn, and 750 have already enrolled for the course (DES press release,

[12] Elective summer mathematics courses were organised by the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL) in the University of Limerick to meet the growing professional development needs of teachers. Materials from the summer courses are available at http://www.nce-mstl.ie.

September 22, 2012). The National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL) based in the University of Limerick (www.nce-mstl.ie) leads its delivery of this course, which is fully funded by the Department of Education and Skills.

2.2. What are the Existing Views/Findings on Project Maths?

As of yet, no research on the impact of Project Maths, e.g. on student achievement, has been published. However, an interim report on Project Maths, based on research commissioned by the NCCA and conducted by the National Foundation for Educational Research (NFER, UK) will include information on students' attitudes and achievement, and is expected in November 2012. Also, when the results of PISA 2012 become available at the end of 2013, it will be possible to look at both the achievements and attitudes of PISA students in the context of when Project Maths was implemented in their schools. Again, it should be borne in mind that we are currently in the early stages of the full implementation of Project Maths.

The remainder of this section offers a brief review of the research and commentary on Project Maths, up to the time of writing of this report (November 10, 2012).

A survey of mathematics teachers in the initial schools was carried out through meetings with these teachers by staff of the NCCA in December 2011, with follow-up meetings in April 2012. It sought information from teachers on the impact of Project Maths on teaching practices, mathematics departments and students' experiences (Kelly, Linney & Lynch, 2012). The authors identified six themes emerging from the interviews with school staff: *new roles; supporting change and using resources; issues of assessment; time; issues of change;* and *feedback on syllabus strands*.

Key findings from Kelly et al. (2012) may be summarised as follows. First, teachers struggled with the *new role* of facilitating students as active learners, and reported that it was common to revert to the traditional examination preparation techniques as the State Examinations approached. Indeed, teachers reported that the examinations were impacting negatively on the new teaching and learning approaches. They also underlined their need for appropriate support and resources to allow them to continue to develop in this new role. Second, some teachers commented positively on the *changes in their teaching* and collaboration between teachers was viewed as valuable. They also reported a general increase in the use of ICTs and other resources during teaching, and with this, less emphasis on textbooks. Third, *time* was highlighted as an issue by teachers, who commented on the difficulties posed by the time required to meet and plan, cover the syllabus, and to use different kinds of assessment.

Kelly et al. (2012) also reported that tests, homework and sample examination questions were cited as the principal forms of assessment, and teachers commented that they needed support in using alternative methods of assessment in class. There was a view among teachers that the syllabus was too long, and that further consideration needed to be given to its length, particularly in light of the increased emphasis on problem-solving and context-based tasks. However, comments from some of the teachers suggested that, as teachers develop their familiarity with the connections between the strands, they can make more efficient and effective use of their time. It is too early to make this conclusion confidently though – the issue will become clearer as implementation of all five strands progresses.

Some commentary on Project Maths has come from the third-level sector[13]. A report from the School of Mathematical Science in University College Cork has cautioned against the 'unrealistic expectations' of, and 'the exaggerated claims' being made about, Project Maths (Grannell, Barry, Cronin, Holland & Hurley, 2011, p. 3). The authors express concerns generally about the ensuing mathematical knowledge and skills of third-level entrants, and more specifically about the removal of core material that was included on the pre-Project Maths syllabus, particularly vectors. They are also concerned about the burden that has been placed on teachers.

The report of the Taskforce on Education of Mathematics and Science at Second Level (Engineers Ireland, 2010), includes the following observations: first is the low level of take-up of Higher level mathematics for the Leaving Certificate along with mediocre mathematics standards internationally; second, the 'generally untapped resource' (p. 1) that Transition Year represents; third, the major challenges or 'quantum leap required in the transitioning of teaching methods' (p. 2); and fourth, the broad issue of adequate resourcing of Project Maths.

The lack of textbooks to support Project Maths has been highlighted by some commentators (e.g. Engineers Ireland, 2010; Grannell et al. 2011). However, the Project Maths website (www.projectmaths.ie) cautions against over-reliance on textbooks, and encourages teachers to use supplementary resources. Lubienski (2011) has argued that the Project Maths leaders 'appear to be circumventing textbooks as opposed to leveraging them' (p. 45) and, comparing two of the textbooks in common use at the time, comments that: 'one text [was] presenting traditional boxed formulas and examples for students to follow and the other text [was] structuring a sequence of investigations through which students derive the formulas'. Lubienski suggests that instead of circumventing textbooks, Project Maths leaders should assist teachers in critically analysing the contents of texts and selecting the most appropriate to their own needs and the goals of Project Maths.

Lubienski (2011) considered Project Maths from a US perspective. Her findings are based on interviews with members of the Project Maths Development Team (PMDT) and the NCCA, and visits to three of the initial schools. She comments positively on the collaborative nature of the initiative; its adherence to the timeline; responsiveness to feedback from the initial schools; teacher professionalism; and changes in teachers' practices. She also highlights some key difficulties raised by the interviewees. The first is the decision to implement Project Maths at both junior and senior cycles at the same time. Lubienski (2011, p. 31) comments that this was 'the subject of the majority of complaints… from Irish teachers.' The second was the lack of availability of sample papers at the time of her study, while the third was the length and difficulty of the statistics strand, particularly for senior cycle students.

Lubienski (2011) also raises two 'high-level' issues in her review. First is the high emphasis in Ireland that is placed on the Leaving Certificate examination, which, in her view, constrains instruction and places teachers in the role of 'exam coach'. This stands in contrast to the US, where students sit the independently-administered Scholastic Aptitude Test (SAT) or American College Test (ACT). She comments that the examinations-driven approach in Ireland may give rise to teaching and learning that emphasises form over substance (or procedural over conceptual knowledge), and a blurring in the distinction between instruction and assessment. Second, time pressure appears to stem from

[13] It should also be noted that the third-level sector has representation on the NCCA's course committees through the Irish Universities Association (see NCCA, 2012).

two system-level or structural sources – pressure to cover the syllabus (partly, she notes, with the inclusion of Religious Education and Irish as core subjects), and short class periods (35-40 minutes) relative to the US (45-50 minutes).

Since September 2008 (when Project Maths was first introduced), there have been over 500 media reports on Project Maths. Common themes in these reports are concerns over the 'dumbing down' of the subject, the content of the revised syllabi (e.g. too much emphasis on problem-solving, not enough on formal or pure mathematics), and effects of Project Maths on the level of preparedness of students for third-level courses in mathematics, science, engineering and technology.

Some media reports have commented on the immediate effects of the awarding of bonus points for Higher Level mathematics, noting that there has been a marked increase, from 16% to 22% in the number of students taking Higher level mathematics for the 2012 Leaving Certificate (e.g. Irish Independent, August 15, 2012). Some express concerns that the bonus points scheme may affect the CAO points requirements for college entry in a very general way, with an increase in points required for entry to many courses, some of which do not require Higher level mathematics (e.g. Irish Times, August 16, 2012).

A review of the recommendations made in the report of the Project Maths Implementation Support Group (DES, June 2010) indicates that already, attempts are being made to address some areas of concern. First, the report recommended that schools allocate a minimum of one mathematics class per day for all students. This was included in a Circular sent to schools in September 2012 (Circular Number 0027/2012) asking that every effort be made to provide students with a mathematics class every day, particularly at junior cycle. One would also hope that, as the Framework for Junior Cycle (DES, 2012) is implemented (see the next section), the reduction in the numbers of subjects taken by students, together with the specification of a minimum amount of instructional hours for English, Irish and mathematics, will help to further alleviate time pressures reported by teachers. Second, the Implementation Support Group report recommended encouraging rather than discouraging students to take Higher Level mathematics at Leaving Certificate level, and to award excellence in mathematics (as is already done in schools for English and Irish during prize-giving ceremonies). This may go part (but by no means all) of the way in helping more students achieve their full potential in mathematics (recall that in Chapter 1, we noted the relatively low performance of students in Ireland at the upper end of the PISA mathematics achievement distribution). Third, it recommends a review of third level entry processes and requirements, including bonus points for Higher Level mathematics. As noted earlier, bonus points were awarded for the first time in 2012, coinciding with an increase in the percentages taking Leaving Certificate mathematics at higher level. Fourth, it contains recommendations for addressing gaps in teacher qualifications and professional development. Also as noted, the new Professional Diploma in Mathematics for Teaching commenced in autumn 2012, and Project Maths has included the delivery of fairly intensive CPD by the PMDT and NCTE.

2.3. Project Maths in the Wider Context of Educational Reform

We have already commented that, at the time of teacher survey that formed part of PISA 2012 in Ireland, Project Maths was at a relatively early stage of implementation. Project Maths is also occurring within a wider context of educational reform. The National Strategy to Improve Literacy and Numeracy Among Children and Young People, 2011-2020 (DES, 2011) may be regarded as a key reference for the broader educational context at this time. Although Project Maths began before the Strategy was published, its objectives fit well into its overarching framework.

In the Strategy, numeracy and mathematics appear to be used interchangeably. It states that 'Numeracy encompasses the ability to use mathematical understanding and skills to solve problems and meet the demands of day-to-day living in complex social settings' (DES, 2011, p. 8). The Strategy places the development of numeracy within the role of all teachers, not just teachers of mathematics. It sets out the following five goals and targets for outcomes at post-primary level that are relevant to mathematics/numeracy (DES, 2011, p. 18):

- Ensure that each post-primary school sets goals and monitors progress in achieving demanding but realistic targets for the improvement of literacy and numeracy skills;
- Assess the performance of students at the end of second year in post-primary education, establish the existing levels of achievement, and set realistic targets for improvement;
- Increase the percentage of 15-year old students performing at or above Level 4 (i.e. at the highest levels) in PISA reading and mathematics tests by at least 5 percentage points by 2020;
- Halve the percentage of 15-year old students performing at or below Level 1 (the lowest level) in PISA reading and mathematics tests by 2020; and
- Increase the percentage of students taking the Higher Level mathematics examination at the end of junior cycle to 60 per cent by 2020, and increase the percentage of students taking the Higher Level mathematics examination at Leaving Certificate to 30 per cent by 2020.

In order to achieve these targets, the Strategy sets out a number of supportive actions. With respect to initial teacher education, it proposes changes to both the content and length of the courses. It also sets out ways to better support newly-qualified teachers, and recommends focusing continuing professional development (CPD) on literacy, numeracy and assessment, with a minimum participation of 20 hours every five years. The Strategy specifies CPD and resource materials for school principals and deputy principals for effective teaching approaches, assessment, and self-evaluation. It emphasises the importance of assessment in informing current standards and identifying areas for improvement at individual, school and national levels, and notes that assessment for learning (AfL) 'is not used sufficiently widely in our schools and we need to enable teachers to improve this practice' (DES, 2011, p. 74). It notes that AfL needs to be combined with AoL (assessment of learning), chiefly in the form of standardised tests, and highlights the lack of standardised mathematics tests currently in place at post-primary level. The Strategy specifies the development of standardised tests for use in post-primary schools in 2014, with the requirement that post-primary schools administer these tests at the end of second year in 2015. It specifies how schools should use the results of these assessments for individual learning, reporting to parents, and school self-evaluation. It is also intended that the results of these assessments will be used to monitor trends in achievement nationally. To complement this, the Strategy recommends continued participation in international assessments, in order to benchmark national achievement levels against international ones.

In discussing the mathematics curriculum, the Strategy is supportive of the recommendations made by the Project Maths Implementation Support Group (DES, 2010), and indicates that Project Maths is designed to address many of the long-standing concerns about mathematics teaching and learning at post-primary level. It notes, however, that '…the adoption of this radically new approach to the subject is challenging for teachers and has to be supported by extensive continuing professional development' (DES, 2011, p. 52).

The *Framework for Junior Cycle* (DES, 2012) follows from *Innovation and Identify: Ideas for a New Junior Cycle* (NCCA, 2010) and *Towards a Framework for Junior Cycle* (NCCA, 2011). The framework highlights the lack of progress made by some students in English and mathematics in the earlier stages of post-primary school, as well as the dominant influence of the Junior Certificate examination on the experiences of junior cycle students. It describes reforms to both the content of the junior cycle curriculum, and 'most particularly to assessment' (p. 1).

Eight principles underpin the new junior cycle: quality, wellbeing, creativity and innovation, choice and flexibility, engagement and participation, inclusive education, continuity and development, and learning to learn (DES, 2012, p. 4). Four of the 24 statements of learning in the framework are of particular relevance to mathematics, though almost all have some relevance (DES, 2012, pp. 6-7). The four are that the student:

- recognises the potential uses of mathematical knowledge, skills and understanding in all areas of learning;
- describes, illustrates, interprets, predicts and explains patterns and relationships;
- devises and evaluates strategies for investigating and solving problems using mathematical knowledge, reasoning and skills; and
- makes informed financial decisions and develops good consumer skills.

The Framework identifies 18 junior cycle subjects (DES, 2012, p. 11), along with seven short courses. It is planned that there will be a reduction in the number of subjects taken by students, with most taking 8-10 subjects in total. Short courses will count as half of a subject. The Framework specifies that a minimum of 240 hours of instruction be provided for English, Irish and mathematics, a minimum of 200 hours for other subjects, with 100 hours for up to four short courses.

It is envisaged that students will study a mix of subjects and short courses. Subjects are to be revised over a period of about five years, starting with English in 2014-2015, with no revisions to the new mathematics curriculum until 2017-2018. All subjects and short courses will be described in specification documents, which are to include the following elements: aims and rationale; links with statements of learning, literacy, numeracy, and other key skills; overview (strands and outcomes); expectations for students; and assessment and certification.

Literacy and numeracy are recognised as key skills, along with managing self, staying well, communicating, being creative, working with others, and managing information and thinking (DES, 2012, p. 9).

Aside from these substantial changes to the content and specifications of the curriculum, assessment in junior cycle is seen as the 'most significant change' (DES, 2012, p. 18). The Junior Certificate examination is to be phased out, and replaced by school-based assessment (culminating in a School Certificate). Given the proposed scale of this reform, the SEC will continue to be involved in the initial stages, particularly with respect to English, Irish and mathematics, and the timeline for the changes to assessment will mirror that for the revision of subjects and courses (see DES, 2012, p. 25 and p. 39). English, Irish and mathematics will continue to be assessed at both Higher and Ordinary levels, while other subjects will be assessed at Common level.

2.4. Conclusions

There can be little doubt that Project Maths is a highly ambitious curricular reform initiative, and it is too early yet to expect to observe its effects on mathematics education, particularly students' mathematics achievement, since implementation (in the form of examination of all five syllabus strands) will not be complete until 2014 (at Leaving Certificate)/2015 (at Junior Certificate).

There has been a considerable amount of commentary on Project Maths, some of it is based on opinion rather than fact, and of course dependent on the particular stage of implementation of the initiative. We suggest that commentary on Project Maths is best interpreted in the broader context of educational reform, i.e. the implementation of the new junior cycle framework, and the overarching strategy to improve literacy and numeracy.

In reviewing the research conducted on Project Maths to date, we have noted the lack of empirical data, particularly achievement data, and data from parents, though the forthcoming interim report from the NFER (due before the end of 2012) can be expected to provide some information on the opinions and mathematics achievements of students. Additional data on achievement will be analysed and reported on in the international and national reports on PISA 2012 in December 2013 (see Chapter 1).

Commentary on the omission of some aspects of mathematics from senior cycle raises concerns about its suitability for candidates who want to enter third-level courses which have high mathematics or mathematics-related content. We suggest, however, that the changes brought about by Project Maths at post-primary level should be managed as a two-way process across both the post-primary and third-level sectors (see Chapter 7).

Views from the teachers themselves, particularly regarding the time required to become familiar with and implement the revised syllabus, and the constraints imposed on them by the examinations should also be treated with concern, though the reform of the junior cycle can be expected to alleviate some of the time pressure experienced by teachers. Further, while the full impact of the introduction of CAO bonus points for Higher Level mathematics may not yet be apparent, we have concerns that introducing bonus points could have the unintended consequence of a focus on Higher level uptake and grades attained, to the detriment of due consideration of actual mathematics standards achieved by all students. We note, however, that a review of the provision of bonus points is expected in 2014 (DES, personal communication, September 2012).

3. Survey Aims, Questionnaires and Respondents

3.1. Aims of the Survey and Content of Questionnaires

The teacher and mathematics school co-ordinator[14] questionnaires are national instruments, administered only in Ireland as part of PISA 2012. Their content was established and finalised on the basis of discussions with the PISA national advisory committee (membership of which is shown in the Acknowledgements to this report), the literature review (see Chapter 1), and analyses of the field trial data, which were conducted in March 2011.

The aims of administering the questionnaires were fourfold:

1. To obtain a reliable, representative and up-to-date profile of mathematics teaching and learning in Irish post-primary schools.
2. To obtain empirical (numeric) and qualitative (text) information on the views of a nationally representative sample of teachers on the implementation of Project Maths; and to compare this information across teachers in initial schools and teachers in other schools.
3. To obtain information on aspects of Transition Year mathematics.
4. To make findings available to teachers and school principals, the DES, NCCA, and partners in education in an accessible format and timely manner.

With respect to the second aim, it is our view that, since Project Maths was implemented in an earlier timeframe in the initial schools, comparisons between initial schools and other schools could provide some indication of any issues or changes to do with the implementation of Project Maths in initial and later stages, though it should be borne in mind that national roll-out of Project Maths was informed by the experiences of the initial schools.

With respect to the third aim, the results from questions on Transition Year mathematics are reported in a separate ERC publication *(Transition Year Mathematics: The Views of Teachers from PISA 2012)*. Transition Year has been highlighted as being in need of review, particularly in light of Project Maths and educational reform more generally (e.g., DES, 2010; Engineers Ireland, 2010).

With respect to the fourth aim, to expedite the dissemination of the results from these national questionnaires, it was decided to publish reports on them prior to the availability of students' achievement scores and other PISA 2012 data. However, the examination of the data discussed in this report with respect to students' mathematics achievement in PISA 2012 will be a next step. As noted in Chapter 1, students' mathematics achievement will be available in December 2013.

The mathematics teacher questionnaire consisted of five sections as follows:

- Background information (gender, teaching experience, employment status, qualifications, teaching hours, participation in CPD)
- Views on the nature of mathematics and teaching mathematics
- Teaching and learning of students with differing levels of ability
- Views on Project Maths
- Teaching and learning in Transition Year mathematics (if applicable to the teacher).

[14] Mathematics school co-ordinators may also be referred to as 'mathematics subject heads' or 'mathematics department heads'.

Most of the information from the survey was numeric (i.e. consisting of pre-coded ‘tick-box’ responses); however, teachers also wrote comments on Project Maths and on the use of differentiated teaching practices. This report includes the results from both numeric and written responses.

The mathematics school co-ordinator questionnaire was considerably shorter than the teacher questionnaire and asked about the following:

- Organisation of base and mathematics classes for instruction
- Distribution of students across mathematics syllabus levels
- Arrangements for Transition Year mathematics (if available/taught in the school).

It is important to note that the content of the questionnaires that were administered impacts on what this report does and does not cover. In particular, this report does not examine teachers’ views on Applied Mathematics (taken at Leaving Certificate level by about 2.5% of students; www.curriculumonline.ie, www.examinations.ie). Results do not address the opinions of other groups such as principals, students and parents, or if they do, they do so indirectly (e.g. teachers’ views on the opinions of students and parents). Also, while the teacher questionnaire does consider various aspects of the content and skills underlying the revised syllabi, the results cannot be viewed as a review of the revised curriculum. Finally, since a majority of students taking part in PISA are in junior cycle, some of the questions on Project Maths are targeted specifically to junior cycle: there is no equivalent, specific focus on teachers’ views at senior cycle.

3.2. Demographic Characteristics of Mathematics Teachers and School Co-ordinators

Tables 3.1 and 3.2 show some of the characteristics of the teachers and mathematics school co-ordinators who participated in the survey, which was conducted in schools in Ireland that participated in PISA 2012.

Table 3.1. Demographic characteristics of teachers participating in the PISA 2012 mathematics teacher survey

Characteristic	N	%
Gender		
Female	844	65.2
Male	451	34.8
Years Teaching Experience		
One to two	83	6.3
Three to five	207	15.7
Six to ten	287	21.8
Eleven to twenty	334	25.4
Twenty one or more	405	30.8
Employment Status		
Permanent	852	66.0
Fixed term > 1 year	201	15.6
Fixed term < 1 year	238	18.4

Note. Data are weighted to reflect the population of teachers.

Table 3.2. School-related characteristics of mathematics teachers and school co-ordinators participating in the PISA 2012 teacher survey

Characteristic	Teachers		Co-ordinators	
	N	%	N	%
Sector/Gender Composition				
Community/Comprehensive	219	16.6	95	13.4
Vocational	330	25.0	232	32.9
Secondary all boys	226	17.1	111	15.6
Secondary all girls	298	22.6	132	18.7
Secondary mixed	248	18.8	137	19.4
DEIS/SSP Status				
No	1041	78.8	506	71.5
Yes	280	21.2	202	28.5
Initial Project Maths School				
No	1267	95.9	684	96.7
Yes	54	4.1	23	3.3
Fee Pay Status				
No	1207	91.3	656	92.7
Yes	114	8.7	52	7.3
School Size				
Small (<400)	275	20.8	289	40.9
Medium (401-600)	481	36.4	226	31.9
Large (601-800)	370	28.0	124	17.6
Very Large (>801)	195	14.7	68	9.6

Note. Data are weighted to reflect the population of teachers/co-ordinators.

The schools were sampled at random, and are nationally representative of the population of post-primary schools. In each school, all teachers of mathematics were selected to participate. All results are weighted.[15] Overall, 80.3% of teachers returned a questionnaire, and 93.4% of school co-ordinators returned a questionnaire. Sixty-five percent of mathematics teachers were female (Table 3.1). This is consistent with the profile of teachers who participated in the OECD's TALIS survey, in which 69% were female (Gilleece, Shiel, Perkins & Proctor, 2008). About three-tenths of teachers indicated having 21 or more years of experience, 47.2% had between six and 20 years of experience, 15.7% between three and five years, and 6.3% reported having fewer than two years of teaching experience. Years of experience reported by mathematics teachers is again broadly similar to those reported in TALIS (Gilleece et al., 2008), as well as in a recent national survey (Uí Ríordáin & Hannigan, 2009).

Two-thirds of teachers (66.0%) were permanently employed; of the remaining respondents, similar proportions of teachers were on fixed-term contracts of more than a year (15.6%) and on fixed-term contracts of less than a year (18.4%). The proportion of permanently employed teachers is less than the figure of 74% reported in TALIS (Gilleece et al., 2008) while the number of teachers with fixed-term contracts of more than a year is somewhat higher (8% in TALIS).

[15] See the Technical Appendix for information on response rates the computation of the sampling weights used in analyses for this report.

A quarter of teachers were in vocational schools, 22.6% in all girls' secondary schools, 18.8% in mixed secondary schools, 17.1% in all boys' secondary schools, and 16.6% in community/comprehensive schools. One-fifth of teachers (21.2%) were in DEIS (SSP) schools[16] and four percent of teachers responding were working in Project Maths initial schools (recall that we sampled all 23 initial schools). Just under a tenth of teachers were based in fee-paying schools. Most schools (64.4%) had student enrolments of between 401 and 800 students, one fifth of schools were small (<400) and the remaining 14.7% were very large schools of over 800 students.

The characteristics of mathematics co-ordinators were broadly similar to those of mathematics teachers (Table 3.2).

3.3. Conclusions

As part of PISA 2012 in Ireland, mathematics teachers and mathematics school co-ordinators completed questionnaires which provide information on the contexts for teaching and learning mathematics, views on mathematics, and specifically on Project Maths. This is a nationally representative sample of teachers, so results can be generalised to teachers of mathematics nationally. Once the achievement data of students in PISA 2012 are available in 2013, the information gathered from teachers will help us to contextualise and better understand achievement.

[16] DEIS, Delivering Equality of Opportunity in Schools, provides additional, targeted resources to primary and post-primary schools that have high concentrations of disadvantage, under the School Support Programme (SSP) (DES, 2005).

4. General Characteristics of Mathematics Teachers and Organisation of Mathematics

4.1. Teacher Background and Qualifications

This section describes the qualifications of mathematics teachers who took part in the survey, including work in other professions prior to entering teaching.

The Teaching Council (2012) specifies that in order to teach mathematics at post-primary level, teachers should have completed at least a primary degree in which mathematics was a major subject (minimum of 30% of the period of the degree) and that the breadth and depth of the syllabi undertaken are such as to ensure competence to teach mathematics to the highest level in post-primary education.

Three-fifths of teachers overall had completed a primary degree that incorporated mathematics up to final year, a proportion which was almost identical across Project Maths initial and other schools (Table 4.1). Only three percent of teachers overall had completed a primary degree that did not include mathematics as a subject. The remainder (35.4%) had completed a primary degree with mathematics in first year/first and second year only.

Table 4.1. Percentage of teachers who hold primary degrees with varying quantities of mathematics content: Overall, and in initial and other schools

Degree Content	Overall	Initial Schools	Other Schools
Primary degree with mathematics up to final year	60.0	60.2	60.0
Primary degree with mathematics in first and second year	20.1	15.4	20.3
Primary degree with mathematics in first year only	15.3	21.5	15.0
Primary degree that did not include mathematics as a subject	3.3	1.3	3.4
None of the above	1.2	1.8	1.2

Table 4.2 shows the type of primary degree held by teachers. The most commonly-held primary degree was a BA or BSc with mathematics (58.3%). About 13% of teachers held a B Comm/Business degree, and the same proportion held a BA or BSc without mathematics. Just 6.3% held a B Ed with mathematics. The distribution of primary degree types was similar in initial schools and other schools (Table 4.2).

Table 4.2. Percentage of teachers who hold primary degrees of various types: Overall, and in initial and other schools

Degree Type	Overall	Initial Schools	Other Schools
B Comm or Business degree	13.0	19.1	12.7
B Eng	3.0	2.5	3.0
BA or BSc with mathematics	58.3	63.3	58.1
BA or BSc without mathematics	13.1	8.2	13.3
B Ed with mathematics	6.3	3.1	6.4
B Ed without mathematics	2.4	0.4	2.5
Other	3.9	3.5	3.9

The most common postgraduate qualification, held by 56.3% of teachers, was a Higher or Postgraduate Diploma in Education (H Dip/PGDE) that included a specific focus on mathematics education (Table 4.3). The percentage of teachers with this qualification was slightly lower in initial schools than other schools, though initial schools had a slightly higher percentage of teachers with a H Dip/PGDE without a specific focus on mathematics education (29.1% vs. 22.1%). Ten percent of teachers reported having no postgraduate qualification. Of these, 75.6% indicated that they had a primary degree which included mathematics for two years or more.

Table 4.3. Percentages of teachers with various postgraduate qualifications: Overall, and in initial and other schools

Postgraduate qualification	Overall	Initial Schools	Other Schools
No postgraduate qualification (includes B Ed)	10.3	7.3	10.7
Postgraduate degree related to mathematics (but not the teaching of mathematics)	4.7	6.4	4.7
Postgraduate degree related to the teaching of mathematics	5.2	8.7	5.1
Postgraduate degree unrelated to mathematics or the teaching of mathematics	11.2	10.8	11.4
Higher Diploma in Education/Postgraduate Diploma in Education *with* Mathematics	56.3	49.3	57.7
Higher Diploma in Education/Postgraduate Diploma in Education *without* Mathematics	22.0	29.1	22.1

Note. Teachers could hold more than one postgraduate qualification.

The manner in which teachers were asked about their qualifications does not allow us to make a direct comparison with Teaching Council guidelines. However, it is likely that 15.5% of mathematics teachers (i.e., those with a BA, BSc or B Ed *without* mathematics) would not meet the requirements[17]. No inferences can be made about a further 16.9% (i.e., those with a B Comm or Business degree, or 'other' primary degree), as the mathematical content of these degree types is unknown and may vary from institution to institution. The remaining 67.6% (i.e., those with a B Eng, or a BA, BSc or B Ed *with* mathematics) are likely to meet the criteria, though again this is impossible to determine definitively from the data. Therefore, our best estimate from the information available is that somewhere between 68% and 85% of mathematics teachers surveyed were qualified to teach mathematics according to Teaching Council guidelines.

Overall, 76.3% of mathematics teachers reported that they had studied mathematics teaching methods at some point in their preservice teacher preparation. Note that this percentage does not tally exactly with the information presented in Tables 4.2 and 4.3, as the former concern qualifications, while the latter concerns material studied. The percentage of teachers in initial schools who had studied mathematics teaching methods in their teacher preparation was just slightly lower (73.9%) than those in other schools (76.4%).

Teachers were asked whether or not they had worked in a profession prior to teaching, and three in ten indicated that they had. More teachers in vocational schools and community/comprehensive schools indicated that they had worked in another profession prior to teaching than in other school types. Slightly more teachers in DEIS (SSP) than in non-SSP schools, as well as in initial than in other schools, reported having worked in another profession prior to teaching (Table 4.4).

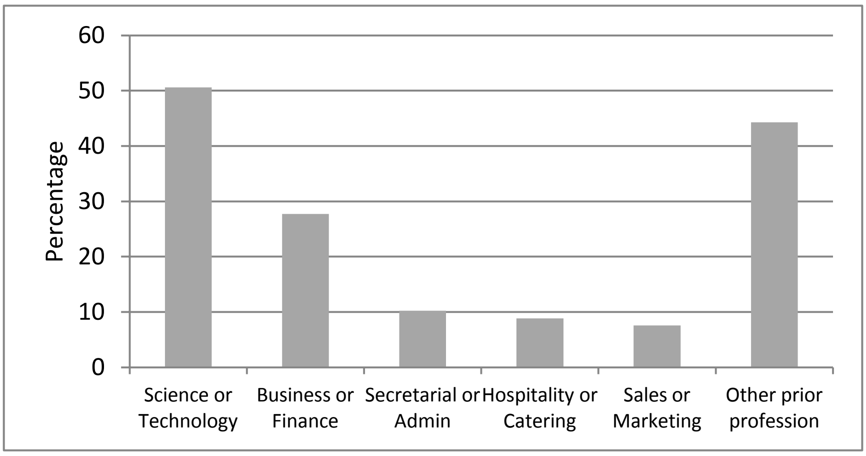
[17] It is not possible to be definitive about this, as some of this group may hold qualifications that feature substantive mathematics content, e.g., science.

Table 4.4. Percentage of mathematics teachers who worked in a profession in another field prior to teaching: Overall, and by school characteristics

	Yes	No
Overall	29.9	70.1
School Sector/Gender Composition		
Community/Comprehensive	35.0	65.0
Vocational	36.3	63.7
Boys' Secondary	25.8	74.2
Girls' Secondary	26.8	73.2
Mixed Secondary	24.3	75.7
DEIS		
No	28.4	71.6
Yes	35.3	64.7
Initial Project Maths School		
No	29.7	70.3
Yes	33.2	66.8

Of those teachers who had worked in another profession before teaching, the most frequently reported fields were science or technology (50.6%), business or finance (27.7%), and other (44.3%), with a sales or marketing background being least frequent (7.6%) (Figure 4.1). A little under a third of teachers who had worked in a different profession prior to teaching (29.4%) had done so for five years or more, nearly half (46.2%) had worked in another profession for two to four years, while about a quarter of this group (24.2%) had about a year's experience.

Figure 4.1. Teachers' professional backgrounds prior to teaching



Note. Teachers could select more than one prior profession. Percentages apply to the 29.9% of teachers who indicated that they had worked in a prior profession.

Teachers were asked to indicate how adequate they thought their qualifications were for preparing them to teach mathematics in post-primary schools. Table 4.5 shows overall levels of agreement/disagreement across five aspects of their qualifications (for all teachers, regardless of qualification type). In general, there was agreement that teachers' courses of study had prepared them to teach mathematics, with between 62% and 78% agreeing or strongly agreeing with the five aspects under consideration. 'Mathematical content' and 'General teaching methods/pedagogy' were the most strongly endorsed aspects (77-78% agreed or strongly agreed with these two) while the highest level of disagreement was in relation to 'Teaching methods/pedagogy of mathematics' and 'Assessment of mathematics' (33.5% disagreed or strongly disagreed with the former, and 37.6% with the latter).

Table 4.5. Perceived adequacy of qualifications for preparing mathematics teachers to teach mathematics in post-primary schools

Aspect of qualification	Strongly Disagree	Disagree	Agree	Strongly Agree
Mathematical content	5.2	17.8	46.5	30.5
Teaching methods/pedagogy of mathematics	6.6	26.9	44.9	21.6
Assessment of mathematics	6.6	31.0	45.1	17.3
General teaching methods/pedagogy	4.9	17.2	49.9	28.0
Assessment in general	5.2	21.5	50.0	23.3

4.2. Teaching Hours and Classes/Levels Taught

Teachers were asked how many hours per week they spent teaching mathematics to each year level as well as hours each week spent teaching all other subjects. On average, teachers reported teaching a total of 9.2 hours of mathematics a week, with 9.8 hours spent teaching other subjects (Table 4.6). Note that the average number of hours taught at each year level correspond to teaching hours, not class periods per week. These figures are similar to those reported in a recent survey of mathematics teachers (Uí Ríordáin & Hannigan, 2009). In the current study, approximately 52% of all teaching time per week was spent teaching mathematics. The percentage of teaching time spent teaching mathematics was slightly higher for teachers in initial schools (59.8%) than those in other schools (51.1%) with teachers in initial schools spending on average 10.5 hours per week teaching mathematics compared to 9.2 hours per week in other schools.

Table 4.6. Average hours spent teaching per week: Overall, and in initial and other schools

Year Levels*	All		Initial Schools		Other Schools	
	Mean	SD	Mean	SD	Mean	SD
First Year mathematics	2.87	0.71	2.95	0.68	2.87	0.71
Second Year mathematics	2.92	0.59	3.01	0.57	2.91	0.59
Third Year mathematics	3.00	0.60	3.06	0.70	2.99	0.59
Transition Year mathematics	2.33	0.70	2.37	0.73	2.33	0.70
Fifth Year mathematics	3.28	0.70	3.20	0.75	3.29	0.69
Sixth Year mathematics	3.40	0.70	3.37	0.63	3.40	0.71
Other levels/programmes e.g. Repeat LC or PLC - mathematics	2.44	0.85	2.35	0.67	2.45	0.86
Total hours teaching mathematics per week	9.2	5.2	10.5	5.1	9.2	5.2
Hours teaching all other subjects	9.8	6.4	8.1	6.5	9.9	6.4
Total hours teaching per week	18.9	4.5	18.6	4.0	18.9	4.6
Percentage of all teaching time spent teaching mathematics	51.5	29.3	59.8	30.3	51.1	29.2

*Teachers who indicated that they did not teach any hours at a given year level were excluded from the calculation of means and standard deviations for that year level. The total number of hours spent teaching mathematics is based on the sum of hours across year levels.

Table 4.7 shows the percentage of teachers teaching mathematics at junior cycle at each syllabus level. Note that percentages exceed 100% as teachers could teach more than one level. Most teachers indicated teaching Ordinary or Higher level, with a third of mathematics teachers teaching Foundation level. One in eight teachers (12.2%) did not teach junior cycle at the time of the survey. Proportions were very similar across initial and other schools.

Table 4.7. Percentages of teachers teaching Junior Cycle mathematics levels since 2009: Overall, and in initial and other schools

	All		Initial Schools		Other Schools	
Level	Yes	No	Yes	No	Yes	No
Foundation Level	30.6	69.4	27.3	72.7	30.7	69.3
Ordinary Level	76.2	23.8	76.9	23.1	76.1	23.9
Higher Level	70.1	29.9	69.1	30.9	70.1	29.9

Note: Teachers could select more than one level. 8.3% of respondents were missing data on this question.

4.3. Teaching and Classroom Activities

Teachers were asked to indicate how much emphasis they placed on various teaching and classroom activities in a typical week in teaching mathematics to Third Year students (Table 4.8). The response options were none, low, medium or high emphasis. About 42% of teachers were not teaching Third Years at the time of the survey, and these are excluded from the analysis. Overall highest emphasis was placed on whole class teaching activities (72.9% indicated placing high emphasis on this) followed by keeping order in the classroom (45.0%) and individual student learning activities (41.7%). Ten percent of teachers reported placing no emphasis on group learning activities.

Table 4.8. Percentages of teachers placing no, low, medium or high emphasis on various teaching/classroom activities in Third Year

Activity	None	Low	Medium	High
Whole class teaching activities	0.6	4.8	21.7	72.9
Individual student learning activities	1.7	17.9	38.7	41.7
Student group learning activities	10.0	41.4	34.4	14.1
Student assessment activities	1.2	11.0	49.9	37.9
Keeping order in the classroom (maintaining discipline)	7.2	25.8	22.0	45.0
Administrative tasks, e.g. recording attendance	1.5	48.7	21.2	28.6

Note. 7.6% to 8.2% of respondents were missing data on these items.

Table 4.9 shows levels of emphasis placed on various teaching and classroom activities in initial and other schools. Responses for the ‘none’ and ‘low’ categories have been collapsed into one category to make three: little to no emphasis, medium emphasis, and high emphasis. Teachers in both initial and other schools placed highest emphasis on whole class teaching. However, across other teaching/classroom activities, teachers in initial and other schools appeared to have slightly different profiles. More teachers in other than in initial Project Maths schools placed little to no emphasis on student group learning activities (52% vs. 44.5% respectively) and student assessment (12.2% vs. 5.9%).

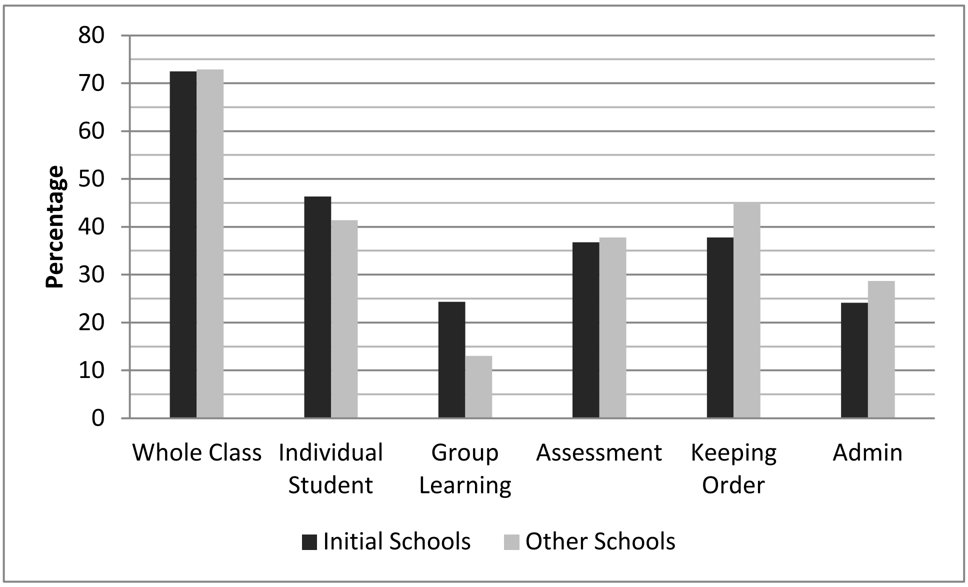
Table 4.9. Emphasis on each teaching/classroom activity in Third Year: Initial schools and other schools

Activity	Initial Schools			Other Schools		
	None/Low	Medium	High	None/Low	Medium	High
Whole class teaching activities	5.6	21.5	72.9	1.2	26.3	72.5
Individual student learning activities	19.9	38.7	41.4	18.2	35.6	46.3
Student group learning activities	52.0	35.0	13.0	44.5	31.2	24.3
Student assessment activities	12.2	49.9	37.8	5.9	57.3	36.8
Keeping order in the classroom	33.2	21.7	45.1	32.5	29.7	37.8
Administrative tasks, e.g. recording attendance	50.5	20.8	28.7	45.8	30.2	24.1

Note. 7.6% to 8.2% of respondents were missing data on these items.

Figure 4.2 shows teaching and classroom activities on which teachers placed high emphasis in initial and other schools. More teachers in initial schools than in other schools reported placing a high emphasis on individual student learning (a difference of 5%), and on group learning activities (a difference of 11.3%). Teachers in initial schools placed lower emphasis than teachers in other schools on keeping order in the classroom (a difference of 7.3%) and administrative tasks (a difference of 4.6%).

Figure 4.2. Percentages of teachers in initial schools and other schools who reported placing a *high* emphasis on various teaching and classroom activities when teaching mathematics to Third Year students



4.4. Ability Grouping for Mathematics Classes

Figure 4.3 shows the percentages of schools that group students by ability for their base classes[18] and mathematics classes for each year level. This information is based on responses from mathematics co-ordinators. Ability grouping for mathematics is the focus of this section; grouping for base classes is provided for comparative purposes. In First Year, the frequency of ability grouping is quite low, at 17.3% and 14.3% for base and mathematics classes respectively. Ability grouping for base classes increases in Second, Third, Fifth and Sixth Years, but remains at around 40% in all cases. In contrast, ability grouping for mathematics classes increases to 80.9% in Second Year and remains

[18] A 'base class' is usually identified for administrative purposes, e.g. recording attendance.

above 90% in Third, Fifth and Sixth Years. Ability grouping for mathematics is lower in Transition year (43.5%), but higher than ability grouping for base classes at this year level (13.6%).

Figure 4.3. Ability grouping for base and mathematics classes, by year level

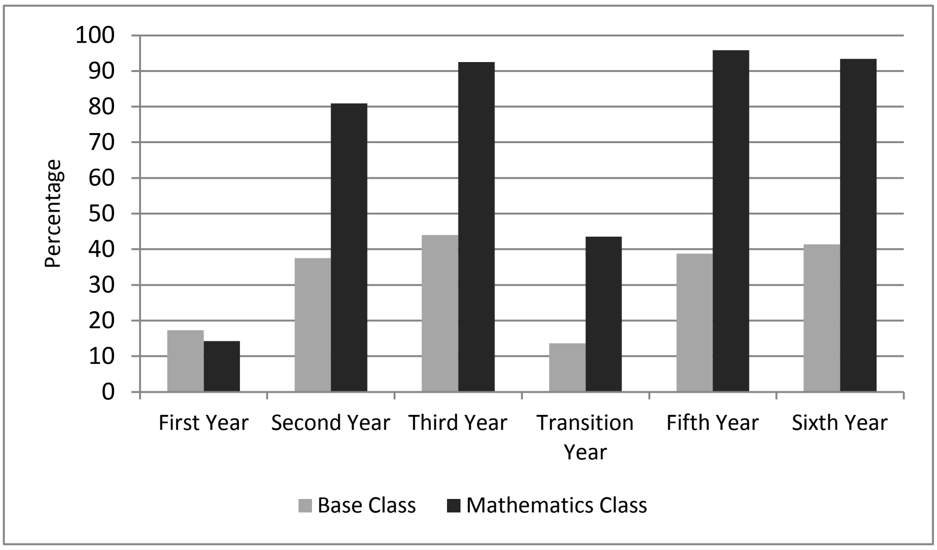


Table 4.10 shows the prevalence of ability grouping for mathematics classes, and for students' base classes, by the total enrolment size of the school.

Table 4.10. Prevalence of ability grouping for base classes and mathematics classes by school enrolment size: First to Sixth Year

Year level/Class	Very Small (300 or fewer)	Small (301-400)	Medium (401-600)	Large (601-800)	Very large (801 or more)
Base class	%	%	%	%	%
First year	28.1	14.1	15.2	10.6	20.5
Second year	50.2	35.5	39.4	27.4	28.7
Third year	62.4	37.0	49.8	28.3	28.7
Transition year	15.0	5.4	22.4	5.1	9.4
Fifth year	47.4	22.4	47.7	32.8	32.2
Sixth year	47.4	29.5	50.5	32.8	35.8
Mathematics class	%	%	%	%	%
First year	15.9	10.2	14.1	13.8	19.4
Second year	80.0	70.8	82.2	87.1	87.2
Third year	100.0	75.8	94.1	92.9	100.0
Transition year	23.3	38.8	50.3	53.6	44.6
Fifth year	100.0	88.7	93.3	100.0	100.0
Sixth year	79.6	95.5	96.5	100.0	100.0

Note. 0.5% to 6.2% of respondents were missing data on these items.

One might expect that the approach taken to grouping students into classes for instruction would be related to the enrolment size: in smaller schools, ability grouping into separate classes might be less prevalent, as there would be fewer students and hence fewer class groups at each year level. However, this is not generally the case, either for assignment to students' base classes, or to their mathematics classes. Across schools of all enrolment sizes, the prevalence of ability grouping for mathematics classes rises dramatically between First and Second Year, and remains high up to Sixth Year, with the exception of Transition Year. It should be noted that the pattern of grouping might be different in schools with extremely small enrolment sizes, i.e. for some of the schools in the 'Very Small' group in Table 4.10, but this is not examined in further detail here.

Table 4.11 shows the prevalence of ability grouping for base and mathematics classes across school characteristics (DEIS/SSP status, initial/other schools, and school sector/gender composition). Ability grouping of base classes appears to be consistently more common in DEIS than non-DEIS schools for all years, though especially in First Year (39.4% in DEIS and 8.0% in non-DEIS schools). However, this pattern appears to be reversed for mathematics, with ability grouping after First Year less prevalent in DEIS than non-DEIS schools. Initial schools were less likely to report ability grouping of base classes than other schools across all years. Differences between initial and other schools were not notable for mathematics ability grouping.

Patterns of ability grouping varied a little across school type. Base class ability grouping tended to be more prevalent than the overall averages across most years in boys' secondary schools. The opposite pattern was true for girls' secondary schools, where ability grouping of base classes tended to be less prevalent than the overall averages. Ability grouping for base classes differed between some school types by more than 30% from Third Year upwards. For example, prevalence of base class ability grouping in senior cycle ranged from 20.6% (girls' secondary schools) to 53.4% (mixed secondary schools). The range was smaller for mathematics class ability grouping which was very prevalent (over 85%) from Third Year upwards across all school types, with the exception of Transition Year for which there was a range of 26.2% to 54.5% across the school characteristics considered in Table 4.11.

Table 4.11. Prevalence of ability grouping for base classes and mathematics classes: Overall, and by DEIS status, pilot status, and school sector/gender composition

Year level/Class	All	Non - DEIS	DEIS	Initial	Other	Comm/ Comp	Vocat- ional	All boys sec	All girls sec	Mixed sec
Base class										
First year	17.3	8.0	39.4	9.5	17.5	24.6	16.0	25.7	13.7	10.7
Second year	37.6	31.3	52.3	28.6	37.9	37.4	40.2	45.7	25.4	38.6
Third year	43.9	36.7	60.9	33.3	44.3	37.4	49.4	53.9	25.4	49.9
Transition year	13.6	11.2	22.1	5.0	14.0	11.4	8.7	33.1	2.3	16.5
Fifth year	38.8	37.9	40.9	28.6	39.2	21.1	45.9	50.7	20.6	47.1
Sixth year	41.4	38.4	48.2	28.6	41.9	23.8	48.9	50.7	20.6	53.4
Mathematics class										
First year	14.3	8.5	28.5	9.5	14.4	22.2	10.1	21.2	14.3	10.0
Second year	80.9	83.5	74.6	85.7	80.8	89.2	70.1	84.1	83.7	88.1
Third year	92.5	96.3	83.0	95.2	92.4	91.7	86.0	92.3	96.7	100.0
Transition year	43.5	44.5	40.2	40.0	43.7	54.5	26.2	49.4	51.0	41.4
Fifth year	95.8	98.6	88.7	95.2	95.8	91.7	94.8	100.0	99.2	93.6
Sixth year	93.4	93.8	92.3	100.0	93.2	91.7	86.5	93.1	100.0	100.0

Note. 0.5% to 6.2% of respondents were missing data on these items.

Overall, the widespread prevalence of ability grouping for mathematics in Third, Fifth and Sixth Years regardless of various school characteristics suggests that this practice may be led by broad, system-level features, such as the structure of the syllabus and examinations, rather than school-level structures or policies.

4.5. Patterns of Mathematics Syllabus Uptake

Mathematics co-ordinators were asked to estimate the percentage of students in their schools studying mathematics at each syllabus level during the 2011-2012 school year. Table 4.12 shows a steady decrease in the number of students studying Higher level mathematics from 51.2% in Second Year to 20.3% in Sixth Year, while the percentages taking Ordinary/Foundation level mathematics increases from 35.5% in Second Year to 75.6% by Sixth Year. Of note is the drop in the percentage of students studying mathematics in senior cycle, from 31.9% to 20.3%, implying that about a third of students who begin senior cycle studying mathematics at Higher level end up taking Ordinary or Foundation level. First Years tend to study mathematics at Common level (in line with the implementation of the Common Introductory Course), though 10.6% were reported to be taking Higher level and 5% taking Ordinary/Foundation level.

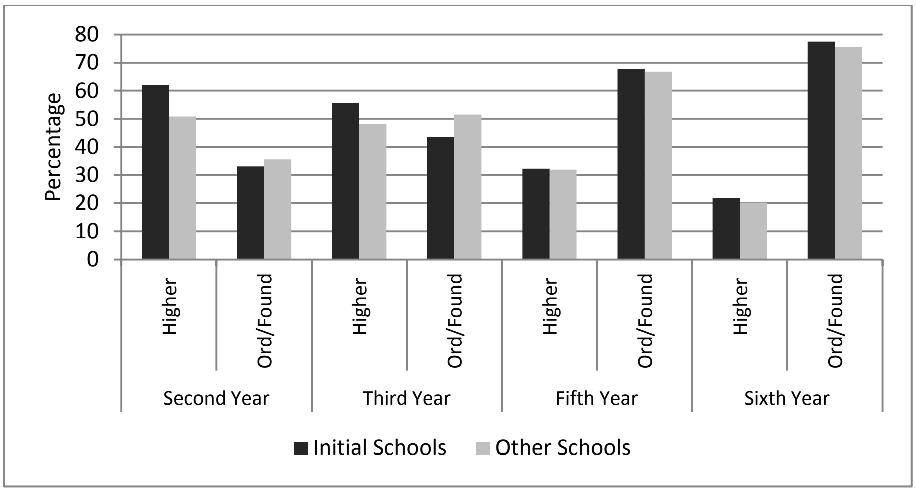
Table 4.12. Percentages of students studying mathematics at each syllabus level, by year level (mathematics school co-ordinators' estimates)

Year/Syllabus level	Higher		Ordinary/Foundation		Common	
	Mean	SD	Mean	SD	Mean	SD
First year	10.6	26.8	5.0	15.3	84.4	36.2
Second year	51.2	29.1	35.5	25.3	12.8	33.1
Third year	48.5	20.8	51.3	20.8	0.2	1.3
Fifth year	31.9	15.6	66.7	15.4	1.4	5.7
Sixth year	20.3	15.3	75.6	18.2	4.1	15.7

Note. 4.7% to 10.5% of respondents were missing data on these items.

As noted in Chapter 1, one objective of Project Maths is to increase uptake of Higher level mathematics for both the Junior and Leaving Certificates. Patterns of syllabus level uptake in initial and other schools (again as estimated by mathematics school co-ordinators) are shown in Figure 4.4.

Figure 4.4. Percentages of students studying mathematics at each syllabus level by year level (mathematics school co-ordinators' estimates): Initial schools and other schools

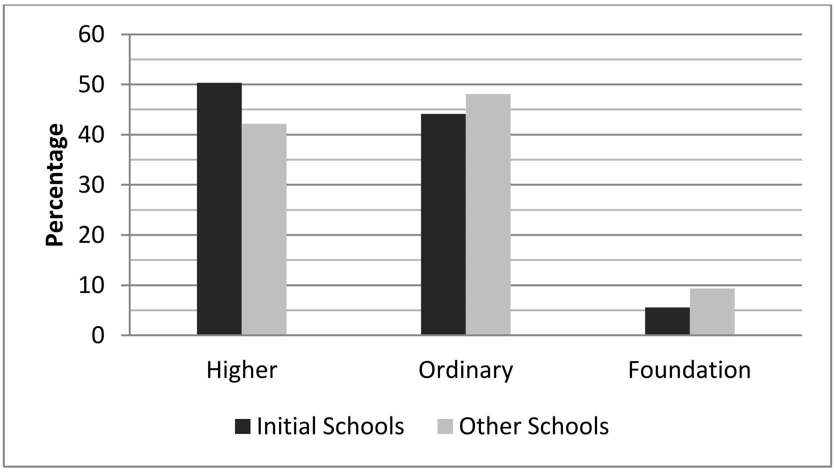


Note. The difference is statistically significant for Higher level uptake in Second Year.

For clarity, the graph only displays Higher and Ordinary/Foundation levels (i.e. excludes common level). First Year is also excluded. A general pattern of slightly more frequent Higher level and slightly lower Ordinary/Foundation level uptake in initial schools than other schools emerges. Though differences are slight in most years, the pattern is more pronounced in Second and Third Years. The only statistically significant difference in Higher-level uptake is at Second Year level.

Figure 4.5 shows the percentage of students who *actually took* the Junior Certificate mathematics examination in 2011 at each syllabus level, for initial and other schools (as estimated by school co-ordinators). Fewer than 1% did not sit the examinations and for clarity these are not included on the graph. In initial schools, a significantly higher proportion of students sat the Junior Certificate mathematics examination at Higher level than in other schools. Though the differences were not statistically significant for the other syllabus levels, it can be noted that marginally fewer students in initial schools sat the examinations at Ordinary and particularly Foundation level. The data in Figure 4.5 are comparable to those for the 2011 Junior Certificate mathematics examination overall, in which 46% sat Higher level, 47% sat Ordinary level, and 7% took Foundation level (www.examinations.ie).

Figure 4.5. Percentages of students who took the Junior Certificate mathematics examination in 2011 at each syllabus level (mathematics school co-ordinators' estimates): Initial schools and other schools



Note. The difference is statistically significant for Higher level uptake.

4.6. Continuing Professional Development (CPD)

Teachers were asked to indicate the number of hours of continuing professional development (CPD) relating to mathematics in which they had engaged, how much of this was outside school time, and what obstacles they had encountered in attending CPD related to mathematics education. When answering this question, teachers were advised that CPD was intended to cover both formal and informal activities. It should also be noted that the model of support for initial schools is much different to other schools, with workshops delivered in a shorter space of time, and in-school support available from a designated RDO. On average, mathematics teachers reported spending 45.2 hours (SD = 25.9) engaging in CPD in the last three years[19]. Thirteen percent had attended less than 16 hours of CPD, 68.3% had attended between 16 and 64 hours and the remaining 17.9% had attended over 64 hours.

[19] The hours of CPD discussed in this section should be treated as broad estimates, since they are values that were recoded from the original response categories as follows: None=0; 1-8=4; 9-16=12; 17-24=20; 25+=28.

Table 4.13 shows the average number of hours of participation in different kinds of CPD in the last three years for all mathematics teachers, as well as the averages for teachers in initial and other schools. Overall, the highest levels of participation were for formal CPD on Project Maths (20.2 hours) and self-directed CPD (study of Project Maths materials; books or journals on mathematics education etc.) (14.2 hours). The least time was spent on formal courses designed to address a gap in qualifications to teach mathematics (1.5 hours), formal postgraduate study that included mathematics or mathematics education (1.6 hours) and formal CPD relating to the Junior Certificate mathematics syllabus (other than Project Maths) (1.8 hours).

Table 4.13. Hours of CPD participation in the last three years: Overall, and in initial schools and other schools

Type of CPD	All			Initial Schools			Other Schools		
	Mean	SE	SD	Mean	SE	SD	Mean	SE	SD
Formal CPD on Project Maths	20.2	0.34	9.6	21.9	0.74	9.6	20.1	0.35	9.6
Formal CPD on the Junior Certificate mathematics syllabus other than Project Maths	1.8	0.18	5.2	1.9	0.34	4.8	1.8	0.19	5.2
A formal CPD course designed to address a gap in your qualifications to teach mathematics	1.5	0.19	5.7	2.9	0.56	7.9	1.4	0.20	5.6
In-school professional development activities relating to mathematics	3.0	0.23	5.7	7.2	1.41	9.1	2.8	0.23	5.4
Self-directed CPD, e.g. study of Project Maths materials; of books or journals on mathematics education	14.2	0.34	11.4	18.2	1.74	11.5	14.1	0.34	11.4
External meetings relating to mathematics, e.g. the Irish Maths Teachers Association	2.9	0.23	5.9	3.7	0.65	6.3	2.9	0.24	5.9
Formal postgraduate study that included mathematics or mathematics education (e.g., M.A., M.Ed.)	1.6	0.19	6.3	2.1	1.62	7.4	1.6	0.19	6.3
Total CPD Hours	45.2	0.87	25.9	57.9	4.92	29.4	44.7	0.87	25.6

Note. Grey shading indicates a statistically significant difference ($p < .05$).

There were some significant differences[20] between teachers in initial and other schools in the average number of CPD hours undertaken during the three years preceding the survey. Teachers in the initial schools spent slightly more time than teachers in the other schools attending formal CPD on Project Maths (21.9 vs. 20.1 hours), formal CPD courses designed to address a gap in qualifications (2.9 vs. 1.4 hours) and self-directed CPD (18.2 vs. 14.1 hours). The largest differences observed between initial-school and other-school teachers were in the amount of in-school professional development activities relating to mathematics (7.2 vs. 2.8 hours) and the total number of CPD hours (57.9 vs. 44.7), with teachers in initial schools engaging in more hours than their counterparts in other schools for both categories. This may reflect the more widespread provision in and encouragement of CPD in schools in which Project Maths was introduced earlier.

[20] Formal tests of significance are carried out on group mean differences, but not on group percentages, in this report.

Teachers also indicated whether they had participated in CPD during or outside of school time. Responses were quite varied: 19.3% indicated that none of their CPD had taken place outside of school time; 29.4% indicated that a minority was outside of school time; while about a quarter (25.2%) indicated that about half of CPD was outside of school time; a similar percentage indicated that all or a majority occurred outside of school time. There were no substantial differences between teachers in initial schools and other schools in response to this question. It may be recalled from Chapter 2 that CPD for Project Maths occurred both during and outside of school hours.

Teachers were asked to indicate, which obstacle(s), if any, prevented them from participating in mathematics-related CPD (see Table 4.14). The most frequent reasons cited were a lack of time, both outside of school hours (45.9%) and during school hours (24.2%). Other obstacles included location of courses (18.4%), not being informed of courses (16.7%), lack of availability of courses (16.5%), lack of incentive (financial or otherwise) (14.3%) and lack of personal resources to pay for CPD (13.4%). The least frequently indicated obstacle was lack of school resources to pay for CPD (5.6%). Approximately a third of teachers (31.6%) indicated that nothing had prevented them taking part in CPD[21].

Teachers in the initial and other schools identified broadly similar obstacles to CPD attendance that they indicated, as shown in Table 4.14. Teachers in initial schools, however, were less likely (a difference of 5% or more) to indicate that not being informed of courses and a lack of time outside of school hours had affected their participation in CPD attendance, and more likely than teachers in other schools to indicate that location of courses had prevented CPD participation.

Table 4.14. Percentage of teachers indicating factors that prevented participation in CPD related to mathematics education: Overall, and in initial schools and other schools

Factor	Overall	Initial Schools	Other Schools
Lack of availability of courses	16.5	15.5	16.6
Not being informed of courses	16.7	10.8	16.9
Location of courses	18.4	23.6	18.2
Lack of time during school hours	24.2	28.3	24.0
Lack of time outside of school hours	45.9	40.5	46.2
Lack of incentive (financial or otherwise)	14.3	15.3	14.3
Lack of school resources to pay for CPD	5.6	9.5	5.5
Lack of personal resources to pay for CPD	13.4	15.8	13.3
None of the above (nothing prevented me)	31.6	33.3	31.5

Note. Teachers could select more than one obstacle to CPD attendance.

[21] In interpreting these results, it should be noted that teachers are assigned to workshops in one of the 21 education centres, each of which is associated with a cluster of schools. Also, for national roll-out workshops, an invitation for all named teachers with dates and venues is sent by the PMDT to the principal along with a request to update the database if teachers have changed. A general notice is also on www.projectmaths.ie and emailed to the 4,200 teachers who have signed up for the e-newlsetter. For modular courses, education centres contact local schools and run the course on demand (DES, personal communication, October 2012).

4.7. Key Findings and Conclusions

Based on a broad comparison between teachers' responses and Teaching Council guidelines, we estimate that two-thirds to five-sixths of the mathematics teachers who took part in our survey are qualified to teach their subject. These figures suggest that teachers who took part in the present study were better-qualified to teach mathematics than those who took part in the UL survey (Uí Ríordáin & Hannigan, 2009), in which 48% were described as not having a mathematics teaching qualification. However, the manner in which teachers were asked about their qualifications, as well as the sampling, differed somewhat across the two studies. The profile of teacher qualifications presented in this chapter is closer to a more recent survey conducted by the Teaching Council. On the basis of responses of about 3,300 teachers in 420 schools, this more recent survey found that two-thirds were fully qualified to teach mathematics, about three in ten had undergone some studies in mathematics, and only 2.5% had no third level qualifications/studies in mathematics (DES press release, September 29, 2011).

In addition to teachers who may not hold required qualifications, a further, unknown percentage of teachers are likely to benefit from the opportunity to upskill their existing qualifications. It is noteworthy that a sizeable minority of teachers in our study felt that their qualifications were inadequate in helping them to prepare for teaching mathematics, particularly in the areas of assessment and teaching methods. Hence, the new Postgraduate Diploma, available from the autumn of 2012 and running each year for three years, is unlikely to be adequate on its own in addressing teachers' professional needs. Continued, concentrated efforts will be required to provide appropriate, accessible CPD and support to mathematics teachers in the medium to long term.

About three in 10 mathematics teachers had worked in a field other than teaching prior to entering the teaching profession, many of them in the science and technology and/or business and finance sectors. These are likely to have valuable and relevant prior experience that could be brought to bear on their work as teachers, and some may already be doing this. Also, the fact that 30% of mathematics teachers worked in another profession prior to teaching suggests that it is possible to attract individuals with a diversity of skills and experience into the teaching profession.

Teachers on average reported spending about nine hours per week, or just over half of their teaching hours, teaching mathematics. Teachers in initial schools spent proportionately more of their total teaching time on mathematics than teachers in other schools (10.5 vs. 9.2 hours per week). It should be noted that these figures reflect teaching hours rather than hours of instruction received by students. However, results from the PISA 2012 student questionnaire will include information on the amount of mathematics instruction received by students. These results will be available from December 2013, and it is planned to analyse them with reference to how they vary across school types and student characteristics, including year level.

Ability grouping ('streaming'/'setting') for mathematics class is very common. Our results suggest that it is occurring in over 90% of schools in Third, Fifth and Sixth Years. Although there is some variation across schools in the extent to which students are grouped into different classes by ability for mathematics instruction, the overall picture points to an issue that is structural or systemic. We also found that there is a steady decrease in the percentages of students studying Higher Level mathematics, from almost half (48.5%) in Third Year, to just 20% in Sixth Year. This sharp decline underlines the need for further encouragement of students to take Higher Level mathematics in senior cycle. Furthermore, the finding that there was a drop in numbers taking Higher Level, from about 32% in Fifth Year to 20% in Sixth Year, indicates that somewhere in the region of one-third of the students who initially study Higher Level mathematics in senior cycle end up taking Ordinary or Foundation level. The PISA results for mathematics (described in Chapter 1) also suggest a need to further challenge higher achievers, and to motivate and support students more in achieving their full potential in mathematics.

Teachers in our study reported placing a high emphasis on whole class and individual work during mathematics class with their Third Years. However, almost twice as many teachers in initial schools (24%) compared with other schools (13%) reported that they placed a high emphasis on engaging students in group learning activities during mathematics class, which can be regarded as a positive finding.

Finally, teachers reported having participated in an average of 45 hours of CPD, both formal and informal, over the past three years. The majority of CPD was on Project Maths, or self-directed in nature. It is likely that the information concerning CPD presented in this chapter, together with some of the results presented in Chapters 5 and 6, will be of relevance to future planning and delivery of CPD related to Project Maths.

5. Teaching and Learning Mathematics: Teachers' Views and Practices

5.1. General Views on the Teaching and Learning of Mathematics

The teacher questionnaire included a set of 12 statements on the nature of mathematics and the teaching and learning of mathematics. Teachers indicated their level of agreement/disagreement with each, and their responses are shown in Table 5.1 (on six items that indicate support for a more 'fixed' view of mathematics) and Table 5.2 (on six items that indicate support for a constructivist/applied view of mathematics).

A majority of teachers (88.0%) agreed or strongly agreed that some students have a natural talent for mathematics, while others do not, and that an effective approach for students who are having difficulty is to give them more practice by themselves in class (57.3%). In contrast, two-thirds of teachers disagreed or strongly disagreed that mathematics is a difficult subject for most students, while over 80% disagreed/strongly disagreed that mathematics is primarily an abstract subject, and that learning mathematics mainly involves memorising.

Table 5.1. Teachers' levels of agreement/disagreement with six statements indicating a fixed view of mathematics and the teaching and learning of mathematics (all teachers)

Statement	Strongly disagree	Disagree	Agree	Strongly agree
Some students have a natural talent for mathematics and others do not	2.1	9.9	67.3	20.8
If students are having difficulty, an effective approach is to give them more practice by themselves during the class	5.8	36.9	51.6	5.7
Mathematics is a difficult subject for most students	6.4	60.6	28.4	4.6
Few new discoveries in mathematics are being made	12.8	54.8	30.1	2.3
Mathematics is primarily an abstract subject	19.5	61.7	17.1	1.7
Learning mathematics mainly involves memorising	30.6	59.5	8.5	1.4

Table 5.2 shows that in general, a large majority of teachers agreed or strongly agreed with the six statements indicative of a constructivist or applied view of mathematics. In fact, 80% or more of teachers agreed or strongly agreed with all six statements, and this exceeded 95% for the first two (i.e., that there are different ways to solve most mathematical problems, and that more than one representation should be used in teaching a mathematics topic).

There are few differences in the response patterns of teachers in initial schools and other schools on the items assessing fixed views of mathematics, with one exception: teachers in initial schools were more inclined to disagree or strongly disagree that an effective approach for students with difficulties is to give them more practice by themselves (54.9%), compared to teachers in other schools (42.2%).

Responses of teachers in initial and other schools were also similar for the six items assessing constructivist/applied views of mathematics. However, teachers in initial schools were more inclined to disagree or strongly disagree that there are different ways to solve most mathematical problems (12.3%) than teachers in other schools (3.0%); they were also more likely to disagree or strongly disagree that to be good at mathematics at school, it is important for students to understand how mathematics is used in the real world (24.8% vs. 15.4%). These two differences are counter-intuitive:

one might expect teachers in initial schools to be somewhat more positively disposed towards a constructivist approach to the teaching and learning of mathematics (such an approach underpins Project Maths).

Table 5.2. Teachers' levels of agreement/disagreement with six statements indicating support for a constructivist or applied view of mathematics and the teaching and learning of mathematics (all teachers)

Statement	Strongly disagree	Disagree	Agree	Strongly agree
There are different ways to solve most mathematical problems	0.8	2.5	58.3	38.3
More than one representation (picture, concrete material, symbols, etc.) should be used in teaching a mathematics topic	0.9	2.6	49.9	46.6
Solving mathematics problems often involves hypothesising, estimating, testing and modifying findings	1.0	10.8	66.2	22.1
Modelling real-world problems is essential to teaching mathematics	0.5	11.7	63.1	24.6
To be good at mathematics at school, it is important for students to understand how mathematics is used in the real world	1.1	14.8	54.6	29.6
A good understanding of mathematics is important for learning in other subject areas	0.7	19.3	65.8	14.1

The items shown in Tables 5.1 and 5.2 were used to form two scales, each of which have an overall mean of 0 and standard deviation of 1[22]. The first scale can be interpreted as a measure of *fixed views of mathematics*, while the second is an indicator of *constructivist/applied views of mathematics*. Means on these scales did not differ significantly between teachers in initial schools and other schools; nor did they differ by DEIS status, school sector/gender composition, or teacher gender. In all cases, differences between sub-groups were less than 0.15 scale points (about one-seventh of a standard deviation).

5.2. Sources Used in Establishing Teaching Practices

Teachers were asked to rate the extent to which a range of sources influenced their decisions about the teaching practices that they use in mathematics lessons. Responses are shown in Table 5.3. A majority of teachers used sample/past examination papers and syllabus documents a lot (71.5% and 62.8%, respectively), while close to half of teachers made 'a lot' of reference to students' needs and interests and to textbooks (47.9% and 46.3%, respectively). Only three in 10 teachers (30.7%) reported drawing on CPD a lot, while 26.5% reported that they derived their teaching practices from or with other teachers in their school a lot. Use of the remaining resources was less common: in particular, 45.1% of teachers never used information on teaching practices in other countries. This pattern of results suggests that more 'traditional' sources of information (e.g. examination papers) are used more widely than more novel ones (such as information gained in CPD, or from books and journals).

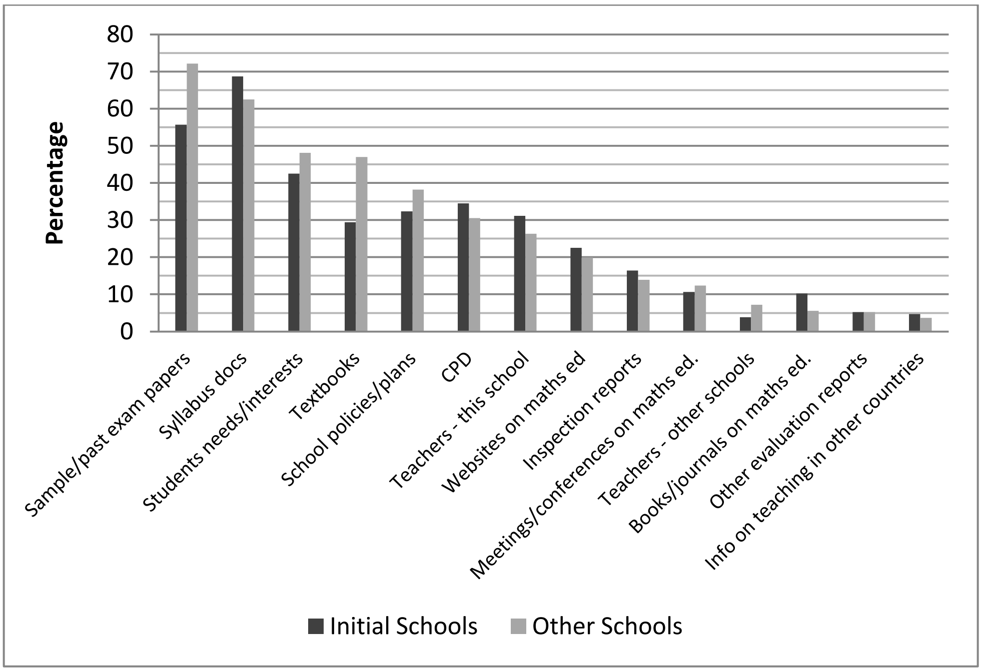
[22] The items are grouped in the manner shown in Tables 5.1 and 5.2 following the results of a principal components analysis. This technique shows how items 'cluster' together in terms of the response patterns. If items cluster together, it can be inferred that they are measuring a common underlying construct. Note, however, that these are not necessarily the best items for measuring views on mathematics; alternatively-worded questions may have provided a better measure. In particular, the scale indicating fixed views of mathematics has a low reliability of .42, so results should be interpreted with caution. See the Technical Appendix for information on how the scales were constructed.

Table 5.3. Extent to which teachers use various sources to influence decisions about teaching practices in mathematics lessons (all teachers)

Source	Not at all	A little	To some extent	A lot
Sample or past examination papers	0.3	3.7	24.5	71.5
Syllabus documents	1.8	10.0	25.5	62.8
Students' needs and interests	0.9	11.4	39.8	47.9
Textbooks	1.1	9.1	43.5	46.3
In-school policies and plans	3.7	17.2	41.2	37.9
Continuing Professional Development	2.9	17.3	49.1	30.7
Other teachers in this school	4.2	20.2	49.1	26.5
Websites on mathematics education	7.7	30.1	42.3	20.0
Inspection reports on teaching mathematics	11.8	32.0	42.1	14.0
Meetings/conferences on mathematics education	21.6	34.9	31.2	12.3
Teachers in other schools	31.4	37.2	24.4	7.0
Books and journals on mathematics education	29.8	41.2	23.2	5.8
Other evaluation reports	22.0	39.2	33.6	5.2
Information about teaching practices in other countries	45.1	38.0	13.1	3.8

Figure 5.1 shows the percentages of teachers in initial and other schools who reported using each of the sources listed in Table 5.3 a lot. Some differences are apparent. Teachers in other schools reported making more use of sample or past examination papers and textbooks, and, to a lesser extent, of in-school policies or plans and students' interests, compared to teachers in initial schools. On the other hand, teachers in initial schools reported slightly higher use of syllabus documents, other teachers in their school, books and journals on mathematics education, and CPD, than teachers in other schools.

Figure 5.1. Percentages of teachers in initial schools and other schools indicating that they used various sources 'a lot' when making decisions about teaching practices in their mathematics classes



5.3. Use of ICTs in the Teaching and Learning of Mathematics

Table 5.4 shows the frequency with which teachers reported using six ICT resources during their mathematics classes. Note that although 'data projector' did not explicitly refer to an interactive whiteboard, it would be reasonable to assume that many teachers would have included use of an interactive whiteboard under this category. The most commonly-used resources were a PC/laptop and a data projector, with 60% or more of teachers using these at least once a week. Spreadsheet packages were used much less frequently (48.9% of teachers never used these), and use of Internet sites, general and mathematics-specific software was intermediate.

Table 5.4. Teachers' use of ICT resources during mathematics classes

ICT Resource	Hardly ever/never	About once a term	About once a month	At least once a week
PC or laptop	12.4	8.2	17.1	62.3
Data projector	15.4	8.5	16.1	60.0
Internet sites	16.7	15.1	28.5	39.7
General software (e.g. PowerPoint, Word)	26.4	15.0	21.4	37.2
Mathematics-specific software (e.g. Geometer's Sketchpad, Geogebra, Logo)	25.3	21.4	28.3	25.0
Spreadsheets (e.g. Excel)	48.9	24.5	19.1	7.5

Just over 5% of teachers reported using all six resources at least once a week, and a further 24.5% of teachers reported using four or five of them with this frequency. These 29.7% of teachers may be regarded as *high users of ICT* during mathematics classes. At the other extreme, 6.0% of teachers indicated that they never or hardly ever used any of the resources shown in Table 5.4. A further 7.3% hardly ever or never used four or five of these resources, and these 13.3% may be regarded as *low users of ICT* during mathematics classes. Other teachers were categorised as medium ICT users.

There are substantial differences between the usage of ICTs by teachers in initial schools and other schools (Figure 5.2): 49.5% of teachers in initial schools were high users of ICTs, compared with 28.9% of teachers in other schools. Teachers in initial schools were more likely to report using each form of ICT at least once a week. In particular, teachers in initial schools were more likely to report using mathematics-specific software at least once a week than teachers in other schools (42.5% vs. 24.2%); they were more likely to report using general software at least once a week (50.3% vs. 36.7%). Use of spreadsheets was quite low in both groups, however (Figure 5.3).

Figure 5.2. Percentages of low, medium and high users of ICTs during mathematics classes: Teachers in initial schools and other schools

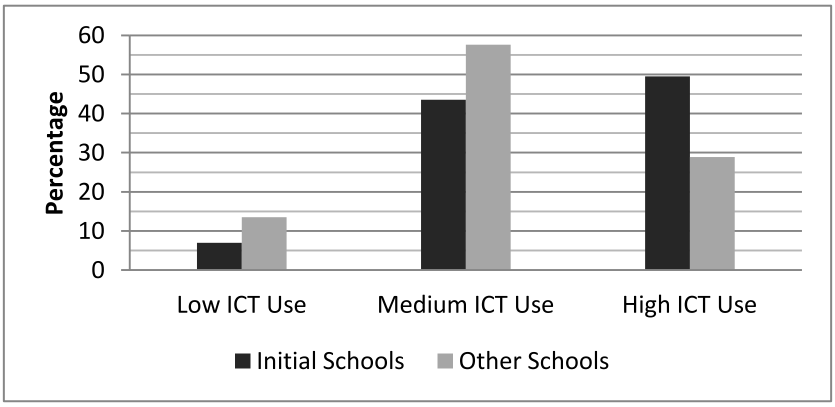
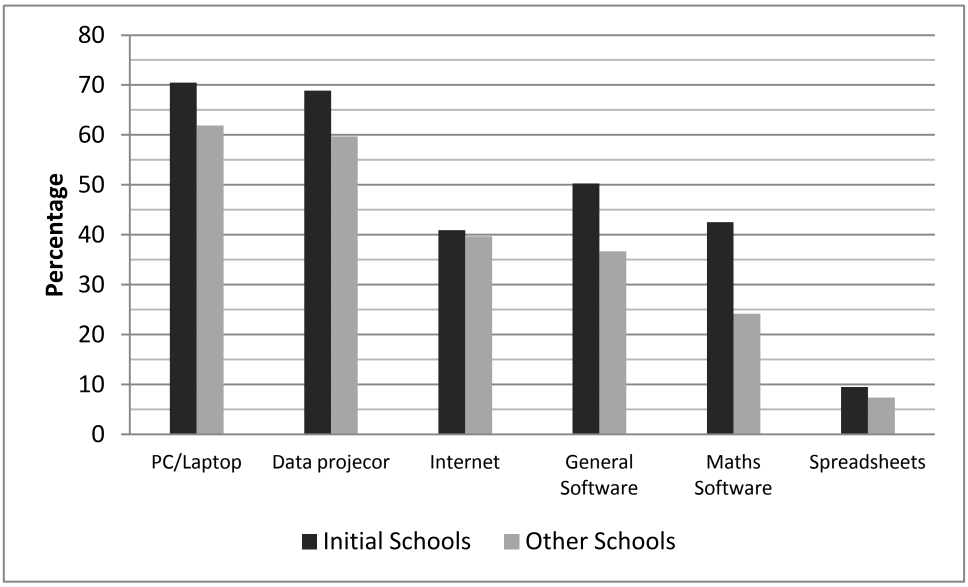


Figure 5.3. Percentages of teachers in initial schools and other schools who report using various ICTs at least once a week during mathematics classes



5.4. Ability Grouping for Mathematics

5.4.1. Teachers' Views on Ability Grouping

Teachers were asked about the extent to which they agreed or disagreed with 12 statements concerning ability grouping for mathematics classes at junior cycle level. When answering these questions, teachers were provided with the following definition:

Class-based ability grouping refers to the allocation of students of differing ability levels to different class groups for mathematics. This may be done on the basis of a standardised test, base class, or some other means, and generally reflects school policy.

Their responses are shown in Table 5.5 (on eight items that indicate an endorsement of ability grouping into different class groups) and Table 5.6 (on four items that suggest that ability grouping can have negative effects on some students). Note that we did not ask questions about the timing of class formation, i.e. when ability groupings are made.

There were very high rates of agreement with four of the statements in Table 5.5 (with over 75% of teachers agreeing or strongly agreeing). These were:

- 'Allocating students to mathematics classes based on some measure of academic ability is, overall, a good practice'
- 'Class-based ability grouping for mathematics facilitates a more focused teaching approach'
- 'Class-based ability grouping for mathematics accelerates the pace of learning for all students', and
- 'The best way to teach the mathematics curriculum effectively is in class-based ability grouped settings'.

Consistent with this, there was somewhat less widespread agreement with the statement 'Mixed-ability teaching in mathematics 'drags down' the performance of higher achievers' (64.0% agreed or strongly agreed). There were also mixed views on the statement that 'Mixed-ability teaching in mathematics is beneficial to lower-achieving students' (46.5% agreed or strongly agreed). A minority of teachers agreed with the remaining two statements: 'It is possible to teach the mathematics curriculum in mixed-ability settings without compromising on the quality of learning' (30.4%) and a negatively worded statement, 'Class-based ability grouping is not particularly beneficial for teaching and learning mathematics' (11.9%).

Table 5.5. Teachers' levels of agreement/disagreement with eight statements indicating support for ability grouping in the teaching and learning of mathematics at junior cycle level

Statement	Strongly disagree	Disagree	Agree	Strongly agree
Allocating students to mathematics classes based on some measure of academic ability is, overall, a good practice	0.4	4.6	56.2	38.8
Class-based ability grouping for mathematics facilitates a more focused teaching approach	0.6	5.1	58.1	36.1
Class-based ability grouping for mathematics accelerates the pace of learning for all students	1.2	21.6	53.8	23.4
Class-based ability grouping is not particularly beneficial for teaching and learning mathematics*	28.1	60.1	10.0	1.9
Mixed-ability teaching in mathematics is beneficial to lower-achieving students	10.7	42.9	41.1	5.4
Mixed-ability teaching in mathematics 'drags down' the performance of higher achievers	4.1	31.9	48.4	15.6
It is possible to teach the mathematics curriculum in mixed-ability settings without compromising on the quality of learning	18.0	51.7	27.9	2.5
The best way to teach the mathematics curriculum effectively is in class-based ability grouped settings	2.0	14.7	55.2	28.2

*This statement is negatively worded, meaning that higher agreement is indicative of lower endorsement of ability grouping.

Table 5.6 indicates that 55.5% of teachers agreed or strongly agreed that ability grouping can have a negative impact on some students' self-esteem, which indicates that there is awareness that, despite widespread support for ability grouping in general (previous table), it can be damaging in some specific respects. About two-fifths of teachers (38.7%) agreed or strongly agreed that ability grouping was more beneficial for higher achievers than for lower achievers, which again indicates some awareness of the potential differential effectiveness of this practice. About three in ten teachers agreed with the remaining two statements in Table 5.6 ('Class-based ability grouping for mathematics slows the pace of learning of lower-achieving students' - 28.2%; and 'Class-based ability grouping results in lower expectations by teachers of the mathematical abilities of lower-achieving students'- 31.2%).

Table 5.6. Teachers' levels of agreement/disagreement with four statements indicating an awareness of the negative effects of ability grouping on some students in the teaching and learning of mathematics at junior cycle level

Statement	Strongly disagree	Disagree	Agree	Strongly agree
Class-based ability grouping for mathematics has a negative impact on some students' self-esteem	5.5	39.0	47.8	7.7
Class-based ability grouping for mathematics slows the pace of learning of lower-achieving students	12.3	59.5	24.4	3.8
Class-based ability grouping results in lower expectations by teachers of the mathematical abilities of lower-achieving students	14.0	54.8	26.6	4.6
Class-based ability grouping for mathematics benefits higher-achieving students more than lower-achieving students	11.3	49.9	25.8	12.9

These response patterns suggest that, although a vast majority of teachers support ability grouping in mathematics *in general* (e.g. with 95.0% agreement with the first statement in Table 5.5), there is less widespread consensus on the practice of ability grouping with respect to effects for *specific groups,* i.e. low and high achievers.

It should be borne in mind that these questions asked about teachers' views on ability grouping for mathematics in a general sense; their views may vary depending on the year level or topic being taught. Also, we did not ask teachers for their views on the relationship they may perceive between the structure of the mathematics syllabus and examinations on the one hand, and the need to group students by ability for mathematics on the other.

The items shown in Tables 5.5 and 5.6 were used to form two scales, each of which has an overall mean of 0 and standard deviation of 1[23]. The first scale can be interpreted as *support for ability grouping*, while the second is a measure of *awareness of the potential negative effects of ability grouping*, particularly with respect to low achievers.

Means on these scales did not differ between teachers in initial and other schools; nor did they differ by DEIS status or school sector/gender composition. However, female teachers had a significantly higher mean score (0.06) than male teachers (-0.11) on the scale measuring support for ability grouping. Female teachers also had a significantly lower mean score (-0.06) than male teachers (0.10) on the scale measuring awareness of the potential negative effects of ability grouping[24].

5.4.2. Views on Ability Grouping and Schools' Practices on Ability Grouping

In Chapter 4 (Section 4.4), we described the extent to which classes were grouped by ability for mathematics in the schools that participated in PISA 2012. Ability grouping for mathematics is very common after First Year: for example, 81% of mathematics co-ordinators reported that mathematics classes for Second Years were grouped by ability, and this rose to 93% in Third Year.

Table 5.7 compares the means on the two scales indicating *support for ability grouping* and *awareness of the potential negative effects of ability grouping* for teachers in schools which do and do not group students by ability for mathematics at each year level. Mean scores on the *support for ability grouping* scale tend to be lower for teachers in schools where mathematics classes are *not* grouped by ability, and this is statistically significant with respect to Second, Third and Transition Years. The differences at Fifth and Sixth Year levels are not significant. This arises because the standard errors at these class levels are large, mainly due to the small numbers of schools that do not practise ability grouping at these year levels.

Mean scores on the *awareness of potential negatives of ability grouping* scale are higher for teachers in schools where ability grouping for mathematics is *not* practiced. This is significant only at Second and Transition Year levels, however, again due to large standard errors.

Overall, these results suggest that school-level policy and practice on ability grouping may influence teachers' own views on ability grouping.

[23] The fourth item in Table 5.5 was reverse coded for this analysis. See the Technical Appendix for information on how the scales were constructed.
[24] In both cases, $p < .05$ but $> .01$. For details on how comparisons of means were made, see the Technical Appendix.

Table 5.7. Scale means (support for ability grouping and potential negatives of ability grouping) of teachers in schools that group and do not group students by ability for mathematics, First to Sixth Years

Year level	% of teachers in schools with grouping	Support for ability grouping				Potential negatives of ability grouping			
		Maths grouped		Maths not grouped		Maths grouped		Maths not grouped	
		Mean	SE	Mean	SE	Mean	SE	Mean	SE
First year	14.8	0.133	0.095	-0.024	0.039	-0.080	0.073	0.027	0.040
Second year	83.5	0.054	0.036	-0.286	0.092	-0.016	0.039	0.154	0.079
Third year	94.1	0.021	0.036	-0.381	0.203	-0.002	0.035	0.245	0.170
Transition year	45.7	0.119	0.054	-0.068	0.056	-0.081	0.048	0.071	0.056
Fifth year	96.3	0.010	0.038	-0.288	0.314	0.009	0.037	0.076	0.234
Sixth year	96.6	0.013	0.037	-0.390	0.309	0.008	0.037	0.107	0.212

Note. Teachers are missing 6.7% of data on the questions on ability grouping for mathematics, 11.1% of data on the support for ability grouping scale, and 6.3% on the potential negatives of ability grouping scale. Cells marked in bold indicate a significant difference (p < .05).

5.5. Use of Differentiated Teaching Practices

Teachers were asked how they provide different teaching and learning experiences for students of differing ability levels *within* their Third Year mathematics classes. Responses are shown in Table 5.8. In interpreting these, it should be noted that class groups may already reflect ability grouping between classes, and hence, there may be more limited opportunity for differentiated approaches. Two-thirds of teachers (65.5%) indicated that they taught Third Years at the time of completing the questionnaire, and the responses shown in Table 5.8 are based on these teachers only.

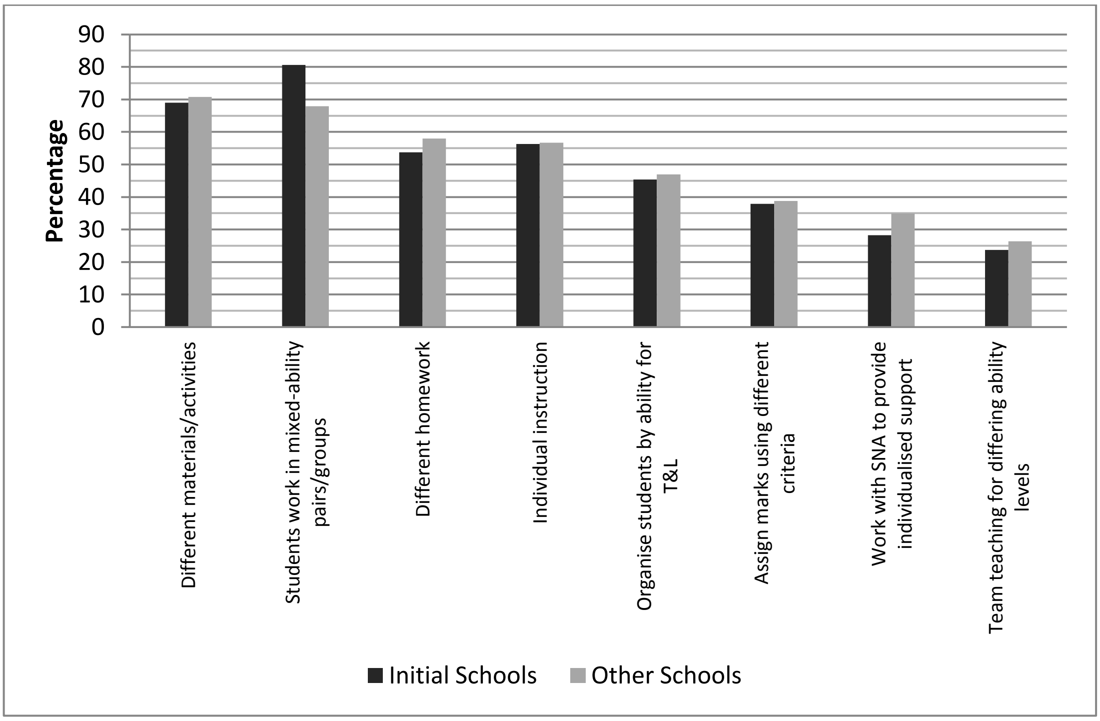
The four most commonly-used strategies (with 55-70% of teachers reporting using these sometimes or often) were providing different class materials or activities, having students work in mixed-ability pairs or groups, providing different homework tasks, and providing planned or structured (one-to-one) instruction. Team teaching was used considerably less frequently (with 61.1% never using this), as was working with a Special Needs Assistant (54.4% reported never using this). The use of these latter two approaches may be partly related to the availability of other staff to support their implementation. The remaining two strategies, organising students by ability for teaching and learning, and assigning grades on the basis of differing criteria, were used with moderate frequency.

A comparison of the extent to which teachers in initial schools and other schools used each of the strategies listed in Table 5.8 indicates that, in general, teachers use these practices with similar levels of frequency. However, there are two exceptions. Figure 5.4 shows that teachers in initial schools were more likely to report having students work in mixed-ability groups or pairs sometimes or often (81.0%) when compared to teachers in other schools (66.6%). Also, teachers in other schools were more likely to report working with an SNA sometimes or often (34.2%) than teachers in initial schools (26.0%).

Table 5.8. Frequency with which teachers use differentiated teaching and learning approaches within their Third Year mathematics classes: All teachers of Third Years

Strategy	Never	Rarely	Sometimes	Often
I provide different class materials or activities to students of differing ability levels	9.2	21.8	50.6	18.3
I get students to work in mixed-ability pairs or small mixed-ability groups	12.3	20.5	45.8	21.4
I assign different homework tasks to students of differing ability levels	14.2	29.1	37.6	19.0
I provide planned or structured individual (one-to-one) instruction that is embedded into whole-class teaching	20.9	22.9	35.7	20.4
Within a class group, I organise students by ability for teaching and learning activities	24.8	30.2	34.3	10.7
I assign grades or marks for homework, assessments or project work on the basis of differing criteria	27.0	35.8	25.9	11.2
I work with a Special Needs Assistant to provide individualised support during my mathematics class(es)	54.4	11.8	18.2	15.7
I participate in team teaching that caters for differing ability levels	61.1	14.7	15.4	8.8

Figure 5.4. Percentages of teachers using differentiated teaching and learning approaches 'sometimes' or 'often' within their Third Year mathematics classes: Teachers in initial schools and other schools



An additional 24.2% of teachers indicated that they used a strategy other than that listed in Table 5.8. However, only 8.8% or 116 teachers described these practices in written comments. These were subjected to content analysis, whereby comments addressing similar themes or topics were grouped into specific categories. The categorisation of comments was conducted initially by one researcher, and subsequently validated by a second researcher. In some cases, teachers' comments were subdivided if they fell under different categories. In total, 157 comments (or 1.35 comments per teacher) were identified for analysis. Of the 157 comments, 16.6% were from teachers in initial schools, and 83.4% from teachers in other schools. About one-third of comments (35.0%) were deemed not to concern differentiated teaching and learning strategies specifically, while the remaining 65.0% did. Table 5.9 shows the distribution of the themes for the sample overall (as a percentage of teachers who made comments), and by teachers in initial schools and other schools. Data in Table 5.9 are unweighted and should be interpreted in a broad, general sense.

Table 5.9. Types of strategies for differentiated teaching within classes identified in teachers' comments: All teachers, and teachers in initial schools and other schools

Theme	All Responses	Initial Schools	Other Schools
Peer-to-peer activities	22.9	23.1	22.9
Differentiation by task/teaching strategy	19.7	15.4	20.6
Use of tools and resources*	14.6	19.2	13.7
Extra classes or time/Withdrawal for extra support*	10.8	11.5	10.7
Use of practical materials/real life examples*	8.3	11.5	7.6
Student-led teaching and learning	6.4	3.8	6.9
One-to-one work: teacher and student	5.7	7.7	5.3
Differentiation by outcome	4.5	3.8	4.6
General/Other comments	7.0	3.8	7.6

Note. Data are unweighted. Frequencies are based on 8.8% of the entire teacher sample, i.e. only those teachers (n=116) who made written comments on this question.
*Some of the comments in these categories did not refer explicitly refer to differentiated teaching strategies.

It should be noted that two of the categories identified in the comments made by teachers overlap substantially with the 'closed' parts of this question (comparing Tables 5.8 and 5.9): i.e. *peer-to-peer activities* is similar to the fifth item in the set *(I get students to work in mixed-ability pairs or small mixed-ability groups),* while *one-to-one work* is similar to the eighth item: *(I provide planned or structured individual (one-to-one) instruction that is embedded into whole-class teaching).*

The most commonly-occurring category was *peer-to-peer activities* (almost 23% of all comments). Responses in this category referred to mixed-ability work in pairs or small groups, co-operative and collaborative learning, and peer learning and assessment. Typical teachers' comments in this category included the following:

Let students work in twos regardless of ability and let them help each other and explain how they thought a solution could be achieved.

I have used co-operative learning with small class groups.

About one-fifth of comments concerned *differentiation by task/teaching strategy.* These referred to setting students different tasks based on ability, and employing or modifying teaching strategies to encompass the range of abilities/interests in the class. Two examples of this category are shown below.

Have different targets/levels for students to reach, i.e. classwork/homework. Higher achieving do only one/two of easier questions and select more of the challenging questions. Weaker students get all the first (easier) questions.

If there is more than one method to teach a solution to a problem I demonstrate these methods to students, being conscious of the fact that students learn in different ways.

About 15% of comments referred to the use of various *tools and resources*. Teachers in initial schools made comments in this category slightly more frequently than teachers in other schools (19.2% vs. 13.7%). The responses in this category mentioned ICT-based or other resources that they used in their teaching, and references to these tended to be fairly specific, but not necessarily related to differentiated teaching. Examples include:

The use of ICT - students practice charts in Excel, students use Geogebra, students access materials from www.projectmaths.ie.

We talk through examples in PowerPoint which students take down into their notes. They use these as a guide to help them with more difficult questions for homework.

The fourth theme that was identified in teachers' comments concerned the provision of *additional support* to students, either through extra time outside of normal mathematics classes, or withdrawal of some students for more individualised instruction during mathematics classes. Examples include:

Individual small groups during lunchtime or other 'non-pressurised' times of the day, e.g. we eat and learn and get through a lot of work.

Low ability students are withdrawn by resource teacher to work on basic concepts of a topic when more higher-order material is being covered in mixed-ability setting.

About 8% of comments referred to the use of *practical materials or real-life examples*. Again, these were not necessarily related to differentiated teaching practices. Two examples from this category are shown below.

Use concrete materials as much as possible in the classroom.

Practical exercises: real life mathematics outside of classroom and inside classroom.

Several teachers (6.4%) made comments that referred to s*tudent-led teaching and learning activities*, such as *I allow students who understand a topic to teach others in the class during their work.* There is some overlap thematically between this category and *peer-to-peer activities*, referred to above. A similar percentage of teachers referred to individual work with students during class time; typically these would be students that the teacher perceives to be struggling with the material, e.g. *If a student has a difficulty with a concept (when the rest of the class is busy) I give her some one-to-one help and try to present the concept in a different way.* A small number of teachers referred to *differentiation by outcome*, e.g. *Demand different standards of homework within range of group.* Finally, 7% of comments were of a very general nature and/or didn't easily lend themselves to classification under the other categories.

5.6. Key Findings and Conclusions

Teachers in our study strongly endorsed items that are consistent with constructivist views on teaching and learning mathematics. For example, 80% or more of teachers agreed that there are different ways to solve most problems, that more than one representation should be used in teaching a mathematics topic, and that it is important to understand how mathematics is used in the real world. On the other hand, 88% agreed that some students have a natural talent for mathematics, while others do not, although only one-third of teachers agreed that mathematics is a difficult subject for most students.

We asked teachers about their views on ability grouping for mathematics; that is, the practice of grouping students into separate class groups on the basis of ability. The set of items was designed to tap two (not necessarily mutually exclusive) views – one supporting ability grouping, the other indicative of an awareness of the negatives of ability grouping for some students, particularly those of lower ability. There was high overall support for the practice of ability grouping. For example, 83% agreed with the statement 'The best way to teach the mathematics curriculum effectively is in class-based ability grouped settings'. On the other hand, there was awareness of the potential negatives of this practice: for example, 39% agreed that 'Class-based ability grouping for mathematics benefits higher-achieving students more than lower-achieving students'.

Teachers' views on ability grouping did not vary appreciably across school sector/gender composition, DEIS/SSP status, and initial school status. Small, though statistically significant, differences in the views of male and female teachers were apparent. Views did, however, vary substantially depending on whether teachers were in a school that grouped students by ability for their mathematics classes or not. For example, there was a difference of a third of a standard deviation on the scale measuring support for ability grouping for mathematics between teachers in schools where Second Year students were grouped for mathematics classes, compared to teachers in schools that did not group their Second Years. These findings suggest that school-level policy on ability grouping may have a direct effect on teachers' own views on this practice (or vice versa), more so than the other characteristics considered.

In considering the results relating to general views on mathematics and ability grouping for mathematics, it is useful to bear the overarching context of the Junior and Leaving Certificate examinations in mind. For example, teachers may indicate that they agree with constructivist approaches to teaching mathematics, but this will not necessarily translate into practice; similarly, views on ability grouping for mathematics are influenced by what material is to be covered in class and how it is to be examined or assessed. High levels of support for constructivist approaches coupled with low reported usage of such approaches were also found in TALIS (Gilleece et al., 2009).

Teachers were asked about their use of differentiated teaching practices *within* mathematics classes. The four most commonly-used strategies were providing different materials/activities, having students work in mixed-ability pairs/groups, providing different homework tasks, and structured individual instruction. Team teaching and working with a Special Needs Assistant were used considerably less frequently (perhaps because these are contingent on staff availability, and in the case of the latter, on whether there were students in the class with special educational needs). Teachers in initial schools reported having students work in mixed-ability pairs/groups somewhat more frequently than teachers in other schools, which is a positive finding, since research points to the benefits of these kinds of approaches (see Smyth & McCoy, 2011).

About 9% of teachers wrote down additional differentiated teaching strategies that they used. The most common of these were peer-to-peer activities (e.g. co-operative learning, paired learning tasks)/student-led teaching and learning; differentiation by task or teaching strategy; the use of tools and resources to support differentiated teaching (including practical materials and real-life examples); and extra classes or time allocated to struggling students.

In preparing for their junior cycle mathematics classes, teachers reported relatively high usage of sample examination papers, syllabus documents, and textbooks, though students' interests were also frequently taken into account. There was somewhat less frequent use of sample examination papers and textbooks by teachers in initial schools than by teachers in other schools. Teachers in the initial Project Maths schools reported greater use of syllabus documents, CPD, other teachers in their school, websites and inspection reports than teachers in other schools. These results, overall, are consistent with results of the OECD's TALIS survey, in which relatively low incidences of exchange, co-ordination and collaboration between teachers were found (Gilleece et al., 2009). They also point to the dominant influence of the examinations (e.g. with more reliance on sample or past examination papers than on CPD and mathematics education websites).

There were large differences between teachers in initial schools and other schools in the extent to which ICTs were incorporated into mathematics classes. In particular, teachers in initial schools reported using both general and mathematics-specific software in class at least once a week to a greater extent than teachers in other schools. This is a positive finding in that one can infer that the increased use of ICTs by teachers in initial schools is occurring as a direct result of the CPD that they received (see Chapter 2); however, without information on the relative effectiveness of various types of ICT usage, caution should be exercised in drawing any general conclusions about this finding. This finding also points to discrepancies between teaching and learning activities and modes and methods of classroom assessment on one hand, and the structure and format of the Junior Certificate mathematics examination on the other.

6. Teachers' Views on Project Maths at Junior Cycle

Teachers of junior cycle were asked to complete a section in the questionnaire concerning Project Maths. Sections 6.1 to 6.4 in this chapter concern only the teachers who were teaching junior cycle mathematics at the time of PISA 2012, i.e. 88.8% of all teachers surveyed. Section 6.5, which examines comments made by teachers on Project Maths, includes all teachers who wrote comments, whether they taught junior cycle or not at the time of the survey. In considering the results presented in this chapter, it may be borne in mind that the findings are based on teachers' reports. The views of other stakeholders, particularly students, would provide a more complete picture on the implementation of Project Maths. In interpreting the results in this chapter, it should be borne in mind that the manner in which Project Maths was introduced set a challenging context (see Chapter 2), and this will come to bear on any appraisal of the initiative.

6.1. General Views on the Implementation of Project Maths

About half (50.2%) of respondents indicated that they had been teaching Project Maths at junior cycle; 45.3% for two years, and a small minority (4.6%) for longer than two years. Teachers were asked to indicate, overall, whether or not they agreed that Project Maths was having a positive impact on students' learning of mathematics (Table 6.1). What is striking about the results is that close to half of teachers (47.5%) indicated that they did not know if Project Maths was having a positive impact. This indicates, not unexpectedly, that it is too early in the implementation of Project Maths for teachers to have an informed opinion[25].

Slightly fewer teachers disagreed (22.8%) than agreed (29.7%) with the statement. A comparison of the responses of teachers in initial and other schools indicates that more teachers in initial schools were inclined to agree with the statement, and fewer teachers in initial schools indicated that they didn't know.

Table 6.1. Responses of teachers to the statement 'Overall, Project Maths is having a positive impact on students' learning of mathematics': All teachers, and teachers in initial and other schools

	All	Initial Schools	Other Schools
Strongly disagree	7.5	4.0	7.6
Disagree	15.3	12.5	15.4
Don't know	47.5	38.4	48.0
Agree	23.3	35.0	22.7
Strongly agree	6.4	10.1	6.3

Note. 8.4% of respondents were missing data on this question.

Teachers were asked to respond to a series of 19 statements on specific aspects of Project Maths. Table 6.2 shows their overall levels of agreement/disagreement. There is considerable variation in the responses, although there was only one statement with which a majority of teachers disagreed. This was 'Introducing the syllabus strands in three phases was a good idea' (62.2% disagreed or strongly disagreed).

[25] In Chapter 1, it was noted that first examination of all five strands of the revised curriculum does not take place until 2014 for the Leaving Certificate and 2015 for the Junior Certificate.

Table 6.2. Teachers' levels of agreement with 19 general statements on Project Maths (junior cycle only)

Statement	Strongly disagree	Disagree	Don't know	Agree	Strongly agree
The professional development workshops were useful to me	2.2	7.6	7.9	56.1	26.2
I find the www.projectmaths.ie website useful	1.9	7.0	6.9	64.6	19.5
The Common Introductory Course for First Year is a good idea	1.6	5.4	12.7	61.3	19.0
When planning mathematics lessons I use the syllabus published by the NCCA/DES	2.0	10.7	4.2	64.4	18.7
My students now have to do more 'thinking' in mathematics class	1.5	11.3	11	57.8	18.3
I now use a greater range of teaching and learning resources in my mathematics classes	1.5	11.8	5.8	63.3	17.7
The Bridging Framework to promote continuity between primary and post primary is a good idea	1.3	2.1	24	56.2	16.5
In my classroom I now encourage a greater level of discussion about mathematics	1.5	14.1	6.5	62.2	15.7
I find the www.ncca.ie/projectmaths website useful	2.2	11.6	16.0	60.2	10.0
The syllabus learning outcomes are clear	4.0	19.6	10.2	57.5	8.7
I find the new geometry course for post-primary schools useful	2.0	11.8	24.6	53.5	8.1
I find the NCCA student resource material for strand 1 useful	1.4	9.6	24.9	57.1	7.1
I find the NCCA student resource material for strand 2 useful	1.4	9.2	26.5	55.8	7.0
The in-school support for implementing the syllabus changes was adequate[26]	10.0	25.9	15.1	42.9	6.2
Support from the *Project Maths* development team (RDOs) was effective[27]	4.7	15.9	20.2	53.1	6.1
Introducing the syllabus strands in three phases was a good idea	38.5	23.6	10.4	22.5	4.9
The new textbooks support the *Project Maths* approach appropriately	13.3	32.1	14.6	38.0	2.1
Students welcomed the new approach to mathematics teaching and learning	7.6	30.8	31.2	28.4	1.9
Parents welcomed the new approach to mathematics teaching and learning	5.2	14.1	65.4	13.8	1.5

Note. 6.6% to 8.7% of respondents were missing data on these items.

Levels of agreement were high (in excess of 80%) for five of the statements, which covered the website at www.projectmaths.ie (84.2% agreed or strongly agreed that it was useful); using the syllabus in planning lessons (83.1% agreed that they used it); the usefulness of professional development workshops (82.3%); use of a greater range of resources in class (81.0%); and the view that the Common Introductory Course[28] in First year is a good idea (80.3%). Between 70% and 80% of teachers agreed with four further statements: that they now encourage a greater level of discussion in class (77.9% agreed or strongly agreed); that students now have to do more 'thinking'

[26] Only initial schools received in-school support from the PMDT.
[27] Only initial schools had a designated RDO.
[28] This is the minimum course to be covered by all students at the start of junior cycle (NCCA/DES, 2011a, Appendix).

in class (76.1%); that the Bridging Framework[29] is a good idea (72.6%); and that the website at www.ncca.ie/projectmaths is useful (70.2%).

Teachers were most inclined to express disagreement (with 20% or more disagreeing or strongly disagreeing) with the following five aspects of Project Maths: that it was a good idea to introduce the syllabus strands in three phases (62.2% disagreed or strongly disagreed); that the new textbooks support the Project Maths approach appropriately (45.3%); that students welcomed the new approach (38.5%); that the syllabus learning outcomes are clear (23.6%); and that support from the Project Maths development team was effective (20.6%).

Examining the extent to which teachers indicated that they didn't know (or didn't have an opinion on) the items in Table 6.2 can give an indication of aspects of Project Maths that may take longer to become established, or those with which teachers are less familiar. Teachers were particularly unsure whether or not parents welcomed the new approach to mathematics (65.4% indicated that they didn't know), and were also unsure if students welcomed it (31.2% didn't know). Further, around a quarter of teachers didn't know if they found the resource materials for Strands 1 and 2, and the new geometry course, useful. Also, although the level of agreement was high with the statement on the Bridging Framework, 24.0% of teachers indicated that they didn't know if this framework was useful. Similarly, 20.2% of respondents didn't know if support from the Project Maths team was useful.

Table 6.3 compares the responses of teachers in initial and other schools to the items shown in Table 6.2. The 'strongly agree' and 'agree' categories have been combined, as have the 'disagree' and 'strongly disagree' categories.

Levels of disagreement differed by more than 10 percentage points between the two groups of teachers on three of these items:

- teachers in other schools were more inclined to disagree that introducing the syllabus strands in three phases was a good idea (63.0% compared with 44.9%);
- teachers in initial schools were more inclined to disagree that parents welcomed the new approach (49.8% vs. 17.9%); and
- teachers in initial schools were more inclined to disagree that students welcomed the new approach (62.7% vs. 37.3%).

Teachers in initial schools tended to agree more than teachers in other schools that:

- the Common Introductory Course is a good idea (90.9% vs. 79.8%); and
- students now have to do more 'thinking' in class (86.7% compared with 75.6%).

On three items, teachers in other schools were more inclined than teachers in initial schools to respond that they didn't know: these were that

- parents welcomed the new approach (66.8% compared with 35.8%);
- students welcomed the new approach (31.9% vs. 17.8%); and
- the support from the Project Maths team was effective (20.6% compared with 10.0%).

[29] The Bridging Framework describes how content areas and concepts covered at 5th/6th class at primary level map onto the revised junior cycle syllabus (http://action.ncca.ie/curriculum-connections/bridging-documents.aspx).

Table 6.3. Levels of agreement with 19 general statements on Project Maths (junior cycle only): Teachers in initial schools and other schools

Statement	Initial Schools			Other Schools		
	Disagree/ Strongly disagree	Don't know	Agree/ Strongly Agree	Disagree/ Strongly disagree	Don't know	Agree/ Strongly Agree
The professional development workshops were useful to me	12.5	11.0	76.5	9.7	7.8	82.5
I find the www.projectmaths.ie website useful	12.7	11.5	75.9	8.7	6.7	84.6
The Common Introductory Course for First Year is a good idea	4.7	4.4	90.9	7.1	13.1	79.8
When planning mathematics lessons I use the syllabus published by the NCCA/DES	8.9	1.9	89.2	12.9	4.3	82.8
My students now have to do more 'thinking' in mathematics class	4.9	8.4	86.7	13.2	11.2	75.6
I now use a greater range of teaching and learning resources in my mathematics classes	6.9	9.7	83.3	13.6	5.6	80.9
The Bridging Framework to promote continuity between primary and post primary is a good idea	3.4	23.6	73.1	3.4	24.0	72.6
In my classroom I now encourage a greater level of discussion about mathematics	10.3	9.0	80.6	15.8	6.4	77.8
I find the www.ncca.ie/projectmaths website useful	20.3	18.9	60.8	13.5	15.9	70.6
The syllabus learning outcomes are clear	29.7	7.3	63.0	23.3	10.3	66.4
I find the new geometry course for post-primary schools useful	15.6	23.0	61.5	13.8	24.6	61.6
I find the NCCA student resource material for strand 1 useful	15.6	23.3	61.1	10.7	24.9	64.4
I find the NCCA student resource material for strand 2 useful	16.3	23.5	60.2	10.3	26.7	63.0
The in-school support for implementing the syllabus changes was adequate	42.4	10.6	47.0	35.5	15.4	49.2
Support from the *Project Maths* development team (RDOs) was effective	28.7	10.0	61.3	20.2	20.6	59.1
Introducing the syllabus strands in three phases was a good idea	44.9	11.6	43.5	63.0	10.4	26.6
The new textbooks support the *Project Maths* approach appropriately	38.2	13.5	48.3	45.7	14.6	39.7
Students welcomed the new approach to mathematics teaching and learning	62.7	17.8	19.5	37.3	31.9	30.8
Parents welcomed the new approach to mathematics teaching and learning	49.8	35.8	14.4	17.9	66.8	15.3

Note. 8.0% to 12.9% of teachers in other schools were missing responses on these items. Rates of missing data for teachers in initial schools were less than 5%.

6.2. Perceived Changes in Students' Learning

Teachers were asked to indicate, for a set of 17 statements relating to students' learning of mathematics, whether they perceived that there had been a change, ranging from a large negative one, to a large positive one, with the implementation of Project Maths. These responses were recoded as follows: large negative change: -2; moderate negative change: -1; no change: 0; moderate positive change: +1; and large positive change: +2. Thus, a negative score on an item signifies a perceived negative change, while a positive score signifies a perceived positive change. Scores at or close to zero indicate no perceived change.

Table 6.4 shows the mean scores on each of these items overall, and for teachers in initial schools and in other schools. The 'Diff' column shows the difference in the average rating or response on each item between teachers in initial schools and teachers in other schools. Differences that are statistically significant (p <.05) are shaded in grey[30].

Table 6.4. Perceived changes in 17 areas of learning:
All teachers, and teachers in initial schools and other schools (junior cycle only)

Area of Learning	All			Initial Schools			Other Schools			Diff (initial-other)
	Mean	SD	SE	Mean	SD	SE	Mean	SD	SE	
Students' understanding of key concepts in Statistics and Probability	0.718	0.572	0.023	0.798	0.510	0.066	0.713	0.576	0.024	0.085
Students' levels of awareness of the relevance of mathematical applications in other disciplines	0.608	0.575	0.019	0.715	0.467	0.058	0.602	0.580	0.020	0.113
Students' understanding of key concepts in Geometry and Trigonometry	0.565	0.596	0.023	0.589	0.650	0.072	0.564	0.593	0.024	0.025
Students' ability to solve real-life problems involving mathematics	0.546	0.631	0.021	0.614	0.620	0.045	0.542	0.632	0.021	0.073
Students' ability to work collaboratively in groups	0.520	0.595	0.023	0.689	0.507	0.053	0.511	0.598	0.024	**0.178**
Students' problem-solving strategies	0.494	0.627	0.022	0.537	0.696	0.045	0.492	0.623	0.023	0.045
Students' ability to explain how they solved mathematics problems	0.476	0.672	0.026	0.625	0.615	0.063	0.468	0.674	0.028	**0.157**
Students' ability to try different strategies to solve a problem	0.473	0.622	0.027	0.704	0.550	0.065	0.460	0.623	0.028	**0.244**
Students' grasp of fundamental mathematical concepts and principles	0.437	0.639	0.023	0.654	0.581	0.051	0.426	0.640	0.024	**0.228**
Sense of challenge experienced by higher-achieving students	0.367	0.718	0.029	0.573	0.644	0.039	0.356	0.720	0.030	**0.216**
Students' understanding of the vocabulary and language of mathematics	0.358	0.879	0.032	0.395	0.914	0.115	0.356	0.878	0.034	0.039
Students' interest in mathematics	0.338	0.616	0.023	0.366	0.611	0.110	0.336	0.617	0.024	0.029
Students' ability to work independently in mathematics classes	0.321	0.615	0.022	0.318	0.607	0.086	0.321	0.616	0.023	-0.003
Students' confidence in their mathematics skills	0.280	0.670	0.028	0.124	0.817	0.164	0.288	0.660	0.029	-0.165
Performance of students in class tests you have administered	0.260	0.629	0.024	0.229	0.721	0.114	0.262	0.624	0.025	-0.032
Conceptual learning experienced by lower-achieving students	0.250	0.730	0.030	0.292	0.778	0.109	0.248	0.728	0.031	0.044
Students' ability to persist when they have difficulty solving a problem	0.191	0.651	0.026	0.244	0.648	0.055	0.188	0.652	0.027	0.056

Note. 13.8% to 16.8% of teachers were missing responses on these items. In the 'Diff' column, values in shaded in grey are statistically significant (p < .05).

[30] The standard errors have been corrected for sampling error using the replicate weights, as described in the Technical Appendix.

Across all teachers (first column of data), there has been a perceived positive overall change on all aspects of students' learning considered in this question. The largest positive overall changes were associated with students' understanding of key concepts in statistics and probability (0.72); awareness of the relevance of mathematics in other disciplines (0.61); understanding of key concepts in geometry and trigonometry (0.57); ability to solve real-life problems (0.55); ability to work collaboratively in groups (0.52); and problem-solving strategies (0.49). The smallest (positive) changes are associated with students' confidence in their mathematics skills (0.28); performance on class tests (0.26); conceptual learning experienced by lower achievers (0.25); and ability to persist when having difficulty in solving a problem (0.19).

Generally, teachers in initial schools reported larger positive changes than teachers in other schools. These differences are statistically significant on five of the 17 items: collaborative group work, students explaining how they solved a problem, students trying different strategies, their grasp of fundamental concepts and principles, and the sense of challenge experienced by higher achievers.

6.3. Levels of Confidence in Teaching Aspects of Project Maths

Teachers were asked to rate their levels of confidence in teaching 14 aspects of Project Maths at junior cycle. Responses of teachers overall are shown in Table 6.5.

Table 6.5. Levels of confidence in teaching 14 aspects of Project Maths: All teachers (junior cycle only)

Aspect of Project Maths	Not at all confident	Not too confident	Moderately confident	Very confident
Teaching statistics	0.8	4	39.1	56.1
Teaching geometry and trigonometry	0.6	6.7	41.8	50.9
Teaching probability	1.5	7.4	41.4	49.7
Providing feedback to students about their performance in mathematics	0.8	7.3	55.9	35.9
Teaching students to solve problems in real-life settings	1.2	6.9	59.1	32.9
Engaging students in practical mathematics activities	0.8	11	56.4	31.7
Assessing how students are performing in mathematics	2.2	12.7	54.8	30.3
Preparing students for the revised Junior Certificate mathematics examination	4.1	20.2	49.2	26.4
Catering for students of varying mathematical ability	1.5	15.1	59	24.4
Organising classes so that students can use concrete materials	2.1	18.2	55.9	23.8
Supporting students with learning difficulties in mathematics	2.6	21.3	54.6	21.5
Facilitating students' independence in problem solving/doing mathematics	1.9	16.7	61	20.4
Analysing students' problem-solving strategies	1.9	21.5	58.8	17.8
Engaging students in assessing their own progress/performance in mathematics	2.4	27.4	53.4	16.8

Note. 9.4% to 10.1% of teachers were missing responses on these items.

Overall, teachers indicated high levels of confidence in teaching the 14 aspects. Reported confidence levels were highest for teaching statistics, and geometry and trigonometry, with 50% or more indicating that they were very confident in teaching these topics. Also, between 30% and 50% of teachers reported being very confident in providing feedback to students on their performance, teaching students to solve problems in real-life settings, engaging students in practical mathematics, and assessing students' performance on mathematics.

However, a sizeable minority of teachers reported lower levels of confidence in teaching five of the 14 aspects listed: 20-30% indicated that they were not at all or not too confident in the following: organising classes to facilitate the use of concrete materials (20.3%), analysing students' problem-solving strategies (23.4%), supporting students with learning difficulties (23.9%), preparing students for the revised Junior Certificate examination (24.3%), and engaging students in assessing their own progress or performance (29.8%).

Table 6.6 compares the confidence levels reported by teachers in initial schools with teachers in other schools for the same items shown in the previous table. The categories 'not at all confident' and 'not too confident' have been collapsed, since very few teachers selected the 'not at all confident' category.

Table 6.6. Levels of confidence in teaching 14 aspects of Project Maths: Teachers in initial schools and other schools (junior cycle only)

Aspect of Project Maths	Initial Schools			Other Schools		
	Not at all/Not too confident	Moderately confident	Very confident	Not at all/Not too confident	Moderately confident	Very confident
Teaching statistics	2.0	37.6	60.3	5.0	39.1	55.9
Teaching probability	3.4	40.3	56.4	9.2	41.4	49.4
Teaching geometry and trigonometry	1.3	54.7	44.0	7.5	41.2	51.3
Engaging students in practical mathematics activities	16.4	43.3	40.3	11.7	57.1	31.3
Teaching students to solve problems in real-life settings	11.9	57.0	31.1	7.8	59.2	33.0
Providing feedback to students about their performance in mathematics	11.7	61.5	26.7	8.0	55.6	36.4
Organising classes so that students can use concrete materials	18.5	56.0	25.5	20.5	55.8	23.7
Assessing how students are performing in mathematics	23.3	54.9	21.8	14.4	54.8	30.8
Preparing students for the revised Junior Certificate mathematics examination	28.4	51.6	19.9	24.1	49.1	26.8
Facilitating students' independence in problem solving/doing mathematics	18.3	62.5	19.2	18.6	60.9	20.5
Catering for students of varying mathematical ability	18.7	66.8	14.6	16.5	58.6	24.9
Supporting students with learning difficulties in mathematics	31.2	55.9	12.8	23.5	54.6	21.9
Analysing students' problem-solving strategies	20.5	69.5	10.0	23.6	58.2	18.2
Engaging students in assessing their own progress/performance in mathematics	36.0	55.4	8.6	29.5	53.3	17.3

Note. 9.8% to 10.1% of teachers in other schools were missing responses on these items. Rates of missing data for teachers in initial schools were less than 5%.

Perhaps unexpectedly, more teachers in other schools reported being very confident in teaching eight of these aspects (i.e. with a difference of 5 percentage points or more) compared to teachers in initial schools. These included catering for students of varying ability, providing feedback, supporting students with learning difficulties, and assessing students. Teachers in initial schools reported being more confident than teachers in other schools on just two of the items (again with a difference of 5 percentage points or more), i.e. teaching probability and engaging students in practical activities.

6.4. Perceived Challenges in the Implementation of Project Maths

Teachers indicated the level of challenge for 12 aspects associated with the implementation of Project Maths in their schools. Results for all junior cycle teachers are shown in Table 6.7. Note that we cannot infer from these results which of these are perceived challenges that have been overcome, or that are possible to overcome, and those that might represent significant obstacles in implementation with no obvious solution. For example, while 41.7% cited availability of assessment materials, and 31.6% cited teaching materials, as major challenges, there are some resources at www.projectmaths.ie. Having said this, five of these aspects may be regarded as significant difficulties, since 40% or more of teachers indicated that they were a major challenge. These are the time required to prepare for classes and for group work/investigations, the staggered or phased implementation of Project Maths, the literacy demands of the new courses, the rate of implementation, and available assessment materials. In addition, 31.6% of teachers indicated that available teaching materials such as textbooks were a major challenge, while a similar percentage (29.0%) indicated that funds or resources were a major challenge.

Table 6.7. Perceived challenges in the implementation of Project Maths: All teachers (junior cycle only)

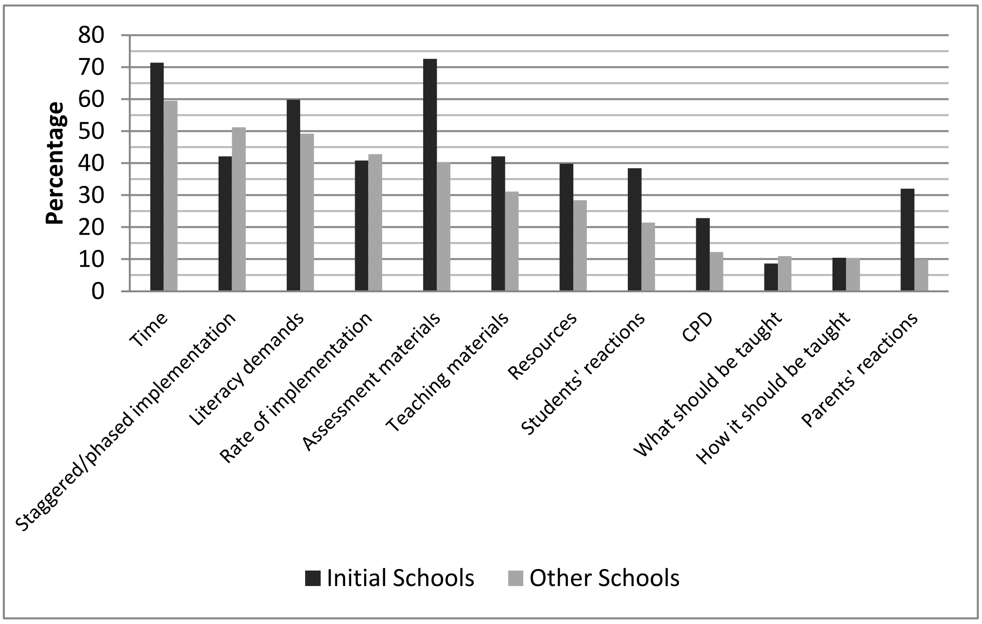
Perceived Challenge	A major challenge	A challenge	Not a challenge
Time, for example to become familiar with coursework, to prepare classes, for group work and investigations	59.8	36.7	3.4
The staggered/phased implementation of Project Maths	51.0	34.5	14.5
Literacy demands of the new courses	49.7	36.1	14.2
The rate of implementation of Project Maths	42.8	44.7	12.5
Assessment materials, for example sample examination papers and guidelines on assessing students' progress	41.7	39.7	18.6
Teaching materials, for example the content and range of textbooks available	31.6	45.8	22.6
Resources, for example funds to buy materials, facilities or equipment	29.0	42.3	28.8
Students' reactions to Project Maths	22.0	45.1	32.9
Continuing Professional Development, for example training opportunities available, content covered	12.6	45.6	41.8
Parents' reactions to Project Maths	10.7	36.2	53.1
How the Project Maths approach relates to my views on what mathematics content should be taught	10.6	39.3	50.1
How the Project Maths approach relates to my views on how mathematics should be taught	10.2	38.5	51.4

Note. 7.1% to 11.4% of teachers were missing responses on these items.

In contrast, four of the 12 aspects were not considered to represent significant difficulties, with 15% or fewer teachers indicating that CPD, parents' reactions to Project Maths, how Project Maths relates to teachers' beliefs on what should be taught, and how Project Maths relates to teachers' beliefs on how mathematics should be taught, were major challenges.

Figure 6.1 compares the percentages of teachers in initial schools and in other schools who indicated that each of the 12 aspects shown in Table 6.7 is, in their view, a major challenge. Responses diverge considerably between the two groups (by 10 percentage points or more) on eight of the items. In all eight cases, teachers in initial schools were more inclined than teachers in other schools to rate them as a major challenge. These were: the assessment materials available at the time of the survey (72.6% compared with 40.3%), parents' reactions (32.0% vs. 9.7%), students' reactions (38.4% vs. 21.2%), time available (71.4% compared with 59.3%), resources available (39.8% vs. 28.4%), teaching materials (42.1% compared with 31.1%), CPD available or attended (22.8% vs. 12.1%), and the literacy demands of the new courses (59.8% compared with 49.2%).

Figure 6.1. Percentages of teachers in initial schools and other schools indicating that each of 12 aspects of the implementation of Project Maths is 'a major challenge' (junior cycle only)



Across both groups, however, three aspects of the implementation of Project Maths emerged as significant challenges (appearing among the top four in terms of the percentages rated as being a major challenge) (Table 6.8). These were the time available, the phased implementation of Project Maths, and the literacy demands of the new courses. Also, both groups shared the view that the following four aspects of Project Maths posed less of a challenge in its implementation: parents' reactions, CPD available/attended, Project Maths in terms of their own views on what should be taught, and Project Maths in terms of their views on how it should be taught.

Table 6.8. Rankings of 12 aspects of the implementation of Project Maths as 'a major challenge': All teachers, and teachers in initial schools and other schools (junior cycle only)

Perceived Challenge	All %	All Rank	Initial Schools %	Initial Schools Rank	Other Schools %	Other Schools Rank
Time available	59.8	1	71.4	2	59.3	1
Phased implementation	51.0	2	42.5	4	51.4	2
Literacy demands	49.7	3	59.8	3	49.2	3
Rate of implementation	42.8	4	40.8	6	42.9	4
Assessment materials available	41.7	5	72.6	1	40.3	5
Teaching materials	31.6	6	42.1	5	31.1	6
Resources available	29.0	7	39.8	7	28.4	7
Students' reactions	22.0	8	38.4	8	21.2	8
CPD available/attended	12.6	9	22.8	10	12.1	9
Parents' reactions	10.7	10	32.0	9	9.7	12
Project Maths and what should be taught	10.6	11	8.6	12	10.7	10
Project Maths and how it should be taught	10.2	12	10.4	11	10.2	11

Note. 7.1% to 11.4% of teachers were missing responses on these items. Rates of missing data for teachers in initial schools ranged between 3.4% and 9.2%; 7.3% to 11.6% of teachers in other schools were missing responses on these items.

6.5. Teachers' Comments on Project Maths

6.5.1. Analysis of Comments

Teachers were provided with space in the questionnaire to make written comments about their experiences of/views on Project Maths. The question was pitched at a general level (i.e. *please make any further comments on Project Maths in your work as a teacher in the space below, if you wish*). Of all respondents, 34.5% wrote comments. About the same percentages of teachers in initial schools and other schools made written comments (35.7% and 34.7%, respectively).

Responses were split in cases where they pertained to more than one discrete theme or aspect of Project Maths. On average, teachers made 1.65 comments, yielding a total of 757 separate comments or pieces of text. Teachers in initial schools had a slightly higher average number of comments (1.78) than teachers in other schools (1.63).

Comments were subjected to a detailed content analysis, and classified along three dimensions:

1. Overall tone of the comment: positive, negative, or mixed[31].
2. Which part of the education system the comment referred to (junior cycle, senior cycle, or both).
3. The content of the comment itself: eight themes were identified. These are described in detail in the next section. Some of these themes overlap with one another. In addition, 5.6% of comments were classified under 'other' as they did not readily fit under the main themes.

The content analysis was conducted initially by one researcher, and then validated by a second. The content of the comment was discussed by the two researchers and re-classified in a small number of cases. Note that analyses in this section are not weighted. As such, results should be interpreted in a broad and general sense.

[31] Example of a general, positive comment: *I like the concept of Project Maths. I see how children learn from one another;* example of a general, negative comment: *Introducing this change on top of dealing with very large classes is ridiculous;* example of a general, mixed comment: *There is a good understanding of the concepts but it is difficult to prepare the for the junior cert exam.*

6.5.2. Main Themes Emerging

Table 6.8 shows the distribution of comments in terms of their overall tone and the level to which they pertained (junior cycle, senior cycle, or both). A large majority (87%) were negative in tone, and the percentages of negative comments were similar in initial and other schools. A further 8% were mixed in tone, and just 5% were positive. However, it is possible that teachers may have thought it more important to record reservations than to re-assert positive opinions, which other parts of the questionnaire gave them plenty of opportunities to express.

A majority of comments (81%) covered both junior and senior cycles. Teachers in other schools were slightly more inclined than teachers in initial schools to comment on senior cycle or junior cycle separately.

In interpreting the tone of the comments, and indeed their content, we were aware that there was a possibility that those teachers who were negatively disposed towards Project Maths may have been more inclined to make written comments, in which case we could not say that these comprised a representative sample of views. However, a comparison of teachers' agreement/disagreement with the statement 'overall, Project Maths is having a positive impact on students' learning of mathematics' with whether or not they made written comments, indicated that written comments were equally prevalent, whether teachers agreed, didn't know, or disagreed (Table 6.9). As such, written comments do not appear to have come predominantly from teachers who have negative overall views on Project Maths.

Table 6.8. Distribution of teachers' comments on Project Maths by tone and level: Overall, and from teachers in initial schools and other schools

Tone	All comments	Initial Schools comments	Other Schools comments
Positive	4.8	4.4	4.8
Negative	87.3	87.7	87.2
Mixed	8.0	7.9	8.0
Level			
Junior cycle	4.6	0.9	5.3
Senior cycle	14.3	7.9	15.5
Both	81.0	91.2	79.2

Note. Data are unweighted. Percentages are based on a total of 757 comments.

Table 6.9. Cross-tabulation of teachers' overall view on Project Maths with whether or not they made written comments on Project Maths

Comment	'Overall, Project Maths is having a positive impact on students' learning of mathematics'		
	Agree	Don't know	Disagree
No written comment	21.6	48.8	29.6
Written comment	24.9	45.4	29.7
All	22.8	47.5	29.7

Table 6.10 shows the distribution of teacher comments across the themes identified through the content analysis for the sample overall, and separately for teachers in initial schools and in other schools. In two of the eight themes, *syllabus and assessment* and *resources,* there are three sub-themes, since these seemed to reflect related but distinct aspects of the overarching theme.

To a large extent, the distribution of comments across content areas is similar for teachers in initial schools and teachers in other schools, with two exceptions. Other schools were more inclined to make comments on the *phased implementatio*n of Project Maths, while initial schools tended to comment more frequently on *examinations*. These differences can be related to the fact that the initial schools are ahead of the other schools in their experiences of Project Maths.

The four most commonly-occurring themes are *phased implementation, literacy and ability, syllabus,* and *time*. Each of the themes is described in the remainder of this section, with illustrative examples of comments made by the teachers who participated in PISA 2012. Comments are transcribed verbatim in order to retain as much of respondents' original intent as possible.

Table 6.10. Frequencies of teachers' comments on Project Maths by theme: All comments, and comments from teachers in initial schools and other schools

Theme	Sub-Theme	All		Initial Schools		Other Schools	
		%	Rank	%	Rank	%	Rank
Syllabus and Assessment	Syllabus	12.2	3	11.4	3	12.3	3
	Time	9.2	4	6.1	6	9.8	4
	Examinations	8.5	5	20.2	1	6.4	6
Resources	Textbooks	4.4	7	3.5	10	4.5	7
	Professional Development	3.6	8	5.3	7	3.3	8
	Resources in general	3.0	9	4.4	8	2.8	10
Phased implementation		26.8	1	19.3	2	28.2	1
Literacy and Ability		13.1	2	8.8	4	13.9	2
Methodology		2.8	10	1.8	11	3.0	9
Change		2.1	11	4.4	9	1.7	11
Communication		1.3	12	0.9	12	1.2	12
General Comments		7.5	6	7.9	5	7.5	5
Other		5.6	-	6.1	-	5.5	-

Note. Data are unweighted. Percentages are based on a total of 757 comments.

Theme 1 – Syllabus and Assessment

Theme 1a – Syllabus

One-eighth (12.2%) of teachers commented on aspects of the revised syllabus, and 92.4% of these comments were negative in tone. A number of teachers felt the course was too long, with too much content, and reported difficulty in being able to cover the syllabus[32]. Some teachers felt that statistics and probability posed a challenge for students, especially in senior cycle; others felt there was a reduction in level of difficulty in the revised curriculum compared to the one previously in place. This theme overlaps with the *examinations* theme (below) insofar as teachers felt more pressure to cover the entire course with choice removed from the examinations.

[32] It may be borne in mind that, at the time of the survey, most teachers were dealing with the implementation of part of the new syllabus, while maintaining part of the old syllabus (see Table 2.1, Chapter 2).

Examples of comments made on Syllabus:

If the goal was to provide time to allow teachers and students to explore topics in greater depth and detail then project maths will not succeed. The curriculum is too overloaded for this. Some topics have doubled in size. Teachers are intimidated by the amount of new material and methods recommended.

Project maths has some great ideas but unfortunately the department of education has fallen back to its love of content. There is too much content to allow teachers the luxury of exploring concepts/topics in the detail they may want to, especially at leaving cert level. Less content and more time for exploration would produce a more valuable course.

…the course is much longer for leaving cert now as there is a large amount of statistic/probability which is on new junior cert which was not [on the old junior cert.] for them.

There seems to be a reduction in level of difficulty in questions, need to keep a high standard.

Theme 1b – Time

About 9% of teachers' comments mentioned time being an issue for the successful implementation of Project Maths, and again these comments were mostly (92.9%) negative. From the comments received, it can be inferred that teachers were referring both to instructional time, and time outside of teaching hours. Many teachers who commented on time felt they did not have enough time to cover the course. Some teachers reported spending evenings and weekends doing extra work in order to prepare students. They also felt this extra work had resulted in other subjects suffering. These comments also relate to the *syllabus* theme and the view that there is not enough time to cover the amount of content in the new course. Teachers also reported that the lack of time limited the amount of group/practical work that could be implemented.

Examples of comments made on Time:

Time constraints make it more difficult to employ new teaching methodologies.

There is a major problem in Project Maths. Practical work and investigations take up a lot of time. Using it in the classroom can be time consuming also and is not always effective. Preparation for project maths takes up a lot of my time at night and I feel my other subjects have suffered as a result.

There is a serious problem with time. It's well and good devising these experiments but we have NO time to get them done if we want to get course finished. It's ridiculous.

…the course appears to be very long, explorative/investigative methods are 'ideal' but only if the course can be covered in time! I have concerns about this.

Theme 1c – Examinations

Comments that came under the theme of examinations (8.5% of all comments, 93.8% of which were negative in tone) covered the structure, content and layout of examination papers and marking schemes. Comments on examinations were more prevalent among teachers in initial schools compared with teachers in other schools. Teachers were generally unhappy with the removal of question choice from the examination papers. Some even felt it may discourage students from taking the examination at Higher level. Others commented that the removal of choice resulted in them being under too much pressure to cover the course and adequately prepare students. Some felt that the layout and structure of the sample papers and marking schemes lacked clarity. Teachers also voiced dissatisfaction with the lack of availability of sample papers and marking schemes, and were of the view that aspects of the examination (including the marking) were aiding the ‘dumbing down’ of maths. A few teachers noted a discrepancy between the problem-solving and group work approach of Project Maths and the prescribed nature of the Leaving Certificate examination.

Examples of comments made on Examinations:

I would question the notion of ‘no choice’ of leaving cert papers - this will discourage some students from pursuing higher level course, instead they will pick a perceived ‘easy subject’ with choice on paper.

While I agree with the aims of Project Math, the pupils find it very difficult to translate what they learn in class to what is asked in sample exam papers - the jump is too big and the language varies from book to book and paper to paper.

No exam papers for students to practise is going to be a major factor in June, and hence the results. The mock exams gave a marking scheme, giving 6/10 for even attempting a graph - crazy marking scheme, which won't be replicated in marking in June [2012] I'm sure?

<u>Theme 2 – Resources</u>

Theme 2a – Textbooks

A small percentage (4.4%) of comments made by teachers referred to textbooks, all of them negative in tone. Some teachers commented that the textbooks contained content unrelated to Project Maths and/or the syllabus, and/or being of poor quality; others indicated a lack of funds to purchase textbooks. It was also reported that there is a lack of textbooks for certain years and levels: for example, Foundation level; Third Year ‘honours’ level. A few teachers noted that the style of material covered by the textbooks differed to what was covered in CPD for Project Maths.

Examples of comments made on Textbooks:

From my observation schools continue to use ‘old’ text books which do not contain the new approaches to maths. When asked why they are using ‘old’ books the common reply is financial due to lack of money in the school to buy the new books. This is a particular shame especially for children with learning difficulties as most of these aren't even in colour! The weaker students lose out. The stronger students will get by.

The main problems I see are the text books do not have the style of material that was covered at the inservices from project Maths. This is misleading for students and teachers.

There’s no book for foundation level maths. Students in our school like the ‘This is how to do it’ approach.

Theme 2b – Professional Development

The majority of teachers who made comments on professional development (3.6% of all comments concerned this theme, 88.9% of them negative in tone) reported inservice training being of poor quality. However, the fact that a small number of teachers made comments to this effect indicates that teachers’ experience of inservice development was largely positive, and negative experiences were relatively isolated. Some teachers reported wanting more inservice and some felt that it did not adequately prepare them to teach Project Maths. Teachers suggested that inservice for junior and leaving cert should have been separated into different days. Teachers also commented on marking schemes being unavailable at the time of inservice. A small proportion of comments were positive in nature. Overall, these comments contrast with data shown in Table 6.2, where 82% of teachers agreed that they found the professional development workshops useful.

Examples of comments made on Professional Development:

The inservice was very poor. It did not prepare you for the course.

Some things done at inservice can be unrealistic to achieve in class.

Inservice was excellent. I have always used problem solving techniques in my teaching of maths - use of maths competencies and quizzes and fun type problems solving. If students enjoy the subject math then it is easy to learn.

In services should be split into two groups for teachers of junior cert and teachers of leaving cert maths. As a teacher of junior cert ord level maths not all inservices are relevant to me.

Theme 2c – Resources in General

A majority of comments under this theme (which comprised 3.0% of all comments, 78.3% negative in tone) concerned funds available to spend on resources. Some teachers commented that their school had limited resources whilst others reported having many resources available. A number of teachers made positive comments on the usefulness and quality of resources.

Examples of comments made on Resources in General:

Lack of resources available to me to use and how to use them means that I cannot try new things.

As a trainee teacher, I feel that project maths and the project maths web site, provides me with an abundance of highly useful resources. Without these resources, I would have a lot less confidence in my ability to teach maths.

Once the implementation process is over we (teachers) really need the Project Maths Development Team to continue their work. The materials they produce are excellent and regardless of teachers views on Project Maths everyone I have talked to comments on the quality, standard and usability of the resources.

<u>Theme 3 – Phased Implementation</u>

Over one-quarter (26.8%) of all comments referred to the *phased nature of the implementation of Project Maths*, with 92.1% of such comments being negative in tone. Comments on implementation were more prevalent among teachers in other schools (28.2%) compared with teachers in initial schools (19.3%).

Most teachers who made comments on implementation disagreed with Project Maths being implemented in a phased manner. Teachers viewed implementation in this way as being unfair on senior cycle students who may not have acquired the knowledge or skills needed for the new course during junior cycle. Teachers generally felt it would have been better to introduce Project Maths initially to first years (and implement it upwards from there).

Examples of comments made on Implementation:

The way in which project maths is being introduced is proving to be a major challenge. If it had been introduced in first year only it would have been more manageable as it would give the students the chance to use the terminology from the beginning.

It is difficult to implement at senior cycle when students have not had the grounding in certain topics at junior cycle. It would have been better to have started in 1st yr only.

The staggered nature of the implementation of Project Maths course is the single most challenging thing I have endured since beginning teaching. It's highly confusing for students and teachers!

Better if introduced for 1st years only. V. good for junior students.

Theme 4 – Literacy and Ability

About one in eight of the comments (13.1%) concerned literacy levels and differences in students' ability more generally. A high proportion (91.8%) of these comments was negative in tone. Some teachers expressed concerns about the use of language in the revised curriculum. Teachers felt that weaker students, students with special needs and non-national students were struggling with comprehension of the material and the wordy nature of some of the questions. They were of the view that Project Maths was a good approach for students of higher ability; however, they felt that some higher-ability mathematics students were now struggling as they also needed good literacy skills in order to read, understand and answer questions. Some teachers perceived a neglect of foundation level in the development of syllabus and CPD materials and resources.

Examples of comments made on Literacy and Ability:

The increased use of language is a big disadvantage, in theory it was a good idea but not in practice.

The language used when phrasing a question poses a major problem for students whose literacy skills would be weak, they can therefore not answer a question they are mathematically capable of doing! This is a major issue! It is something which needs to be addressed if students are to be examined.

Project maths is a good approach for students of good ability and middle of the road students'.

'Total neglect of foundation disgraceful.

Theme 5 – Methodology

These comments relate to teachers' views on the methodologies espoused in Project Maths (2.8% of all comments were under this theme and 61.9% of the comments were negative in tone). Some teachers reported that it was difficult to implement certain methodologies within time and other constraints; other teachers were happy with 'hands-on' approach; and some commented on difficulties in implementing the constructivist approach underlying Project Maths.

Examples of comments made on Methodology:

While the teaching methods advocated with the introduction of Project Maths are very good I frequently had myself reverting back to traditional methods because I'm under a great deal of pressure to get the course covered in the allocated time.

Overall I do think the problem solving aspect of project maths is a very positive step, mixing topics in questions, retaining set procedures to solve questions.

I think the lesson plans given by Project maths are very idealistic. I have found that students are not having the 'eureka' moment. I spent some time with my 1st years on the teaching and learning plan (constructivist approach) for multiplication of fractions which took 2-3 lessons and at the end the students just asked 'why did you not just give us the rule? multiply the top line and multiply the bottom line'. I will not give up but there is more time needed to allow for the discovery learning expected in Project Maths.

Theme 6 – Change

Teachers felt the implementation of Project Maths was a huge change all at once for both teachers and students, particularly those in senior cycle. Many comments under this theme (which comprised 2.1% of all comments, three-quarters of which were negative in tone) indicated that senior cycle students were finding it difficult to adapt to new ways of thinking and new methodologies. Positive comments indicated that teachers welcomed the possibilities offered through Project Maths.

Examples of comments made on Change:

I feel that Project Maths is a very positive development for maths in secondary school. I feel it will really challenge students to think about maths from a broader perspective and change the mentality that maths is one-dimensional - i.e. only one correct answer, method, interpretation. They will learn more also as they will have to be able to explain/defend their answers/method.

At senior cycle students are reluctant to change the way they approach maths and at times find it difficult to work within new methodologies.

I agree with the principles of project maths but the change of content, assessment and teaching methodologies all at once is radical.

Theme 7 – Communication

A small number of comments (1.3%, all negative in tone) concerned communication involving teachers, ranging from informational to consultative. Some of these comments suggested poor communication between teachers and others in the development of Project Maths. Others suggested a lack of information given to teachers and parents on the curriculum changes being made.

Examples of comments made on Communication:

I think parents (and students) are concerned at changes and a lack of information on the curriculum changes, and that little useful public information has been given.

Suggesting that schools were part of 'pilot project' and did little to listen to the comments and suggestions of teachers was a disgrace.

Theme 8 - General Comments

This 'theme' (though not strictly speaking a theme as such) relates to general overall comments about Project Maths. This theme covered 7.5% of comments, 31.6% of which were positive, 54.4% were negative, and 14.0% mixed.

Examples of General Comments made:

I find project maths to be very beneficial in the junior cycle. Students are enthusiastic about it and learn well.

This is a worthwhile initiative, giving students a more practical take on mathematics.

Mathematics is a practical subject. Having innate ability is only a small part of it. Having the interest and confidence to try and work out problems is the key to success. Enjoying homework is very important. Project maths is making a good effort at promoting these gifts.

Whole idea is all over the place. Not impressed by project maths at all.

As a teacher I feel completely let down by the decision makers – education (real education) was very poorly served.

I reserve judgement on Project Maths in terms of it improving mathematical ability. It is more engaging at times for students but I'm not, as yet, convinced of its benefits.

6.6. Key Findings and Conclusions

This chapter presented information on teachers' views on the implementation of Project Maths at junior cycle. Data were collected when the implementation of the new syllabus was in transition, due to its phased design. Comparisons between teachers in initial schools and other schools shows some differences in perceptions about Project Maths. It can be argued that the 'piloting' experience in these initial schools may have brought key issues to the fore, particularly with examination classes, and that these are the issues that need to be addressed in any future evaluation or review of Project Maths.

Close to half of the teachers in our survey (48%) did not know if Project Maths was having a positive impact on students' learning of mathematics (i.e. they responded 'don't know' to the statement 'overall, Project Maths is having a positive impact on students' learning of mathematics'). However, fewer teachers in initial schools indicated that they didn't know (38%), and 45% of these teachers agreed with the statement, compared to 29% in other schools. Overall, these data show that it is too soon for teachers to form an opinion about the impact of Project Maths. Some of the other findings in this chapter (for example, lower levels of confidence with aspects that may be regarded as key to the Project Maths approach, such as problem-solving strategies, differentiated instruction, and student self-assessment) also indicate that any judgements about the success or otherwise of Project Maths will need to be made in the longer term. A counter position for this is that the identification of these issues represents a call for action sooner rather than later.

When asked about their views on specific aspects of Project Maths, teachers were least positive about the phased implementation of Project Maths, new textbooks, clarity of the syllabus learning outcomes, support from the Project Maths Development Team (PMDT), and students' responses to Project Maths. In contrast, teachers generally had positive views on the websites at www.projectmaths.ie and www.ncca.ie/projectmaths, using the syllabus in lesson planning, the Common Introductory Course, the Bridging Framework, and the professional development workshops. They also reported using a greater range of resources in class, and that students now have to do more 'thinking' in class than previously.

Teachers were unsure about some of the specific aspects of Project Maths; in particular, parents' and students' views, resource materials for Strands 1 and 2, the new geometry course, and the Bridging Framework. However, teachers in initial schools were less unsure, and more negative, in their views on students' and parents' responses to Project Maths. On the other hand, teachers in initial schools had more positive views than teachers in non-Project Maths schools on the Common Introductory Course. Teachers in initial schools also indicated that students now have to do more 'thinking' in class to a greater degree than teachers in other schools. That teachers were unsure about students' and parents' views on Project Maths underlines the need for more information from (and for) these two groups. The NCCA is shortly to publish an interim report on Project Maths[33] that will examine students' reactions to the initiative; however, the opinions of parents are not expected to be covered in this review.

Teachers were of the opinion that there had been positive changes in a number of aspects in students' learning, in particular, their understanding of key concepts in statistics and probability, and in geometry and trigonometry, their level of awareness of the relevance of mathematics to other disciplines, their ability to solve real-life problems, and their ability to work collaboratively with one another. Teachers in initial schools perceived significantly greater improvements in five aspects of students' learning. These were their ability to work collaboratively, to explain how they solved problems, to try different strategies, their grasp of fundamental concepts and principles, and the sense of challenge experienced by higher-achieving students. There were no perceived declines in any of the 19 aspects of students' learning that were included in the questionnaire. These early indications of students' learning can be cautiously interpreted as a positive finding.

Confidence levels were high in teaching the statistics, geometry and trigonometry parts of the revised course. Teachers also reported being relatively confident in providing feedback to students, teaching problem-solving in real-life settings, engaging students in practical mathematics, and assessing students. Confidence levels were lower in facilitating the use of concrete materials, analysing problem-solving strategies, supporting students with learning difficulties, engaging students in assessing their own progress, and preparing students for the revised Junior Certificate examination. In a general sense, teachers' reported levels of confidence appear to be rooted in the more traditional aspects of mathematics, i.e. specific syllabus topics, and are lower in areas that reflect newer concerns in the teaching and learning of mathematics.

We also found, perhaps unexpectedly, that teachers in other schools tended to perceive themselves as 'very confident' to a greater degree than teachers in initial schools on a majority of the 14 aspects of teaching Project Maths that we asked about. Why this is the case is not clear. It could be that participation in Project Maths during its initial phase gave rise to a process of questioning and reflection, and with that, some self-doubt, which did not occur to the same degree in schools that did not participate in the developmental phase of Project Maths.

When asked about the challenges that they perceived in implementing Project Maths in their schools (based on a list of 12 aspects), teachers in both initial schools and other schools indicated that the time available (both inside and outside of mathematics classes), the phased implementation, the rate of implementation, assessment materials available, and the literacy demands of the revised courses presented difficulties, in that at least 40% of teachers in both initial

[33] This report is expected in November 2012.

and other schools rated these as significant challenges. In addition, between 20% and 40% of teachers indicated that teaching materials and resources available and students' reactions to Project Maths were major challenges. Generally, teachers in initial schools were more inclined than teachers in other schools to identify these aspects as major challenges, though particularly so with respect to time, assessment materials, and parents' and students' reactions. Overall, these results indicate that the most significant challenges faced by teachers focus on organisational aspects, with the exception of literacy demands of the revised syllabi.

Just over one-third of teachers (35%) made written comments on Project Maths, and these were categorised into 12 themes/sub-themes, the content of some of which overlapped. The five most common themes identified were *phased implementation, literacy and ability, syllabus, time,* and *examinations.* The content of these mirror the challenges identified by teachers (previous paragraph). A large majority of these comments (87%) were negative in tone, despite the fact that teachers with overall positive and negative views of Project Maths were equally likely as one another to make written comments. However, other parts of the questionnaire gave teachers ample opportunity to indicate positive views on the initiative. In using these results for future planning for mathematics teaching and learning, it would seem important to focus on those that are pedagogical rather than structural or organisational. For example, future CPD should include a focus on the literacy needs of students, the nature of mathematical literacy, and the value of extracting mathematical information from overall context, which is an important (if not essential) part of the process of mathematical modelling.

Teachers were almost unanimous in their view that implementation should not have been done on a phased basis, with a strong preference for starting at First Year, and working upwards from there. There was concern that the current cohort of senior cycle students lacked the foundation skills to tackle the new course. There was also relatively widespread concern about the challenges that the literacy demands of the new course presented to students of lower ability, with special needs, and with another first language. Some teachers commented on the lack of resources for Foundation level. Many teachers were of the view that the revised syllabus was too long and that the removal of choice from the examinations put them under pressure to cover material. The general view here was that the balance between quantity, or breadth, and quality, or depth, had not been achieved. Other themes to emerge were some negative views on textbooks (e.g. no coloured texts for lower-ability students; differences between texts and material covered during CPD for Project Maths), support for the teaching methodologies of Project Maths with an acknowledgement that these were difficult to implement under time and examination-related constraints, the large and radical changes necessitated by Project Maths, and the need for continued support from the PMDT to implement Project Maths.

Taken together, the quantitative and qualitative findings concerning Project Maths indicate that while teachers regard some aspects of the implementation of the initiative as being positive, particularly as they relate to teaching and learning practices (e.g. use of ICTs, small group work), significant challenges remain. Many of the challenges reported by teachers can be related to structural aspects of the education system and the perception that it would have been preferable to implement Project Maths on a non-phased basis.

7. Conclusions and Recommendations

7.1. Introduction

This report is based on a survey of a nationally representative sample of mathematics teachers and mathematics school co-ordinators (a co-ordinator being the staff member with overall responsibility for mathematics in the school). Just over 1,300 teachers in 180 schools took part in the survey, which was implemented as part of Ireland's administration of PISA 2012.

The survey aimed to provide a reliable, representative and up-to-date profile of mathematics teaching and learning in Irish post-primary schools; to obtain quantitative and qualitative information on teachers' views on the implementation of Project Maths; and to make findings available in an accessible format and timely manner.

While achievement data for the students who took part in PISA 2012 will not be available until December 2013, the results in this report aim to paint a picture of mathematics education in Irish post-primary schools at a time of considerable curricular change. When achievement results do become available, it will be possible to link them with the data presented in this report, and to compare the achievements of students in the 23 schools that participated in the initial phase of Project Maths ('initial schools') with other schools. We will also be able to compare students' performance in PISA 2012 with achievements in previous cycles of PISA, in order to examine whether or not the decline in performance levels found in 2009 relative to previous cycles has continued through to 2012. However, it is too early to expect that Project Maths will have had a systematic impact with respect to PISA mathematics achievement, given that its implementation is still underway: it will be 2017 before the first cohort of students studies mathematics under the new curriculum right through post-primary school (from First to Sixth Year).

This chapter does not attempt to provide a summary of main findings; readers are referred instead to the end parts of the previous chapters. Rather, the aim here is to draw findings together in order to make some conclusions and recommendations. Some of the recommendations are aimed at teachers and school principals, while others are aimed in a more general way at the level of the system.

7.2. Conclusions and Recommendations

There are three major aspects of Project Maths that, in our view, run throughout this report:

- the implementation of Project Maths at the same time at both junior and senior cycles within a relatively short timeframe;
- the content of the new curriculum (including what is not included); and
- the assessment and certification of students under the new curriculum.

These may be worth keeping in mind as we draw conclusions under the headings of *implementation and time; grouping, syllabus and assessment; professional development for teachers; literacy; use of tools and resources;* and *parents and other stakeholders.*

Some of our recommendations are made with the expectation that Project Maths will be subject to review and refinement as implementation progresses, and in light of issues raised in this report (see Chapters 1 and 2) and elsewhere.

In reflecting on our recommendations, it should be borne in mind that these arise from the literature review and our survey of teachers. Views of other stakeholders, particularly students and parents, should also be taken into account as the implementation of Project Maths progresses. Empirical data on the mathematics achievement of students studying under the new curriculum will be necessary in order to fully evaluate the efficacy of the Project Maths initiative. The NCCA-commissioned interim report (expected in November 2012) will provide some of this information by describing the achievements and attitudes of students in initial schools and other schools. Also, as noted in Chapter 6, how Project Maths was introduced (at both junior and senior cycles and in the wider context of economic recession) can be expected to impact on any overall views on the initiative.

7.2.1. Implementation and Time

The decision to implement Project Maths at both junior and senior cycles at the same time remains a very unpopular one among teachers, and arguably colours views on the initiative as a whole. Lubienski (2011) noted this in her study, and it emerged again as a major theme in the current survey. It appears that this two-prong implementation strategy has given rise to several secondary problems, some of which are related to a lack of time. The NCCA has pointed out that previous experience with changes made to Junior Certificate mathematics (from 2000) did not result in sustained changes in classroom practice and where change did take place, it tended not to have an impact at Leaving Certificate. It was recognised that, unless the desired change in emphasis in teaching and learning approaches was reflected in a corresponding change in the examinations at both junior and senior cycles, it was unlikely to succeed (NCCA, 2006).

Consistent with the literature review in Chapter 2, teachers in our survey reported considerable time pressures, not only to come to grips with new materials and teaching approaches at both junior and senior cycles, but also to cover a course that they perceived to be too long and too broad. However, the implementation of the new framework for junior cycle (DES, 2012), in which the numbers of subjects taken by students is to be capped, and the numbers of teaching hours for mathematics can be expected to increase, is likely to go some way towards addressing this issue. The perceived lack of availability of textbooks and sample papers that was noted by teachers in our survey and elsewhere (e.g. Lubienski, 2011), could also be traced back to the demands on the system – not only on schools but also the NCCA and the SEC – to adhere to this 'two-prong' approach within a short timeframe.

Although implementation is already well underway in a wider context of significant financial constraints, any future changes of this nature and scale may benefit from a reflection on the experiences of the implementation of Project Maths from the viewpoints of teachers and students, as well as those responsible for implementation, while at the same time maintaining a sense of realism with respect to time, financial and other constraints. In this sense, Project Maths offers a model for change which can be refined and built on.

1. We recommend that lessons be drawn from the implementation of Project Maths with respect to any future policies or initiatives that entail changes to curriculum and assessment, and particularly with the implementation of the new framework for junior cycle. *(System)*

A second way in which time arises in our consideration of our results is the tension that emerges between implementing new, active teaching approaches within the instructional time that is available. This was identified by teachers in our survey as a major challenge, although there is emerging evidence (discussed in Chapter 2) that, as familiarity with the revised courses increases, teachers' use of time is perceived to be more efficient and effective. In line with the *National Strategy to Improve Literacy and Numeracy Among Young People* (DES, 2011), the Department of Education and Skills sent circulars to schools in 2011 and September 2012 (Circular Numbers

0058/2011 and 0027/2012) asking that every effort be made to provide students with a mathematics class every day, particularly at junior cycle.

2. It is recommended, in addition to the objective to increase the number of mathematics classes to five or more per week as specified in the *National Strategy to Improve Literacy and Numeracy Among Young People* (DES, 2011), that timetabling arrangements for mathematics in post-primary schools are reviewed with a view to establishing whether or not longer single or double class periods would be the most appropriate way in which to deliver the mathematics curriculum. A review of timetabling arrangements should also be accompanied by changes in what is done during the increased time with respect to teaching and learning approaches. *(Schools and System)*

7.2.2. Grouping, Syllabus and Assessment

Ability grouping ('streaming' or 'setting') for mathematics class is very widespread in the Irish post-primary education system, according to the results of our survey. The overall picture points to an issue that is structural or systemic. The consequences of ability grouping have been well-documented elsewhere, with strong national and international evidence pointing to negative consequences for lower-ability bands, with few corresponding gains for higher-ability groups (e.g. Smyth & McCoy, 2011; Smyth, Dunne, Darmody & McCoy, 2007). Research on the use of mixed-ability teaching approaches for mathematics (e.g. Boaler, 2008; Linchevski & Kutscher, 1998) provides promising evidence for the benefits of mixed-ability peer learning outcomes in mathematics (relative to more traditional approaches). However, building up a co-operative learning culture can be difficult to achieve, and easy to dismantle (Boaler, 2009). Teachers in initial schools reported more frequent use of these kinds of approaches than teachers in other schools, which is a positive finding, and is an early indication that some of the challenges presented by Project Maths may be possible to overcome with time.

3. We recommend that a better balance be struck between ability grouping for mathematics classes and the strategic use of mixed-ability teaching approaches. Such teaching approaches can be promoted through dissemination of practices that have been found to be effective in Ireland (for example in initial Project Maths schools), as well as through an examination of practices suggested in international research. *(Schools)*

In our view, the widespread nature of ability grouping for mathematics also points to a need to review the content and length of the syllabus on one hand, and the structure of the mathematics Certificate examinations on the other. This is particularly so if it is an aim of Project Maths to increase the numbers of students studying mathematics at Higher level. However, aiming to increase Higher-level uptake is, in our view, insufficient in the absence of a focus on increasing mathematics standards across all ability levels. The SEC has described the setting of standards for the Leaving Certificate as follows (SEC, personal communication, August 2012):

> The approach taken is sometimes described as a 'college of professionals' approach. In the first instance, a group of people who are deemed to have an expert knowledge of what the students in the target audience ought to be able to achieve in the subject concerned reach a consensus regarding the content standards of the syllabus. This is achieved through the various committees in the National Council for Curriculum and Assessment (NCCA), which are representative of teachers and other subject experts including third level and industry. […] These… are then put into effect as a set of performance standards by the State Examinations Commission (SEC), through the preparation of sample papers [in collaboration with the NCCA and DES] and subsequent examinations.

There is evidence that this approach to standard-setting is not sufficient, as it appears to result in some anomalies. For example, it was found in PISA 2003 that 10.5% of students taking Higher level mathematics for the Junior Certificate had a PISA mathematics score that was at or below Level 2, which is considered to be below a minimal level of competency (Cosgrove et al., 2005). While we would not expect perfect alignment with these two measures of mathematical achievement, it is nonetheless of concern that a small proportion of students taking Higher level mathematics performed at a minimal level on the PISA assessment of mathematics.

Our survey results also recorded a substantial drop in the percentages of students studying higher level mathematics in Fifth Year (31%) compared with Sixth Year (20%). Furthermore, a review of the PISA results for mathematics in 2003 and 2009 suggest a decline in mathematics achievement that is more marked among high achievers than those at the lower end of the achievement distribution. Thus, apart from a potential misalignment between syllabus levels studied and standards associated with the syllabus levels, there is also some evidence that high-achieving students in Ireland are not achieving their full potential in mathematics.

It is likely that some aspects of Recommendations 4, 5 and 6 will occur with the implementation of the new framework for junior cycle. However, it is important that the focus on mathematics is not 'lost' within the wider junior cycle reform agenda.

4. The Junior and Leaving Certificate mathematics examinations should be systematically reviewed in light of the implementation of all five syllabus strands, in 2014 for Leaving Certificate, and 2015 for Junior Certificate, and ideally on an ongoing basis. The review should concentrate on (i) the match between syllabus content (both concepts and skills) and its assessment, with the aim of ensuring that these are in line with one another, (ii) the extent to which senior cycle mathematics builds smoothly and successfully on what is covered during the junior cycle, and (iii) what improvements might be made to delivering the curriculum in classrooms. To support this, consideration should be given to the production of a Chief Examiner's report at both Junior and Leaving Certificate levels on an annual basis in the short to medium term. *(System, with input from Schools)*
5. To ensure continued consistency in the standards associated with the Junior and Leaving Certificate mathematics examinations, ongoing comparisons between examination performance and standardised measures of mathematics achievement, including, but not limited to, PISA mathematics, should be made, and, where appropriate, discrepancies in performance should be identified and examined. The proposed implementation of standardised testing of Second Years under *National Strategy to Improve Literacy and Numeracy Among Young People* (DES, 2011) is a further potential data source with respect to this recommendation. *(System)*
6. Students should receive active encouragement from junior cycle onwards to achieve their potential in mathematics. The decision to take mathematics at Ordinary level should be made with care and consideration not only of students' abilities and interests, but also with respect to their future plans for education and work. The Department of Education and Skills should develop guidelines to help schools allocate students to the most appropriate syllabus level at junior cycle that are based on both needs/interests, and objective evidence, such as performance on a standardised test. Schools should develop a policy to promote take-up of Higher Level mathematics in senior cycle that includes active encouragement and support for students in Fifth Year. *(Schools and System)*

Some of the commentary on Project Maths that was described in Chapter 2 has been critical of aspects of the content of the Project Maths syllabus. In broad terms, this boils down to a perceived over-emphasis on real-life, everyday problem solving, and too little emphasis on more formal or technical mathematics, such as topics covered in calculus, vectors and matrices. Some of the comments from teachers in our survey would support this view. There is a concern that the revised course will not adequately prepare students who wish to enrol in third-level courses with more specialised or applied mathematical content. It was also noted that only about 2.5% of the Leaving Certificate cohort take Applied Mathematics as a Leaving Certificate subject (www.examinations.ie).

7. We recommend that an overall priority in moving forward is to obtain further clarity with respect to the purposes of mathematics education at post-primary level. The review process proposed under recommendation 4 should be extended to reconsider the content and skills underlying the revised mathematics syllabus with a view to ascertaining the appropriateness of the balance between everyday and formal mathematics. The review should gather information on the mathematical demands of some of the most popular third-level courses to determine whether a better match between post-primary and third-level mathematics is possible or desirable; it should also consider what third-level institutions are doing in order to adapt to the changes at post-primary level in order to improve delivery of their courses. The review will need to consider the place of Applied Mathematics within post-primary mathematics education in general. *(System, with input from Schools)*

7.2.3. Professional Development for Teachers

Our findings indicate that somewhere between 15% and 32% of teachers who currently teach mathematics may lack the appropriate qualifications to do so effectively. This issue has already been flagged by researchers at the University of Limerick (Uí Ríordáin & Hannigan, 2009), though results of a recent Teaching Council survey of teachers suggest that the problem may not be as widespread as suggested in the UL report (DES press release, September 29, 2011). The results of our survey are closer to the Teaching Council survey than to those in the UL report.

As noted in Chapter 2, a welcome development is the commencement of a new Professional Diploma in Mathematics for Teaching which is funded by the Department of Education and Skills. The course is expected to run each year over three years, and already, the figures point to its need, with almost two teachers (750) applying for every one place on the course (390) (DES press release, September 22, 2012). It is also noteworthy that consecutive teacher education is to be extended from one to two years from 2014 in light of the Literacy and Numeracy Strategy (DES, 2011).

Our survey also found that high numbers of teachers were of the view that their initial teacher training/third-level studies did not adequately prepare them for some aspects of their work as mathematics teachers, particularly in the areas of the mathematics assessment and mathematics teaching methods; literacy also emerged as an aspect of the teaching and learning of mathematics in need of more attention (see also the next section). It is reasonable to argue that substantial support is required to make the changes to teaching and learning suggested by Project Maths. There is plenty of evidence that supports the importance of high-quality teacher education (e.g., Gilleece et al., 2009; Smyth & McCoy, 2011). Some of the reported difficulties in attending formal continuing professional development (CPD) courses could be circumvented through the provision of flexible online courses. Another important trend in CPD is the engagement of teachers in professional development activities within their schools e.g., they identify an issue and then seek to find solutions (e.g., Gilleece et al., 2009).

8. Future CPD opportunities should include a focus on mathematics teaching methods, the assessment for and of mathematics, mathematical literacy, and the importance of extracting mathematical information from context as part of the overall process of mathematical modelling. As many as possible of these should be offered in the form of flexible online resources and training modules. *(System and Schools)*
9. Teachers of mathematics should be encouraged to identify gaps in their professional development and/or understanding of mathematics teaching, learning and assessments, and schools should seek to support them in addressing these gaps. *(Schools, with input from System)*

7.2.4. Literacy

A major theme to emerge in this study, more so, perhaps than in existing research and commentary on Project Maths, concerns the perceived literacy demands of the revised mathematics syllabus, which challenges teachers and students alike. Teachers' concerns for students focused on those with lower achievement, learning difficulties, and/or a first language other than English or Irish. In general, teachers felt that middle- to high-ability students would be able to manage the revised syllabus. Some drew attention to the fact that there is a lack of resource materials for students studying Foundation level mathematics, though it was noted in Chapter 2 that while the revised junior cycle curriculum now no longer includes a Foundation Level syllabus, the Junior Certificate Foundation Level examination has been retained.

10. It is recommended that increases in the amount of instructional time as described in *National Strategy to Improve Literacy and Numeracy Among Young People* (DES, 2011) be accompanied by a strategic approach to organising mathematics instruction within the allocated time that incorporates teaching mathematical literacy (i.e., the language and procedures of mathematics and mathematical problems; communicating mathematical thinking and ideas) to students who need it. Mathematics teachers should have primary responsibility for this. *(Schools)*
11. The DES/NCCA should clarify the role and purpose of Foundation level mathematics at both junior and senior cycles, and review its provision of guidance and materials specifically as they relate to students with lower levels of literacy. *(System)*

7.2.5. Use of Tools and Resources in Delivering Project Maths

Our survey found that teachers tended to use textbooks to a considerable degree in planning and conducting their teaching and learning activities; they also commented, consistent with previous research reported in Chapter 2, that appropriate textbook resources were lacking. The NCCA recommends against the over-reliance on textbooks as teaching and learning resources, and the Project Maths website (www.projectmaths.ie) includes a range of teaching resources, including handbooks, learning plans, student CDs, example questions and tasks, and reference books and websites. However, information on how these resources might be used in an integrated way is lacking.

12. It is recommended that the DES/NCCA further clarify how the resources available to teachers and students may be used with one another and in conjunction with textbook resources. Some re-organisation of these resources may be required to achieve this. *(System)*

Teachers in Project Maths initial schools reported markedly higher usage of ICT resources during mathematics classes, particularly software, both general and mathematics-specific. It was noted in

Chapter 2 that some of the CPD emphasised the use of ICTs in teaching and learning, and it is very encouraging that teachers in initial schools appear to have incorporated these tools into their classroom practices quickly, and in a manner that can only be described as widespread. However, we do not know which tools and practices are associated with more and less effective teaching and learning approaches.

13. It is recommended that the use of ICTs in teaching mathematics be examined carefully with a view to identifying those tools and strategies that are most effective in achieving teaching and learning goals, and that these are worked into the suite of resources available to all mathematics teachers. *(System and Schools)*

7.2.6. Parents and Other Stakeholders

We noted that the views of parents on Project Maths were absent from existing research and commentary on Project Maths. The fact that teachers, particularly those in the initial schools, were of the view that a large minority of parents and students may not be happy with aspects of Project Maths, is a potential cause for concern. There are only limited resources for parents at present. The NCCA's website includes information for parents under Project Maths FAQs[34]; there are also introductory courses for parents (e.g. www.careerguidance.ie), though these are not available on a widespread basis. It may well be that many parents do not yet have an informed opinion on Project Maths, and/or are unsure about its content and objectives, and how best to support their children's mathematics learning. Media coverage of the initiative, some of it negative, may be an influence here.

There is also the potential for more collaboration between the post-primary and third-level sectors with respect to achieving the objectives of Project Maths, particularly in the promotion of interest in mathematics and an awareness of the importance of mathematics across a range of third-level disciplines. There are already some instances of this which could be built on further. For example, Engineers Ireland's STEPS programme, established in 2000, works in partnership with the DES to encourage positive attitudes towards science, technology, engineering and mathematics (STEM) disciplines, and increase awareness about these disciplines (see www.steps.ie).

14. We recommend that a campaign be implemented for parents, as one of the key stakeholders in education, whereby: (i) they are informed about Project Maths – its aims and objectives; (ii) they have an opportunity to voice their opinions about Project Maths, and have these opinions heard; (iii) they are encouraged to play an active role in their children's mathematics education through the promotion and dissemination of practical tips and examples; and (iv) schools encourage and facilitate parental involvement in their children's mathematics education in ways that suit local needs. *(System and Schools)*
15. It is recommended that the DES develops a strategy to mobilise and utilise support from the third-level education sector in order to further develop the aims and objectives of Project Maths, particularly in fostering an interest in and awareness of the importance of mathematics, and in the provision of clear, relevant information on the mathematics content and skills requirements of various STEM disciplines. *(System)*

[34] http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Project_Maths/Information_on_Project_Maths/Parents_info_note.pdf.

References

Boaler, J. (2009). *The elephant in the classroom: Helping children to learn and love maths.* London: Souvenir Press.

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal, 34,* 167-194.

Brick, J.M., Morganstein, D., & Valliant, R. (2000). *Analysis of complex sample data using replication.* Rockville, MD: Westat.

Cartwright, F. (2011). *PISA in Ireland, 2000-2009: Factors affecting inferences about changes in student proficiency over time.* Dublin: Educational Research Centre.

Close, S. (2006). The junior cycle curriculum and the PISA mathematics framework. *Irish Journal of Education, 37,* 53-78.

Conway, P.F. & Sloane, F.C. (2006). *International trends in post-primary mathematics education: Perspectives on learning, teaching and assessment*. Dublin: NCCA.

Cosgrove, J. (2011). *Does student engagement explain performance on PISA? Comparisons of response patterns on the PISA tests across time.* Dublin: Educational Research Centre.

Cosgrove, J., Shiel, G., Sofroniou, N., Zastrutzki, S., & Shortt, F. (2005). *Education for life: The achievements of 15-year-olds in Ireland in the second cycle of PISA.* Dublin: Educational Research Centre.

Cosgrove, J., Shiel, G., Archer, P., & Perkins, R. (2010). *Comparisons of performance in PISA 2000 to PISA 2009: A preliminary report to the Department of Education and Skills.* Dublin: Educational Research Centre.

Department of Education and Science (2005). *DEIS (Delivering Equality of Opportunity in Schools). An action plan for educational inclusion.* Dublin: Author

Department of Education and Skills (2010). *Report of the Project Maths Implementation Support Group.* Dublin: Author.

Department of Education and Skills (2011). *Literacy and numeracy for learning and life: The National Strategy to Improve Literacy and Numeracy Among Children and Young People 2011-2020.* Dublin: Author.

Department of Education and Skills (2012). *A framework for junior cycle.* Dublin: Author.

DeVellis, R.F. (1991). *Scale development*. Newbury Park, NJ: Sage Publications.

Engineers Ireland. (2010). *Report of the task force on education of mathematics and science at second level*. Dublin: Engineers Ireland.

Faulkner, F., Hannigan, A., & Gill, O. (2010). Trends in the mathematical competency of university entrants in Ireland by Leaving Certificate mathematics grade. *Teaching Mathematics and its Applications, 29,* 76-93.

Gilleece, L., Shiel, G., Perkins, R., & Proctor, M. (2008). *Teaching and Learning International Survey (2008): Report for Ireland.* Dublin: Educational Research Centre.

Grannell, J. J., Barry, P. D., Cronin, M., Holland, F., & Hurley, D. (2011). *Interim report on project maths.* Cork: University College Cork.

Hutcheson, G., & Sofroniou, N. (1999). *The multivariate social scientist: Introductory statistics using generalized linear models*. London: Sage.

Kelly, A., Linney, R. & Lynch, B. (2012). T*he challenge of change – experiences of Project Maths in the initial group of schools.* Paper presented at SMEC 2012, Dublin City University, Ireland, 7th-9th June, 2012.

LaRoche, S., & Cartwright, F. (2010). *Independent review of the PISA 2009 results for Ireland: Report prepared for the Educational Research Centre at the request of the Department of Education and Skills.* Dublin: Department of Education and Skills.

Linchevski, L., & Kutscher, B. (1998). Tell me with whom you're learning, and I'll tell you how much you've learned: Mixed-ability versus same-ability grouping in mathematics. *Journal for Research in Mathematics Education, 29,* 533-554.

Lubienski, S. (2011). Mathematics education and reform in Ireland: An outsider's analysis of project maths. *Irish Mathematical Society Bulletin, 67*, 27-55.

Lyons, M., Lynch, K., Close, S., Sheerin, E., & Boland, P. (2003). *Inside classrooms: The teaching and learning of mathematics in social context.* Dublin: IPA.

National Council for Curriculum and Assessment (2005). *Review of mathematics in post-primary education: A discussion paper.* Dublin: Author.

National Council for Curriculum and Assessment (2006). *Review of mathematics in post-primary education: Report on the consultation.* Dublin: Author.

National Council for Curriculum and Assessment (2012). *Membership of NCCA committees and boards of studies.* Dublin: Author.

National Council for Curriculum and Assessment /Department of Education and Science (2002). *Junior Certificate mathematics: Guidelines for teachers.* Dublin: Stationery Office.

National Council for Curriculum and Assessment /Department of Education and Skills (2011a). *Junior Certificate mathematics syllabus: Foundation, ordinary and higher level.* Dublin: NCCA/DES.

National Council for Curriculum and Assessment /Department of Education and Skills (2011b). *Leaving Certificate mathematics syllabus: Foundation, ordinary and higher level.* Dublin: NCCA/DES.

Organisation for Economic Co-operation and Development (2009). PISA data analysis manual: SPSS (2nd ed.). Paris: Author.

Organisation for Economic Co-operation and Development (in press). *PISA 2012 mathematics framework.* Paris: Author. Accessed at http://www.oecd.org/pisa/pisaproducts/46961598.pdf

Shiel, G., Moran, G., Cosgrove, J., & Perkins, R. (2010). *A summary of the performance of students in Ireland on the PISA 2009 test of mathematical literacy and a comparison with performance in 2003: Report requested by the Department of Education and Skills.* Dublin: Educational Research Centre.

Smyth, E., Dunne, A., Darmody, M, & McCoy, S. (2007). *Gearing up for the exam? The experiences of junior certificate students*. Dublin: Liffey Press/ESRI.

Smyth, E., & McCoy, S. (2011). *Improving second-level education: Using evidence for policy development (Renewal Series Paper 5).* Dublin: ESRI.

Stacey, K. (2012). The international assessment of mathematical literacy: PISA 2012 framework and items. *Paper presented at the 12th International Congress on Mathematics Education, COEX, Seoul, Korea, July 8-15.*

Teaching Council (2012). *Post-primary registration: Post-primary curricula subject criteria and list of recognised teacher degree/teacher education programmes*. Dublin: Author.

Uí Ríordáin, M., & Hannigan, A. (2009). *Out-of-field teaching in post-primary mathematics education: An analysis of the Irish Context.* Limerick: NCE-MSTL, University of Limerick.

Technical Appendix

This Appendix contains technical background information on the analysis procedures used to report results. It is likely to be of relevance to readers with an interest in the analysis methodologies underlying the results.

A.1. Sample Design, Response Rates and Computation of Sampling Weights

Like any large-scale educational assessment, it is important that the sampled schools, teachers and students are representative of their respective populations. Schools were sampled first, with probability proportional to size (with larger schools having a higher likelihood of being sampled). Prior to sampling, schools were grouped by the enrolment size of PISA-eligible (15-year-old) students and school sector (community/comprehensive, secondary, and vocational). Small schools had 40 or fewer PISA students enrolled; medium ones had 41-80 students enrolled, and large schools had 81 or more students enrolled. In addition, all 23 schools that participated in the initial stage of Project Maths were included in the sample. This resulted in ten strata or clusters of schools:

- Size 41-80 / Community/Comprehensive
- Size > 80 / Community/Comprehensive
- Size <=40 / Secondary
- Size 41-80 / Secondary
- Size > 80 / Secondary
- Size <=40 / Vocational
- Size 41-80 / Vocational
- Size > 80 / Vocational
- Project Maths initial schools.

Within each cluster, schools were sorted by the percentage of students whose families are eligible for a medical card (split into quartiles), and the percentage of female students enrolled (also split into quartiles).

Once schools were sampled, students were sampled at random within each school. However, the focus of this section is a description of the sample of teachers and mathematics school co-ordinators, so the remainder discusses these respondents, rather than the students that participated.

The sample of mathematics teachers was defined as *all teachers of mathematics in the school.* Therefore this included mathematics teachers of both junior and senior cycles, although the teacher questionnaire focused on junior cycle in some sections, since the majority of PISA students were in junior cycle at the time of the assessment. At the beginning of the administration of PISA, school principals were asked to provide the ERC with the total number of mathematics teachers in the school, and the numbers of questionnaires were sent out were based on this information. However, it emerged that, in 32 of the 183 participating schools, more teachers returned questionnaires than expected (i.e. the total number of returns was more than the expected number of mathematics teachers). In these schools, the total number of mathematics teachers was adjusted to equal the total number of returns, or else the response rate would have exceeded 100% for those schools.

It is estimated, therefore, that there were 1645 mathematics teachers in participating schools. Of these, 1321 returned a questionnaire, which constitutes an acceptable response rate of 80.3%. On average, 7.2 questionnaires were returned per school, and school-level teacher response rates ranged from 7% to 100%.

In all analyses of the teacher questionnaire, data are weighted by a teacher weight. This consists of four components, and ensures that the reported results are representative of the population of mathematics teachers in Ireland. The first component, the school base weight, is the reciprocal of the schools' probability of selection. The second, school non-response adjustment, is an adjustment that is applied to account for the fact that two of the 185 sampled schools did not participate. The third component is an adjustment to take the over-sampling of initial schools into account; if this were not done, initial schools would contribute disproportionately to estimates for the sample as a whole. The fourth component is a teacher non-response adjustment. Since each mathematics teacher has a selection probability of 1, it is necessary only to compute the non-response adjustment, which is the number of returned questionnaires divided by the number of expected questionnaires. In summary, the teacher weight = school base weight * school non-response adjustment * oversampling adjustment for initial schools * teacher non-response adjustment. For analyses in this report, the normalised teacher weight is used; that is, the population weight adjusted in order to return the same N as the number of respondents. The normalised rather than the population weight is used in order to avoid artificially inflating the power of analyses.

The sample of mathematics school co-ordinators (and hence the computation of the weights) is more straightforward than that of mathematics teachers, since there was only one co-ordinator per school. In total, 171 co-ordinators returned a questionnaire, which constitutes a highly satisfactory response rate of 93.4%. The mathematics school co-ordinator weight was computed as the school base weight * co-ordinator non-response adjustment. As with the analyses of the teacher questionnaire data, the normalised school co-ordinator weight is used in all analyses in this report.

A.2. Correcting for Uncertainty in Means and Comparisons of Means

We surveyed a sample of mathematics teachers rather than the whole population of mathematics teachers, estimates are prone to uncertainty due to sampling error. The precision of these estimates is measured using the standard error, which is an estimate of the degree to which a statistic, such as a mean, may be expected to vary about the true (but unknown) population mean. Assuming a normal distribution, a 95% confidence interval can be created around a mean using the following formula: Statistic ± 1.96 standard errors. The confidence interval is the range in which we would expect the population estimate to fall 95% of the time, if we were to use many repeated samples. For example, the mean perceived change in students' interest in mathematics shown in Chapter 6, Table 6.4 of this report is 0.338, with a standard error of 0.023. Therefore, it can be stated with 95% confidence that the population mean for perceived changes in students' interest in mathematics lies within the range of 0.293 to 0.383.

To correct for the uncertainty or error due to sampling, we have used SPSS® macros developed by the Australian Council for Educational Research (ACER). The standard errors were computed in a way that took into account the complex, two-stage, stratified sample design. The macros incorporate sampling error into estimates of standard errors by a technique known as variance estimation replication. This technique involves repeatedly calculating estimates for N subgroups of the sample and then computing the variance among these replicate estimates. The particular method of variance estimation used was Jackknife N. Variance estimation replication is generally used with multistage stratified sample designs, and usually has two units (in this case, schools) in each variance stratum. In the case of the teacher data, there were 90 variance strata, and there were 85 such strata for the mathematics co-ordinator data. Using the particular Jackknife method, half of the sample is weighted by 0, and the other half is weighted by 2. For more information on this and related techniques, see Brick, Morganstein, and Valliant (2000); the PISA data analysis manual (second edition) also provides a good overview of the rationale and implementation of this family of methods (OECD, 2009).

A.3. Constructing Questionnaire Scales from Responses to Individual Questions

In Chapter 5 of this report, we presented results relating to four scales which we constructed on the basis of teachers' responses to individual items on the teacher questionnaire. Each scale has an overall mean of 0 and a standard deviation of 1. These scales were created using principal components analysis in SPSS® (see, e.g. Hutcheson & Sofroniou, 1999), initially through exploring the characteristics of the item batteries as a whole, then establishing which items 'fit together' best with each other. Table A1 shows the factor loadings and reliabilities for two scales concerning general views on mathematics (fixed views of mathematics and constructivist/applied views of mathematics; see also Tables 5.1 and 5.2), while Table A2 shows the factor loadings and reliabilities for two scales concerning views on ability grouping (support for ability grouping and awareness of potential negative effects of ability grouping). It should be noted that the scale reliability for the fixed view scale (.42) is low, while the reliability for the constructivist/applied view scale is acceptable (.69) (Table A.1); scale reliabilities for the two scales on ability grouping are acceptable to good (.81 for the support for ability grouping scale and .68 for the potential negatives of ability grouping scale; Table A.2) (see DeVellis, 1991).

Table A.1. Factor loadings and scale reliabilities for the two scales concerning general views on mathematics

Items on fixed views of mathematics scale	Factor Loading	Items on constructivist/applied views of mathematics scale	Factor Loading
Some students have a natural talent for mathematics and others do not	.414	There are different ways to solve most mathematical problems	.520
If students are having difficulty, an effective approach is to give them more practice by themselves during the class	.347	More than one representation (picture, concrete material, symbols, etc.) should be used in teaching a mathematics topic	.587
Mathematics is a difficult subject for most students	.496	Solving mathematics problems often involves hypothesising, estimating, testing and modifying findings	.587
Few new discoveries in mathematics are being made	.609	Modelling real-world problems is essential to teaching mathematics	.730
Mathematics is primarily an abstract subject	.525	To be good at mathematics at school, it is important for students to understand how mathematics is used in the real world	.711
Learning mathematics mainly involves memorising	.613	A good understanding of mathematics is important for learning in other subject areas	.609
Scale reliability (Cronbach's alpha)	.419	Scale reliability (Cronbach's alpha)	.691

Table A.2. Factor loadings and scale reliabilities for the two scales concerning views on ability grouping

Items on support for ability grouping scale	Factor Loading	Items on potential negatives of ability grouping scale	Factor Loading
Allocating students to mathematics classes based on some measure of academic ability is, overall, a good practice	.764	Class-based ability grouping for mathematics has a negative impact on some students' self-esteem	.661
Class-based ability grouping for mathematics facilitates a more focused teaching approach	.743	Class-based ability grouping for mathematics slows the pace of learning of lower-achieving students	.696
Class-based ability grouping for mathematics accelerates the pace of learning for all students	.703	Class-based ability grouping results in lower expectations by teachers of the mathematical abilities of lower-achieving students	.772
Class-based ability grouping is not particularly beneficial for teaching and learning mathematics*	.623	Class-based ability grouping for mathematics benefits higher-achieving students more than lower-achieving students	.727
Mixed-ability teaching in mathematics is beneficial to lower-achieving students	.585		
Mixed-ability teaching in mathematics 'drags down' the performance of higher achievers	.540		
It is possible to teach the mathematics curriculum in mixed-ability settings without compromising on the quality of learning	.615		
The best way to teach the mathematics curriculum effectively is in class-based ability grouped settings	.712		
Scale reliability (Cronbach's alpha)	.810	Scale reliability (Cronbach's alpha)	.679

*Item was reverse coded for the scale

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ISBN: 978 0 900440 38 0

9 780900 440380

PROJECT MATHS

Responding to current debate

October 2012

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Contents

1. Introduction

In September 2012 all schools are implementing the five strands of the revised mathematics syllabuses. This marks a significant milestone in *Project Maths* – an initiative which began in 2008 to address issues in syllabuses, teaching, learning and assessment of post-primary mathematics.

Project Maths is a work in progress, significant both because of the immense practical, cultural and symbolic legacy that it attempts to address, as well as the weight that is attached to what it aspires to achieve. The aim of mathematics education in schools has always been highly contested, especially when new curricula are being developed or disseminated through a national education system. It is not surprising then, that *Project Maths* is the subject of much debate in Ireland at present. There are always tensions between those who believe the new mathematics syllabus should prepare students for their further study in mathematics and mathematical related disciplines, and those who believe it should prepare students to participate fully in modern society. The former may well be how Leaving Certificate Mathematics, particularly at higher level, used to be viewed by both teachers and students. Smyth and Hannan (2002) noted many students who achieved high grades in mathematics at Junior Certificate did not continue their study of the subject at higher level in Leaving Certificate unless they felt it was needed for third level.

Mathematics is studied by the vast majority of students at upper secondary level in Ireland, which contrasts with the situation in many other jurisdictions where it is an optional subject. As a consequence, mathematics at Leaving Certificate focuses on preparing all students to participate fully in modern society while, at the same time, ensuring that those who wish to progress to further study have the conceptual understanding they need to serve as a foundation for further and deeper engagement in mathematics and related disciplines or careers.

This paper has been prepared in response to current comment on *Project Maths*, notably the UCC *Interim Report on Project Maths*[1] and a paper by Kirkland, Stack et al[2] in which they pointed to what they consider to be major flaws in the revised syllabus. The paper begins by revisiting the origins of the initiative; it sets out the background to, and beginnings of, *Project Maths* – including the research evidence that informed its development. Specific criticisms and issues of current debate are then considered in detail.

1 https://www.ucc.ie/en/media/academic/maths/ProjectMathsInterimReport_Nov2011.pdf

2 http://media.tcm.ie/media/documents/m/MathsCurriculumReport.pdf

2. The origins of *Project Maths*

Context for the review

In 2005 the NCCA undertook a review post-primary mathematics. This was not simply an exercise in syllabus revision but more a fundamental evaluation of the appropriateness of the mathematics that students engage with in school and its relevance to their needs. A number of studies had highlighted concerns (Smyth et al., 2004; Lyons et al., 2003; Elwood and Carlisle, 2003; Smyth and Hannan, 2002).

Key findings from this research were that

- post-primary mathematics education in Ireland featured a highly didactic pedagogy with mathematics being taught in a procedural fashion with relatively little emphasis on problem solving (Lyons, *et al*., 2003).
- the 'new' mathematics curricular culture, with its elevation of abstraction as a core principle, had dominated post-primary mathematics teaching for the last forty years (Oldham, 2001).
- over three cycles of PISA, Ireland had been ranked in the middle of OECD countries in mathematical literacy (OECD PISA reports in 2003, 2005, 2008).

The proportion of students opting for HL and their performance in the state examinations was also a serious concern. Chief Examiners' reports on state examinations in mathematics over a number of years had consistently pointed to the over-reliance by candidates on rote-learned procedures and the lack of deeper understanding of basic mathematics concepts. There was evidence that students were not able to apply mathematical knowledge and skills, except in the most practised way and in familiar contexts.

> *As identified in various previous Chief Examiners' reports, candidates' conceptual understanding of the mathematics they have studied is inferior to that which one would hope for and expect at this level* [LC-HL]*. Whereas procedural competence continues to be adequate, any question that requires the candidates to display a good understanding of the concepts underlying these procedures causes unwarranted levels of difficulty* (Chief Examiner's Report 2005, page 72).

O'Donoghue (2002) pointed to another worrying trend: the low level of mathematical knowledge and skills shown by some students proceeding to further and higher education, and their inability to cope with basic concepts and skill requirements in the mathematical aspects of their courses.

Another element of concern lay in the attitudes towards mathematics in Irish society generally. In a country which has aspirations towards becoming a high-skilled economy, it was a matter of some concern that, culturally, poor attitudes towards and performance in mathematics were accepted as the norm. Many people were negative about mathematics, many children were turned off by it and their school experiences of the subject remained with them throughout their lives.

As part of its review of post-primary mathematics curriculum and assessment, the NCCA commissioned research into international trends in mathematics education[1]. In their research, Conway and Sloane (2006) observed that there was no single template for reforming post-primary mathematics education. There are, however, trends of which the move towards a more 'real-life' focus and emphasis on the development of problem-solving skills is the most distinctive and significant shift in mathematics education in many countries. In addition, they point to a key insight from the TIMSS 1999 video study (Kawanaka, Stigler & Hiebert), where the nature of the links between procedural skills and conceptual knowledge are highlighted as a critically important dimension of high-quality mathematics education.

In addition to the challenges in mathematics education, the broader challenges of a system featuring two external examinations have been well documented. Since the early 1980's there have been several reviews of research on the impact of high-stakes testing on the curriculum (Linn et al., 1982; Crooks, 1988; Koretz, 1988; Koretz et al., 1991; Shepard, 1991; Kellaghan et al., 1996; Black & Wiliam, 1998; Stiggins, 1999; Linn, 2000). There is a strong common theme in the findings: high stakes testing has a backwash effect into daily learning and teaching. This high-stakes impact is universally found to be associated with the practices prevalent in Irish senior cycle mathematics classrooms: teachers focusing on the content of the exams, administering repeated practice tests, training students in the answers to specific questions or types of question, and adopting transmission styles of teaching. In such circumstances, teachers make little use of formative assessment to help the learning process (Broadfoot et al., 1998; Reay & Wiliam, 1999; Osborn et al., 2000; Pollard et al., 2000).

The study of junior cycle mathematics education, *Inside Classrooms* (Lyons, et al., 2003), corroborated these findings and found that these transmission practices also extend to the junior cycle where the stakes are lower. The OECD's TALIS study (2009) found that, while teachers in Ireland reported that they favour a constructivist approach to teaching and

[1] See http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Review_of_Mathematics/Review_of_Mathematics.html

learning, there was greater classroom emphasis on the use of structured didactic practices and less on student-oriented practices and activities that would require students to explain their reasoning, which are a feature of developments in other countries.

The Eurydice report in 2011 indicates that the trends identified by Conway and Sloane (2006) have continued to be influential across Europe:

> *The research relating to different approaches and methods suggests that there is no one correct way of teaching mathematics, with some researchers arguing that different methods work in different contexts, and others that teachers ought to select the most appropriate method for their context and for a particular learning outcome, and that there may be complex relationships between what works. The conclusion would seem to be that professional development for teachers in a range of different methods, and allowing them to make decisions about what can be applied, when and why, is the best approach for improving teaching.* (Eurydice Network (2011), page 52)

and

> *The use of problem-based learning, exploration and investigation is the focus in a number of countries, as is the use of real life contexts to make mathematics more relevant to the students' own experience.* (Eurydice Network (2011), page 52)

Consultation and discussion

A consultation on the identified issues was held in 2005/2006 with an open invitation for anyone to respond to the range of issues presented in a discussion paper (NCCA, 2005). This paper identified particular areas of concern that needed to be addressed as part of the review, including the uptake of mathematics at the different syllabus levels and the performance of candidates in the state examinations and international assessments. The consultation documentation was made available online and, to facilitate feedback, a consultation questionnaire was prepared that was based on the main issues identified in the discussion paper. An online version of this questionnaire was also made available and a free text messaging service (SMS) was established to encourage as wide an audience as possible to participate in the consultation. Consultation information and documentation was circulated widely, including to all post-primary schools and to the education partners. In particular, third-level education departments and the science/engineering departments of Universities and Institutes of Technology were invited to respond. The NCCA also held discussions with a number of focus groups, including parents' representative groups, the council of the Irish Mathematics Teachers' Association, and at an open meeting in UCC.

A report on this consultation was published in April 2006[2]. Respondents commented on the importance of mathematics education to the individual and to society, as well as its significance as a foundation for other subjects, especially science and technology subjects. Particular areas of concern included

- the over-emphasis on procedural skills and rote learning to the detriment of understanding and application to problem solving
- the erosion of time allocated to mathematics in schools, particularly in the junior cycle
- a declining interest in mathematics and commitment on the part of students to making the effort required to understand the subject
- low levels of understanding on the part of students progressing to third-level education, even among those who had studied mathematics at higher level
- a lack of in-depth knowledge by some teachers of mathematics, who tended to operate in a narrow ‘comfort zone’
- the significant and complex challenge that changes in methodology will present for *the system and the need to provide long-term, continuous support for such change.*

Beginning *Project Maths*

Informed by the consultation and the commissioned research, and following consideration of a number of possible approaches, the NCCA proposed the *Project Maths* initiative in 2007. The aim was to improve the mathematical experience of students in the classroom by retaining and reinforcing the central elements and mathematical rigour of previous syllabuses, while at the same time changing the approach and emphasis in teaching, learning and assessment. *Project Maths* calls for more student sense-making, problem solving, engagement in rich learning activities, and conceptual understanding to accompany procedural skill.

The NCCA maths committees were convened to progress syllabus and assessment revision under the *Project Maths* initiative. These committees comprised representatives of second-level teachers, school management bodies and third level institutions, as well as the State Examinations Commission and the Department of Education and Skills. Where further expertise was desirable, additional members were co-opted by Council. This facility was availed of, for example, when the NCCA Board of Studies for Mathematics was established (see the Appendix for membership of this board).

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[2] The discussion paper and the report on the consultation are available at http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Review_of_Mathematics/Review_of_Mathematics.html

The initiative placed the teachers at the centre of the curriculum development process and, in order to adapt the developments in light of feedback from the classroom, it began initially with a small number of schools where changes in the syllabus and examination were phased in. The five syllabus strands were introduced in three stages, as were the corresponding changes in the examinations. Since Project Maths was as much about changing teaching and learning practice as it was about changing syllabus content, it was considered desirable to introduce the changes simultaneously in first year and fifth year. This enabled teachers to embed the changed teaching approaches at both junior cycle and senior cycle simultaneously. It also enabled teachers to see the coherence of the subject across second level and the progression in the development of knowledge and skills as students continue their study of the subject at senior cycle.

Skills development

Skills have become the focus of developments at all levels of education systems around the world. There is a lot of debate about the need for schools to help learners to develop 21st century skills, to create new knowledge and to navigate their way through change, uncertainty and opportunity. Initiatives on the teaching and assessment of 21st century skills originate in the widely-held belief shared by several groups – teachers, educational researchers, policy makers, politicians, employers – that the current century will demand a very different set of skills and competencies from people for them to function effectively at work, as citizens and in their leisure time (Forfás, 2007; Forfás, 2009; OECD, 2009).

Arising from the senior cycle consultations which commenced in 2002, the NCCA set about developing a key skills framework for senior cycle education. The framework sets out five skills identified as central to teaching and learning across the curriculum at senior cycle. These are *information processing*, *being personally effective*, *communicating*, *critical and creative thinking* and *working with others*. These key skills were identified during the review as being important for all students to achieve to the best of their ability, both during their time in school and into the future, and to participate fully in society, in family and community life, in the world of work and in lifelong learning. By engaging with key skills learners enhance their ability to learn, broaden the scope of their learning and increase their capacity for learning. *Project Maths*, informed by international trends, develops key skills by promoting a ‘collaborative’ culture where mathematics is seen as a network of ideas which teacher and students construct together. Learning is seen as a social activity in which students are challenged and arrive at understanding through discussion. Teaching is seen as a non-linear dialogue in which meanings and connections are explored, misunderstandings are recognised, made explicit and students learn from them.

Cooperative approaches to support learning with understanding

Much research has shown the positive effects of co-operative learning on student achievement (Whicker et al. 1997; Bernero, 2000; Walmsley, 2003; Yamarik, 2007).Two of the ten research-based strategies for improving student achievement in mathematics promoted in the International Academy of Education (IAE) handbook, support this approach:

> *using small groups of students to work on activities, problems and assignments can increase students' mathematical achievements (page 21)…focussing instruction on the meaningful development of important mathematical ideas increases the level of student learning (page 14).*

Knapp et al (1995) identified the following characteristics of teaching mathematics for understanding, one of the goals of *Project Maths*:

- emphasising connections between mathematical ideas
- exploring the mathematics that is embedded in rich, 'real life' situations
- encouraging students to find multiple solutions and focusing students' attention on links between the solution processes used
- creating multiple representations of ideas.

The central role that mathematical thinking should play in mathematics education has been receiving attention, both among educators and in the research community, since as far back as the mid 1980's (e.g. Schoenfeld, 1985a: Silver, 1985). As Schoenfeld says,

> *You understand how to think mathematically when you are resourceful, flexible, and efficient in your ability to deal with new problems in mathematics* (Schoenfeld 1985a page 2)

The alignment of mathematics learning with mathematical thinking is an ongoing feature of mathematics education. According to Lutfiyya (1998)

> *mathematical thinking involves using mathematically rich thinking skills to understand ideas, discover relationships among the ideas, draw or support conditions about the ideas and their relationships and solve problem involving the ideas*.(page 56)

It is not surprising then that the development of mathematical thinking should be an important goal of schooling and of *Project Maths*. Mathematical thinking plays a significant role in the development of mathematical literacy. Mathematical literacy, as defined by PISA, is the ability to use mathematics for everyday living, for work and for further study; the PISA assessments present students with problems set in realistic contexts. The framework used by PISA shows that mathematical literacy involves many components of mathematical thinking, including reasoning, modelling, and making connections between ideas. The UCC *Interim report* appears to adopt a strongly critical view of PISA, and some of the criticisms of *Project Maths* appear to stem from the authors' concerns about PISA. However, where PISA has a particular focus on mathematical literacy rather than on curriculum, *Project Maths* is concerned with curriculum, teaching and learning, and assessment.

3. Current debate about *Project Maths* – a response

Now that the full syllabus strands in mathematics at both Junior Certificate and Leaving Certificate are being introduced in all schools and we have had a number of years of examination results, there has been increased public debate about mathematics education at second level. Particular areas of concern have arisen in the public discourse and this paper addresses the main points that have been raised in recent times. These have predominantly related to Leaving Certificate mathematics – and almost exclusively focus on higher level. This, in itself, is problematic since it fails to recognise the role that mathematics education plays at earlier stages in the students' experiences. It also narrows the debate to one of mathematics education as preparation for further and higher education. By focusing on the most advanced practitioners, the broader purposes of second-level mathematics education are lost, as are the issues related to general mathematical literacy and the need to overcome the cultural and historical attitudes that have given rise to many of the problems facing mathematics education in Ireland today.

The depth/breadth trade-off

There has been much debate about particular content topics that have been removed from the syllabus. In line with other subjects, the syllabus for Leaving Certificate Mathematics is designed for 180 hours of class contact – typically five class periods per week. Because of the number of subjects taken by a student in Leaving Certificate, this time allocation is considerably less than the time devoted to mathematics in upper secondary education in other countries, where students generally take fewer subjects and a smaller proportion of students generally take mathematics. Since the last change in the mathematics syllabus at Leaving Certificate in the mid-90s, experience in schools has shown that the syllabus, particularly at higher level, was both long and time-demanding. Students recognised this and the less than expected take-up at higher level is at least partially due to their choosing to seek CAO points elsewhere, while achieving the minimum matriculation requirements in respect of mathematics by studying it at ordinary level. The recent surge in uptake at higher level, arising from the awarding of 25 bonus CAO points for achieving a minimum of Grade D in higher-level mathematics, shows that a greater number of students were capable of studying higher-level mathematics at Leaving Certificate.

The breadth of the syllabus needed to be reduced to leave time in the classroom for the development and assessment of conceptual understanding and of the higher-order thinking and problem-solving skills that are so valued in the 21st century, yet which were noticeably absent in candidates' work:

> *Weaknesses, by and large, relate to inadequate understanding of mathematical concepts and a consequent inability to apply familiar techniques in anything but the most familiar of contexts and presentations* (Chief Examiner's Report 2005, page 49).

In deciding on the topics to be retained or removed, care was taken to ensure that students would get a sufficient foundation in the basic mathematical concepts in all of the strands to enable them to build on these at a later stage.

The elimination of choice

A decision was taken to eliminate choice from the examination because previous examination choice had resulted in parts of the course being omitted, with the result that students were unable to make important connections between topics. Another consequence of this choice, allied to the options available in the syllabus, was that students could omit significant topics from their study of mathematics at Leaving Certificate, unaware of the consequences that this could have when they progressed to third-level education. The acquisition of good mathematical skills and understanding was considered more desirable than a procedural treatment of a large range of content. The removal of choice in the examination had been successfully introduced in Junior Certificate Mathematics as part of the syllabus change in 2000.

Issues and concerns

Concern has been expressed by commentators in respect of the following specific topics that were removed from the syllabus:

Vectors and Matrices: In the past, the study of vectors was an option at ordinary level that was avoided by many students; matrices was not part of the syllabus at this level. At higher level, the syllabus content relating to vectors and matrices was treated in a procedural manner with little application or connection to other areas of mathematics. To deal fully with these topics would require several months of study and, following discussion with third-level and engineering interests in particular, it was decided to remove these topics entirely and to focus on pedagogical practices that promote the development of skills and conceptual understanding in topics that underpin these areas of mathematics.

Calculus: The treatment of this topic in the revised syllabus has come in for particular criticism, with some commentators suggesting that it has been omitted entirely. A glance at Strand 5 of the revised syllabus, dealing with functions and calculus, will show that the basic principles of both differential and integral calculus have been retained, including differentiation from first principles, the derivatives and integrals of specified functions, and applications of both differentiation and integration. Some predominantly procedural aspects of calculus, including techniques such as integration by substitution, have been omitted from the revised syllabus at higher level. However, it should be borne in mind that the manner in which these were taught and learned in the past was far from ideal in promoting understanding and application. There is evidence from successive Chief Examiner reports that students were experiencing only a procedural treatment of calculus.

> *Answering indicated that candidates can competently execute the technique of differentiating by rule, (as evidenced by success in part (i)), but are not able to apply their knowledge with any degree of understanding, (as evidenced by their failure to engage meaningfully with part (ii)).* (Chief Examiner's Report 2005, page 32).

> *Other than part (i), this was very poorly handled by the majority, despite the fact that establishing an iterative rule for approximating a square root is a standard application of the Newton-Raphson method, and has been dealt with before on examination papers. The widespread inability to handle this question stands in sharp contrast to the fact that candidates normally have little difficulty applying the same method to find a specific approximation of a specific polynomial (a well-rehearsed routine algorithm). This indicates that candidates are not achieving the stated aim of the syllabus that they should be able to apply their knowledge in the context of "the ability to solve problems, abstract and generalise".* (Chief Examiner's Report 2005, page 61).

The need for conceptual understanding was again highlighted in the 2011 examination with students (and many teachers) reporting that they were unable to do a question that required them to recognise a graphical representation of functions of equal derivatives, which was a move away from the predictable type of question asked in the past. Since the ability to move fluently between different function representations is important in the development of conceptual understanding, time to develop this fluency was considered more critical than 'covering' large amounts of predominantly procedural content.

It should also be noted that treatment of some aspects of calculus has been given greater emphasis in the revised syllabus. For example, foundational issues such as limits, continuity, differentiability, as well as careful treatment of function definitions such as injective, bijective and surjective are explicitly mentioned in learning outcomes in the revised syllabus. Emphasis is given to analysis of functions and to applications of calculus in problems that are purely mathematical as well as in applied contexts.

A recent advertisement placed by the Dublin Institute of Advanced Studies for a pre-college course in calculus was critical of the 'severe' reduction in the calculus taught at Leaving

Certificate higher level. The advertised course was aimed at higher-level Leaving Certificate Mathematics students to prepare them for

> *third level courses in Mathematics, Science and Engineering (as well as Economics) and in order to give students with an aptitude for mathematics the opportunity to prepare themselves better for further study* (from DIAS website, www.dias.ie)

The course lists ten ‘subjects’ to be covered, the majority of which are included in the revised syllabus (under Strand 5: functions and calculus). One advertised topic/subject – Primitive functions and the Fundamental Theorem of Calculus – was not included in the previous Leaving Certificate Mathematics syllabus.

Cognitive challenge in mathematics

Much media comment on *Project Maths* has referred to the ‘dumbing down’ of the syllabus. Kirkland,Stack et al in their aforementioned paper, refer to the ‘watering down’ of the Leaving Certificate higher-level syllabus and its knock-on effect at third level. It is a matter of fact that content has been removed, although not to the degree suggested by the Dublin Institute for Advanced Studies among others, but these criticisms do not take into account the changed emphasis on conceptual understanding and the development of problem-solving skills which is advocated under *Project Maths*. Classroom activities that facilitate the development of conceptual understanding and problem-solving skills are not only more cognitively challenging than practising routine procedures to solve predictable problems, but they also promote the development of key skills which are essential for lifelong learning. Students learn to think for themselves, to make connections between different mathematical ideas, and to reason. Jo Boaler, professor of mathematics education at Stanford University in California, recently made the point that young people rarely experience real mathematics:

> *Instead of posing questions, solving real and interesting problems, using and applying methods, investigating patterns and relationships, students spend their time watching a teacher demonstrate methods and then practising them. These kinds of activities involve higher-order thinking and problem-solving skills which are essential for life in modern society* (Opinion piece, The Irish Times, 29th August 2012).

Teacher professional development

The authors of the UCC report refer to the ‘enormous burden’ of up-skilling that will be placed on teachers as a consequence of *Project Maths*. In planning the initiative, the professional development of teachers was phased to coincide with the introduction of the different strands. However, the thinking behind the initiative is that teachers should continue to engage in professional development as part of lifelong learning to enrich their professional knowledge, understanding and capabilities throughout their careers. Building the capacity of maths teachers to adopt new approaches in their classrooms and building their confidence and professional expertise are fundamental to educational progress.

As *Project Maths* was rolled out nationally, the programme of professional development provides for ten full-day workshops over five years, with the focus on methodology. In response to requests from teachers for additional support in content knowledge, these workshops were complemented by a range of optional evening courses, run in local Education Centres, which dealt mainly with mathematics topics (content) and/or with using ICT in the teaching and learning of mathematics. These supplementary courses, which were attended by significant numbers of mathematics teachers, were facilitated by trained teachers who were supported in their role and who were drawn mainly from the membership of the Irish Mathematics Teachers' Association.

In addition, elective summer maths courses have been organised at the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL) based in the University of Limerick, to meet the growing professional development needs of teachers. While these were run for teachers from the initial schools and the support personnel, the materials used and developed during these courses were made available to all teachers via a CD (first course in 2009) and on the NCE-MSTL website. Feedback from teachers in the initial schools indicated that collaborating formally and informally with colleagues was the most valuable support in helping them change their practice. As a result, the development of communities of practice is the priority for the support in these schools in the coming year and this development will inform the programme of support nationally in subsequent years. The project originally gave a commitment to supporting professional development of teachers for a minimum of five years; the response of teachers to date shows an enthusiasm for continued engagement in the process of change and for availing of the opportunities that are available to support them in doing so.

In September, a Professional Diploma in Mathematics for Teaching was launched to provide a university accredited qualification at level 8 for 'out-of-field' teachers of mathematics. This two-year, part-time course is managed by the NCE-MSTL and funded by the Department of Education and Skills. Almost 400 teachers are taking the course in its first year, and another two-year cycle will be run again from September 2013.

Text-book issues

Traditionally, mathematics textbooks in Ireland have supported the instrumental, structured approach to teaching and learning described by Lyons et al (2003). An important feature of *Project Maths* is the reduction in emphasis on practising routine or procedural questions and solutions based on illustrative examples, with more emphasis being given to students engaging in problem-solving approaches and justifying or explaining their solutions. Students are encouraged to think about their strategies, to explore possible approaches and evaluate these, and so build up a body of knowledge and skills that they can apply in both familiar and unfamiliar situations.

At all stages of the curriculum and assessment development process associated with *Project Maths*, publishers were briefed on what was emerging. Some of the publishers opted to develop supplementary texts according as the various strands were being phased in; others waited to see the full syllabus before publishing revised texts. In their feedback to the NCCA[3], while critical of the lack of dedicated textbooks for the revised syllabuses, teachers in the initial schools reported that their classroom practice now relies less heavily on using the textbook as the sole teaching resource.

Inter-connection of topics

The UCC report is critical of the lack of inter-connection between the various topics in the various strands. However, this report is based on early drafts of the syllabus strands and this time-position needs to be taken into account. While there may have been less emphasis on inter-connection between topics in the first year of introduction, when just two strands of revised topics were involved, the syllabus makes clear that students should be encouraged to make appropriate connections within and between strands, but also with other areas of learning. One of the main principles of *Project Maths* is that mathematics should be taught in contexts that allow students to make connections within mathematics, between mathematics and other subjects, and between mathematics and its application in the real world. Establishing connections across the syllabus proved a challenge for teachers due to the phased nature of the change, as reported in their feedback at school meetings with NCCA personnel. A strong emphasis is now being placed on developing connections between topics and strands, and this is made explicit in the syllabus. Teacher support is focusing on how best to organise the sequence of teaching to allow students to develop the connections. This approach was also taken in developing the Common Introductory Course as a minimum set of topics across the different syllabus strands to be studied in first year.

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[3] *Project Maths: Reviewing the project in the initial group of 24 schools* – report to Council on school visits. See www.ncca.ie/projectmaths

Information meetings for third-level personnel

Meetings have taken place with those involved in initial teacher education in the third level colleges so that their courses can be adapted to fully prepare newly qualified teachers in both the content knowledge and pedagogical knowledge required to engage with the revised syllabuses. Information/discussion meetings have also been held in third-level institutions so that academic staff could be informed about the developments in post-primary mathematics and have the opportunity to consider the implication of these for students who are progressing to third-level education – whether that be in courses that contain significant study of mathematics or courses in which there is little mathematical content. Further meetings of this nature are being planned.

4. Concluding remarks

As indicated at the outset, *Project Maths* is a work in progress. It has proved to be a challenging experience for students, teachers and schools. The comfort of well-rehearsed classroom practice that has served well over the years, the style and predictability of examination questions, and the relatively steady stream of ‘good’ results achieved have meant that the scale of change required under *Project Maths* came as a shock to the system. When teachers in the initial group of schools were asked to consider how their classroom practice has changed as a result of engagement in the project, many noted that they had achieved significant change, while others saw themselves as having a long way to go[4]. While they recognised that the road ahead is a long one, increasingly, it is a road that they can see the point of being on. The can see improved engagement on the part of their students and, in general, would not want to go back to ‘the old way’.

Students, too, have found the change difficult and disorientating, not least the unpredictability of examination questions and the context-based problems. However, on the other hand, they have become more involved in discussing mathematics problems, in making sense of the maths they are studying and in some cases deepening their mathematical knowledge and building their confidence.

Project Maths as a curriculum and assessment initiative has also proved challenging for the system more generally. The developmental nature of the syllabus revision process, the phasing of the introduction of the syllabus strands, the unprecedented scale of teacher professional development and support, the challenge of generating examination papers that really probe the kinds of understanding which *Project Maths* is trying to achieve have all proved to be a significant challenge. However, there are encouraging signs of progress towards achieving the vision of a change in mathematics education that is fundamentally for the better, and for all.

Like any significant change initiative, it requires some time before its impact can be evaluated. The NCCA is continuing to monitor the progress of the project, through its engagement with schools, with the team involved in professionally supporting teachers, and through its committees. The National Foundation for Educational Research (NFER) based in the UK has been commissioned to evaluate the impact of the project on student learning, achievement and motivation. There are four main phases to the NFER research:

[4] *Project Maths: Reviewing the project in the initial group of 24 schools* – report to Council on school visits. See www.ncca.ie/projectmaths

- internationally comparable assessment of students' achievement in all strands of the revised mathematics curriculum, based on indicator items administered to two separate cohorts of Junior Certificate and Leaving Certificate students in Spring 2012 and Autumn 2012 (examination classes of 2012 and 2013, respectively)
- attitude surveys exploring students' experience of the revised mathematics curriculum and their confidence and motivation in mathematics, administered to two separate cohorts of Junior Certificate and Leaving Certificate students in Spring 2012 and Autumn 2012 (examination classes of 2012 and 2013, respectively)
- ongoing data-rich case studies in eight of the initial 24 schools, and eight of the national rollout schools, exploring in depth students' and teachers' experiences of the revised mathematics curriculum
- qualitative analysis of students' work exploring trends in the processes being promoted in the revised mathematics curriculum and its impact upon individual students' progress, which will be conducted in Autumn 2012 focusing on the Junior Certificate and Leaving Certificate examination classes of 2013.

An interim report on this DES-funded NFER research will be published in Autumn 2012, and the final report is due to be published in May 2013.

In cooperation with the Higher Education Authority, the NCCA plans to hold a conference on mathematics education early in 2013, so that the debate can extend beyond discussion of specific curricular issues at second level and lead to a clearer and shared understanding of the role of mathematics in the broader sphere of education and the contribution it makes to the development of the individual and of society.

5. References:

Anderson, W. L. (2005) Comparison of student performance in cooperative learning and traditional lecture-based biochemistry classes. *Biochemistry and Molecular Biology Education*, 33(6), 387-393.

Bernero, J. (2000) *Motivating students in math using cooperative learning*. ERIC Document Reproduction Service No. ED446 999.

Black, P. & Wiliam, D. (1998) Assessment and classroom learning in *Assessment in Education* 5(1), 7–74.

Broadfoot, P., Pollard, A., Osborn, M., McNess, E. & Triggs, P. (1998) *Categories, standards and instrumentalism: theorizing the changing discourse of assessment policy in English primary education*, paper presented at the Annual Meeting of the American Educational Research Association, 13–17 April, San Diego, California, USA.

Crooks, T. (1988) The impact of classroom evaluation practices on students, *Review of Educational Research*, 58, pp. 438–481.

Conway, P. and Sloane, F. (2005) *International trends in post-primary mathematics education*. Dublin: NCCA.

Elwood, J. and Carlisle, K. (2003) *Examining Gender: Gender and Achievement in the Junior and Leaving Certificate Examinations 2000/2001.* NCCA Research Report No. 1. Dublin, National Council for Curriculum and Assessment.

European Commission Eurydice Network (2011). Mathematics in Education in Europe: Common Challenges and National Policies. Education, Audiovisual and Culture Executive Agency (EACEA).

Forfás (2007) and Forfás (2009) Forfás annual reports. Dublin: author

Gillies, R. M. (2004) *The effects of cooperative learning on junior high school students during small group learning.* Learning and Instruction, 14(2), 197-213.

Gordon, S. & Rees, M. (1997) *High-stakes testing: worth the price?*, Journal of School Leadership, 7,345–68.

Grannell, J. J., Barry, P. D., Cronin, M., Holland, F., & Hurley, D. (2011). Interim report on project maths. Cork: University College Cork.

Kawanaka, T., Stigler, J.W., & Hiebert, J. (1999) *Studying mathematics classrooms in Germany, Japan, and the United States: Lessons from the TIMSS videotape study*. London: Falmer Press.

Kellaghan, T., Madaus, G. F. & Raczek, A. (1996) *The use of external examinations to improve student motivation* Washington, DC, American Educational Research Association.

Kirkland, Stack, et al *Major Flaws in Project Maths*. The Irish Examiner, August 13th 2012.

Knapp, M., with Adelman, N., Marder, C., McCollum, H., Needels, C., Padilla, C., Shields, P.,Turnbull, B., & Zucker, A. (1995) *Teaching for meaning in high-poverty classrooms.* NewYork: Teachers College Press

Kohn, A. (2000) *The case against standardized testing* Portsmouth, NH, Heinemann.

Koretz, D. (1988) *Arriving at Lake Wobegon: are standardized tests exaggerating achievement and distorting instruction?,* American Educator, 12(2), 8 – 15, 46 – 52.

Koretz, D., Linn, R. L., Dunbar, S. B. & Shepard, L. A. (1991) *The effects of high-stakes testing on achievement: preliminary findings about generalization across tests,* paper presented at the annual meeting of the American Educational Research Association, 11 April, Chicago.

Linn, R. L., (1990) *The Nation's report card goes home: Good news and bad about trends in achievement*. Phi Delta Kappan, 72 (2), October, 127-133;

Lutfiyya, L. A. (1998) *Mathematical thinking of high school students in Nebraska*. International Journal of Mathematics Education and Science Technology. Vol. 29 (1), 55 − 64.

Lyons, M., Lynch, K., Close, S., Sheeran, E., and Boland, P. (2003) *Inside Classrooms: a Study of Teaching and Learning*. Dublin: Institute of Public Administration.

NCCA (2005) *Review of Mathematics in Post-Primary Education: a discussion paper*, Dublin: NCCA.

OECD (2009) *Creating effective teaching and learning environments: First results from TALIS. Main report*. Paris: Author.

O'Donoghue, J. (2002) *Mathematics transition from second level to university*; Fr. Ingram memorial lecture to the Irish Mathematics Teachers' Association, November 2002.

O'Keeffe, L., O'Donoghue, J. (2011) *A review of school text books for Project Maths.* NCE-MSTL Limerick.

Oldham, E. (2001) The culture of mathematics education in Ireland: keeping the faith? *Irish Educational Studies*, 20, 266-277.

Pollard, A., Triggs, P., Broadfoot, P., McNess, E. & Osborn, M. (2000) *What Pupils Say: changing policy and practice in primary education* London, Continuum.

Reay, D & Wiliam, D (1999) *I'll be a nothing': structure, agency and the construction of identity through assessment [1]',* British Educational Research Journal, Vol. 25, No. 3, pp. 343-354.

Schoenfeld, A. H. (1992) *Learning to Thinking Mathematically: Problem Solving, Metacognition and Sense−Making in Mathematics.* In D. Grouws (Ed), Handbook of Research on Mathematics Teaching and Learning, pp.334 − 370, New York:MacMillan.

Smyth, E. and Hannan, C. (2002) Who Chooses Science? Subject Take-up in Second-Level Schools. Dublin: Liffey Press/ESRI

Smyth, E., McCoy, S., and Darmody, M. (2004). Moving Up: *The experiences of First-Year Students in Post-Primary Education*. Dublin: Liffey Press/ESRI

Stiggins, R. J. (1999) *Assessment, student confidence and school success,* Phi Delta Kappan, 81(3), 191–8.

The Teaching Council (2012) *Policy on the Continuum of Teacher Education.* [online] Available online at: http://www.teachingcouncil.ie/_fileupload/Teacher%20Education/FINAL%20TC_Policy_Paper_SP(1).pdf.

Walmsley, A. L. (2003) *Cooperative learning and its effects in a high school geometry classroom.* Mathematics Teacher, 96(2), 112-116.

Whicker, K.M., Bol, L., & Nunnery, J. A. (1997) Cooperative learning in the secondary mathematics classroom. Journal of Education Research, 91(1), 42-48.

Yamarik, S. (2007) *Does cooperative learning improve student learning outcomes*? Journal of Economic Education, 38(3), 259-277.

6. Appendix

NCCA Board of Studies for Mathematics 2009-2011

Name			Nominated by
Mr	John	McGuinness	ACCS
Ms	Brigid	Cleary	ASTI
Mr	Christy	Maginn	ASTI
Dr	Seán	Delaney	Co-opted
Dr	David	Flannery	Co-opted
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UNIVERSITY OF MANCHESTER

Leaving Certificate Mathematics: a comparative analysis

Author: Dr. Sue Pope, University of Manchester

1st March 2013

Report commissioned by NCCA

Leaving Certificate Mathematics: a comparative analysis

This report compares the Leaving Certificate Mathematics syllabus for examination from 2015 with five other countries/ jurisdictions.

Contents

Introduction

This work was commissioned by Ireland's National Council for Curriculum and Assessment (NCCA) in Spring 2013, with a view to seeing how Ireland's Leaving Certificate Mathematics syllabus under the *Project Maths* initiative compares with curriculum expectations around the world.

The following countries/jurisdictions were used in the comparison: Finland, Massachusetts, New Zealand, Singapore and Scotland. They represent a subset of the countries analysed by Hodgen et al. (2013) in the Nuffield study: *Towards universal participation in post-16 mathematics: lessons from high-performing countries* which looked in depth at countries with relatively high participation rates identified in their 2010 study: *Is the UK an outlier?* (Hodgen et al. 2010a) These countries are most similar to Ireland in terms of participation in upper secondary mathematics. The table below shows how the selected countries compare with Ireland[1].

Extract from table 6: What are the participation rates in upper secondary mathematics education? (Hodgen et al. 2010a: 38)

Country	Not studying mathematics	Studying any mathematics	Studying advanced mathematics	Compulsory/ proportion of curriculum
Ireland (NCCA data)	Negligible: 0-5%	All: 95-100%	Low: 0-15%	N, one of seven
Finland	Negligible: 0-5%	All: 95-100%	Medium: 16-30%	Y, one of seven or eight
Massachusetts (USA)	Few: 6-20%	Most: 81-94%	Medium: 16-30%	Y, one of seven
New Zealand	Some: 21-50%	Many: 51-80%	High: 31-100%	Y, one of eight
Scotland	Many: 51-80%	Some: 21-50%	Medium: 16-30%	N, one of five
Singapore	Some: 21-50%	Many: 51-80%	High: 31-100%	N, one of six

Three of these six countries include mathematics as a compulsory subject in the upper secondary curriculum. Of those that don't include mathematics as a compulsory subject in the upper secondary curriculum, two of them (Scotland and Singapore) have a narrower curriculum than the other countries.

This report briefly sets out the provision in each country under the following themes:

- Curriculum Aims/Objectives
- Proportion of curriculum
- Strands and areas of study
  - depth of treatment
  - progression across levels
- Evidence of problem-solving approach
- Assessment of syllabus objectives

[1] Note that the data for Ireland has been provided by NCCA.

These are then compared with Ireland's Leaving Certificate Mathematics. The final section of this report summarises the findings from the comparisons and includes:

- Observations of what is included/not included in Ireland's Leaving Certificate Mathematics
- The extent to which connections between topics are made
- Problem-solving approaches
- Differentiation and progression issues
- Clarity of the strands of study and the learning outcomes

Only the publicly available documentation concerned with the curriculum and assessment for upper secondary level were scrutinised for this report.

The countries/jurisdictions

Ireland

The Leaving Certificate Mathematics curriculum aims to develop knowledge, skills and understanding for study, life and work. The objectives for Leaving Certificate Mathematics are:

- the ability to recall relevant mathematical facts
- instrumental understanding ("knowing how") and necessary psychomotor skills (skills of physical co-ordination)
- relational understanding ("knowing why")
- the ability to apply mathematical knowledge and skill to solve problems in familiar and unfamiliar contexts
- analytical and creative powers in mathematics
- an appreciation of mathematics and its uses
- a positive disposition towards mathematics (NCCA, 2012: 2)

Students study mathematics alongside six other subjects. The vast majority study mathematics as it is essential for entry to higher education.

The content is set out in five strands: Statistics and Probability; Geometry and Trigonometry; Number; Algebra and Functions, at three different levels: Foundation, Ordinary and Higher. The levels are progressive and a student working at ordinary level is expected to be competent at foundation level material. At higher level students are expected to have engaged with learning outcomes at both foundation and ordinary level. The content for each level is helpfully set out alongside one another, which is unique to Ireland at this level of study amongst the countries investigated.

The development of synthesis and problem-solving skills is articulated through each content strand of the curriculum:

- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts
- analyse information presented verbally and translate it into mathematical form
- devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions (NCCA 2012: 19)

At each of the three levels students are assessed by two examination papers, for which calculators are allowed, comprising two sections: section A: core mathematics topics with a focus on concepts and skills, section B: context-based applications (NCCA, 2012: 85).

In addition there is an Applied Mathematics qualification that is taken by very few students and assesses material beyond the Leaving Certificate Mathematics syllabus. As it is currently under review it has not been included in this comparison.

Finland

The upper secondary curriculum for mathematics has two pathways: Advanced and Basic, set out in just ten pages. They have the following objectives:

Objectives of Basic syllabus	Objectives of Advanced syllabus
• be able to use mathematics as an aid in everyday life and social activities; • obtain positive learning experiences when working with mathematics and learn to trust their own abilities, skills and thinking; find courage to engage in experimental, exploratory and inventive learning; • acquire such mathematical skills, knowledge and capabilities that will create a sufficient foundation for further studies; • internalise the significance of mathematics as a tool which can be used to describe, explain and model phenomena and to draw conclusions; • form an overview of the nature of mathematical knowledge and its logical structure; • gain practice in receiving and analysing information provided by the media in a mathematical form and in assessing its reliability; • acquaint themselves with the significance of mathematics in the development of culture; • learn to use figures, formulae and models in support of thinking. (Finnish National Board of Education, 2003: 129)	• become accustomed to persistent work, thus learning to trust their own mathematical abilities, skills and thinking; • find courage to adopt experimental and exploratory approaches, discover solutions and assess these critically; • understand and be able to use mathematical language, so as to be capable of following mathematical presentations, reading mathematical texts and discussing mathematics, and learn to appreciate precision of presentation and clarity of argumentation; • learn to perceive mathematical knowledge as a logical system; • develop their skills to process expressions, draw conclusions and solve problems; • gain practice in processing information in a way characteristic of mathematics, become accustomed to making assumptions, examining their validity, justifying their reasoning and assessing the validity of their arguments and the generalisability of the results; • gain practice in modelling practical problem situations and making use of various problem-solving strategies; • know how to use appropriate mathematical methods, technical aids and information sources. (Finnish National Board of Education, 2003: 123)

All students have to study mathematics as part of a curriculum which includes six or seven other subjects. The basic course comprises six compulsory courses and two specialisation courses. The advanced mathematics course comprises ten compulsory courses and three specialisation courses. It is possible for students to move from the advanced course to the basic course (see the mapping of core courses in the summary table below).

Advanced courses (core)	Basic Courses (core, includes mapping)
MAA1 Functions and Equations	MAB1 Expressions and Equations
MAA2 Polynomial Functions	
MAA3 Geometry	MAB2 Geometry
MAA4 Analytical Geometry	
MAA5 Vectors	
MAA6 Probability and Statistics	MAB5 Statistics and Probability
MAA7 The derivative	MAB4 Mathematical Analysis
MAA8 Radical and logarithmic functions	MAB3 Mathematical models I
MAA9 Trigonometric functions and number sequences	
MAA10 Integral calculus	
	MAB6 Mathematical models II
Advanced courses (specialisation)	**Basic Courses (Specialisation)**
MAA11 Number theory and logic	MAB7 Commercial mathematics
MAA12 Numerical and algebraic methods	MAB8 Mathematical models 3
MAA13 Advanced differential and integral calculus	

As part of the Finnish Matriculation examination, mathematics is assessed at either the Basic or Advanced level through a 15 item, six hour test based on all the compulsory and specialisation courses. Students are expected to complete ten questions and may use calculators and formulae booklets.

Massachusetts (USA)

The Massachusetts curriculum framework for mathematics was revised in 2011. It defines eight standards for mathematical practice that describe mathematical proficiency:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning (Massachusetts Curriculum Framework for Mathematics, 2011: 15-17)

All students have to study mathematics as one of seven subjects. For high school mathematics (grades 9 to 12, ages 15 to 18) the mathematics standards are presented by conceptual category:

- Number and quantity
- Algebra
- Functions
- Modelling
- Geometry
- Statistics and probability (Massachusetts Curriculum Framework for Mathematics, 2011: 66)

Except for modelling, the conceptual categories are further subdivided into domains with a unique code under which there will be a number of individual standards. The modelling cycle is included and aspects of the other standards relevant to modelling are starred. The table below summarises the domains and number of associated standards (summary of Massachusetts Curriculum Framework for Mathematics, 2011: 67-94).

Conceptual Category	Domain	Code	Standards
Number and Quantity	The real number system	N.RN	3
	Quantities*	N.Q	3
	The complex number system	N.CN	9
	Vector and matrix quantities	N.VM	12
Algebra	Seeing structure in expressions	A.SSE	4
	Arithmetic with polynomials and rational expressions	A.APR	7
	Creating equations*	A.CED	4
	Reasoning with equations and inequalities	A.REI	12
Functions	Interpreting functions*	F.IF	10
	Building functions*	F.BF	5
	Linear, quadratic and exponential models*	F.LE	5
	Trigonometric functions	F.TF	9
Geometry	Congruence	G.CO	13
	Similarity, right triangles and trigonometry	G.SRT	11
	Circles	G.C	5
	Expressing geometric properties with equations	G.GPE	7
	Geometric measurement and dimension	G.GMD	4
	Modelling with geometry*	G.MG	4
Statistics and Probability	Interpreting categorical and quantitative data*	S.ID	9
	Making inferences and justifying conclusions*	S.IC	6
	Conditional probability and the rules of probability*	S.CP	9
	Using probability to make decisions*	S.MD	7

These standards are arranged into model courses: traditional (Algebra I, Geometry, Algebra II) and integrated (Mathematics I, II, III). In addition there are two advanced model courses: Pre-calculus and Quantitative methods.

The table below shows the intended progression and coherence by mapping the conceptual categories, domains and standards to the integrated mathematics and advanced courses (summary of Massachusetts Curriculum Framework for Mathematics, 2011: 119-152).

	Number and quantity	**Algebra**	**Functions**	**Geometry**	**Statistics and probability**
Mathematics I	N.Q 1-3	A.SSE 1 A.CED 1-4 A.REI 1,3,5, 10-12	F.IF 1-10 F.BF 1-3 F.LE 1-5	G.CO 1-8,12 G.GPE 4,5,7	S.ID 1-3,5-9
Mathematics II	N.RN 1-3 N.CN 1,2,7-9	A.SSE 1-3 A.APR 1 A.CED 1,2,4 A.REI 4,7	F.IF 4-8,10 F.BF 1,3,4 F.LE 3 F.TF 8	G.CO 9-11 G.SRT 1-8 G.C 1-5 G.GPE 1-4,6 G.GMD 1,3	S.CP 1-9 S.MD 6,7
Mathematics III	N.CN 8,9	A.SSE 1,2,4 A.APR 1-7 A.CED 1-4 A.REI 2,11	F.IF 4-9 F.BF 1,3,4 F.LE 4		S.IC 1-6 S.MD 6,7
Pre-calculus	N.CN 3-6,8,9 N.VM 1-12	A.APR 5-7 A.REI 8,9	F.IF 7 F.BF 1,4,5 F.TF 3,4,6,7,9	G.SRT 9-11 G.C 4 G.GPE 3 G.GMD 2,4	
Quantitative reasoning	N.VM 1-12	A.APR 5 A.REI 8,9	F.TF 3,4,5,7,9	G.SRT 11 G.C 4 G.GPE 3 G.GMD 2,4 G.MG 3,4	S.ID 9 S.IC 4-6 S.CP 8,9 S.MD 1-7

The Massachusetts Comprehensive Assessment System (MCAS) assesses high school mathematics in grade 10. MCAS requires that Number & quantity, Algebra & functions, Geometry and Statistics & probability are weighted 2:3:3:2. The vast majority of students take MCAS, which comprises external tests in mathematics, English language arts (ELA) and one of four science or engineering technology. MCAS grade 10 is a component of the high school diploma and is the basis of school accountability. The MCAS grade 10 test comprises two one hour papers: one without a calculator comprising multiple choice and short response items and one with a calculator comprising short response and open response items. In order to enter higher education, students must successfully complete the high school courses Mathematics I, II and III or the traditional alternative.

New Zealand

The New Zealand curriculum was revised in 2007 following a comprehensive review of international research (Anthony & Walsh, 2007). Mathematics and statistics is one of eight learning areas for years 1 to 13 of compulsory schooling (age 5 to 18). The objectives are:

1. Inspire thinking

Mathematics and statistics help make sense of information, experience, and ideas by engaging students to think:

- flexibly and creatively
- critically and effectively
- strategically and logically

2. Stimulate creativity and curiosity

Mathematics and statistics open the door to a world of beauty, mystery, and awe. They provide students with the enjoyment of intellectual challenge: opportunities to explore ideas and to wrestle with interesting problems. Mathematics and statistics provide ways of connecting abstract ideas with real world thinking.

3. Equip students for the 21st century

Mathematics and statistics equip students with the knowledge and skills to be global citizens in the 21st century. Effective citizens have the ability and inclination to use mathematics and statistics at home, at work, and in the community by:

- using problem-solving strategies
- using mathematical and statistical models to solve problems
- making sensible estimates
- using and interpreting data
- evaluating mathematical and statistical information
- communicating ideas (http://seniorsecondary.tki.org.nz/Mathematics-and-statistics/Rationale)

Content requirements are set out as achievement objectives in eight levels. Levels 6, 7 and 8 comprise the upper secondary expectations. At level 6 there are three strands for mathematics and statistics: Number and algebra (NA), Geometry and measurement (GM) and Statistics (S), and just two strands at levels 7 and 8: Mathematics (M) and Statistics (S).

Mathematics and statistics is assessed as part of the National Certificate of Educational Achievement (NCEA) available at levels 1 to 3 (roughly equivalent to achievement objectives levels 6, 7 and 8 respectively). The NCEA comprises short coherent units at the three levels which are largely internally assessed. At level one there is an externally set one hour, non-calculator mathematics common assessment task (MCAT) taken by most students in year 11. The other level one units comprise nine internally assessed units and three externally assessed units. At level two there are three externally assessed units and eleven internally assessed units, and at level three of the 15 available units, six are externally assessed. The units allow students to build up a personal profile of mathematical achievement that contributes to their overall upper secondary education achievement. Apart from MCAT, externally assessed units are sat three at a time in a three hour sitting. MCAT is unusual in that it is internally marked and the marking is externally moderated. Internally assessed units are subject to both internal and external moderation.

NCEA assessment units for mathematics and statistics (assessed internally and **externally**) (summarised from http://ncea.tki.org.nz/Resources-for-aligned-standards/Mathematics-and-statistics)

Level 1	Level 2	Level 3
Apply numeric reasoning in solving problems	Apply co-ordinate geometry methods in solving problems	Apply the geometry of conic sections in solving problems
Apply algebraic procedures in solving problems (MCAT)	Apply graphical methods in solving problems	Apply linear programming methods in solving problems
Investigate relationships between tables, equations and graphs	Apply sequences and series in solving problems	
Apply linear algebra in solving problems	Apply trigonometric relationships in solving problems	Apply trigonometric methods in solving problems
Apply measurement in solving problems	Apply network methods in solving problems	Use critical path analysis in solving problems
Apply geometric reasoning in solving problems	**Apply algebraic methods in solving problems**	**Apply the algebra of complex numbers in solving problems**
Apply right-angled triangles in solving measurement problems	**Apply calculus methods in solving problems**	**Apply differentiation methods in solving problems**
		Apply integration methods in solving problems
Apply knowledge of geometric representations in solving problems	Design a questionnaire	Investigate time series data
		Investigate bivariate measurement data
Apply transformation geometry in solving problems	Use statistical methods to make an inference	Use statistical methods to make a formal inference
Investigate a given multivariate data set using the statistical enquiry cycle	Conduct an experiment to investigate a situation using statistical methods	Conduct an experiment to investigate a situation using experimental design principles
Investigate bivariate numerical data using the statistical enquiry cycle	Evaluate a statistically based report	**Evaluate statistically based reports**
Demonstrate understanding of chance and data	**Apply probability methods in solving problems**	**Apply probability concepts in solving problems**
Investigate a situation involving elements of chance	Investigate a situation involving elements of chance using a simulation	**Apply probability distributions in solving problems**
	Apply systems of equations in solving problems	Apply systems of simultaneous equations in solving problems

Scotland

The curriculum in Scotland has undergone considerable revision with the recent introduction of the Curriculum for Excellence. The overarching aims are that each young person becomes a successful learner, a confident individual, a responsible citizen and an effective contributor. Numeracy and mathematics is one of eight subject areas, set out in terms of experiences and outcomes at early, first, second, third and fourth stages (typically age 16). The curriculum is written in learner-friendly language including the objectives: My learning in mathematics enables me to:

- develop a secure understanding of the concepts, principles and processes of mathematics and apply these in different contexts, including the world of work
- engage with more abstract mathematical concepts and develop important new kinds of thinking
- understand the application of mathematics, its impact on our society past and present, and its potential for the future
- develop essential numeracy skills which will allow me to participate fully in society
- establish firm foundations for further specialist learning
- understand that successful independent living requires financial awareness, effective money management, using schedules and other related skills
- interpret numerical information appropriately and use it to draw conclusions, assess risk, and make reasoned evaluations and informed decisions
- apply skills and understanding creatively and logically to solve problems, within a variety of contexts
- appreciate how the imaginative and effective use of technologies can enhance the development of skills and concepts (Numeracy and Mathematics experiences and outcomes, available at http://www.educationscotland.gov.uk/learningteachingandassessment/curriculumareas/mathematics/eandos/index.asp)

Upper secondary students usually study five subjects. Although mathematics is not compulsory it is a popular choice, with just under half of all students studying some mathematics in upper secondary and just over half of these studying advanced mathematics (Higher). A much smaller proportion study the Advanced Higher which goes beyond the expectations of most other countries in this comparison. Students studying the Advanced Higher typically do so with just two other subjects and may progress directly to the second year of the four-year degree courses that are typical of Scottish universities (Hodgen et al, 2013)

The table below summarises the content strands in the Scottish curriculum (http://www.educationscotland.gov.uk and http://www.sqa.org.uk):

Early – Fourth Stage	Intermediate	Higher	Advanced Higher
Number, money and measure Shape, position and locations Information handling	Arithmetic Algebra Geometry Trigonometry Statistics	Algebra Geometry Trigonometry Statistics Elementary calculus	Algebra Geometry Calculus

All of the mathematics qualifications (http://www.sqa.org.uk) have the same problem solving objectives:

- Interpret the problem and consider what might be relevant
- Decide how to proceed by selecting an appropriate strategy
- Implement the strategy through applying mathematical knowledge and understanding, and come to a conclusion
- Decide on the most appropriate way of communicating the solution to the problem in an intelligible form

At age 16 most students take the Standard Grade (SG) qualification which is available at three levels: Foundation, General and Credit. For the fifth stage or senior phase (upper secondary) the curriculum is set out in the qualifications: Intermediate 1 (SG general), Intermediate 2 (SG credit), Higher and Advanced Higher. There are also Access qualifications for those working below the level expected for most students at the end of the fourth stage. Each qualification comprises three units assessed internally (usually through a closed book test taken under controlled conditions, available from the National Assessment Bank) and an externally set and marked exam. The table below summarises the duration and papers.

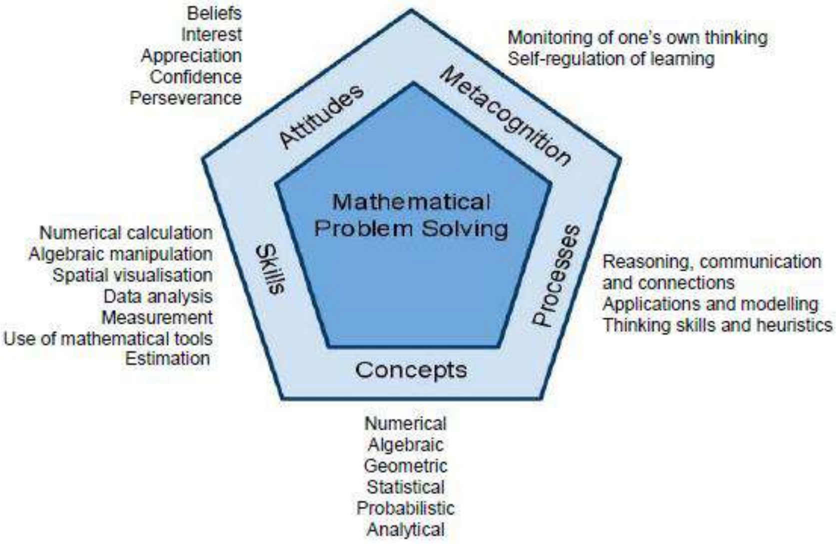
	Intermediate 1	**Intermediate 2**	**Higher**	**Advanced Higher**
Total duration	1h 30min	2h	2h 40min	3h
Papers	1. Non-calculator 2. Calculator	1. Non-calculator 2. Calculator	1. Non-calculator 2. Calculator	1. Calculator

Singapore

The curriculum in Singapore has undergone considerable change recently. The changes are being introduced year by year through the four years of the secondary phase and will be complete by 2016. The broad aims for the mathematics curriculum are:

- Acquire and apply mathematical concepts and skills
- Develop cognitive and metacognitive skills through a mathematical approach to problem solving
- Develop positive attitudes towards mathematics (Teaching and Learning syllabus, Chapter 1)

Approximately two thirds of all students study mathematics during upper secondary education and about 40% study advanced mathematics (H1 or H2). Students usually study six subjects in upper secondary and it is expected that there is breadth through a contrasting subject. Problem solving is at the heart of the Singapore curriculum and has been since 1990. The problem solving framework (below) stresses conceptual understanding, skills proficiency and mathematical processes and is used by all teachers of mathematics.



(Teaching and Learning Syllabus, Chapter 2)

The content strands in the new secondary curriculum are set out as Number and algebra; Geometry and measurement; and Statistics and probability with a Mathematical processes strand which emphasises modelling, problem-solving and making connections. (Teaching and Learning Syllabus, Chapter 3). These strands are used for O level. At A level pure mathematics and statistics are distinct strands.

	H1	H2	H3
Pure mathematics	Functions and graphs Calculus	Functions and graphs Sequences and series Vectors Complex numbers Calculus	Functions and graphs, Sequences and series, and Calculus Differential equations as mathematical models
Statistics		Permutations, combinations and	Combinatorics

	Probability Binomial and normal distributions Sampling and hypothesis testing Correlation and regression	probability Binomial, Poisson and normal distributions Sampling and hypothesis testing Correlation and regression	

Singaporean qualifications are hierarchical and summarised below (adapted from Teaching and Learning Syllabus, Chapter 1):

	Academic		**Vocational**
Pre-university		A level H3	
	A level H1	A level H2	
Secondary		O level Additional mathematics	
	O level		
	National (academic) level		National (technical) level

For post-16 academic students, the A level qualifications are available at H1, H2 and H3. H1 is a subset of H2 and is not taken by students who have completed O level Additional Mathematics – they can progress directly to H2. H3 is aimed at the highest attainers and develops content from H2 and extends the use of differential equations to model real phenomena including mechanics and populations. This content is beyond that of other countries.

All of the qualifications are externally assessed. In all examinations technology is expected: calculators at O level and graphing calculators at A level. The table below summarises duration and number of items.

	O level	**A level H1**	**A level H2**	**A level H3**
Paper 1	2h 25 short answer items – fundamental skills and concepts	3h A pure 5 items B statistics 6-8 items	3h pure 10-12 items	3h Entire syllabus About 8 items
Paper 2	2h 30mins 10-11 items - higher order thinking		3h A pure 3-4 items B statistics 6-8 items	

Comparing the countries/jurisdictions

Comparing curriculum aims and objectives

The table below summarises the curriculum aims and objectives for the six countries. Note that the aims and objectives for comparator countries have been rephrased and reordered to match those of Ireland.

Ireland	Finland	Massachusetts	New Zealand	Scotland	Singapore
factual recall					
knowing how		use appropriate tools strategically; attend to precision	carry out procedures flexibly and accurately	essential numeracy skills	skills, processes
knowing why	mathematics as a logical system	reason abstractly and quantitatively, develop understanding; look for and make use of structure and pattern	estimate with reasonableness, understand when results must be interpreted with a degree of uncertainty	secure understanding of mathematics	concepts
application to problems	mathematics for modelling and problem solving	make sense of problems and persevere in solving them	wrestle with interesting problems	apply skills and understanding creatively and logically to solve problems	mathematical problem solving
analytical and creative powers	draw conclusions, assess reliability of information; assess validity, justify reasons	stimulate curiosity, construct viable arguments and critique the reasoning of others	think creatively, critically, strategically and logically	interpret information, draw conclusions, assess risk, reasoned evaluations, informed decisions	thinking skills, metacognition
appreciation of uses	contribution of mathematics to culture and society	model with mathematics	practical applications in everyday life, other learning areas and the workplace	understand the application of mathematics now and in the past; financial capability	mathematical problem solving

positive disposition	trust their own abilities; persistence & courage	create enjoyment of mathematics	enjoy intellectual challenge	firm foundations for further specialist learning	positive attitudes towards mathematics
				use of technology	

There is considerable commonality: problem solving, procedural competence, relational understanding, positive dispositions, and analytical and critical thinking. Only Ireland has factual recall as an objective for mathematics education at this level. Only Scotland explicitly mentions technology but Ireland, Finland and Singapore expect technology use in all exams. Massachusetts and New Zealand have some external assessments without a calculator. Most countries have an objective around appreciation of mathematics but only Scotland explicitly mentions the history of mathematics and Finland mentions the contribution of mathematics to culture. Scotland also includes personal financial capability as an objective for mathematics. Although this is not an objective for other countries, many include arithmetic related to personal finance in the content for lower levels (see the content analysis).

Comparing content strands

The table below summarises the main content strands for each country. Note that the aims and objectives for comparator countries have been rephrased and reordered to match those of Ireland.

Ireland	Finland	Massachusetts	New Zealand	Scotland	Singapore
Statistics and probability	Probability and statistics	Statistics and probability	Statistics	Statistics and probability	Statistics and probability
Geometry and trigonometry	Geometry	Geometry	Geometry and measurement	Geometry	Geometry and measurement
Number	Commercial arithmetic (B) Number theory (A)	Number and quantity	Number and algebra	Arithmetic	Number and algebra
Algebra	Algebra	Algebra	level 7 & 8 Mathematics includes functions and algebra	Algebra	
Functions and calculus	Functions and calculus	Functions-*not calculus* - includes trigonometry		Algebra & trigonometry	Calculus (A level)
Synthesis and Problem solving skills	Mathematical modelling	Mathematical modelling	Problem-solving and modelling	Problem-solving	Mathematical processes

All countries include Statistics and Probability. New Zealand's Statistics incorporates Probability. Finland and Scotland have relatively little Statistics in their curricula. New Zealand has Mathematics and Statistics as the named subject area (rather than Mathematics) and, like Ireland and Massachusetts, has a greater emphasis on statistical literacy. New Zealand (level 8), Scotland (H) and

Singapore (H1) include linear regression. New Zealand and Singapore combine Number and Algebra into a single strand. Some countries like Ireland combine Geometry and Trigonometry into a single strand but others include Trigonometry as part of Functions (e.g. Scotland).

There is a detailed mapping of the different countries' content against the strands and levels of the Ireland curriculum in the appendix. As evidenced in the mapping many countries have a similar progression to Ireland. The main areas of difference are in Statistics and Geometry. Finland and Scotland have relatively little statistics and Scotland and Singapore have relatively little synthetic geometry. Finland is most different with little of its Basic syllabus mapping to Ireland's Foundation level. Finland's progression is also distinctive, for example geometric and arithmetic progressions are in the Basic syllabus whereas they are in Ireland's Higher level. Ireland is consistent with most of the other countries.

Ireland and Massachusetts have a greater emphasis on synthetic geometry at all levels compared with Finland, New Zealand, Scotland and Singapore. The latter three countries have a similar emphasis at the lower levels but not in their advanced mathematics curriculum.

Massachusetts is unique in not including calculus in its senior high school curriculum. At the advanced level, many countries go beyond Ireland in the study of calculus. Finland (advanced), New Zealand (level 8), Scotland (Advanced Higher) and Singapore (H2, H3) all include differential equations. Finland (advanced) includes volumes of integration. Singapore (H2, H3) and Scotland (AH) include MacLaurin series and Scotland (AH) includes calculus of inverse trigonometric functions.

All of the countries considered except Finland include complex numbers although most introduce them at a level beyond that of Ireland (OL), and the extent of development is less than in Ireland. For example, only Scotland (AH) and Singapore (H2) include De Moivre's theorem for a very small proportion of students.

Although Ireland (OL) includes applying differentiation to rates of change, it does not explicitly mention speed-time and distance-time unlike Scotland (Intermediate 1) and Singapore (O level). Linear programming is included in Finland (MAB6) and Massachusetts refers to using inequalities for modelling (A.CED 3). Transformation of functions is explicitly mentioned in the curriculum for Massachusetts (F.BF 3), Scotland (A, AH) and Singapore (H2). Conic sections are included in the Massachusetts curriculum (G.GPE – advanced courses only)

New Zealand (levels 7 and 8) includes network diagrams and critical path analysis. Scotland also includes networks (Intermediate 1 and 2). Finland has an advanced specialisation number theory course which includes Euclid's algorithm; Euclid's algorithm is also included in Scotland's Advanced Higher.

Unlike Ireland, Finland, Massachusetts, Scotland and Singapore include vectors, although Massachusetts only deals with vectors in advanced courses. Massachusetts (advanced courses) and Scotland (AH) include matrices and using matrices to solve systems of simultaneous linear equations. Singapore includes matrices and sets at O level.

The detailed mapping shows that there is considerable consensus amongst the six countries around the content of upper secondary mathematics. Some of the exceptions could be attributed to the breadth of the upper secondary curriculum and whether or not mathematics is compulsory. For example, Singapore (H3) and Scotland (AH) have a relatively narrow curriculum and allow greater mathematics specialisation than countries where mathematics is compulsory or a minimum of seven subjects is studied (Ireland, Finland, Massachusetts and New Zealand).

Comparing approaches to problem-solving

The table below summarises the key features of each country's approach to problem-solving. Note that Finland, Massachusetts and New Zealand place greater emphasis on modelling than on problem-solving. There is considerable consistency between each country's aspirations for their learners to be able to use their mathematics for making sense of situations and reaching justifiable conclusions.

Ireland	**Finland**	**Massachusetts**	**New Zealand**	**Scotland**	**Singapore**
explore patterns	exploratory approaches	try different approaches	seek patterns and generalisation		simplify a problem, consider special cases
formulate conjectures	discover solutions	develop conjectures	conjecture	interpret the problem	make inferences
justify conclusions	justify reasoning, assess validity of arguments and generalisability of results	construct viable arguments, review conclusions	justify and verify solutions and conclusions		justify solutions
communicate mathematics	use mathematical language	communicate results and respond to others' arguments	process and communicate information	communicate the solution	write mathematical arguments
solve problems in familiar and unfamiliar contexts	model practical problem situations	solve progressively deeper, broader and more sophisticated problems	wrestle with interesting problems	implement the problem-solving strategy	solve problems in a variety of contexts
translate information into mathematical form	use appropriate mathematical methods, technical aids and information sources	model with mathematics	create and use mathematical and statistical models to solve problems	interpret the problem	mathematical modelling process
devise, select and use appropriate mathematical models, formulae or techniques		try different approaches		select an appropriate strategy	use appropriate mathematics
draw relevant conclusions	draw conclusions				draw logical conclusions

Comparing approaches to assessment

There is considerable variation across the countries in how assessment is carried out. Like Ireland, Singapore is 100% external assessment. In Scotland there is unit assessment that is internally administered and assessed, alongside an externally assessed exam. New Zealand has a unitised system: most units are internally assessed and some are externally assessed through one hour exams that are usually taken in a three hour sitting. New Zealand's unitised system allows for personalisation of learning, which has increased participation in mathematics by more than 10% (Hodgen et al., 2010b). Students in Finland take a six hour test that is part of the Matriculation examination. The tests are marked both internally and externally. Massachusetts has a grade 10 exam that all students must complete to gain their school diploma. In addition they must successfully complete the high school mathematics courses (Mathematics I, II, III or Algebra I, II and geometry) in order to apply for college or university.

All of the countries allow calculators in at least part of their assessment. Like Ireland, Finland and Singapore expect them to be used in all the assessments. In Singapore graphing calculators are required for A levels (H1, H2, H3). Examinations comprise a range of item types including multiple choice (Massachusetts), short response (Ireland, New Zealand, Scotland, Singapore) and extended response.

Conclusions

This report is based on the publicly available documentation about curriculum and assessment for upper secondary mathematics education in Ireland, Finland, Massachusetts, New Zealand, Scotland and Singapore. There is considerable commonality around the aims and objectives, curriculum content and progression, and aspirations for problem-solving.

The mathematics expectations in Ireland are comparable to those in the other countries that offer a broad upper secondary curriculum. There are some distinctive features, as discussed in the report. The NCCA may wish to consider the following points in the future development of its Leaving Certificate Mathematics curriculum:

- Should factual recall be the first objective for leaving certificate mathematics?
- Should there be more emphasis on modelling alongside problem-solving?
- Should complex numbers be included at Ordinary level?
- Should speed-time and distance-time graphs be explicitly mentioned?
- Should transformations of functions be explicitly mentioned?
- Should vectors and matrices be introduced into the curriculum?
- Should differential equations be included in the development of calculus?

Obviously any additional content may be at the expense of synthetic geometry, proof and the emphasis on statistics; some distinctive elements that Ireland may consider worth retaining. Some of this additional content, in particular differential equations and vectors and matrices, may be worth considering in the review of the Applied Mathematics qualification.

A wider range of assessment types might allow greater personalisation of the curriculum (as In New Zealand) which may increase participation in advanced mathematics.

References[2]

Anthony, G. & Walshaw, M. (2007) *Effective Pedagogy in Mathematics/Pàngarau: Best Evidence Synthesis Iteration*, Wellington: New Zealand Ministry of Education http://www.educationcounts.govt.nz/publications/series/2515/5951

Finnish National Board of Education (2003) *National Core Curriculum for Upper Secondary School* Helsinki: Finnish National Board of Education

The Finnish Matriculation Examination http://www.ylioppilastutkinto.fi/en/

Hodgen, J., Pepper, D., Sturman, L. & Ruddock, G. (2010a) *Is the UK an Outlier? An international comparison of upper secondary mathematics* London: Nuffield Foundation

Hodgen, J., Pepper, D., Sturman, L. & Ruddock, G. (2010b) *An international comparison of upper secondary mathematics: 24 country profiles* London: Nuffield Foundation

Hodgen, J., Marks, R. & Pepper, D. (2013) *Towards universal participation in post-16 mathematics: lessons from high-performing countries* London: Nuffield Foundation

Massachusetts Department of Elementary and Secondary Education http://www.doe.mass.edu/omste/ca.html

Massachusetts Department of Higher Education *Admissions Standards for the Massachusetts State University System and the University of Massachusetts* http://www.mass.edu/shared/documents/admissions/admissionsstandards.pdf

Ministry of Education Singapore http://www.moe.gov.sg/education/syllabuses/sciences/

NCCA (2012) *Leaving Certificate Mathematics* Dublin: NCCA

New Zealand Assessment Objectives for Upper Secondary Mathematics and Statistics http://seniorsecondary.tki.org.nz/Mathematics-and-statistics/Achievement-objectives/AOs-by-level

New Zealand Qualification Authority http://www.nzqa.govt.nz/qualifications-standards/qualifications/ncea/understanding-ncea/

Scotland's Curriculum for Excellence http://www.educationscotland.gov.uk/thecurriculum

Scottish Qualifications Authority http://www.sqa.org.uk

Singapore Examinations and Assessment Board http://www.seab.gov.sg/

[2] All websites were accessed during March 2013

	Finland			Massachussetts			New Zealand			Scotland			Singapore		
	FL	OL	HL	FL	OL	HL	FL	OL	HL	FL	OL	HL	FL	OL	HL
Statistics and probability															
1.1 Counting		MAB5 combinatorics; MAA6 combinatorics	MAB5 combinatorics; MAA6 combinatorics		S.CP 9 permutations	S.CP 9 permutations	S L4 possible outcomes	M L8 permutations & combinations	M L8 permutations & combinations		H & AH n! and combinations			H1 4.1 n! and permutations; H2 6.1 permutations	H2 6.1 combinations; H3 2 cominatorics
1.2 Concepts of probability	MAB5 concept of probability	MAB6 rules of calcuating probabilities; MAA6 expected value of discrete distributions	MAA6 mathematical and statistical probability; rules for calculating probabilities; expected value of discrete distributions	S.CP1 sample space; S.CP4 approximate conditional probabilites from data	S.CP2&3 statistical independence; S.MD 2 calculate expected value	S.CP 6, 7 & 8 apply multiplication and addition rules	S L6 theoretical and experimental probability, sample size	S L8 expected value	S L8 probabilities of independent, combined and conditional events	Int.1& 2 determine probabilities; H sample space	H addition and multiplication of probabilities	H laws of probability	O3.3 probability	H1 3.1 probability	H1 4.1 binomial distibution and expected values; H1 4.2 Normal distribution; H2 6.2 probability
1.3 Outcomes of random processes			MAB5 Normal distribution and the standardisation of distributions; MAA6 discrete and continuous probability distributions; distribution parameters; Normal distribution	S.MD1 use experiments to determine probabilities		S.MD 3 & 4 generate probability distributions and expected values; S.ID 4 estimate areas under the normal curve	S L7 two-way tables, tree diagrams	S L8 binomial distribution	S L8 applying Normal, Poisson and Binomial distributions, CLT	H sample space	H expected value	H random variable, probability distribution for discrete and continuous variable			H1 5.1 distribution of sample means, Central Limit theorem; H2 7.1 Binomial and Poisson distributions; H2 7.2 Normal distribution; H2 8.1 distribution of sample means, Central limit theorem
1.4 Statistical reasoning with an aim to becoming a statistically aware consumer				S.IC 6 evaluate reports based on data			S L1-8 statistical literacy, L7 samples	S L8 critiquing causal relationship claims							
1.5 Finding, collecting and organising data					S.IC different types of studies	S.IC 1.5 randomisation	S L1-8 using the statistical enquiry cycle, S L7 random sampling	S L7 random sampling, experiments and existing data sets, S L8 experimental design principles	S L8 conducting experiments, surveys, evaluating all stages of the cycle				O3.1 data collection methods		H1 5.1 concept of population and sample, sampling methods; H2 6.1 concept of population and sample, sampling methods
1.6 Representing data graphically and numerically			MAB5 determining the parameters of continuous and discrete statistical distributions; MAA6 discrete and continuous statistical distributions; distribution parameters	S.ID 1 represent data including histograms	S.ID 5&6 represent bivariate data on a scatterplot; S.ID 9 distinguish between correlation and causation	S.ID 7 interpret a linear model; S.ID 8 compute a correlation coefficient	S L5 use multiple displays and recategorise data to find patterns, variations, relationships and trends, S L6 evaluate statistical reports, S L6 statistical measures	S L8 critiquing causal relationship claims, S L7 point estimates of population parameters	S L6 justify findings using displays and measures (outliers and percentiles not mentioned)	Int. 1&2 & H simple graphs, charts, tables; find mean, median, mode	Int. 1&2 & H scattergraphs; quartiles; Int. 2 standard deviation; line of best fit on a scattergraph	H outliers; correlation coefficient	O3.1 data presentation methods; O3.2 mean, median, mode, interpretation of representations	O3.2 interquartile range and standard deviation; H1 6.1 & H2 9.1 scatter diagram and correlation coefficient	H1 6.1 & H2 9.1 correlation coefficient
1.7 Analysing, interpreting and drawing inferences from data				S.IC 1&2 statistics for making inferences and testing models		S.IC 4 margin of error; S.IC 2 use sampling distributions for informal inference	S L1-8 using the statistical enquiry cycle, S L6 making informal inferences about populations from sample data, justifying findings	S L7 recognising the effect of sample size on the variability of an estimate	S L8 determining estimates and confidence intervals for means, proportions and differences (no mention of Z-tables, two-tailed Z-test, p-values)					H1 5.2 & H2 8.2 concept of hypothesis testing	H1 5.1 & H2 8.1 calculation of unbiased estimates of population mean and variance; H1 5.2 & H2 8.2 hypothesis testing, p-values, level of significance
Geometry and trigonometry															
2.1 Synthetic geometry	MAB2 draw plane figures			G.CO 12&13 make formal constructions, construct equilateral triangle, square and regular hexagon; G.C.4 construct a tangent from a point to a circle	G.CO 10 prove theorms about triangles; G.SRT 2&3 similarity; G.SRT 6 similarity and trigonometric ratios; G.C 3 inscribed and circumscribed circles in a triangle	G.CO 11 prove theorems about parallelograms; G.SRT 4&5 prove theorems concerning ratios (similarity)	GM L5 construct simple loci	GM L5 create accurate nets, deduce angle properties; GM L6 deduce and use angle properties related to circles; (no formal proof)	GM L5 apply Pythagoras' theorem in 2D, GM L6 use Pythagoras' theorem in 2D and 3D (no formal proof)		Int. 2 properties of circle, angle in a semi circle; concurrency of triangle medians, angle bisectors, altitudes; properties of straight lines		O2.1 angles, triangles and polygons; O2.2 congruent and similar figures	O2.3 properties of circles	
2.2 Co-ordinate geometry	MAB2 geometrical methods in the coordinate system; MAA4 equation of a line	MAA4 equation of a circle	MAA4 equation of a circle; distance of a point from a straight line		G.CPE 1 equation of a circle; G.GPE 4&5 slope criteria for parallel and perpendicular	G.GPE 6 divide a line segment in a given ratio; conic sections	NA L6 relate rate of change to the gradient of a graph; graphs, tables and equations of linear relationships	GM L6 areas contained by two or more loci on a coordinate plane; M L8 apply the geometry of conic sections	M L7 coordinate geometry techniques to points and lines; M L8 apply the geometry of conic sections	Int. 1&2 straight line graphs; Int. 2 gradient of a straight line graph		H equation of a circle	O2.6 coordinate geometry (not parallel and perpendicular)		
2.3 Trigonometry	MAB2 Pythagoras' theorem	MAB2 right-angled-triangle trigonometry; MAA3 sine and cosine rules; MAA9 radians	MAA8 solve trigonometric equations; radians; MAA9 solving trigonometric equations	G.SRT 8 solve problems using Pythagoras' theorem (and trigonometric ratios)	G.SRT 7 sine and cosine for complementary angles; G.SRT 9 area of triangle with trigonometry; G.SRT 10&11 sine and cosine rules; G.C 5 arc length and sector area; F.TF 3 accurate trigonometric ratios involving surds	F.TF 1&2 radians; F.TF 4 trigonometric functions; F.TF 7 solve trigonometric equations; F.TF 8&9 derive and apply trigonometric formulae	GM L5 apply Pythagoras' theorem in 2D	GM L5 perimeters and areas of circles and composite shapes; M L7 apply trigonometric relationships, including sine and cosine rules, in 2D and 3D; form and use simple trigonometric equations (no mention of surds)	M L8 form and use trigonometric equations	Int. 1 Pythagoras' theorem	Int. 1 right angled triangle trigonometry; Int. 2 arc of a circle and sector area; sine, cosine, tangent of all angles; area of a triangle using trigonometry; sine and cosine rules; H exact trigonometric ratios	Int. 2 & H graphs of trigonometric functions; solve simple trigonometric equations; H radians; solve trigonometric equations and use trigonometric identities	O2.4 Pythagoras' theorem	O2.4 extend sine and cosine to obtuse angles; use trigonometry to find the area of a triangle; sine and cosine rule; O2.5 arc length and sector area	O2.5 radians
2.4 Transformation geometry, enlargement	MAB2 similarity of figures			G.SRT 1 enlargement (called dilation)			GM L5 describe and use transformations			Int.1 scale drawing			O2.2 enlargement, scale drawing		
Number															
3.1 Number systems	MAB6 number sequences; MAA9 number sequences; recursive number sequences	MAB6 arithmetic progressions and sums; MAA9 arithmetic progressions and sums	MAB6 geometric progressions and sums; MAB7 mathematical models applicable to economic situation, using number sequences and sums; MAA9 geometric progressions and sums; MAA12 limits of number sequences	N.RN 3 rational and irrational numbers	N.CN 1&2 complex numbers; N.CN 3 find complex conjugate; N.CN 4 represent complex numbers on an Argand diagram; N.CN 5 addition and subtraction of complex numbers	N.CN 4 polar form; N.CN 5 multiplying complex numbers	NA L5 use prime numbers, common factors, multiples; understand operations on fractions, decimals, percentages and integers; know commonly used fraction, decimal and percentage conversions; know and apply standard form; NA L3 & L5 number patterns	M L8 manipulate complex numbers and present them graphically	M L7 use arithmetic and geometric sequences and series; M L8 identify limits of functions (no proof of irrationality or proof by induction)	Int. 1 integers; standard form	AH complex numbers; Argand diagram; complex conjugate	H sequences, limits of sequences; AH arithmetic and geometric sequences and series, limits of sequences; proof by induction	O1.1 numbers and the four operations; O1.5 number sequences	H2 2.1 concept of sequences and series; H2 4.1 complex numbers; H2 4.2 Argand diagram	H2 3.1 summation of series; proof by induction; H2 2.2 arithmetic and geometric series; H3 sequences & series
3.2 Indices		MAA1 roots and fractional powers; MAA8 radical functions and equations; exponential functions and equations	MAB3 solve exponential equations using logarithms; solve power equations; MAA1 solve power equations; MAA8 logarithmic functions and equations		A.SSE use the properties of indices (exponents)	F.BF 5 exponents and logarithms as inverse functions; F.LE 4 use logarithms to solve equations related to exponential models	NA L6 extend powers to includer integers and fractions	NA L6 extend powers to include integers and fractions	NA L7 manipulate logarithmic algebraic expressions		Int. 2 simplify surds; use the laws of indices	H laws of logarithms			H1 1.1 laws of logarithms
3.3 Arithmetic	MAB7 money transaction	MAB7 money transaction		N.Q 1 use units consistently; N.Q 3 choose an appropriate level of accuracy	N.Q 3 describe the effects of approximate error		NA L5 know and apply significant figures, rounding and decimal place value (nothing on checking); use rate and ratios; GM L5 convert between metric units (nothing on Imperial)	NA L6 apply everyday compounding rates;(nothing on error)	NA L6 apply everyday compounding rates	Int. 1 basic calculations; calculations in social contexts; Int. 2 calculations involving percentages; round calculations; calculations in a social context	Int. 2 appreciation/depreciation		O1.1 approximation and estimation; O1.2 ratio, rate and proportion; O1.3 percentage; O2.5 conversion between metric units	O1.9 applications of mathematics in practical money situations (not profit and loss)	
3.4 Length, Area and Volume		MAB2 determining areas and volumes; MAA3 geometry of circles; calculate lengths, angles, areas, and volumes		G.GMD 1 circumference and area of a circle, volume of a cylinder, pyramid and cone	G.GMD 3 use volume formulae to solve problems		GM L5 deduce and use formulae to find the perimeters and areas of polygons; find areas and perimeters of composite shapes; create accurate nets; GM L6 calculate volume of prisms, pyramids, cones and spheres; M L8 integration and anti-differentiation by numerical methods	GM L5 find areas and perimeters of circles and composite shapes; GM L6 calculate volume of prisms, pyramids, cones and spheres; (no surface area)		Int. 1 surface area of solids; nets; Int. 2 volumes of solids			O2.5 area of polygons, volume of solids	O2.5 volume and surface area of composite solids	

	Finland			Massachussetts			New Zealand			Scotland			Singapore		
	FL	OL	HL	FL	OL	HL	FL	OL	HL	FL	OL	HL	FL	OL	HL
Algebra															
4.1 Expressions		MAB2 linear equations with two variables; MAA2 factorisation of quadratics	MAA2 products of polynomials and binomial theorem	A.SSE 1&2 understand and use the structure of an expression to rewrite it (expand and simplify)	A.SSE 1, 2 & 3a understand and use the structure of an expression to rewrite it (factorise); A.APR 1 arithmetic with polynomials; A.CED 4 rearrange formulae	A.APR 5 binomial theorem; A.APR 6 rewrite rational expressions; A.APR 7 arithmetic of rational functions	M L7 manipulate rational, exponential and logarithmic algebraic expressions	M L7 manipulate rational, exponential and logarithmic algebraic expressions	M L8 manipulate trigonometric expressions (no binomial theorem and surds)	Int. 1 evaluate formulae, expand and simplify expressions; Int.2 use formulae	Int. 1 factorise expressions, multiply algebraic expressions, factorise algebraic expressions; Int. 2 manipulate algebraic fractions, rearrange formulae, simplify surds	AH binomial theorem; manipulate rational functions; factorise polynomials with real coefficients	O1.5 algebraic representation and formulae; O1.6 algebraic manipulation	O1.5 manipulation of algebraic fractions	H1 4.1 binomial theorem; H2 2.1 binomial theorem
4.2 Solving equations	MAB1 solving equations graphically and algebraically; solving quadratic equations	MAB3 solve power equations; MAB6 solving pairs of linear equations	MAA7 rational equations	A.REI 3 solve linear equations; A.REI 6 solve a pair of linear simultaneous equations; A.REI 10 &11 use graphs to solve equations	A.CED 1 create and solve equations in one variable; A.CED 2 create and solve equations in two variables and graph them; A.REI 1 solve equations; A.REI 4 solve quadratics, choosing between inspection, factorisation, completing the square or use of the formula; A.CED7 solve simultaneous equations when one is linear and the other quadratic; A.REI 10 &11 use graphs to solve equations	A.APR 2&3 apply the remainder theorem (related to the factor theorem) and identify zeroes related to factors; A.REI 10 &11 use graphs to solve equations; N.CN 7&8 solve quadratic equations with real coefficients that have complex roots	NA L5 solve linear and simple quadratic equations	NA L6 form and solve linear equations, quadratic equations, simultaneous equations with two unknowns; M L7 form and use pairs of simultaneous equations, one of which may be non-linear, manipulate rational expressions	M L8 form and use trigonometric, polynomial and other non-linear equations (no factor theorem)	Int. 1 solve linear inequalities	Int. 2 solve two simultaneous linear equations algebraically and graphically, solve quadratic equations graphically and algebraically; H solve quadratic equations;	H factor theorem, solve simultaneous equations with one linear and one of order 2	O1.8 solving linear and quadratic equations	O1.8 solving quadratic equations and simple fractional equations; H1 1.2 solving quadratic equations and simultaneous equations	H2 1.3 solving equations
4.3 Inequalities		MAB6 solving inequalities with two variables graphically	MAA2 polynomial inequalities; MAA4 solving absolute value inequalities; solving equation systems; MAA7 rational inequalities	A.REI 3 solve linear inequalities; A.REI 12 use graphs to solve linear inequalities	A.CED 1 create and solve inequalities in one variable; A.CED 2 create and solve inequalities in two variables and graph them; A.REI 12 use graphs to solve linear inequalities		NA L6 form and solve inequations	NA L6 form and solve inequations		Int. 1 solve linear equations	H solve quadratic inequalities		O1.8 solving linear inequalities	H1 1.2 solving inequalities graphically	H2 1.3 solving inequalities
4.4 Complex numbers						N.CN 4 polar form		M L8 manipulate complex numbers and present them graphically	M L8 manipulate complex numbers and present them graphically (no De Moivre's theorem)			AH Cartesian & polar form; De Moivre's theorem			H2 4.2 complex numbers in polar form; De Moivre's theorem
Functions															
5.1 Functions	MAB1 quadratic functions	MAA1 power functions; MAA4 equation of a parabola; MAB1 quadratic functions	MAB8 trigonometric functions; MAA1 exponential functions; MAA2 roots of a quadratic equation; MAA7 limits and continuity of functions; MAA8 inverse functions; MAA9 trigonometric functions; MAA12 continuity and differentiability of functions, limits of functions	F.BF 1 describe functions		F.BF 4 inverse functions; F.LE 3 construct linear and exponential models; A.SSE use the completed square form of a quadratic to identify features	NA L6 relate graphs, tables and equations to linear, quadratic and simple exponential relationships	M L7 sketch the graphs of functions; M L8 identify limits of functions	M L8 identify discontinuities and limits of functions (no formal development of functions), form and use trigonometric, polynomial and other non-linear equations	Int. 2 graphs of quadratic functions, including roots	H composite functions	H functions and graphs, complete the square, exponential and logarithmic functions; AH functions and their propoerties, inverse functions	O1.7 functions and graphs	H2 1.1 functions, composite functions	H1 1.1 concept of function, exponential and logarithmic functions; H2 1.1 inverse functions; H2 1.2 graphing techniques; H3 1 functions
5.2 Calculus		MAB4 derivatives of polynomial equations, examining the behaviour and determining maxima and minima; graphical and numerical methods MAA7 differentiation of polynomial functions; examining the behaviour and determining extreme values	MAA7 differentiation of products and quotients of functions; MAA8 derivatives of composite functions, radical, exponential and logarithmic functions; MAA9 derivatives of trigonometric functions; MAA10 integral functions; the definite integral; calculating areas					M L7 sketch the graphs of functions and their gradient functions, apply differentiation techniques to polynomials	M L8 choose and apply a variety of differentiation, integration and anti-differentiation techniques to functions, using both analytical and numerical methods			H differentiation from first principles, interpret the gradient function, basic integration, integration as anti-differentiation; AH differentiate sums, differences, products, quotients and compositions, integration		O1.7 estimate gradient of a curve using a tangent; H1 2.1 differentiation of polynomials; H2 5.1 differentiation	H1 2.2 integration of polynomials; H2 5.3 integration techniques; H2 5.4 definite integral; H3 1 calculus
Additional content		MAB6 linear programming	MAA5 vectors			G.GPE 2&3 conic sections			S L8 linear regression for bivariate data	Int. 1 speed, distance, time graphs		H&AH transformation of functions	O1.4 speed; O1.9 distance-time and speed-time graphs		H1 4.2 manipulating Normal variables
		MAB7 commercial mathematics	MAA10 integration to find volumes			G.MG modelling with geometry			M L8 linear simultaneous equations in three unknowns	Int. 1 & 2 logic diagrams (i.e. networks)		H recurrence relations for sequences	O1.10 set language and notation		H1 6.1 linear regression
		MAB8 mathematical models III - vectors	MAA11 number theory and logic includes Euclid's algorithm			F.BF 3 transformation of functions			M L7 choose appropriate networks to find optimal solutions; M L8 develop network diagrams to find optimal solutions, including crtical path	Int. 1&2 spreadsheets		H discriminant of a quadratic equation	O1.11 matrices		H2 1.2 transformation of functions
			MAA12 numerical and algebraic methods			F.LE 1 distinguish between linear and exponential models; F.LE 3 distinguish between a range of models; F.LE 5 interpret parameters related to the context modelled; F.TF 4,5&6 use trigonometric functions to model real phenomena			M L8 form differential equations and interpret the solutions	Int. 1 3 figure bearings, compass points		H&AH vectors in 3D	O2.7 vectors in 2D		H2 3D vectors
			MAA13 advanced differential and integral calculus - improper integrals			A.CED 3 represent constraints as equations and inequalities in a modelling context				Int. 2 loans, APR, credit cards		H $a\cos\theta + b\sin\theta$			H2 5.2 MacLaurin's series
						A.REI 9&10 use matrix methods to solve systems of linear equations						H linear regression			H2 5.5 differential equations
						N.VM vector and matrix quantities						AH derivative of sec, cosec and cot			H3 1 MacLaurin series
						N.Q2 define appropriate quantities for models.						AH integration by sustitution			H3 3 differential equations
												AH use matrices to solve systems of linear equations			
												AH differentiate inverse trigonomtric functions; parametric functions			
												AH integration related to inverse trignometric functions; integration by parts			
												AH differential equations			
												AH other methods of proof			
												AH matrix algebra			
												AH MacLaurin series			
												AH Euclid's algorithm			

Report

First interim report for the Department of Education and Skills and the National Council for Curriculum and Assessment

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Research into the impact of Project Maths on student achievement, learning and motivation

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November 2012

The views expressed in this report are those of the authors and do not necessarily reflect the views or policy of the Department of Education and Skills or the National Council for Curriculum and Assessment.

Published in November 2012
by the National Foundation for Educational Research,
The Mere, Upton Park, Slough, Berkshire SL1 2DQ
www.nfer.ac.uk

How to cite this publication: Jeffes, J., Jones, E., Cunningham, R., Dawson, A., Cooper, L., Straw, S., Sturman, L. and O'Kane, M. (2012). *Research into the impact of Project Maths on student achievement, learning and motivation: first interim report.* Slough: NFER.

Contents

Acknowledgements

The authors would like to thank the schools who have participated in this research to date, especially the students who have given their time to share their views and experiences with us. We would like to extend our gratitude to the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), for the opportunity to undertake this work, and particularly to Rachel Linney and Anne Looney for their helpful guidance and feedback. In addition, we would like to acknowledge the valuable contribution to this research made by Máirín Wilson at the Church of Ireland College of Education.

Executive summary

The National Foundation for Educational Research (NFER) has been commissioned by the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), to explore the impact of Project Maths on student achievement, learning and motivation in:

- the initial post-primary schools (phase one schools), which introduced the revised mathematics syllabuses in September 2008
- post-primary schools (non-phase one schools), which introduced the revised mathematics syllabuses in September 2010

This first interim report presents the key findings arising from the first two elements of this research, drawing on the findings of **an internationally comparable assessment of student achievement** and **survey of student attitudes, motivation and confidence**, administered to Junior Certificate and Leaving Certificate students in Spring 2012.

About the students

This part of the research involved phase one and comparison group students at Junior Certificate level, and phase one and non-phase one students at Leaving Certificate level, all of whom were in the examination year of 2012. They were, therefore, reaching the end of their studies at the time of participating in the research.

As the revised mathematics syllabuses are being introduced incrementally in schools, this cohort of phase one Junior Certificate students had studied Strands 1-4 of the revised mathematics syllabus. The same cohort of students in non-phase one schools had followed the previous mathematics syllabus introduced in 2000, and was therefore included as a comparison group for this research.

Phase one Leaving Certificate students had studied all five strands of the revised mathematics syllabus. Students in non-phase one schools were part of the first national cohort of the revised mathematics syllabus. These students had followed revised syllabuses for Strands 1 and 2, and for the remainder of their studies had followed the previous mathematics syllabus. Whilst they were not, therefore, a comparison group, students in this group had been less immersed in the revised syllabus than their phase one counterparts.

Key research findings

Assessment of students' achievement in mathematics shows that, overall, students are performing well in many aspects of the revised mathematics syllabus, and there are many parallels between students' achievement and their attitudes, suggesting that students are reflective about their experiences of learning mathematics, and able to identify their own areas of strengths and weaknesses.

Whilst the research does not reveal any discernible differences between the skills of students following the revised mathematics syllabus and their peers following the previous syllabus, students following the revised syllabuses reported that they are regularly engaging with a range of teaching techniques central to the aims of the new syllabus. This includes activities such as: the application of mathematics to real-life situations; making connections and links between mathematics topics; using mathematical language and verbal reasoning to convey ideas; and planning and conducting investigations.

Achievement of Junior Certificate students

Performance of phase one schools

In general, **students appear to be performing well in many aspects of the revised syllabus**. In relation to Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry), there were no items which students appeared to have found particularly difficult. Students also performed well on the majority of items relating to Strand 3 (Number) and Strand 4 (Algebra), although there were some specific items that students appeared to find more challenging. However, students showed a wide variation of abilities within each area of the syllabus, suggesting that students following the revised syllabus struggled with particularly demanding questions, rather than a specific topic or theme.

The performance of the students following the revised syllabus on the Junior Certificate item indicator booklets suggests that, in general, **items requiring higher order skills (such as reasoning and an ability to transfer knowledge to new contexts) are found more difficult** than those which are more mechanical in demand.

Comparison of student performance between phase one and comparison group schools

The **performance of students following the revised syllabus and those following the previous syllabus at Junior Certificate level is similar**. Whilst, in general, students following the revised syllabus performed better than their comparison group peers, this difference is only statistically significant in relation to a particular item which explores students' abilities in Strand 1, Statistics and Probability (assessing students' understanding of the outcomes of simple random processes).

Whilst it is encouraging that Junior Certificate students are performing well, it is reasonable to conclude that engagement with the revised syllabus has not yet positively or negatively impacted on the performance of students following the revised syllabus at Junior Certificate level relative to their peers following the previous syllabus.

Comparison of student performance with international standards

In general, students following the revised syllabus scored well on the majority of items in comparison to international standards. In particular, **these students have shown a strong performance on items assessing Strand 1 (Statistics and Probability)**. The high performance of phase one students on the items in this strand is encouraging and suggests that the implementation of this part of the new syllabus is working well.

However, **students following the revised syllabus appear to find Strand 4 (Algebra), and 'Examining algebraic relationships' in particular, to be especially difficult.** Overall, their knowledge on subject areas relating to Strand 2 (Geometry and Trigonometry) and Strand 3 (Number) appear to be similar to that of students internationally.

Attitudes of Junior Certificate students

Students' experiences of mathematics lessons

The revised mathematics syllabus encourages approaches such as applying mathematics to real-life situations; conducting investigations; and participating in discursive and collaborative activities, such as group work. A high proportion of students following the revised mathematics syllabus **are regularly undertaking many of these approaches**.

Furthermore, in many areas, there was a higher proportion of positive responses from students following the revised syllabus than from those following the previous syllabus. This suggests that such approaches are being effectively translated into classroom practice. However, similarly high proportions of phase one and comparison group students reported that they regularly participated in activities aimed at developing their mathematical thinking skills, suggesting that the introduction of the revised syllabus does not appear to have had an impact on students' experiences in this area.

Although there are positive indications that the approaches promoted through the revised syllabus are being reflected in the classroom, there remains a high proportion of phase one pupils who report that **they participate in activities associated with more traditional approaches** to mathematics teaching and learning, such as copying what their teacher writes on the board and using textbooks in lessons. Whilst the reasons for this remain to be explored in subsequent phases of this research, it is possible that schools have concerns about the content and format of examinations for the revised syllabus, leading them to use more familiar methods of supporting young people to achieve examination success, whilst simultaneously promoting and delivering many of the features of the revised syllabus.

Students' attitudes towards learning mathematics

Both phase one and comparison group students were **broadly confident in their abilities in topics spanning all strands of the revised syllabus**. This was particularly notable in relation to Strand 1 (Statistics and Probability), an area in which students generally performed well in the assessment part of this research. Although the majority of students in both phase one and comparison groups reported that they were confident in relation to Strand 4 (Algebra), a lower proportion of phase one students reported that this was the case, relative to the comparison group. Again, this reflects the findings of the assessment phase. The reasons for this will be explored further during the case-study phase. For example, the two groups may take different approaches interpreting this type of question, which could explain any differences in students' confidence.

Across all strands, **phase one and comparison group students were confident in their synthesis and problem-solving abilities**. Interestingly, however, phase one students appeared to feel somewhat less confident than their comparison group peers in solving problems based on real-life situations, despite phase one students reporting to have done so more frequently. One possible explanation for this is that, as phase one students do this more frequently than their comparison group counterparts, they have been encouraged to test out and challenge their skills in this area to a greater degree.

Students' attitudes towards careers involving mathematics

Both phase one and comparison group students were in broad agreement that **mathematics was important in a range of contexts outside of the classroom** (e.g. in daily life, and to enable them to access further education and jobs). The majority of students felt that mathematics was important for a range of career types including, for example, business management, accountancy, engineering, and retail. There were no substantial differences between the views of students following the revised syllabus and those following the previous syllabus. This suggests that the introduction of the revised mathematics syllabus has not, to date, had any discernible impact on students' appreciation of the application of mathematics.

Achievement of Leaving Certificate students

Performance of phase one schools

Student performance was mixed across the different strands of the syllabus. Overall, the results echoed those of Junior Certificate students, in that **items requiring higher order skills were found to be more difficult**.

A number of items required a multiple choice response followed by a 'show your working' section. Many students did not attempt to justify their answers, suggesting either that they were daunted by this request, or did not realise they needed to complete this section to be awarded full marks. This was surprising given the emphasis placed on this type of approach within the revised syllabus. Leaving Certificate students appeared to find items relating to Strand 5 (Functions) of the revised syllabus, relatively difficult, even amongst Higher Level students.

Comparison of student performance between phase one and non-phase one schools

The performance of phase one and non-phase one students was broadly similar on Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry), which is to be expected, as both groups of students had been studying Strands 1 and 2 of the revised syllabus for the same amount of time.

There were some indications of differences in the performance of the two groups in specific aspects of each strand, these tended not to span a whole strand. There were some indications that phase one students performed better than their non-phase one peers in **analysing verbal geometric information and translating it into mathematical form**. However, in other items relating to this area there was no discernible difference between the two groups.

Comparison of student performance with international standards

Phase one students performed much better than international students on many of the items relating to Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry), and the majority of these fall within Strand 1. The high performance of phase one students on the items in this strand is encouraging and suggests that the implementation of this part of the new syllabus is working well. In general, phase one students performed better than expected on items where the solution strategy is clear, and where diagrams, if applicable, are provided. They performed less well on multi-step items.

It is important to note that comparisons between the performance of students following the revised syllabus and the available international data are confounded by a range of factors, including differences in age, but provide some baseline indicators. Using this analysis as a baseline measure will enable comparative analysis of students following the revised syllabus with international performance in the subsequent stage of the evaluation.

Attitudes of Leaving Certificate students

Students' experiences of mathematics lessons

Leaving Certificate students gave a similar pattern of responses to their Junior Certificate peers, with a **higher proportion of phase one students reporting that they regularly engaged in activities promoted within the revised syllabus** (for example, applying mathematics to real-life situations; conducting investigations; and participating in discursive and collaborative activities). Again, although there are indications that the approaches promoted through the revised syllabus are being reflected in the classroom, there remains a high proportion of phase one pupils who report that they participate in more traditional teaching and learning approaches, such as **copying from the board and working from textbooks**.

Students' attitudes towards learning mathematics

Both phase one and non-phase one students **had similar high levels of confidence in items relating to Statistics and Probability (Strand 1)** which, as both groups have studied this strand, is a positive indicator of its impact on students. Students were also largely confident in relation to Strand 2 (Geometry and Trigonometry) and Strand 3 (Number), albeit to a lesser extent than Strand 1.

Although the responses of both groups were broadly positive in relation to Algebra (Strand 4), as with the Junior Certifcate, **phase one students appeared slightly less confident in**

using algebra than their non-phase one peers. Again, the reasons for this will be explored further during the case-study phase. Both phase one and non-phase one students were highly confident in relation to Strand 5 (Functions), although the phase one cohort had considerable difficulty with this strand in the testing part of the research.

In general, both phase one and non-phase one students reported that they were **confident to use mathematics to solve problems based on real-life situations**. Like their Junior Certificate peers, however, Leaving Certificate phase one students appeared to feel somewhat less confident than non-phase one students. As with the Junior Certificate, this may be because students following the revised syllabus have a greater understanding of the complexities of this type of activity as a result of doing so more frequently than their peers.

For Leaving Certificate students, unlike Junior Certificate students, there was a statistically significant difference between phase one and non-phase one groups in their confidence to synthesise what they have learned in more than one topic, and apply it to solving a range of mathematical problems, with non-phase one students appearing to feel less confident.

Students' attitudes towards careers involving mathematics

Like Junior Certificate students, both groups of Leaving Certificate students reported that mathematics was important in a range of contexts outside of the classroom, but shared their views regarding the scope and range of careers which may involve mathematics.

Many Leaving Certificate students (in both phase one and non-phase one schools) were planning to pursue further study and/or careers in mathematics, favouring professions such as accountancy and business management

Discussion and next steps

In Autumn 2012, a further round of attitude surveys and assessments of student performance will be conducted with Junior Certificate and Leaving Certificate students in the examination classes of 2013. This will enable comparisons to be drawn between year groups as the revised syllabus becomes further embedded and developed in schools.

1. Introduction

The National Foundation for Educational Research (NFER) has been commissioned by the Department of Education and Skills, Ireland, and the National Council for Curriculum and Assessment (NCCA), to undertake research into the impact of Project Maths on student achievement, learning and motivation in:

- the initial schools (phase one schools), which introduced the revised mathematics syllabuses in September 2008
- all other post-primary schools (non-phase one schools), which introduced the revised mathematics syllabuses in September 2010

This first report to NCCA presents the key findings of the first assessment of student achievement and survey of student attitudes, motivation and confidence, administered to Junior Certificate and Leaving Certificate students in both groups of schools in Spring 2012.

1.1 Background and context

Project Maths is a major national reform of the post-primary mathematics syllabus in the Republic of Ireland for both junior and senior cycles. Introduced in 24 phase one schools in September 2008, and rolled out to all post-primary schools in September 2010, Project Maths was designed to change not just *what* students learn about mathematics, but *how* they learn and how they are assessed. Project Maths represents a philosophical shift in Irish post-primary education towards an investigative, problem-focused approach to learning mathematics, emphasising its application in real-life settings and contexts.

A recent report produced by the Eurydice Network exploring the common challenges and national policies for teaching mathematics in Europe found that the use of such approaches is the focus of a number of European countries. Mathematics teaching and learning in all countries involved in the study feature problem-focused learning and, when applying mathematics to real life contexts, a wide range of approaches are taken. Some countries emphasise contexts which are familiar to students so as to provide a meaningful frame of reference for their learning (for example, this approach is taken in Spain, Poland and Italy). In other countries (for example, Estonia), students are encouraged to participate in outdoor learning, relating their mathematical knowledge to architecture and visual arts. Similarly, active learning and critical thinking is advocated in many jurisdictions (for example Belgium, the Czech Republic, Slovenia and Spain), encouraging 'pupils to participate in their own learning through discussions, project work, practical exercises and other ways that help them reflect upon and explain their mathematics learning' (Eurydice, 2011:56).

Existing research into the impacts of mathematics teaching and learning clearly show that the ways in which mathematics is taught in schools can have a considerable impact on student attainment. A report into the mathematical needs of learners produced by the Advisory Committee on Mathematics Education (ACME) in the UK has shown that teaching strategies based on deepening and enriching students' mathematical understanding can have a positive impact on achievement. Conversely, when schools focus primarily upon results and disregard the mathematical understanding of the pupils, this can have a negative impact on the ability of young people to apply their mathematical knowledge in later life (ACME, 2011). There is also evidence to suggest that a more innovative and stimulating mathematics syllabus may have a disproportionately positive effect on lower achieving students, thus helping to close the gap in achievement and enthuse mathematics students of all abilities – not simply the higher achieving groups (JMC, 2011).

Whilst research has shown that many different approaches to mathematics can be effective according to the varying needs of learners, the 'use of higher order questions, encouraging reasoning rather than 'answer getting', and developing mathematical language through communicative activities' (Swan *et al.*, 2008:4) have been found to be particularly effective in developing students' conceptual understanding; investigation and problem-solving strategies; and fluency and appreciation of mathematics. Likewise, engaging students in mathematical discussion; encouraging them to make connections between different areas of mathematics; and encouraging them to participate in open-ended, investigative mathematics activities are particularly conducive to mathematical success (Hiebert and Grouws, 2009).

There remain, however, some areas promoted in the national policies of many countries, which remain underdeveloped in the teaching of mathematics. Data from the Trends in International Mathematics and Science Studies (TIMSS) has shown, for example, that students work in small groups less frequently than they work individually. This is challenging for the development of mathematics teaching and learning, given the importance of group work as a forum for discussion and collaboration. Similarly, whilst information and communication technology (ICT) is heavily prescribed in many jurisdictions, the use of computers in mathematics lessons is relatively rare (Eurydice, 2011). In the UK, research from the Joint Mathematical Council shows that despite heavy investment in digital materials for schools, the use of technology as a teaching resource for mathematics is greatly under-exploited, with teachers tending to use ICT as a visual aid rather than as an instrument to assist with mathematical thinking and reasoning (JMC, 2011).

Within this context, Project Maths occupies an important position within the Irish government's commitment to the central role of mathematics education as a necessary precursor to innovation, competitiveness and economic growth. In its plan for a 'smart economy', the Irish government identified the need to improve the mathematical skills of post-primary school leavers as a step towards realising the vision of Ireland as a hub of innovation (Department of the Taoiseach, 2008). This was echoed by the Innovation Taskforce (2010), who considered improvement in mathematics attainment to be a necessary force for driving the science, technology and engineering disciplines at third and fourth levels. Furthermore, the Taskforce identified mathematics as ‘crucial’ to developing creativity and problem-solving skills, which are prerequisites for an innovative workforce.

1.2 Aims and objectives of the new mathematics syllabus

Whilst the specific objectives of the revised mathematics syllabuses are multifaceted, reflecting the wide range of learning that takes place within each age group and ability range, their core aims are to equip students at both Junior Certificate and Leaving Certificate levels with:

- the mathematical knowledge, skills and understanding they need to succeed in education, work and daily life
- the skills to use mathematics in context, and to solve problems with a range of real-life applications
- a lifelong enthusiasm for mathematics.

To achieve this, the revised mathematics syllabuses at Junior Certificate level aim to build upon students’ experiences of learning mathematics at primary school. Among the objectives in the syllabuses are that learners develop an in-depth knowledge of, and enthusiasm for, the reasons and processes underpinning mathematics, as well as the ability to recall mathematical facts and techniques. It is envisaged that by fostering students’ sense of creativity, they will be confident and able to apply their mathematical knowledge in a range of contexts.

The revised mathematics syllabuses at Leaving Certificate level seek to develop these skills further, encouraging students to engage with the connections between mathematics and other subjects, and to think creatively and effectively about the many ways that mathematics can be used and applied. In addition to the core mathematical knowledge necessary to succeed at this level, the revised syllabuses aim to meet many of the wider outcomes associated with Leaving Certificate studies, including, for example, communication skills and working with others.

1.3 Structure of the revised mathematics syllabus

The structure of the revised mathematics syllabus is underpinned by two key features. Firstly, development of the syllabus in all post-primary schools has built directly upon the experiences of the 24 phase one schools, using the insights from the experiences of those schools to inform and refine the initiative on an ongoing basis. Secondly, the new programme has been designed to allow schools a phased introduction of syllabuses in both junior cycle and senior cycle simultaneously, gradually building up the number of strands of the new syllabus taught each year. Each phase is accompanied by associated incremental changes in assessment and professional development.

Both the Junior Certificate and Leaving Certificate syllabuses are divided into five strands, as follows:

- Strand 1: Statistics and Probability
- Strand 2: Geometry and Trigonometry
- Strand 3: Number
- Strand 4: Algebra
- Strand 5: Functions.

Across all of these strands, students are encouraged to test out and apply their knowledge to meaningful real-life contexts, and to take responsibility for their own learning through, for example, setting goals and developing action plans. Centred around the individual learner, the revised syllabuses are designed to offer a continuous learning experience for students throughout junior and senior cycles, building upon the foundations of mathematical knowledge acquired at primary school.

The timetable for introduction of the revised syllabuses in both phase one and non-phase one schools is detailed in Tables 1.1 and 1.2.

Table 1.1: Introduction of revised mathematics syllabus strands in junior cycle

	Cohort	Years of study	Syllabus strands
Initial 24 schools (phase one)	**1**	**2008 - 2011**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry
	2	**2009 - 2012**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra
	3	**2010 - 2013**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra Strand 5: Functions
All schools (non-phase one)	**1**	**2010 - 2013**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry
	2	**2011 - 2014**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra
	3	**2012 - 2015**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra Strand 5: Functions

Table 1.2: Introduction of revised mathematics syllabus strands in senior cycle

	Cohort	Years of study	Syllabus strands
Initial 24 schools (phase one)	1	**2008 - 2010**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry
	2	**2009 - 2011**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra
	3	**2010 - 2012**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra Strand 5: Functions
All schools (non-phase one)	1	**2010 - 2012**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry
	2	**2011 - 2013**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra
	3	**2012 - 2014**	Strand 1: Statistics and Probability Strand 2: Geometry and Trigonometry Strand 3: Number Strand 4: Algebra Strand 5: Functions

2. About the evaluation

This chapter sets out the overall approach to the evaluation of the impact of Project Maths on student achievement, learning and motivation including:

- the overarching aims of the evaluation, and key research questions
- the methodology for the evaluation, and details of the specific research activities presented in this report.

2.1 Aims of the research

The overarching aim of the research is to explore the impact of Project Maths on students' achievement, learning and motivation in mathematics, in both phase one and non-phase one schools. Table 2.1 sets out the key research themes for this study, mapped against each element of the research (outlined in more detail in section 2.2).

Table 2.1: Key research themes

Research theme	Assessment of student performance	Student attitude surveys	Analysis of students' work	Case studies
Students' achievement in mathematics, across each individual strand of the revised mathematics syllabuses	✓			
Comparison of students' performance in mathematics with international standards	✓			
Students' motivations and attitudes to mathematics, in general and in relation to the revised mathematics syllabuses		✓		✓
Students' opinions on the effectiveness of different strands and approaches used in the revised mathematics syllabuses		✓		✓
Students' perceptions of their knowledge, understanding, confidence and achievement in mathematics		✓		✓
Students' aspirations to further study of mathematics		✓		✓
Students' views of the relevance and application of mathematics more generally		✓		✓
Students' understanding of the processes being promoted in the revised mathematics syllabuses			✓	✓
The impact of the revised mathematics syllabuses on individual students' progress and standards			✓	
Trends in students' approaches to, and performance in, the revised mathematics syllabuses	✓		✓	
Challenges associated with teaching and learning of the revised mathematics syllabuses				✓
Facilitating factors associated with teaching and learning of the revised mathematics syllabuses				✓

2.2 Methodology

The methodology for this research comprises four main phases:

- **internationally comparable assessment of student achievement** in all strands of the revised mathematics syllabuses, based on indicator items administered to two separate cohorts of Junior Certificate and Leaving Certificate students in Spring 2012 and Autumn 2012 (focusing on the examination classes of 2012 and 2013, respectively)
- **attitude surveys exploring students' experiences** of the revised mathematics syllabuses and their confidence and motivation in mathematics, administered to two separate cohorts of Junior Certificate and Leaving Certificate students in Spring 2012 and Autumn 2012 (focusing on the examination classes of 2012 and 2013, respectively)
- **ongoing, data-rich case studies** in eight phase one schools, and eight non-phase one schools, exploring in depth students' and teachers' experiences of the revised mathematics syllabuses
- **qualitative analysis of students' work** exploring trends in the processes being promoted in the revised mathematics syllabuses and its impact upon individual students' progress, which will be gathered from the case-study schools in Autumn 2012 focusing on the Junior Certificate and Leaving Certificate examination classes of 2013.

This report presents the key findings arising from the first two elements of this research, drawing on the findings of the first assessment of student achievement and survey of student attitudes, motivation and confidence, administered to Junior Certificate and Leaving Certificate students in Spring 2012.

2.2.1 About the sample

This part of the research involved students at both Junior Certificate and Leaving Certificate level, who were in the examination year of 2012. They were, therefore, reaching the end of their mathematics syllabus at the time of participating in the research. The target number of students for this study is outlined in Table 2.2:

Table 2.2: Target number of students and schools

Phase one	JC	**Class of 2012** 500-700 students included in collection of attitudes and achievement data from all 24 phase one schools
	LC	**Class of 2012** 500-700 students included in collection of attitudes and achievement data from all 24 phase one schools
Non-phase one	JC	**Class of 2012** 2,000-3,000 students included in collection of attitudes data from a sample of 100 schools, of which 700-1,000 included in collection of achievement data (from 36 of the sampled schools)
	LC	**Class of 2012** 2,000-3,000 students included in collection of attitudes data from a sample of 100 schools, of which 700-1,000 included in collection of achievement data (from 36 of the sampled schools)

To achieve this sample, an initial population of 301 non-phase one schools was drawn, to arrive at a representative sample stratified by:

- school type
- school size
- schools included in the Delivering Equality of Opportunity in Schools (DEIS) programme, which aims to address educational disadvantage
- gender of students.

The sample was also drawn to be representative of geographical location: all 26 counties in the Republic of Ireland were included. Students were selected so that, across the sample, the distribution of predicted examination levels was broadly based on previous State Examination Commission (SEC) entry patterns. Table 2.3 provides a breakdown of the number of schools and students who participated in the research, by phase and level of study.

Table 2.3: Details of schools and students participating in the research

Phase one	JC	**Class of 2012** 375 students included in collection of attitudes data, and 303 included in the collection of achievement data from 19 of the phase one schools
	LC	**Class of 2012** 299 students included in collection of attitudes data, and 370 included in the collection of achievement data from 19 of the phase one schools
Non-phase one	JC	**Class of 2012** 2,375 students included in collection of attitudes data from 125 schools, of which 910 included in collection of achievement data (from 52 of the sampled schools)
	LC	**Class of 2012** 2,004 students included in collection of attitudes data from a sample of 125 schools, of which 722 included in collection of achievement data (from 52 of the sampled schools)

Table 2.3 shows that a total of 674 students in 19 out of 24 phase one schools (375 at Junior Certificate, and 299 at Leaving Certificate) took part in the attitude survey, and 673 in the assessment of student performance (303 at Junior Certificate and 370 at Leaving Certificate). Five of the phase one schools were unable to take part in this phase of the research due to other teaching and learning commitments (for example, timetabling issues, and participation in PISA 2012). However, these schools will be invited to participate in subsequent phases of this research. A total of 4,379 students in 124 non-phase one schools (2,375 at Junior Certificate, and 2,004 at Leaving Certificate), participated in the survey, and 1,632 students in 52 schools (910 at Junior Certificate, and 722 at Leaving Certificate) participated in the assessment of student performance. Overall, this represents approximately 82 per cent of the target response rate of students. Responses were drawn from a greater number of non-phase one schools than initially anticipated as the number of returns from individual schools was, in general, lower than expected.

The use of comparative data, to measure the impact of the revised mathematics syllabuses relative to the previous ones, is central to the research design. However, as this research commenced in January 2012, the revised syllabuses have been rolled out nationally to most cohorts of students (see Tables 1.1 and 1.2). Therefore, involvement of non-phase one Junior Certificate students in the examination class of 2012 represents the only comparison group included in this research. In place of a comparison group, it is intended that examinations data from previous years will be used later in the study to contextualise the findings from this research.

In Autumn 2012, a further round of attitude surveys and assessments of student performance will be conducted with Junior Certificate and Leaving Certificate students in the examination classes of 2013. This will enable comparisons to be drawn between year groups as the revised syllabus becomes further embedded and developed in schools.

2.2.2 Assessment of student performance

As detailed in Table 2.1, assessment of student achievement aims to gather quantitative data, focusing on:

- students' achievement in mathematics, across each individual strand of the revised mathematics syllabuses
- comparison of students' performance in mathematics with international standards
- trends in students' approaches to, and performance in, the revised mathematics syllabuses

Assessment of student performance at Junior Certificate level

In order to assess Junior Certificate students' performance in each individual strand of the revised mathematics syllabus, two item indicator booklets were created, each containing two syllabus strands. Table 2.4 shows the syllabus strands covered by each booklet, and the number of items in each booklet.

Table 2.4: Item indicator booklets for the Junior Certificate

Item indicator booklet	Syllabus strand	Syllabus area	Number of items	Number of pupils completing booklet
JC1/2	Statistics and Probability	• concepts of probability • outcomes of random processes • statistical reasoning with an aim to becoming a statistically aware consumer • representing data graphically and numerically • analysing, interpreting and drawing conclusions from data	11	1094
	Geometry and Trigonometry	• synthetic geometry • transformation geometry • co-ordinate geometry • trigonometry	10	
JC3/4	Number	• number systems • indices • applied arithmetic • applied measure	11	186
	Algebra	• representing situations with tables, diagrams and graphs • finding formulae • examining algebraic relationships • relations without formulae • expressions • equations and inequalities	10	

The booklets were made up of items from two international surveys: 'released' items[1] from TIMSS 2007 (Trends in International Mathematics and Science Study, a survey

[1] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS

of 13-14 year olds) and sample items[2] from PISA 2000, 2003 and 2006 (Programme for International Student Assessment, a survey of 15 year olds). Further explanation of the suitability of the international surveys for this research is detailed below. The use of the TIMSS items has allowed a direct comparison to be made between international results and the results of the phase one (and comparison group) students. Appendix A, Table 1 shows the origin of each item in the five item indicator booklets.

It should be noted that the number of students, at both Junior Certificate and Leaving Certificate, completing each of the item indicator booklets varies. Facilities based on relatively small numbers of pupils taking each item are not estimated to a high level of precision so should be treated with a degree of caution. To estimate facility with a reasonable degree of precision we would usually need to sample around 400 pupils in each group to be reported.

Assessment of student performance at Leaving Certificate level

In the Leaving Certificate examination, students are assessed in five revised mathematics syllabus strands. One item indicator booklet was created for each of these strands. Table 2.5 shows the syllabus strands covered by each booklet, and the number of items in each booklet.

Table 2.5: Item indicator booklets for the Leaving Certificate

Item indicator booklet	Syllabus strand	Syllabus area	Number of items	Number of pupils completing booklet
SPLC1	Statistics and Probability	• concepts of probability • outcomes of random processes • statistical reasoning with an aim to becoming a statistically aware consumer • representing data graphically and numerically	9	902
GTLC2	Geometry and Trigonometry	• synthetic geometry • co-ordinate geometry • trigonometry	10	899
NLC3	Number	• number systems • length, area and volume • problem solving and synthesis skills	10	185

[2] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no data available.

ALC4	Algebra	• expressions • solving equations • inequalities • complex numbers	7	186
FLC5	Functions	• functions • calculus	9	180

The booklets were made up of items from three international surveys: released items[3] from the Trends in International Mathematics and Science Study (TIMSS - 2007, 8th grade and TIMSS Advanced, 2008), and sample items[4] from the Programme for International Student Assessment (PISA) surveys of 2000, 2003, and 2006. Leaving Certificate items were specifically selected to match the revised mathematics syllabus and to assess a variety of mathematical skills. These items were also designed for use with students in their final year of secondary school, which matches the stage of schooling of the Leaving Certificate sample in this evaluation. However, some items were also drawn from the same sources as the Junior Certificate items. This was done for three main reasons. First, these items matched the revised syllabus for the Leaving Certificate. Second, the use of common items across both Junior and Leaving Certificate allowed for some comparison to be made across years. Finally, in PISA there is a focus on context and real-world applications of mathematics, which is a key element of the Project Maths initiative.
Their use has allowed a direct comparison to be made between international results and the results of the phase one (and non-phase one) students. Appendix A, Table 2 shows the origin of each item in the five item indicator booklets.

Suitability of TIMSS and PISA studies for this research

TIMSS and PISA are international comparison studies (four-yearly and three-yearly respectively), evaluating students' achievement in, and attitudes towards, mathematics and science (PISA also assesses reading). While the students who participate in PISA are of a similar age to those participating in the Junior Certificate, they are younger than those taking the Leaving Certificate examinations. The TIMSS students are younger than both groups: those taking the Leaving Certificate and those taking the Junior Certificate. These studies might, therefore, seem to be inappropriate choices as comparative studies for this evaluation. However, they are the only two major international comparison studies which evaluate students' achievement in, and attitudes towards, mathematics and were, therefore, selected as the most relevant international comparison studies for this purpose. It is important to bear in mind that the TIMSS items are designed for a wide range of students, with

[3] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS.
[4] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no data available.

some designed to challenge more able students and others to be more widely accessible. This range made them relevant for this evaluation, despite the age differences outlined above. Differences in age and stage of schooling are taken into account in this report in the discussion of students' performance on the item indicator booklets.

The international studies give some initial indicators of how the achievement of Ireland's students in the revised syllabus strands compares with achievement internationally on assessment items related to those strands. In addition and more importantly, the data gathered from the first administration of these 'indicator' items will provide a baseline against which the achievement of Irish students' can be benchmarked internationally in future. As a first step towards measuring change over time, the TIMSS assessment items answered by the evaluation sample in Spring 2012 will be administered to a further sample of students in Autumn 2012, and this will allow comparison of results over that period, as the revised syllabus begins to become more established in schools.

Because the TIMSS items were selected to match the revised mathematics syllabus, they are drawn from across different TIMSS assessment booklets. As a result, the items provide an indication of how the performance of students following the revised syllabus on those items compares with that of students internationally, but they do not allow conclusions to be drawn about students' performance on a syllabus domain as a whole. This would require a test of representative syllabus coverage that had been trialled and analysed for whole-test reliability, rather than a collection of separate but connected items. It is important to bear this in mind when interpreting the outcomes from the item indicator booklets.

Administration and marking of the assessment booklets

The item indicator booklets were administered to students by their teachers, in the 19 phase one schools and in 52 comparison group schools. Booklets were returned to NFER before being marked using the NFER's own on-line system. Multiple-choice items were double marked by the NFER's data capture staff. The remainder of the items were marked by the NFER team. Teachers in the participating schools were not involved in marking the item indicator booklets.

Versions of the revised mathematics syllabus referenced in this report

Several interim mathematics syllabuses have been created as the Project Maths initiative has been developed. All references to the syllabus in this report have been taken from the Junior Certificate Mathematics syllabus for examination in June 2015 and the Leaving Certificate Mathematics syllabus for examination in 2014 only. It was felt that consistency in the labelling of the sub-sections of each strand would make comparisons easier both between the Junior Certificate and the Leaving Certificate stages of school and between the different phases of assessment that are being carried out for this research. Use of the latest syllabus provides this consistency. Where a sub-section of a strand is referred to, the relevant numbering from the syllabus is provided in brackets. For example, ‘representing data graphically and numerically’ is labelled as sub-section 1.6 of Strand 1, Statistics and Probability.

A technical note about the international data

As previously mentioned, international data is only available for those items which were sourced from either TIMSS 2007 or TIMSS Advanced 2008. To compare the performance of the Irish students to international standards, TIMSS international item level data was downloaded from the TIMSS international database. This included the TIMSS year 8 dataset and the TIMSS Advanced year 12 database from 2008. There were 48 countries involved in TIMSS 2007, and 20 countries were involved in TIMSS Advanced 2008. This means the number of pupils taking each item varied across items. In order to compare common items with the international outcomes, facilities[5] were calculated by weighting item raw scores by total student weight (TOTWGT). This provides a more robust international estimate for item facility and allows comparison to the revised syllabus item results.

2.2.3 Student attitude surveys

As detailed in Table 2.1, the student attitude survey aims to gather quantitative data, focusing on:

- students’ motivations and attitudes to mathematics, in general and in relation to the revised mathematics syllabus
- students’ opinions on the revised mathematics syllabus, including the effectiveness of different strands and approaches
- students’ perceptions of their knowledge, understanding, confidence and achievement in mathematics
- students’ aspirations to further study of mathematics
- students’ views of the relevance and application of mathematics more generally.

Surveys were administered to students by their teachers, and contained questions relating to students’ experiences of, and attitudes towards, mathematics in general, as well as questions relating to individual strands of the revised syllabus. As both groups of Leaving Certificate students, and phase one Junior Certificate students were studying between two and five strands of the revised mathematics syllabus, strand-specific questions were asked of all students, in order to determine the impact of the revised syllabus on motivation, attitudes and confidence.

[5] Facility’ is a measure of the difficulty of an item, expressed as the percentage gaining credit for their answer.

Part A

Achievement, learning and motivation of Junior Certificate students

3. About the Junior Certificate students

This chapter describes the profile of the Junior Certificate students who participated in this research, as a basis for further exploration in subsequent chapters. The findings from the assessment of student achievement are presented in Chapter 4, and from the survey of student attitudes in Chapter 5.

3.1 About the students

In total, 375 students from 19 phase one schools, and 2,375 students from 125 comparison group schools, completed the Junior Certificate student attitude survey. A total of 303 students from the same phase one schools, and 910 students from 52 of the comparison group schools, completed the Junior Certificate assessment of student performance.

3.2 Syllabus strands studied

The Junior Certificate students who participated in the research were in the examination class of 2012: they were, therefore, in their final year of their Junior Certificate studies. Having commenced junior cycle in September 2009, participating students in the phase one schools were in the second cohort of those following the revised mathematics syllabus. These students had studied the following four strands of the revised syllabus:

- Strand 1: Statistics and Probability
- Strand 2: Geometry and Trigonometry
- Strand 3: Number
- Strand 4: Algebra.

The comparison group[6] was composed of Junior Certificate students in non-phase one schools, also in their examination year. This cohort of students was following the previous mathematics syllabus introduced in 2000, and had not studied any strands of the revised syllabus.

[6] The revised syllabus was introduced to non-phase one schools in September 2010. As this group of students commenced junior cycle in September 2009, they did not follow the revised mathematics syllabus.

4. Achievement of Junior Certificate students

This chapter presents the findings of the assessment of Junior Certificate student achievement in phase one schools across all four strands of the revised mathematics syllabus studied. These findings are compared to the achievement of comparison group students, and to international standards. Key messages are highlighted in each of these sections.

4.1 Overview of achievement patterns

The performance of the phase one students on the Junior Certificate item indicator booklets showed some clear patterns, which are described in detail in section 4.2. For example, it can be noted that **items requiring higher order skills (such as reasoning and an ability to transfer knowledge to new contexts) are found more difficult than those which are more mechanical in demand**. Comparison group students (those in non-phase one schools) also showed this same pattern. The performance of phase one and comparison group students is similar with no statistically significant difference on any item. Therefore, it would appear that there is **no discernible difference in the students' skills** as measured by the items contained in the item indicator booklets.

The pattern of performance when compared with the international data centres on the different topic areas of the syllabus, rather than on overarching skills that can be applied more generally. These comparisons are affected by differences in the sample between TIMSS and this study, with the TIMSS students being somewhat younger. Whilst **phase one students scored well on the majority of items, it seems that they find Algebra (Strand 4) and 'Examining algebraic relationships' in particular to be especially difficult.** Facilities[7] for items assessing this area of the syllabus are low, and in some cases, below the international average. This is noteworthy since we would expect phase one students to perform above the available international averages on these items (due to the differences in the samples, as discussed in section 4.2). By contrast, phase one **students have shown a strong performance on items assessing Statistics and Probability (Strand 1)**. Although phase one students perform above the international average in Geometry and Trigonometry (Strand 2) and Number (Strand 3) this is to be expected given the sampling differences.

[7] 'Facility' is a measure of the difficulty of an item, expressed as the percentage gaining credit for their answer.

4.2 Performance in detail: phase one schools

Key messages

In relation to Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry) of the revised syllabus, there were no items which students appeared to have found particularly difficult. Students also performed well on the majority of items relating to Strand 3 (Number) and Strand 4 (Algebra), although there were some specific items that students appeared to find more challenging. However, students showed a wide variation of abilities within each area of the syllabus, suggesting that **phase one students struggled with particularly demanding questions, rather than a specific topic or theme.**

The performance of the phase one students on the Junior Certificate item indicator booklets suggest that, in general, **items requiring higher order skills** (such as reasoning and an ability to transfer knowledge to new contexts) are found more difficult than those which are more mechanical in demand.

4.2.1 Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry)

A total of 184 students from phase one schools completed item indicator booklet JC1/2 (for further details on the methods used, see chapter 2). This booklet contained items assessing statistics, probability, trigonometry and geometry (Strands 1 and 2). Column four of Table 4.1 below shows the performance of phase one students completing JC1/2[8]. For two mark items, the table shows the proportion of students who achieved just one mark, and the proportion who received full credit. For each item, the table also gives the broad syllabus area assessed and a summary of the task (see Appendix A, Table 3 for performance on each item matched to the specific numbered area of the revised syllabus, as referred to in the following commentary).

[8] The table also shows the performance of non-phase one students.

Table 4.1: Item indicator booklet JC 1/2 – Student performance and summary of items

Item	Syllabus area	Item summary	Phase one students		Non-phase one students	
			1 mark (%)	2 marks (%)	1 mark (%)	2 marks (%)
1	Probability	Estimate and compare probabilities (numbered tickets)	87		87	
2	Interpreting data	Interpret data (bar chart)	96		95	
3	Representing data	Transform data (pie chart to bar chart)	6	62	8	60
4	Probability	Estimate probability (coloured marbles)	86		76	
5	Representing data	Match tabulated data to corresponding line graph	95		94	
6a	Interpreting data	Use bus timetables to plan travel according to time constraints	17	47	21	40
6b	Interpreting data	Use bus timetables to plan travel according to time constraints	24	33	27	28
6c	Interpreting data	Draw conclusions from tabulated data	47		47	
7a	Representing data	Find and compare means from tabulated data	73		74	
7b	Interpreting data	Draw conclusions from data in scatter graph	35	41	37	37
8	Probability	Find number of coloured beads (from probability of selection)	71		70	
9	Statistical reasoning	Understand how data points relate to their average	22		19	
10	Probability	Estimate size of sectors on coloured spinner (from experimental data)	60		52	
11	Statistical reasoning	Recognise that a graph is potentially misleading	37	17	32	13
12	Coordinate geometry	Identify a point given coordinates	87		82	
13	Synthetic geometry	Find size of angle (using congruent triangles & sum to 180)	59		52	
14	Synthetic geometry	Find size of angle formed by diagonals of hexagon	67		59	
15	Synthetic geometry	Find size of angle (using straight angle)	64		68	
16	Synthetic geometry	Find size of angle (using vertically opposite angles & isosceles triangle)	37		40	
17	Transformation geometry	Rotate 3-D shape	76		81	
18	Coordinate geometry	Identify coordinates of top vertex of isosceles triangle	73		65	
19	Synthetic geometry	Construct obtuse & acute angles	51		46	
20	Synthetic geometry	Find size of angle (using bisectors & straight angle)	35		41	
21	Synthetic geometry	Find size of angles (using alternate angles or exterior angle)	65		56	

Table 4.1 shows that the range of facilities for JC 1/2 is from 22 per cent to 96 per cent with students performing well on the majority of items. There are no items which students have found exceptionally difficult (facility <20 per cent) but both item 2 and item 5 were very easy with facilities higher than 90 per cent (96 per cent and 95 per cent respectively).

Item 2 assesses ‘Analysing, interpreting and drawing conclusions from data’ (1.7). It requires students to read and manipulate data depicted on a bar graph. The same area of the syllabus is assessed by items 6a, 6b, 6c and 7b although these are based on timetables and a scatter graph rather than a bar chart. The facilities for these items are in the range of 45-58 per cent so it is likely that the simplicity of the bar chart in item 2 is the cause for the high facility rather than a particular proficiency in this syllabus area.

Item 5 assesses ‘Representing data graphically and numerically’ (1.6). Students are asked to select (from four options) the graph which shows the information given in a table. The same area of the syllabus is assessed by items 3 and 7a. The lower facilities of these items (1m: 6 per cent, 2m: 62 per cent[9] for item 3 and 73 per cent for item 7a) are likely to be due to the greater demand necessitated by the need to manipulate the data. By comparison item 5 more simply requires transposing the data to a different form.

4.2.2 Strand 3 (Number) and Strand 4 (Algebra)

Item indicator booklet JC3/4 was completed by 186 phase one students. It contains items assessing number and algebra. Table 4.2 shows the performance of phase one students completing JC3/4, as well as the broad syllabus area assessed and a summary of the task (see Appendix A, Table 3 for performance on each item matched to the specific numbered area of the revised syllabus).

[9] For items worth two marks, facilities are expressed as the percentage gaining exactly one mark and the percentage gaining full credit (two marks).

Table 4.2: Item indicator booklet JC 3/4 - Student performance and summary of items

Item	Syllabus area	Item summary	Phase one students[10]	
			1 mark (%)	2 marks (%)
1	Number: percentages	Estimate percentage of four digit number	77	
2	Scientific notation	Evaluate number written in scientific notation	90	
3	Number: ratio	Find number of boys in a class given boy:girl ratio	69	
4	Number: fractions	Add and subtract simple fractions	50	
5	Applied measure	Find distance travelled in given time	83	
6	Number: operations	Perform division with negative number	72	
7	Number: prime factors	Recognise prime factors of four digit number	59	
8	Applied measure	Compare value for money of two pizzas based on surface area	3	0
9	Applied measure	Interpret graph (speed of racing car on track)	93	
10a	Applied arithmetic	Currency conversion with given exchange rate	74	
10b	Applied arithmetic	Explain benefit of lower exchange rate	34	
11	Number: proportion	Understand proportional relationship (cost of apartment based on floor area)	10	
12	Representing situations	Use numerical methods to extend pattern of matches	12	17
13a	Representing situations	Complete table of number of trees by expanding systematic pattern	8	66
13b	Equations	Solve equation with quadratic term	29	
13c	Algebraic relationships	Understand that squared terms increase more quickly than linear terms	12	9
14	Graphical relations	Interpret graph of motion (moving walkway)	14	
15	Inequalities	Solve linear inequality	38	
16	Expressions	Simplify linear expression with two variables	73	
17	Finding formulae	Express unknown length in terms of two variables	65	
18	Expressions	Evaluate expression with two variables	54	
19	Equations	Solve linear equation (shipping charges)	57	
20	Algebraic relationships	Determine which point is on a line (given equation)	36	
21	Finding formulae	Derive formula for linear relation between two variables	46	

[10] Non-phase one students are not included in this table as they did not complete booklet JC 3/4.

Table 4.2 shows that the proportions of students scoring one or both marks range from 0 per cent to 93 per cent and students performed well on the majority of items. However, item 9 proved to be very easy for the students (facility, 93 per cent) and some items (8, 11, 12, 13c and 14) were particularly difficult (facilities <25 per cent).

Items 8 and 9 assess the same area of the syllabus (Applied measure, 3.4) yet have the lowest and highest facilities in the booklet (1m: 3 per cent, 2m: 0 per cent; and 93 per cent respectively). This indicates that the topic area itself is not the main factor in determining performance, but rather the demand of the item. Item 9 asks students to interpret a graph that shows the speed of a vehicle over time. It therefore also requires basic skills in 'Analysing, interpreting and drawing conclusions from data' (1.7), a skill students have already demonstrated in item 2 of JC1/2 that they are proficient in. Item 8 on the other hand requires students to calculate and compare the surface area of two pizzas. Very few students attempted to do this and instead incorrectly reasoned that the pizzas offered the same value for money (39 per cent).

Item 11 (facility, 10 per cent) assesses 'Number systems' (3.1) which is also the main focus of items 1, 3, 4, 6 and 7. The facilities of these latter items range from 50 to 77 per cent, again indicating that the difficulty of item 11 is due to the specific item demand rather than the topic area. The main difference between item 11 and the other comparable items is that students are not required to make a particular calculation. Rather, they must reason and make generalisations using their knowledge of proportions within the given topic of purchasing an apartment building. Item 12 (facility, 1m: 12 per cent, 2m: 17 per cent) assesses 'Representing situations with tables, diagrams and graphs' (4.2). However, for item 13a which assesses the same area, 66 per cent of students achieved full credit. It is likely that differing demands of the items are affecting performance. In addition to assessing 'Representing situations', item 12 also assesses students' reasoning and their ability to find a simple formula (4.3), which makes the item more challenging.

Item 13c (facility 1m:12 per cent, 2m:9 per cent) assesses elements of 'Examining algebraic relationships' (4.4) as does item 20 (facility 36 per cent). It is not surprising that item 13c has a lower facility than item 20 as it asks students to contrast a linear and quadratic relationship in order to explain whether the number of apple trees (quadratic) or the number of conifer trees (linear) will increase more quickly. Item 20 by contrast is a multiple choice item requiring numbers to be substituted into a simple algebraic equation. The generally low facilities in this area of the syllabus suggest that students may be finding it more difficult than other topic areas. Interestingly 'Examining algebraic relationships' also includes proportional relationships which was assessed in item 11 (facility, 10 per cent).

Item 14 is the final item that has a low facility (14 per cent). It assesses 'Relations without formulae' (4.5). It asks students to draw a line on a graph to show the distance travelled by a person standing still on a moving walk way in comparison to people walking on the walk way or on the ground. This area of the syllabus has not been assessed by any other item in the indicator booklet. Therefore it is difficult to make generalisations regarding the cause of the low facility.

The majority of omission rates are below 15 per cent for items in both the JC1/2 and JC3/4 indicator item booklets. Items with omission rates greater than 15 per cent are all open response items. This is not unexpected as the multiple choice format allows students to make an informed guess at an answer even if the item is difficult. A difficult open response item, however, provides less structure for students who are not sure how to answer.

4.3 Comparison of student performance between phase one and comparison group schools

Key messages

The performance of phase one and comparison group students at Junior Certificate level is similar. Whilst, in general, phase one students performed better than their comparison group peers, this difference is only statistically significant in relation to a particular item exploring students' abilities in Strand 1, Statistics and Probability (assessing students' understanding of the outcomes of simple random processes). It is therefore reasonable to conclude that **engagement with the revised syllabus has not yet influenced the performance of phase one students** at Junior Certificate level relative to their peers following the previous syllabus.

Table 4.1 also presents the scores of students from comparison group schools who completed JC1/2 (N=910). This allows for a basic comparison of performance between phase one and non-phase one students. Appendix A, Table 3 presents further analysis of this comparison, using the statistical method of differential item functioning analysis.

This table shows that, in some cases, the performance of students from comparison group schools follows a similar pattern to that of the phase one students. For example, items 2 and 5 were completed easily and item 9 proved more difficult. However, many items have facilities that are between five and ten percentage points different from those of the phase one students. Phase one students scored five to ten percentage points higher on eleven of the 29 items or item parts. These are highlighted in orange in Appendix A, Table 3. The difference in performance of all but one of these marks is likely to be due to sample differences (significance five per cent or higher). Only on item 4 do phase one students perform statistically better than their comparison group peers. Item 4 assesses students' understanding of 'Outcomes of simple random processes' (1.3). The students are asked to identify the true statement regarding the probability of which colour marble will be picked out of a bag next. This item has the largest difference in facility (10 percentage points) between the two groups of students. The omission rate for both groups of students is the same (1 per cent). As a result, the options that the students chose were studied to identify any difference in the pattern of responses. The first option is the correct one and as noted above was chosen less often by students in the comparison group schools. However, the pattern of selection of the other options is similar to that for the phase one students, as presented in Table 4.3 below.

Table 4.3: Answers given for item 4 by phase one and comparison group students

Answer option	Phase one students choosing each answer option (%)	Comparison group students choosing each answer option (%)
A	86	76
B	1	5
C	10	13
D	2	5
Omitted	1	1
Total	100	100

Table 3 (Appendix A) shows that comparison group students performed better than their peers by five or more percentage points on two items. The difference in facilities is significant beyond the five per cent level for item 20. This item assesses 'Synthetic geometry' (2.1) which is also the focus for items 13, 14, 15, 16, 19 and 21. It is unclear why comparison group students have performed better on item 20 as it is similar in demand to several of the other items in that it asks students to calculate the size of an angle.

While phase one students have scored higher on more items than the comparison group students, the difference in performance for each item is not statistically different. On the basis of this analysis, it seems reasonable to conclude that engagement in the revised syllabus has not influenced the performance of the students in the phase one schools at Junior Certificate level.

4.4 Comparison of student performance with international standards

Key messages

In general, phase one students scored well on the majority of items in comparison to international standards. In particular, phase one students have shown a **strong performance on items assessing Statistics and Probability (Strand 1)**.

However, phase one students appear to find **Algebra (Strand 4) and 'Examining algebraic relationships' in particular to be especially difficult**. Overall, their knowledge on subject areas relating to Strands 2 and 3 appear to be similar to that internationally.

The Junior Certificate item indicator booklets were constructed using material from two international surveys: 'released' items[11] from TIMSS 2007 (Trends in International Mathematics and Science Study, a survey of 13-14 year olds) and sample items[12] from PISA 2000, 2003 and 2006 (Programme for International Student Assessment, a survey of 15 year olds). For the majority of the TIMSS items, the international average facility is available. However, no international data could be sourced for item 9 of JC1/2. This item was not included in the analysis of TIMSS 2007 data, and therefore is not included in the available dataset. Comparative international data for the PISA items is also not available as these are sample items, which were not used in a live test. As the performance of the phase one and comparison group students is similar and the comparison group students did not complete item indicator booklet JC3/4, international data is only compared with that of the phase one students.

There are three factors that should be highlighted and considered when comparing the performance of the phase one students with that internationally. Firstly, and most importantly, as outlined earlier, there is a substantial difference in the average age of the two groups of students. The TIMSS items were originally used to collect data about the performance of students in the 8^{th} grade. These students would be 13 to 14 years old. The students completing the Junior Certificate, however, are mostly 14 or 15 years old. They are one school year older than the students who completed the TIMSS assessments. It is not possible to quantify the effect that this difference in age and additional schooling has had. However, it is plausible that there would be some effect and it should be anticipated that the Irish students would, therefore, achieve higher scores.

A second factor to consider when comparing the performance is the stage of the students' schooling. Students who completed the Junior Certificate item indicator booklets were preparing for the live Junior Certificate examinations which would require considerable revision and preparation. While the situation would vary internationally, it is unlikely that the international students would have been preparing for or had just completed examinations of this kind. Both the amount of revision and the students' familiarity with the exam situation may have influenced the difference in performance. Although it is not possible to quantify, this suggests that the Irish students may be at an advantage in answering the TIMSS items.

Thirdly, the TIMSS international averages are derived from a range of countries. Some are highly developed and high-performing countries, while others are less well developed and less highly performing. As such, they provide a useful baseline measure of the achievement of the phase one students' performance on the TIMSS

[11] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS
[12] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no data available.

items, but data should be interpreted with this context in mind. Appendix A, Table 4 repeats the average scores of the phase one students (given in Table 3) and compares them with the international average scores in the 2007 TIMSS study. As expected, phase one students appear to have generally scored more highly than the international average.

4.4.1 Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry)

Comparative data is available for 22 out of 24 items or item parts of JC1/2, and the average facility for each item is higher for the phase one students than the international students on all items and item parts. Table 4.4 shows the number of items with differences in facility that fall within the three performance bands as described above.

Table 4.4: Number of items in JC1/2 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	3	9	10	22
International students score more highly	0	0	0	0

The differences in sample in terms of age, length of schooling and preparedness for exams have led us to expect the phase one students to achieve higher scores. Items which show little difference (0-9 percentage points) in performance (and where it is possible that the difference could be due to sampling effects) might indicate areas in which the Irish students are performing less well than anticipated. There are three such items (15, 16 and 20) which all assess 'synthetic geometry' (2.1). They are similar in demand to items 13, 14 and 21. All six of these items ask students to calculate the size of an angle. Interestingly, items 13, 14 and 21 have differences in facility of 16-26 percentage points, higher than items 15, 16 and 20. It is unclear why these six items should show such a large variation in comparative performance given that their construct is so similar.

Phase one students have performed much better (≥25 percentage points) than international students on many of the items in JC1/2. The majority of these, however, fall within syllabus Strand 1 (Statistics and Probability). Within this strand, the phase one students have consistently outperformed the international average by at least 20 percentage points.

There is some historical evidence of proficiency in this content area among Irish students compared with their peers internationally. In TIMSS 1995, Irish students scored an average of 69 per cent correct in items assessing ‘Data representation, analysis and probability’, while the international average was 62 per cent (IEA Third International Mathematics and Science Study, 1994-95).[13] However, the elevated performance of phase one students in the present evaluation may also reflect the increased emphasis on Statistics and Probability in the revised syllabus. While the old syllabus included elements of statistics and data analysis, the Project Maths initiative has made Statistics and Probability a key component of mathematics at both Junior and Leaving Certificate level. This strand covers basic skills such as calculating probabilities and measures of central tendency. However, there is also a focus on higher order skills, including an emphasis on ‘Statistical reasoning with an aim to becoming a statistically aware consumer’. The high performance of phase one students on the items in this strand is encouraging and suggests that the implementation of this part of the new syllabus is working well.

4.4.2 Strand 3 (Number) and Strand 4 (Algebra)

Comparative data is available for 15 out of 24 items/item parts of JC3/4. As for JC1/2, the average facilities show that phase one students are generally performing better than the international students. Table 4.5 below shows the number of items with differences in facility that fall within the three performance bands.

Table 4.5: Number of items in JC3/4 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	2	8	2	12
International students score more highly	1	1	1	3

The elevated performance of the phase one students is generally as expected given the differences in sample in terms of age, length of schooling and preparedness for exams. However, there are some exceptions where international students have scored more highly by more than 10 percentage points) or similarly (0-9 percentage points) to the phase one pupils. (Any differences in this latter group of items could possibly be due to sampling effects.) There are five such items in JC3/4 and these

[13] These results are for the Grade 8 students (in Ireland this corresponded to students in their second year of secondary school).

are likely to be items on which phase one students are not performing as well as might be expected. International students are performing considerably better than phase one students on items 15 and 20. Item 20 assesses an area of the syllabus (Examining algebraic relationships) that is not assessed by any other available internationally comparable item. It asks students to identify the graph point which lies on a line with a given equation. It does not particularly require the use of higher order skills such as problem solving and logical reasoning and may simply reflect a lack of confidence in this area of the syllabus as item 13c, which also assesses this area of the syllabus was also found difficult by phase one students (facility ≥1m: 21 2m: 9).

Item 15 assesses 'expressions' as do items 16, 18 and 19. Of these, items 15 and 16 ask students to identify an equivalent equation by simplifying the one given. Items 18 and 19 on the other hand require students to substitute a value into an equation. The pattern in the difference in performance is difficult to explain, as items 16 and 19 show little difference in performance while item 18 appears to favour phase one students. Item 18 requires two values to be substituted and there are brackets in the equation while for item 19, students must decide which of the values they are substituting for (and which they are calculating) and also there is some simplification required too.

Item 21 is the final item in JC3/4 on which phase one students do not substantially outperform the international sample (the difference is just two percentage points in favour of phase one students). It assesses 'Finding formulae' an area of the syllabus also assessed by item 17. These items differ slightly in that students can check their answer to item 21 by substituting values into the equation and checking that the answer matches the data given in a table. This is not possible for item 17 which also requires students to interpret a relationship described in the text. It is likely to require a greater depth of understanding than item 21 which enables students to use their data handling skills, an area in which they have already demonstrated they are proficient (in JC1/2).

Items assessing Strand 3 of the syllabus (Number) are generally showing performance as anticipated, with phase one students scoring higher than the international average. Item 4 was completed particularly well by phase one students It assesses 'Number systems' (3.1) an area of the syllabus that is assessed by several other items which can be compared with international data (1, 3, 6 and 7). The items are mostly similar in style with a description of a scenario and a calculation to be carried out in answer to the question. It is not clear why phase one students are performing particularly well on item 4, compared with the other items addressing this area.

It is clear from the evidence in Table 4 that phase one students are generally performing better in comparison with the international 2007 TIMSS sample. Twenty-nine (78 per cent) of the 37 Junior Certificate items for which international data is available show differences in facility of more than 10 percentage points in favour of phase one students. The corresponding proportion on which the international students score higher is just five per cent (two items). Six items (16 per cent) show little difference in performance. Given the wide breadth of the revised mathematics syllabus it is difficult to cover all topic areas comprehensively enough to draw conclusions on the topic areas or skills that phase one students are most likely to excel at. However, it is likely that phase one students have more secure knowledge in statistics, but are perhaps less knowledgeable in algebra than the international average. They are of a similar average standard as the (younger) international students in geometry, trigonometry and number. Table 4.6 gives the average per cent correct in each Mathematics content area for the eighth grade Irish and international samples in the 1995 TIMSS survey.

Table 4.6: Average per cent correct in each mathematics content area

TIMSS 1995 Sample	Mathematics overall %	Fractions and number sense %	Geometry %	Algebra %	Data represent-ation analysis and probability %	Measurement %	Proportion-ality %
Ireland	59	65	51	53	69	53	51
Internat-ional average	55	58	56	52	62	51	45

The most recent TIMSS data for Ireland comes from the 1995 survey (TIMSS 2011 outcomes, including those for Ireland, will be published in December 2012) and it provides some historical context with which the performance of phase one schools can be compared. In the 1995 survey Irish students performed better than the international average in mathematics overall and in data representation, analysis and probability. The phase one students have also shown a strong performance in statistics. Previously Irish students did less well than the international average in Geometry. With the advantage of age and an extra year of schooling, phase one students have scored more highly in Geometry. Algebra has continued to be an area that Irish students find difficult. In 1995 Irish students showed a strong performance on Number items in comparison to the international average. Phase one students are also performing higher than the international average, but this is with the advantages mentioned previously.

5. Junior Certificate student attitude survey

This section presents the findings of the first survey of Junior Certificate students' attitudes towards mathematics, for both phase one and comparison group schools. It explores:

- their experiences of mathematics lessons
- their attitudes towards learning mathematics
- their views and perspectives on careers involving mathematics.

Key messages are highlighted in each of these sections, and in relation to individual strands of the revised syllabus.

5.1 Students' experiences of mathematics lessons

To contextualise students' attitudes towards mathematics, and to compare the learning experiences of phase one students with the comparison group, participants were asked about how mathematics is taught in school. Students were asked about the frequency with which they participated in a range of activities which feature in the revised mathematics syllabus. This included how often they:

- **apply their learning in mathematics**, including how often they apply what they learn in mathematics to real-life situations, and make links between different mathematics topics
- **consider the processes underpinning mathematics** (the 'how' and the 'why' of mathematics), including how often they think about mathematics problems and plan how to solve them, and show their working to justify their answers
- **participate in discursive, collaborative activities**, including how often they work together in groups or pairs, and talk about their ideas using the language of mathematics
- **participate in investigative, practical activities**, including how often they plan and conduct investigations to solve mathematics problems, and use IT in mathematics lessons
- **actively engage in their own learning and progress**, including how often they set goals and targets about their mathematics learning.

Students were also asked about the frequency with which they participate in activities which are **not** intended to feature in the revised syllabus, instead representing more traditional approaches to mathematics teaching and learning (for example, learning by rote and teaching for success in particular types of examination question). This included how often students:

- use textbooks in lessons then later practise what they have learned
- practise examination questions in class
- copy what their teacher writes on the board then practice using examples.

An overview of Junior Certificate students' perspectives in relation to each of these areas is presented in Figure 5.1. Phase one students are presented alongside those of the comparison group so that similarities and differences are immediately apparent.

Figure 5.1: Proportion of Junior Certificate students reporting that they 'often' or 'sometimes' take part in the specified mathematics teaching and learning activities

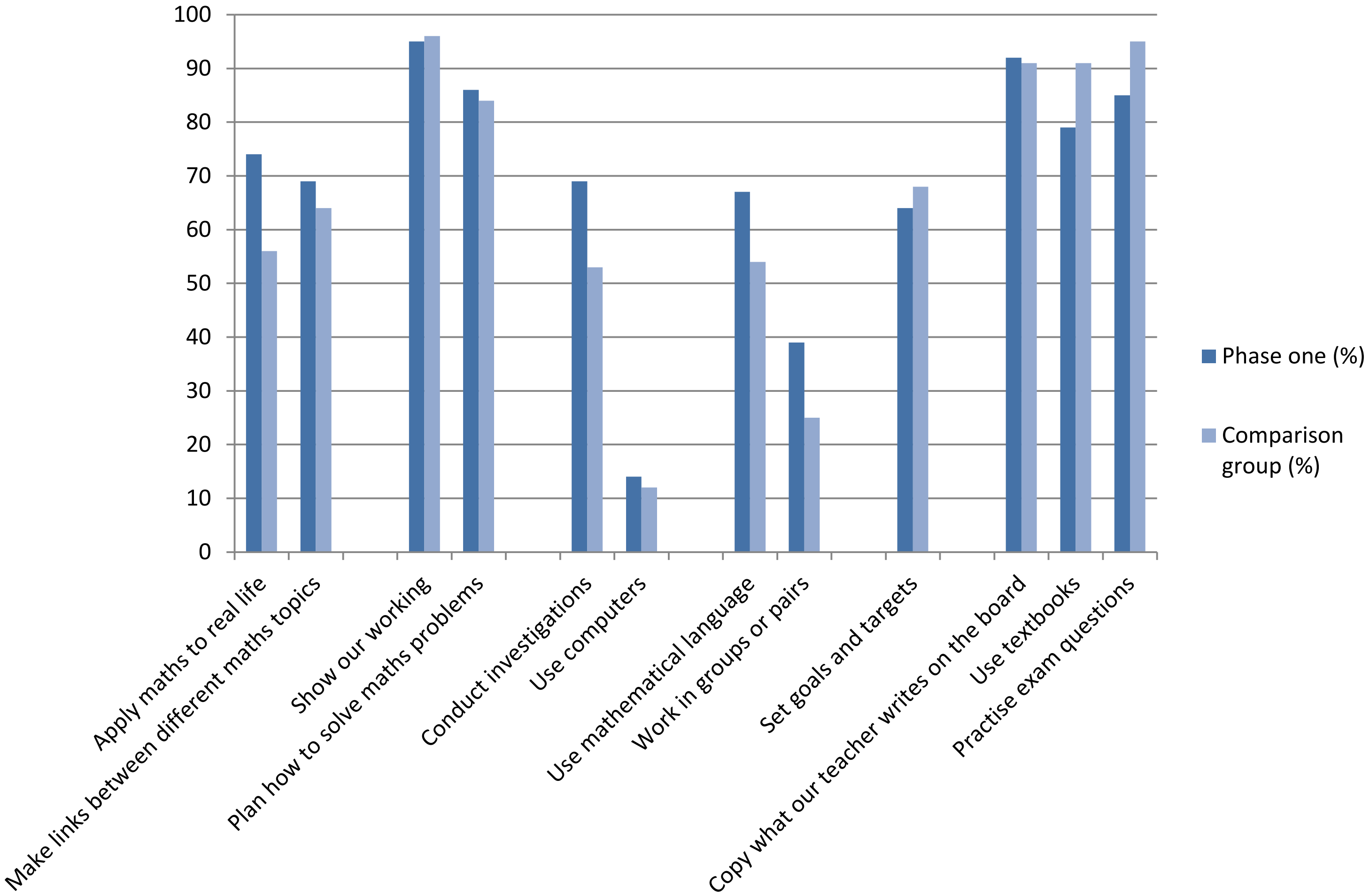

Figure 5.1 generally shows the expected pattern. In many of the mathematics teaching and learning approaches that are promoted within the revised syllabus, there are a higher proportion of positive responses from the phase one group than the comparison group. Correspondingly, in the more traditional areas there tends to a higher proportion of positive responses from the comparison group. Further analysis also reveals that, in many areas, these are statistically significant differences, indicating that there are meaningful variations in the types of mathematics activities

that each group has participated in. Students' perspectives in relation to each of these areas are discussed in further detail below.

5.1.1 Students' perspectives on learning approaches characteristic of the revised syllabus

This section explores students' perspectives on the learning approaches they have experienced in their mathematics lessons.

Applying mathematics

Key messages

Students following the revised mathematics syllabus appear to **apply their learning to real-life situations** more frequently than those following the previous syllabus.

They also appeared to **make connections between mathematics topics** more frequently than their peers, suggesting that the revised syllabus is encouraging students to develop their synthesis skills.

Phase one students reported particularly strongly, relative to their comparison group peers, they that regularly **applied their learning**, to both real-life situations and to other mathematics topics. This reflects the prominence of students' ability to situate their mathematical knowledge within realistic contexts as a learning outcome of the revised syllabus.

Table 5.1 shows, for example, that almost three-quarters of phase one students (74 per cent) reported that they **applied their learning in mathematics to real-life situations** 'sometimes' (45 per cent) or 'often' (29 per cent), compared to just over half (56 per cent) of comparison group students who reported that they do this 'sometimes' (45 per cent) or 'often' (11 per cent).

Table 5.1: We apply what we learn in maths to real life situations

	Phase one %	Comparison group %
Often	29	11
Sometimes	45	45
Rarely	17	33
Never	6	8
No Response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

This is a statistically significant difference and demonstrates that, whilst the same proportion of students in both groups feel they sometimes apply their learning in mathematics to real-life situations, a much higher proportion of phase one students, relative to the comparison group, feel that they do this often. As the application of mathematics to real-life contexts is a central feature of the revised mathematics syllabus, this provides a clear indication that such approaches are being effectively translated into classroom practice, and that students recognise this as a distinct aspect of their learning.

Likewise, when asked how frequently they **made links between different mathematics topics**, there was a statistically significant difference between the responses of phase one and comparison group students. The findings are presented in Table 5.2:

Table 5.2: We make links between different maths topics

	Phase one %	Comparison group %
Often	27	19
Sometimes	42	45
Rarely	21	23
Never	6	10
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.2 shows that whilst similar proportions of phase one and comparison group students reported that they make links between different mathematics topics 'sometimes' (42 per cent of phase one students, and 45 per cent of comparison group students), a considerably higher proportion of phase one students reported that they did this 'often' (27 per cent of phase one students, compared to 19 per cent of comparison group students). This suggests that the revised syllabus is having a positive impact in encouraging phase one students to synthesise their mathematical knowledge, and to draw on a range of different topics and contexts to solve mathematical problems.

Knowledge of the processes underpinning mathematics

Key messages

The vast majority of students following the revised syllabus reported that they regularly take part in activities aimed at developing their knowledge and understanding of the **processes underpinning mathematics**. This included activities such as **showing their working to justify their answers**, and **planning how to solve mathematics problems**.

A similarly high proportion of students following the previous syllabus also reported that this was the case. This suggests that the approaches promoted throughout the revised syllabus are complementing, rather than replacing, established techniques within school.

Figure 5.1 also shows that students reported strongly that they had participated in those teaching and learning activities aiming to support their understanding of **the processes underpinning mathematics**, although in this area there was less of a distinction between phase one and comparison group students. For example, whilst the vast majority (95 per cent) of phase one students reported that they **show their working to justify their answers** 'sometimes' or 'often', a similar proportion of comparison group students (96 per cent) also reported that this was the case. This is not a statistically significant difference, indicating that the revised syllabus has not impacted upon students in relation to this area. However, as the proportion of students in both groups reporting that they regularly show their working to justify their answers is very high, it appears that this approach is already well embedded within schools (since the introduction of the previous syllabus in 2000, students at Junior Certificate level have been required to show their working in examination questions to support their answers, but not necessarily to justify their conclusions) (Appendix B, Table 1).

As shown in Table 5.3, the vast majority of phase one students also reported that they **think about mathematics problems and plan how to solve them** in lessons, reflecting the emphasis placed on exploring the 'how' and the 'why' of mathematics within the revised syllabus.

Table 5.3: We think about maths problems and plan how to solve them

	Phase one %	Comparison group %
Often	55	50
Sometimes	31	34
Rarely	9	12
Never	2	4
No Response	3	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.3 shows that the vast majority (86 per cent) of phase one students reported that they think about mathematics problems and plan how to solve them 'sometimes' (31 per cent) or 'often' (55 per cent), compared to 84 per cent of comparison group students who reported that they do this 'sometimes' (34 per cent) or 'often' (50 per cent). Whilst there is a statistically significant difference between the two groups, both phase one and comparison group students gave a similar distribution of responses. This suggests that students following both the revised and previous versions of the mathematics syllabus are encouraged to participate in this type of approach during lessons.

Participation in investigative, practical activities

Key messages

Students following the revised syllabus appear to be taking a **hands-on approach** to learning mathematics. The majority of students following the revised syllabus reported that they **regularly conduct investigations** to solve mathematical problems, and appeared to do so more often than those following the previous syllabus.

Use of **information technology (IT) in mathematics lessons appeared to be limited** amongst students following both syllabuses, although those following the revised syllabus appeared to use computers in mathematics lessons to help them solve problems more often than their peers.

Similarly, Figure 5.1 shows that a higher proportion of phase one students reported that they regularly take part in investigations and practical activities in mathematics lessons, than their comparison group peers. This is an encouraging indication that the hands-on emphasis of the revised syllabus, characterised by participation in interactive, problem-solving activities, is being translated in the classroom. However, there was considerable variation in the extent to which different aspects of this approach were being applied.

For example, Table 5.4 shows that the majority of students in the phase one group (69 per cent) reported that they conduct investigations to solve mathematical problems ‘sometimes’ (34 per cent) or ‘often’ (35 per cent). Comparison group students reported that they did this considerably less frequently: just over half (53 per cent) reported that they did this ‘sometimes’ (30 per cent) or ‘often’ (23 per cent).

Table 5.4: We do investigations to solve maths problems

	Phase one	Comparison group
	%	%
Often	35	23
Sometimes	34	30
Rarely	19	26
Never	7	19
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

This difference is statistically significant and demonstrates that whilst similar proportions of phase one and comparison group students feel that they sometimes conduct investigations to solve mathematics problems, a higher proportion of phase one students feel that they do this often.

By contrast, as shown in Table 5.5, when asked how often they **used computers in mathematics lessons to help them solve problems**, far fewer students reported that they had taken this approach. This suggests that the use of information technology (IT) as a classroom learning tool in mathematics is, in general, limited.

Table 5.5: We use computers in maths lessons to help us solve problems

	Phase one	Comparison group
	%	%
Often	3	3
Sometimes	11	9
Rarely	27	17
Never	57	69
No Response	3	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.5 shows that there was a statistically significant difference between the responses of phase one and comparison groups students when asked how frequently they used computers in mathematics to help them solve problems, with phase one students reporting that they used computers more frequently than their comparison group counterparts. However, just 14 per cent of phase one students reported that they used computers in mathematics ‘sometimes’ (11 per cent) or ‘often’ (three per cent). By contrast, 12 per cent of comparison group students reported that they did this ‘sometimes’ (nine per cent) or ‘often’ (three per cent). This suggests that, whilst the revised syllabuses appear to have had a positive impact on the frequency of students’ use of IT in mathematics, there is still considerable room for the use of such resources to be increased.

Participation in discursive and collaborative activities

Key messages
There is considerable variation in the frequency with which students participate in discursive and collaborative activities. Students following both syllabuses appear more likely to **participate in discussion about mathematics as a whole class**, than they are to work together in small groups or pairs. However, students following the revised syllabus report that do both types of activity more frequently than those following the previous syllabus. This indicates that such approaches are growing in prominence.

Figure 5.1 also shows that there is considerable variation in the regularity with which students participate in discursive and collaborative activities. The findings suggest that across both phase one and comparison groups, students are more likely to participate in discussion about mathematics as a whole class, than they are to work collaboratively in small groups or pairs. However, phase one students report that they participate in both of these activities more frequently than their comparison group peers, indicating that such approaches have become more prominent with the introduction of the new syllabus.

For example, as shown in Table 5.6, phase one students reported that they **talk about their ideas using the language of mathematics** in lessons more frequently than comparison group students.

Table 5.6: We talk about our ideas using the language of maths

	Phase one %	Comparison group %
Often	30	19
Sometimes	37	35
Rarely	20	27
Never	10	18
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.6 shows that over two-thirds (67 per cent) of phase one students talk about their ideas using the language of mathematics ‘sometimes’ (37 per cent) or ‘often’ (30 per cent), compared to just over half (54 per cent) of comparison group students who reported that they do this ‘sometimes’ (35 per cent) or ‘often’ (19 per cent).

Again, this difference is statistically significant, and demonstrates that whilst similar proportions of phase one and comparison group students feel they sometimes talk about their ideas using the language of mathematics in lessons, a much higher proportion of phase one students, relative to the comparison group, feel that they do this often. This suggests that students following the revised syllabus are more frequently engaging in discursive activities in the classroom, and are being encouraged to develop their mathematical reasoning skills.

A considerably lower proportion of students in both phase one and comparison groups, however, reported that they regularly **work together in small groups or pairs**, relative to other areas. The findings are presented in Table 5.7:

Table 5.7: We work together in small groups or pairs

	Phase one	Comparison group
	%	%
Often	14	8
Sometimes	25	17
Rarely	31	30
Never	28	44
No Response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.7 shows that, whilst phase one students reported that they work together in small groups or pairs more frequently than their comparison group peers, this remains relatively uncommon: almost three-fifths (59 per cent) of phase one students reported that they ‘rarely’ (31 per cent) or ‘never’ (28 per cent) do this, compared to just under three-quarters (74 per cent) of comparison group students who reported that they do this ‘rarely’ (30 per cent) or ‘never’ (44 per cent). As this approach is actively promoted within the revised syllabus, it is encouraging that phase one students report they are doing this with more regularity than students in the comparison group. However, the approach does not yet appear to be particularly widespread.

Becoming active learners

Key messages

The majority of students following the revised and previous syllabuses reported that they regularly **set goals and targets** about their mathematics learning.

This is a highly positive reflection of students' experiences of learning mathematics. However, there does not appear to be any discernible difference between the frequency with students following the revised syllabus undertake this activity and their peers.

The majority of phase one students (64 per cent) reported that they **set goals and targets about their mathematics learning** 'sometimes' or 'often', compared to 68 per cent of comparison group students. Whilst this suggests that students following the revised syllabus actually do this slightly less frequently than their comparison group peers, the difference between the two groups is not statistically significant. This indicates that the revised syllabus has not impacted upon students in relation to this particular approach (Appendix B, Table 2).

5.1.2 Students' perspectives on learning approaches characteristic of a more traditional syllabus

Key messages

Whilst students appear to be using many of the approaches promoted through the revised syllabus, a high proportion of students following the revised syllabus also report that they **regularly participate in activities associated with more traditional approaches to mathematics teaching and learning**.

This includes, for example, copying what their teacher writes on the board and working from textbooks in lessons. In some cases, students following the revised syllabus appeared to take part in these types of activities less frequently than those following the previous syllabus, and in other cases more so. This suggests a **degree of variability in the extent to which the revised syllabus has impacted upon the use of traditional approaches** to the teaching and learning of mathematics in the classroom.

Figure 5.1 also shows that, although there are positive indications that the approaches promoted through the revised syllabus are being reflected in the classroom, there remains a high proportion of phase one pupils who report that they participate in activities associated with more traditional approaches to mathematics teaching and learning.

Most notably, as shown in Table 5.8, when students were asked how often they **copy what their teacher writes on the board then practise using examples**, phase one students reported that they did this more commonly than their comparison group peers.

Table 5.8: We copy what our teacher writes on the board then practice using examples

	Phase one	Comparison group
	%	%
Often	62	68
Sometimes	30	23
Rarely	4	6
Never	2	1
No Response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.8 shows that 92 per cent of the phase one group reported that they did this 'sometimes' (30 per cent) or 'often' (62 per cent), compared to 91 per cent of the comparison group who reported that they did this 'sometimes' (23 per cent) or 'often' (68 per cent). There is a statistically significant difference between the responses of these two groups (although it is not clear where this statistical significance lies), which appears to be inconsistent with the approaches promoted throughout the revised syllabus, as well as with students' perceptions about such aspects of their learning.

A reasonably high proportion of students in both phase one and comparison groups reported that they **use textbooks in lessons and then practise what they have learned, either in class or for homework**; although phase one students appeared to do this less frequently than their comparison group peers. The findings are presented in Table 5.9:

Table 5.9: We use textbooks in lessons then practise what we have learned in class and/or for homework

	Phase one	Comparison group
	%	%
Often	60	77
Sometimes	19	14
Rarely	12	6
Never	6	2
No Response	3	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.9 shows that a total of 79 per cent of phase one students reported that they do this 'sometimes' (19 per cent) or 'often' (60 per cent). By contrast, 91 per cent of comparison group students reported that they do this 'sometimes' (14 per cent) or 'often' (77 per cent). This difference is statistically significant. Similarly, as shown, in Table 5.10, a lower proportion of phase one students, relative to the comparison group, reported that they regularly **practise examination questions in class**.

Table 5.10: We practise exam questions in class

	Phase one	Comparison group
	%	%
Often	55	80
Sometimes	30	15
Rarely	10	3
Never	2	1
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.10 shows that overall, 85 per cent of phase one students reported that they did this 'sometimes' (30 per cent) or 'often' (55 per cent), compared to 95 per cent of comparison group students who reported that they did this 'sometimes' (15 per cent) or 'often' (80 per cent). Again, this is a statistically significant difference.

As such approaches are considered to be characteristic of a more traditional perspective on mathematics teaching and learning, rather than of the revised syllabus, it is encouraging to find that, generally, a lower proportion of phase one students report that they regularly participate in these activities. Nonetheless, phase one students' participation in these activities remains high. This may be underpinned by a range of possible explanations: one such possibility may be that as the revised syllabus is relatively new, there remain concerns about examination content and format, leading schools to use more familiar methods of supporting young people to achieve examination success, whilst simultaneously promoting and delivering many of the features of the revised syllabus. This will be explored in further detail during the case-study phase.

5.1.3 Students' perspectives on mathematics teaching approaches

Key messages

Students who were following both the revised and previous versions of the mathematics syllabus were **highly positive about the mathematics teaching they had experienced**, and expressed broadly similar views.

Whilst this suggests that the revised syllabus does not appear to have impacted significantly on students' perceptions of the ways that their teachers support their learning, it positively reflects students' general satisfaction with their classroom teaching.

Given that the introduction of the revised syllabus marks a considerable shift in the way that mathematics is taught, requiring schools to familiarise themselves with a range of new topic areas and teaching approaches, it would not have been surprising to see a decrease in students' satisfaction with their learning experiences. It is therefore **encouraging that students' views remain so positive**.

In addition to questions exploring their perspectives on the approaches used to facilitate their learning, students were also asked about how their teachers were helping and supporting them in their mathematics classes, as an indicator of their experiences of the *teaching approaches* promoted throughout the revised syllabus. This question aimed to elicit students' views on the ease with which teachers have been able to apply the principles of the revised syllabus, rather than to assess or judge individual teacher quality. Specifically, students were asked about their teachers':

- capacity to set them work that reflected their abilities and interests, and to challenge and improve their skills
- ability to explain to students what they expect them to do, and support them in areas they are finding difficult
- ability to present mathematics in a way that is interesting
- confidence in students' abilities.

The responses of both phase one and comparison group students were highly positive, reflecting students' general satisfaction with their classroom teaching. However, the revised syllabus does not appear to have had impacted significantly on students' perceptions of the ways that their teachers are able to support them in their learning. However, given the many changes in teaching approaches that schools have implemented following the introduction of the revised mathematics syllabus (as identified throughout this section), it is to phase one schools' credit that their students' positive perceptions of teaching closely reflect those of the comparison group, who are following a syllabus with more established teaching approaches (Appendix B, Tables 3-10).

5.1.4 Discussion

The responses of students following the revised mathematics syllabus provide an encouraging indication that in many areas, the approaches promoted throughout the Project Maths initiative are being felt by students. The survey cannot tell us, however, the extent to which students' experiences are shaped by differences in their approaches to *learning* (underpinned by the structure, format and content of the revised mathematics syllabus), or whether it is instead a reflection of the nature of the *teaching* they receive. For example, are students finding connections between different mathematics topics as a result of their own learning process, or are they instead replicating knowledge passed on by their teachers, who already understand and appreciate these links?

Additionally, this section raises interesting questions regarding the extent to which schools are *able* to implement some of the approaches promoted throughout the revised syllabus. For example, students' use of IT in mathematics lessons may be affected by a range of factors, such as availability of facilities or resources specifically relating to the Project Maths initiative. These issues, and others relating to students' experiences of learning mathematics, will be explored further during the case-study phase.

5.2 Students' attitudes towards learning mathematics

This section explores students' attitudes towards learning mathematics, both generally and in relation to the individual strands of the revised mathematics syllabus.

5.2.1 Attitudes towards individual strands of the revised mathematics syllabus

To gain a more in-depth understanding of students' confidence in relation to the individual strands of the revised syllabus, students were asked about how confident they would feel when undertaking a range of different activities during their mathematics lessons. This includes students' confidence in relation to:

- Strand 1, measured by their confidence in working out the probability of an event occurring, and in drawing charts to display data, including pie charts and bar charts
- Strand 2, measured by their confidence in solving problems using trigonometry, making different shapes, and solving problems using the properties of different shapes
- Strand 3, measured by their confidence in understanding indices, and using formulae to solve problems in measurement
- Strand 4, measured by their confidence in solving problems using algebra,
- Strand 5, measured by their confidence to represent relationships between numbers graphically.

Students were also asked about their confidence to undertake mathematics activities embedded within all strands of the revised syllabus. This included students' confidence in using mathematics to solve problems based on real-life situations; solving mathematics problems using what they have learned in more than one mathematics topic; and gathering together information from a range of sources, and applying it to solving mathematical problems.

An overview of students' perspectives in relation to each of these areas is presented in Figure 5.2:

Figure 5.2: Proportion of Junior Certificate students reporting that they would find it 'very easy' or 'easy' if they were asked to solve problems in each of the following areas

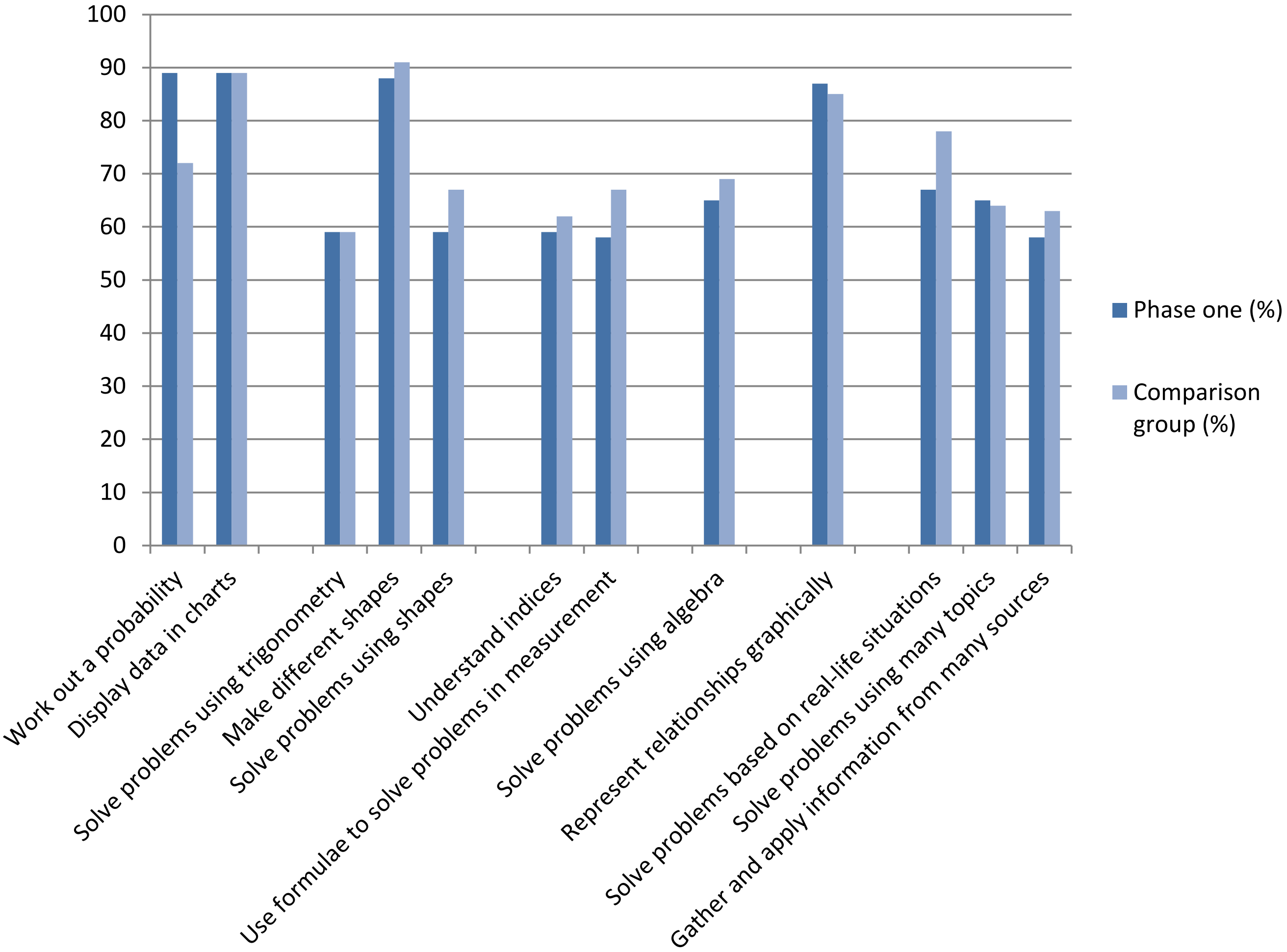

Figure 5.2 shows that both phase one and comparison group students are broadly confident in their abilities in topics spanning all strands of the revised syllabus. Furthermore, it indicates that in most areas, the views of phase one and comparison group students are similar. However, there is a notable difference in relation to calculating probabilities, which is a key aspect of Strand 1, Statistics and Probability. Here, phase one students appear to be more confident than their comparison group peers. These findings, and others, are explored by individual strand in the following sections.

5.2.2 Strand 1: Statistics and Probability

Key messages

Junior Certificate students who had following the revised syllabus, as well as their peers following the previous syllabus, **appeared highly confident in items relating to Strand 1, Statistics and Probability**.

Students who had followed the revised syllabus appeared more confident than their peers in **calculating the probability of an event occurring**, with over a quarter more students reporting that they would find it 'very easy' to do this. This is an encouraging finding, indicating that the approaches used to support young people in their understanding are being successfully applied.

Students were asked how confident they would feel to work out the probability of an event occurring. The findings are presented in Table 5.11.

Table 5.11: If I were asked to work out the probability of something happening...

	Phase one %	Comparison group %
I would find it very easy	61	37
I would find it easy	28	35
I would find it a little difficult	7	21
I would find it very difficult	2	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.11 shows that phase one students are highly confident at calculating the probability of an event occurring, and more confident than their comparison group peers:

- the vast majority of phase one students (89 per cent) reported that they would find it 'easy' (28 per cent) or 'very easy' (61 per cent) to calculate the probability of an event occurring

- by contrast, just under three-quarters (72 per cent) of comparison group students reported that they would find it ‘easy’ (35 per cent) or ‘very easy’ (37 per cent)
- whilst just nine per cent of phase one students reported that they would find it ‘a little difficult’ (seven per cent) or ‘very difficult’ (two per cent) to calculate the probability of an event occurring, over one-quarter (27 per cent) of comparison group students reported that this was case, with 21 per cent reporting they would find it ‘a little difficult’ and six per cent ‘very difficult’.

These differences are statistically significant, and demonstrate considerable positive impacts of the revised syllabus for this strand. The revised syllabus emphasises probability to a far greater extent than the previous mathematics syllabus, so it is encouraging that phase one students are highly confident in this area, both as an absolute measure, and relative to the comparison group.

However, there was no statistically significant difference between the responses of phase one and comparison group students in relation to other aspects of Strand 1, for example, confidence in **drawing charts to display data, including pie charts and bar charts,** although responses were largely positive (over 89 per cent of both phase one and comparison group students reported that they would find this ‘easy’ or ‘very easy’, with over half (54 per cent of both groups) reporting that they would find this ‘very easy’ (Appendix B, Table 11).

5.2.3 Strand 2: Geometry and Trigonometry

Key messages

Again, Junior Certificate students appeared to be **broadly confident in their understanding in relation to Strand 2, Geometry and Trigonometry**. Both groups of students, however, reported lower levels of confidence than they did for Strand 1, Statistics and Probability.

In some areas, Junior Certificate students following the revised syllabus appeared to be less confident than those who had followed the previous syllabus: for instance, to make different shapes, and to solve problems using the properties of different shapes. This suggests that there may be room for students’ confidence in Strand 2 to be further developed, albeit from a high baseline.

In general, students in both phase one and comparison groups were less confident in relation to Strand 2 of the revised syllabus than they were in Strand 1.

In relation to trigonometry, students were asked how confident they would feel to **solve problems using trigonometry**. The findings revealed that there was no statistically significant difference between the responses of phase one students and their comparison group peers, with just under three-fifths (59 per cent) of both phase one and comparison group students reporting that they would find it ‘easy’ or ‘very easy' to solve problems using trigonometry. This suggests that the revised syllabus has not yet positively influenced students' trigonometry skills (Appendix B, Table 12).

There were, however, statistically significant differences between phase one and comparison groups when asked how confident they would feel to **make different shapes** (for example, to draw a triangle with sides of length 3cm, 5cm and 8cm). The findings are presented in Table 5.12.

Table 5.12: If I were asked to make different shapes...

	Phase one %	Comparison group %
I would find it very easy	62	69
I would find it easy	26	22
I would find it a little difficult	9	7
I would find it very difficult	1	1
No response	2	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.12 shows that both groups of students are, overall, highly confident in this area. However, phase one students are statistically less confident than their comparison group peers in this area: 88 per cent of phase one students reported that they would find it 'easy' (26 per cent) or 'very easy' (62 per cent). By contrast, 91 per cent of comparison group students reported that they would find this 'easy' (22 per cent) or 'very easy' (69 per cent).

Taking this one step further, students were then asked how confident they would feel to **solve problems using the properties of different shapes** (for example, to find the surface area and volume of a range of solids). The findings are presented in Table 5.13:

Table 5.13: If I were asked to solve problems using the properties of different shapes...

	Phase one %	Comparison group %
I would find it very easy	22	30
I would find it easy	37	37
I would find it a little difficult	33	27
I would find it very difficult	8	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.13 shows that, again, students in both groups were generally less confident about their ability to solve problems using the properties of different shapes than they were to approach the mathematics problems posed in relation to Strand 1, and phase one students were less confident than their comparison group peers:

- just under three-fifths (59 per cent) of phase one students reported that they would find it 'easy' (37 per cent) or 'very easy' (22 per cent) to solve problems using the properties of different shapes
- just over two-thirds (67 per cent) of comparison group students reported that they would find this 'easy' (37 per cent) or 'very easy' (30 per cent).

Again, this is statistically significant and suggests that, to date, the revised syllabus has not yet had a positive impact on students' confidence in relation to geometry, and specifically in the use of shape. Therefore, there may be a need for further development in this area.

5.2.4 Strand 3: Number

Key messages

In general, confidence was high amongst both groups of Junior Certificate students in items relating to Strand 3 (Number), although again, they did not appear to have the same degree of confidence exhibited in items relating to Strand 1 (Statistics and Probability).

There were some areas in which students following the revised syllabus appeared to be less confident than those who had followed the previous syllabus. Students following the revised syllabus reported that they were less confident, for example, in relation to their ability to use formulae to solve problems in measurement.

Students' responses to questions relating to their confidence in Strand 3 of the revised syllabus suggest that, in some areas, it does not appear to have made any difference to students' confidence in relation to Number. In other areas, however, phase one students appeared less confident than their comparison group peers.

Students were asked how confident they felt at **understanding indices**, for example, simplifying the equation $5^3 \times 5^4$. The findings reveal that the difference between phase one students and comparison group students is not statistically significant. Around three-fifths of students (59 per cent of phase one students, and 62 per cent of comparison group students) reported that they would find it 'easy' or 'very easy' to understand indices. This suggests that the revised syllabus has not positively influenced students' confidence in this area (Appendix B, Table 13).

Students were also asked how confident they would feel to **use formulae to solve problems in measurement**: for example, to find the speed of a car that travelled a distance of 100km in 1.5 hours, and the differences were statistically significant. The findings are presented in Table 5.14.

Table 5.14: If I were asked to use formulae to solve problems in measurement...

	Phase one %	Comparison group %
I would find it very easy	21	30
I would find it easy	37	37
I would find it a little difficult	34	26
I would find it very difficult	5	6
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.14 shows that:

- just under three-fifths (58 per cent) of phase one students reported that they would find it 'easy' (37 per cent) or 'very easy' (21 per cent) to use formulae to solve problems in measurement
- just over two-thirds (67 per cent) of comparison group students reported that they would find this 'easy' (37 per cent) or 'very easy' (30 per cent)

The findings show that phase one students are, in relation to this particular part of Strand 3, less confident than their comparison group peers.

5.2.5 Strand 4: Algebra

Key messages

Although the **majority of Junior Certificate students in both groups reported that they were confident to solve problems using algebra, a lower proportion of those following the revised syllabus reported that this was the case**. This indicates that students who have studied Strand 4 (Algebra), as part of the revised syllabus are finding this more challenging. This reflects the findings from the assessment part of this research, which suggest that students find algebra to be difficult relative to other areas.

Students were asked how confident they would feel to **solve problems using algebra**: for example, to find the value of x when 4x+3 = 2x+11. The findings are presented in Table 5.15.

Table 5.15: If I were asked to solve problems using algebra…

	Phase one %	Comparison group %
I would find it very easy	29	39
I would find it easy	36	30
I would find it a little difficult	24	20
I would find it very difficult	10	10
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.15 shows that:

- just under two-thirds (65 per cent) of students in phase one schools reported that they would find it 'easy' (36 per cent) or 'very easy' (29 per cent) to solve problems using algebra
- by contrast just over two-thirds (69 per cent) of students in comparison group schools, reported that they would find this 'easy' (30 per cent) or 'very easy' (39 per cent)

Whilst these findings overall positively reflect students' confidence in solving problems using algebra, there is a statistically significant difference between the two groups. This indicates that, in general, phase one students found algebra more challenging than students in the comparison group. The reasons for this will be explored further during the case-study phase. For example, the two groups may take different approaches interpreting this type of question, which could explain any differences in students' confidence.

5.2.6 Strand 5: Functions

Key messages

Neither of the Junior Certificate groups responding to the survey had studied Strand 5 of the revised syllabus, Functions. Despite this, **both groups of students reported that they were highly and similarly confident to represent relationships between numbers graphically**. This suggests that schools can feel confident in their students' capabilities when introducing Strand 5 of the syllabus.

In relation to Strand 5 of the revised syllabus, students were asked how confident they would feel to **represent relationships between numbers graphically**. Both groups of students reported that they were confident in this area, despite neither group having studied Strand 5 of the revised mathematics syllabus (although there is considerable overlap between this strand and Strand 4, Algebra). Overall, 87 per cent of phase one students reported that they would find this 'easy' (32 per cent) or 'very easy' (55 per cent) Overall, 85 per cent of comparison group students reported that they would find this 'easy' (31 per cent) or 'very easy' (54 per cent). However, there was no statistically significant difference between the responses of the two groups. This is to be expected given the similarity of their experiences in this area, as none of the students responding to the survey had studied this strand of the revised syllabus (Appendix B, Table 14).

5.2.7 All strands: Synthesis and problem solving

Key messages

In general, both phase one and comparison group students reported that they were **confident in solving problems based on real-life situations**. Phase one students, however, appeared to feel somewhat less confident than their comparison group peers.

Similarly, the majority of phase one students reported that they were confident in their ability to synthesise what they have learned in more than one topic and apply it to solving a range of mathematical problems, and to gather together information from a range of sources, and apply it to solving mathematical problems, with comparison group students reporting similar levels of confidence. The **findings suggest that the revised mathematics syllabus has not yet positively impacted on phase one students' abilities in each of the areas described**.

This is particularly notable in relation to the application of mathematics to real-life situations, as students in the phase one group reported that they had applied mathematics to real-life situations much more commonly than the comparison group. The reasons for this disparity are not yet understood, and will therefore be explored further during the case-study phase.

Across all strands of the revised syllabus, students are expected to be able to **use mathematics to solve problems based on real-life situations**. The findings relating to this area are presented in Table 5.16:

Table 5.16: If I were asked to use mathematics to solve problems based on real-life situations...

	Phase one %	Comparison group %
I would find it very easy	22	31
I would find it easy	45	47
I would find it a little difficult	29	19
I would find it very difficult	4	2
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5.16 shows that:

- in general, both phase one and comparison group students reported that they were confident in this area. Phase one students, however, appeared to feel somewhat less confident than their comparison group peers
- just over two-thirds (67 per cent) of phase one students reported that they would find this ‘easy’ (45 per cent) or ‘very easy’ (22 per cent), compared to almost four-fifths (78 per cent) of comparison group students who reported that they would find this ‘easy’ (47 per cent) or ‘very easy’ (31 per cent).

This is a statistically significant difference, and is particularly notable as students in the phase one group reported that they applied mathematics to real-life situations much more commonly than the comparison group. One possible explanation for this is that, as phase one students have done this more regularly than their comparison group counterparts, they have been encouraged to test out and challenge their skills in this area, and apply their knowledge to more complex real-life contexts, more so than comparison group students. Likewise, it may also be possible that phase one and non-phase one students have differing understandings of what is meant by ‘problem-solving’, perhaps associating it with the application of newly learned techniques rather than real-life contexts (see Meehan and Paolucci (2009) for a further discussion of this issue).

Similarly, across all strands of the revised syllabus students are expected to demonstrate their ability to synthesise what they have learned in more than one topic, and apply it to solving a range of mathematical problems. Whilst just under two-thirds (65 per cent) of phase one students reported that they would find it 'easy' (43 per cent) or 'very easy' (22 per cent) to solve **mathematics problems using what they have learned in more than one mathematics topic**, there was no statistical difference between the responses of phase one students and their comparison group peers (64 per cent of whom reported that they would find this 'easy' (41 per cent) or 'very easy' (23 per cent) (Appendix B, Table 15). Findings presented earlier in this report showed that phase one students more commonly make links between mathematics topics than the comparison group. Therefore, despite phase one students doing this type of activity more often, there is no apparent difference in student confidence between the two groups.

In both phase one and comparison group schools, students also appeared to find it challenging to **gather together information from a range of sources, and apply it to solving mathematical problems.** Overall, 58 per cent of phase one students report that they would find this 'easy' (43 per cent) or 'very easy' (15 per cent). By contrast, 63 per cent of comparison group students reported that they would find this 'easy' (45 per cent) or 'very easy' (18 per cent). However, there was no statistical difference between groups, suggesting that the revised syllabuses have not had an impact on students' skills in this area (Appendix B, Table 16).

5.2.8 General attitudes towards mathematics

Key messages

Both groups of Junior Certificate students appeared to have **highly positive attitudes towards mathematics in general**. This is indicative that the revised mathematics syllabus has been received positively by students, given that the approach is new to both students and teachers.

In order to understand students' perceptions of their own abilities and levels of engagement with mathematics, participating students were asked to comment on the extent of their agreement with a range of statements about learning mathematics. The areas explored included students':

- confidence in their own mathematical ability, and in their ability relative to their peers
- enjoyment of mathematics, and the process of learning mathematics
- interest in studying more mathematics in school.

Overall, students in both phase one and comparison groups reported similarly positive views about learning mathematics and in most areas, there were no statistically significant differences between the two groups (Appendix B, Tables 17-23), with the exception of students' **confidence in their mathematical ability relative to their peers**. In this case, phase one students appeared to be less confident than comparison group students in their mathematical ability relative to their peers. Just over two-fifths (43 per cent) of phase one students agreed either 'a little' (27 per cent) or 'a lot' (16 per cent) when asked if mathematics was more difficult for them than many of their classmates. By contrast, just over one-third (37 per cent) of comparison group students agreed either 'a little' (23 per cent) or 'a lot' (14 per cent) that this was the case. Nonetheless, it should be recognised that, in both groups, the students who lacked confidence relative to their peers were in the minority: in most cases, students had a positive view of their abilities in this as well as other regards (Appendix B, Table 24).

This is indicative that the revised mathematics syllabus has been received positively by students and, given that the approach is new to both students and teachers, it is encouraging that students' confidence has not diminished in the early stages of the syllabus's implementation. This is particularly promising when situated within the context of students' perspectives on the difference in difficulty between mathematics at primary and Junior Certificate level: when students were asked how often they felt that the way they learned mathematics at Junior Certificate level was harder than mathematics in primary school, phase one students appeared to feel this more frequently than their comparison group peers. The vast majority (90 per cent) of phase one students reported that they found mathematics at Junior Certificate level to be harder than mathematics at primary school 'sometimes' (18 per cent) or 'often' (72 per cent), compared to 86 per cent of comparison group students who reported this 'sometimes' (23 per cent) or 'often' (63 per cent). There is a statistically significant difference between the two groups, and it is therefore encouraging that, although phase one students appear to have found the transition from primary school to junior cycle more challenging than comparison group students, their overall confidence levels remain high (Appendix B, Table 25).

5.2.9 Discussion

This section highlights that students feel confident in their abilities in many aspects of the revised syllabus, particularly in relation to Strand 1 (Statistics and Probability). It is not possible to determine from the survey findings, however, the ways in which schools have been able to foster such positive impacts for this strand. Therefore, a key issue for further exploration is to more fully understand how schools have arrived at such impacts, so that this learning may be transferred into other strands.

The data explored in this section indicates that in some areas, the frequency with which students participate in particular mathematics learning approaches do not always result in positive impacts on their confidence (for example, phase one students reported that they regularly apply their learning to real-life situations, but do not appear to be any more confident in doing so than their non-phase one peers). As discussed earlier in this chapter, it is possible that students following the revised syllabus are now more aware of the complexities and challenges of applying mathematics to real-life contexts, and are therefore more cautious in estimating their abilities in this area. The case-study phase will allow for further consideration of this issue.

5.3 Students' attitudes towards careers involving mathematics

Key messages

Both groups of Junior Certificate students were in broad agreement that **mathematics was important in a range of contexts outside of the classroom** (e.g. in daily life). However, the revised syllabus does not appear to have significantly impacted on students' perspectives about the wider applications of mathematics, expressing relatively narrow perceptions of the range of careers involving mathematics.

To gain an understanding of students' attitudes towards careers involving mathematics, the survey explored students' knowledge of, and perspectives on:

- the wider application of mathematics beyond the classroom
- the range of jobs and career pathways involving mathematics.

5.3.1 Students' understanding of the wider application of mathematics

To ascertain the students' views on the broader application of mathematics beyond the classroom, they were asked to comment on the extent to which they perceived it to be useful in the following ways:

- to help in daily life
- to aid learning in other school subjects
- to enable them to get into the university of their choice
- to enable them to get the job of their choice.

The findings showed that, whilst both groups of students were in broad agreement that mathematics was important in each of these areas (between 70 per cent and 85 per cent of phase one students agreed 'a little' or 'a lot' that this was the case, as did between 69 per cent and 83 per cent of comparison group students), there were no statistically significant differences between phase one students and their comparison group peers in any of these areas. This suggests that the revised syllabus has not significantly impacted on phase one students' perspectives about the wider applications of mathematics (Appendix B, Tables 26-29).

5.3.2 Students' understanding of jobs involving mathematics

To explore students' understanding of jobs and career pathways involving mathematics, they were provided with a list of ten different professions, all involving mathematics in a variety of different ways. Students were then asked to select which of these roles involved using mathematics. These professions, in rank order according to the proportion of students indicating positively that they involve mathematics, are shown in Table 5.18.

Table 5.18: Proportion of students indicating that mathematics is involved in each profession

	Phase one students	Comparison group students
80-100 per cent	Owning a business	Owning a business
	Accountant	Accountant
	Engineer	Engineer
	Scientist	Scientist
	Sales assistant	Sales assistant
	Working with technology	Working with technology
40-60 per cent	Doctor	Doctor
	Fashion designer	Dietician
	Dietician	Fashion designer
< 40 per cent	Nurse	Nurse

Table 5.20 shows that there were no substantial differences between the students' views on which of these roles involve using mathematics. Perhaps unsurprisingly, students in both groups reported most strongly that this was the case for jobs involving a clear mathematical component (for example, **accountancy, or owning a business**): over 90 per cent of students in both phase one and comparison group schools identified that this was the case.

Next, students in both phase one and comparison groups strongly identified that careers in other science, technology, engineering and mathematics (STEM) subjects involved mathematics. This was reported most strongly in relation to **engineering and science** (where over 85 per cent of students in both phase one and comparison group schools identified that this was the case) and, to a slightly lesser extent, working with technology (over 80 per cent reported that that was the case in both groups). Students were less convinced, however, that careers in design involved using mathematics (for example, becoming a **fashion designer**): just less than half of students reported that this was the case (48 per cent of phase one students, and 49 per cent of comparison group students).

Overall, students did not appear to perceive that careers in the **medical profession** involved using mathematics. Whilst just over half (57 per cent of both phase one and comparison groups) reported that being a doctor would require mathematics, just 37 per cent of both groups felt that the same would be true for nursing (Appendix B, Tables 30-39).

The similarities in the responses of phase one and comparison group students suggest that the revised syllabus has not, as yet, broadened students' perspectives on the range of professions that involve mathematics. This may, therefore, be an area which would benefit from further development.

5.3.3 Discussion

The findings presented in this section indicate that the introduction of the revised mathematics syllabus has not, to date, had any discernible impact on students' appreciation of the application of mathematics outside of the classroom (although in general, students in both phase one and non-phase one groups had broadly positive views in this regard). This suggests that, although phase one students reported having applied their learning to a range of contexts, this may not yet have impacted upon their appreciation of mathematics in the real world.

In addition, students' responses indicate a relatively narrow perception of the range of careers involving mathematics. One possible explanation for this may be that schools and students have not had sufficient opportunity within the syllabus timetable to discuss these career pathways, or to engage with individuals occupying these roles. This may, therefore, merit further exploration in subsequent parts of this research.

Part B

Achievement, learning and motivation of Leaving Certificate students

6. About the Leaving Certificate students

This section describes the profile of Leaving Certificate students who participated in the research, as a basis for further exploration in subsequent chapters. Chapter 7 presents the findings from the assessment of Leaving Certificate student achievement, and Chapter 8 the findings of the survey of Leaving Certificate students' attitudes towards mathematics

6.1 About the students

In total, 299 students from 19 phase one schools, and 2,004 students from 125 comparison group schools, completed the Leaving Certificate student attitude survey. A total of 370 students from the same phase one schools, and 722 students from 52 of the comparison group schools, completed the Leaving Certificate assessment of student performance.

6.2 Syllabus strands studied

Like the Junior Certificate students, participating Leaving Certificate students were in the examination class of 2012, and in the final year of their studies. Students had commenced their Leaving Certificate in September 2010: phase one students were, therefore, part of the third cohort of those following the revised mathematics syllabus. This group of students had studied all five strands of the revised syllabus, as follows:

- Strand 1: Statistics and Probability
- Strand 2: Geometry and Trigonometry
- Strand 3: Number
- Strand 4: Algebra
- Strand 5: Functions.

Students in non-phase one schools were part of the first national cohort of the revised mathematics syllabus. These students had followed revised syllabuses for Strands 1 and 2, and for the remainder of their studies had followed the previous mathematics syllabus. Whilst they were not, therefore, a comparison group, students in this group had been less immersed in the revised syllabus than their phase one counterparts.

7. Achievement of Leaving Certificate students

This section presents the findings of the assessment of Leaving Certificate student achievement in phase one schools across all five strands of the revised mathematics syllabus. These findings are compared to the achievement of comparison group students, and to international standards. Key messages are highlighted in each of these sections.

As noted earlier for the Junior Certificate section of the report, when reading this section, it is important to bear in mind the earlier discussion about differences in the ages of the Irish students and the students participating in the international studies. The students who participated in PISA and TIMSS 2007 (two of the studies from which the evaluation's Leaving Certificate indicator items were taken) are younger (15 years and 13-14 years old respectively) than the Leaving Certificate students. Conversely, those who participated in TIMSS Advanced 2008 (a third study from which some items were drawn) are older than the Leaving Certificate students. However, PISA and TIMSS are the only major international comparison studies of mathematics achievement and, between them, they assess a range of mathematical concepts at appropriate and, in some cases, challenging levels of difficulty. In addition and despite the age differences, the test items administered in this stage of the evaluation will provide a useful baseline measure of students' achievement over time, as the evaluation progresses and the revised approaches become more established in schools.

7.1 Overview of achievement patterns

Student performance was mixed across the different strands of the syllabus. **Phase one students displayed proficiency in Strand 1, Statistics and Probability**: most of the items in this strand had high facilities.[14] **In contrast, performance was slightly weaker on Geometry and Trigonometry** and many items had low facilities. However, this may be partly due to the item styles used in this booklet. A number of Leaving Certificate items required a multiple choice response followed by a 'show your working' or 'explain your answer' section (while one of these multiple choice items was also in the Junior Certificate booklet, only the Leaving Certificate booklet contained the additional explanation or working section). Many students did not attempt to justify their answers, suggesting either that they were daunted by this request, or did not realise they needed to complete this to be awarded full marks.

[14] 'Facility' is a measure of the difficulty of an item, expressed as the percentage gaining credit for their answer.

Performance on Strand 3 (Number) and Strand 4 (Algebra) was mixed, with a wide range of facilities. Low item facilities on these two strands may be due to a range of factors. For example, some items assess a concept in a complex way that may have been beyond many students (but were included in order to provide appropriate challenge for the more advanced students). One example of this is item 6 in booklet ALC4 which assesses 'expressions' using a composite function. While composite functions are covered in the revised syllabus, simplifying this type of expression is more demanding than working with a single function. Leaving Certificate students appeared to find items relating to Strand 5 (Functions) of the revised syllabus, relatively difficult, even amongst Higher Level students.

Overall, **the results echoed the finding among Junior Certificate students, in that items requiring higher-order skills were more difficult**. In addition, even where an item assessed a concept that all students should be familiar with, they struggled if required to view the problem from a different viewpoint. An example of this is item 7 in booklet ALC4 which requires students to construct a quadratic function by working backwards from its roots.

The performance of **phase one and non-phase one students was broadly similar on Strands 1 and 2** (non-phase one students were not tested on Strands 3 to 5). However, this is to be expected, as both groups of students had been studying Strands 1 and 2 of the revised syllabus for the same amount of time. Phase one students scored significantly higher on an item that requires analysing verbal geometric information and translating it into mathematical form. However, it is unclear whether the two groups truly differ on this skill, as a similar item showed no differential performance.

Comparisons between the performance of phase one students and the available international data are confounded by a range of factors including, as outlined earlier, differences in age. The effects of these are difficult to quantify. However, given the characteristics of the student samples, we would broadly expect phase one Leaving Certificate students to outperform the international average on TIMSS 8th Grade items, and to perform somewhat less well than the international average on TIMSS Advanced items. In general, the results followed this pattern, though there were some exceptions. Phase one students performed better than expected on items where the solution strategy is clear, and where diagrams, if applicable, are provided. By contrast, they performed less well on multi-step items.

7.2 Performance in detail: phase one schools

Key messages

Student performance was mixed across the different strands of the syllabus. Overall, the results echoed those of Junior Certificate students, in that **items requiring higher order skills were found to be more difficult**.

A number of items required a multiple choice response followed by a 'show your working' section. **Many students did not attempt to justify their answers**, suggesting either that they were daunted by this request, or did not realise they needed to complete this section to be awarded full marks.

7.2.1 Strand 1: Statistics and Probability

In the phase one schools, 178 students completed the Strand 1 item indicator booklet (referred to throughout this section as SPLC1), which covered Statistics and Probability (for further details on the methods used, see chapter 2). Table 7.1 shows the performance of phase one students completing SPLC1[15]. For two mark items, the table shows the proportion of students who only achieved one mark, and the proportion who received both available marks. For each item, the table also provides the broad syllabus area assessed, and a summary of the task (see Appendix A, Table 5 for item performance on each item matched to the specific numbered area of the revised syllabus). The indicator item booklet covered the following aspects of the syllabus: 'concepts of probability' (1.2), 'outcomes of random processes' (1.3), 'statistical reasoning with an aim to becoming a statistically aware consumer' (1.4), and 'representing data graphically and numerically' (1.6). In general, students performed well on the items in this booklet. The range of facilities for all but one of the one mark items was between 49 and 80 per cent. However students performed very poorly on item 6 which showed a facility of one per cent.

[15] The table also shows the performance of non-phase one students.

Table 7.1: Item indicator booklet SPLC1 (Statistics and Probability) – student performance and summary of items

Item	Syllabus area	Item summary	Phase One Students		Non-phase One Students	
			1 mark (%)	2 marks (%)	1 mark (%)	2 marks (%)
1	Probability	Estimate probability of two independent events	61		62	
2	Probability	Interpret long-term probability of earthquake	67		62	
3	Statistical Reasoning	Recognise graph as potentially misleading	30	28	30	21
4	Probability	Estimate size of sectors on coloured spinner (from experimental data)	70		61	
5	Representing Data	Understand why bar graph is unsuitable for given data	66		56	
6	Statistical reasoning	Understand how data points relate to their average	1		1	
7a	Representing Data	Calculate and compare means from tabulated data	80		75	
7b	Representing Data	Draw conclusions from data in graphical form	37	42	34	39
8	Statistical reasoning	Compare quality of polls based on sampling methods	5	58	9	49
9	Statistical reasoning	Use graph to make mathematical argument	49		38	

Item 6 assesses 'statistical reasoning' (1.4), as do items 3, 8 and 9. The fact that students did much better on items 3, 8 and 9, (facilities of 1m: 30 2m: 28[16]; 1m: 5 2m: 58; and 49 per cent respectively), suggests that the difficulty of item 6 is not due to the topic in general, but is likely to be due to question content. In order to score one mark for item 6, students had to decide whether each of five statements was conclusive or not. Although there were high facilities on some of the individual statements, the last two statements were classified correctly by only 11 per cent and 19 per cent of students. As a result, very few students managed to get the correct answers for all five statements. This is a complex task as it requires students to fully understand how individual data points do or do not affect the average. Students must also be able to construct counter-examples for each of the five statements to realise that each conclusion cannot be drawn. Furthermore, their performance may have been influenced by the need to classify all five statements as *not* conclusive in order to gain full credit; students tend to expect such yes/no classification item types to have at least one response in each classification group and this may have confused those less confident in their knowledge.

7.2.2 Strand 2: Geometry and Trigonometry

In the phase one schools, 179 students completed the Strand 2 item indicator booklet (referred to throughout this section as GTLC2). This booklet contains items on Geometry and Trigonometry. In particular, the following syllabus areas were covered: 'synthetic geometry' (2.1), 'co-ordinate geometry' (2.2), and 'trigonometry' (2.3). Column four of Table 7.2 shows the performance of phase one students completing GTLC2, as well as the broad syllabus area assessed, and a summary of the task[17] (see Appendix A, Table 5 for performance on each item matched to the specific numbered area of the revised mathematics syllabus).

Facilities for these items show a slightly weaker performance overall than for statistics and probability (SPLC1). The range of facilities for the one mark items in this booklet was between 12 and 77 per cent. There were a number of items that students found particularly difficult and some of the two mark items showed very low facilities. Items with low facilities included: item 2b (facility 1m: 2 per cent 2m: 18 per cent), item 7 (facility 17 per cent), item 8b (facility 13 per cent), item 10a (facility 12 per cent), and item 10b (facility 1m: 10 per cent 2m: 1 per cent). Item 2b assesses 'synthetic geometry' (2.1). A stronger performance on other items covering this area of the syllabus indicates that the low facility may be due to question content. Phase one students found part a of this item much easier (facility 77 per cent). Item 2a is the original multiple choice TIMSS item, which requires students to determine the size of an angle inscribed in a hexagon. It requires students to show their working for item 2a. Students seemed to find the requirement to explain working difficult in this context.

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[16] For items worth two marks, facilities are expressed as the percentage gaining exactly one mark and the percentage gaining full credit (two marks).
[17] The table also shows the performance of non-phase one students.

Table 7.2: Item indicator booklet GTLC2 (Geometry and Trigonometry) – student performance and summary of items

Item	Syllabus area	Item summary	Phase One Students		Non-phase One Students	
			1 mark (%)	2 marks (%)	1 mark (%)	2 marks (%)
1	Synthetic Geometry	Match complex description of shapes to diagram	59		48	
2a	Synthetic Geometry	Size of angle formed by diagonals of hexagon	77		68	
2b		Show working for 2a	2	18	3	10
3	Synthetic Geometry	Size of angle (sum to 180; vertically opposite angles)	51		47	
4	Synthetic Geometry	Size of angles (alternate angles; exterior angle theorem)	67		68	
5	Synthetic Geometry	Length of median of isosceles triangle	28		28	
6	Coordinate Geometry	Sum of slopes of equilateral triangle	34		41	
7	Coordinate Geometry	Investigate whether two lines are parallel	17		18	
8a	Trigonometry	Solve for x given value of sin 2x	31		25	
8b		Show working for 8a	13		14	
9	Coordinate Geometry	Prove two lines intersect at a common midpoint (diagonals of parallelogram)	8	22	6	30
10a	Trigonometry	Find the length of a chord of a circle (width of window in semi-circular room)	12		18	
10b		Show working for 10a	10	1	13	0

It is possible that some students merely estimated the size of the angle based on the diagram and therefore could not justify their choice. It may also have been the case that students could do the working out for part a in their head and so did not attempt part b. There was a high omission rate for part b of 49 per cent: the fact that so many students omitted this question so early in the assessment booklet suggests either that students found this style of question difficult or that they did not realise that a separate mark was awarded for working (the number of marks per question was not indicated in the assessment booklet). The norm in other tests is for partial credit to be awarded for working, albeit often only in cases where the final answer is incorrect.

A similar situation seems to be happening in item 8a and b, where there is also an original TIMSS multiple choice item followed by a new requirement to show working. Part a has a facility of 31 per cent, but this drops to 13 per cent for part b. However, it is likely that the topic area for item 8 is also less familiar to students. Item 8 and item 10 are the only items in the indicator booklet that assess ‘trigonometry’ (2.3) and both items show weak performance overall. Item 10 also consists of a TIMSS multiple choice item followed by a new request to ‘show your working’, but in this item, the facilities are even lower. Twelve per cent of students received a mark for 10a. This is even lower than the proportion expected by chance (for a multiple choice item of this type there is a one in four chance of guessing the correct answer). This suggests that this item has very effective distractors.[18] For part 10b, 11 per cent of students received at least one mark for partial working: they showed that 180 degrees divided by 10 is 18 degrees, thus establishing the first step in a trigonometry approach. However, only one per cent of students received two marks by following through with the remaining steps needed to work out the problem. This is a complex item, with a number of steps. It requires students to calculate the width of a flat window in a semi-circular room. This equates to calculating the length of a chord of a circle. In order to do this, students must recognise the need to bisect a sector of the circle to form two right-angled triangles. They must then apply the ratio for the sine of an angle to calculate the relevant length. Therefore, this item requires thorough knowledge of trigonometric ratios and the geometric properties of triangles. This item also benefits from the ability to construct an accurate diagram. The combination of all of these factors makes this a difficult item, which was clearly beyond the capabilities of the vast majority of phase one students.

The other item with a low facility was item 7, which assesses ‘co-ordinate geometry’ (2.2). Items 6 and 9, which also covered this topic area, had higher facilities (approximately 30 per cent), although as the facilities across all items on this topic are relatively low, it may be the case that this is also a topic area with which students are less familiar. Item 7 requires students to establish whether two lines are parallel. The low facility may be due in part to the fact that the slopes of the lines are deliberately very close in value and therefore appear parallel. Some students may

[18] ‘Distractors’ are the incorrect response options in a multiple choice item. These may include one or more responses related to common misconceptions, or errors that students are likely to make.

have relied on the appearance of the lines, rather than calculating the slopes using the appropriate formula.

7.2.3 Strand 3: Number

In the phase one schools, 185 students completed the Strand 3 booklet (referred to as NLC3). This booklet assesses students' understanding of Number, covering 'number systems' (3.1), 'length, area and volume (3.4)', and 'synthesis and problem solving skills' (3.5). Table 7.3 shows the performance of phase one students completing NLC3, as well as the broad syllabus area assessed, and a summary of the task[19] (see Appendix A, Table 5 for performance on each item matched to the specific numbered area of the revised syllabus). Facilities were mostly within the 20-80 per cent range. Students found two items more difficult: item 7 (facility 18 per cent) and item 10 (facility 14 per cent). Both item 7 and item 10 cover the same area of the syllabus: 'number systems' (3.1). The same topic is covered in items 2, 6, 8a, and 8b. Overall, the students did better in these four items (facilities 29-59 per cent), but generally performance appears to be quite weak in this topic area. Item 7 asked students to give the sum of an infinite geometric series. Many students selected option 2 (48 per cent of students), rather than the correct option 4 (18 per cent of students). The sum in option 2 used 3 as a denominator, which also featured in the question stem, and it may be a sign that students were using guess work for this question. (None of the other options included the number 3.)

Table 7.3: Item indicator booklet NLC3 (Number) – Student performance and summary of items

Item	Syllabus area	Item summary	Phase One Students	
			1 mark (%)	2 marks (%)
1	Area	Find area of square floor	82	
2	Percentages	Design system of coins under set conditions	29	
3	Area	Compare surface area of regular and irregular shapes	42	
4	Length	Estimate perimeter of regular shapes	33	14
5	Problem-solving	Find how many bookshelves can be made from constituent parts available	71	
6	Operations	Find thickness of paper folded multiple times	35	
7	Geometric series	Find sum of infinite geometric series	18	
8a	Patterns	Recognise pattern present in flashes of lighthouse	59	
8b	Patterns	Construct pattern of lighthouse flashes under set conditions	11	25
9	Area	Estimate how many people fit in a field of given dimensions	23	
10	Induction	State the steps required for proof by induction	14	

[19] Non-phase one students only completed the first two booklets. Therefore they are not included in this table or in the tables for booklets ALC4 or FLC5.

Item 10 required students to describe the necessary steps to prove a mathematical statement by induction. The item has a high omission rate: 61 per cent of students did not respond to this question, although the fact that this question was at the end of the assessment booklet might partially explain this. In addition, this open style of questioning may be more difficult for students, albeit a core element of the revised syllabus.

7.2.4 Strand 4: Algebra

In the phase one schools, 186 students completed the Strand 4 booklet (referred to as ALC4), focusing on Algebra. Table 7.4 below shows the performance of phase one students completing ALC4, as well as the broad syllabus area assessed, and a summary of the task (see Appendix A, Table 5 for performance on each item matched to the specific numbered area of the revised mathematics syllabus). The syllabus strands covered in this booklet include: ‘expressions’ (4.1), ‘solving equations’ (4.2), ‘inequalities’ (4.3), and ‘complex numbers’ 4.4). The items in this booklet were difficult for many students: out of the nine items, four had overall facilities under 20. The items with these low facilities were items 3, 5, 6 and 7. These four items each cover a different syllabus area, so there does not seem to be a link to topic area.

Item 6 assesses ‘expressions’ (4.1), and the facility was 16 per cent for this item. The other items assessing this have higher facilities: items 1a and b (facility 82 per cent and 25 per cent), and 2a (facility 1m: 8 2m: 76) and 2b (1m: 13 2m: 34). This suggests that it is question content that is proving difficult. Items 1 and 2 require students to substitute values and manipulate a single expression. In contrast, item 6 involves a composite function and asks students to determine its minimum value. It is therefore a more complex item, and as a result, the lower facility is not surprising.

Table 7.4: Item indicator booklet ALC4 (Algebra) – student performance and summary of items

Item	Syllabus area	Item summary	Phase One Students	
			1 mark (%)	2 marks (%)
1a	Expressions	Evaluate expression with four variables	82	
1b	Expressions	Construct expression to meet specified conditions	25	
2a	Equations	Solve equation with two variables given value of one	8	76
2b	Equations	Solve equation with two variables given value of one	13	34
3	Complex numbers	Divide real number by complex number (using complex conjugate)	13	
4	Inequalities	Solve inequality (quadratic)	26	
5	Inequalities	Solve inequality (algebraic fraction)	8	
6	Expressions	Find minimum value of composite function	16	
7	Equations	Form quadratic function given points of intersection with both axes	5	

Item 7 assesses ‘solving equations’ (4.2), and the facility was very low at 5 per cent. Item 7 is an open question which may have contributed to its lower facility, as other examples of open questions have shown generally weaker performance. However, there are no other items in the booklet covering the same syllabus area so it is unclear whether the low facility is particular to this type of item, or represents a wider lack of knowledge in this strand. This item requires students to form a quadratic function from its graph. The graph shows the points at which the function cuts both axes. While all students following the revised syllabus study quadratic equations, it may be that they are more familiar with solving equations to find the roots, rather than working backwards as this item requires.

Item 5, which assesses ‘inequalities’ (4.3), showed a facility of eight per cent. This area is also assessed by item 4, which had a higher facility of 26 per cent. Item 5 is open response, whereas item 4 is multiple choice and this may have contributed to the lower outcome for item 5. For item 4, it is possible that some students simply substituted in the values given in the multiple choice options and deduced the answer, rather than solving the inequality itself. This was not a possibility for item 5, as it is an open question. Furthermore, the inequality in item 5 contains an algebraic fraction, which requires students to square both sides before solving the inequality. It may be that some students were not aware of this necessary step.

Item 3, which assesses ‘complex numbers’ (4.4) showed a facility of 13 per cent. This item is a multiple choice question. Complex numbers are not covered on the Foundation Level syllabus, so not all students in the sample would be familiar with this topic. This item requires students to divide a real number by a complex number, and involves multiplying the numerator and denominator by the complex conjugate. It seems that only a small proportion of students were aware of this correct method. It may be the case that students are generally weak on this topic, but as this is the only item in the booklet that assesses this area, it is difficult to judge.

7.2.5 Strand 5: Functions

In the phase one schools, 180 students completed the Strand 5 booklet (FLC5). This booklet was based on Functions, and covered ‘functions’ (5.1) and ‘calculus’ (5.2) only. Table 7.5 below shows the performance of phase one students completing FLC5, as well as the broad syllabus area assessed, and a summary of the task (see Appendix A, Table 5 for performance on each item matched to the specific numbered area of the revised syllabus). This seemed to be a topic that students struggled with. The highest facility (item 3) was 77 per cent, although out of the 11 items, six showed facilities of under 20 per cent. All of these six items were based on syllabus area 5.2 (calculus), suggesting that this is an area that students may need to cover in more depth. It should be noted that calculus is not covered in the Foundation Level course, and integration in particular is only covered at the Higher Level. For this reason, additional analysis has been carried out to determine the item facilities of the different student level groups. Appendix A, Table 9 details the facilities of the Ordinary Level and Higher Level students on the seven items in FLC5 which assess a construct that is only taught at the Higher Level. This includes five of the six items with facilities below 20 per cent. While the Higher Level students have performed better than their Ordinary Level counterparts, the facilities for four items remain below 20 per cent for both Ordinary and Higher Level groups, confirming that calculus is a topic that challenges students following the revised syllabus.

Of the six items with very low facilities, five of them are open response (there were five open response items in the assessment booklet in total). It is likely that students do not have sufficient understanding to cope with the extra demands of open response questions based on calculus. Omission rates were high on these open response questions, ranging from 33 per cent to 46 per cent. The percentage of students who omitted questions in booklet FLC5 was the highest overall omission rate of all the booklets. This also suggests relative unfamiliarity with this topic amongst students.

Table 7.5: Item indicator booklet FLC5 (Functions) – student performance and summary of items

Item	Syllabus area	Item summary	Phase One Students 1 mark[20] (%)
1	Functions	Match story of phenomenon to graph (rising level of water in tank)	34
2	Calculus	Apply differentiation to find stopping distance of car	9
3	Functions	Match story of phenomenon to graph (height of feet above ground while swinging)	77
4a	Calculus	Find where function of order four cuts x-axis	2
4b	Calculus	Find maxima and minima of function (differentiate)	1
5	Calculus	Link slope of trigonometric function to its derivative	3
6	Functions	Find number of integer coordinates on graph of fractional function	21
7a	Calculus	Find values where function is not continuous (given graph)	15
7b	Calculus	Find values where function is not differentiable (given graph)	2
8	Calculus	Find value of definite integral (given area between function and x-axis)	28
9	Calculus	Integrate exponential function	24

[20] This booklet had no two mark items.

7.3 Comparison of student performance between phase one and non-phase one schools

Key messages

The performance of phase one and non-phase one students was broadly similar on Strands 1 and 2, which is to be expected, as both groups of students had been studying Strands 1 and 2 of the revised syllabus for the same amount of time.

Whilst there were some indications that phase one students performed better than their non-phase one peers in **analysing verbal geometric information and translating it into mathematical form**, in other items relating to this area there was no discernible difference between the two groups.

Tables 7.1 and 7.2 also present the scores of non-phase one students who completed booklets SPLC1 and GTLC2 (non-phase one students did not sit booklets NCL3, ALC4 or FLC5). This allows for a basic comparison of performance between phase one and non-phase one students. Appendix A, Table 5 presents further analysis of phase one and non-phase one students' performance, comparing their average scores on each item using the statistical analysis of differential item functioning.

7.3.1 Strand 1: Statistics and Probability

In the non-phase one schools, 725 students completed SPLC1. As shown in Appendix A, Table 5, performance of non-phase one students follows a similar pattern to that of phase one students with the same items found difficult or hard. Item 6, for example, continues to be the hardest item in the booklet with a facility of one per cent for both groups. Differences in facilities between individual items ranged from between zero to 11 percentage points. Seven out of a total of 10 items have a difference in facility of five percentage points or greater. In all cases, phase one students scored more highly than non-phase one students. These items are coloured orange in Table 5. However, the apparent difference in performance on individual items is likely to be due to sample differences as there is no significant difference between the phase one and non-phase one students for any of the items in the booklet.

7.3.2 Strand 2: Geometry and Trigonometry

In the non-phase one schools, 720 students completed GTLC2. Appendix A, Table 5 shows that, like SPLC1, the performance of non-phase one students follows a similar pattern to that of phase one students with the same items found difficult or hard. Those items where the percentage difference is five or more percentage points, and where phase one students achieved a higher performance than non-phase one students, are likewise highlighted in orange. The differences in facilities between phase one and non-phase one students ranged from zero to 11 percentage points. In addition, phase one students performed significantly better than non-phase one students on item 1 (significance at the 1 per cent level). This item has the largest difference in facility (11 percentage points) between the two groups of students. This item assesses synthetic geometry (2.1), but also calls on students' 'synthesis and problems solving skills' (2.5). In particular, it requires students to analyse information presented verbally and translate it into mathematical form. This skill is not specific to this strand (Geometry and Trigonometry) and appears in the syllabus for all five strands of the revised syllabus. This sample of phase one students was studying four strands of the revised syllabus, while the non-phase one students were only studying two. It is possible that phase one students were more immersed in this aspect of the revised syllabus, leading to a higher performance on this type of item. That said, item 5 also requires the translation of verbal information into mathematical form and did not show differential performance. Thus, it is unclear whether the difference in performance on item 1 represents a systematic difference between phase one and non-phase one students.

There were three items in which performance of the non-phase one students was better than that of the phase one students by five or more percentage points. These were items 6, 9 and 10a and they have been highlighted in green in Table 5. The differences in percentage points for items 6 and 10a were seven and six respectively. However, only item 9 showed a statistically significant difference, indicating that the difference is less likely to be due to chance. The reason for this difference is unclear from the item content, as item 9 is very similar to item 7 which did not show differential performance. Both are open questions that assess 'coordinate geometry' and both require the use of a formula for the correct solution (the formulae for the slope and the midpoint of a line). As a result, these two items are very similar in demand. Given that there was no difference in performance on item 7, and that the difference was very small for item 9, there is not sufficient evidence to suggest a systematic difference in student proficiency in coordinate geometry.

7.3.3 Common Junior Certificate and Leaving Certificate items

Seven items/item parts were common to both the Junior Certificate item indicator booklet JC1/2 and either SPLC1 or GTLC2. The performance data for these items is drawn together in Tables 6 and 7 in Appendix A. Given the differences between the two samples in age and years of schooling it should be expected that higher facilities would be seen in the Leaving Certificate indicator item booklets. For phase one students, this is true of items 3, 4 and 7a in SPLC1, and also of items 2a and 3 in GTLC2. Item 7b of SPLC1 shows little difference in performance. It asks students to use the information presented in a graph to help them classify three statements as true or false. Although it requires a certain amount of reasoning, this is also true of item 3 (SPLC1) in which students must explain why a reporter's statement is or is not a reasonable interpretation of a graph. Item 4 of GTLC2 also shows little difference in performance between the Junior Certificate and Leaving Certificate students. It is not clear why this should be so as it is very similar in demand to items 2a and 3 in GTLC2.

Comparison of the results of the common items for non-phase one students shows a slightly different pattern. Leaving Certificate students performed better on items 3 and 4 in SPLC1 and all the items (2a, 3 and 4) in GTLC2. Both items 7a and 7b in SPLC1 show little difference in performance between the Junior Certificate and Leaving Certificate students. Item 7a was relatively easy for both groups, while item 7b was challenging for both. Item 7a involves a simple calculation of two mean scores and a comparison between them. This is a basic skill that is covered as early as primary school and is clearly grasped by the majority of students by that stage. Conversely, the second part of this item (7b) was challenging for both groups. It requires students to answer a series of true or false statements based on a graph. The graph is essentially a scatter plot but is presented similarly to a coordinate geometry grid. It is unclear why Leaving Certificate students found this item just as difficult as Junior Certificate students. It may be that this particular graph was an equally unfamiliar way of presenting information for both groups.

7.4 Comparison of student performance with international standards

Key messages

Phase one students performed much better than international students on many of the items relating to Strand 1 (Statistics and Probability) and Strand 2 (Geometry and Trigonometry), and the majority of these fall within Strand 1.

The high performance of phase one students on the items in this strand is encouraging and suggests that the **implementation of this part of the new syllabus is working well**. In general, phase one students performed better than expected on items where the solution strategy is clear, and where diagrams, if applicable, are provided. They performed less well on multi-step items.

The Leaving Certificate booklets were constructed using material from three international surveys: released items[21] from the Trends in International Mathematics and Science Study (TIMSS - 2007, 8th grade and TIMSS Advanced, 2008), and sample items[22] from the Programme for International Student Assessment (PISA) surveys of 2000, 2003, and 2006. International data is available for all TIMSS 2007 and TIMSS Advanced items, but not for PISA items, as these have not been used in a live test. As the performance of the phase one and non-phase one students was broadly similar, and the non-phase one students only completed two of the five booklets, the international data will only be compared with that of the phase one students.

[21] Released items are those that have been made public following administration of the survey, in contrast to secure items, which are kept secure for use in evaluating trends in performance in later cycles of TIMSS.

[22] Sample items exemplify the type of material included in a PISA assessment, but have not been used in a live test and so have no data available.

As with the Junior Certificate comparisons, a number of factors should be considered when comparing the performance of phase one students with performance internationally. For the TIMSS 2007 data, the primary caveats relate to age, stage of schooling and exam readiness. The TIMSS 2007 students were mostly 13 or 14 years of age, whereas most Leaving Certificate students are 17 or 18 years of age. While the TIMSS 2007 students were in 8th Grade (which equates to the second year of secondary school in Ireland), the phase one students tested in this evaluation were in their final year of secondary school (sixth year). The phase one students were also preparing for the Leaving Certificate examination, which is very high stakes (admission to tertiary education depends on their results). The combination of these factors places the phase one students at an advantage over the TIMSS 2007 students. For the TIMSS 2008 Advanced data, the situation is reversed. These students were of a similar age and stage of schooling as Leaving Certificate students. However, TIMSS Advanced collects data from students who have studied advanced mathematics in specialist tracked courses, with a view to further mathematics learning at tertiary level. These students represent only a subset of all secondary school students of that age. In contrast, the revised mathematics syllabus is compulsory for all Irish secondary school students. It is also designed to be inclusive and span a range of abilities (including Foundation, Ordinary and Higher Level). Therefore, we would not expect the average performance of phase one students to be quite as high as the international average in TIMSS Advanced.

Appendix A, Table 8 repeats the average scores of the phase one students (given in Table 5) and compares them to the international average scores in the 2007 TIMSS and 2008 TIMSS Advanced studies. This table informs the discussion below.

7.4.1 Strand 1: Statistics and Probability

Comparative data is available for three of the items in SPLC1: items 4, 7a and 7b. The remaining items were released PISA items and so, no international data is available. Table 7.6 below shows the number of items with differences in facility that fall within the three performance bands as described above.

Table 7.6: Number of items in SPLC1 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	0	1	2	3
International students score more highly	0	0	0	0

All of these items were sourced from TIMSS Grade 8. Therefore, the advantages for phase one students in terms of age, stage of schooling and exam readiness apply here. Both phase one and non-phase one students performed well on this booklet. Phase one students also performed well relative to the international average; on two of the three items (items 4 and 7a) the difference in facilities was greater than 25 percentage points. Item 4 requires students to estimate the area of three coloured sectors on a spinner, given data on how many times the pointer stops in each sector. Apart from any differences in the characteristics of the student samples, the high performance on this item may be due to the focus in the revised mathematics syllabus on the 'outcomes of random processes'. This syllabus area (1.3) includes the principle of equally likely outcomes and specifies working with processes such as coins, dice and spinners.

Item 7a requires calculating two means from tabulated data. The data represents students' popularity ratings for two school subjects: mathematics and history. The item also requires students to judge which subject is more popular. The relatively higher performance of phase one students on this item is not surprising for two main reasons. Firstly, it has already been pointed out that students in the international sample were substantially younger and at an earlier stage in their schooling. Secondly, the syllabus area of 'representing data graphically and numerically' (1.6) places explicit emphasis on the use of measures of central tendency, including the mean. This is included in the syllabus from Foundation Level upwards, so all students should be familiar with this concept. The second part of this item (7b) requires students to interpret a graph where the ratings for each subject are plotted against each other. Students are asked to indicate whether each of three statements about the ratings is true or false. While the achievement of phase one students was higher than the international average, the difference was not as large as for the other items in this strand. The relatively poorer performance on this item may be due to the graph used. The ratings were plotted in the style usually used in coordinate geometry. This may have been unfamiliar to students in a data interpretation context. In addition, as discussed earlier in this report, coordinate geometry is an area that the Leaving Certificate sample found difficult.

7.4.2 Strand 2: Geometry and Trigonometry

Comparative data is available for nine of the 13 items/item parts in GTLC2. Table 7.7 below shows the number of items with differences in facility that fall within the three performance bands.

Table 7.7: Number of items in GTLC2 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	0	1	2	3
International students score more highly	3	2	1	6

As Table 7.7 shows, phase one students performed above the international average on three items (2a, 3 and 4). These items were sourced from the TIMSS Grade 8 survey, and all assess the syllabus area of synthetic geometry (2.1). Given the advantages of phase one students over the international sample in terms of age and experience, these differences are to be expected. The pattern is very different for the remaining items, which were taken from the 2008 TIMSS Advanced survey. On item 5 in particular, there was a large difference in facility of 40 percentage points in favour of the international sample. This item also assesses synthetic geometry but includes the concept of a median of a triangle. While this term is included in the revised mathematics syllabus, it is possible that some students failed to recall its meaning. In addition, this item gave a verbal description and did not include a diagram. It may be that phase one students are less proficient at visualising a geometric situation than their international peers.

Items 6, 7 and 9 assess coordinate geometry. For items 7 and 9, performance was very similar to the international average with differences of 7 per cent. Both of these items provide a diagram and can be solved by using a formula (for the midpoint and the slope of a line). In contrast, for item 6, the difference in facility was 20 percentage points in favour of the international students. For this item there is no diagram and the solution strategy is less clear. It requires students to apply their knowledge of slopes to a triangle. It is possible that not all students following the revised syllabus are achieving the aim of applying knowledge to unfamiliar and less procedural problems. Items 8a and 10 assess trigonometry.

For item 8a, performance was very similar among phase one students and international sample. However, performance was somewhat different on item 10a, with a difference in achievement of 14 per cent in favour of the international students. This item required a number of steps to find the solution. It may be that phase one students are less able to handle multi-step problems than their counterparts internationally.

7.4.3 Strand 3: Number

Comparative data are available for three of the items/item parts in NLC3. All three were taken from TIMSS Advanced and assess 'number systems' (3.1). Table 7.8 below shows the number of items with differences in facility that fall within the three performance bands.

Table 7.8: Number of items in NLC3 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	0	0	0	0
International students score more highly	1	2	0	3

Item 10 requires students to describe, but not perform, the steps needed for proof by induction. The difference in facility for this item was relatively small (9 percentage points) and could be due to sampling error. For items 6 and 7 there were moderate differences in facility, favouring the international average (16 per cent and 21 per cent). Item 6 requires students to interpret verbal information and translate it into mathematical form. While this skill is explicitly addressed in the revised syllabus, phase one students do not appear to be demonstrating this as well as the TIMSS Advanced students. Item 7 requires students to find the sum of an infinite geometric series.

7.4.4 Strand 4: Algebra

Comparative data are available for five of the seven items in ALC4. All of these items were sourced from TIMSS Advanced 2008. Table 7.9 shows the number of items with differences in facility that fall within the three performance bands.

Table 7.9: Number of items in ALC4 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentage points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	0	0	0	0
International students score more highly	1	3	1	5

As Table 7.9 shows, the performance of phase one students was mixed in comparison with the international sample. For item 3, performance was very similar, with both samples finding it difficult. This item involves dividing a real number by a complex number. However, the performance of phase one students on this item is perhaps better than expected, as complex numbers are not part of the revised syllabus at Foundation Level. Items 4, 6 and 7 displayed a moderately higher facility for international students, which is the expected finding. Item 6 requires students to simplify an expression consisting of a composite function, and determine its minimum value. Item 7 assesses students' understanding of the link between the graphical and numerical form of a quadratic function. The solution strategies required for this item are not covered at Foundation Level, and this may have contributed to the lower average performance among phase one students. Item 4 requires students to solve a quadratic inequality. However, because this was a multiple choice item, students could also have substituted in the values given in the response options and deduced the answer. The only item that displayed a much higher facility for the international sample was item 5.

7.4.5 Strand 5: Functions

Comparative data are available for nine of the items/item parts of FLC5. All of these items were sourced from TIMSS Advanced 2008. As discussed earlier in this report, Irish students found this booklet particularly difficult. On many items, their performance was also poor compared with the international average. Table 7.10 shows the number of items with differences in facility that fall within the three performance bands.

Table 7.10: Number of items in FLC5 showing facility differences in each performance band

Difference in facility between phase one and international students (percentage points)	Number of items with a score difference of 0-9 percentage points	Number of items with a score difference of 10-24 percentag e points	Number of items with a score difference of ≥25 percentage points	Total
Phase one students score more highly	0	0	0	0
International students score more highly	2	3	4	9

As Table 7.10 shows, phase one students did not outperform the international sample on any item. However, given the characteristics of the TIMSS Advanced students mentioned earlier, this would not necessarily be expected. For items 8 and 9, the difference in facility was only 7 per cent. Both of these were calculus items involving integration (5.2). This similar performance is noteworthy here, as integration is only covered at Higher Level in the revised syllabus. Therefore, only a subset of the phase one students would have been familiar with this concept. Additional analysis for items 8 and 9 (Appendix A, Table 9) confirms that the performance of the Higher Level students is similar to or better than that of the international students who scored two percentage points higher and 10 percentage points lower for these items respectively. The remaining calculus items (2, 4, 5 and 7) displayed moderate to large differences in facilities. In particular, calculus performance was very different for items 4a, 5 and 7a. Item 4a requires students to find the roots of a function of order four. It may be that phase one students are more comfortable with functions of order two and three. One of the solution strategies for item 5 involves differentiating a trigonometric function, a skill which is only included in the Higher Level syllabus. Item 7a asks students to indicate the values for which a function is continuous. It is possible that phase one students were not as familiar with this terminology as the TIMSS Advanced students. Because the construct for items 4, 5 and 7 is contained in the syllabus for the Higher Lever only, additional analysis has been carried out to calculate the facilities of the Higher Level students on these items. Appendix A, Table 9 confirms that the Higher Level students found these items more difficult than the international students.

Item 6 also showed a large difference in facility in favour of the international sample. This item assesses functions, and requires students to deduce the number of integer coordinates on the graph of a function. However, the graph is not provided. As in Strand 2 (Geometry and Trigonometry), this suggests a weakness among Irish students in visualising or constructing graphical representations of verbal information, compared with students of a similar age internationally.

8. Leaving Certificate student attitude survey

This chapter presents the findings of the first survey of Leaving Certificate students' attitudes towards mathematics, for both phase one and comparison group schools. It explores:

- their experiences of mathematics lessons
- their attitudes towards learning mathematics
- their views and perspectives on careers involving mathematics.

Key messages are highlighted in each of these sections, and in relation to individual strands of the revised syllabus.

8.1 Students' experiences of mathematics lessons

Following the same approach taken within the Junior Certificate survey, Leaving Certificate students were asked about how mathematics is taught in school in order to compare the learning experiences of phase one students with the non-phase one group. The areas explored echoed those of the Junior Certificate survey, detailed in section 5.1. An overview of Leaving Certificate students' perspectives in relation to each of these areas is presented in Figure 8.1. Phase one students are presented alongside those of the comparison group so that similarities and differences are immediately apparent.

Figure 8.1: Proportion of Leaving Certificate students reporting that they 'often' or 'sometimes' take part in mathematics teaching and learning activities

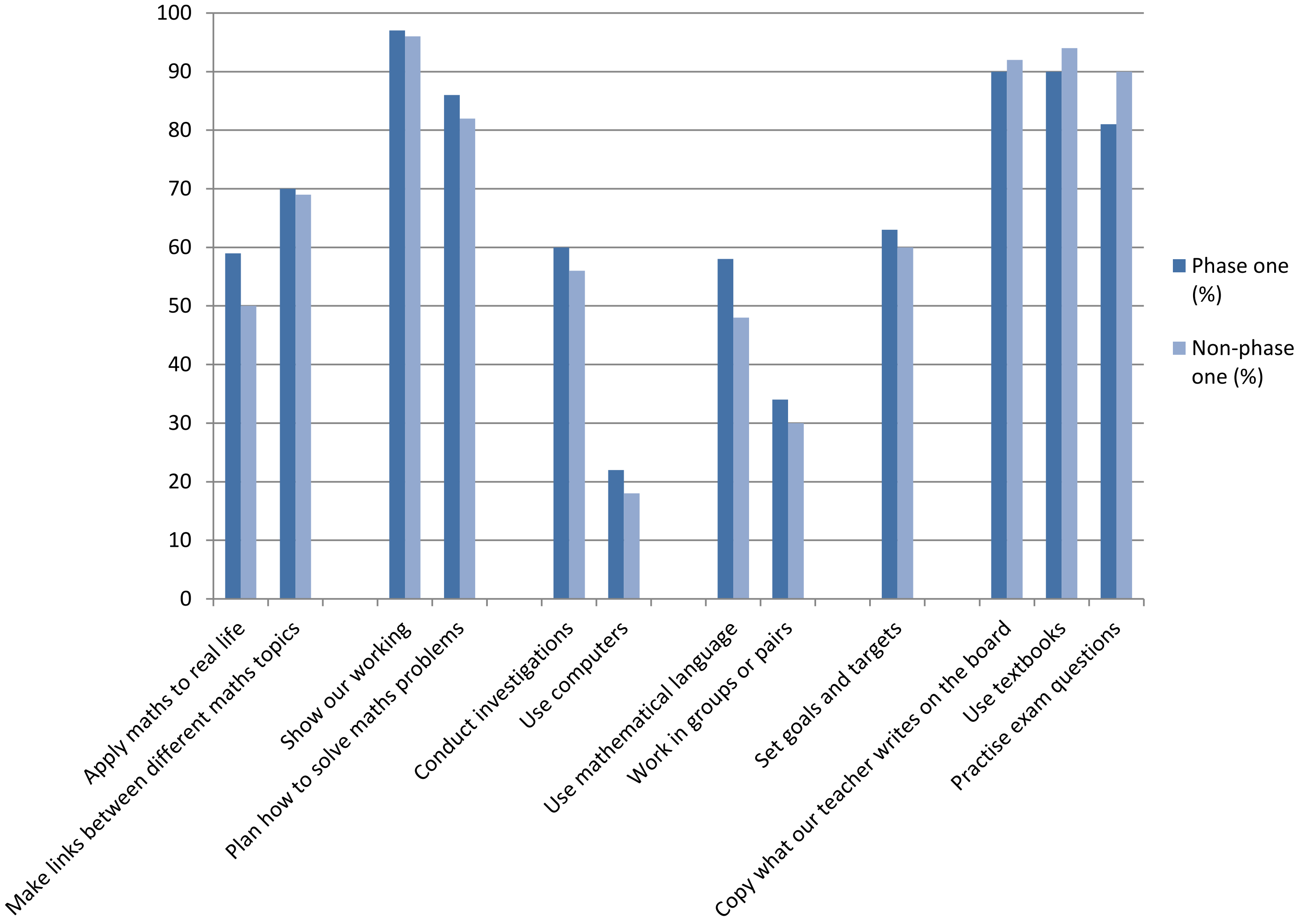

Figure 8.1 shows that Leaving Certificate students gave a similar pattern of responses as their Junior Certificate peers, with a higher proportion of phase one students reporting positive responses in many of the areas promoted by the revised syllabus.

8.1.1 Students' perspectives on learning approaches characteristic of the revised syllabus

This section explores students' perspectives on the learning approaches they have experienced in their mathematics lessons.

Applying mathematics

Key messages

Leaving Certificate students following all strands of the revised syllabus reported particularly strongly, relative to those who were just studying Strands 1-2, they that **regularly applied their learning to real-life situations**. Likewise, whilst high proportions of both groups 'sometimes' make **connections between different mathematics topics**, a higher proportion of phase one students do this 'often'. This suggests that the approaches promoted through the revised syllabus become increasingly apparent as students become more immersed.

Overall, both groups of students reported that they regularly **apply what they learn in mathematics to real-life situations less frequently than the Junior Certificate students**, indicating that this approach may not be as well established at Leaving Certificate level.

Similar to the responses given by Junior Certificate students, phase one students at Leaving Certificate reported particularly strongly, relative to their non-phase one peers, they that regularly **applied their learning** to real-life situations and to other mathematics topics.

The Leaving Certificate survey showed, for example, that there were statistically significant differences between the frequency with which students **applied their learning in mathematics to real-life situations** between phase one and non-phase one groups. The findings are presented in Table 8.1:

Table 8.1: We apply what we learn in maths to real life situations

	Phase one %	Non-phase one %
Often	21	11
Sometimes	38	39
Rarely	31	36
Never	9	13
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.1 shows that just under three-fifths of phase one students (59 per cent) reported that they apply their learning to real-life situations 'sometimes' (38 per cent) or 'often' (21 per cent). By contrast, half (50 per cent) of the non-phase one group reported that they do this 'sometimes' (39 per cent) or 'often' (11 per cent).

This demonstrates that a higher proportion of phase one students, relative to the non-phase one group, regularly apply their learning in mathematics to real-life contexts, indicating that such approaches are being implemented at Leaving Certificate, as well as Junior Certificate, levels. The absolute difference between the two groups is less pronounced that at Junior Certificate level, which is perhaps to be expected as non-phase one students have themselves studied part of the revised syllabus. This is indicative that the frequency with which students make use of real-life contexts in their mathematics increases with the number of strands studied. This suggests that the use of such contexts is being applied consistently across all strands of the revised syllabus.

However, it should also be noted that, overall, both phase one and non-phase one groups reported that they regularly apply what they learn in mathematics to real-life situations less frequently than the Junior Certificate groups, indicating that this approach is not as embedded at Leaving Certificate level. Indeed, a substantial minority of phase one students (40 per cent) also reported that they 'rarely' (31 per cent) or 'never' (nine per cent) apply what they learn in mathematics to real-life situations. This suggests that, despite the positive findings indicated above, there remains room for development in this area.

Similarly, as shown in Table 8.2, when asked how frequently they **made links between different mathematics topics**, there was a statistically significant difference between the responses of phase one and non-phase one students.

Table 8.2: We make links between different maths topics

	Phase one	Non-phase one
	%	%
Often	31	23
Sometimes	39	46
Rarely	20	20
Never	9	9
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.2 shows that, like the Junior Certificate survey, whilst broadly similar proportions of Leaving Certificate students in phase one and non-phase one schools 'sometimes' make connections between mathematics topics (39 per cent of phase one, and 46 per cent of non-phase one students), a higher proportion of phase one students do this 'often' (31 per cent of phase one, and 23 per cent of non-phase one students). This is perhaps to be expected as phase one students, having studied a greater number of revised syllabus strands, have a wider range of syllabus topics to link together, and therefore potentially greater opportunities to do this. Nonetheless, it is an encouraging indication that such approaches are being effectively translated into classroom practice.

Knowledge of the processes underpinning mathematics

Key messages

The findings suggest that **Leaving Certificate students following the revised syllabus are being encouraged to consider the 'how' and the 'why' of mathematics in lessons**, and that this increases with the number of syllabus strands studied. For example, a higher proportion of Leaving Certificate students following all strands of the revised syllabus, relative to those following Strands 1-2 of the syllabus, reported that they regularly **think about mathematics problems and plan how to solve them in lessons**, although the majority of both groups reported that they did this 'sometimes' or 'often'.

Interestingly, the vast majority of both groups of Leaving Certificate students reported that they regularly **show their working to justify their answers**. However, assessment of student achievement suggested that students did not show routinely their working, indicating that **there may be some discrepancies between students' attitudes and abilities in this area**.

Figure 8.1 shows that Leaving Certificate students in both phase one and non-phase one groups had, more strongly than any other aspect of the revised syllabus, regularly participated in teaching and learning activities aiming to develop their knowledge of **the processes underpinning mathematics**. In some areas, the experiences of phase one and non-phase students appeared to be similar: there was, for example, no statistically significant difference in the frequency with which phase one and non-phase one students **show their working to justify their answers** (97 per cent of phase one students, and 96 per cent of comparison group students reported that they do this 'sometimes' or 'often') (Appendix B, Table 40).

However, as shown in Table 8.3, a higher proportion of phase one Leaving Certificate students, relative to the non-phase one group, reported that they regularly **think about mathematics problems and plan how to solve them** in lessons.

Table 8.3: We think about maths problems and plan how to solve them

	Phase one	Non-phase one
	%	%
Often	54	46
Sometimes	32	36
Rarely	8	13
Never	5	3
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.3 shows that the vast majority (86 per cent) of phase one students reported that they do this ‘sometimes’ (32 per cent) or ‘often’ (54 per cent). By comparison, 82 per cent of non-phase one group students reported that they do this ‘sometimes’ (36 per cent) or ‘often’ (46 per cent). This is a statistically significant difference, which indicates that the revised syllabus has positively impacted on students’ learning experiences in this area. Again, this finding indicates both that students are being encouraged to consider the ‘how’ and the ‘why’ of mathematics lessons, and that the extent to which students take part in this type of activity increases according to the number of revised syllabus strands studied.

Participation in investigative, practical activities

Key messages

The findings suggest that the **frequency with which Leaving Certificate students participate in investigative, practical activities increases with the number of revised syllabus strands studied**, reflecting the importance placed on these approaches within the revised syllabus. For example, Leaving Certificate students following all strands of the revised syllabus appear to conduct investigations to solve mathematics problems more frequently than those following Strands 1-2 (although the majority of both groups reported that they did so regularly).

Although Leaving Certificate students following all strands reported that they regularly used computers in mathematics to help them solve problems more frequently than those following Strands 1-2, a high proportion of both groups reported that they 'rarely' or 'never' do this. Use of IT in mathematics lessons may, therefore, be an area for further development.

Figure 8.1 tells us that, like the Junior Certificate students, a higher proportion of phase one students studying for their Leaving Certificate regularly take part in investigations and practical activities in mathematics, than their comparison group peers. Again, however, there was considerable variation in the extent to which this appears to be occurring.

The findings show that there was a statistically significant difference between phase one and non-phase one students, in terms of the frequency with which they **conduct investigations to solve mathematics problems,** with phase one students tending to undertake investigations more frequently than their non-phase one peers. The findings are presented in Table 8.4:

Table 8.4: We do investigations to solve maths problems

	Phase one	Non-phase one
	%	%
Often	29	21
Sometimes	31	35
Rarely	28	28
Never	10	15
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 8.4 shows that the majority of students in the phase one group (60 per cent) reported that they conduct investigations to solve mathematical problems 'sometimes' (31 per cent) or 'often' (29 per cent). By contrast, this was reported by 56 per cent of non-phase one students: of whom, a slightly higher proportion reported that they did this 'sometimes' (35 per cent), and a slightly smaller proportion reported that they did this 'often' (21 per cent). Again, this finding reflects the high degree of emphasis placed on investigative, problem-solving approaches in the revised syllabus, and demonstrates that the frequency with which students participate in such activities appears to be increasing with the number of revised syllabus strands studied.

Similar to the findings of the Junior Certificate survey, use of information technology (IT) in the classroom as a tool for teaching mathematics was, at Leaving Certificate level, more limited. As shown in Table 8.5, although phase one students were more likely than non-phase one to **use computers in mathematics to help them solve problems** on a regular basis, a high proportion of both groups reported that they 'rarely' or 'never' do this (77 per cent of phase one students reported that this was the case, with 55 per cent reporting 'never', compared to 81 per cent of non-phase one students, with 62 per cent reporting 'never').

Table 8.5: We use computers in maths lessons to help us solve problems

	Phase one %	Non-phase one %
Often	4	5
Sometimes	18	13
Rarely	22	19
Never	55	62
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

These are statistically significant differences and, as with the Junior Certificate survey, suggest that whilst it is encouraging that use of IT appears to be increasing with the number of strands studied, there may be room for further development. More promisingly, however, use of computers in mathematics appears to have increased between Junior and Leaving Certificate levels, indicating that there is scope for schools to increase the range of ways that they use IT in mathematics lessons. This suggests that use of IT may continue to rise as schools become increasingly familiar with its application in the revised syllabus.

Participation in discursive and collaborative activities

Key messages

Over half of both groups of Leaving Certificate students reported that they regularly **talk about their ideas using the language of mathematics** in lessons. However, a greater proportion of students studying all strands of the revised syllabus appeared to do so regularly. Again, this suggests that the level at which students engage in discursive, collaborative and investigative activities increases according to the number of revised syllabus strands studied.

Relatively few Leaving Certificate students regularly work together in small groups or pairs, although again a greater proportion of students following all strands appear to do this regularly. This suggests that such activities are increasing as the revised syllabuses become further embedded within schools

.

Figure 8.1 also shows that, like the Junior Certificate survey, there are differences in the frequency with which students participate in different types of discursive and collaborative activities. As shown in Table 8.6, for example, phase one students reported that they **talk about their ideas using the language of mathematics** in lessons more frequently than non-phase one students.

Table 8.6: We talk about our ideas using the language of maths

	Phase one %	Non-phase one %
Often	22	15
Sometimes	36	33
Rarely	24	31
Never	16	20
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 8.6 shows that just under three-fifths (58 per cent) of phase one students talk about their ideas using the language of mathematics 'sometimes' (36 per cent) or 'often' (22 per cent), compared to just under half (48 per cent) of non-phase one students who reported that they do this 'sometimes' (33 per cent) or 'often' (15 per cent). This difference is statistically significant, and demonstrates that a higher proportion of phase one students, relative to the non-phase one group, feel that they regularly use mathematical language to convey their ideas. This suggests that the level at which students engage in discursive, collaborative and investigative activities increases according to the number of revised syllabus strands studied. This, in conjunction with the findings from the Junior Certificate survey, affirms that such techniques are being readily and continuously applied in the classroom.

By contrast, a far lower proportion of Leaving Certificate students in both phase one and non-phase one groups reported that they regularly **work together in small groups or pairs**, relative to other areas. The findings are presented in Table 8.7:

Table 8.7: We work together in small groups or pairs

	Phase one	Non-phase one
	%	%
Often	12	8
Sometimes	22	22
Rarely	36	32
Never	29	37
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

However, Table 8.7 shows that relatively few students regularly work together in small groups or pairs, although phase one students appear to do this more frequently than their non-phase one peers. Whilst overall, 65 per cent of phase one students, and 69 per cent of non-phase one students, reported that they do not regularly work in this way, a greater proportion of non-phase one students reported that they 'never' do this (29 per cent of phase one students, and 37 per cent of non-phase one students reported that this was the case),

This finding is statistically significant, and suggests that classroom activities which involve working in groups or pairs are relatively uncommon at Leaving Certificate level. However, the fact that this happens more frequently in phase one than non-phase one schools suggests that such activities are increasing as the revised syllabuses become further embedded within schools. This may, therefore, lead to a general increase in pair and group work over time.

Becoming active learners

Key messages

The majority of Leaving Certificate students reported that they **regularly set goals and targets about their mathematics learning**, and the degree to which this occurred was similar between the two groups. It is encouraging that aspect of the revised syllabus appears to have been applied in the classroom.

The majority of both phase one and non-phase students reported that they frequently **set goals and targets about their mathematics learning** (63 per cent of phase one students, and 60 per cent of comparison group students reported that they do this 'sometimes' or 'often') (Appendix B, Table 41). Whilst it is encouraging that high proportions of both phase one and non-phase one students routinely use these approaches, as with the Junior Certificate group this is not a statistically significant difference. This indicates that the revised syllabus has not yet had an impact in relation to this particular approach.

8.1.2 Students' perspectives on learning approaches characteristic of a more traditional syllabus

Key messages

Whilst there are many positive indications that the approaches promoted through the revised syllabus are being reflected in the classroom, **there remains a high proportion of phase one pupils who report that they participate in activities associated with more traditional approaches to mathematics teaching and learning** (for example, using textbooks in lessons and copying from the board).

Unlike Junior Certificate students, however, in general **Leaving Certificate students studying all strands of the revised syllabus appeared to be participating in these types of activities less frequently** than those studying Strands 1-2. This is an encouraging finding, as it suggests that such approaches are becoming less common as the revised syllabus is increasingly embedded within schools.

Again following a similar pattern to the Junior Certificate survey, Figure 8.1 shows that although there are positive indications that the approaches promoted through the revised syllabus are being reflected in the classroom, there remains a high proportion of phase one pupils who report that they participate in activities associated with more traditional approaches to mathematics teaching and learning.

Unlike Junior Certificate students, phase one Leaving Certificate students were statistically less likely to **copy what their teacher writes on the board then practise using examples** than their non-phase one peers. The findings are presented in Table 8.8:

Table 8.8: We copy what our teacher writes on the board then practise using examples

	Phase one	Non-phase one
	%	%
Often	69	70
Sometimes	20	22
Rarely	6	6
Never	4	1
No response	2	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.8 shows that overall, 89 per cent of the phase one group reported that they did this 'sometimes' (20 per cent) or 'often' (69 per cent), compared to 92 per cent of the non-phase one group who reported that they did this 'sometimes' (22 per cent) or 'often' (70 per cent). Whilst it is encouraging that the frequency with which students undertake this type of activity reduces with the number of revised syllabus strands studied, the overall proportion of both groups reporting that they do this regularly remains high. Additionally, a greater proportion of phase one Leaving Certificate students report that they do this 'often' compared to their Junior Certificate counterparts.

A lower proportion of phase one Leaving Certificate students also reported that they **use textbooks in lessons and then practise what they have learned, either in class or for homework**, than their non-phase one peers. The findings are presented in Table 8.9:

Table 8.9: We use text books in lessons then practise what we have learned in class and/or for homework

	Phase one %	Non-phase one %
Often	75	80
Sometimes	15	14
Rarely	5	4
Never	4	2
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 4.14 shows that:

- a total of 90 per cent of phase one students reported that they use text books in lessons then practise what they have learned in class or for homework ‘sometimes’ (15 per cent) or ‘often’ (75 per cent)
- by contrast, 94 per cent of non-phase one students reported that they do this ‘sometimes’ (14 per cent) or ‘often’ (80 per cent).

This is a statistically significant difference, and suggests that whilst those students studying a greater number of revised syllabus strands are using textbooks less frequently than those following a mixed syllabus, use of textbooks amongst phase one students remains high. Furthermore, phase one students at Leaving Certificate appear to be using textbooks considerably more often than their Junior Certificate phase one peers, despite studying an additional strand of the revised syllabus. It may, therefore, be valuable to explore ways of supporting schools to develop more varied approaches in this area.

Similarly, a lower proportion of phase one students, relative to the comparison group, reported that they regularly **practise examination questions in class**. The findings are presented in Table 8.10:

Table 8.10: We practise exam questions in class

	Phase one %	Non-phase one %
Often	49	71
Sometimes	32	19
Rarely	12	6
Never	6	3
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.10 shows that the vast majority (81 per cent) of phase one students reported that they did this 'sometimes' (32 per cent) or 'often' (49 per cent), compared to 90 per cent of comparison group students who reported that they did this 'sometimes' (19 per cent) or 'often' (71 per cent). This is a statistically significant difference, and suggests that the extent to which students' practise examination questions in class reduces as the number of revised syllabus strands increases. This may, in part, be because there are fewer examination papers relating to the revised syllabus currently available, but is a positive indicator that schools are using a wider range of techniques to prepare their students for examinations, rather than relying on practice papers.

8.1.3 Students' perspectives on mathematics teaching

Key messages

Leaving Certificate students in both groups were **highly positive about their experiences of mathematics teaching**, suggesting that their teachers were able to help and support them effectively. This is a positive indicator of the success of the teaching approaches promoted through the revised syllabus.

Students were also asked about how their teachers were helping and supporting them in their mathematics classes, as an indicator of their experiences of the teaching approaches promoted throughout the revised syllabus. Leaving Certificate students, in both phase one and non-phase one schools, and similar to their Junior Certificate peers, were highly positive about the mathematics teaching they had experienced.

The only statistically significant difference between phase one and non-phase one students related to the extent of their agreement that their teacher **is easy to understand**, as shown in Table 8.11:

Table 8.11: My teacher is easy to understand

	Phase one	Non-phase one
	%	%
Agree a lot	49	53
Agree a little	29	29
Disagree a little	14	11
Disagree a lot	7	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 299 phase one students, and 2,004 comparison group students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.11 shows that:

- the vast majority (81 per cent) of phase one students agreed either ‘a little’ (38 per cent) or ‘a lot’ (43 per cent) that their teacher is easy to understand
- a slightly greater proportion (84 per cent) of non-phase one students agreed either ‘a little’ (32 per cent) or ‘a lot’ (52 per cent) that their teacher is easy to understand
- whilst, overall, both groups of students had positive views in this area, phase one students found it more challenging to understand their teachers than non-phase one students. This could, perhaps, be attributable to the challenges facing teachers in teaching all five strands of the revised syllabus for the first time.

In other areas, no statistically significant differences were found between students in phase one and non-phase one schools (Appendix B, Tables 42-48), including in relation to the extent which students felt their teacher sets them work to suit their abilities and interests (which was found to be significant at Junior Certificate level).

8.1.4 Discussion

Again, this section identifies many interesting findings arising from the research, and largely affirms that students have had similar experiences of mathematics lessons at Junior Certificate and Leaving Certificate levels. Additionally, the Leaving Certificate findings show that the positive impacts on students' experiences appear, in many cases, to increase with the number of strands studied. This suggests that the approaches to mathematics promoted throughout the mathematics syllabuses will continue to grow and develop as they become more embedded within schools.

In subsequent stages of the research, it will be valuable to explore further the benefits and challenges of providing students with this type of learning experience. For example, whilst there are positive differences in the extent to which phase one and non-phase one students participate in more traditional teaching and learning approaches, for example practising for examinations, suggesting that this becomes less frequent in time, it remains a fairly prominent feature of students' experiences. Possible areas for further investigation include, then, consideration of why this is the case.

8.2 Students' attitudes towards learning mathematics

This section explores Leaving Certificate students' attitudes towards learning mathematics, both generally and in relation to the individual strands of the revised syllabus.

8.2.1 Attitudes towards individual strands of the revised mathematics syllabus

Students were asked about how confident they would feel when undertaking a range of different activities during their mathematics lessons, to gain a further insight into their attitudes towards specific areas pertinent to the revised syllabus. Leaving Certificate students were asked about the same aspects of individual strands as their Junior Certificate counterparts, as detailed in section 5.2.

An overview of students' perspectives in relation to each of these areas is presented in Figure 8.2:

Figure 8.2: Proportion of Leaving Certificate students reporting that they would find it 'very easy' or 'easy' if they were asked to solve problems in each of the following areas

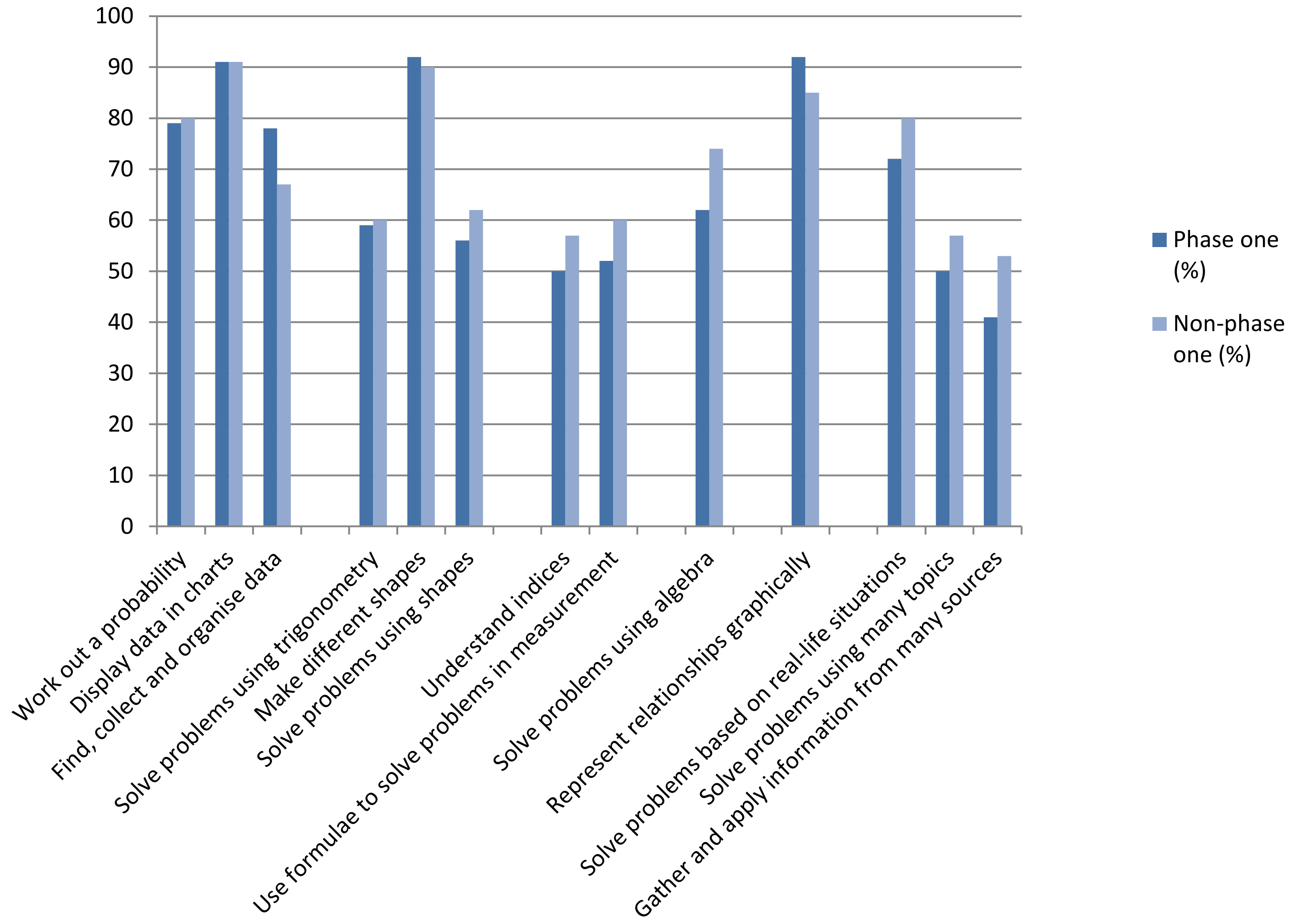

Figure 8.2 shows that both phase one and comparison group students are broadly confident in their abilities in topics spanning all strands of the revised syllabus, although, as with the Junior Certificate survey, there is considerable variation both within and between individual strands. This is explored more fully in the following sections.

8.2.2 Strand 1: Statistics and Probability

Key messages

Both groups of Leaving Certificate students appeared to be highly confident in items relating to Strand 1, Statistics and Probability. Overall, the vast majority of students in both groups reported that they would be **confident to calculate the probability of an event occurring**, and **to display their data using charts, including pie charts and bar charts**. The similarity between the two groups is perhaps to be expected, given that both groups of students have studied this strand of the revised syllabus, and it is encouraging that such high proportions of students feel confident to undertake these types of activities.

Students following all strands of the revised syllabus appeared to feel somewhat more confident, however, than those following Strands 1-2 at **finding, collecting and organising data**, although again responses were highly positive amongst students in both groups.

Students were asked how confident they would feel in working out the probability of an event occurring. Overall, the vast majority of students in both groups (79 per cent of phase one students and 80 per cent of non-phase one students) reported that they would find it ‘easy’ or ‘very easy’ to calculate the probability of an event occurring and there was no statistically significant difference between the two (Appendix B, Table 49).

Equally, there was no statistically significant difference between the different groups of students in relation to their confidence to **display their data using charts, including pie charts and bar charts** (91 per cent of both phase one and non-phase one students reported that they would find this ‘easy’ or ‘very easy’). These findings are perhaps to be expected, given that both groups of students have studied this strand of the revised syllabus, and it is encouraging that such high proportions of both phase one and non-phase one students feel confident to undertake these types of activities (Appendix B, Table 50).

There was, however, a statistically significant difference between the two groups in terms of their confidence to **find, collect and organise data** (for example, to time each person in their class while they estimated the length of minute and subsequently organise their answers into sequence order). However, both groups provided positive responses. The findings are presented in Table 8.13.

Table 8.13: If I were asked to find, collect and organise data…

	Phase one %	Non-phase one %
I would find it very easy	34	29
I would find it easy	44	38
I would find it a little difficult	16	26
I would find it very difficult	4	5
No response	2	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 299 phase one students, and 2,004 comparison group students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.13 shows that:

- phase one students appeared to feel somewhat more confident than their non-phase one peers at finding, collecting and organising data
- over three-quarters (78 per cent) of phase one students reported that they would find this ‘easy’ (44 per cent) or ‘very easy’ (34 per cent)
- by contrast just over two-thirds (67 per cent) of non-phase one students reported that they would find this ‘easy’ (38 per cent) or ‘very easy’ (29 per cent).

8.2.3 Strand 2: Geometry and Trigonometry

Key messages

Overall, Leaving Certificate students in both groups were confident in their responses to items relating to Strand 2. Like Junior Certificate students, however, Leaving Certificate students in both groups appeared to be slightly less confident in this strand than they were in items relating to Strand 1.

The majority of students in both groups reported that they would be confident in **solving problems using trigonometry**. Students following all strands of the syllabus, however, appeared to be slightly more confident than those following Strands 1-2, in their use of shape. Given that both groups of students had studied this strand, this indicates that students may feel more confident within individual strands of the revised syllabus when they are following a greater number of strands overall.

In general, Leaving Certificate students in both phase one and non-phase one groups were, like Junior Certificate students, less confident in relation to Strand 2 of the revised mathematics syllabus than they were in relation to Strand 1.

When students were asked how confident they would feel to **solve problems using trigonometry**, around three-fifths of both phase one and non-phase one students (59 per cent, and 60 per cent, respectively) reported that they would find this 'easy' or 'very easy'. Again, this is perhaps to be expected, given that both groups of students have studied this strand of the revised syllabus (Appendix B, Table 51).

There was, however, a statistically significant difference between phase one and non-phase one groups when asked how confident they would feel to **make different shapes** (for example, to draw a triangle with sides of length 3cm, 5cm and 8cm). The findings are presented in Table 8.14:

Table 8.14: If I were asked to make different shapes...

	Phase one %	Non-phase one %
I would find it very easy	66	64
I would find it easy	27	26
I would find it a little difficult	4	7
I would find it very difficult	0	2
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 299 phase one students, and 2,004 comparison group students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.14 shows that:

- both phase one and non-phase one students were highly confident at making different shapes, although phase one students were more confident than their non-phase one peers
- 93 per cent of phase one students reported that they would find this 'easy' (27 per cent) or 'very easy' (66 per cent)
- by contrast, 90 per cent of non-phase one students reported that they would find this 'easy' (26 per cent) or 'very easy' (64 per cent).

There was no statistical difference in terms of Leaving Certificate students' confidence to **solve problems using the properties of different shapes** (for example, to find the surface area and volume of a range of solids), although in general students appeared less confident in this area than other topics within Strand 2 (56 per cent of phase one students reported that they would find this 'easy' or 'very easy', compared to 62 per cent of non-phase one students). Again, this similarity is perhaps to be expected as both groups have studied this strand of the revised syllabus (Appendix B, Table 52).

8.2.4 Strand 3: Number

> **Key messages**
>
> Across both groups, **Leaving Certificate students had mixed views about their confidence in items relating to Strand 3, Number**. They expressed similar levels of confidence in understanding indices, and using formulae to solve problems in measurement. As the two groups have followed different syllabus pathways in relation to the topics covered in this strand, this suggests that the revised syllabus has had little impact on students within this strand of learning at Leaving Certificate level.

Despite phase one and non-phase one groups having followed different syllabus pathways in relation to the topics covered in this strand (with phase one students following the revised mathematics syllabus, and non-phase one students the previous mathematics syllabus) there were no statistically significant differences in their confidence to approach mathematical problems relating to number. Students in both groups had mixed views about their confidence in this area: half (50 per cent) of phase one students reported that they would find it 'easy' or 'very easy' to **understand indices**, for example, compared to 57 per cent of non-phase one students. Similarly, just over half (53 per cent) of phase one students reported that they would find it 'easy' or 'very easy' to **use formulae to solve problems in measurement**, compared to three-fifths (60 per cent) of non-phase one students (Appendix B, Tables 53-54).

These findings suggest that the revised syllabus has had little impact on students within this strand of learning at Leaving Certificate level. This contrasts with the findings of the Junior Certificate survey, where the revised syllabus appears to have had a slightly downward impact on particular aspects of this strand.

8.2.5 Strand 4: Algebra

> **Key messages**
>
> Although the responses of both groups of Leaving Certificate students were broadly positive in items relating to Strand 4, Algebra, those following all strands of the revised syllabus appeared slightly less confident than those following only Strands 1-2.
>
> This suggests that **students following this strand of the revised syllabus may have found the topics covered more difficult to grasp than those experiencing the more established teaching and learning approaches of the previous syllabus**. This is reflected in the findings of the assessment part of this research, which showed that students following this strand of the revised syllabus appeared to find items relating to Strand 4 more challenging than other areas.

Students were asked how confident they would feel to solve problems in using algebra: for example, to find the value of x when $4x+3 = 2x+11$. The findings are presented in Table 8.15:

Table 8.15: If I were asked to solve problems using algebra…

	Phase one %	Non-phase one %
I would find it very easy	28	42
I would find it easy	34	32
I would find it a little difficult	26	17
I would find it very difficult	11	8
No response	2	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 299 phase one students, and 2,004 comparison group students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.15 shows that:

- non-phase one students were more confident at solving problems using algebra than phase one students. However, the findings for both groups were positive
- just under two-thirds (62 per cent) of phase one students reported that they would find it ‘easy’ (34 per cent) or ‘very easy’ (28 per cent)
- almost three-quarters of phase one students (74 per cent) reported that they would find it ‘easy’ (32 per cent) or ‘very easy’ (42 per cent).

There is a statistically significant difference between the two groups, which indicates that, in general, phase one students found algebra more challenging than students in the non-phase one group. The reasons for this will be explored further during the case-study phase. As with the Junior Certificate students, the two groups may take different approaches interpreting this type of question, which could explain any differences in students’ confidence.

8.2.6 Strand 5: Functions

Key messages

Both groups of Leaving Certificate students were highly confident in relation to Strand 5, Functions, as measured by an item exploring their confidence in their ability to use graphs to represent information. A greater proportion of Leaving Certificate students following all strands of the syllabus, however, reported that they would be confident to approach this task (although, interestingly, they had considerable difficulty with this strand in the testing part of the research).

This indicates that the revised mathematics syllabus is positively influencing students' confidence in relation to functions, albeit from a relatively high baseline: students who had not followed this strand were also highly confident.

In relation to Strand 5 of the revised syllabus, students were asked how confident they would feel to **represent relationships between numbers graphically**. The findings are presented in Table 8.16:

Table 8.16: If I were asked to represent this relationship in a graph…

	Phase one %	Non-phase one %t
I would find it very easy	51	50
I would find it easy	41	35
I would find it a little difficult	6	11
I would find it very difficult	2	2
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 299 phase one students, and 2,004 comparison group students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.16 shows that:

- both phase one and non-phase one students were highly confident in using graphs to represent information.
- phase one students were slightly more confident with 92 per cent responding they would find it 'easy' (41 per cent) or 'very easy' (51 per cent).

- by contrast, 85 per cent of non-phase one students responded that they would find it 'easy' (35 per cent) or 'very easy' (50 per cent) to represent this relationship in a graph.

This finding is statistically significant, and indicates that the revised mathematics syllabus is positively influencing students' confidence in relation to functions. However, as detailed in section 7.4.5, phase one students had considerable difficulty with this strand in the testing part of the research, suggesting that their may be a mismatch between their confidence and abilities in this area.

8.2.7 All strands: Synthesis and problem solving

Key messages

In general, both groups of Leaving Certificate students appeared confident in their abilities to apply their mathematics to real life situations, synthesise their mathematical learning across more than one strand of learning. Furthermore, a greater proportion of students following all strands of the syllabus reported that they were confident in undertaking these tasks: this is a highly encouraging reflection of the positive impact of the revised syllabus.

Students tended, however, to lack confidence in their ability to **solve mathematics problems using what they have learned in more than one mathematics topic**, with students following all strands of the revised syllabus appearing less confident than those studying Strands 1 and 2. This suggests that whilst students feel confident that they can effectively make connections between different mathematics topics, they do not yet feel as confident that they can directly apply this knowledge.

Across all strands of the revised syllabus, students are expected to be able to **use mathematics to solve problems based on real-life situations**. In general, both phase one and non-phase one students reported that they were confident in this area. Like their Junior Certificate peers, however, Leaving Certificate phase one students appeared to feel somewhat less confident than non-phase one students. Almost three-quarters (72 per cent) of phase one students reported that they would find it 'easy' (43 per cent) or 'very easy' (29 per cent) to use mathematics to solve problems based on real-life situations, compared to four-fifths (80 per cent) of non-phase one students who reported that they would find this 'easy' (45 per cent) or 'very easy' (35 per cent) (Appendix B, Table 55). This is a statistically significant difference, and, like the Junior Certificate survey, is particularly notable as students in the phase one group reported that they applied mathematics to real-life situations much more commonly than the non-phase one group. As suggested earlier in this report, one possible explanation for this is that, as phase one students do this more frequently than their non-phase one counterparts, they have been encouraged to test out and challenge their skills in this area to a greater degree. Likewise, as discussed in relation to the findings of the Junior Certificate survey, it may also be possible that there are competing conceptions of what is meant by 'problem-solving' in this context.

Similarly, across all strands of the revised syllabus, students are expected to demonstrate their ability to **synthesise what they have learned in more than one topic, and apply it to solving a range of mathematical problems**. For Leaving Certificate students, unlike Junior Certificate students, there was a statistically significant difference between phase one and non-phase one groups. Whilst half (50 per cent) of phase one students reported that they would find it 'easy' (42 per cent) or 'very easy' (eight per cent) to gather all the information available, and then use it to solve a particular mathematics problem, non-phase one students appeared to feel more confident. Overall, 57 per cent of non-phase one students reported that they would find this 'easy' (43 per cent) or 'very easy' (14 per cent) (Appendix B, Table 56).

There was no statistically significant difference, however, in students' confidence to solve **mathematics problems using what they have learned in more than one mathematics topic**, although in general, students in both groups appeared to lack confidence in this area. Just over two-fifths (41 per cent) of phase one students reported that they would find this 'easy' or 'very easy', compared to 53 per cent of non-phase one students (Appendix B, Table 57). This suggests that whilst students felt confident that they can effectively make connections between different mathematics topics, they do not yet feel as confident that they can directly apply this knowledge.

8.2.8 General attitudes towards mathematics

Key messages

Both groups of Leaving Certificate students held similarly positive attitudes towards mathematics in general. Whereas Junior Certificate students who had followed the revised syllabus reported that they felt less confident in their mathematical ability relative to their peers, compared to students who followed the previous syllabus, there was no such distinction between Leaving Certificate groups.

In order to understand students' perceptions of their own abilities and levels of engagement with mathematics, participating students were asked to comment on the extent of their agreement with a range of statements about learning mathematics. The areas explored included students':

- confidence in their own mathematical ability, and in their ability relative to their peers
- enjoyment of mathematics, and the process of learning mathematics
- interest in studying more mathematics in school.

Overall, Leaving Certificate students in both phase one and non-phase one groups reported similarly positive views about learning mathematics and, in most areas, there were no statistically significant differences between the two groups. This includes students' **confidence in their mathematical ability relative to their peers**, an area in which Junior Certificate phase one students felt significantly less confident than their comparison group peers. This indicates that, overall, the revised syllabus has not influenced students' general attitudes towards mathematics, despite the many changes associated with the implementation of a new syllabus, which in itself is a positive finding (Appendix B, Tables 58-64). Indeed, just 42 per cent of phase one students reported that the way they learned mathematics at Leaving Certificate was the same as Junior Certificate, compared to 60 per cent of Junior Certificate students (Appendix B, Table 65).

There was, however, a statistically significant difference in relation to the extent to which Leaving Certificate students agreed that they **would like to take more mathematics in school**. The findings are presented in Table 8.17:

Table 8.17: I would like to take more maths in school

	Phase one %	Non-phase one %
Often	11	17
Sometimes	24	26
Rarely	28	28
Never	36	27
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 8.17 shows that whilst, in general, students in both groups were not especially positive that they would like to study more mathematics in school, phase one students were less likely to report that this was the case than non-phase one students. Just 35 per cent of phase one students agreed either 'a little' (24 per cent) or 'a lot' (11 per cent) that they would like to take more mathematics in school, compared to 43 per cent of non-phase one students who agreed either 'a little' (26 per cent) or 'a lot' (17 per cent) that this was the case

8.2.9 Discussion

Again, this section highlights the many areas in which phase one students appear to feel confident in their mathematics learning. It is of particular interest that within Strands 1 and 2, there are areas where phase one students are more confident than their non-phase one peers, despite having followed the same syllabus as their non-phase one peers. This may require further exploration in subsequent phases of this research to determine whether there are school-based factors contributing to this difference (e.g. the revised syllabus is more embedded), or whether instead following a greater number of strands leads to benefits in all strands of the revised syllabus.

8.3 Students' attitudes towards careers involving mathematics

> **Key messages**
>
> Like Junior Certificate students, both groups of Leaving Certificate students reported that **mathematics was important in a range of contexts outside of the classroom**, but shared their views regarding the scope and range of careers which may involve mathematics.
>
> Nonetheless, many Leaving Certificate students were planning to pursue further study and/or careers in mathematics, **favouring professions such as accountancy and business management.**

To gain an understanding of students' attitudes towards careers involving mathematics, the survey explored students' knowledge of, and perspectives on:

- the wider application of mathematics beyond the classroom
- the range of jobs and career pathways involving mathematics.

8.3.1 Students' understanding of the wider application of mathematics

To ascertain Leaving Certificate students' views on the broader application of mathematics, they were asked to comment on the extent to which they perceived it to be useful in the following ways:

- to help in daily life
- to aid learning in other school subjects
- to enable them to get into the university of their choice
- to enable them to get the job of their choice.

The findings showed that, whilst both groups of students were in broad agreement that mathematics was important in each of these areas, albeit to a lesser extent than their Junior Certificate counterparts (between 51 per cent and 67 per cent of phase one students agreed ‘a little’ or ‘a lot’ that mathematics was important in each of these areas, as did between 49 per cent and 69 per cent of comparison group students), there were no statistically significant differences between phase one and non-phase one students in any of these areas (Appendix B, Tables 66-69).

8.3.2 Students’ understanding of jobs involving mathematics

To explore students’ understanding of jobs and career pathways involving mathematics, they were provided with a list of ten different professions, all involving mathematics in a variety of different ways. Students were then asked to select which of these roles involved using mathematics. These professions, in rank order according to the proportion of students indicating positively that they involve mathematics, are shown in Table 8.18:

Table 8.18: Proportion of Leaving Certificate students indicating that mathematics is involved in each profession

	Phase one students	Non-phase one students
80-100 per cent	Accountant	Accountant
	Engineer	Engineer
	Owning a business	Owning a business
	Scientist	Scientist
	Working with technology	Sales assistant
	Sales assistant	Working with technology
50-70 per cent	Doctor	Doctor
	Dietician	Dietician
40-50 per cent	Fashion designer	Fashion designer
	Nurse	Nurse

Table 8.18 shows that, like the Junior Certificate survey, there are no substantial differences between students’ views on which of these roles involve using mathematics. Again, students in both groups reported most strongly that this was the case for jobs involving a clear mathematical component (for example, **accountancy, or owning a business**): over 90 per cent of students in both phase one and non-phase one schools identified that this was the case. Over 90 per cent of both groups at Leaving Certificate level also identified that **engineering** also involved mathematics.

Next, students in both phase one and non-phase one schools strongly identified that careers in other STEM subjects involved mathematics, including **science and technology** (over 85 per cent of students in both phase one and non-phase one schools identified that this was the case).

Whilst overall, Leaving Certificate students did not perceive as strongly that careers in the **medical profession** involved using mathematics, they did so more than their Junior Certificate peers: 68 per cent of Leaving Certificate phase one students, and 59 per cent of non-phase one students, reported, for example, that being a doctor would require mathematics. This reflected a generally higher recognition of the role of mathematics in all professions, than was present at Junior Certificate (Appendix B, Table 70-79).

8.3.3 Interest in a career in mathematics

Leaving Certificate students were asked about their future plans to study and pursue careers in mathematics, to ascertain the extent to which the revised syllabus is having an impact on students' aspirations in this area.

The vast majority of students (91 per cent from both phase one and non-phase one schools) reported that that they were considering further study after finishing their Leaving Certificate (Appendix B, Table 80), and almost half of these students (49 per cent of phase one students, and 47 per cent of non-phase one students) indicated that they would be **going to university to do a course involving 'some' or 'a lot' of mathematics**. Around one-third of students (33 per cent of phase one students, and 35 per cent of non-phase one students) reported they would be doing a course that did not involve mathematics at university (Appendix B, Table 81). The remainder of students in both phase one and non-phase one schools were planning to take a technical or vocational course, an apprenticeship, or training for a variety of different careers (Appendix B, Table 82).

Just over one-third of students from both phase one and non-phase one schools (34 per cent, and 32 per cent, respectively) were considering doing a job that involves mathematics in the future (Appendix B, Table 83). Most commonly, these students reported that these jobs may include teaching; finance and accountancy; business and management; and science (Appendix B, Table 84).

8.3.4 Discussion

The findings presented in this section indicate that, as found in the Junior Certificate survey, the introduction of the revised mathematics syllabus has not, to date, had any discernible impact on students' appreciation of the application of mathematics outside of the classroom (although again, in general, students in both phase one and non-phase one groups had broadly positive views in this regard).

Leaving Certificate students' responses also echo those of their Junior Certificate peers regarding their perceptions of the range of professions involving mathematics.

Nonetheless, the findings positively indicate that many Leaving Certificate students are keen to go on to further study and careers in mathematics. It may, therefore, be valuable to explore ways in which Leaving Certificate students can be encouraged to broaden their understanding of the ways that mathematics can be applied in the workplace, to support them in making informed decisions about their future study and career choices.

9. Overview and next steps

This section provides a brief overview and discussion of the assessment and survey findings, as a basis for further exploration in subsequent phases of the research.

Assessment of students' performance reveals that, overall, students are performing well in many aspects of the revised mathematics syllabus. Furthermore, parallels between the assessment of students' performance and findings of the attitude survey suggest that students are reflective about their experiences of learning mathematics, and in most cases able to identify their own areas of strengths and weaknesses[23]. Both Junior Certificate and Leaving Certificate students are performing particularly highly in relation to Strand 1, Statistics and Probability, for example, which is reflected in the high degree of confidence reported in relation to this strand. By contrast, students who had followed the revised syllabus appeared to find Strand 4, Algebra, more difficult which is, again, identified as an area in which students lack confidence, relative to their comparison group peers.

There do not yet appear to be any discernible differences in skills of students following the introduction of the revised mathematics syllabus, relative to their peers, as measured by the items contained in the indicator item booklets. At Leaving Certificate level, this is perhaps to be expected given that both groups of students have studied Strand 1 and 2 of the revised mathematics syllabus (although there were particular areas, for example analysing verbal geometric information and translating it into mathematical form, in which phase one students appeared more proficient, suggesting that they had benefited from greater immersion in the revised syllabus). At Junior Certificate level, however, there is no discernible difference despite students having followed different syllabus pathways: this suggests that the revised syllabus is not, as yet, having a significant impact on students' performance.

The student attitude survey of both Junior Certificate and Leaving Certificate students, however, shows that those who have studied the revised mathematics syllabus positively identify a range of teaching techniques central to the aims of the new syllabus, including the application of mathematics to real-life situations; making connections and links between mathematics topics; using mathematical language and verbal reasoning to convey ideas; and planning and conducting investigations. Whilst many students report that they have found it challenging to adapt to the new approaches to learning mathematics promoted though the revised syllabus (and again, this is corroborated by the assessment data which reveals that at both Junior Certificate and Leaving Certificate level, higher order skills, such as reasoning and an ability to transfer knowledge to new contexts, are found more difficult than those

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[23] Leaving Certificate students' gave a more mixed picture in relation to Strand 5, Functions. Whilst students appeared highly confident in relation to this strand, they experienced some difficulties in the assessment part of the research.

which are more mechanical in demand), it is positive that they have remained confident in their mathematical abilities and skills throughout. This is particularly notable for Leaving Certificate students, who did not have any experience of the teaching approaches promoted in the revised syllabus at Junior Certificate level. Furthermore, students who are following the revised syllabus appear to be acquiring a growing knowledge of the application of mathematics outside of the classroom, and within a range of different professions.

9.1 Next steps

The early findings outlined in this report provide a sound basis for further exploration throughout this research, which includes:

- attitude surveys and assessment of performance with a further cohort of Junior Certificate and Leaving Certificate students in Autumn 2012
- ongoing, in-depth case studies in eight phase one, and eight non-phase one schools: this includes further exploration of many of the specific issues arising from this phase of the research
- qualitative analysis of students' work in Autumn 2012, exploring the processes being promoted in the revised syllabus.

References

Advisory Committee on Mathematics Education (2011) *Mathematical Needs- The Mathematical Needs of Learners*. Available online: http://www.acme-uk.org/media/7624/acme_theme_a_final%20(2).pdf [Accessed 20 August 2012]

Department of the Taoiseach (2008). *Building Ireland's smart economy: A framework for sustainable economic renewal*. Dublin: The Stationery Office [online]. Available: http://www.taoiseach.gov.ie/attached_files/BuildingIrelandsSmartEconomy.pdf [accessed 18 November 2011]

Eurydice (2011) *Mathematics Education in Europe: Common Challenges and National Policies*. Available online: http://eacea.ec.europa.eu/education/eurydice/documents/thematic_reports/132EN.pdf [Accessed 20 August 2012]

Hiebert, J. and Grouws, D. (2009) 'Which teaching methods are most effective for maths?' *Better: Evidence-based Education*, 2 (1), pp. 10-11 [Online] Available at http://content.yudu.com/A1i1c9/BetterFall09US/resources/index.htm?referrerUrl= [Accessed 20 August 2012]

Innovation Taskforce (2010). *Innovation Ireland: Report of the Innovation Taskforce*. Dublin: The Stationery Office [online]. Available: http://www.taoiseach.gov.ie/eng/Innovation_Taskforce/Report_of_the_Innovation_Taskforce.pdf [accessed 18 November 2011]

Joint Mathematical Council of the United Kingdom (2011) *Digital Technologies and mathematics education.* Available online: http://cme.open.ac.uk/cme/JMC/Digital%20Technologies%20files/JMC_Digital_Technologies_Report_2011.pdf [Accessed 20 August 2012]

Meehan, M. and Paolucci, C. (2009). *Proceedings of Third National Conference on Research in Mathematics Education.* Available online: http://main.spd.dcu.ie/main/academic/education/staff_details/documents/proceedings_mei_09.pdf#page=256 [Accessed 24 July 2012]

Mullis, I., Martin, M., and Foy, P. In collaboration with Olson, J., Preushoff, C., Erberber, E., Arora, A. and Galia, J. (2008) *TIMSS 2007 International Mathematics Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. Boston. Available: http://timssandpirls.bc.edu/TIMSS2007/PDF/TIMSS2007_InternationalMathematicsReport.pdf

Swan, M., Lacey, P. and Mann. S. (2008) *Mathematics Matters: Final Report.* [pdf] Available online:

https://www.ncetm.org.uk/public/files/309231/Mathematics+Matters+Final+Report.pdf
[Accessed 20 August 2012]

Appendix A

Student achievement data tables, Spring 2012

Table 1: Origin of items used in the Junior Certificate item indicator booklets

Indicator Item Booklet	Item	Syllabus area assessed	Source of items	
			TIMSS	PISA
JC1/2	1	1.2	TIMSS 2007 - Grade 8	
	2	1.7	TIMSS 2007 - Grade 8	
	3	1.6	TIMSS 2007 - Grade 8	
	4	1.3	TIMSS 2007 - Grade 8	
	5	1.6	TIMSS 2007 - Grade 8	
	6a	1.7	TIMSS 2007 - Grade 8	
	6b	1.7	TIMSS 2007 - Grade 8	
	6c	1.7	TIMSS 2007 - Grade 8	
	7a	1.6	TIMSS 2007 - Grade 8	
	7b	1.7	TIMSS 2007 - Grade 8	
	8	1.3	TIMSS 2007 - Grade 8	
	9	1.4	TIMSS 2007 - Grade 8	
	10	1.3	TIMSS 2007 - Grade 8	
	11	1.4		✓
	12	2.3	TIMSS 2007 - Grade 8	
	13	2.1	TIMSS 2007 - Grade 8	
	14	2.1	TIMSS 2007 - Grade 8	
	15	2.1	TIMSS 2007 - Grade 8	
	16	2.1	TIMSS 2007 - Grade 8	
	17	2.2	TIMSS 2007 - Grade 8	
	18	2.3	TIMSS 2007 - Grade 8	
	19	2.1	TIMSS 2007 - Grade 8	
	20	2.1	TIMSS 2007 - Grade 8	
	21	2.1	TIMSS 2007 - Grade 8	
JC3/4	1	3.1	TIMSS 2007 - Grade 8	
	2	3.2	TIMSS 2007 - Grade 8	
	3	3.1	TIMSS 2007 - Grade 8	
	4	3.1	TIMSS 2007 - Grade 8	
	5	3.4	TIMSS 2007 - Grade 8	
	6	3.1	TIMSS 2007 - Grade 8	
	7	3.1	TIMSS 2007 - Grade 8	
	8	3.4		✓
	9	3.4		✓
	10a	3.3		✓
	10b	3.3		✓
	11	3.1		✓
	12	4.2	TIMSS 2007 - Grade 8	
	13a	4.2		✓
	13b	4.4		✓
	13c	4.4		✓
	14	4.5		✓

	15	4.7	TIMSS 2007 - Grade 8	
	16	4.6	TIMSS 2007 - Grade 8	
	17	4.3	TIMSS 2007 - Grade 8	
	18	4.6	TIMSS 2007 - Grade 8	
	19	4.6	TIMSS 2007 - Grade 8	
	20	4.4	TIMSS 2007 - Grade 8	
	21	4.3	TIMSS 2007 - Grade 8	

Table 2: Origin of items used in the Leaving Certificate item indicator booklets

Indicator Item Booklet	Item	Syllabus area assessed	Source of items	
			TIMSS	PISA
SPLC1	1	1.2		✓
	2	1.2		✓
	3	1.4		✓
	4	1.3	TIMSS 2007 - Grade 8	
	5	1.6		✓
	6	1.4		✓
	7a	1.6	TIMSS 2007 - Grade 8	
	7b		TIMSS 2007 - Grade 8	
	8	1.4		✓
	9	1.4		✓
GTLC2	1	2.1		✓
	2a	2.1	TIMSS 2007 - Grade 8	
	2b			
	3	2.1	TIMSS 2007 - Grade 8	
	4	2.1	TIMSS 2007 - Grade 8	
	5	2.1	TIMSS 2008 (Advanced)	
	6	2.2	TIMSS 2008 (Advanced)	
	7	2.2	TIMSS 2008 (Advanced)	
	8a	2.3	TIMSS 2008 (Advanced)	
	8b			
	9	2.2	TIMSS 2008 (Advanced)	
	10a	2.3	TIMSS 2008 (Advanced)	
	10b			
NLC3	1	3.4		✓
	2	3.1		✓
	3	3.4		✓
	4	3.4		✓
	5	3.5		✓
	6	3.1	TIMSS 2008 (Advanced)	
	7	3.1	TIMSS 2008 (Advanced)	
	8a	3.1		✓
	8b			
	9	3.4		✓
	10	3.1	TIMSS 2008 (Advanced)	
ALC4	1a	4.1		✓
	1b			
	2a	4.1		✓

	2b			
	3	4.4	TIMSS 2008 (Advanced)	
	4	4.3	TIMSS 2008 (Advanced)	
	5	4.3	TIMSS 2008 (Advanced)	
	6	4.1	TIMSS 2008 (Advanced)	
	7	4.2	TIMSS 2008 (Advanced)	
FLC5	1	5.1		✓
	2	5.2	TIMSS 2008 (Advanced)	
	3	5.1		✓
	4a	5.2	TIMSS 2008 (Advanced)	
	4b			
	5	5.2	TIMSS 2008 (Advanced)	
	6	5.1	TIMSS 2008 (Advanced)	
	7a	5.2	TIMSS 2008 (Advanced)	
	7b			
	8	5.2	TIMSS 2008 (Advanced)	
	9	5.2	TIMSS 2008 (Advanced)	

Junior Certificate indicator items: comparison of phase one and non-phase one schools

Table 3 compares the performance of phase one and non-phase one schools, including analysis using the statistical method of differential item functioning analysis. This highlights item-by-item differences, where comparison group students did better, or less well, than their phase-one peers.

Differential item functioning is analysed using the Logistic Regression Approach. The basic purpose of this approach is to calculate the probability of particular groups of students (in this case phase one or comparison group students) getting each item correct, in relation to the probability of the whole sample getting those items correct. The output of differential item function analysis is in the form of a coefficient and the significance of the coefficient is calculated, i.e. the probability that such a value could have arisen by chance and that there is in reality no difference between the two groups. Three measures of significance for differential functioning are given:

- significance at the 5 per cent level ($p < 0.05$): less than 5 per cent probability that the difference is due to chance;
- significance at the 1 per cent level ($p < 0.01$): less probable that the difference arose by chance;
- significance at the 0.1 per cent level ($p < 0.001$): improbable that the difference arose by chance.

It should be noted that similar findings may not occur with a different sample. Past experience suggests that this is particularly the case for those differences which are significant only at the 5 per cent level.

Table 3: Junior Certificate indicator items – comparison of phase one and non-phase one schools

Indicator Item Booklet	Item	Syllabus area assessed	Phase One Students		Non-phase One Students		Significant difference (%)
			Facility (%)	% Omit	Facility (%)	% Omit	
JC1/2	1	1.2	87	1	87	0	none
	2	1.7	96	0	95	1	none
	3	1.6	≥1m: 68 2m: 62	3	≥1m: 68 2m: 60	4	none
	4	1.3	86	1	76	1	**1**
	5	1.6	95	1	94	1	none
	6a	1.7	≥1m: 64 2m: 47	14	≥1m: 61 2m: 40	17	none
	6b	1.7	≥1m: 57 2m: 33	17	≥1m: 55 2m: 28	17	none
	6c	1.7	47	16	47	17	none
	7a	1.6	73	2	74	1	none
	7b	1.7	≥1m: 76 2m: 41	1	≥1m: 74 2m: 37	1	none
	8	1.3	71	1	70	2	none
	9	1.4	22	2	19	3	none
	10	1.3	60	5	52	10	none
	11	1.4	≥1m: 54 2m: 17	10	≥1m: 45 2m: 13	12	none
	12	2.3	87	2	82	2	none
	13	2.1	59	4	52	3	none
	14	2.1	67	3	59	4	none
	15	2.1	64	5	68	4	none
	16	2.1	37	4	40	4	none
	17	2.2	76	4	81	4	5
	18	2.3	73	4	65	5	none
	19	2.1	51	19	46	17	none
	20	2.1	35	14	41	15	**1**
	21	2.1	65	5	56	8	none
JC3/4	1	3.1	77	1	-	-	-
	2	3.2	90	0	-	-	-
	3	3.1	69	2	-	-	-
	4	3.1	50	9	-	-	-
	5	3.4	83	1	-	-	-
	6	3.1	72	2	-	-	-
	7	3.1	59	2	-	-	-
	8	3.4	≥1m: 3 2m: 0	11	-	-	-
	9	3.4	93	2	-	-	-
	10a	3.3	74	3	-	-	-
	10b	3.3	34	11	-	-	-
	11	3.1	10	3	-	-	-
	12	4.2	≥1m: 29 2m: 17	8	-	-	-
	13a	4.2	≥1m: 74 2m: 66	2	-	-	-
	13b	4.7	29	37	-	-	-
	13c	4.4	≥1m: 21 2m: 9	31	-	-	-
	14	4.5	14	18	-	-	-
	15	4.7	38	4	-	-	-
	16	4.6	73	4	-	-	-
	17	4.3	65	4	-	-	-
	18	4.6	54	5	-	-	-
	19	4.7	57	4	-	-	-

	20	4.4	36	5	-	-	-
	21	4.3	46	5	-	-	-

Junior Certificate indicator items: comparison with international performance

Table 4 presents the average scores of the phase one students and compares them with the international average scores in the 2007 TIMSS study. The difference in item facilities is also shown. These differences are indicative, as significance tests could not be carried out.

Items have been shaded to ease comparison. If the difference is less than 10 percentage points, the item has not been shaded as it is possible that any difference in performance is due to sampling effects. Green denotes items on which phase one students have substantially higher facilities and orange indicates items on which the international students have scored considerably more highly. If the difference is 10-24 percentage points the item has a pale shading while differences of 25 percentage points and more have darker shading.

Comparative data is available for 22 out of 24 items or item parts of JC1/2.

Table 4: Junior Certificate indicator items – comparison with international performance (TIMSS items only)

Indicator Item Booklet	Item	Syllabus area assessed	Phase One Facility (%)	International Facility (%)	Difference in facility (percentage points)
JC1/2	1	1.2	87	63	-24
	2	1.7	96	64	-32
	3	1.6	≥1m: 68 2m: 62	≥1m: 34 2m: 29	≥1m: -34 2m: -33
	4	1.3	86	44	-42
	5	1.6	95	75	-20
	6a	1.7	≥1m: 64 2m: 47	≥1m: 29 2m: 17	≥1m: -35 2m: -30
	6b	1.7	≥1m: 57 2m: 33	≥1m: 24 2m: 10	≥1m: -33 2m: -23
	6c	1.7	47	23	-24
	7a	1.6	73	41	-32
	7b	1.7	≥1m: 76 2m: 41	≥1m: 47 2m: 20	≥1m: -29 2m: -21
	8	1.3	71	49	-22
	9	1.4	22	-	-
	10	1.3	60	31	-29
	11	1.4	≥1m: 54 2m: 17	-	-
	12	2.3	87	68	-19
	13	2.1	59	33	-26
	14	2.1	67	51	-16
	15	2.1	64	59	-5
	16	2.1	37	32	-5
	17	2.2	76	51	-25
	18	2.3	73	57	-16
	19	2.1	51	39	-12
	20	2.1	35	28	-7
	21	2.1	65	42	-23
JC3/4	1	3.1	77	64	-13

	2	3.2	90	70	-20
	3	3.1	69	47	-22
	4	3.1	50	19	-31
	5	3.4	83	63	-20
	6	3.1	72	55	-17
	7	3.1	59	44	-15
	8	3.4	≥1m: 3 2m: 0	-	-
	9	3.4	93	-	-
	10a	3.3	74	-	-
	10b	3.3	34	-	-
	11	3.1	10	-	-
	12	4.2	≥1m: 29 2m: 17	≥1m: 19 2m: 9	≥1m: -10 2m: -8
	13a	4.2	≥1m: 74 2m: 66	-	-
	13b	4.4	29	-	-
	13c	4.4	≥1m: 21 2m: 9	-	-
	14	4.5	14	-	-
	15	4.7	38	64	26
	16	4.6	73	70	-3
	17	4.3	65	47	-18
	18	4.6	54	19	-35
	19	4.6	57	63	6
	20	4.4	36	55	19
	21	4.3	46	44	-2

Leaving Certificate indicator items: comparison of phase one and non-phase one schools

Table 5 presents the scores of phase one and non-phase one students who completed booklets SPLC1 and GTLC2 (non-phase one students did not sit booklets NCL3, ALC4 or FLC5). This allows for a basic comparison of performance between phase one and non-phase one students. Their average scores on each item are compared using the statistical analysis of differential item functioning. This highlights item-by-item differences, where non-phase one students did better, or less well, than their phase one peers.

Differential item functioning is analysed using the Logistic Regression Approach. The basic purpose of this approach is to calculate the probability of particular groups of students (in this case phase one or non-phase one students) getting each item correct, in relation to the probability of the whole sample getting those items correct. The output of differential item function analysis is in the form of a coefficient and the significance of the coefficient is calculated, i.e. the probability that such a value could have arisen by chance and that there is in reality no difference between the two groups. Three measures of significance for differential functioning are given:

- significance at the 5 per cent level ($p < 0.05$): less than 5 per cent probability that the difference is due to chance;
- significance at the 1 per cent level ($p < 0.01$): less probable that the difference arose by chance;
- significance at the 0.1 per cent level ($p < 0.001$): improbable that the difference arose by chance.

It should be noted that similar findings may not occur with a different sample. Past experience suggests that this is particularly the case for those differences which are significant only at the 5 per cent level.

Table 5: Leaving Certificate indicator items - comparison of phase one and non-phase one schools

Indicator Item Booklet	Item	Syllabus area assessed	Phase One Students		Non-phase One Students		Significant difference (%)
			Facility (%)	% Omit	Facility (%)	% Omit	
SPLC1	1	1.2	61	2	62	2	none
	2	1.2	67	3	62	3	none
	3	1.4	≥1m: 58 2m: 28	3	≥1m: 51 2m: 21	7	none
	4	1.3	70	2	61	5	none
	5	1.6	66	3	56	7	none
	6	1.4	1	2	1	4	none
	7a	1.6	80	2	75	6	none
	7b		≥1m: 79 2m: 42	2	≥1m: 73 2m: 39	7	none
	8	1.4	≥1m: 63 2m: 58	8	≥1m: 58 2m: 49	18	none
	9	1.4	49	17	38	25	none
GTLC2	1	2.1	59	4	48	3	**1**
	2a	2.1	77	5	68	6	5
	2b		≥1m: 20 2m: 18	49	≥1m: 13 2m: 10	53	5
	3	2.1	51	2	47	2	none
	4	2.1	67	5	68	4	none
	5	2.1	28	9	28	6	none
	6	2.2	34	8	41	9	5
	7	2.2	17	9	18	16	none
	8a	2.3	31	19	25	19	none
	8b		13	50	14	50	none
	9	2.2	≥1m: 30 2m: 22	31	≥1m: 36 2m: 30	32	**1**
	10a	2.3	12	27	18	29	none
	10b		≥1m: 11 2m: 1	70	≥1m: 13 2m: 0	70	none
NLC3	1	3.4	82	5	-	-	-
	2	3.1	29	13	-	-	-
	3	3.4	42	3	-	-	-
	4	3.4	≥1m: 47 2m: 14	2	-	-	-
	5	3.5	71	4	-	-	-
	6	3.1	35	6	-	-	-
	7	3.1	18	14	-	-	-
	8a	3.1	59	7	-	-	-
	8b		≥1m: 36 2m: 25	24	-	-	-
	9	3.4	23	9	-	-	-
	10	3.1	14	61	-	-	-
ALC4	1a	4.1	82	2	-	-	-
	1b		25	12	-	-	-
	2a	4.1	≥1m: 84 2m: 76	9	-	-	-
	2b		≥1m: 47 2m: 34	17	-	-	-
	3	4.4	13	10	-	-	-
	4	4.3	26	10	-	-	-
	5	4.3	8	19	-	-	-
	6	4.1	16	11	-	-	-

	7	4.2	5	38	-	-	-	
FLC5	1	5.1	34	3	-	-	-	
	2	5.2	9	12	-	-	-	
	3	5.1	77	3	-	-	-	
	4a	5.2	2	34	-	-	-	
	4b		1	46	-	-	-	
	5	5.2	3	33	-	-	-	
	6	5.1	21	10	-	-	-	
	7a	5.2	15	35	-	-	-	
	7b		2	46	-	-	-	
	8	5.2	28	17	-	-	-	
	9	5.2	24	16	-	-	-	

Table 6: Comparison of Common Items – phase one students

Indicator Item Booklet	Item	Indicator Item Booklet	Item	Syllabus area assessed	Phase One JC Students		Phase One LC Students	
					Facility (%)	% Omit	Facility (%)	% Omit
JC1/2	7a	**SPLC1**	7a	1.6	73	2	80	2
	7b		7b	1.7	≥1m: 76 2m: 41	1	≥1m: 79 2m: 42	2
	10		4	1.3	60	5	70	2
	11		3	1.4	≥1m: 54 2m: 17	10	≥1m: 58 2m: 28	3
	14	**GTLC2**	2a	2.1	67	3	77	5
	16		3	2.1	37	4	51	2
	21		4	2.1	65	5	67	5

Table 7: Comparison of common items – non-phase one students

Indicator Item Booklet	Item	Indicator Item Booklet	Item	Syllabus area assessed	Non-phase One JC Students		Non-phase One LC Students	
					Facility (%)	% Omit	Facility (%)	% Omit
JC1/2	7a	**SPLC1**	7a	1.6	74	1	75	6
	7b		7b	1.7	≥1m: 74 2m: 37	1	≥1m: 73 2m: 39	7
	10		4	1.3	52	10	61	5
	11		3	1.4	≥1m: 45 2m: 13	12	≥1m: 51 2m: 21	7
	14	**GTLC2**	2a	2.1	59	4	68	6
	16		3	2.1	40	4	47	2
	21		4	2.1	56	8	68	4

Leaving Certificate indicator items: comparison with international performance

Table 8 presents the average scores of the phase one students and compares them to the international average scores in the 2007 TIMSS and 2008 TIMSS Advanced studies. The difference in item facilities is also shown and each item has been shaded to ease comparison. If the difference is less than 10 percentage points, the item has not been shaded as it is possible that any difference in performance is due to sampling effects. Green denotes items on which phase one students have substantially higher facilities and orange indicates items on which the international students have scored considerably more highly. If the difference is 10-24 percentage points the item has a pale shading while differences of 25 percentage points and more have darker shading.

Comparative data is available for three of the items in SPLC1: items 4, 7a and 7b. The remaining items were released PISA items and so, no international data is available. Table 7.6 below shows the number of items with differences in facility that fall within the three performance bands as described above.

Table 8: Leaving Certificate indicator items - comparison with international performance

Indicator Item Booklet	Item	Syllabus area assessed	Phase One Facility (%)	International Facility (%)	Difference in facility (percentage points)
SPLC1	1	1.2	61	-	-
	2	1.2	67	-	-
	3	1.4	≥1m: 58 2m: 28	-	-
	4	1.3	70	31	-39
	5	1.6	66	-	-
	6	1.4	1	-	-
	7a	1.6	80	41	-39
	7b		≥1m: 79 2m: 42	≥1m: 47 2m:20	≥1m: -32 2m:-22
	8	1.4	≥1m: 63 2m: 58	-	-
	9	1.4	49	-	-
GTLC2	1	2.1	59	-	-
	2a	2.1	77	51	-26
	2b		≥1m: 20 2m: 18	-	-
	3	2.1	51	32	-19
	4	2.1	67	42	-25
	5	2.1	28	68	40
	6	2.2	34	54	20
	7	2.2	17	24	7
	8a	2.3	31	38	7
	8b		13	-	-
	9	2.2	≥1m: 30 2m: 22	≥1m: 36 2m:29	≥1m: 6 2m: 7

	10a	2.3	12	26	14
	10b		≥1m: 11 2m: 1	-	-
NLC3	1	3.4	82	-	-
	2	3.1	29	-	-
	3	3.4	42	-	-
	4	3.4	≥1m: 47 2m: 14	-	-
	5	3.5	71	-	-
	6	3.1	35	51	16
	7	3.1	18	39	21
	8a	3.1	59	-	-
	8b		≥1m: 36 2m: 25	-	-
	9	3.4	23	-	-
	10	3.1	14	23	9
ALC4	1a	4.1	82	-	-
	1b		25	-	-
	2a	4.1	≥1m: 84 2m: 76	-	-
	2b		≥1m: 47 2m: 34	-	-
	3	4.4	13	16	3
	4	4.3	26	45	19
	5	4.3	8	54	46
	6	4.1	16	26	10
	7	4.2	5	26	21
FLC5	1	5.1	34	-	-
	2	5.2	9	25	16
	3	5.1	77	-	-
	4a	5.2	2	41	39
	4b		1	15	14
	5	5.2	3	30	27
	6	5.1	21	54	33
	7a	5.2	15	52	37
	7b		2	18	16
	8	5.2	28	35	7
	9	5.2	24	31	7

Student achievement on Leaving Certificate Function items by student level

Table 9 compares the performance of the Ordinary Level and Higher Level students. Although five (3%) Foundation Level students completed the items in Table 9, none achieved a mark. Facilities for Foundation Level students are therefore not presented.

Generally the Higher Level students achieve higher average scores on each item than the Ordinary Level students, as might be anticipated. Since these facilities are based on relatively small numbers of pupils taking each item, they are not estimated to a high level of precision so should be treated with a degree of caution. This is even more so with the Higher-level pupils since there were only 63 of them. To estimate facility with a reasonable degree of precision we would usually need to sample around 400 pupils in each group to be reported.

In order to determine whether differences in facility between Ordinary and Higher-level students were significant, chi-squared tests were carried out. Levels of significance can be summarised as follows:

- significance at the 5 per cent level ($p < 0.05$): -less than 5 per cent probability that the difference is due to chance;
- significance at the 1 per cent level ($p < 0.01$): less probable that the difference arose by chance;
- significance at the 0.1 per cent level ($p < 0.001$): improbable that the difference arose by chance.

Table 9: Student achievement on Leaving Certificate Function items by student level

Item in Indicator Booklet FLC5	Achievement of Ordinary Level Students (%)	Achievement of Higher Level Students (%)	Significance (%)
4a	0	6	5
4b	0	2	Not significant
5	0	10	0.1
7a	8	29	0.1
7b	1	5	Not significant
8	27	33	Not significant
9	16	41	0.1

Ordinary Level, N=111 (62%); Higher Level, N = 63 (35%)

Appendix B

Attitude survey data tables, Spring 2012

Junior Certificate

How often do you do these things in your maths lessons?

Table 1: We show our working to justify our answers

	Phase one	Comparison group
	%	%
Often	86	84
Sometimes	9	12
Rarely	1	2
Never	1	1
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 2: We set goals and targets about our maths learning

	Phase one %	Comparison group %
Often	25	29
Sometimes	39	39
Rarely	23	21
Never	10	9
No Response	4	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

How much do you agree with these statements about your maths lessons?

Table 3: My teacher sets me work to suit my abilities and interests

	Phase one %	Comparison group %
Agree a lot	14	17
Agree a little	42	38
Disagree a little	30	25
Disagree a lot	13	18
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,375 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012

Table 4: My teacher gives me work that will challenge me to improve my skills

	Phase one %	Comparison group %
Agree a lot	61	59
Agree a little	31	33
Disagree a little	6	6
Disagree a lot	2	2
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 5: I know what my teacher expects me to do

	Phase one %	Comparison group %
Agree a lot	59	62
Agree a little	29	29
Disagree a little	7	7
Disagree a lot	3	2
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 6: My teacher helps me to understand if I am finding something difficult during a maths lesson

	Phase one %	Comparison group %
Agree a lot	65	67
Agree a little	22	22
Disagree a little	9	7
Disagree a lot	4	3
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 7: My teacher thinks I can do well in maths

	Phase one %	Comparison group %
Agree a lot	59	61
Agree a little	33	29
Disagree a little	7	5
Disagree a lot	1	3
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 8: My teacher will decide if I should do Foundation Level, Ordinary Level or Higher Level

	Phase one %	Comparison group %
Agree a lot	18	19
Agree a little	35	29
Disagree a little	23	25
Disagree a lot	22	25
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 9: My teacher explains maths in ways that make it interesting

	Phase one %	Comparison group %
Agree a lot	30	28
Agree a little	35	36
Disagree a little	20	20
Disagree a lot	14	15
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 10: My teacher is easy to understand

	Phase one %	Comparison group %
Agree a lot	49	53
Agree a little	29	29
Disagree a little	14	11
Disagree a lot	7	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 375 phase one students, and 2,366 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

How confident would you feel when doing the following types of activities during maths lessons?

Table 11: If I were asked to draw charts to display my data

	Phase one %	Comparison group %
I would find it very easy	54	54
I would find it easy	35	35
I would find it a little difficult	9	9
I would find it very difficult	2	1
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 12: If I were asked to solve problems using trigonometry

	Phase one %	Comparison group %
I would find it very easy	23	28
I would find it easy	31	31
I would find it a little difficult	34	31
I would find it very difficult	11	9
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012

Table 13: If I were asked to understand indices

	Phase one %	Comparison group %
I would find it very easy	26	28
I would find it easy	33	34
I would find it a little difficult	31	29
I would find it very difficult	10	8
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012

Table 14: If I were asked to represent this relationship in a graph

	Phase one %	Comparison group %
I would find it very easy	55	54
I would find it easy	32	31
I would find it a little difficult	10	11
I would find it very difficult	2	3
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 15: If I were asked to solve mathematics problems using what I have learned in more than one mathematics topic…

	Phase one %	Comparison group %
I would find it very easy	22	23
I would find it easy	43	41
I would find it a little difficult	32	30
I would find it very difficult	3	3
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 16: If I were asked to gather all the information available, and then use it to solve a particular mathematics problem…

	Phase one %	Comparison group %
I would find it very easy	15	18
I would find it easy	43	45
I would find it a little difficult	34	31
I would find it very difficult	6	5
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 374 phase one students, and 2,364 comparison group students, gave at east one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

How much do you agree with these statements about learning maths?

Table 17: I usually do well in maths

	Phase one %	Comparison group %
Agree a lot	25	30
Agree a little	47	45
Disagree a little	20	17
Disagree a lot	8	7
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 18: I would like to take more maths in school

	Phase one %	Comparison group %
Agree a lot	20	21
Agree a little	29	29
Disagree a little	29	26
Disagree a lot	20	23
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 19: I enjoy learning maths

	Phase one %	Comparison group %
Agree a lot	22	24
Agree a little	37	38
Disagree a little	22	22
Disagree a lot	17	15
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 20: Maths is not one of my strengths

	Phase one %	Comparison group %
Agree a lot	30	27
Agree a little	24	26
Disagree a little	24	25
Disagree a lot	21	20
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.

A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 21: I learn things quickly in maths

	Phase one %	Comparison group %
Agree a lot	19	21
Agree a little	37	39
Disagree a little	30	28
Disagree a lot	13	11
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 22: Maths is boring

	Phase one %	Comparison group %
Agree a lot	17	19
Agree a little	30	27
Disagree a little	29	31
Disagree a lot	23	22
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 23: I like maths

	Phase one %	Comparison group %
Agree a lot	25	26
Agree a little	37	37
Disagree a little	19	19
Disagree a lot	18	16
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 24: Maths is more difficult for me than many of my classmates

	Phase one %	Comparison group %
Agree a lot	16	14
Agree a little	27	23
Disagree a little	35	35
Disagree a lot	20	27
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 373 phase one students, and 2,365 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 25: The way we learn maths at Junior Certificate level is harder than maths in primary school

	Phase one %	Comparison group %
Often	72	63
Sometimes	18	23
Rarely	5	7
Never	1	5
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 367 phase one students, and 2,361 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

How much do you agree with these statements about maths?

Table 26: I think learning maths will help me in my daily life

	Phase one %	Comparison group %
Agree a lot	38	37
Agree a little	38	42
Disagree a little	17	14
Disagree a lot	6	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 372 phase one students, and 2,355 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012

Table 27: I need maths to learn other school subjects

	Phase one %	Comparison group %
Agree a lot	24	24
Agree a little	46	45
Disagree a little	21	23
Disagree a lot	8	7
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 372 phase one students, and 2,355 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 28: I need to do well in maths to get into the university of my choice

	Phase one %	Comparison group %
Agree a lot	52	52
Agree a little	33	31
Disagree a little	11	11
Disagree a lot	3	4
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 372 phase one students, and 2,355 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 29: I need to do well in maths to get the job I want

	Phase one %	Comparison group %
Agree a lot	39	42
Agree a little	36	34
Disagree a little	19	16
Disagree a lot	5	7
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 372 phase one students, and 2,355 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Which of these jobs do you think involve doing maths?

Table 30: Engineer

	Phase one %	Comparison group %
Yes	87	89
No	11	9
No Response	2	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 31: Doctor

	Phase one %	Comparison group %
Yes	57	57
No	39	39
No Response	4	4
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 32: Sales Assistant

	Phase one %	Comparison group %
Yes	87	88
No	10	10
No Response	4	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 33: Scientist

	Phase one %	Comparison group %
Yes	87	89
No	10	9
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

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Table 34: Working with technology

	Phase one %	Comparison group %
Yes	81	82
No	15	15
No Response	4	3
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 35: Accountant

	Phase one %	Comparison group %
Yes	94	95
No	3	3
No Response	3	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 36: Nurse

	Phase one %	Comparison group %
Yes	37	37
No	58	57
No Response	6	5
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 37: Dietician

	Phase one %	Comparison group %
Yes	45	51
No	50	45
No Response	6	4
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Table 38: Fashion Designer

	Phase one %	Comparison group %
Yes	48	49
No	47	46
No Response	5	4
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

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Table 39: Owning my own business

	Phase one %	Comparison group %
Yes	95	96
No	3	2
No Response	2	2
Total	100	100

Due to rounding percentages may not sum to 100.
A total of 371 phase one students, and 2,359 comparison group students, gave at least one response to these questions.
Source: NFER survey of Junior Certificate student attitudes, Spring 2012.

Leaving Certificate

How often do you do these things in your maths lessons?

Table 40: We show our working to justify our answers

	Phase one %	Non-phase one %
Often	79	83
Sometimes	18	13
Rarely	1	2
Never	1	0
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 41: We set goals and targets about our maths learning

	Phase one %	Non-phase one %
Often	22	21
Sometimes	41	39
Rarely	26	26
Never	10	12
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,991 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

How much do you agree with these statements about your maths lessons?

Table 42: My teacher sets me work to suit my abilities and interests

	Phase one %	Non-phase one %
Agree a lot	14	16
Agree a little	41	39
Disagree a little	24	26
Disagree a lot	20	19
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 43: My teacher gives me work that will challenge me to improve my skills

	Phase one %	Non-phase one %
Agree a lot	55	50
Agree a little	36	40
Disagree a little	7	7
Disagree a lot	2	3
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 44: I know what my teacher expects me to do

	Phase one %	Non-phase one %
Agree a lot	61	57
Agree a little	28	33
Disagree a little	8	7
Disagree a lot	2	2
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 45: My teacher helps me to understand if I am finding something difficult during a maths lesson

	Phase one %	Non-phase one %
Agree a lot	67	67
Agree a little	23	22
Disagree a little	7	7
Disagree a lot	3	3
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 46: My teacher thinks I can do well in maths

	Phase one %	Non-phase one %
Agree a lot	49	51
Agree a little	39	37
Disagree a little	7	8
Disagree a lot	3	3
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 47: My teacher will decide if I should do Foundation Level, Ordinary Level or Higher Level

	Phase one %	Non-phase one %
Agree a lot	12	11
Agree a little	22	20
Disagree a little	25	24
Disagree a lot	40	44
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 48: My teacher explains maths in ways that make it interesting

	Phase one %	Non-phase one %
Agree a lot	24	21
Agree a little	35	39
Disagree a little	25	24
Disagree a lot	14	15
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

How confident would you feel when doing the following types of activities during maths lessons?

Table 49: If I were asked to work out the probability of something happening

	Phase one %	Non-phase one %
I would find it very easy	42	43
I would find it easy	37	37
I would find it a little difficult	16	16
I would find it very difficult	3	3
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 50: If I were asked to draw charts to display my data

	Phase one %	Non-phase one %
I would find it very easy	59	57
I would find it easy	32	34
I would find it a little difficult	8	7
I would find it very difficult	0	1
No response	1	2
Total	100	100

Due to rounding percentages may not sum to 100
A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 51: If I were asked to solve problems using trigonometry

	Phase one %	Non-phase one %
I would find it very easy	27	29
I would find it easy	32	31
I would find it a little difficult	30	29
I would find it very difficult	10	10
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100
A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.
Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 52: If I were asked to solve problems using the properties of different shapes

	Phase one %	Non-phase one %
I would find it very easy	23	24
I would find it easy	33	38
I would find it a little difficult	36	31
I would find it very difficult	6	6
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 53: If I were asked to understand indices

	Phase one %	Non-phase one %
I would find it very easy	19	22
I would find it easy	31	35
I would find it a little difficult	38	32
I would find it very difficult	12	8
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 54: If I were asked to use formulae to solve problems in measurement

	Phase one %	Non-phase one %
I would find it very easy	17	23
I would find it easy	36	37
I would find it a little difficult	38	32
I would find it very difficult	8	7
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 55: If I were asked to solve problems based on real-life situations

	Phase one %	Non-phase one %
I would find it very easy	29	35
I would find it easy	43	45
I would find it a little difficult	23	16
I would find it very difficult	4	3
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at l east one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 56: If I were asked to gather all the information available, and then use it to solve a particular maths problem

	Phase one %	Non-phase one %
I would find it very easy	8	14
I would find it easy	42	43
I would find it a little difficult	40	36
I would find it very difficult	7	5
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 57: If I were asked to solve maths problems using what I have learned in more than one maths topic

	Phase one %	Non-phase one %
I would find it very easy	15	17
I would find it easy	36	39
I would find it a little difficult	42	38
I would find it very difficult	6	5
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,996 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

How much do you agree with these statements about your maths lessons?

Table 58: I usually do well in maths

	Phase one %	Non-phase one %
Agree a lot	20	22
Agree a little	48	47
Disagree a little	22	21
Disagree a lot	9	9
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 59: Maths is more difficult for me than many of my classmates

	Phase one %	Non-phase one %
Agree a lot	13	13
Agree a little	23	25
Disagree a little	38	36
Disagree a lot	26	26
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 60: I enjoy learning maths

	Phase one %	Non-phase one %
Agree a lot	18	19
Agree a little	34	37
Disagree a little	27	23
Disagree a lot	20	20
No response	0	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 61: Maths is not one of my strengths

	Phase one %	Non-phase one %
Agree a lot	33	32
Agree a little	24	26
Disagree a little	26	24
Disagree a lot	16	17
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012

Table 62: I learn things quickly in maths

	Phase one %	Non-phase one %
Agree a lot	13	17
Agree a little	43	39
Disagree a little	29	30
Disagree a lot	14	14
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 63: Maths is boring

	Phase one %	Non-phase one %
Agree a lot	24	24
Agree a little	26	25
Disagree a little	30	30
Disagree a lot	19	20
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 64: I like maths

	Phase one %	Non-phase one %
Agree a lot	19	21
Agree a little	35	37
Disagree a little	22	20
Disagree a lot	23	21
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 65: The way we learn maths at Leaving Certificate Level is the same as how we learned maths for the Junior Certificate

	Phase one %	Non-phase one %
Often	12	21
Sometimes	30	39
Rarely	29	24
Never	26	14
No response	2	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 1,995 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

How much do you agree with these statements about maths?

Table 66: I think learning maths will help me in my daily life

	Phase one %	Non-phase one %
Agree a lot	16	23
Agree a little	50	43
Disagree a little	21	22
Disagree a lot	11	10
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 297 phase one students, and 1,988 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 67: I need maths to learn other school subjects

	Phase one %	Non-phase one %
Agree a lot	14	13
Agree a little	38	36
Disagree a little	26	30
Disagree a lot	21	20
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 297 phase one students, and 1,988 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 68: I need to do well in maths to get into the university of my choice

	Phase one %	Non-phase one %
Agree a lot	38	39
Agree a little	29	30
Disagree a little	21	16
Disagree a lot	11	14
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 297 phase one students, and 1,988 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 69: I need to do well in maths to get the job I want

	Phase one %	Non-phase one %
Agree a lot	19	23
Agree a little	32	30
Disagree a little	27	25
Disagree a lot	21	21
No response	1	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 297 phase one students, and 1,988 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Which of these jobs do you think involve doing maths?

Table 70: Engineer

	Phase one	Non-phase one
	%	%
Yes	95	96
No	3	3
No Response	2	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 71: Doctor

	Phase one	Non-phase one
	%	%
Yes	68	59
No	28	38
No Response	4	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 72: Sales Assistant

	Phase one	Non-phase one
	%	%
Yes	88	88
No	10	10
No Response	2	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 73: Scientist

	Phase one %	Non-phase one %
Yes	94	89
No	5	9
No Response	1	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 74: Working with technology

	Phase one %	Non-phase one %
Yes	89	87
No	9	10
No Response	2	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 75: Accountant

	Phase one %	Non-phase one %
Yes	95	96
No	3	2
No Response	2	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 76: Nurse

	Phase one %	Non-phase one %
Yes	46	40
No	50	56
No Response	4	3
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 77: Dietician

	Phase one %	Non-phase one %
Yes	60	56
No	37	41
No Response	3	3
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 78: Fashion Designer

	Phase one %	Non-phase one %
Yes	45	45
No	52	51
No Response	3	4
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 79: Owning my own business

	Phase one %	Non-phase one %
Yes	94	96
No	3	2
No Response	3	1
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 1,990 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 80: Are you currently thinking of going on to further study when you finish your Leaving Certificate?

	Phase one %	Non-phase one %
Yes	91	91
No	5	6
No response	5	3
Total	100	100

Due to rounding percentages may not sum to 100

A total of 298 phase one students, and 2,003 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 81: If yes, please tick the box that best describes the further study you plan to do after finishing your Leaving Certificate

	Phase one %	Non-phase one %
University, doing a course that will involve a lot of maths	12	14
University, doing a course that will involve some maths	37	33
A technical or vocational course that will involve maths	6	6
University, doing a course that won't involve maths	33	35
Other	9	11
No response	4	2
Total	100	100

Due to rounding percentages may not sum to 100

A total of 271 phase one students, and 1,816 non-phase one students, gave at l east one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 82: Other, please specify

	Phase one	Non-phase one
Technical or vocational course that does not include maths	4	14
Post-leaving certificate course/college	8	7
Other - vocational training (e.g. apprenticeship)	16	2
Unspecified further study – medicine and healthcare	4	9
Unspecified further study - sports sciences	0	4
Unspecified further study – music and arts	8	6
Unspecified further study – childcare and education	8	5
Unspecified further study - law and social sciences	0	6
Unspecified further study – science and technology	12	6
Unspecified further study – business and economics	0	4
Unspecified further study - veterinary and animal care	0	2
Unspecified further study- tourism and hospitality	0	4
Unspecified further study – computing and IT	0	1
Unspecified further study - agriculture	4	2
Unspecified further	4	4

study – fashion, hair and beauty		
Unspecified further study - social care	0	1
Unspecified further study - archaeology	4	0
Uncodeable	12	2
No response	16	22
N =	35	200

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 83: Are you currently thinking of doing a job that involves maths?

	Phase one %	Non-phase one %
Yes	34	32
No	60	63
No response	6	4
Total	100	100

Due to rounding percentages may not sum to 100

A total of 299 phase one students, and 2,003 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

Table 84: Other, please specify

	Phase one	Non-phase one
Architecture	2	2
Veterinary	3	1
Physiotherapy	1	1
Biomedical Science	1	0
Science	7	8
ICT/Computer Science/Computing	5	9
Software Development	3	3
Computer Game Design/Development	6	2
Chemical engineering	1	1
Civil engineering	1	1
Engineering (general/unspecified)	4	8
Systems engineering	1	1
Biomedical engineering	1	1
Mechanical engineering	6	5
Structural engineering	1	1
Business/management	11	9
Law	2	0
Finance/Accounting	10	12
Economist	3	1
Teaching (general)	15	11
Teaching (maths)	3	2
Teaching (science)	3	1
Agriculture	1	1
Sound/Audio Visual engineering	2	1
Design engineering	1	1
Psychology	2	1
Military	2	1
Aeronautic engineering	1	1
Marketing	1	1
Medicine/Health Sciences	0	5
Pharmacy/Pharmaceutical Science	0	1
Sports science/fitness	0	2
Electrical engineering	0	2
Hospitality	0	1

Other relevant/vague comment	1	0
Irrelevant/Uncodeable comment	1	2
No response	9	7
N =	101	649

More than one answer could be put forward so percentages may sum to more than 100.

A total of 92 phase one students, and 601 non-phase one students, gave at least one response to these questions.

Source: NFER survey of Leaving Certificate student attitudes, Spring 2012.

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NCCB

Maths in Practice

Report and recommendations

June 2014

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Preface

This report comes at a time when the final syllabus revisions under the Project Maths initiative get under way in schools. The report contains a number of recommendations; work has already commenced on

- additional online resource material for teachers at www.ncca.ie/projectmaths and www.projectmaths.ie
- new composite guidelines by the Project Maths Development Team (PMDT) that link the resources available on www.projectmaths.ie with the syllabus
- planning additional PMDT support for 2014/2015.

The recommendations will also inform

- a planned review of Leaving Certificate Mathematics following the June 2015 examinations; this review will also be informed by a Chief Examiner's Report to be prepared by the State Examinations Commission in autumn 2015
- preparation of the new mathematics specification as part of the junior cycle developments; the new specification is scheduled to be published in autumn 2017, a year before it's introduced in schools.

You may email projectmaths@ncca.ie if you wish to comment on any aspect of this report.

Contents

1. Introduction

This report presents a summary of the discussions and recommendations arising from a series of meetings of a Maths in Practice group convened to consider how teachers of mathematics could be further supported in engaging with the new mathematics syllabuses, particularly at Leaving Certificate.

Project Maths began on a phased basis1 in 24 schools in September 2008 and rolled out to all schools nationally two years later. In September 2012 a notable landmark was reached when phase 3 began in all schools. This will see all five strands of the mathematics syllabus examined at Leaving Certificate in June 2014, and at Junior Certificate in June 2015.

For an interim period—arising from the experience in the initial 24 schools—some Leaving Certificate Mathematics syllabus material in Strand 1 (statistics and probability) was deferred and an element of choice was allowed in the synthetic geometry section of Strand 2 (geometry and trigonometry). The final syllabus introduced in September 2013 marks the ending of these interim arrangements.

When the syllabuses were presented for approval to Council in June 2013, concern was expressed that some teachers were experiencing considerable difficulty interpreting the syllabus expressed in learning outcomes and that more support was needed in this area. It was agreed that a group of teachers should be established which would explore how such support might be provided; a similar arrangement had worked well in respect of issues which had arisen around Leaving Certificate foundation-level mathematics. The outcomes of these discussions would inform a review, following a complete cycle of implementation. The first examination of the revised syllabuses which include the previously deferred material will take place in June 2015.

[1] The *Project Maths* initiative involved the phased development and introduction of new syllabuses in Mathematics for both Junior Certificate and Leaving Certificate students. In each case, the 5 strands of the revised syllabuses were introduced in three phases. Phase 1 involved the introduction of Strand 1 (statistics and probability) and Strand 2 (geometry and trigonometry); Phase 2 involved the additional introduction of Strand 3 (number) and Strand 4 (algebra); and Phase 3 involved the additional introduction of Strand 5 (functions). The content of the previous syllabus was phased out as the new material was phased in and corresponding changes were made to the examinations at each level.

2. *Maths in Practice* group meetings

Following contact with teacher unions (ASTI and TUI), the Irish Mathematics Teachers' Association (IMTA) and the Project Maths Development Team (PMDT), a Maths in Practice group was convened, comprising nominees from these organisations as well as representation from the DES and the SEC. Appendix 1 lists the membership of the group.

Four meetings were held in the period September 2013 to January 2014 and preparatory or follow-up work was undertaken between meetings, and following the final meeting, based on points which arose at the meetings. A record was kept of the main points considered at each meeting and the suggested actions to follow, which facilitated collating the various points of discussion into a set of themes from which this report and recommendations emerged.

Overall, the group regarded the opportunities for engaging in discussion about the syllabus and its assessment to be beneficial. They acknowledged the wide range of supports for teachers which had been developed and sought to ensure that these were as accessible as possible in a manner which would best support teachers in planning and conducting their mathematics classes.

At the final meeting of the group, six themes were identified as a means of capturing the ideas and issues which had arisen for discussion over the course of the four group meetings:

1. Time
2. Syllabus and assessment
3. Support for teachers engaging with the syllabus
4. Teaching and learning
5. Professional development
6. External influences.

Each of these themes is considered in more detail in the next section of the report.

Following the final meeting of the group, an initial collation of the main points arising under each theme was prepared and circulated. Following feedback, a draft report was prepared for the group to comment on. The report was then finalised and re-circulated to the group.

3. Themes emerging from group discussions

3.1 Time

The time demand presented by the revised syllabus—with its emphasis on developing student understanding of concepts and their ability to apply knowledge and skills in solving problems—arose frequently in the discussions. This is particularly an issue at Leaving Certificate higher level. The design of the examination paper that requires candidates to answer all questions also contributes to the time demand inasmuch as it requires students and teachers to cover the course. While acknowledging that the need to fill in background knowledge and skills for students who had not experienced the new approaches and topics in junior cycle contributed to the time demand in the initial years, consideration may need to be given to a reduction in syllabus content if the aim of ensuring greater understanding of mathematics is to be achieved. Teachers have reported instances of additional class periods being required in order to 'cover the course', allied to the pressure to focus on exam preparation so students can cope with the changed style of questions, especially those of a problem-solving nature.

A timeline or timing guide would be helpful. In order to develop guidance of this nature, feedback from the schools is needed on their experience of how long different syllabus topics/sections take with different classes and levels. The prior learning that students bring to a lesson or set of lessons and their progress with learning is unique to each class group, which further complicates the task of providing a definitive guide on time allocation to topics. There is a tension between allowing time for student engagement in activities linked to the new approaches and emphasis, and the time needed to ensure that essential mathematical knowledge and skills are being developed. Possible ways to manage time include a 'flipped classroom' approach, where activities that have a high time demand could be undertaken as preparatory or follow-on homework, allowing class time to focus on the underlying concepts and principles. The use of technology to support such activities is also worth considering.

Recommendations

The Leaving Certificate syllabus should be reviewed in light of the experience in schools over a complete cycle. This opportunity will arise following the examinations of 2015.

Evidence-based timing guides, outlining the composition and prior learning experience of particular class groups and levels, should be made available to assist in lesson and course planning. Consideration should be given to providing examples of student work which illustrate how specified learning outcomes are being achieved.

3.2 Syllabus and assessment

The presentation of the syllabus in the form of learning outcomes has given rise to some uncertainties for teachers used to lists of content, as presented in the previous syllabus. The examination papers over a period of years provided exemplification of the standards of knowledge and processes required. In the revised syllabuses, there is a greater focus on developing student understanding of concepts as well as their ability to apply their mathematical knowledge and skills in both familiar and unfamiliar problems and contexts. The separation of foundation-level Leaving Certificate Mathematics from the other two syllabus levels and the inclusion of a description of topics, which arose from the work of the foundation level group established to consider this issue, is seen by teachers as particularly helpful.

Many of the queries about the syllabus reflect a general tendency to view the syllabus through the lens of the examinations. Participants reported concerns from teachers that they feel the examination papers are not fully aligned with the syllabus and, in some cases, questions seemed to extend what the syllabus requires. This perception is related to the uncertainties mentioned previously and to the inclusion of problem-solving questions of an unfamiliar nature in the ‘contexts and applications’ section of the examination paper.

Concern was also expressed about what many teachers and students seemed to believe is a requirement to learn a lot of definitions, and that this perception was having a detrimental effect on teaching and learning, with students engaging in rote learning to an extent that is neither intended nor desirable. Since only one learning outcome in the syllabus requires students to give a definition, it would seem that an appropriate distinction is not being made between explaining and defining. Students should understand terminology that arises in the course of their study; asking them to explain in their own words the terms they have encountered and the concepts they have studied is one way in which such understanding can be tested appropriately. Nonetheless, the perception that there is a requirement to rote-learn many definitions or explanations is of itself a cause of concern and should be addressed.

Another area of concern expressed by teachers was the requirement in some examination questions to use a particular solution strategy, while the syllabus expects students to be able to choose suitable strategies to solve problems: numeric, algebraic, graphical or mental. They believe that the requirement to use a specific strategy does not reflect the variety of approaches and methods that the syllabus seems to suggest or allow, nor the ideas and processes which are contained in the resources which have been made available. As an example, the use of ‘hence’ in an examination question has the effect of narrowing solutions to one required strategy, whereas the use of ‘hence, or otherwise’, would allow alternative strategies to be employed. While a directive to use a particular approach arises infrequently in examination questions, in some instances the task can be properly narrowed in order to assess particular skills or procedures specified in the syllabus or to require that previous work be continued. If the approach to be adopted is always left open to candidates, there is a risk that only one approach will be considered in class and used in the examination, thereby depriving students of experience with a range of approaches as required by the syllabus. Nonetheless, it is clear that teachers would prefer to see the choice of solution method left open in all cases.

Other points of discussion related to the format of the examination papers, including increased use of contexts, and the breakdown of marks allocation. There is still a degree of uncertainty about the length of the written answer required; some teachers considered that this is not always clear from the space allowed for answering and that more guidance was needed. It was pointed out that, generally, additional answer space is provided at the back of the booklet and extra sheets are also available in the examination centres, although in recent years candidates have used this extra space only when offering additional solutions.

Recommendations

While recognising that there will be instances where it is appropriate to direct students to carry out a task in a particular way, the examinations should seek to maximise the opportunities for candidates to select their own appropriate strategies for solving problems.

In light of the need for questions testing the new syllabus to be more text-heavy than was traditionally the case in mathematics examinations, care should continue to be taken to ensure that the text used in questions is as simple, straightforward and concise as possible in the context of the learning outcomes being tested, and that it is appropriate to the examination level.

Teachers should receive further guidance on the importance of developing in candidates a good understanding of concepts and terms through discussion and application, so that students will be confident that they can explain the terms they use and the concepts and processes they have studied. Teachers would benefit from supplementary material or examination-style questions exemplifying how such understanding can be assessed and the standard that is expected at the different syllabus levels.

3.3 Support for teachers engaging with the syllabus

Much of the discussion at each meeting centred on this main theme, with a number of practical proposals being explored, developed and adapted. Some of these related to the syllabus itself, some to resources which are currently available on the NCCA and the PMDT websites, while others are seen as being noted for consideration as part of the revised specification for mathematics which will arise under the Junior Cycle Developments.

Syllabus documents

While accepting that the current syllabus documents will apply for the next two years, an improvement in the presentation of the syllabus would lead to greater clarity for teachers and facilitate teacher planning of lessons. The links between different topics/strands in the syllabus are not always obvious. The progression from junior cycle to senior cycle and the lines of distinction between ordinary level and higher level—particularly where the same learning outcomes apply at both levels—are sometimes unclear.

In the discussions about the syllabuses, suggestions were made in relation to

- making more visible the connections within and across syllabus strands and topics
- drawing attention to the kinds of skills which are both developed by and support the study of these topics
- having the syllabuses in an accessible format which facilitates planning; for example, using a numbering/reference system for the different learning outcomes which is both consistent and convenient and making the syllabuses available in a usable text format which would allow teachers to develop their own documentation
- indicating through exemplification how specified learning outcomes are being achieved; this would also provide clarity for teachers about what achieving the same learning outcome represents at different syllabus levels.

As a short-term measure, the FAQ section of the NCCA website could provide information on the 'Big Ideas' in mathematics—such as algebra, functions, calculus—and also address specific concerns though illustrative questions and answers. An online version of the syllabus could facilitate the use of embedded links to highlight connections between topics as well as to tag relevant teaching and learning resources. Examples of lesson ideas and/or student work to illustrate the kinds of standards expected could also be referenced in an electronic version of the syllabus—something which is not possible with hard-copy documents due to their 'flat' nature.

A model of how the syllabus might be re-presented was developed and refined. This involves a referencing/numbering system which would allow teachers to navigate and access syllabus information more efficiently, as well as a layout which provides clearer grouping of similar or related statements and learning outcomes. Appendix 2 illustrates how this might look when applied to one section of Strand 3 (Number) at Leaving Certificate.

Development of a composite online resource

A further discussion took place around the potential for combining this syllabus re-formatting approach with generating a teacher's guide which would link particular topics in the syllabus with available resources, whether on the NCCA website or the PMDT website.

The PMDT are in the process of updating reference material on their website. A listing of resources on the NCCA website has been compiled, which includes reference to relevant syllabus sections. It is proposed to convene a small working group of NCCA/PMDT personnel to explore how a more comprehensive reference resource might be generated which would allow teachers, when planning lessons, to easily access specific learning outcomes in the syllabus and relevant support material such as Teaching and Learning Plans, examples of student work, or suggested/recommended activities to support learning.

Recommendations

Explore the feasibility of using the curriculumonline approach for the LC Maths syllabus, maximising the features of an online environment to identify the connections within and across the strands and to provide exemplification of standards. The NCCA should also consider developing a curriculumonline resource for JC Maths ahead of the proposed schedule for introduction of the specification under the Junior Cycle Developments.

Re-design and re-format the LC Maths syllabus to make referencing the learning outcomes more convenient and to provide a clear sequence and progression of sub-topics across ordinary and higher level.

Develop and publish additional 'Big Ideas' materials and expand the FAQs on the NCCA website to help clarify areas of syllabus uncertainty for teachers.

The NCCA and the PMDT should collaborate to develop a composite reference resource that associates the various support materials with the relevant syllabus strands and learning outcomes. Consideration should also be given to using one site for all support materials so that what has been developed remains available in the longer term.

3.4 Teaching and learning

While much of the initial professional development for teachers of mathematics focused on individual strands and topics, due mainly to the phased nature of the syllabus changes, later workshops drew attention to making connections across the strands. By engaging with the ‘Big Ideas’ about mathematics education, as mentioned above, teachers can be supported in maximising the linkage between syllabus strands and topics when adopting and refining changed approaches to teaching and learning in their classes. However, teachers still need support in developing and using appropriate tasks that exploit the connections between topics rather than planning for individual lessons that focus on isolated areas of mathematics in a linear fashion.

The changed approach and emphasis in teaching and learning requires more detailed planning for, and integration of, knowledge and skills development. Teachers would benefit from support in the form of teaching and learning suggestions for topics which illustrate how the connections across the syllabus strands can be managed. This could be a focus for the next phase of professional development for mathematics teachers. Ideas for tasks which are seen as effective in promoting student development of thinking and problem-solving skills would be welcome. Annotated examples of student work would assist teachers with both the initial planning of lessons and the evaluation of student progress.

The issue of the time demanded to ensure development of student understanding also has implications for planning the sequence and pace of learning. Evidence-based timing guides which reflect the make-up and prior learning experiences of differing class groups would assist a school maths department to plan more effectively. Encouragement for schools to adopt a flexible approach to timetabling would allow for opportunities to engage in a more in-depth treatment of topics where this is required or desirable.

Annotated examples of student work would help teachers to develop their capacity to assess the level of conceptual understanding being achieved by students as well as the opportunity to identify the kinds of misconceptions that can arise—which may impede the student’s progress and development in later topics. For example, algebra is seen as a significant topic in its own right, but also as underpinning many other topics and strands in mathematics. A firm foundation for developing both knowledge and skills in the area of algebra is laid down in learning about number, number operations, number patterns, etc. Teachers and students would benefit from knowing why they do a particular activity and how that learning fits with the continuum of conceptual development.

Recommendations

Develop ‘Big Ideas’ documents which outline the continuum of key concept development and the end point of this continuum for different levels in a post-primary context.

Provide support for teachers in using the ‘Big Ideas’ to maximise the linkage across syllabus strands and topics, and in developing teaching and learning strategies that enable students to understand and appreciate the interconnectedness of different areas of mathematical knowledge and skills.

Develop a range of topics and examples with approaches that will promote conceptual understanding, support the development of key knowledge and skills, and allow the teacher to gain insights into, and address, student misconceptions. These could also help to highlight connections between topics.

3.5 Professional development

The current programme of workshops for mathematics teachers is coming to a conclusion, but support will continue to be provided by the Project Maths Development Team. The Professional Diploma in Mathematics for Teaching affords opportunities for out-of-field mathematics teachers who successfully complete the diploma to meet the Teaching Council’s requirements for registration as a mathematics teacher. However, teachers who already hold a recognised qualification in mathematics would also benefit from further study in the area of mathematics education.

Design-based research and reflective practice offer opportunities for teachers to engage in continuing professional development and this should be encouraged. While the Professional Diploma in Mathematics for Teaching, which is currently funded by the DES and offered by UL, enables out-of-field teachers of mathematics to improve their mathematical and pedagogical content knowledge, the group felt that other teachers would benefit from a third level qualification in maths pedagogy. The use of design-based research as a focus for professional development could enable teachers to gain a third-level post-graduate qualification, ideally at Masters level. Such research would be extremely beneficial for mathematics teachers and could contribute significantly to improving maths pedagogy in Ireland. A post-graduate course along these lines could be run by one or more institutions through local education centres.

Recommendations

Explore the potential of design-based research as a focus of continuing professional development for mathematics teachers, including its potential for achieving a post-graduate qualification at Masters level.

The next phase of the NCCA research grant scheme might consider inclusion of teachers of mathematics, so that their research experiences could inform policy in relation to developing effective teaching and learning practice in mathematics.

3.6 External influences

The syllabus and assessment changes under *Project Maths* and associated developments have contributed to increased pressure on teachers. There is a tension between parents' and students' expectations on the one hand—mainly in relation to achievement in the examinations—and the expectations of the 'system' on the other hand, in terms of student development of knowledge and skills in mathematics. Media attention to mathematics, and particularly in relation to *Project Maths*, has added to the pressure on teachers to 'cover the course' and to ensure that students are well prepared for the examinations.

The change in emphasis which has accompanied the changes in the syllabus and its assessment was not sufficiently explained to students, parents and the wider public. Students still expect to be shown how to solve problems and then given additional work of this kind so that they can practise their skills. Despite the changed nature of the examinations, there remains an expectation that every possible type of problem can be experienced in class so that there is no 'surprise' in the examination papers. This has added to the time demand on teachers, in a situation where the syllabus has not been reduced to any significant extent. At the same time, there is criticism in relation to the inclusion of some new topics, such as in statistics, at the expense of others, such as parts of the calculus section, which have not been retained.

A reconceptualization of mathematics teaching and learning is needed so that there is a clear understanding of the changed emphasis and approach brought about by the *Project Maths* initiative and how this is reflected in the changed style of the examination papers. Students are no longer engaged in a teacher-centred endeavour where they cover a fixed body of knowledge in a linear fashion; instead, students are challenged to engage with an interconnected body of ideas and reasoning processes, and to work collaboratively with their teacher and their peers in developing their mathematical knowledge and skills.

The awarding of bonus points for achieving a minimum of a D grade in higher-level Leaving Certificate Mathematics has indeed increased the numbers of students at this level, but it has also increased the challenge for teachers in terms of course completion and the pace of learning in class where there is a greater diversity of ability than was the case in the past.

4. Conclusion

In the course of the discussions by the Maths in Practice group, a number of other developments took place. The final report of the NFER research into the impact of *Project Maths* on student achievement, learning and motivation was published, as were the OECD and national reports on PISA 2012. A background paper on the review of the Applied Mathematics syllabus at Leaving Certificate was being prepared, which takes into consideration the mathematics syllabus changes already in place under *Project Maths*. The last of the ten scheduled professional development workshops for teachers of mathematics is currently under way.

The NFER research report drew attention to aspects of the initiative which have not yet come to full fruition:

- Although there is evidence that students are engaging in activities associated with the revised syllabuses, more traditional approaches continue to be widespread.
- It is possible that teachers are currently emphasising the content of the revised syllabuses rather than the processes promoted within it.
- Students are demonstrating mastery of mathematical procedures, but there is less evidence of problem solving and making connections between mathematical topics.

While the overall performance of Irish students in PISA 2012 was significantly above the OECD average, the scores of higher-achieving students did not differ significantly from the corresponding OECD average. In the four content area subscales, Irish students obtained significantly higher mean scores on Change and Relationships, Quantity, and Uncertainty and Data subscales compared to the OECD average scores; however, they performed significantly less well on the Space and Shape subscale.

In order to progress the recommendations contained in this report, the NCCA, the PMDT, the DES Inspectorate and the SEC should collaborate in planning the steps and resources necessary to achieve them. A design/planning team, along the lines of the group which was established for the development of the programme of workshops, may be an effective means of ensuring a coordinated approach.

As the revised mathematics syllabuses bed in and students progress to Leaving Certificate having experienced the changed syllabus in the junior cycle, relevant recommendations contained in this report can be taken into consideration as part of a planned review at Leaving Certificate and also in the development of the mathematics specification under the Junior Cycle Developments.

Appendix 1: Membership of *Maths in Practice* group

Mr	Robert	Chaney	ASTI
Ms	Ann	Piggott	ASTI
Mr	Jerry	McCarthy	ASTI
Mr	Paddy	Flood	IMTA
Ms	Catherine	Kierans	IMTA
Ms	Pauline	Gallery	TUI
Mr	Ciaran	Duffy	TUI
Ms	Imelda	Moloney	TUI
Mr	Tom	O'Connor	SEC
Mr	Seamus	Knox	DES
Ms	Sheelagh	Clowry	PMDT
Ms	Aoife	Kelly	NCCA
Ms	Rachel	Linney	NCCA
Mr	Bill	Lynch	NCCA

Appendix 2: Illustration of a more useful syllabus layout

Strand 3: Number – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
3.1 Number systems	– recognise irrational numbers and appreciate that **R** ≠ **Q** – work with irrational numbers – revisit the operations of addition, multiplication, subtraction and division in the following domains: • **N** of natural numbers • **Z** of integers • **Q** of rational numbers • **R** of real numbers and represent these numbers on a number line – investigate the operations of addition, multiplication, subtraction and division with complex numbers **C** in rectangular form a+ib – illustrate complex numbers on an Argand diagram – interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate – develop decimals as special equivalent fractions strengthening the connection between these numbers and fraction and place-value understanding – consolidate their understanding of factors, multiples, prime numbers in **N** – express numbers in terms of their prime factors – appreciate the order of operations, including brackets – express non-zero positive rational numbers in the form $a \times 10^n$, where $n \in \mathbf{Z}$ and $1 \leq a < 10$ and perform arithmetic operations on numbers in this form	– geometrically construct √2 and √3 – prove that √2 is not rational – calculate conjugates of sums and products of complex numbers – verify and justify formulae from number patterns – investigate geometric sequences and series – prove by induction • simple identities such as the sum of the first n natural numbers and the sum of a finite geometric series • simple inequalities such as $n! > 2^n$, $2^n \geq n^2$ $(n \geq 4)$ $(1+x)^n \geq 1+nx$ $(x > -1)$ • factorisation results such as 3 is a factor of $4^n - 1$ – apply the rules for sums, products, quotients of limits – find by inspection the limits of sequences such as $\lim_{n \to \infty} \frac{n}{n+1}$; $\lim_{n \to \infty} r^n$, $\|r\| < 1$ – solve problems involving finite and infinite geometric series including applications such as recurring decimals and financial applications, e.g. deriving the formula for a mortgage repayment – derive the formula for the sum to infinity of geometric series by considering the limit of a sequence of partial sums

Strand 3: Number – Ordinary level and Higher level

Students learn about	Students working at OL should be able to	In addition, students working at HL should be able to
3.1 Number systems (continued)	– appreciate that processes can generate sequences of numbers or objects – investigate patterns among these sequences – use patterns to continue the sequence – generalise and explain patterns and relationships in algebraic form – recognise whether a sequence is arithmetic, geometric or neither – find the sum to *n* terms of an arithmetic series	

This example is based on the current Leaving Certificate Maths syllabus. The illustration shows a section of the syllabus as it is currently set out, while that on the opposite page shows how the same text might appear in a re-formatted and number-referenced layout which facilitates grouping of related learning outcomes at ordinary level and higher level, as well as internal reference to other sections of the syllabus.

3.1 Number Systems

Students working at OL should be able to:	*In addition, students working at HL should be able to:*
a) recognise irrational numbers and appreciate that $R \neq Q$	
b) work with irrational numbers	c) geometrically construct $\sqrt{2}$ and $\sqrt{3}$ d) prove that $\sqrt{2}$ is not rational
e) revisit the operations of addition, multiplication, subtraction and division in the following domains: a. $\boldsymbol{N}$ of natural numbers b. $\boldsymbol{Z}$ of integers c. $\boldsymbol{Q}$ of rational numbers d. $\boldsymbol{R}$ of real numbers and represent these numbers on a number line	
f) investigate the operations of addition, multiplication, subtraction and division with complex numbers $\boldsymbol{C}$ in rectangular form $a+ib$	
g) illustrate complex numbers on an Argand diagram	
h) interpret the modulus as distance from the origin on an Argand diagram and calculate the complex conjugate *- see also Strand 4, section 4.4*	i) calculate conjugates of sums and products of complex numbers *- see also Strand 4, section 4.4*
j) develop decimals as special equivalent fractions, strengthening the connection between these numbers and fraction and place-value understanding	
k) consolidate their understanding of factors, multiples, prime numbers in $\boldsymbol{N}$	
l) express numbers in terms of their prime factors	
m) appreciate the order of operations, including brackets	
n) express non-zero positive rational numbers in the form: $a \times 10^n$, where $n \in \boldsymbol{Z}$ and $1 \leq a < 10$ and perform arithmetic operations on numbers in this form	

3.1 Number Systems (continued)

Students working at OL should be able to:	*In addition, students working at HL should be able to:*
o) appreciate that processes can generate sequences of numbers or objects	
p) investigate patterns among these sequences	
q) use patterns to continue the sequence	
r) generalise and explain patterns and relationships in algebraic form	s) verify and justify formulae from number patterns
t) recognise whether a sequence is arithmetic, geometric or neither	
u) find the sum to n terms of an arithmetic series	v) investigate geometric sequences and series w) prove by induction: ▪ simple identities such as the sum of the first n natural numbers and the sum of a finite geometric series ▪ simple inequalities such as: $! > 2^n$ $2^n \geq \ ^2 \quad (n \geq 4)$ $(1+x)^n \geq 1+nx \quad (x > -1)$ ▪ factorisation results such as 3 is a factor of $4^n - 1$ x) apply the rules for sums, products, quotients of limits y) find by inspection the limits of sequences such as: $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$; $\lim_{n\to\infty} r^n$, $\lvert r \rvert < 1$ z) solve problems involving finite and infinite geometric series including applications such as recurring decimals and financial applications, e.g. deriving the formula for a mortgage repayment *- see also section 3.3* aa) derive the formula for the sum to infinity of geometric series by considering the limit of a sequence of partial sums

NCCA

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Project Maths: Reviewing the project in the initial group of 24 schools – report on school visits

Contents

1. Introduction and context

Project Maths commenced in an initial group of 24 schools[1] in September 2008. Over 200 schools had applied to participate in the project and the initial group of 24 schools is reflective of the range of all post-primary schools. Changes to mathematics syllabuses and their assessment at both Junior Certificate and Leaving Certificate were phased in with these schools over a three-year period beginning in September 2008, with associated changes to the examinations commencing in 2010 (LC) and 2011 (JC).

Teachers of mathematics in the 24 schools have been supported through professional development workshops conducted by the Project Maths Development Team (PMDT) of Regional Development Officers (RDOs) and through complementary evening courses, with school-based support from the RDOs over the same period. The PMDT developed a range of teaching and learning support materials for teachers and students, which are published on their website (www.projectmaths.ie). The NCCA also developed student resources for the initial school group and these are now available to all students on the updated Project Maths pages of the NCCA website (www.ncca.ie/projectmaths).

A series of ten workshops, to which all maths teachers in the 24 schools were invited, focussed on the changed teaching and learning approaches advocated under Project Maths. Attendance at these workshops was consistently high – often in the 90%+ range. The workshops used specific topics from the different syllabus strands to illustrate a more investigative approach to teaching, learning and assessment and to emphasise the development of student problem-solving skills. The changed emphasis and approach to teaching and learning were also reflected in the examination papers for successive cohorts of students.

The complementary courses, including a series of summer courses by the National Centre for Excellence in Mathematics and Science Teaching and Leaning (NCE-MSTL), addressed areas of mathematics content where the need for support had been identified by teachers and/or the support team. Over 2,000 of almost 6,000 maths teachers nationally attended the complementary evening courses, which were held in local Education Centres. Each year, close to 100 of the approximately 230 teachers from the 24 initial schools, attended the summer courses held in 2009, 2010 and 2011.

[1] In preparation for its amalgamation in September 2011 with another school that was not one of the initial group of schools, Abbey Community College, Wicklow aligned itself with the national roll-out schedule from September 2010. Nonetheless, maths teachers from the former Abbey Community College were included in this review exercise.

Over the period of the project to date, the NCCA has had limited direct contact with the initial 24 schools. The main contact has been the RDO team, who visit the schools on a regular basis to provide individual and group support to maths teachers. Following the selection of the 24 schools in May 2008, regional meetings were held for maths teachers in these schools to outline the project and the planned programme of support. In December 2008 NCCA held a meeting of the 24 school principals to discuss progress and to identify particular issues that needed to be addressed. In May 2009 NCCA convened a meeting for principals/deputy principals and one or two maths teachers from each school, at which the feedback obtained through a teacher questionnaire was presented and discussed. This feedback resulted in adjustments to the initial syllabus drafts for the following year. In subsequent years, NCCA contact with the schools has mainly been in the form of regular Information Notes for teachers, and attendance by NCCA personnel at a selection of workshops in the different regions.

Now that the maths teachers in these schools have completed the set of ten workshops, and some changed syllabus strands have been through a full cycle at both Junior Certificate and Leaving Certificate, the opportunity was availed of by the NCCA to renew direct contact with the 24 schools in order to get teacher feedback on their experience of the process of change that Project Maths introduced and the impact it is having on their teaching practice.

This report presents the feedback that teachers gave in the series of school visits by NCCA personnel to the 24 schools in the period December 2011–January 2012.

2. Meetings with maths teachers

2.1 Arranging and conducting the meetings

The arrangements for the school visits were agreed with each school and the relevant RDO. The December Information Note (see appendix, page 26) to the schools outlined the purpose of the meetings with teachers and the main focus points:

- the impact of Project Maths on their practices as a maths teacher
- the impact of Project Maths on the school's maths department
- the impact of Project Maths on their students' experiences of maths.

By agreement with the schools, these meetings were limited to two class periods. While it was not possible to meet with all of the maths teachers in each of the schools, as many teachers as possible were included in the meetings that took place. In some schools, which had a small number of maths teachers, all of them attended the meeting. In a few schools, two meeting sessions were arranged so that disruption to class schedules was minimised. While principals were invited to attend the meetings, none did so. In a small number of schools the deputy principal, who was also a maths teacher, attended the meeting. In all, over 150 maths teachers attended the school-based meetings in the 24 schools.

Each of the review meetings was conducted by one of the NCCA's Project Maths personnel, who was accompanied by the RDO who normally works with that school. Teachers were assured of the confidentiality of the process; no one teacher or school would be identified or associated with particular comments. A series of question prompts was used to which the teachers were asked to respond and their responses were recorded on flip-charts. Teachers were free to amend or elaborate on the recorded responses, or to re-visit earlier responses in light of later discussion. They were informed that their responses would be collated and analysed, with feedback to be given to the schools at a general meeting later in the year. They were also informed that a report on the feedback from the teachers would be presented to the Council of the NCCA and would inform any further refinement of the syllabuses being finalised for national roll-out in September 2012, when all five strands would be in place for all schools.

2.2 Focus questions

The main areas of focus during the school visits were as follows.

1. **The impact of Project Maths on teaching, learning and assessment practice in mathematics classes**
   Teachers were asked to consider their classroom practices prior to the syllabus changes and at present, and also to consider what elements of practice they saw, and now see, as most valuable. They were asked to consider what forms of assessment they previously used, and now use, to assess student learning and progress in mathematics.

2. **The tools and resources that teachers find most beneficial**
   Having identified these resources, teachers were asked to elaborate on why they valued them.

3. **The impact of Project Maths on the school's maths department**
   As with individual teacher practices, they were asked to consider the functioning of the school's mathematics department (i) prior to the syllabus changes and (ii) at present and also to consider what aspects of the maths department and its role they saw, and now see, as most valuable to them as teachers.

4. **The impact of Project Maths on the student experience of maths**
   Teachers were asked, from their perspective, to identify the most significant change for their mathematics students as a result of Project Maths.

In addition, teachers were invited to give feedback on the syllabuses, focusing on strands 3, 4 and 5 (dealing with number, algebra and functions respectively). To facilitate written feedback on this item, a template was included as part of the Information Note sent to schools in advance of the visit (see Appendix, page 26).

3. Collating the responses, identifying themes

When all of the visits were completed, the sets of teacher responses were reviewed to identify emergent themes across the various focus points. The process of analysing and collating the feedback is set out below, together with the main themes to emerge from this feedback.

3.1 Thematic Analysis

The analysis of the feedback from teachers in the 24 schools was undertaken independently of the NCCA team involved in conducting the meetings. All responses gathered from all sessions in the 24 schools were reviewed through a process of data coding. First order coding was a descriptive level of coding and involved organising and categorising the views expressed by the teachers. Since, in many instances, teachers' responses related to more than one of the focus points, second-order coding was used to combine the descriptive first-order codes into meaningful super-ordinate codes. Finally, overarching themes emerging from the data were identified.

3.2 Overarching themes

Theme 1 – New roles

Across the board, there was recognition from teachers that participation in Project Maths calls for a change in the roles of the teacher and the student.

> *We taught in the same way we were taught at school, and now it's different.*
>
> *It was book led – all about ticking off chapters; rote learning, with the result tested in the exam.*
>
> *You just told them that's the rule and they have to learn it.*
>
> *Before, teaching was very much exam focused.*

Teachers recognised that the student now needs to be a more active learner, becoming involved in activity and discovery learning through new classroom practices such as group work, questioning and discussion. However, teachers reported struggling with this new role, which requires using a new skill set and a new set of classroom practices to enable learning for their students.

> *I am uncomfortable about this new role, there is an unknown.*
>
> *You are now a facilitator of learning as opposed to a giver of knowledge - I struggle with that.*

Teachers voiced fears that their own lack of confidence with the new approaches under Project Maths is picked up on by students.

> *I had more confidence – I knew the full story; the exam reflected teaching and there was predictability for both student and teacher.*
>
> *Now, students have less confidence in that they don't know what is expected of them in the exam.*

Some teachers described how students are gaining a different type of understanding in this new learning environment; this is especially true for the more junior classes. However, the feedback from the meetings indicates that, as exams approach, students and teachers value the old ways and there is a pressure to ignore their new role and to revert to previous exam preparation techniques rather than focusing on learning.

> *In the long run, it is a positive process and kids can see the relevance of maths to their lives. It was hard at the start, but once everything settles down I wouldn't go back.*
>
> *It used to be very easy to prepare for the exams using repetition of practice and exam-style questions. We miss the comfort of past papers.*
>
> *In sixth year maths you are pressurized and fall back into the old style of teaching under pressure; the reality is that there is a Leaving Cert that determines children's futures.*
>
> *I am trying more to teach the maths, but at certain times of the year it's just the exams that I concentrate on.*

Theme 2 – Supporting the changed approach; using resources

Teachers emphasised that they need support and resources to assist them in developing the skills and knowledge required for their new and different role. They are learning and developing and there is evidence of their being at different stages on this learning continuum; some have undergone major changes, some are still at the start of this development. As they learn, confidence grows.

> *I have so much learning to do (teacher doing HL after a gap). I'm still not in the comfort zone with strands 3, 4 and 5.*
>
> *My methods have totally changed since Project Maths came in.*
>
> *We're going a little slower. I find we're constantly changing because it might work with one group and not another. I find we need to use different methods for different groups; this is the way it should be, not making one method fit all.*
>
> *I have had to look at maths myself and it's made me improve my teaching practice; it's made me think outside the box.*

It is clear that teachers value and need support during this period of learning and development. However, their comments suggest that over-dependence on various forms of support may become an issue. As more resources become available there is a danger that teachers may lose sight of how intrinsic they are to the change.

> *There are so many (resources) but they're a bit all over the place. The problem is they are coming in dribs and drabs. (We need to) have resources together digitally in strands.*
>
> *All the resources are great but (students) still expect us to have all of the stuff in paper in front of them. The RDO is the saving of us.*
>
> *We struggle to be able to make up questions ourselves; we need the resources of past papers.*
>
> *I can't wait for the definitive book to be produced. I don't care how big it is, I want it.*

Teachers cited collaboration as being valuable to them. They reported that within schools and maths departments there is much more collaboration and support between colleagues than before. It was noted, however, that much of this 'team' work takes place in informal collaborations and that collaborations need to be planned for and supported by the school administration. The issue of time for planning was a recurrent theme and will be dealt with in more detail later in this report.

There is more interaction between teachers and the focus is on maths approaches.

Before, everyone was king of their own castle; now everyone depends on each other.

Sharing means we are learning how to change and it gives us different insights.

Project Maths emphasises student understanding of concepts. There is evidence that teachers need support in making connections within mathematics. They recognise this ability to connect as an advanced skill that only develops after a period of immersion in teaching with the revised syllabuses. It is apparent that being confident in maths supports this identification of connections and their efficient use.

I didn't tend to link topics, but I see that there are more connections between all the strands now.

(Identifying) linkages between strands is challenging, as we didn't really see the linkage at the beginning. We learnt it as we went through and this makes teaching it more difficult.

I am starting to make connections across strands and able to say 'do you remember that we did this at (a certain time)?'

More time is spent on linking different topics together now.

I've learned loads. I never thought of linking slope the way I do now.

I put greater emphasis on several approaches to solving a problem rather than on a singular approach.

Teachers reported that engaging with Project Maths had a positive impact on their teaching approaches in other subject areas. There was a concern from teachers whose first subject was not maths that they were not familiar enough with the maths to teach in the new way.

Business is my subject (but) I don't have the subtleties needed for maths. I have to think about it and engage lots with the material to get that confidence. I am a better business teacher now because of my experience with Project Maths.

There is evidence that the syllabus is now seen as a useful resource and some teachers are becoming less dependent on text books. However, they report that their students value the textbook and they feel under pressure to use the text books in class.

We read the syllabus (now). We have more understanding of the syllabus.

You have a copy of the syllabus in your back pocket now. Before, the book was the syllabus and you followed that.

Trouble is, they (students) respect the book, not worksheets.

They are used to having a book and perceive that it is not a real lesson without it. Having a textbook is a serious advantage – good kids are used to it from other subjects and weaker kids get structure from it.

Some comments point to the need to provide support in interpreting a syllabus now that it is written in terms of learning outcomes.

We had a syllabus that made sense, we didn't have these statements. I don't know the syllabus well and I'm not able to direct students to exams.

The syllabus is too vague; we may be over-teaching some topics and not emphasizing enough in other areas. The (text)book was easy – this isn't.

When asked about the resources they have used and valued teachers cite practical activities, resources and equipment that they have developed themselves or have been provided with through Project Maths. They also commented that having all the necessary materials and resources was challenging.

We need to use hands-on, open-ended activities.

The T& L plan is good – (doing) probability by making a game; statistics through doing their own surveys and presentations.

We did a school census. I used a newspaper article about statistics as a discussion.

(I'd) like to be in a position to have a set of maths equipment in the class; it's very useful to have and getting them is difficult. If we had these on a shelf it would be very helpful. Maths classes need resources like other practical subjects: dice, cards, spinners, geostrips, probability kits; this must be taken seriously.

In almost every school, teachers reported an increased use of IT and this too is valued.

IT is valued because, for example, geogebra allows visualization and very quickly aids understanding. It is good for constructions, graphs, and seeing what differentiation does.

(Students) are used to technology. It brings the maths home to them and they want to do it themselves.

The internet is great, but it is time consuming to get 'tailored material' – one size doesn't fit all and you have to rework it for different classes. It takes a while to gather a portfolio of resources.

Some teachers report that they find group work useful.

In group work you encourage students to learn themselves, making mistakes.

Highly motivated students benefit more from the approach of group work and open-ended activities.

Other teachers report on the challenges that they experience with group work.

In some classes discipline is a problem and this means you can't do group work and things like that.

There is not enough time for it (group work).

(The) effectiveness of group work depends on the size of the group, the ability/nature of group and the motivation of the group.

There's a higher percentage of demotivated kids in OL so I think group work is not suitable with these kids.

Teachers report that they are using questioning and discussion more frequently but, like group work, learning how to use this new learning methodology effectively is a challenge.

It's very difficult to get them to engage; they're not comfortable about sharing solution strategies; some are afraid to give their own opinions.

Sometimes I am met with silence and I end up answering it myself.

No time for this. I can't spend hours on a question and if you can't keep them on task it's disruptive and you'd lose a lot of them.

Questioning gives them an understanding of the vocabulary…..It's brilliant, you know from that that your message has got across.

Teachers had mixed opinions on the value of using open-ended tasks and activities to support learning. Some reported that using open-ended tasks is time consuming and there is evidence that, once again, teachers and students need time to learn how to use these effectively. Some see open-ended tasks as an add-on activity, not a core teaching methodology.

Open ended tasks and activities take time to prepare. It frustrates the hell out of them; depends on their ability ….. weak (students) can't break it into pieces.

(Open-ended tasks) perceived as a doss class by some kids who complain that teachers are not explaining it.

Better able kids can come up with strategies.

With problem solving there are lots of ways of skinning a cat – some of them only want one way of doing something.

Many teachers cite pressure from, and the dominating presence of, the terminal examination (Leaving Certificate) as inhibiting them from using the more student-centered methodologies such as open-ended tasks, discussion and group work.

Ultimately, (the) written exam is the thing, they have to do one – very bright kids answer questions very well in class but fall down in exams.

In class you can use questions to open up a problem but in the exam they need to be able to do this themselves.

I am trying more to teach the maths but at certain times of the year it is just the exam; we want to approach things in Project Maths style but we fail under pressure of exam structures.

Teachers report school issues such as classroom layout, maths-based classrooms and timetabling as being important to them in supporting the changes.

Theme 3 – Issues of assessment

It is clear from comments by teachers that there is still a heavy reliance on 'tests' as a way of assessing learning. There is evidence that teachers need help and support in developing new and trusted ways of assessing, adopting an approach which is reflective and focused on learning, assessing the extent to which learning has been achieved, and refining their teaching to reflect this.

When asked about the methods of assessing student learning that they value most, teachers reported that tests, exam questions and homework were the primary ways they were assessing learning.

> *Tests – assess each individual; they (students) have to be used to a written test.*
>
> *Exam questions and end of topic exams (are) most beneficial, and give practice for the final exams.*
>
> *(Tests) allow you to test a variety of concepts. You get to see where they are going wrong.*
>
> *The only way to check homework is by giving a test.*
>
> *I put more maths questions on tests to get them used to the unexpected.*
>
> *You know it's their own work and it is the way that they will be tested. All the resources are great but the homework and questions are the bread and butter.*
>
> *(Students) don't record enough when they're doing investigations.*
>
> *(Checking homework) gives a fair idea which ones (questions) were the problem, then open forum as to what was the problem and discuss to solve the problem.*
>
> *Swapping homework, marking each other's (work).*

However, even with tests, some teachers find it difficult to adopt a changed approach in marking student work.

> *I haven't a clue how to mark tests; I still use old scheme: +3, etc. Even with mocks I can't decide where marks should be awarded, or what constitutes 'appropriate information'.*
>
> *We need an inservice on marking students' work. I would like to see how a 'fair scheme' for marking works.*
>
> *If you are trying to be innovative, you can't give predictable homework.*

There is some evidence that alternative assessment methods such as project work, assignments, open discussion in class, questions, and examination of students' work are being used, but some teachers expressed the need for support on how to use these methods effectively with all their students.

> *(I use) discussion in open forum, getting them to describe the steps to answering the question.*
>
> *They can look at a question and just think "I can't do it". Some students could sit there doing nothing...They need prompts: Explain how you got there? Elaborate on…, Why did you start with that…?*
>
> *I got my students to make their own questions and give them to each other but I didn't know if that was worthwhile.*
>
> *Individual white boards are useful, you can quiz all students at the same time – it's efficient; they get feedback straight away.*
>
> *We use a folder system; students keep all their work and get graded on it.*
>
> *I listen in on their discussion, absorbing what they are answering.*
>
> *Sites such as ixl.com for online assessment (interactive assessment) give students feedback. A lot of students don't see this as homework. They know the teachers can log in and they are competitive.*

Teachers' comments about their experiences with different approaches to assessment reveal that in some cases the thinking about the purpose of assessment is beginning to change.

> *Formative assessment – it's different, gives you different insights and you can engage with them as you move forward.*
>
> *Extra work/ revision stuff and their attempts inform you.*
>
> *Use students' work to illustrate different solutions – that is useful*.
>
> *Students presenting work and explaining (their) strategy is great because it gives them confidence, they see different ways. If I just show them it gives preference to my thinking. (This way) shows them I am not an expert. I enjoy listening to their ways.*

Teachers have concerns about how well the new teaching methodologies and assessment are supporting the diverse learning needs of their students.

> *There is too much English on the paper. I worry about foreign students and those with weak language ability to interpret questions in an exam. The lack of help and resources for these students is becoming a bigger issue.*
>
> *Assessing the ability to display understanding of maths is an issue.*
>
> *Students with SEN – if they can't get info out of the question then they are at a disadvantage.*

Teachers have concerns about the exams. Currently the exam is impacting on the new teaching and many teachers feel under pressure to revert to old style 'drill and practice' teaching and abandon student-centered, inquiry-based methodologies. Teachers voiced concerns about the length, structure and format of the examination papers and wondered whether they adequately assessed the learning on Project Maths.

> *Exam papers are too long. There is a lack of structure in the (new) exam papers.*
>
> *(Before this) you could always say if you do this and this you'll get attempt marks, but now you can't.*
>
> *I feel sorry for them when we give them Project Maths questions; they don't get a sense of reward or achievement, they are fine working through the resources but then they can't equate what they are doing with exam questions*
>
> *Project Maths needs project-based assessment…there is a need to change the way it is examined.*
>
> *The exam is unpredictable; there's no (a), (b) and (c) parts anymore.*
>
> *Some of the exam questions are unfair, you might not be able to start a question whereas before you knew there were questions that everyone could do.*
>
> *I feel I'm engaging more kids in the class, they enjoy maths more, but they're still not doing well in tests. (LC) students can problem solve but can't do papers.*

Teachers cite fear and anxiety around exams as a feature of the student experience of Project Maths.

> *5th and 6th years are fearful, it's a negative experience.*
>
> *5th years who haven't been through Project Maths can't problem solve – they find it daunting. HL 5th years who engaged with Project Maths in JC are anxious about the LC because they got negative vibes from the last 6th years and are not as confident in the teacher's ability to deliver.*

Theme 4 – Time

This theme is constant throughout the Project Maths experience. Teachers made points about time to meet and plan. They mentioned time in relation to covering the course and using problem solving methodologies. They mentioned time being needed to use different kinds of assessment.

> *Involvement, discussion and activity learning are more time consuming than 'chalk and talk'. We don't have time to explore.*
>
> *You can't afford the time for the hands-on stuff even though the kids enjoy it and get it.*
>
> *In senior cycle I don't have as much time to show them things, whereas in JC you have more time to go through the explanations, investigations and discovery.*
>
> *We need time to teach for and to develop understanding.*

Teachers reported that time pressures inhibited student-centred approaches to learning.

> *Due to the length of the course I'm teaching new material in the old way – drill and practice – it's a time issue.*
>
> *In 6th year maths you are pressurized and fall back into the old style of teaching.*
>
> *Time pressure to get the course done reduces time for questions.*

It was also recognised that it takes time to become familiar and confident with the new syllabus and teaching methods, and teachers report that they find it difficult to know how much time to spend on each topic.

> *I need to be comfortable knowing how much time is available for a topic before I am willing to do the playful stuff.*
>
> *How much time to spend on certain topics is still an issue.*

There is a perception that the syllabus is long and time consuming and that it takes teachers longer to teach the same thing. There is evidence from some teachers' comments, however, that as teachers develop their familiarity with the connections between strands they can make more efficient and effective use of their time.

> *Strands 3 and 4 (take) too long to teach.*
>
> *The HL course is too long; every day you do something new and there's no time to go over stuff.*

The new course is longer, with more material, more depth; we are being asked to teach more to a greater level of understanding.

A small section on the syllabus may take a long time to cover.

Now I spend more time linking different topics together.

There is greater emphasis on several approaches to solving a problem rather than just one way.

Cross-linking, not going chapter by chapter, and looking for different representations takes more time.

There were reports in a small number of schools that teachers are teaching exam classes outside core school hours to cover the syllabus. It was acknowledged that this has lessened as the phasing of Project Maths progressed.

I need 240 hours to come at it (the syllabus) from different perspectives.

I came in after school 2 days a week in the first year. I didn't want to encourage panic and I knew this way we would cover the course.

I'm terrified of not getting the course covered; you can't get sick or you won't get it covered – that wasn't the case before.

Teachers reported that timetabling needs to support the maths learning needs. They also emphasised the pressure that exams exert on how time is used for learning.

35 minute periods are a constraint to new practices. Field work and practical work take more time.

5 class periods are proving inadequate for 5th and 6th years, given that 3 out of 5 classes are the final class of the day.

If maths is to become hands-on it should be treated as a science (a practical subject) with 24 students per class; the ability range in classes is huge.

I am concerned about the time element, it takes more time to get a topic covered and will that be reflected in the paper? I have to finish (the course) by Feb/March because of mock exams.

No revision time any more – 6th years are going into mocks to do material they haven't done since last year.

As noted already, teachers felt strongly about the need for time to be made available for planning and collaboration.

> *I spend more time preparing and thinking about methodology.*
>
> *Maths teachers are involved in other (subject) departments, so can't always meet formally.*
>
> *We have a class a week to meet…but I know we are only getting all this meeting time because we are a pilot school.*
>
> *We're not allowed to use the Croke Park hours, we've agreed on 1hour a term for subject planning, but this is inadequate.*
>
> *We underestimate what time different methodologies require and the amount of preparatory time required, e.g. group work is more than just putting people into groups and throwing stuff at them.*

Theme 5 – Issues of change

Teachers have views about the manner in which Project Maths was introduced simultaneously in first year and fifth year. Many of them felt that the exams were unfair to their students. For some there is still a sense that they don't have ownership of the change. The manner in which Project Maths issues are dealt with in the media impacts on teachers' perceptions.

> *There has been a lot of change in a relatively short time…Maybe it should have started only with first years.*
>
> *You're always having to justify (to students and parents) anything you do that's different as people don't like change; they will blame Project Maths for not doing well in maths.*
>
> *When the first cohort of students went through the exams there should have been more consultation.*
>
> *(We feel) the feedback we gave in June about the exam was ignored.*
>
> *We feel that we're being used as a test – every other school benefits – and that we're guinea pigs.*
>
> *Some of the kids have lost faith. The experience over the first couple of years was stressful for students and still is; they are afraid.*
>
> *The exam last year upset people, especially the students; they lost confidence. It has a lot to do with papers, poor publicity in the media.*

Theme 6 – Syllabus content issues

Teachers made a number of general points about strands 3, 4 and 5 along with other aspects of course content and its assessment. These are summarized in Table 1 below. Their comments suggest that teachers need support in understanding the aims of the syllabus, how to interpret learning outcomes and the purpose of assessment.

Table 1: Specific comments made by teachers about the syllabus

Strand 3	*Generalising a quadratic relationship from a pattern is very difficult, they cannot get the formula. This turned them off patterns.* *I'm happy with identifying a pattern as quadratic and continuing the pattern…but it's a step too far to generalize this.* *Manipulating equations is nightmarish stuff when they have to do procedures – a lot of procedure has gone – they need a balance.*
Strand 4	*Strand 4 is most enjoyable, and provides lots of linking. The different syllabus levels are appropriate.* *Connections - 5th years are actually making connections between different strands and previous work.* *LC - algebra is wider. The 3 nested columns leaves a lot to cover.* *Strands 3 & 4: patterns and algebra are better connected now.* *Standard question algebra style at JC-HL, formulas, etc.* *There is a huge jump from JC-HL algebra standard of the sample paper to that required to study at LC-HL. Problem solving in algebra is a problem…kids are used to algebra as patterns.*
Strand 5	*Differentiation is too short; product, quotient and chain rules are gone, differentiation from first principles is gone, this was always a comfort to the student; they could do it.* *Calculus is a complete change; it's lovely because now it's more applicable, trig the same. Before it was all rules, rules, rules.* *I am disappointed by the amount of integration on LC-HL.*
Foundation	*FL syllabus is needed at JC.*

level	*It's not all about OL and HL and 3rd level; we must think of FL – they are aground completely, with no questions directed for them.* *There is a serious lack of understanding for FL – it needs to be more tightened/specific. It's vague - what depth for FL?*
Common Introductory Course (CIC) in First Year	*Not sure whether the CIC material is working.* *The CIC is so broad and, for good students, it's not challenging enough. I don't know whether they gain anything.*
Comments on exams	*Paper 2 is too long.* *It's difficult to judge the syllabus without a selection of exam papers to see how it is examined.* *The a, b, c structure for questions was better.* *There needs to be a hint to help students identify the differentiation question.* *Top students are unnerved – for some questions they require life experiences beyond their years.* *Title the question – the words make it difficult to know which section this is. There is a need to read and comprehend.* *There is a serious language issue – students ask 'Is this an English test or a Maths test?'*

Note: Teachers comments on specific learning outcomes and other queries on the syllabus were brought to the attention of the relevant course committee. Some additional clarifications are being made in the syllabuses to be issued in September.

4. Learning from the experiences of the 24 schools

The teacher feedback above indicates that the teaching of maths in these 24 schools is changing, albeit at a slow pace, as a result of Project Maths.

For many teachers there has been a change in their role, teaching practices and methods as they have moved away from teacher led and didactic approaches to more student-centered and active methodologies. Many teachers now see themselves as facilitators of learning rather than givers of knowledge. This change has not been easy and many teachers have described a loss of confidence when compared with their familiarity with the previous syllabus and exam. They have also described being very challenged by the increasing time demands of the new syllabus. Using active learning methods, characterised by a higher level of student involvement, classroom discussion and practical work, has proved very time consuming so far and many teachers have reported that covering the whole syllabus is challenging. Some teachers reported that they taught extra classes outside core maths hours to complete the syllabus.

Learning approaches such as group work, classroom discussion and questioning are being used by more teachers. All teachers report that these methods are more time consuming than ‘chalk and talk’ and ‘drill and practice’ methods, and many report that they are challenging to use as not all students are yet comfortable with them. Teachers have indicated a need for support to enable them and their students to develop the skills to use these methods efficiently and effectively. Not all teachers are convinced that these teaching practices offer additional learning benefits over the ‘chalk and talk’ and ‘drill and practice’ approaches that they have relied on in the past.

While embracing new approaches in their teaching, many teachers still focus on the examinations, particularly in sixth year, and want more exam-focused questions and sample papers to use in exam preparation. The practice of striving to finish the full syllabus in time for early mock exams adds additional time pressures. Some teachers have reported that they have reverted to ‘chalk and talk’ teaching methods under these time pressures in sixth year. From teachers’ comments, there would appear to be some danger that, with increased availability of exam-oriented resources, they may revert to old practices and not fully embrace their new role.

Teachers report that increasingly they are using the syllabus as a guide, whereas previously they had used the text book and past exam papers to guide their teaching. However, not all teachers report being comfortable with the language and level of detail provided by the syllabus. If the syllabus is to be a useful guide, then teachers need to be able to read and understand it and it has to be more than simply a list which is ticked off when a topic has been ‘covered’. Reading and understanding a syllabus and using it to design learning activities that ’fit’ the group of students in a class appears to be an important competency for teachers, and an area in which teachers expressed the need for support. This is also a challenge for the NCCA in syllabus design and development.

Collaboration among maths teachers in each of the schools has increased. While most of the schools did have maths department meetings prior to Project Maths, these tended to be more focused on timetabling, exams and sequencing issues. These meetings are now increasingly focused on collaboration around challenging aspects of teaching the syllabus using the new approaches. Teachers report having more meetings and, in particular, more informal meetings, which are often between two or three colleagues and focused on maths. This collegial support has been found to be very valuable by all.

Teachers believe that the student learning experience has changed. They report that students are now engaged in greater discussion, collaboration and activity within their maths classrooms although, as has been reported above, this often changes under the pressure of the exam year. Teachers also report that not all students are comfortable with this new type of learning and it appears that younger students and those not in exam years are most comfortable with the new methods, whilst exam year students are disconcerted by the absence of past exam papers and want teaching geared to answering exam questions. On the other hand, teachers do report that there has been an increase in understanding maths concepts among students. Students who performed best in the previous ‘chalk and talk’ and ‘drill and practice’ learning environment seem to be more challenged by the move to discovery learning and those who were less able in that environment are now performing better.

Another area where teachers need new skills, and support in developing these skills, is the area of assessment. It appears that the majority of teachers used, and many still continue to use, tests as their only assessment tool. The experience of some teachers who have attempted to explore other assessment methods is that it can be challenging. There is some evidence of the insight that teachers have gained into students thinking and learning through changed classroom practices, such as listening as students explain how they solved a problem, or group discussions on different ways of answering a question. These insights can change teachers’ perceptions of student understanding and learning, and also change their perceptions of the efficacy of the newer teaching and learning practices.

It appears that the changes introduced by Project Maths propel both teachers and students on a new learning continuum. Not all teachers are at the same point or proceeding at the same pace along this continuum. From the meetings held in the schools, it would appear that experienced and fully qualified maths teachers who are teaching maths full-time have found the experience of introducing Project Maths less challenging than their colleagues who teach a range of subjects. These teachers have often been able to provide peer assistance to their colleagues and have been an internal source of support within maths departments, which has facilitated collaboration between teachers.

A range of concerns have been voiced about the changed Leaving Certificate exam. Some teachers are worried that it doesn't reflect the type of learning that Project Maths promotes and that there should be a move towards an additional assessment component, such as project work. Others have focused more on issues of exam performance and perceive that students who would previously have got an A1 in maths are now not achieving this high grade and they consider that this is a problem with the exam rather than a reflection of student learning and understanding. Other exam-related concerns include issues of reward for effort through the year, in that a lot of time may be spent teaching a concept that then doesn't feature specifically on the exam.

The experience of these 24 schools has demonstrated that teaching using the approaches in Project Maths is only the starting point in changing the culture of maths teaching and learning within a school. The new syllabus is only one element in this transition. Ongoing supports for teachers, a collaborative maths department, organised and accessible resources, a timetable that supports a discursive learning environment, a classroom infrastructure that supports this type of learning, an assessment methodology that reflects the syllabus learning outcomes, and methodologies and external leadership and support from the educational establishment all have an important role to play. Indeed the experience of the 24 schools to date demonstrates the synergies between these.

Across Europe, it is recognised that professional development opportunities can play a key role in equipping all teachers with the necessary skills to adapt their teaching to changes and developments in mathematics education (*Mathematics Education in Europe: Common Challenges and National Policies;* a report of the Eurydice network, 2011). The report acknowledges the specific reforms in Ireland which target mathematics teachers – one of only two countries where such reforms have been introduced.

5. Next steps

The feedback from the schools will be discussed with the Project Maths Development Team, with a view to planning future support for these schools as they complete the full cycle of changed examinations.

The maths committees were kept informed of emerging issues in relation to the syllabuses and their assessment. From September 2012, all schools nationally will engage with the same syllabuses. This will see the final section of the ‘retained’ syllabus being replaced by strand 5 (functions). In light of the fact that some topics in this section of the LC Maths syllabus will no longer be included, the committee decided not to make any additional adjustment to the length of the syllabus at this stage. As students come through to senior cycle having experienced the revised syllabus and new approaches in the junior cycle, future consideration of syllabus length will be informed by ongoing feedback from the initial schools.

A seminar involving the principals/deputy principals and two maths teachers from each of 24 schools will take place in April, at which the overall findings of the feedback from the school visits will be presented and discussed.

Appendix

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Project Maths
Information Note #14
December 2011

Please ensure that all maths teachers in your school receive this bulletin.

This Information Note provides details about the upcoming Project Maths review meeting in schools.

Project Maths school-based review meeting

One of the NCCA Project Maths team together with your RDO will be visiting your school in the next two weeks. The purpose of this visit is to get feedback from you on your experiences of the Project Maths initiative. On the day of the visit we will be meeting with maths teachers for two class periods; the meeting will be informal and we intend to cover the same ground with all schools. The main focus of the session will be:

- The impact of Project Maths on your practices as a maths teacher
- The impact of Project Maths on your school maths department
- The impact of Project Maths on your students' experience of maths.

Feedback on strands 3,4 and 5

We will also be looking for your feedback as initial participants in the project on strands 3, 4 and 5 so that the committees can consider this before they finalise the syllabus for national roll-out in September 2012. In order to direct your thoughts, and in the event that we are under pressure with time on the day, we would ask that you note any comments on strands 3.4 and 5 in advance of the meeting on the sheet provided. We will take this away with us for discussion with the committees.

NCCA contact details

Contact details for NCCA staff working on Project Maths are set out below.

Email: Rachel.Linney@ncca.ie, Aoife.Kelly@ncca.ie, or Bill.Lynch@ncca.ie

NCCA, 24 Merrion Square,
Dublin 2.
Tel: 01 7996400 Fax: 01 6617180

Strands 3,4 and 5

The four main headings under which we would like to get your feedback are:

- The syllabus topics and learning outcomes

- The appropriateness of the different syllabus levels

- The progression from JC to LC

- The connections between the strands

Community of Practice:

Session Title: Examining student work

Charmaine, Leona, Anna, Hugh and Ciaran were investigating the commutative property for subtraction. The extract below is from their discussion of the following task.

Task: If you switch around the numbers in a subtraction problem will you get the same answer?

With your colleagues think about these students' ideas

- What prior understanding have the learners brought to this task?
- What problem solving strategies were they using?
- What conjecture did they form?
- How did they represent it that will show it is always true?
- What mathematics are the students learning from engaging in this task?

Now think about the learners in ***your*** school

- Are your learners ready for such a task?
- Are they as familiar with using representation to show their ides as these students seem to be?

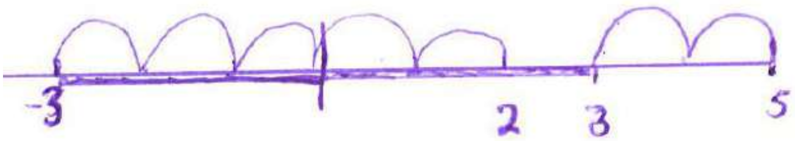
Think about the teacher

- How did the teacher progress the learning?

Together think about how you would manage this task in **your** classrooms

- What students would you do it with?
- How would you introduce it?
- What difficulties do you think your students might have with the task?
- Are your students familiar with making and testing conjectures?
- What do your students understand about mathematical proof?

Charmaine: 5-2 = 3 ..You start here and go back 2 you land on 3
But if you do 2-5 you start here and go back 2 to 0 and then another 3 to -3.



Anna: ye we did it with other numbers too and it's always the same thing

Leona: We used money we thought that if you have some money €10 say and you pay for something that is €7 then you have €3 left so 10-7 =3 ..but if you have €7 and you pay for something that is €10 then you owe €3 cos you'll have to borrow it to pay the whole 10 off….so 7 -10 = -3

Teacher: Can anyone see any pattern when you switch the numbers round in a subtraction problem?

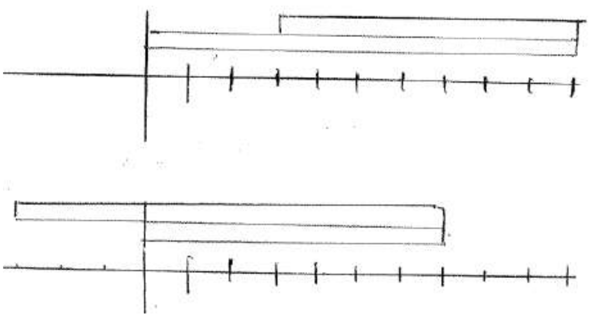
Leona: One is positive and one is negative but the number is the same

Anna: Ye.. that's cos they are both the same distance from 0

Hugh: If the larger number comes first the answer is positive. if the smaller number comes first the answer is negative.

Teacher: Why does this happen? Will it happen for all numbers? Can you explain why?

Ciaran: I think if you use the number line it will prove it ….just get a strip of paper lets say it's to represent 10 then cut a smaller one to represent 7 then place them on the number line like this .



Now to show the switch around just take the same two pieces of paper turn them upside down and push them down the line to show that the answer is the same distance on the other side of 0.These pieces can be any length so it will work for any number

The dialogues that follow are further extracts from the lesson. The teacher is focussing on generalisation in the next extract**.**

Laura: x- y = the difference …so lets say x –y = z

Hugh: ye so y – x = -z

Ciaran: What tells us which is the larger number?

Laura: eh..nothing

Teacher; So what happens if x isn't the larger number?

Leona; Then z is the negative number

Hugh: Well then what is -z?...that's confusing

- What is the issue the learners are struggling with here?
- How would you help your learners resolve this issue?

Teacher: What can you say about z + -z ?

Ciaran: It equals zero because if you give something… +z and then take it away… –z you really have nothing

Teacher: So z + -z = 0 ?......when this happens we say that z =-z are ***additive inverses*** that is because when you add them together the overall effect is that nothing has changed

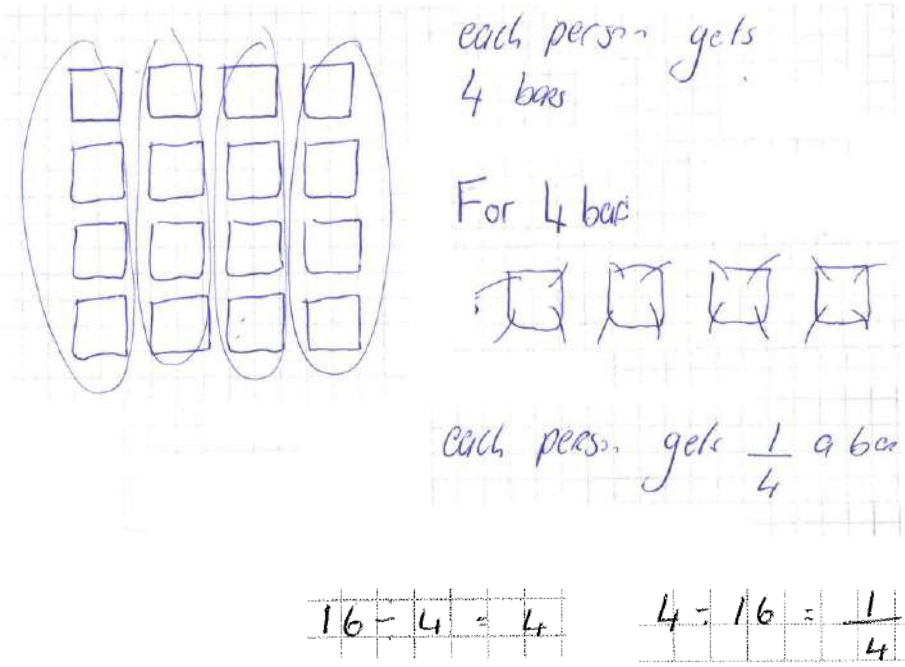
Hugh: Oh ye ..its like when you are doing a big sum on the calculator and you press + something instead of – something you don't have to start again cos you just minus the thing and you are back to where you started.

Teacher: Lets look at this way of writing our idea (x –y) + (y-x) = 0 is this the same idea?

Anna: Mmmm I think we need to try this

Discuss the teacher's move in the discussion above

Here is a piece of work by the same students, this time they were investigating the commutative property of division.



With your colleagues discuss

- How could you progress the learning?
- How could the previous work help these learners make sense of the commutative property of division
- Is their potential for other learning here?
- How would you engage your students in investigations about the commutative property?
- Would you have everyone investigate each operation? Or would you divide the class into groups and give each group a different operation?
- How does the investigation of commutativity support understanding of the additive inverse? The multiplicative inverse? The reciprocal?

Try some ideas in your class and bring some student work to discuss at the next meeting

Problem solving reminder: If you are going to use these tasks remember, answers are important but what is more important is the mathematics students can learn from engaging with the tasks.

Community of Practice:

Session Title: Reflecting on learning

Describe the Task

What mathematics did you want your students to learn from engaging with the task?

How did you manage the task in your class?

Outline a typical interpretation of the task?

Outline an interesting interpretation of the task

How did you progress the learning?

Community of Practice:

Session Title: Developing questioning

These materials have been designed to help you and your colleagues reflect on:

- the ***reasons*** for questioning;
- some ways of making questioning ***more effective***;
- different types of '***thinking questions***' that may be asked in mathematics.

Why ask questions? In their book Questions and prompts for mathematical thinking, 1998, Association of Teachers of Mathematics John Mason and Anne Watson suggest that teachers should ask questions

- To interest, challenge or engage.
- To assess prior knowledge and understanding.
- To mobilise existing understanding to create new understanding.
- To focus thinking on key concepts.
- To extend and deepen learners' thinking.
- To promote learners' thinking about the way they learn.

Can ***you*** think of other reasons ***you*** might ask questions in ***your*** class?

Mason and Watson have classified questions into two categories; **effective** and ***ineffective***

Ineffective Questions are….	**Effective Questions are…**
unplanned with no apparent purpose	planned and related to lesson objectives
mainly closed	mainly open
provide no 'wait time' after asking questions	allow 'wait time'
'guess what is in my head'	ones where the teacher allows collaboration before answering
poorly sequenced	carefully graded in difficulty
ones where the teacher responds immediately	ones where the teacher encourages learners to explain and justify answers
ones where only a few learners participate	ones where all learners participate e.g. using mini-whiteboards
ones where incorrect answers are ignored	ones where both correct and incorrect answers are followed up
all asked by the teacher	asked by learners too

Take an audit of the types of questions you ask in your classroom. Challenge yourselves to replace ineffective questions with effective questions.

In your groups decide

- When you will complete your audit
- When you will share
  - the results of your audit with each other
  - the targets for increasing the number of effective questions you ask

Plan to ask different types of questions, ones that require students to

- Create examples and special cases.
- Evaluate and correct.
- Compare and organise.
- Modify and change.
- Generalise and conjecture.
- Explain and justify.

Example of questions that require students to

1. Create examples and special cases

Show me an example of:

- a number between 1/2 and 3/4;
- a quadrilateral with two obtuse angles;
- a shape with an area of 12 square units and a perimeter of 16 units;
- a number with 5 and 6 as factors
- a set of 5 numbers with a range of 6
  …and a mode of 10
  …and a median of 9
- a linear relationship

2. Evaluate and correct

What is wrong with these statements? How can you correct them?

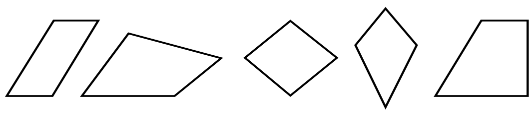
- When you multiply by 10, you add a zero.
- 2/3 + 3/5 = 5/8
- Squaring makes bigger.
- If you double the lengths of the sides you double the area.
- An increase of x% followed by a decrease of x% leaves the amount unchanged.
- Every equation has a solution

3. Compare and organise

What is the same and what is different about these objects?

- Square, trapezium, parallelogram.
- Cone, cylinder, sphere.
- 6, 3, 10, 8.
- 2, 13, 31, 39.
- Δ + 15 = 21, I think of a number, add 3 and the answer is 7, 4 Δ =24,

How can you divide each of these sets of objects into 2 sets?

- 1, 2, 3, 4, 5, 6, 7, 8, 9,10
- 1/2, 2/3, 3/4, 4/5, 5/6, 6/7
- 
- 121, 55, 198, 352, 292, 1661, 24642

4. Modify and Change

How can you change:

- the decimal 0.57 into a fraction?
- the formula for the area of a rectangle into the formula for the area of a triangle?
- an odd number into an even number?

5. Generalise and Conjecture

What are these special cases of?

- 1, 4, 9, 16, 25....
- Pythagoras' theorem.
- A circle.

When are these statements true?

- A parallelogram has a line of symmetry.
- The diagonals of a quadrilateral bisect each other.
- Adding two numbers gives the same answer as multiplying them.

6. Explain and justify

Use a diagram to explain why:

- 27 x 34 = (20 x 30) + (30 x 7) + (20 x 4) + (7 x 4)

Give a reason why:

- a rectangle is a trapezium.

How can we be sure that:

- this pattern will continue: 1 + 3 = 22; 1 + 3 + 5 = 32…?

Convince me that:

- if you unfold a rectangular envelope, you will get a rhombus

In your groups make up your own questions that require students to

- Create examples and special cases.
- Evaluate and correct.
- Compare and organise.
- Modify and change.
- Generalise and conjecture.
- Explain and justify.

Try out some of the questioning strategies suggested in a lesson with your class

- Come to the next session prepared to share your experiences
- Bring examples of the questions you asked and the students' responses to those questions

Framework for assessing Synthesis and Problem solving skills

	Problem- Solving	**Mastery of Mathematical procedures**	**Reasoning and proof**	**Communication**	**Connections**	**Representations**
No evidence	Problem indicated a clear solution strategy.	No evidence of following any basic mathematical procedure	Activity/Task did not require students to give a reason or proof.	Activity/Task did not require students to communicate in any way.	No connections are made.	No attempt is made to construct mathematical representations.
Students working at "novice" level	No strategy is chosen, or a strategy is chosen that will not lead to a solution.	Evidence of some familiarity with a that a basic mathematical procedure	Arguments are made with no mathematical basis. No correct reasoning or justification for reasoning is present.	Everyday familiar language is used to communicate ideas.	No connections are made.	An attempt I**s** made to construct mathematical representations to record and communicate problem solving.
Students working at "practitioner" level	A correct strategy is chosen based on the mathematical situation in the task. Planning or monitoring of strategy is evident. Evidence of solidifying prior knowledge and applying it to the problem solving situation is present. Correct answer is obtained.	Evidence that a basic mathematical procedure was followed but executed inaccurately	Arguments are constructed with adequate mathematical basis. A systematic approach and/or justification of correct reasoning is present. This may lead to clarification of the task exploration of mathematical phenomenon Noting pattern, structures and regularities.	Communication of an approach is evident through a methodical, organised , coherent sequenced and labelled response. Formal mathematical language is used throughout the solution to share and clarify ideas.	Mathematical connections or observations are recognised.	Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions

Framework for assessing Synthesis and Problem solving skills

	Problem- Solving	**Mastery of Mathematical Procedures**	**Reasoning and proof**	**Communication**	**Connections**	**Representations**
Students working at "expert" level	An efficient strategy is chosen and progress towards a solution is evaluated. Adjustments in strategy, if necessary, are made along the way and/or alternative strategies are considered. Evidence of analysing the situation in mathematical terms and extending prior knowledge is present. A correct answer is achieved.	Basic mathematical procedures were followed accurately	Deductive arguments are used to justify decisions and may result in formal proofs. Evidence is used to justify and support decisions made and conclusions reached. This may lead to • testing and accepting or rejecting of a hypothesis or conjecture • Explanation of phenomenon • Generalising and extending the solution to other cases	Communication of argument is supported by mathematical properties. Precise mathematical language and symbolic notation are used to consolidate mathematical thinking and to communicate ideas.	Mathematical connections or observations are used to extend the solution.	Abstract or symbolic mathematical representations are constructed to analyse relationships extend thinking and clarify or interpret phenomenon.

Using the *Assessment Framework for Synthesis and Problem Solving Skills*

Task:

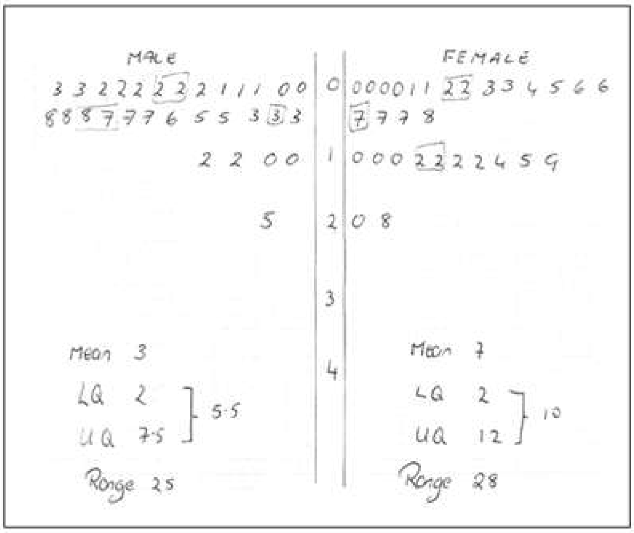
Question: Do Girls spend more time on social networking sites than Boys?

This task was used with a 3rd year class divided into groups of 4. The students were given 2 class periods to work on the task together in their groups, and a further week of independent work at home. Later, in a single class period, under examination conditions they completed their report. The students were allowed to bring charts and calculations to the assessment session. This is Jennifer's work.

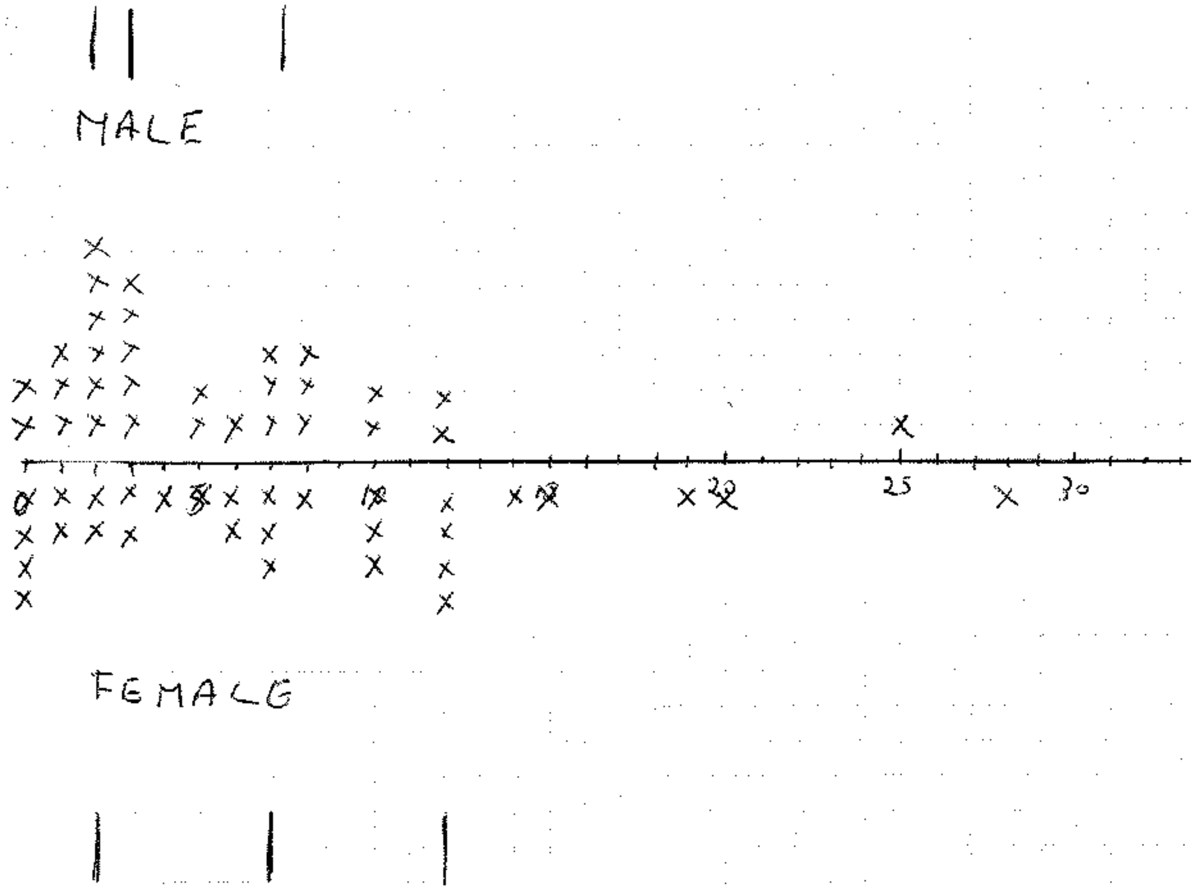
We decided to use the **Census at School** data to see whether girls spend more time than boys on social networking sites.

We used the random data selector facility on the **Census at School** site and selected a sample of size100.Then we decided we needed a quota sample so we took the first 30 boys and the first 30 girls on the list of 100 that was randomly selected.

We plotted back to back stem and leaf plots so that we could compare the samples and we also used them to calculate the lower and upper quartiles and the median.



We wanted to get a look at the shape of the distributions so we plotted back to back line plots.



From the samples I notice that the time spent by girls on social networking sites is shifted further up the scale than the time spent by boys. There is however some overlap of times between the two groups. The median time spent by girls is 7hrs/week over double that of the median time spent by boys.

There is little difference between the ranges for boys (25) and girls (28) but the interquartile ranges for these samples show that the spread within the middle half of the data for girls (10) is greater than the spread in this half for boys (5.5).

I also notice that the distribution for this sample of boys is somewhat skewed right whilst the distribution for this sample of girls is more symmetrical.

I would claim that girls (12-18 yrs) in Ireland tend to spend more time on social networking sites than boys.

Jennifer's teacher sat with colleagues and they used the *Assessment Framework for Synthesis and Problem Solving Skills* to look for evidence in Jennifer's work. The group felt that the work displayed evidence that

- An efficient strategy was chosen and progress towards a solution was evaluated and that adjustments in the strategy were made along the way.
- Evidence of analysing the situation in mathematical terms and extending prior knowledge is present.
- Basic mathematical procedures were followed accurately.
- Evidence was used to justify and support decisions made and conclusions reached.
- Communication of her argument is supported by mathematical properties.
- Symbolic mathematical representations were constructed to analyse relationships, extend thinking and to interpret phenomenon.

This means Jennifer's work is **expert level** in *Problem solving*, *Mastery of Mathematical Procedures, Reasoning and proof*, *Communication* and *Representation.* The teachers decided that there was no evidence that any *Connections* were made

Do you agree with these teachers? Can you find this evidence in Jennifer's work? Use the framework with your colleagues to assess what level *Problem solving, Mastery of Mathematical Procedures, Reasoning and proof, Communication, Connections* and *Representation is evidenced in Jennifer's work.*

Jennifer's teacher was asked to reflect on how useful she found the collaborative exercise. Here is her reflection

….I found this exercise really useful, in fact it made me think a lot about *how* I present a task to the kids. If I structure the task too much I don't give them an opportunity to display evidence of ***problem solving*** and if I don't ask them to give a reason for their answer then they won't see the need to give one and there will be no evidence of ***reasoning and proof*** in the work, same really with ***communications***. I think Jennifer's work is great and I will have to think about how I could have structured the task that would have given her more of an opportunity to display evidence of making ***mathematical connections***. I think I'm going to give the framework to the kids so that they can be aware of what I am looking for in their work. I think strand 1 lends itself to tasks that kids will naturally use ***representations*** and I am really going to have to work with the kids to help them see how useful ***representation*** is in mathematics, concepts can have lots of different representations and I am going to ask more questions that help the kids see this; things like

….solve this problem using a diagram…support your answer with a diagram…represent such and such in different ways…look at the representation can you see such and such..

I was quite impressed with how Jennifer represented the quartiles and the median in her work and I am going to have a discussion with the class about this and ask them to ***evaluate the usefulness*** of each representation the pictorial one and the numerical one.

When I use the text book I'm going to use the questions in different ways

A big ah ha moment for me has been that now I see algebra as just one representation of an idea. I'd love my kids to see this too…

The 5 process skills are just as important as the maths learning outcomes in fact they help the kids understand the maths

	Problem- Solving	**Mastery of Mathematical procedures**	**Reasoning and proof**	**Communication**	**Connections**	**Representations**
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Students working at ***practitioner level***	A correct strategy is chosen based on the mathematical situation in the task. Planning or monitoring of strategy is evident. Evidence of solidifying prior knowledge and applying it to the problem solving situation is present. Correct answer is obtained	Evidence that a basic mathematical procedure was followed but executed inaccurately	Arguments are constructed with adequate mathematical basis. A systematic approach and/or justification of correct reasoning is present. This may lead to clarification of the task exploration of mathematical phenomenon Noting pattern, structures and regularities	Communication of an approach is evident through a methodical, organised, coherent sequenced and labelled response. Formal mathematical language is used throughout the solution to share and clarify ideas.	Mathematical connections or observations are recognised.	Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions

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Understanding Equality

At all levels students should be able to

- consolidate the idea that equality is a relationship expressing the idea that two mathematical expressions hold the same value

Learning equality as a relationship between number sentences is a crucial aspect of learning mathematics. A lack of such understanding is one of the major stumbling blocks in moving from arithmetic towards algebra. This document describes seven different types of tasks that offer teachers ideas of how they can understand and develop their student's understanding of equality and at the same time teach algebra informally.
These tasks are ideal for mixed ability classes as they can be differentiated to suit the learner's needs. As you read through the tasks think

- What mathematics can my students learn from engaging with these tasks?
- How could I use these with ***my*** class?
- How could I adapt these tasks to suit ***my*** class?
- When will I use these tasks with my class?
- What prior learning will I expect them to have had?

Problem solving reminder: If you are going to use these tasks remember, answers are important but what is more important is the mathematics students can learn from engaging with the task.

Box on the left side

These types of tasks are designed to allow learners to construct a greater understanding of the concept of equality. They help learners gain awareness of the fact that the equality symbol does not always come at the end of a number sentence or at the right hand side of the equation. There is no one answer, if you are using tasks like these encourage learners to find more than one way to complete the equality statement and to discuss and justify their solutions with others. Increase the cognitive demand by challenging learners to find as many ways as they can to complete the sentences in a given period of time or include restrictions on the amount of numbers that can appear in the brackets

Task:

Complete the equality sentences in as many ways as you can

(   ) = 64 + 374   (   ) = 376 - 88   ( ) = 45 x 98   (   ) = 24 ÷ 6

Darragh Fifth Class

(438) = 64+374

(60+30+70+4+4)=64+374

(60+378)=64+374

(374+64)=64+374

Boxes on Both sides

The purpose of these types of task is to expand learners understanding of equality by presenting them with the opportunity to think about different statements of equality in complex number sentences. As with the other tasks encourage learners to explain their reasoning.

Task:

Complete the equality sentences in as many ways as you can

26 + (   ) = 12 + (   )   (   ) – 17 = 5 – (   )   (6 x ( )) +5 = (4 X ( )) +13

Symbolising

These tasks help learners build an understanding of letter symbolism in equations.
Usiskin (1997) described algebra as a language which includes unknowns, formulas, generalised patterns, placeholders, and relationships. He added that a number can be represented by a word, a blank, a square, a question mark or a letter, all of them are algebra.

Task:

- What number when added to 12 gives 18?
- Put a number in the square to make this sentence true

  14+ ☐ =25
- a + 2 = 5
  - is this sentence true?.
  - what do you think about this sentence?
  - which one is larger ***a*** or ***5***?

Reading Equation Sentences:

This type of task not only provides learners with the opportunity to reinforce their understanding of the concept of equality but also provides teachers with an opportunity to asses their understanding.

Task: Read the following sentences

( ) = 5 + 32	245 – 29 = ( )	616 = 88 x 7
( ) = 4 x 26	35 ÷ 7 = ( )	( ) = 63÷ 3

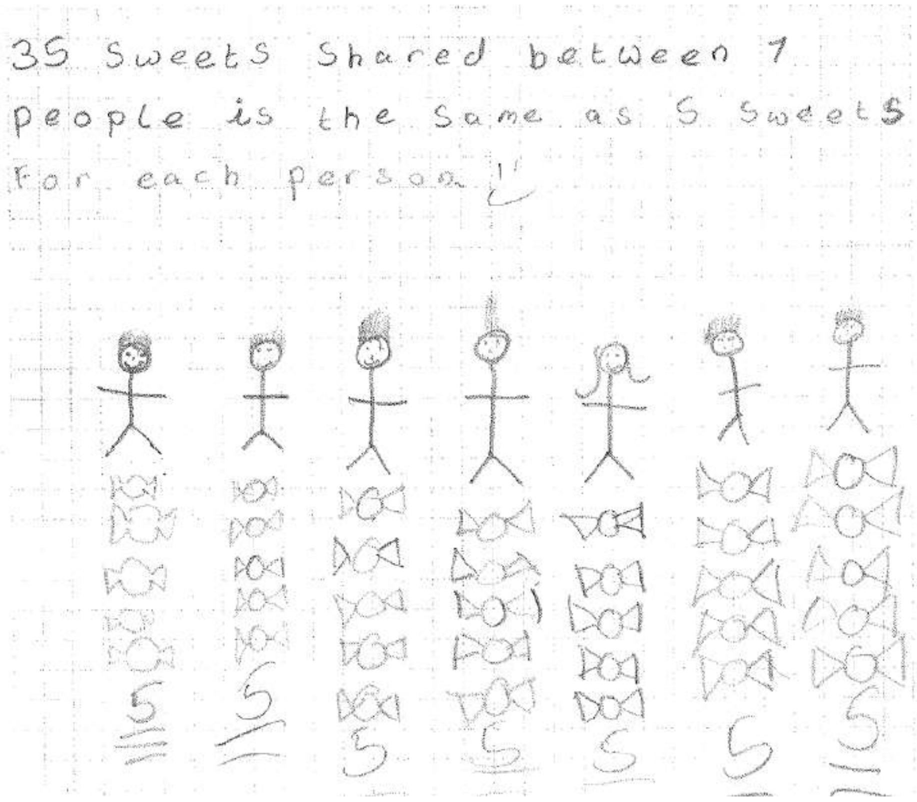
The cognitive demand of this type of task can be increased by asking learners to write story contexts for each sentence and represent the sentence with a diagram or with concrete objects

Task: Write a story context to describe the following arithmetic sentence

35 ÷ 7 =5

Represent the sentence with a diagram or with concrete objects.

Aoibhinn First Year



True/False Statements

This type of task gives teachers an opportunity to assess learners' understanding of the concept of equality.

Task: Decide whether each of the following statements are true or false. Justify your decision 27 + 14 = 41   15 ÷ 3 = 5 x 2   14 - 9 = 5 – 2

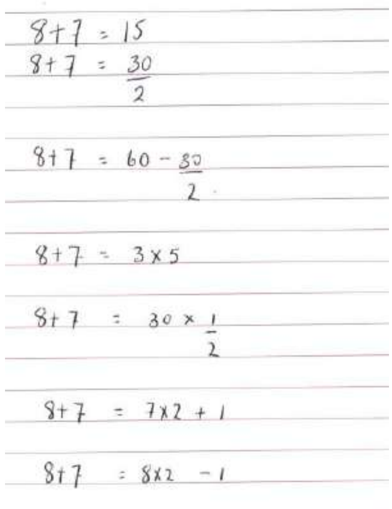
Examining learners' answers to these questions gives teachers the opportunity to assess students' understanding of the concept of equality.

Alternative ways and ***Finding Missing Numbers***

These two types of task focus on representation and encourage learners to write numbers in alternative ways. These tasks not only lead learners to understand the equality concept, but also to understand each number as a composite unit of other numbers. By doing such tasks learners are not only finding arithmetical relationships but are also thinking algebraically.

Task: Solve the following problem and write your answer in as many different ways as possible. **8+7**

Sarah Louise Fist Year



Task: Complete the equality statements

18 = ( ) x ( )

18 = [( ) x ( )] x ( )

18 = [( ) x ( )] ÷ ( )

Summing Up

This task helps learners build up numerical strategies for operating with numbers, it encourages flexible thinking.

Task: The grey box is the place for the sum of the numbers. Complete each of the boxes.

19	20
21	

	48

	110
110	333

Key Concepts in Mathematics – Subitising

If these concepts are not fully developed, students will find it very difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years.

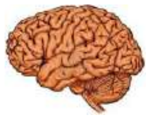
Subitising

Often referred to as *trusting the count*, subitising is the ability to instantaneously recognise the number of objects in a small group without needing to count them.

How does the concept develop?

By about **2yrs** of age children can recognise 1, 2 or 3 objects without being able to count with meaning.

By about **4 yrs** of age mental powers have developed and children can recognise groups of 4 objects without being able to count.



This skill is called ***subitising*** and appears to be based on the mind's ability to form stable mental images of patterns and associate them with a number.

It is thought that the maximum number for ***subitising*** even for most adults is 5.

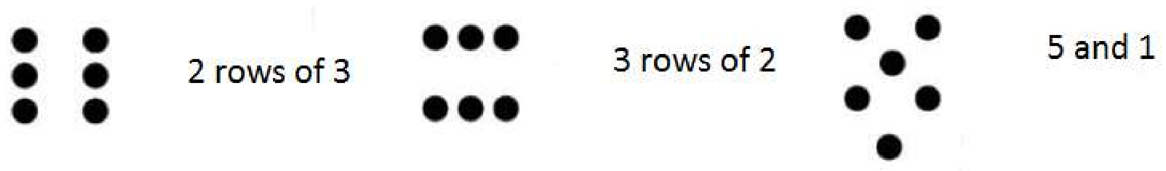
So, for groups of numbers beyond **5** other mental strategies are utilised.

Part–Part–Whole Relationships
i.e. understanding that a number is made up of smaller parts

Together with

Rapid Mental Arithmetic

So, it may be possible to recognise more than 5 objects if they are arranged in a particular way.



What can I do to help?

Encourage Mental strategies ✓ | **Discourage** Simply counting X

How?

Introduce an element of speed into tasks.

Encourage students to reflect and share their strategies.

Why?

- Verbalising brings the strategy to a conscious level and students learn about their own thinking.
- Other students are given the opportunity to pick up a new strategy.
- The teacher is given an opportunity to assess the type of thinking so that they can adjust the teaching accordingly.

Key Concepts in Mathematics – Place Value

If these concepts are not fully developed, students will find it difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years.

Place Value provides a system of new units based on the idea that '10 of these is 1 of those' which can be used to work with and think about larger whole numbers in efficient and flexible ways.

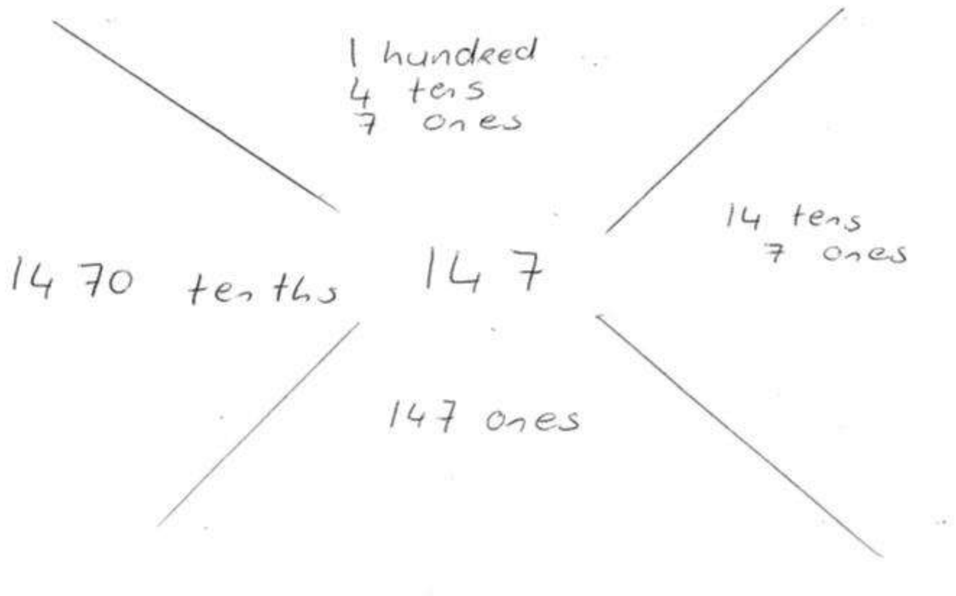
How does the concept develop?

By about First Class children can count by ones to 100 and beyond, read and write numbers to 1000, orally skip-count by twos, fives and tens, and identify place-value parts (e.g., they can say that there are 3 hundreds 4 tens and 5 ones in 345). Being able to re-name numbers in this way does NOT mean that children understand place value; many children who can identify the hundreds, tens and ones, in a number still think about or ***imagine*** these numbers additively as being bunches of ones. That is they ***imagine*** 345 as 300 ones and 40 ones and 5 ones which is 345 ones. This additive ***mental image*** ignores the multiplicative nature of the base ten system which involves counts of different sized groups that are powers of 10.

Children need to move from being able to ***identify*** place-value parts to being able to ***rename*** numbers in terms of their place-value parts and work in place-value parts.

When children are given large collections to count they begin to develop an understanding that the numbers 2 to 10 can be used as countable units and this ability to efficiently count large collections is a sound basis for place value. In addition children also need a well-developed concept of part–part–whole relationships for numbers from 0–10 as well some ***sense*** of numbers beyond 10, e.g. 15 is 10 and 5 more. See the section on ***Subitising*** for more information.

A student's work displaying evidence of a well-developed concept of *Place Value*.



Read the ***case studies*** and ***tasks*** for ideas on how you can support and track your students' development of the concept of Place Value.

Children need a deep understanding of the place-value pattern, 10 of these is 1 of those, to support more efficient ways of working with 2-digit numbers and beyond.

Assessing Place value Understanding

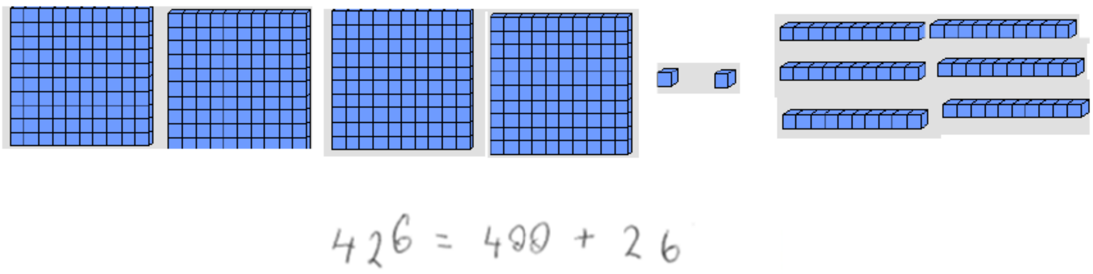
It is an enormous leap from operating with units of one to multi-digit computational procedures that use units of tens, hundreds, thousands and so forth, as well as units of one. To work with units of different values it is necessary to first sort out the complicated ways that each is related to the other.

The ten for one trade structure of our number system is quite complex. Being able to label the tens place and the ones place, or even being able to count by tens, does not, necessarily signal an understanding that 1 ten is simultaneously 10 ones. Becoming mindful of this relationship between tens and ones, or staying mindful of it, is neither simple nor trivial.

Asking students to represent numbers with concrete objects or pictures and carefully examining their use gives an insight into their conceptions.

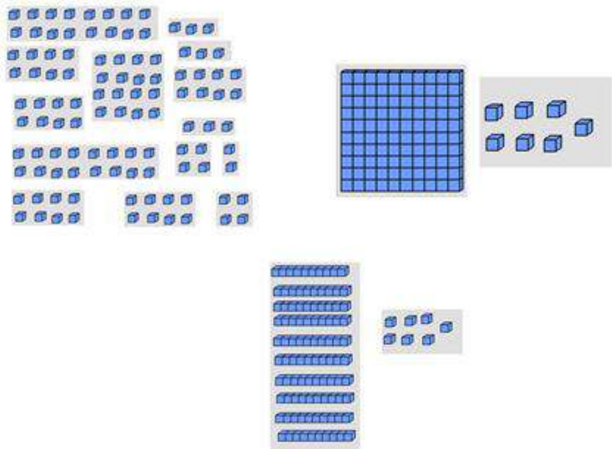
Task: Use Dienes blocks to represent the number 426

Mark First Year



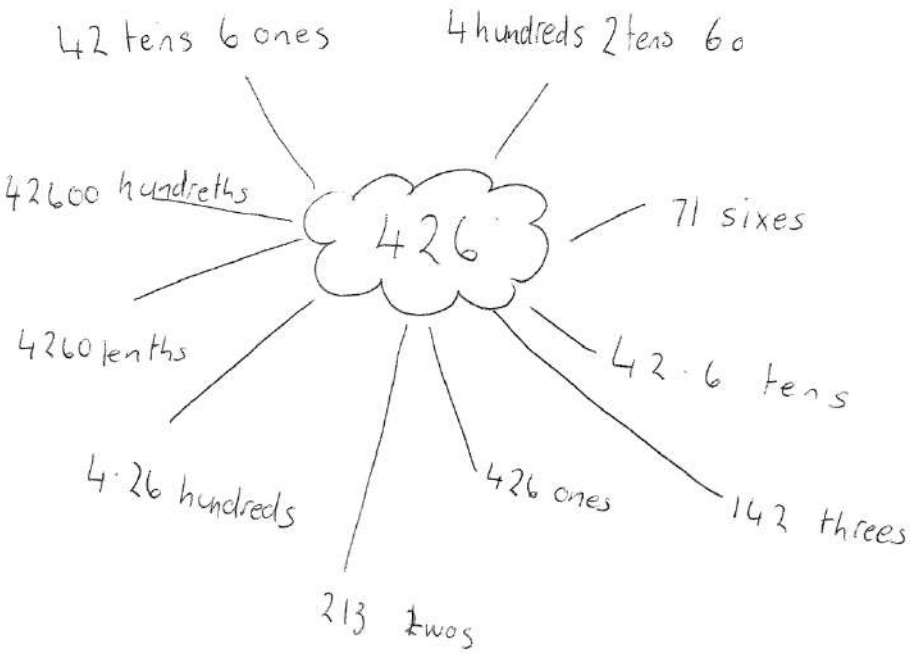
Task: Use Dienes blocks to represent the number 107 in as many ways as you can

Susan First Year



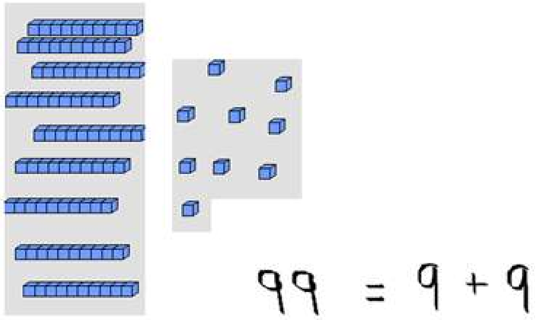
Task: Rename or represent 476 in as many ways as you can

Sarah Louise 1st year



Task: Represent 99 with Dienes blocks

William First Year



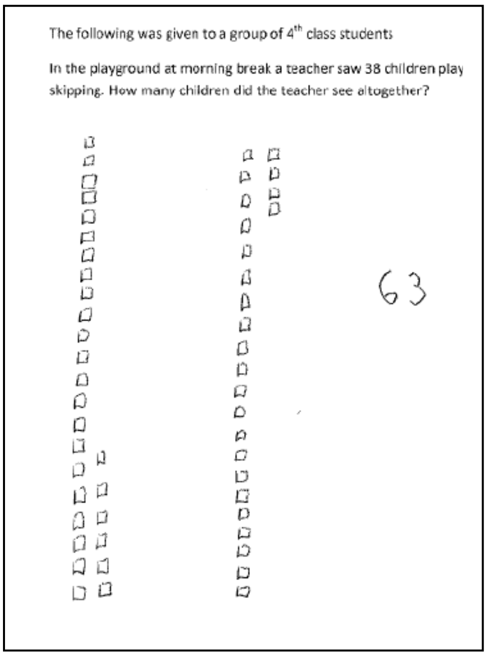
Examine the student work

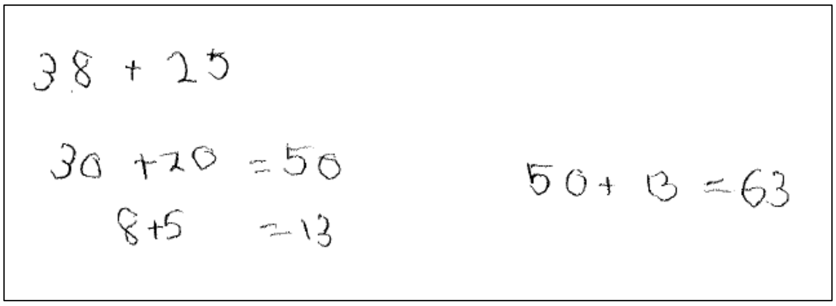
- What does each student's work tell you about their understanding of Place Value?
- What questions would you like to ask each student to find out more about their understanding of Place Value?
- How would ***you*** use Diene's blocks in ***your*** class to help your students develop their understanding of Place Value?
- What questions would you ask to 'dig deeper' into student thinking about place value?

A teacher's reflection

…I was very impressed with Sarah Louise's work.I asked her .*How would you represent the 42600 hundreths and the 213 twos with Dienes blocks. She very quickly held up two little unit blocks and said well this is a '2' so I woud get 213 of them. She had to think a little longer about the 42600 hundreths and said ….**Well I would call this a one [holding up the 100 square] this a tenth [holding up the 10 stick]and this a hundredth [holding up a unit cube] and then I would need 42600 of them but I don't think we'll have enough.**I thought Sarah Louise has a well- developed concept of place value, she is able to look at numbers as separate 'units' and is able to confidently rename; this understanding will be great when we move on to operate with rational numbers..*

As students become adept at breaking apart and recombining numbers, they often invent multi-digit addition and subtraction procedures. These can be the starting places for deeper understanding of the tens structure itself and how it behaves in computation. Consider the first piece of work below, the two students drew out a pile of 38 cubes counting in ones each time. Then they drew another pile of 25 cubes starting again at one. Next they counted both sets together, starting at one until all the cubes were counted. This was in contrast to other students in the class who worked more abstractly with number and made use of groups of ten, generating solutions such as the one shown. Like many students they chose to work with the larger numbers first;in this case tens.





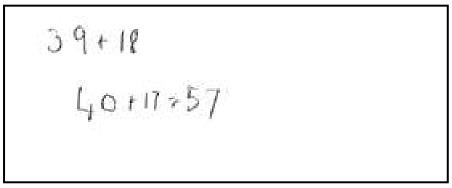
What students do with the objects they use for modelling mathematics situations reflects their understanding of the structure of the situation. In the word problems here, for example, the structure of addition is understood from physically joining quantities.

While the second style solution certainly demonstrates some ability to decompose and recombine numbers using groups of tens and ones, we can’t really tell from this example whether any of these students understand 30 +20 =50 to be equivalent to 3 tens + 2 tens = 5 tens, and therefore to be both similar to, and different from, 3 ones + 2 ones = 5 ones. Certainly the students solving this problem by drawing out cubes and counting them one by one are not looking at numbers in this way. As they grow beyond the need to represent all of the amounts and actions in problems, they no longer rely entirely upon counting to determine the results of joining or separating sets, beginning instead to reason numerically about the quantities involved.

After students have modelled many situations in which they represent all the amounts in the addition and subtraction problem with concrete objects, they develop a more abstract concept of number and begin to use counting up and counting back strategies. Fuson (1992) Carpenter et al (1996)

When students are able to pay attention to how all the amounts in a problem are related to one another, they can combine and separate them more flexibly. They often use strategies based on facts they already know, when they get to this stage they take apart numbers and recombine them to form new quantities that they find easy to work with (Fusion, 1992).

When presented with the problem 39+18 Sarah changed it to involve numbers she found easier to work with.



Problem solving reminder: If you are going to use these tasks remember, answers are important but what is more important is the mathematics students can learn from engaging in the tasks.

The growth from modelling all quantities and actions in a combining or separating problem to abstract reasoning with numbers is not a smooth or consistent transition for students.

Problem solving reminder: If you are going to use this task remember, answers are important but what is more important is the mathematics students can learn from engaging in the tasks

Task:

Students were asked to solve the following without pen, pencil or calculator:

38 + 29

Amy's group said:

67 because 40 + 27 = 57

What was their strategy?

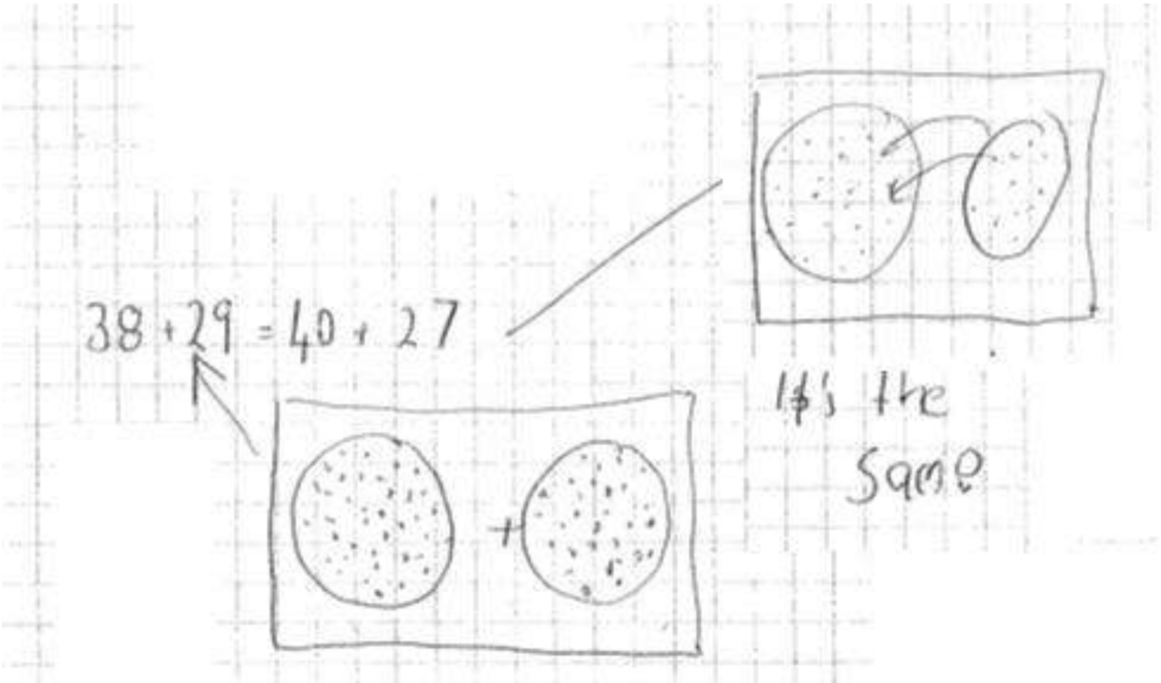
Will this strategy always work?

Justify you answer with representations.

Thoughts for teachers

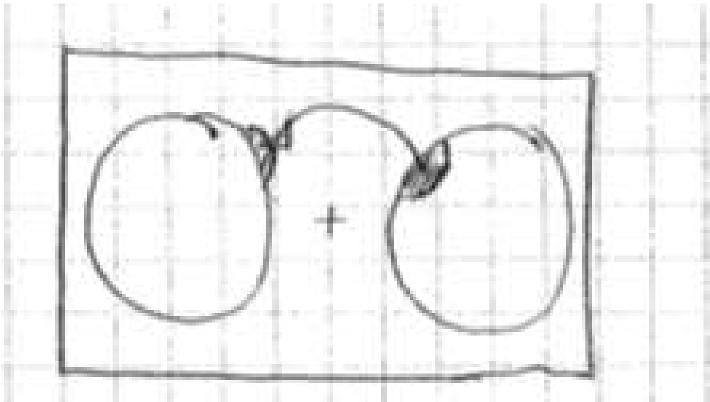
- How would you use this task with your students?
  - What prior experiences should your students have had in order to be able to engage with this task?
  - What misconceptions is this task likely to expose?
  - Would you modify the task in any way for your students?
- What mathematics do you want your students to learn from engaging in this task?
- What questions might you ask as your students as they are working on the task
  - Can you write a mathematical sentence to show Amy's group's strategy?
  - What is Amy's group saying about 38 + 29?
  - Amy's group is saying that 38 + 29 is the same as something else; what is it?
  - Can you write that as a mathematical sentence?
  - Use Amy's strategy to solve this problem…
  - Will Amy's strategy work for all whole numbers? Convince me.

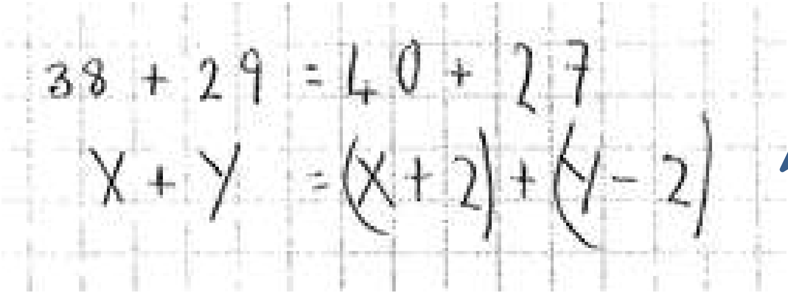
Making sense of the evidence



Teacher: Your picture shows the strategy works for 38+29. Would it work for other numbers too?

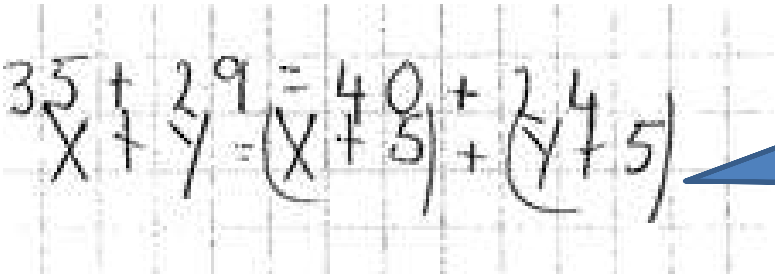
Seosamh: Yes look 'cos it doesn't matter if you are going to put them together if you take 2 from one group and give it to another you'll have the same amount like in the square





What prior knowledge is this student bringing to the task?

Teacher: Would the strategy work for other numbers like say 35+29?



Are your students ready to generalise solution strategies in this way? How would you scaffold your students to generalise their observations?

Extending the learning

Would the strategy work for subtraction? Why? Why not?

Justify your decision

Think: What mathematics do you want your students to learn from extending the task in this way?

Key Concepts in Mathematics – Multiplicative Thinking

If these concepts are not fully developed students' will find it difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years.

Multiplicative Thinking

A capacity to work flexibly with the concepts, strategies and representations of multiplication and division as they occur in a wide range of contexts.

Students who are thinking multiplicatively will be able to

- work flexibly and efficiently with large whole numbers, decimals, common fractions, ratio, and percentages
- recognise and solve problems involving multiplication or division including direct and indirect proportion,
- communicate their solutions effectively in words, diagrams, symbolic expressions, and written algorithms

How does the concept develop? There are several ***ideas*** that support the development of multiplicative thinking. The exploration of these ideas is very important; their development may take many years and according to some researchers, may not be fully understood by students until they are well into their teen years.

1. The ***groups of*** idea. This idea represents an additive model of multiplication and develops when children begin to count large numbers of objects. The one-to-one count becomes tedious and children begin to think about more efficient strategies, they skip count by twos, fives or tens. Some children can find this move from a one-to-one count to a one-to-many count very difficult because they lose sight of what they are actually doing; counting a count. The difficulty is eased if children are given the opportunity to…..
2. Move beyond the ***groups o***f idea to a ***partitioning*** or ***sharing*** idea and focus their attention on the number in each of a known number of shares. Asking children to systematically share collections helps develop this idea. There are documents available outlining tasks that empower children to think about counting by exploring how many ways a number of objects can be shared equally. One of the advantages of the ***sharing*** idea is that it leads to the realisation that a collection may be partitioned in more than one way, e.g. 24 is 2 twelves, 3 eights, 4 sixes, 6 fours, and 12 twos, each of which can be represented more efficiently by an *array* or a *region*.
3. The real strength of the array or region representation is that it provides a basis for understanding fraction diagrams, and leads to the ***area*** idea which is needed to accommodate larger whole numbers and rational numbers. The ***area*** idea is very important and more neutrally represents all aspects of the multiplicative situation, that is, the number of groups, the equal number in each group, and the product. It also demonstrates commutatively of multiplication as well as how multiplication distributes over addition. Read the ***tasks*** and ***case studies*** for ideas on how you can support your students with the ***area idea*** of multiplication.
4. The ***area*** idea generalises to the ***factor-factor-product*** idea which is needed to support fraction representation as well as multiple factor situations such as 24 = 2 x 2 x 2 x 3, exponentiation as in 4 x 4 x 4, and algebraic factorisation as in $(2x + 3)(?) = 2x^2 + 5x + 3$
5. The ***for each*** idea also known as the ***Cartesian Product*** arises in the context of Data in primary school and Strand 1 in post-primary school. It also applies in rate or proportion problems and is evident in the structure of the place-value system, where for example, children need to think about the fact that ***for each*** ten, there are 10 ones, ***for each*** hundred there are10 tens, and ***for each*** one there are 10 tenths and so on. There are documents available outlining tasks that promote the development of the ***for each*** idea.

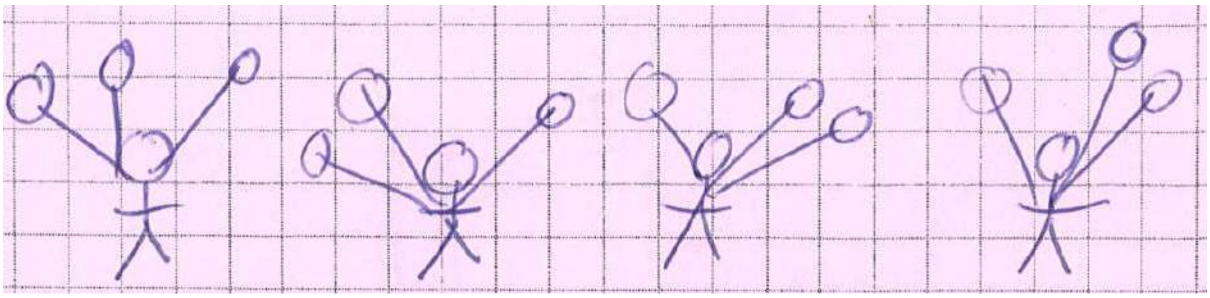
Supporting the shift from a 'groups of' way of thinking about multiplication to an array-based representation.

A group of First Year students were given the following task. The mathematical purpose of the task was to see how students were thinking about multiplication. There are several ***ideas*** that support the development of multiplicative thinking, and the ability to think multiplicatively is very important if students are to engage meaningfully with the ***Strand 3 Number*** in subsequent years. Consequently the development of multiplicative thinking is a major goal of the bridging period.

Task: Solve the following problem using a diagram.

4 people go to a party and they each bring 3 balloons. How many balloons in total do they bring?

The majority of students represented the situation as in the diagram below.



This is a *groups of* model of multiplication. The students are "*accumulating groups of equal size*" to represent the situation. It is a valid representation and learners can easily see the 4 "lots of" or "sets of" 3 balloons and can represent the situation with the arithmetic sentence

$$4 \times 3 = 12$$

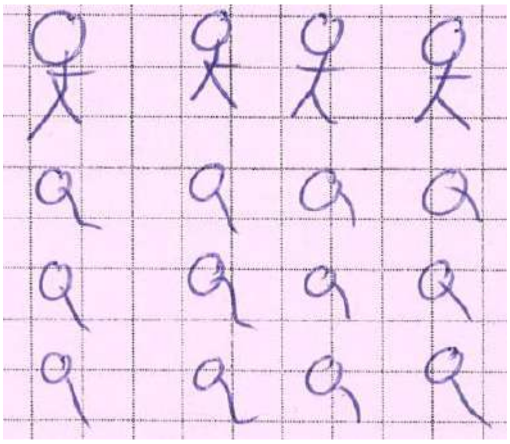
It is however, an additive model of multiplication and students need to move beyond this idea if they are to develop multiplicative thinking. A more useful model for making sense of the operation of multiplication is the *array model*

Why is it important that students make this shift?

Working with array representations enables students to

- simultaneously co-ordinate the number of groups, the number in each group and the total
- recognise commutativity
- relate the two ideas for division partition (or sharing) and quotition (or how many groups in), to multiplication

It also provides a basis for moving from a count of equal groups (eg, 1 three, 2 threes, 3 threes, 4 threes,…) to a constant number of groups (eg, 4 ones, 4 twos, 4 threes, 4 fours, 4 fives …) which supports more efficient mental strategies (eg, 6 groups of anything is double 3 groups or 5 groups and 1 more group).



These learners have arranged the balloons in an array and, as with the above model, they can easily see the 4 “lots of” or “sets of” 3 balloons and can represent the situation with the arithmetic sentence

$$4 \times 3 = 12$$

The array model will only be useful to learners if they fully understand how it can represent the story context and the arithmetic sentence. Learners need time to discuss this model and to reason and make sense of it; hence the initial simple task.

Discuss each group’s answer to the task and encourage learners to see how, of all the representations given, the array model is the most useful.

Useful questions to ask

- What does the “x” symbol represent in the story context? In the array?
- What does the “4” represent in the story context?
- What does the “3” represent in the story context?

Once learners have established the array model as a useful way to represent multiplication you can set further tasks that will allow them to use the model and reason and make sense of the operation of multiplication.

Case study

The students in 5th class were asked to solve the following problem by drawing a clearly labelled diagram.

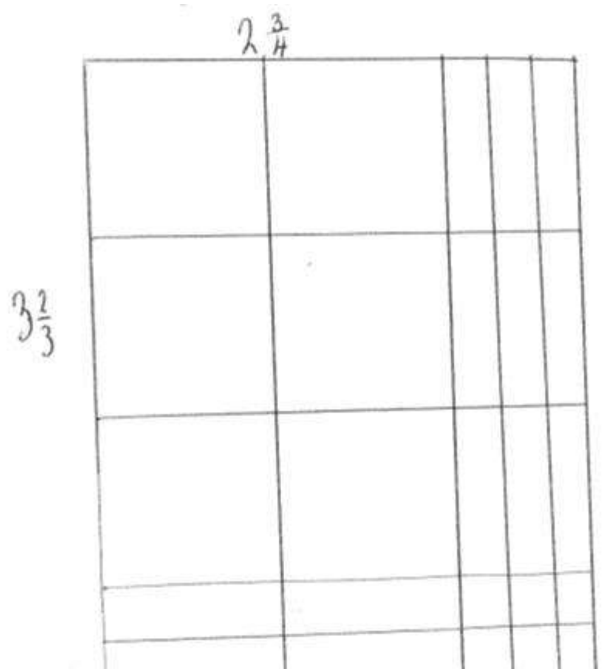
What is the area of a rectangle that has a width of $2\frac{3}{4}$ and a length of $3\frac{2}{3}$?

As I was circulating around the class, listening to and observing the students working on the problem, I overheard Tomás

Tomás: *Usually all you have to do to find the area is to multiply the length by the width, but we can’t do that ‘cos we have fractions.*

We had spent a lot of time working on area and perimeter problems, so the students were familiar with finding the area by counting the amount of units or in the case of rectangles by multiplying the length by the width. Why did Tomás think this method would not apply with fractions?

When we started the whole class discussion Darragh volunteered to come to the board to discuss his strategy for solving the problem. He carefully drew this diagram on the board



Darragh: *You can get some of the area but not all of it.*

Teacher: *What part of the area can you get?*

Darragh: *I know the length times the width is the area so 2 $\times$ 3 =6*

Teacher: *Where is the 2 $\times$ 3 or the 6 in the diagram?*

Darragh: *The big squares are the whole and you can just count 6.The smaller ones you can count too, but…eh they aren't wholes*

Teacher: *Why not?*

Darragh: *Those pieces aren't whole squares the way the other ones are, because of the fractions. So [starts counting rectangles on the top right] there are 9 of those $\frac{1}{4}$ s that is 2 wholes and $\frac{1}{4}$ left. There are 4 of those [points to rectangles at the bottom left] and that is $1\frac{1}{3}$. But I don't know how to count the others.*

Teacher: *Why not?*

Darragh: *I don't know; it's like they are pieces of pieces of something.*

John: *Like fractions of pieces when the pieces are fractions.*

There was a pause in the class and students started to think about that one. I let them discuss it for a minute or two then

Teacher: Can anyone explain what John is saying?

Joseph: *I think what he means is that those pieces [points to the smaller rectangle in the bottom right of Darragh's diagram] are fractions of fractions, but…what is that?*

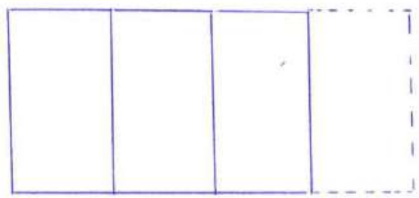
Tomás: *There is $\frac{2}{3}$ on one side and $\frac{3}{4}$on the other*

Padraig: *It's like $\frac{2}{3}$ of $\frac{3}{4}$ but you can't have that*

Tomás: *Yeh, there's no way you could have $\frac{2}{3}$ of $\frac{3}{4}$*

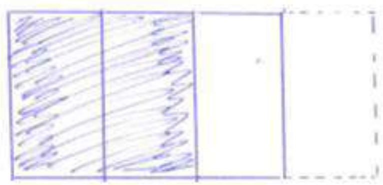
At this stage I decided to introduce an idea that might ease the confusion that the students were having. I used the example that Padraig had posed, but I put the idea into a simple and meaningful context.

Teacher: *Think about this; someone gave me* $\frac{3}{4}$ *of a leftover chocolate bar [I drew a diagram on the board]*



I ate $\frac{2}{3}$ *of that for little break. What part of the whole chocolate bar did I eat?*

Seya: *That much [comes up and shades in the diagram] It's* $\frac{1}{2}$



I said nothing for a few seconds and let the class think about what Seya did. Then Padraig said 'or it could be this…

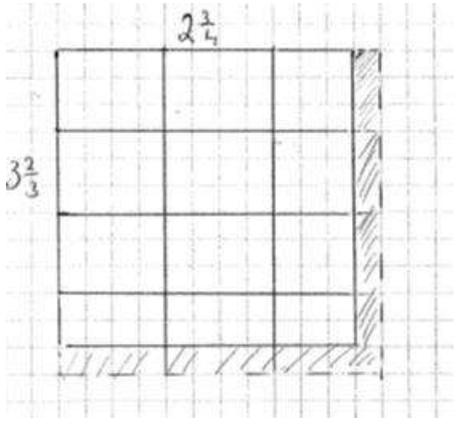
Tomás: *Well I think it's* $\frac{6}{12}$

[comes to the board and draws this]

We spent some time deciding which diagram was “right”. There were interesting discussions which I allowed to continue until the idea that both diagrams were equivalent was more comfortable for them. When we returned to the original problem it was decided unanimously that you could indeed find the area of the entire region by naming all the little bits and counting them.

This is what Darragh had to say.

Darragh: I can see now how you know what the names are ‘cos if you extend the diagram to show all the missing part it’s easy look.



Look it’s so easy 6 [points to the 6 full squares] there’s three $\frac{3}{4}$*’s [points to the top 3 rectangles in the right-hand column]which is* $2\frac{1}{4}$ *and here [points to the 2 bottom right rectangles] two* $\frac{2}{3}$*’s which is* $1\frac{1}{3}$ *and these are the ones I couldn’t do before but it’s easy now:* $\frac{6}{12}$ *‘cos I can see what the whole is.*

So the area is $6 + 2\frac{1}{4} + 1\frac{1}{3} + \frac{6}{12}$

Teacher: *Could we write this in another way?*

There was a lot of discussion.

Tomás: $\frac{6}{12}$ is $\frac{1}{2}$ and that is $\frac{2}{4}$ so the area is $9\frac{3}{4}$ and $\frac{1}{3}$

Teacher: *Can everyone see Tomás's answer in the diagram?*

At this stage there was a lot of discussion as the students tried to show the $9\frac{3}{4}$ and $\frac{1}{3}$ in the diagram. Then Seya said:

Seya: *When you look at the diagram it's easier just to say $\frac{121}{12}$, just one number. Llook it's easier [points to the diagram]*

John: *Yeh, the diagram gives you Seya's and the sums give you Tomás's.*

Thoughts for teachers:

- What prior knowledge should your students bring to the task?
- Are your students ready for this task?
- How would you use this task with your students?
- What mathematics do you want your students to learn from engaging in this task?
- What do you think your students might find difficult about this task?
- What questions might you ask as your students as they are working on the task

Key Concepts in Mathematics – Partitioning

If these concepts are not fully developed, students will find it difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years.

Partitioning: ***A deep understanding of how fractions are made, named and renamed.***

It provides the connection between fractions and the ***sharing*** or ***partititive*** idea of division, and to multiplicative thinking more generally.

How does the concept develop?

Even before they come to school many young children show an awareness of fraction names such as half and quarter. During the first years of schooling, most will be able to halve a piece of paper, identify three-quarters of an orange and talk about parts of recognised wholes (e.g., bars of chocolate, pizzas, cakes, etc). **Beware**! Teachers and parents often think, then, that children understand the relationships inherent in fraction representations. For many children, however, they are simply using these terms to describe and number well-known objects. They may not be aware of or even paying any attention to the **key ideas** involved in a more general understanding of fractions. That is, that

- equal parts are involved
- the number of parts names the parts
- as the number of parts of a given whole is increased, the size of each part (or share) gets smaller.

Partitioning builds on 'region' and 'area' models of multiplication and is a necessary link in building fraction knowledge and confidence. The area model leads to the '**by**' or '**for each'** idea and, more generally, the **factor-factor-product** idea of multiplication and division, which regards multiplication and division as inverse operations. This is the idea needed to support all further work with rational numbers and in algebra.

Partitioning, therefore, is more than just the experience of physically dividing continuous and discrete wholes into equal parts; it also involves generalising that experience so that students can create their own fraction diagrams and representations on a number line and can understand the key ideas mentioned above.

A well-developed capacity to partition regions and lines into any number of equal parts supports fraction renaming and justifies the use of multiplication in this process. The concept of partitioning is best developed when students make their own fraction diagrams rather than interpreting those produced by others. ***Halving***, ***thirding*** and ***fifthing*** are partitioning strategies that students can engage with that facilitate the development of understanding.

Encourage students to reflect and share their strategies, because

- ❖ verbalising brings the strategy to a conscious level and the student learns about their own thinking
- ❖ other students are given the opportunity to pick up a new strategy
- ❖ the teacher is given an opportunity to assess the type of thinking taking place and so can adjust the teaching accordingly.

Read the ***case studies*** and ***tasks*** for ideas on how you can support and track your students' development of the concept of partitioning.

Read an interview with a teacher and find out how she helped her students develop the concept of partitioning by engaging them in a rich task that required them to use representation to help reason and justify their ideas.

Task

Using only the coloured card provided make $3\frac{3}{4}$. Write $3\frac{3}{4}$ in as many ways as you can and justify your naming using your poster.

What mathematics did you want your students to learn from engaging with this task?

Well ultimately I want my students to be able to operate efficiently on rational numbers and to understand the algorithms we commonly use, in order to do this they need a well-developed concept of partitioning. This task gives me an opportunity to assess this conceptual understanding in order to progress the learning whilst at the same time helping students make connections with their previous mathematical experiences. I'm hoping for a few ah ha moments. All the students have dealt with fractions in Primary school but I'm not sure about their conceptual understanding, judging from their errors I suspect it's quite poor

The syllabus learning outcomes the students will be working on are

- investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers
- consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value
- engage with the idea of mathematical proof
- use the equivalence of fractions, decimals and percentages to compare proportions
- explore patterns and formulate conjectures
- explain findings
- justify conclusions
- communicate mathematics verbally and in written form
- apply their knowledge and skills to solve problems in familiar and unfamiliar contexts

How did you manage the task in your class? I divided the class into groups of 4 and gave each group a task. The tasks were similar in so much as each group was required to make a rational number greater than 1 from a piece of A4 coloured card and then rename the rational number in as many ways as they could. The tasks differed in difficulty as each group was given a different rational number to represent.

What did you find interesting about the students' approach to the task? I was surprised at how difficult the students found the task. They really grappled with the concept of 'the unit' I heard comments such as *But we only have one piece of card how can we show* $2\frac{3}{10}$ *with only I piece of card?.*

Many students' work displayed evidence of the fact that they did not understand the concept of equal sizes and students' work lacked precision.

How did you help the students' get over their initial difficulties so that they could access the task?

Some groups were really not in a position to engage with this task and I simplified it for these students by changing the focus fraction to one less than 1. You can read about this task here.

The following is an extract of a discussion with the group of students who were concerned that they needed more sheets of card

Me: Well how many pieces do you need then Josh?

Josh: ehm well 10 ..

Me: Why 10?

Josh: Because then 3 of them would be $\frac{3}{10}$

Me: But you are to make a poster of $2\frac{3}{10}$ So what about the 2

Josh: Then just 2 more

Me: So what would your poster look like then?

Josh: ehm 5 sheets

Me: [To the group] Would you agree with Josh?

When there was no real commitment to an answer from the group I engaged the whole class. I asked Josh to explain his thinking to the class.

Erica: I don't think that is right …..cos that means 3 Sheets are $\frac{3}{10}$ and then 2 sheets are 2 wholes

Me: So what do you think the 5 sheets would be?

Erica: ehm $\frac{5}{10}$

Sorcha: That's $\frac{1}{2}$

There was much discussion about the fact that 5 sheets of paper would be $\frac{1}{2}$ and eventually Josh said

Josh: Ye so 5 is a half and 10 is a whole so I would need 20 sheets to make 2 so I need 23 sheets to make $2\frac{3}{10}$

Me: So what would your poster look like?

Josh: 23 sheets of card

I drew 23 squares on the board each representing a piece of card and said to the whole class

Is it clear that this is a 'picture' of $2\frac{3}{10}$?

The majority of students looked confused and said no. Josh was prepared to defend his work, I was pleased about this.

Josh: It is if you know that 10 sheets is 1 whole..you could do a key.

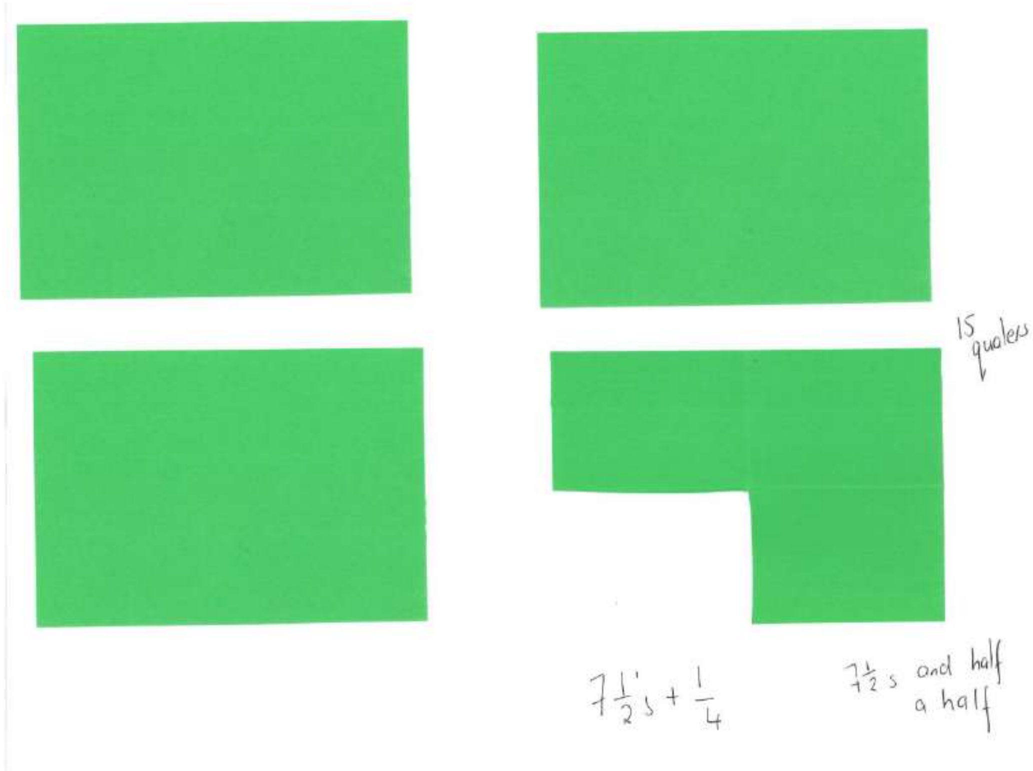
Me: That is true but could we make $2\frac{3}{10}$ in a way that everyone could see it was $2\frac{3}{10}$ without the need for a key?

I left the groups working on this for a while it was interesting to see the struggles , they just could not see how to deal with the 2, they could make $\frac{3}{10}$ and a breakthrough came when Erica spoke.

Erica: Think about what the one is first then make it like this [she folds the paper in four and says this is one and this is one and now I need $\frac{3}{10}$. I'm going to fold to get that.

Here is Erica's poster

I found this piece of work interesting



There was plenty of opportunity for follow on work from this piece of work. I asked the group to write arithmetic sentences to describe the statements written on the poster, this gave a great opportunity to think about the concept of equality. I would use this poster in a later lesson to help students make sense for the algorithm we use for addition and multiplication.

Points for teacher discussion:

- Erica's comment *I don't think that is right …..cos that means 3 Sheets are* $\frac{3}{10}$ *and then 2 sheets are 2 wholes* was a turning point in this lesson. What would you do if your students did not provide this level of understanding? How would you progress the learning?
- Erica provided another turning point when she said *Think about what the one is first then make it.* How would you have progressed the lesson if your students were not thinking like this?
- How could you use the student's poster to help your class make sense of the algorithms we use for addition and subtraction

Key Concepts in Mathematics –

Proportional Reasoning

If these concepts are not fully developed, students will find it very difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years.

Proportional Reasoning *"the ability to recognise, to explain, to think about, to make conjectures about, to graph, to transform, to compare, to make judgements about, to represent, or to symbolize relationships of two simple types … direct … and inverse proportion*" Lamon (1999)

How does the concept develop?

Proportional reasoning has been referred to as the capstone of the primary mathematics and the cornerstone of algebra and beyond. It begins with the ability to understand multiplicative relationships, distinguishing them from relationships that are additive.

Proportional reasoning involves some kind of comparison and, at its core, it requires a capacity to identify and describe what is being compared with what. Recognising what is being compared with what, however, is not always straightforward and it can be further complicated by the types of quantities used, how they are represented, and the number of variables involved.

Research (Van de Walle, 2007) has shown that

- proportional reasoning is best developed in investigative problem solving lessons
- students understand best when multiple strategies are shared and discussed
- many of the most valuable activities to develop proportional reasoning do not involve solving proportions at all but rather reasoning about 'more' in everyday common situations
- problems should start with high content, hands-on situations.

A proportional thinker

- has a sense of covariation, that is, they understand relationships in which two quantities vary together and are able to see how the variation in one coincides with the variation in another
- can recognise proportional relationships as distinct from non-proportional relationships in real-world contexts
- develops a wide variety of strategies for solving proportions or comparing ratios, most of which are based on informal strategies rather than prescribed algorithms
- understands ratios as distinct entities representing a relationship different from the quantities they compare.

Read the ***case studies*** and ***tasks*** for ideas on how you can support and track your students' development of the concept of Proportional Reasoning.

Case Study: The task below was used by a group of teachers from 5th class, 6th class and First Year to help them learn about how their students think. They were particularly interested in whether the students' solution strategies were based on additive thinking or multiplicative thinking. The wording of the task was adapted to suit the students.

Task

The First Year students in Scoil Mhuire are going on an outdoor adventure trip. Each student can choose an activity. The table shows the student's choices.

	Rock Climbing	**Canoeing**	**Archery**	**Zip lining**
Group A	15	18	24	18
Group B	19	21	38	22

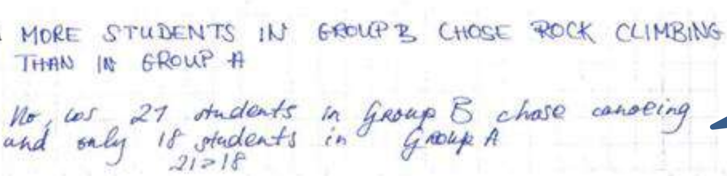
a) What can you say about the choices of Group A and Group B students?
b) The First Year Year Head said that canoeing was more popular with Group A students than Group B students. Do you agree with the Year Head's statement? Use as much mathematics as you can to support your answer

Problem solving reminder: If you are going to use this task remember, answers are important but what is more important is the mathematics students can learn from engaging in the task.

Samples of student solutions

The solution strategies were classified into 3 groups and the samples labelled A, B and C are typical of the solutions in each category.

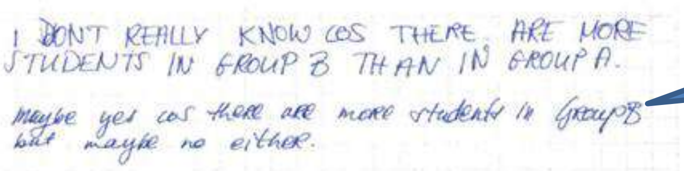
A


MORE STUDENTS IN GROUP B CHOSE ROCK CLIMBING THAN IN GROUP A

No, cos 27 students in Group B chose canoeing and only 18 students in Group A 21>18

Solutions in this category rely on the relative magnitude of the numbers alone. There does not seem to be any awareness of the relevance of proportion.

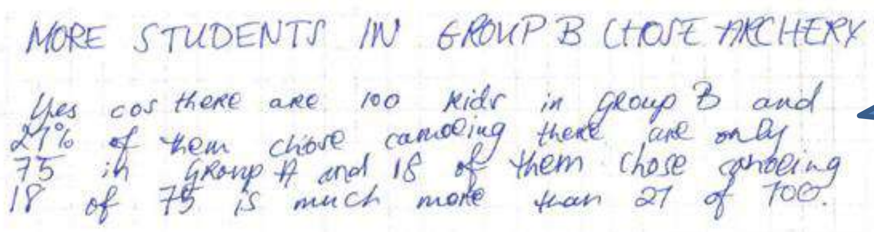
B


I DON'T REALLY KNOW COS THERE ARE MORE STUDENTS IN GROUP B THAN IN GROUP A.

maybe yes cos there are more students in GroupB but maybe no either.

Solutions in this category made at least one observation which recognises the difference in total numbers.

C


MORE STUDENTS IN GROUP B CHOSE ARCHERY

Yes cos there are 100 kids in group B and 27% of them chose canoeing there are only 75 in Group A and 18 of them chose canoeing 18 of 75 is much more than 27 of 100.

Solutions in this category displayed evidence of the awareness of proportion in the situation.

A: Evidence of additive thinking.

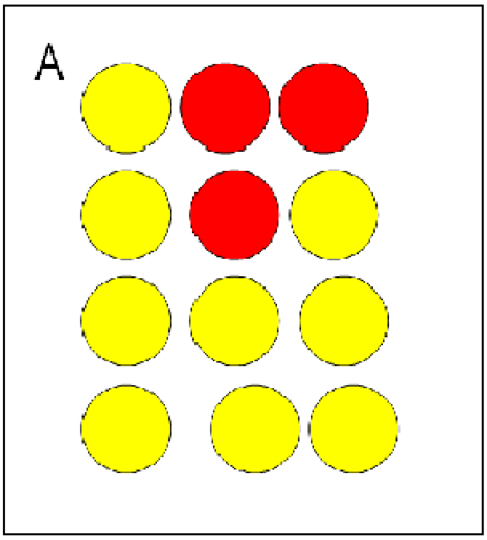
B: Evidence of moving towards multiplicative thinking.

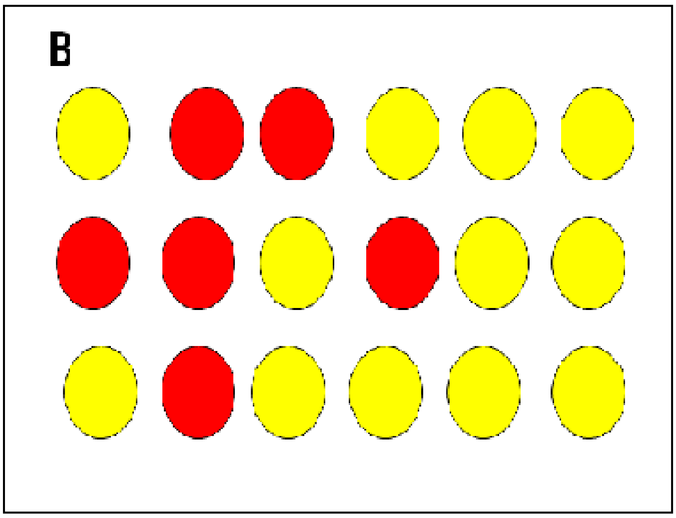
C: Evidence of multiplicative thinking.

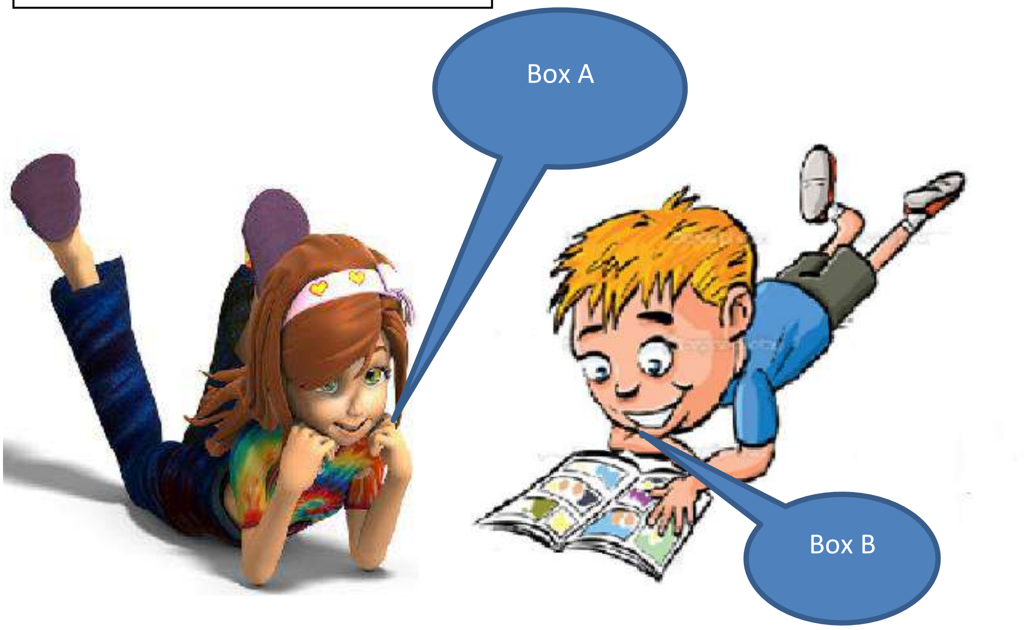
Although more First Year solutions were categorised as C, and so more of these students were thinking multiplicatively since they could sense the relevance of proportion in the situation. There was still a significant majority whose answers relied on the relative magnitude of the numbers alone, they were probably not aware of the relevance of proportion and were working additively. There was some 5th and 6th class solutions categorised as C but the majority of solutions from these students were classified A or B, these students are still thinking additively.

Task: This task is particularly useful in helping students to construct an understanding of the difference between **absolute comparison** and **relative comparison** and to become aware of the relevance of proportion.

Seán and Sinead were asked which has more yellow counters? Box A or Box B?







a) Both Seán and Sinead can justify their thinking.
   What do you think Sinead was thinking when she said Box A has more yellow counters?
b) What do you think Seán was thinking when he said Box B has more yellow counters?

Thoughts for teachers:

- How might a multiplicative (relative) thinker respond to this task?
- How might they justify their reasoning?
- How might an additive thinker explain what Sinead was thinking?
- How might they justify their reasoning?
- How can you help an additive thinker explain what Sineád was thinking?

Here are some ideas to help additive thinkers begin to think multiplicatively

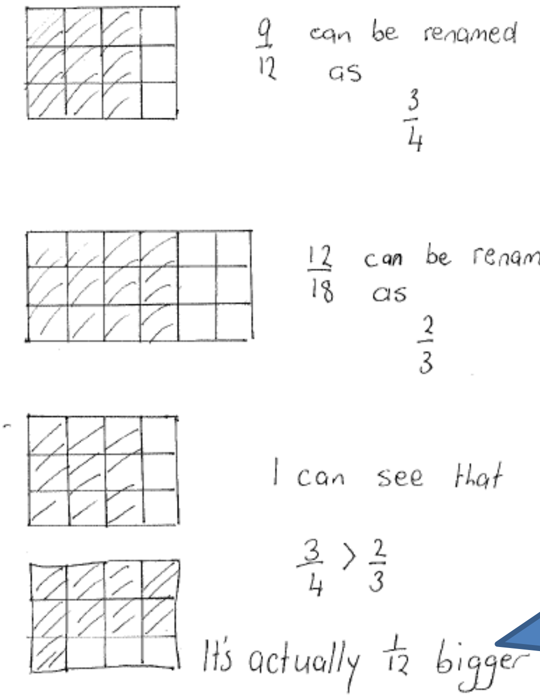
Q. ***What proportion of the box is taken up with yellow counters?***
Encourage answers like

Box A has 9 yellows out of 12 counters whilst Box B has 12 yellows out of 18 counters

Which is bigger?
Encourage learners to draw diagrams to compare these.

Watch the video to see how a student uses partitioning ideas to shows how 9 out of 12 is greater than 12 out of 18.

Here is another example of how a student showed that 9 out of 12 is greater than 12 out of 18.



Think about how you could progress this student's learning. What questions would you like to ask this student?

Think about the prior knowledge each of these student has brought to this task. Both students have developed the concept of partitioning. A well -developed capacity to partition regions and lines into any number of equal parts supports fraction renaming and justifies the use of multiplication in this process. It is clear from the student work that they can easily rename fractions.

When the student was asked how do you know from this diagram that nine twelfths can be renamed as three quarters? The response was; ***Look see the twelfths are the squares and there is 9 of them and the columns are the quarters see there are 3 of them so they are the same*** $\frac{9}{12}$ ***is the same as*** $\frac{3}{4}$***.***

Revisit ***Partitioning*** for tasks that you can do with your students to help develop the concept of partitioning.

Problem solving reminder: If you are going to use this task remember, answers are important but what is more important is the mathematics students can learn from engaging with the task.

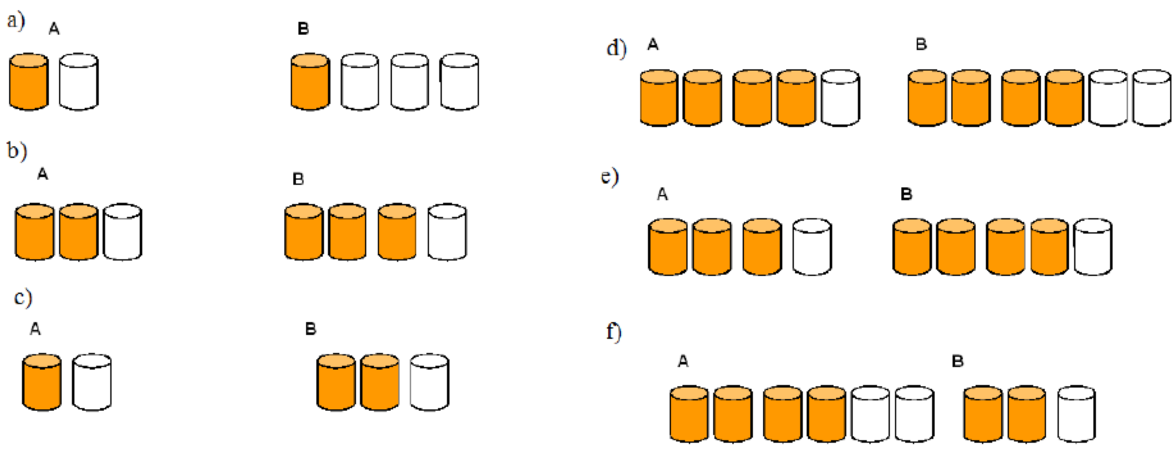
Reasoning about Comparison

This task, set in a real-world context is particularly useful in helping students to construct an understanding of the difference between absolute comparison and relative comparison and to become aware of the relevance of proportion.

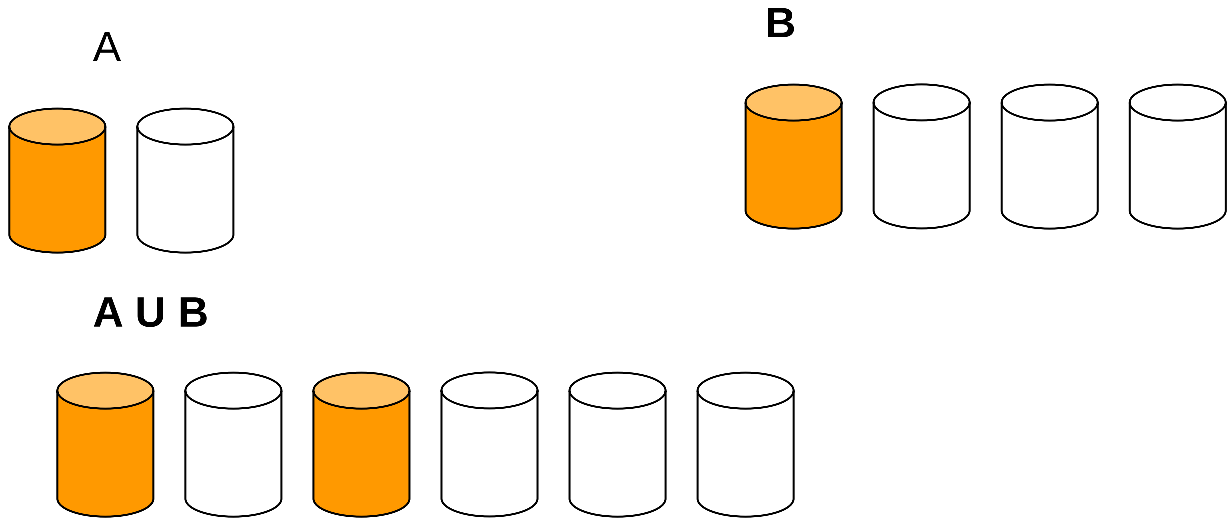
Task: John and Mark were making squash for the school sports. It is important that they get the right strength, they decided to choose a strength by mixing the orange concentrate and clear water together until they get just the right taste then they recorded the shade of orange and this shade is the desirable strength.

The boys have several jugs of liquid, some with orange concentrate and some with clear water. They plan to mix these together in big bowls. Before they mix the liquids, they guess the shade of orange the mixture will be.

In the diagrams below, there are two sets (A and B) of orange-clear combinations to mix. Predict which set will be darker orange, and explain your reasoning.



Challenge: The boys decided to see what would happen if they took two different mixtures and mixed them together. They called this the "union" of the two mixtures. For example, in the first example above, they took the two mixtures A and B and formed the union of the mixtures. Standard mathematical notation for the union of two things (usually sets) is U, so they named their new mixture A U B.



a) In the above example which will be darker orange A, B or AUB?
b) Can A u B ever be more orange than either A or B? Explain

Thoughts for teachers

- How would you use these tasks with your students?
- What mathematics do you want your students to learn from engaging with these tasks?
- What prior knowledge should your students bring to the task?
- What do you think your students might find difficult about this task?
- What questions might you ask as your students as they are working on the task
    - How might an additive thinker answer which is darker orange?
    - How might they justify their reasoning?
    - How might a multiplicative (relative) thinker respond to this task?
    - How might they justify their reasoning? How could you extend those who reason multiplicatively about more?
    - How could you help those who reason additively begin to think multiplicatively about comparison?

Problem solving reminder: If you are going to use these tasks remember, answers are important but what is more important is the mathematics students can learn from engaging with the tasks.

Task:

The activity below is designed to promote discussion around the concept of "more" and to provide an opportunity for learners to reason multiplicatively (**relative comparison**) about comparison.

Task: During dinner at a local restaurant, the five people sitting at Table A and the ten people sitting at Table B ordered the drinks shown below. Later, the waitress was heard referring to one of the groups as the "coke drinkers." To which table was she referring?

Table A

Table B

Thoughts for teachers:

- What mathematics do you want your students to learn from engaging with this task?
- When would you decide to use this task with your students?
- What prior knowledge should your students bring to the task?
- What do you think your students might find difficult about this task?
- What questions might you ask as your students as they are working on the task
    - How might an additive thinker answer which is the coke table?
    - How might they justify their reasoning?
    - How might a multiplicative (relative) thinker respond to this task?
    - How might they justify their reasoning?
- What are the features of this task that make it good for enagaging students in discussion around the idea of '*more*'
- How would you manage this task in a mixed ability setting?

- How could you extend those who reason multiplicatively about '*more*'?
- How could you help those who reason additively begin to think multiplicatively about comparison?

Problem solving reminder: If you are going to use this task remember, answers are important but what is more important is the mathematics students can learn from engaging with the tasks.

Key Concepts in Mathematics - Generalising

If these concepts are not fully developed students' will find it difficult to engage meaningfully with core aspects of the Number, Algebra and Functions strands in later years

Generalising Claiming that something is always true

How does the concept develop?

''*Generalisation is a heartbeat of mathematics. If the teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalisations, then mathematical thinking is not taking place''* Mason (1996) (p. 65).

Students begin to make ***generalisations*** when they begin to address the question ***Does this always work?*** When they begin to justify their own generalisations, they tend to use diagrams, concrete objects and words to do so. As their statements become more complicated they begin to need other ways to point at 'the first number', or 'the bigger number'. This is the beginnings of what later becomes conventional algebraic notation. As they move from particular numbers and actions to patterns of results, they start viewing numbers and operations as a system. This reasoning about operations rather than the notation is part of the work of the bridging period in algebra. Looking for pattern and generalising it, the other area of focus during this period.

Students are ready to engage with the learning outcomes associated with generalisation when they can

- deal with equivalent forms of expressions
- recognise and describe number properties and patterns
- work with the complexities of algebraic text

Difficulties may arise if students

- do not have an understanding of equality as a relationship between number sentences
- have limited access to multiplicative thinking and proportional reasoning

Reasoning about mathematics is an objective of the syllabus and students can begin to show formal reasoning by generalising patterns to fit various situations. In the bridging period we want students to be able to do the following:

- Reason about a problem
- Extend what they already know
- Make a conjecture
- Provide a convincing argument
- Refine their thinking
- Defend or modify their arguments

For many students, this will not be formal proof, but it will help them be better prepared to use proof in a more formal context later in post primary school. More importantly, as students become more adept in explaining and justifying their thinking, the mathematics they are learning will make sense which is what mathematics should be for all students – sensible and reasonable.
Read the ***case studies*** and ***tasks*** for ideas on how you can support and track your students' development of the concept of ***Generalising*** and their ***Understanding of equality***.

Coherence and Continuity

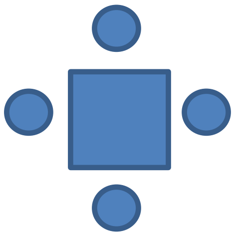
The study of pattern and relationships can support students develop ***multiplicative thinking*** and at the same time lead them smoothly into content traditionally taught at Senior Cycle such as ***functions***, ***sequences and series*** and ***calculus***.

Case Study:

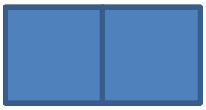
Teacher: I have a real mixed ability class. So I modified one of the *Tasks that promote multiplicative thinking* and gave different versions to different groups. Here are the tasks I used.

Task A:

Scoil Phadraig Naofa is planning a school party. They have lots of small square tables. Each table seats 4 people like this:



a) Make a line with 2 tables like this



How many people will able to sit at it? How many people will be able to sit at a similar line of 4 tables?

b) Make a line of tables that would seat 8 people. How many tables are needed?

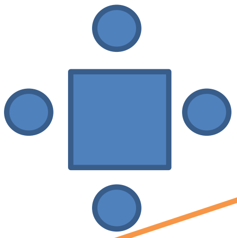
Can you find another way to describe your results so far? Show this in the space below

c) Without making a line of tables how many tables would seat 14 people? Check your answer by making a line of tables.

d) The school can borrow 99 tables. How many people could they seat using 99 tables placed end-to-end? Show your working and explain your answer in as much detail as possible.

Task B:

Scoil Phadraig Naofa is planning a school party. They have lots of small square tables. Each table seats 4 people like this:



I asked the students to "draw" rather than "make" this increases the cognitive demand of the task.

They decide to put the tables in an end-to-end line in the hall to make one big table.

a) Draw a line with 2 tables. How many people will able to sit at it? How many people will be able to sit at a line of 4 tables?
b) Draw a line of tables that would seat 8 people. How many tables are needed?
c) Can you find another way to describe your results so far? Show this in the space below.
d) The school can borrow 99 tables. How many people could they seat using 99 tables placed end-to-end? Show your working and explain your answer in as much detail as possible.

I didn't want to overwhelm the group so I separated out the task.

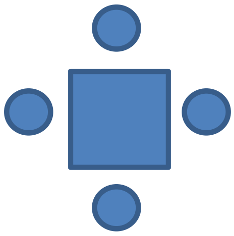
Extension for this group:

The school can borrow tables that seat 6 people.

e) Draw one of these tables showing the people sitting around it.
f) Draw a line of 5 of these rectangular tables placed end-to-end. How many people will be able to sit at it?
g) Explain what happens to the number of people as more rectangular tables are placed end-to end. Describe or show your findings in at least two ways.
h) How many people could be seated if 46 of these rectangular tables were placed end-to-end? Show your working and explain your answer in as much detail as possible.
i) How many of these rectangular tables would you need to place end-to-end to seat 342 people? Show your working and explain your answer in as much detail as possible.

Task: C

Scoil Phadraig Naofa is planning a school party. They have lots of small square tables. Each table seats 4 people like this:



a) How many people will sit at 10 tables if you put them together in a line to form one long table? 100 tables? n tables?

b) The school can borrow 99 tables. How many people could they seat using 99 tables placed end-to-end? Show your working and explain your answer in as much detail as possible.

The school can borrow tables that seat 6 people.

c) How many of these rectangular tables would you need to place end-to-end to seat 342 people? Show your working and explain your answer in as much detail as possible.

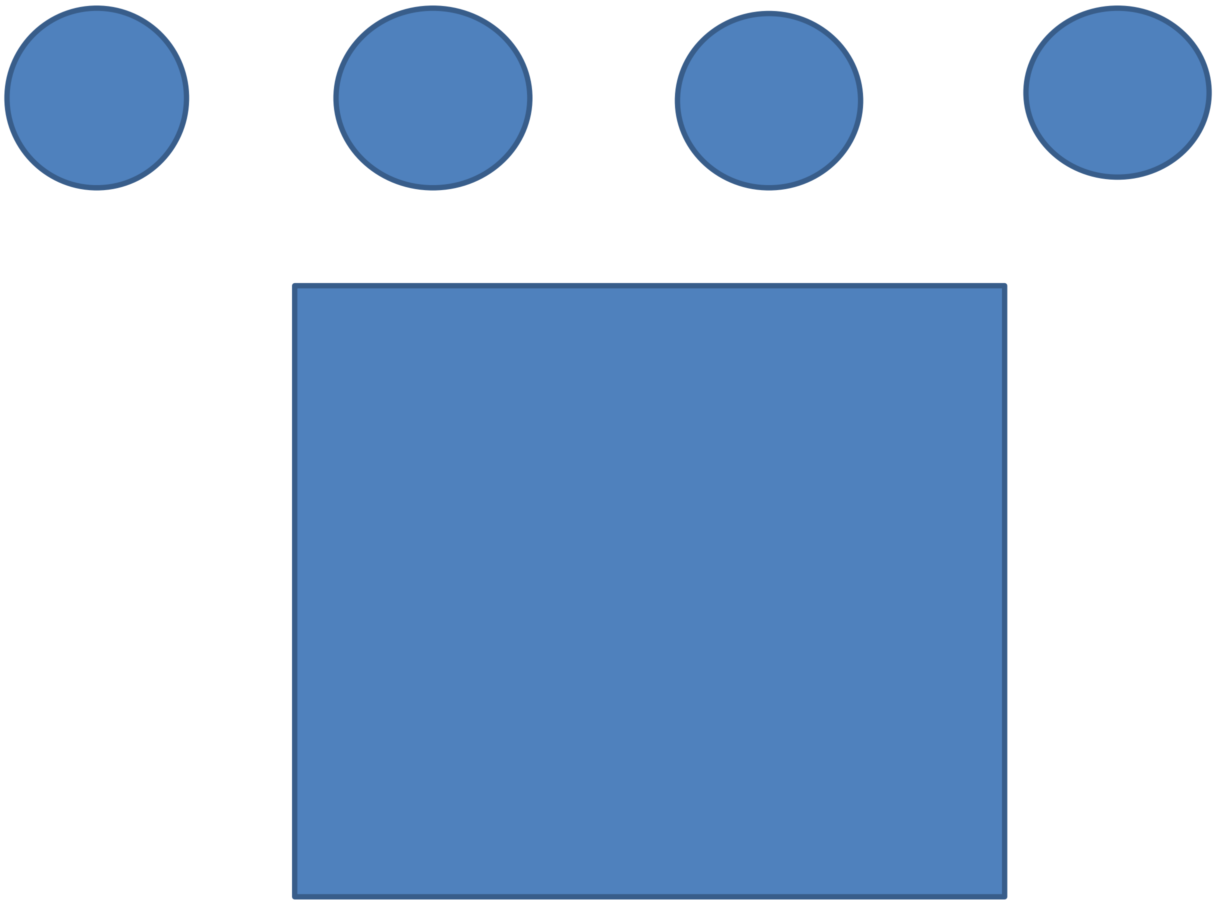
I am using this task for many purposes; to assess the extent of their multiplicative thinking and provide an opportunity to develop it further .I am acknowledging the fact that my students are at different stages of their conceptual development. Their solutions will give me insights into their thinking and will help me plan the next move. Task A; for the additive thinkers and those who are not yet reasoning abstractly, an opportunity to develop multiplicative thinking. Task B; for those who are just beginning to think multiplicatively, an opportunity to further develop. and Task C; for the multiplicative thinkers in the group. I want these students to focus more on the observation of pattern and generalising that pattern, as well as looking at features of the pattern in different representations.

Task A	**Task B**	**Task C**
– investigate models such as, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, and multiplication, in N where the answer is in N – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – explore patterns and formulate conjectures – explain findings – begin to look at the idea of mathematical proof – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions –	– investigate models such as, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, and multiplication, in N where the answer is in N – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – generalise and articulate observations of arithmetic operations – analyse solution strategies to problems – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions – analyse information presented verbally and translate it into mathematical form	– explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions – use tables and diagrams to represent a repeating-pattern situation – use tables, diagrams and graphs as a tool for analysing relations – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence

Student Work

Task B

I gave these students cardboard cards like this so that they could model the situation



The context certainly helped these students access the problem and they could easily see that every time you added a table you added two more people and there were 2 at the ends

Seán: For each table there is 2 added on look [points to the 2 people] and there is always 1 here and 1 here [points to the ends] so for 2 tables its 2+2+1+1=6 people

Me: What would it be if there were 10 tables?

Seán: Counts on fingers[2,4,6,8,10,12,14,16,18,20]+2=22

This was a typical additive strategy from the students working on task B. The context is helping them see the relationship between the number of tables and the number of people. I decided to introduce a ***tabular representation*** to help the shift to multiplicative thinking. I provided the columns and we completed the table together

Seán: Each time I add another table I add 2 more people I can actually see this in the table. The number of tables goes up by 1 and the people go up by 2 and I can actually see this in the table see.

Number of Tables	Number of People
1	4
2	6
3	8
4	10

So it’s 2 for every table

Me: How would you find how many for 10 tables if it’s 2 for every table?

Seán: 2x10 ..20

Me: If there were 100 tables?

Seán: 2x100 ..200 plus the 2 at the ends 202

Me: Can you write that rule in words?

After considerable discussion Seán wrote the generalised expression

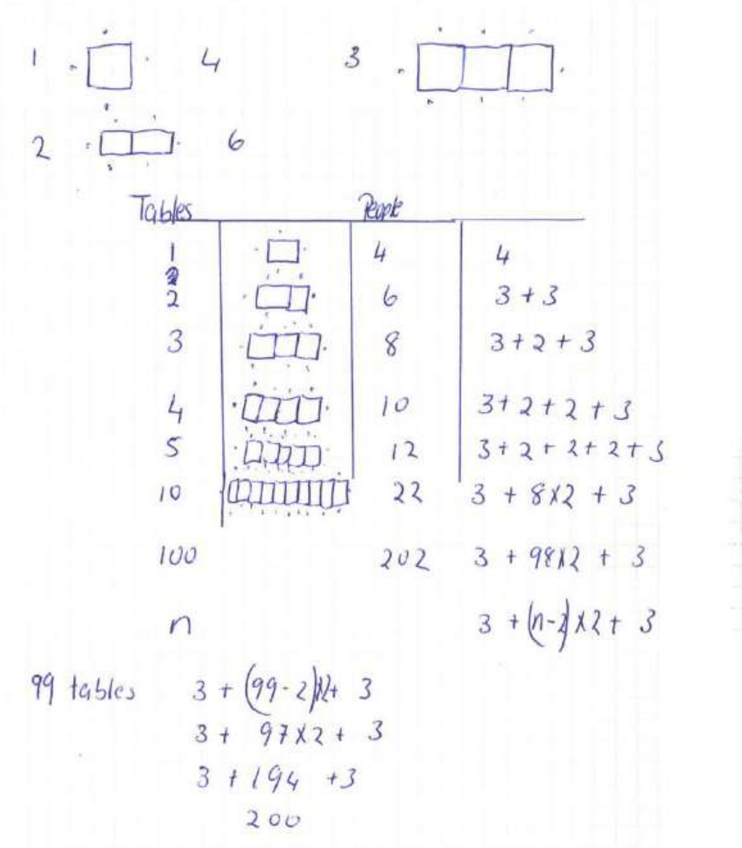
Number of people = 2 x number of tables +2

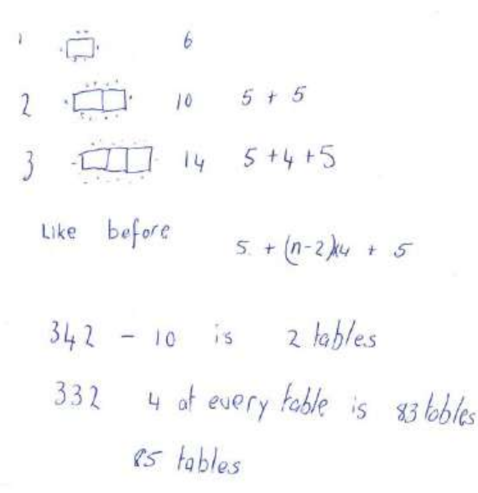
Student work

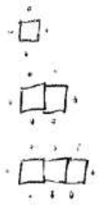
Task C

These two pieces of work allowed for a great discussion and a homework question.

Are 3+2(n-2)+3 and 2n+2 equivalent expressions? Justify your reasoning.







Tables	People	
1	4	4
2	2+4	6
3	2+6	8
4	2+8	10

Everytime you add a table you add 2 people and theres 2 at the ends. So for 10 its $(2 \times 10) + 2 = 22$

for 100 its $(2 \times 100) + 2 = 202$

for 99 its $202 - 2 = 200$

for n its $(2 \times n) + 2$

For 342 people 2 at the ends makes

340 people 2 for each table

If the tables held 6 people it would be 2 at the ends and 4 for every table

$2 + (4 \times \text{tables})$

For 342 people 2 at the ends

340 people 4 for every tables

gives 85 tables

Note to Teachers:

- How would you use these tasks with *your* students?
- Examine how the teacher adjusted the task to suit the needs of the class?
- How would you adjust such a task?
- What mathematics would you want your students to learn from engaging with these tasks?
- Examine how the students generalised the relationship between the number of tables and the number of people at each table. Did the context help? Are there any questions you would like to ask these students? If these were your students what task would you give them next to progress their learning?

Examining homework:

What does each piece of work tell you about the students' understanding of

- the concept of equality ?
- the commutative property?
- the operations of addition and multiplication?

Sample A:

Yes

$$3+2(n-2)+3 = 3+2n-4+3$$

$$= 6-4+2n$$

$$= 2n+2.$$

Sample B:

Yes cos

$$3+2(n-2)+3 = 3+n-2+n-2+3 = 3+3+n-2+n-2$$

$$= 6+n-2+n-2 = 6+n+n-2-2 = 6-2-2+n+n$$

$$= 2+2n$$

$$2n+2 = 2+2n$$

Sample C :

$$3+2(n-2)+3 = 6+2n-2$$

Problem solving reminder: If you are going to use these tasks remember, answers are important but what is more important is the mathematics students can learn from engaging with the tasks.

Generalising with a focus on equivalence

Case Study: I want my students to become flexible in recognising equivalent forms of linear equations and expressions. I am hoping that this flexibility will emerge as they gain experience with multiple ways of representing a contextualised problem. I liked this problem because I think it ticks all the boxes and gives my students an opportunity to develop all the 'bits' of mathematical proficiency it also provides a context in which they can use variables to represent a situation and hopefully gain fluency in using various representations.

Task:

The residents of a town wanted a new swimming pool. They campaigned with the local town councillors and eventually reached a deal. The council agreed to build a pool with an area of 36m^2 but the towns -people had to agree to buy the tiling to make a border around the outside. Money is quite tight in the community so it is important that the tiling bill is as low as possible. What dimensions should the new pool be in order to ensure that the cost of tiling the outside is as low as possible?

I used the ideas from Deborah Ball's video and I first asked the students some questions to make sure they fully understood the problem.

"***If the pool has to have an area of 36 m^2 then what could be the possible dimensions of that pool?'***

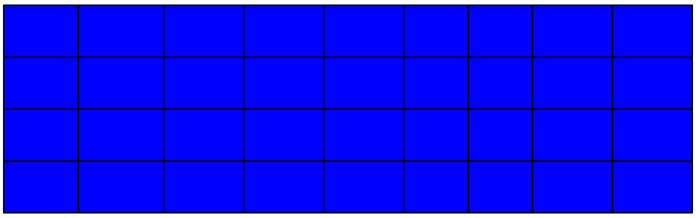
Sophie raised her hand and said 6m x 6m. I said "***any other ideas***? "

Josh said 9m X 4m. I was happy that the class were getting the idea and I said. ***"Keep in mind we have to have an area of 36m^2 any other dimensions****?"*

I wrote the suggestions on the board and intermittently commented ***Are these the only possible dimensions? Is that it****?* ***Did we use every possibility? Is every combination up there****?* I purposely asked about dimensions that couldn't be used. I said ***could we have a pool 5m x 6m? Why not?***

Once I was happy that all combinations were on the board I said "*So supposing each tile is 1m x 1m then how many tiles do you think it will take to go around a 9m x 4m pool? " have a guess*

I posted a cardboard model like this on the board I wanted to see what their guesses would be so I could get an idea of any misconceptions.



I circulated and listened to the conversations. I heard interesting things. I recorded the following conversation as I felt it was very interesting and could be insightful to others who would like to do this lesson

Sean: I think there will be 24

Me: Why do you think that?

Sean: cos I imagined 1 tile in 1 box and keep putting them all around and then count them all up and I get 24.

Me: What do others think? any other ideas?

Sam: I think it's 26 cos I did 9 +9+4+4.

Me: How many others think there are 26? ….A lot of hands went up ..Wow Sam you have a lot of support ..Sam you added up all the tiles around like that what is that called?

Sam The perimeter

Me: very good so you looked at the perimeter and you got 26 tiles…Sean I'm curious is that what you did and you just miscounted?

Sean: No I just think its 24 …see count it [He proceeded to count each tile and counted the edges of the two bottom tiles twice giving 24]

Me: Oh I see where you get the 24 now.

Jessica: But if you wanted to box the whole pool in wouldn't it be 30? Because if you count the corners because you would do 6 0n top 6 on the bottom and 9 on the sides that would be 30 ?

Sam: what do you mean box it in?

I called Jessica to the board and she demonstrated what she meant.

Me: So how many tiles does it take to make a complete border around this pool?

Jessica: 30 tiles

Sam: Oh I see there are 4 corners so it's 4 extra

Sean: Ye I get it now

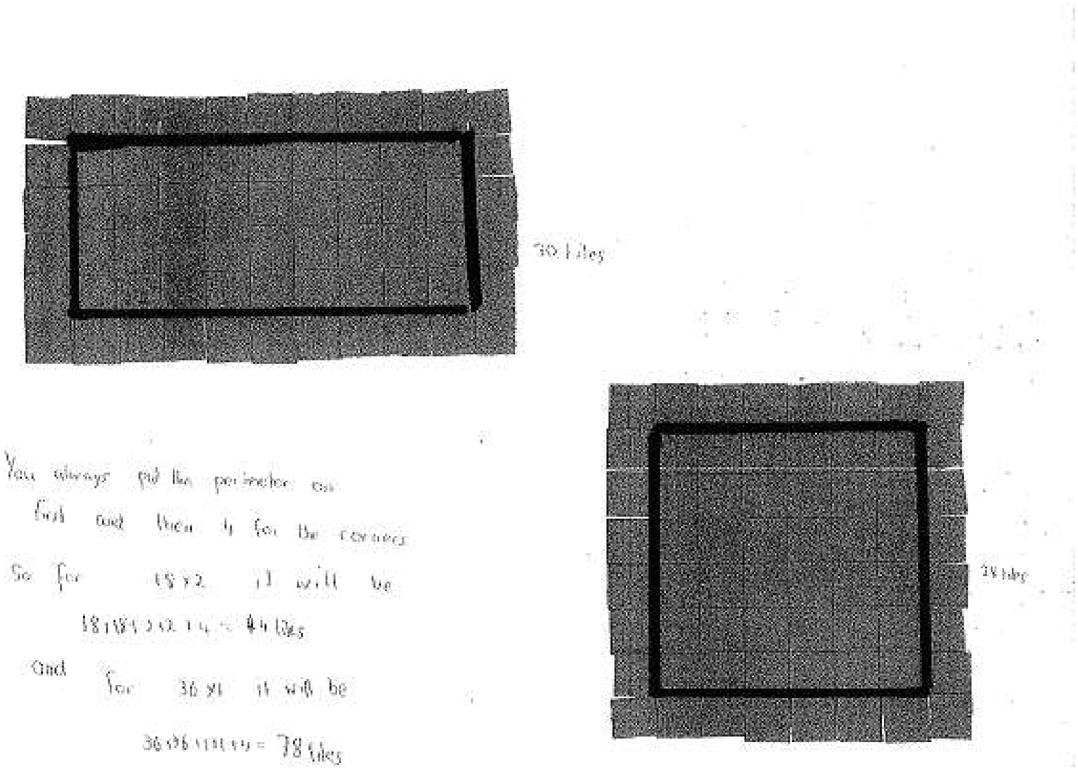
Me: So what I want you to do is to look at all the possibilities that we have for a $36m^2$ pool.

.... lets make a table for what we just saw

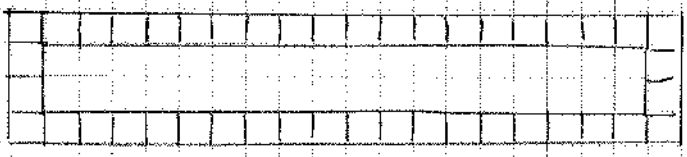
Dimensions of pool	No of tiles
9x4	30

So get into your groups build your own pools of different dimensions and tile the pools. Then look for a pattern to see how the dimensions of the pool relates to the number of tiles needed.

Sample work A:

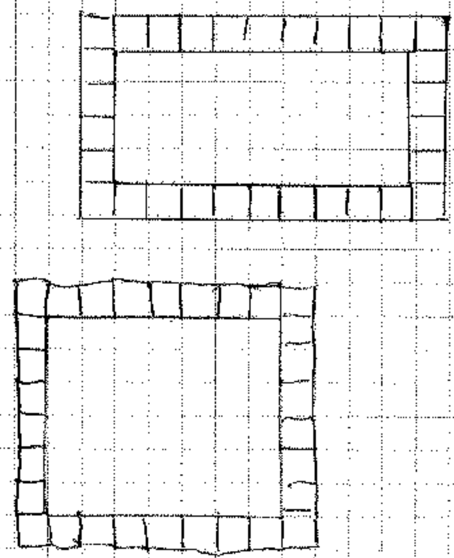


Sample work B



It's (Length + Width) x 2 + 4

Which is the perimeter + 4



Pool	# of tiles
6 x 6	2(6+6)+4 28
9 x 4	2(9+4)+4 30
18 x 2	2(18+2)+4 44
36 x 1	2(36+1)+4 78

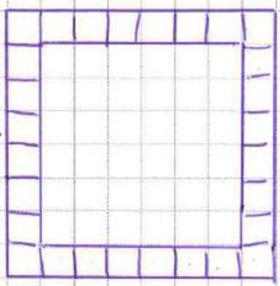
It works

go for a 6m x 6m

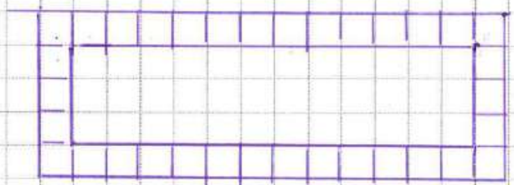
pool it needs

Less tiles

Sample work D

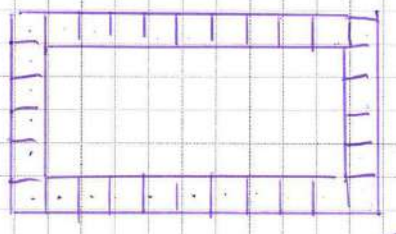


6 x 6 28 tiles



12 x 3 35 tiles

4 x 9



4 x 9 30 tiles

I see a pattern (2L + 2W + 4)

So 36 x 1 is going to need loads 72 + 2 + 4 = 80 tiles

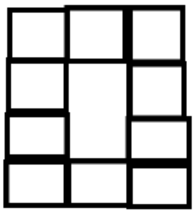
I would advice them to go for a 6x6 pool it's less tiles

The work above gave me an opportunity to look at the two expressions and ask learners to decide whether or not they are equivalent. I also took the opportunity to discuss the difference between an equation and an expression. I asked both groups to write equations to describe their observation rather than expressions.

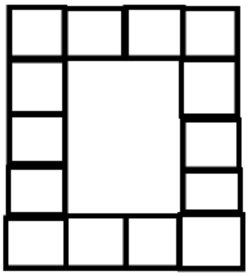
Everyone was in agreement that since the towns-people were on a tight budget they should go with the 6m x 6m pool as it would be cheaper to tile the border. To encourage flexible thinking I said ***if I wanted to do lengths in the pool which one would be best?*** A discussion ensued about how for lengths you would want a larger length and that a 36m x 1m pool would give you the largest length but it would not be very practical.

I would like to discuss the relationship between area and perimeter a bit more but I thought I would save it for another day I wanted to extend this work for now. I decided to pose another problem that is closely related to the original one but yet different so it allows the learners to stretch their thinking and apply what they learned from the first problem to this new problem.

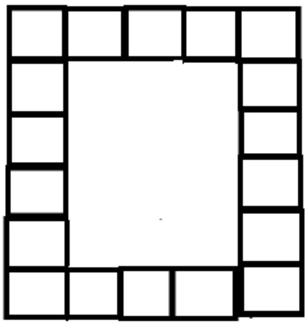
Task: I returned to the pool designer who produced a number of different designs which he numbered 1,2 ,3 as shown.



1



2

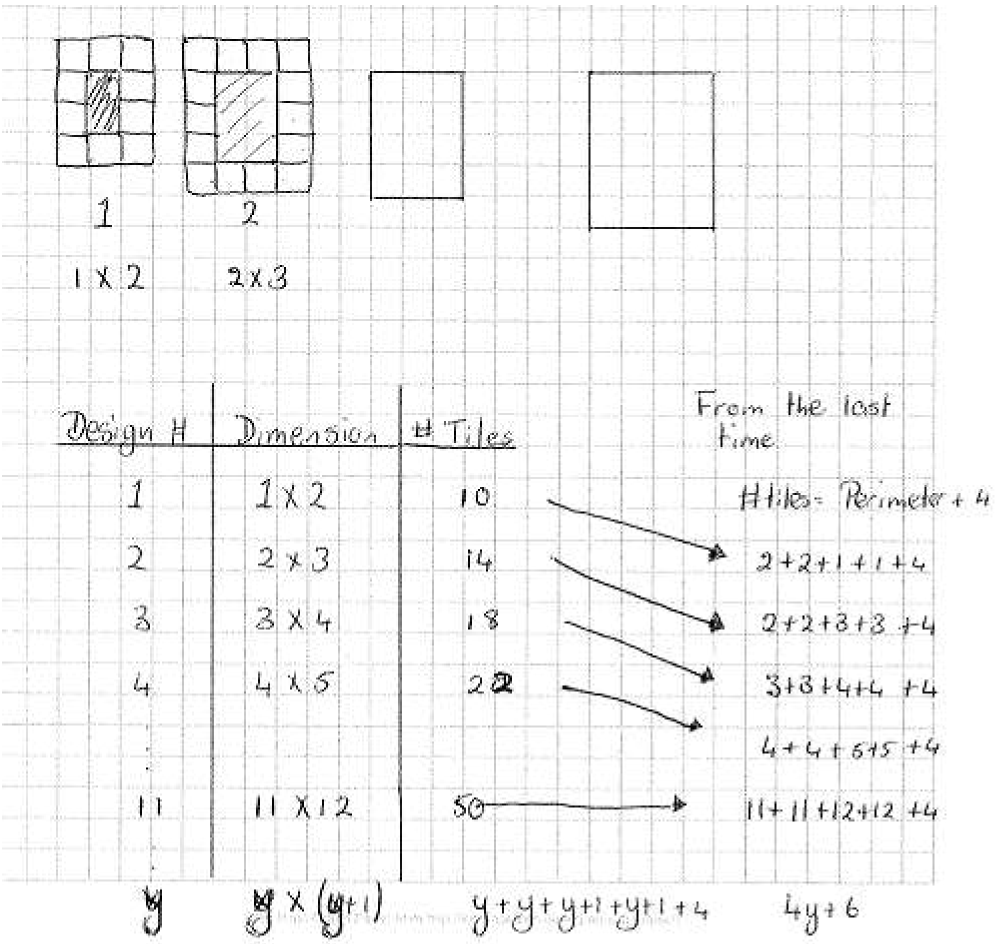


3

With design 1 you get a 1m x 2m pool and with design 2 you can get a 2m x 3m pool

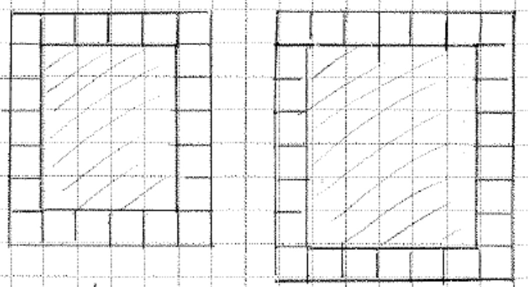
Think about what a design 4 pool would look like draw this out and think about the number of tiles needed for a border.
See can you see a pattern and determine the number of tiles needed for a design 11 pool. Then finally come up with an algebraic expression that relates number of tiles needed to the design number. Below are samples of student work

Sample work E:



Sample work F

Design #	# of tiles	
1	10	
		> +4
2	14	
		> +4
3	18	
		> +4
4	22	
		> +4
5	26	

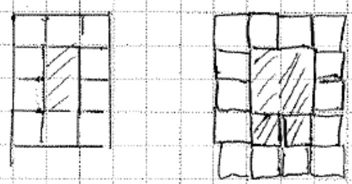


4 5

I see a pattern. It goes up by 4 each time thats the corners

It is like the other pools

in 1 it is 6 + 4 corners then



I see a pattern. It goes up by 4 each time

It is 4 x (Design #) + 6

Tasks for teachers:

Think about your students

- What mathematics would you hope they learn from engaging with the task?
- Which mathematical processes are evidenced in the student work?
    - Problem solving
    - Mastery of mathematical procedures
    - Reasoning and proof
    - Communication
    - Making Connections
    - Representing
- Examine the teacher's role in scaffolding this task to what extent did they help the students attend to the mathematical processes?
- How would you decide if ***your*** students are ready for such a task?
- How would you support students who struggle with this task?

Problem solving reminder: If you are going to use this task remember, answers are important but what is more important is the mathematics students can learn from engaging with the task.

Learning outcomes from the CIC and how they relate to the **key concepts**

Subitising	Place Value	Multiplicative Thinking	Partitioning	Proportional Reasoning	Generalising
	• investigate models such as decomposition, skip counting to make sense of the operations of addition, subtraction, in N where the answer is in N, including the inverse operations	• investigate models such as arranging items in arrays and accumulating groups of equal size to make sense of the operations of multiplication and division in **N** where the answer is in **N**, including the inverse operations • investigate the properties of arithmetic commutative, associative and distributive laws and the relationships between them • investigate models such as the number line to illustrate the operations of addition, subtraction, multiplication and division in **Z** • Consolidate their understanding of factors, multiples and prime numbers in **N**	• investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers • calculate percentages • use the equivalence of fractions, decimals and percentages to compare proportions	• consolidate their understanding of the relationship between ratio and proportion	• engage with the idea of mathematical proof • use tables and diagrams to represent a repeating-pattern situation • generalise and explain patterns and relationships in words and numbers • write arithmetic expressions for particular terms in a sequence

Lesson Study as a form of in-School Professional Development:

Case studies in two post-primary schools

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January 2015

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

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Acknowledgements

This research would not have been possible without the generosity of the Mathematics teachers who gave so freely of their time, expertise, and knowledge throughout this study. Furthermore the principals, staff, and students of the two schools involved were incredibly important in welcoming this educational research to their schools and in supporting their colleagues participating in the project.

This report is based in part on my doctoral dissertation completed at the School of Education, Trinity College Dublin, under the supervision of Dr. Aidan Seery. The research reported here was assisted by the National Council of Curriculum and Assessment who supported the two post-primary schools participating in this research. This study was also sponsored by an Ussher Fellowship postgraduate award from Trinity College Dublin.

Abbreviations

KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
NCCA	National Council for Curriculum and Assessment
PCK	Pedagogical Content Knowledge
PMDT	Project Maths Development Team

Introduction

This research investigated the introduction of a model of in-school professional development which would support mathematics teachers in their practices at a time of curriculum reform. Lesson study, a model of Japanese origin, was a model that would provide teachers with a structure within which they could collaborate and engage with new pedagogical practices encouraged as part of the Project Maths initiative. Twelve teachers in two post-primary schools volunteered to participate in the research which lasted for one academic year.

Qualitative data was generated in both schools through audio recordings of all teacher meetings conducted around lesson study and through interviews with participating teachers. The researcher participated in the investigation as a participant-observer in facilitating the introduction of lesson study to teachers and in participating in meetings as a former Mathematics teacher. The data was analysed thematically in iterative stages utilising a framework of pedagogical content knowledge (PCK) as suggested by Ball, Thames, and Phelps (2008).

Data analysis suggests that as a result of teachers' participation in lesson study, teachers developed their PCK and incorporated reform teaching approaches as part of their classroom practices. A number of teachers reported that their familiarity with and confidence in teaching the new mathematics syllabus improved. Furthermore, teacher communities of both schools developed through their participation in lesson study. Findings related to teacher learning and to the introduction of lesson study in two Irish post-primary schools are reported here.

This report is written as a summary of findings on research into teacher professional development which was supported by the NCCA and also as an introductory guide for teachers interested in lesson study. All references to participating teachers and schools are pseudonyms and any identifiable references have been removed from this report.

Educational Reform and Professional Development

Teachers play a key role in educational reform. Reforms typically impose new demands on the already complex work of teaching and it is therefore important that teachers are provided with professional development supports and opportunities to engage meaningfully with any reforms they are being asked to implement (Charalambos & Philippou, 2010; Hanley & Torrance, 2011; Remillard & Bryans, 2004).

This research was conducted following the introduction of Project Maths. This revised syllabus brought with it a change in content taught to post-primary mathematics students and also encouraged a change in the approach to the teaching and learning of mathematics from a commonly traditional exposition-based classroom (Lyons, Lynch, Close, Sheerin, & Boland, 2003; Oldham, 2001), to one encouraging communication of mathematical thinking with a focus on developing students' higher-order problem-solving skills. While structured in-service professional development opportunities were offered to teachers as part of this particular curriculum reform, this research investigated an alternative model of supporting teachers in their practices through school-based teacher community.

Teacher Community

A challenge in achieving sustainable reform is that most teachers teach alone in isolated classrooms without the opportunity to observe other teachers or reflect on their own practices (Remillard, 2005). While there is much recent focus on the social elements of learning for students, research increasingly refers to the importance of considering the social dimensions of learning for teachers (Grossman, Wineburg, & Woolworth, 2001; Hord, 2004) and it is widely recognised that teacher communities figure among the most important factors for promoting educational change within schools (de Lima, 2001; OECD, 2009). A key rationale for teacher community is that it provides an ongoing venue for teacher learning, provides teachers with social structures for professional collaboration and collegiality, and provides teachers with a means to engage with educational and curricular policies within the environment and realities of their own schools (Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009; Dooner, Mandzuk, & Clifton, 2008; Guskey, 2002; Louis & Marks, 1998; T. H. McLaughlin, 2004; M. W. McLaughlin & Talbert, 2006; van den Akker, 2003).

Teacher Learning and Lesson Study

Lesson study is a form of teacher professional development that is based on teacher collaboration and teacher community. The expression 'lesson study' is a literal translation from the Japanese word *Jugyokenky* where *jugyo* means lesson and *kenkyu* refers to study or research. The translation can be misleading in a sense that lesson study is more than a study of lessons, but rather is an investigation of teachers into their own practices through planning, conducting, observing, and reflecting on research lessons (Conway & Sloane, 2005; C. Fernandez, Cannon, & Chokshi, 2003; Corcoran, 2011a; Fernández & Robinson, 2006; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012; Murata & Takahashi, 2002; Takahashi & Yoshida, 2004).

A cycle of lesson study consists of the following steps portrayed in Figure 1:

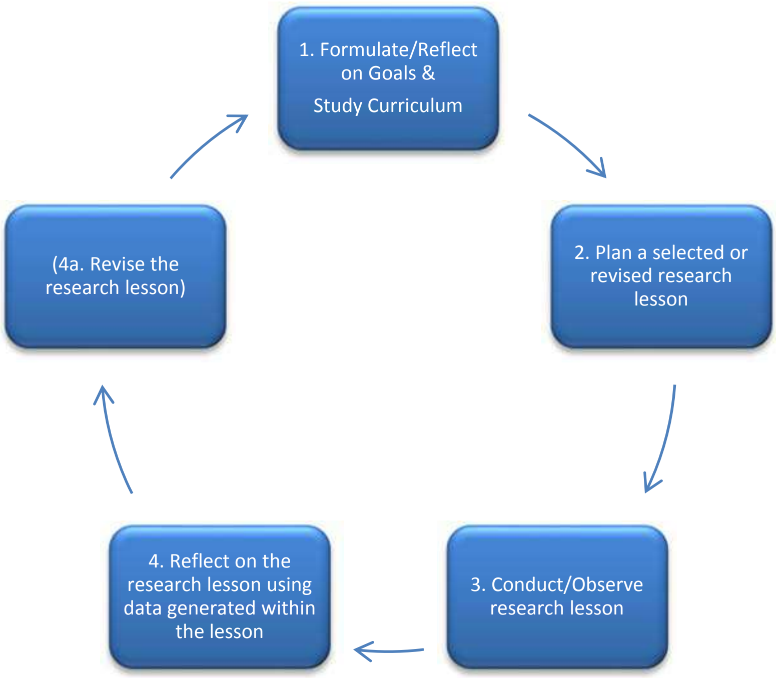


Figure 1 Lesson study cycle adapted from Lewis et al. (2009)

As part of the first step, teachers decide on an overall goal for their teaching which will guide their practices in lesson study. Teachers then directly access the curriculum, decide on a topic to teach within that cycle, and build a lesson plan around particular learning objectives (Lewis, Perry, & Hurd, 2009). Following step 2 of collaboratively planning content and materials for the lesson, one teacher conducts the research lesson while other members of the lesson study community attend and observe that lesson. This observation of the research lesson is an important phase of the cycle which differs to other forms of teacher observation since all teachers have engaged with the planning and observation of the lesson. Teachers then collectively reflect on the lesson and may decide to alter and re-teach it or continue to another cycle of lesson study (Fernandez & Chokshi, 2004).

Since this investigation occurred during a time of reform of the post-primary mathematics curriculum, it was important to incorporate professional development practices which would support teachers in engaging with the reform and in encouraging changes to traditional classroom practices. Remillard (2000) and Hanley & Torrance (2011) note that through engaging directly with curriculum materials and through seeing a curriculum enacted, teachers are encouraged to incorporate changes to their practices as part of curriculum reform. The four stages of lesson study directly reflect these curriculum reform practices of engaging with and enacting curriculum materials. Furthermore, lesson study encompasses observation and reflection which encourages teachers to notice student mathematical thinking during research lessons and structures teachers' reflection on their own pedagogical practices (Corcoran, 2011b; Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es & Sherin, 2008).

Previous research conducted both in Ireland and internationally suggests that participation in lesson study holds potential to develop teachers' content and pedagogical content knowledge (Corcoran, 2011b; C. Fernandez et al., 2003; Leavy, Hourigan, & McMahon, 2010; Lewis, Perry, & Murata, 2006). Murata et al. (2012) found that primary teachers developed in their understanding of student learning through conversations around planning a research lesson. Aligning with Murata et al.'s (2012) research, Cjakler et al. (2013), in their study involving four UK secondary school mathematics teachers, found that teachers began to develop less teacher-centred approaches and focused more on students' thinking through participating in lesson study. In Lewis et al.'s (2009) research they concluded that teacher participation in lesson study impacted on their teaching and learning practices and beliefs and Dudley (2013) found that teachers were more inclined to build in new approaches or take 'risks' in their teaching due to participation in lesson study. These studies suggest that through participation in lesson study, teachers are provided with opportunity to build on their understanding of teaching and learning approaches, to begin to incorporate new practices in their own teaching, and to develop in their approach to building students' mathematical understanding.

Furthermore as a form of professional development, participation in lesson study can help teachers develop a sense of community so that introduction of reforms can feel less daunting and more manageable as a consequence of developing of collaborative pedagogy (Lewis et al., 2009).

Leading from this rich research literature, this study aimed to investigate whether lesson study could be introduced as a form of professional development within Irish post-primary mathematics departments. In addition the research questioned how participation in lesson study would impact on teachers' PCK, referencing new pedagogical approaches highlighted within the Project Maths initiative.

Pedagogical Content Knowledge

Mathematics education research has found that while teachers' subject content knowledge can be an important factor in teaching mathematics (Campbell et al., 2014; Hill, Rowan, & Ball, 2005; Krauss et al., 2008), teachers' PCK also plays a vital role in students' learning experiences (Deborah L. Ball, Thames, & Phelps, 2008; Campbell et al., 2014; Hill, Ball, & Schilling, 2008). Shulman (1976) first referenced 'pedagogical content knowledge' as the knowledge unique to teachers as an amalgam of content and pedagogy which allows teachers to represent information in a number of ways that are particular and relevant to their own students. While there are a number of frameworks of mathematical knowledge for teaching (Rowland, 2014), in this research the multi-dimensional nature of teacher knowledge is incorporated within a framework suggested by Ball et al. (2008) which encompasses teachers' knowledge of mathematics and how to teach that mathematics to their students (Figure 2).

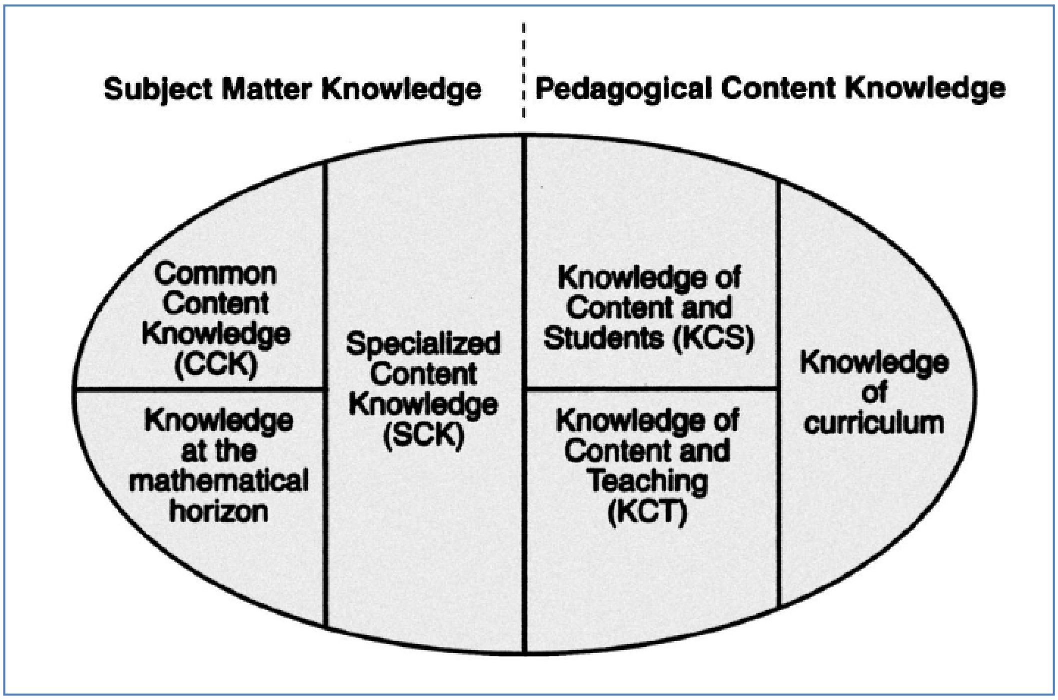


Figure 2 Framework of Mathematical Knowledge for Teaching from Ball et al. (2008)

Focusing on PCK as constructed by Ball et al. (2008), knowledge of content and students (KCS) refers to teachers' understanding of how students learn particular content and includes teachers' knowledge of common misconceptions students have. Knowledge of content and teaching (KCT) combines teachers' knowledge of various approaches and strategies which can be utilised in teaching a particular topic and the various questions which might best illuminate a topic for students. These elements of PCK will be further explored as findings in this research.

Methodology

This investigation was conducted as a double case study in two post-primary schools, Doone and Crannóg, which were involved in separate phases of the national curriculum roll-out. Twelve mathematics teachers volunteered to participate in the research and teachers taught students in all year-groups at all levels of assessment.

Qualitative data was generated in this research through audio recordings of all teacher meetings around lesson study and through individual teacher interviews held at separate stages during the research. Supplementary artefacts such as teacher notes, lesson plans, field notes, samples of student work, and a researcher log formed part of the structuring of data analysis, but were not considered as sources of data within the study. Teachers had autonomy in deciding when and how often they met as a lesson study community and had full independence in deciding the curriculum content to be taught in their research lessons.

The researcher engaged with the teachers in both schools as a participant-observer in facilitating initial phases of lesson study in participating in lesson study as a former mathematics teacher in a phase 1 school and in interviewing participating teachers using semi-structured interview guides. In order to minimise bias, analysis did not commence until all of the data had been generated. Audio files were transcribed and analysis of the data was carried out in four phases guided by a general analytic strategy as suggested by Braun and Clarke (2006). New themes and codes identified as part of the analysis were related to the literature on lesson study and teacher learning and were incorporated within a framework of mathematical knowledge for teaching as suggested by Ball et al. (2008)[1].

Participating Case-study Schools

Doone is a boy's school of approximately 550 students. Five of the nine mathematics teachers chose to participate in the research and all but one of the teachers taught more than one subject. Crannóg has a mixed gender cohort of approximately 800 students. Seven of the ten mathematics teachers at the school chose to participate in this research and of these, five teachers taught mathematics as the only subject in their time-table. These two schools had participated in different phases of the Project Maths curriculum roll-out.

[1] Further detailing of the methodology and analysis can be referenced in the doctoral thesis that is the basis for this report (Ni Shuilleabhain, 2014a)

The lesson study groups participating in both schools included teachers of varying experiences ranging from over thirty years teaching experience to newly qualified teachers. Both schools were urban based as a necessity of this research project. Referencing Ríordáin & Hannigan's (2011) study stating that approximately one half of teachers teaching mathematics at post-primary level are unqualified to do so, two of the twelve teachers were recognised as 'out-of-field'.

As discussed in their initial lesson study meetings, teachers in both schools had a number of reasons to volunteer to participate in the research ranging from feeling unsure and unconfident in teaching the revised syllabus, to wanting to investigate new ways in which they could collaborate as a Mathematics department.

Teachers had authority over when they would meet as a lesson study community, what the content of research lessons would be, what students would be taught, and how many lesson study cycles they would complete during the academic year. Research lessons were therefore developed for both a mix of junior and senior students and over a range of mathematical topics from investigation of quadratic patterns to introducing Pythagoras' Theorem. Teacher meetings were held, on average, once every two weeks and each meeting lasted on average one hour (see Table 1).

Table 1 Lesson study cycles and meeting times

School	Crannóg		Doone	
Lesson Study Cycle	**Lesson Content**	**Duration (Hr:Mins)**	**Lesson Content**	**Duration (Hr:Mins)**
1	Introducing quadratic patterns	0:36 1:19 1:44 1:08 1:23 1:03 (p-l)	Introducing the concept of x-squared	0:44 0:41 1:49 0:35 1:55 1:19 (p-l)
2	Factorising quadratic expressions utilising concrete resources	0:57 1:27 1:01 0:47 (p-l)	Multiplying Fractions: Developing a sense of measure	1:13 1:27 1:16 1:04 0:31 (p-l)
3	Factorising quadratic expressions utilising concrete resources (2)	1:01 (p-l)	Exploring quadratic expressions through application	1:13 1:16 0:30 0:23 (p-l)
4	Differentiation in Calculus: Applications	1:18 1:16 1:57 0:29 (p-l) 0:43	Introducing Pythagoras' Theorem	1:07 0:45 0:38 1:29 0:13 (p-l)

* p-l = post-lesson discussion

Teachers utilised, but were not restricted to, lesson study guidelines for the collaborative planning of research lessons (Appendix A) and to record the lesson plan (Appendix B) adapted from Lewis et al. (2011). During planning meetings resources such as syllabuses, research literature, teacher magazines, teaching and learning plans, and textbooks were utilised (Figure 3). Meetings took place in various classrooms or common teacher areas in the schools and were held during the school day at times where the majority of teachers were free, with a small number of meetings held after school.

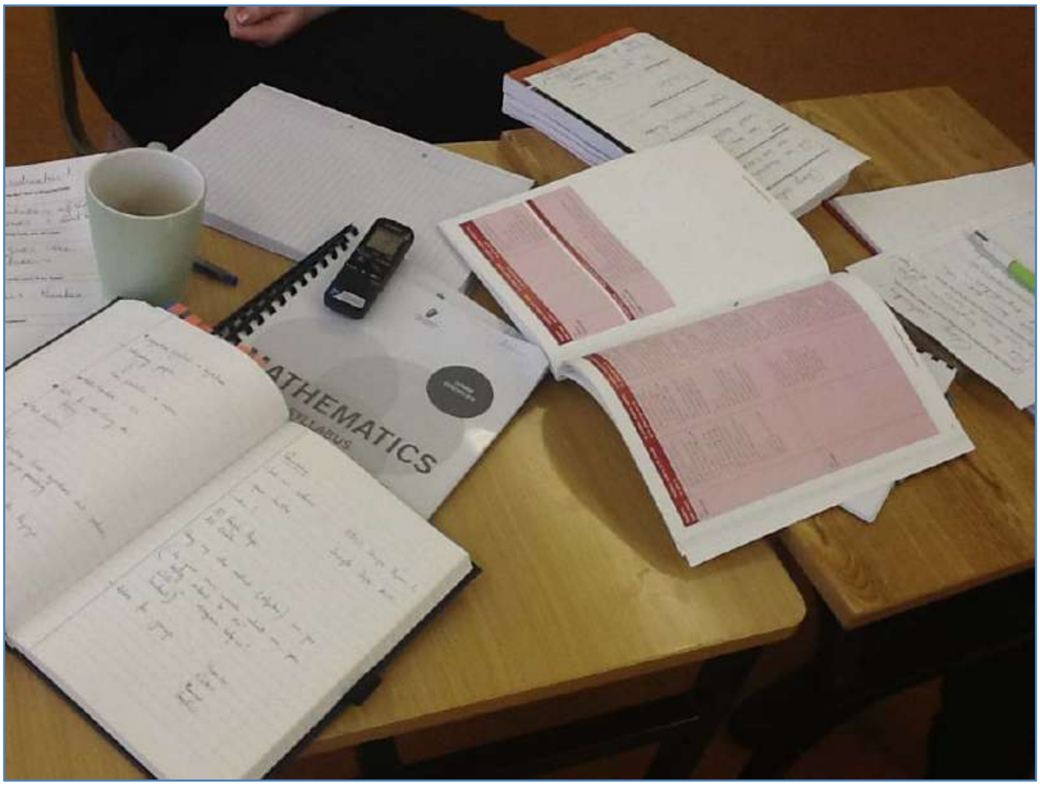


Figure 3 Resources utilised during planning meetings

In the following section lesson study processes, notable findings and items for consideration from the research are detailed.

Findings

A number of findings and contributions are reported here, further details of which may be found in additional research literature (Ni Shuilleabhain, 2014a, 2014b). These are categorised in this report as:

- Lesson study cycle,
- Teacher learning, and
- School-based professional development

Lesson Study Cycle

For the purposes of this report, findings of teachers' engagement in lesson study will initially be reviewed through a framework of the lesson study cycle. It is envisaged that this framework should provide the reader with greater insight into lesson study as a model of school based professional development and teacher community.

Stage 1: Formulate/Reflect on Goals & Study Curriculum

As their overall goal for lesson study, teachers in both Doone and Crannóg identified students' engagement with and attitudes towards mathematics as important classroom issues to address. Following a discussion around their pedagogical experiences, teachers in Doone highlighted that the majority of their students did not identify with mathematics as a meaningful subject and wanted to incorporate these affective issues within their teaching of research lessons during the year. Their overall goal was summarised as the following:

> *For our students to find purpose and meaning in their mathematics that leads to their enjoyment and confidence in the subject.*

In Crannóg's first lesson study meeting, teachers noted that students were often reticent in engaging in the subject and their participation in class was often prohibited by their fear of giving an incorrect answer. After brain-storming a number of ideas within the meeting (Figure 4), teachers agreed on the following over-arching goal:

> *To endeavour to create a culture which fosters independent thinking and fearlessness, leading to autonomous learning.*

These overall goals provided teachers with a philosophical basis for structuring their research lessons and, at the end of the research, teachers reported in their individual interviews that they felt these aims had been achieved through the various lesson study cycles.

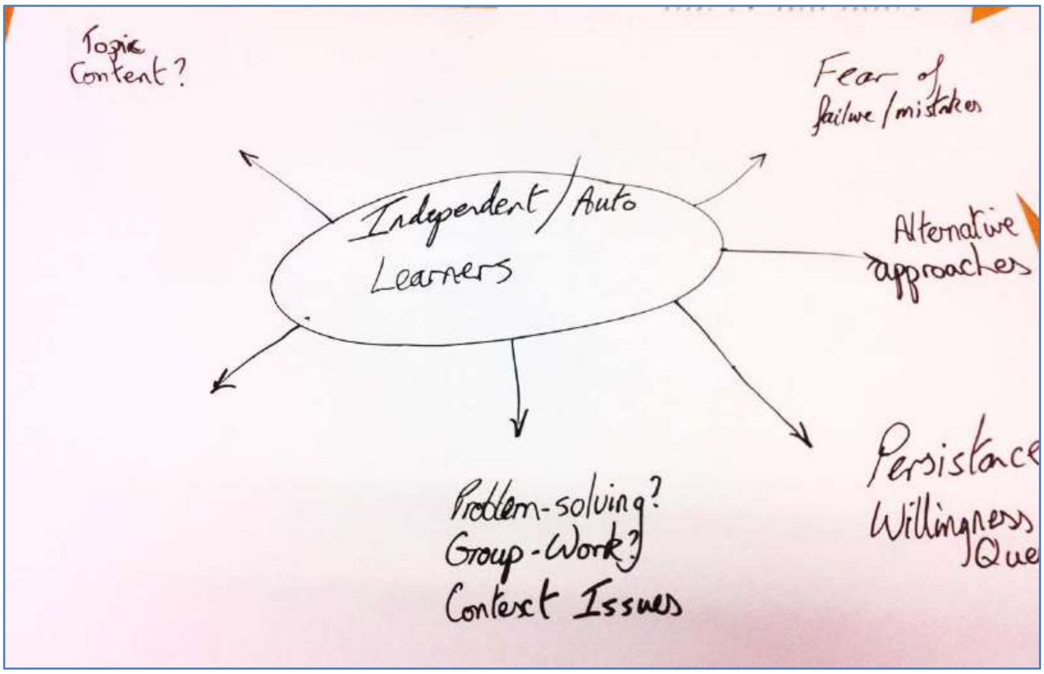


Figure 4 Brain-storm for over-arching goal

In deciding on strands of Mathematics on which to focus, teachers chose areas of the subject which they felt least confident in teaching or identified topics which they felt could be further developed for student understanding (see Table 1). As can be concluded from Table 1, the majority of research lessons were planned either for junior cycle or transition year classes. Due to the nature of the final state assessments, teachers did not wish to interrupt teaching of third year or senior cycle students as research lessons but this is not to suggest that research lessons could not also be conducted within these year groups.

Stage 2: Plan a selected or revised research lesson

In beginning the planning of lessons teachers originally referenced the lesson study guidelines given to them at the beginning of the research (Appendix A). This provided teachers with a structure through which they could collaborate with one another in deciding on the content and sequencing of activities within a research lesson. Following the initial two cycles, teachers became familiar with the processes of planning within their communities and no longer referenced the guidelines for direction.

A sample lesson template provided in the guidelines (Appendix B) was originally utilised by teachers to record their research lesson, but by the final cycles in both schools teachers had adapted these and used their own format to record the details of their research lessons.

In planning the research lessons, it was important that any work involved occurred during the meetings and that teachers did not undertake independent work outside of school. Teachers found that work undertaken outside of meetings often led to incoherence in the research lesson and therefore any activities, worksheets, or questions that were to be incorporated within a research lesson were developed during the meetings.

Materials were designed in a variety of ways during planning meetings utilising research literature, teacher magazines, on-line resources, or textbooks incorporating new and innovative ideas. In Doone, teachers developed contextualised activities through breaking into teams of two or three and devising questions to be incorporated within the lesson. Teachers then shared these questions, attempted these activities in order to anticipate students' thinking, and positively critiqued one another's work ensuring that the language, symbols, and numbers incorporated within problems were relevant to particular groups of students. Where teachers externally sourced activities these were shared with their colleagues during meetings and were modified as relevant, contextualised tasks for students. In Crannóg teachers often collectively built on one another's ideas in designing and incorporating new ideas within research lessons. Developing ideas from research literature such as articles from teacher magazines or books on mathematics education also added to the incorporation of new and innovative ideas within research lessons in both schools.

As a surprising outcome of this research, instead of planning one research lesson in isolation teachers in both schools began independently developing series of lessons within which a research lesson was included. This provided teachers with a better understanding of how students' mathematical thinking could be developed within a topic and highlighted the prior knowledge necessary for students to meet the learning objective within the research lesson (Figure 5). In their final meeting in Crannóg Eileen, an early career teacher, commented that planning a series of lessons was one of the most useful outcomes of lesson study:

Eileen	*For each of them we always managed to come up with a series of lessons, we never just planned one lesson – we always had at least a week or six classes.*
Fiona	*So even though you're looking at a lesson as the focus of it –*

Eileen	*There was always an intro[duction] into it and the afterwards was planned as well.*
Dave	*We had a topic planned as well as a lesson planned.*
Eileen	*You do get more out of that than just one lesson plan. You're getting a section.*

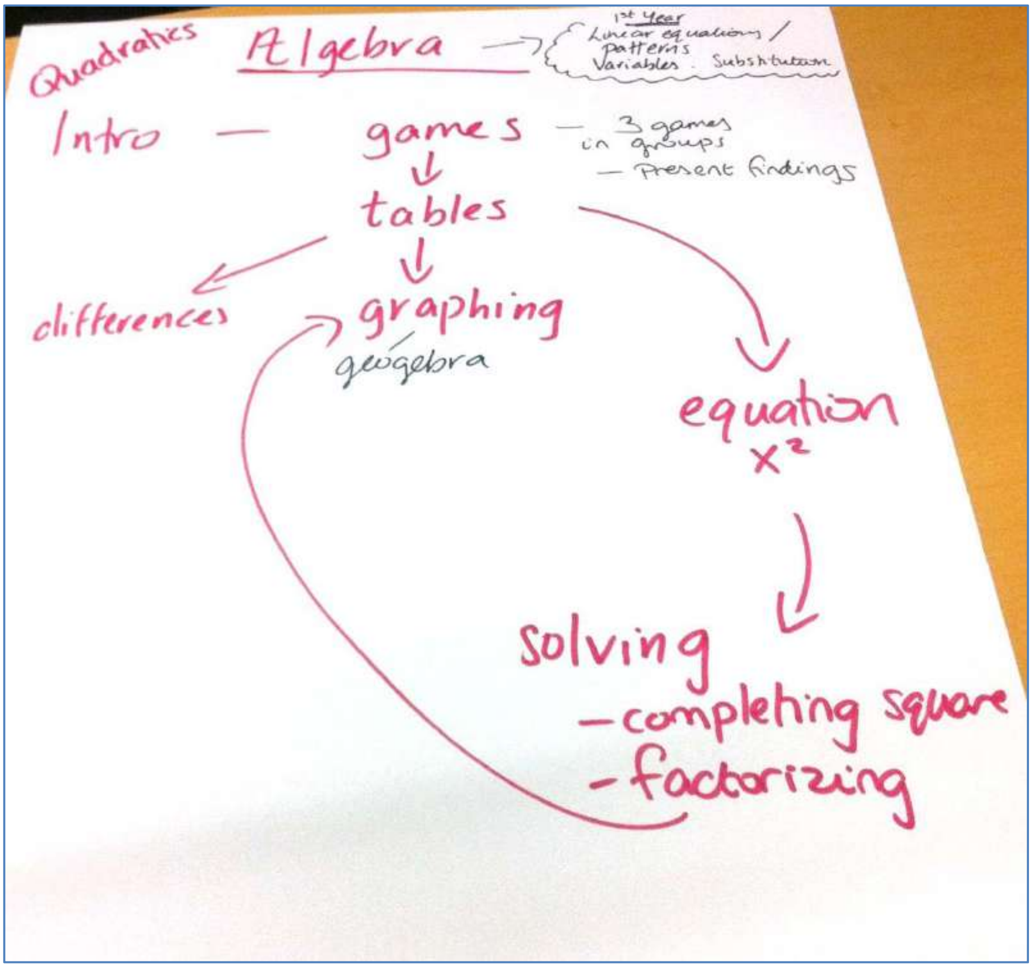


Figure 5 Planning a series of lessons

In Doone's final research cycle teachers mind-mapped their series of lessons and found this format beneficial in ordering their thinking (Figure 6).

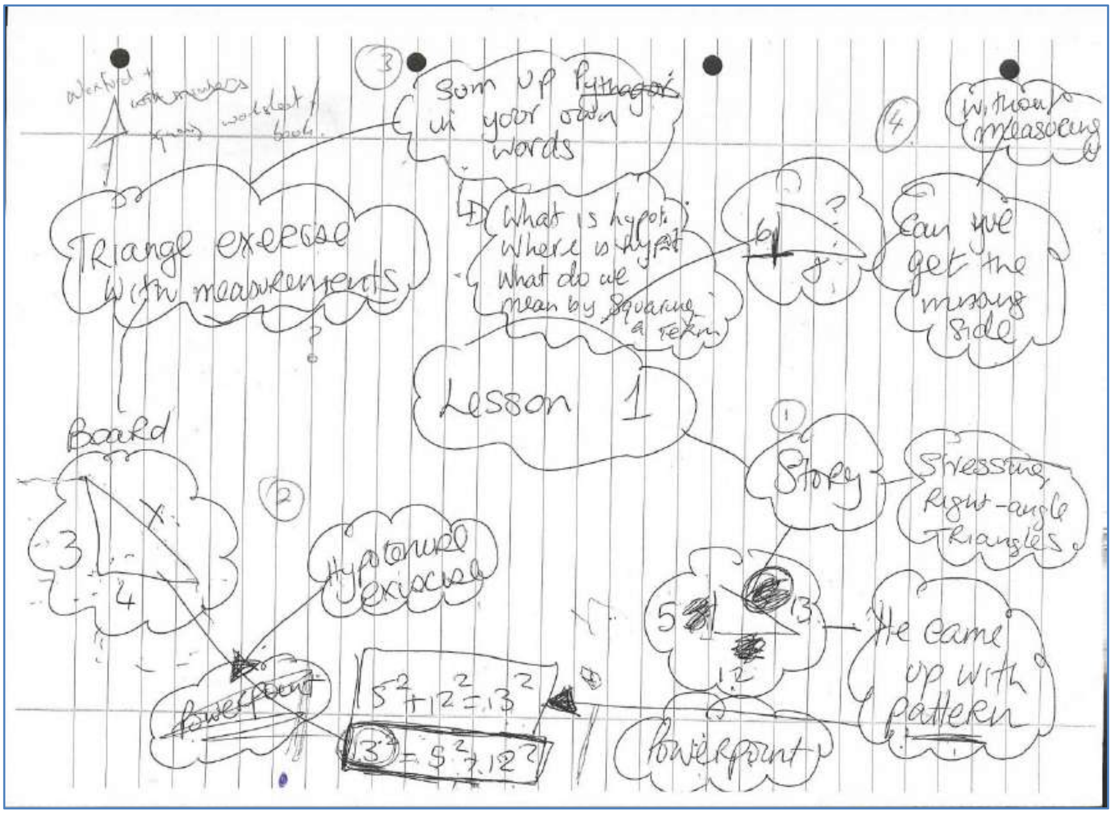


Figure 6 Mind-mapping lesson plan

In Michael's final interview, he extolled the virtues of planning and explained how he felt this stage of the lesson study cycle had benefitted the group.

> Michael *I'm not going to say I like planning but it's useful…It's been a very positive experience for me. I've enjoyed it and I've got a lot out of it so it's been good. It really shows me that planning is not only important but that it's productive as well and it's a very useful thing to do… It's just working in teams, working with the people that you work with, there's just so many potential benefits in it.*

Collaborative planning provided teachers with opportunities to share their teaching experiences with one another but also to build their awareness of different mathematical strategies and develop their focus on students' mathematical thinking.

Stage 3: Conduct/Observe research lesson

While the conducting teacher may, at times, have been nervous about teaching in front of their colleagues, all teachers reflected on this stage of the lesson study cycle as a very positive learning experience. While it was an unusual for teachers to invite colleagues into their classroom to observe a lesson, the collective nature of planning led to a supportive environment for those conducting the lesson as expressed by Kate in her mid-point interview:

> *Kate* *So we were saying this actually ourselves, we were saying it's not that I'm not even nervous about it - I'll probably be a bit nervous the day I have to do it - but we planned the lesson, it's* ***our*** *lesson, so it's kind of like we're teaching it as well… I found that with Owen for instance – I was laughing at things he was saying [in the lesson]. He's just himself in the class, so is Lisa. But I wasn't really looking at them, I was more looking at what they were teaching and how it was said and how the kids were picking up…It was like we've all planned it so if it fails it's all of our faults.*

Not all teachers were free to observe each research lesson, but it was important that a majority of the group observed and reflected on how students engaged with the research lesson.

In observing the research lessons teachers found it most beneficial to track one pair or one group of students for the duration of the lesson in order to better notice and interpret how student thinking developed and how students responded throughout the lesson.

In their individual interviews teachers reported that they found observing students for the duration of a lesson to be incredibly beneficial to their practice, providing them with insight on students' thinking which encouraged teachers to reflect on their own individual classroom practices.

> *Gerald* *It's great to be able to sit there and watch that group working. They were engaged and you could follow the train of thought and give them time because you are not under pressure to run off to another group. If you are a single teacher you can't follow the train of thought around all the groups and if you are delving in and giving assistance it is quite a hard thing to do that effectively.*

Teachers originally utilised observation guides based on Lewis and Hurd's (2011) 'Lesson Study Step by Step: How Teacher Learning Communities Improve Instruction' but then developed their own observation template constructed around recording observations of students' thinking and students' activities (Appendix B). It was not necessary for teachers to maintain their observation records but these notes provided discussion points for teachers in the post-lesson discussion (Figure 7).

Teachers were initially wary of the impact of classroom observation on students' learning but reflected, in their post-lesson discussions, that the presence of a number of teachers within the classroom did not have an adverse effect on students' learning. Aside from some senior students quietly questioning why there were so many people in the room, teachers noted that students seemed surprisingly unperturbed by the larger teacher presence within the classroom.

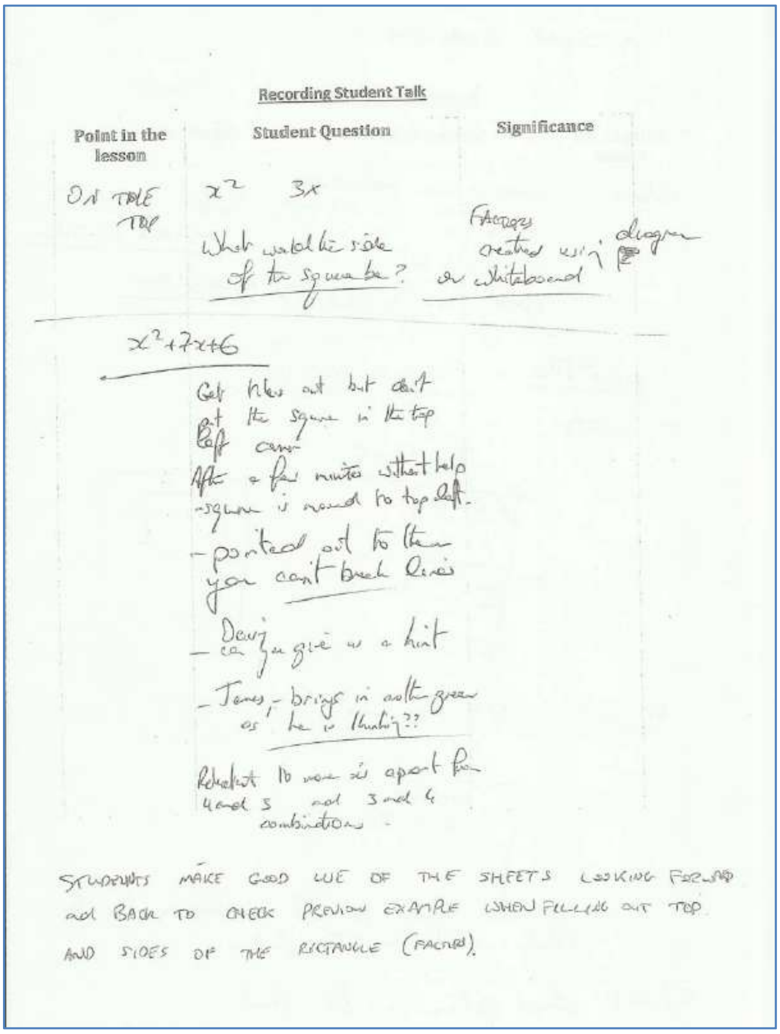


Figure 7 Sample teacher observation sheet

Furthermore, observing how students communicated mathematics with one another during small group and whole class presentations provided teachers with insight into the value of such an approach in the teaching and learning of mathematics and into the features of successfully facilitating class work based around communication of mathematical thinking.

As a point of interest, in observing students during a research lesson teachers found it difficult not to intervene or engage with students while they were 'observing'. At the end of the research a number of teachers reflected that they would learn more from their observations if they did not engage with students in this manner and resolved to maintain their distance as an observer in future research lessons as is recommended by the literature on lesson study.

Stage 4: Reflect on the research lesson

Lewis & Hurd (2011) advise that post-lesson discussions be held at as close a time following the research lesson as is possible. In this research each of the post-lesson discussions was held on the same day as the research lessons where teachers could reference their notes and accurately reflect on students' engagement, the development of the topic, the orchestration of the lesson, and many other elements within the lesson.

In post-lesson discussions teachers often referred to their observation notes but also referenced student work in interpreting how students engaged in activities and in assessing whether the lesson had met its learning objective (Figure 8).

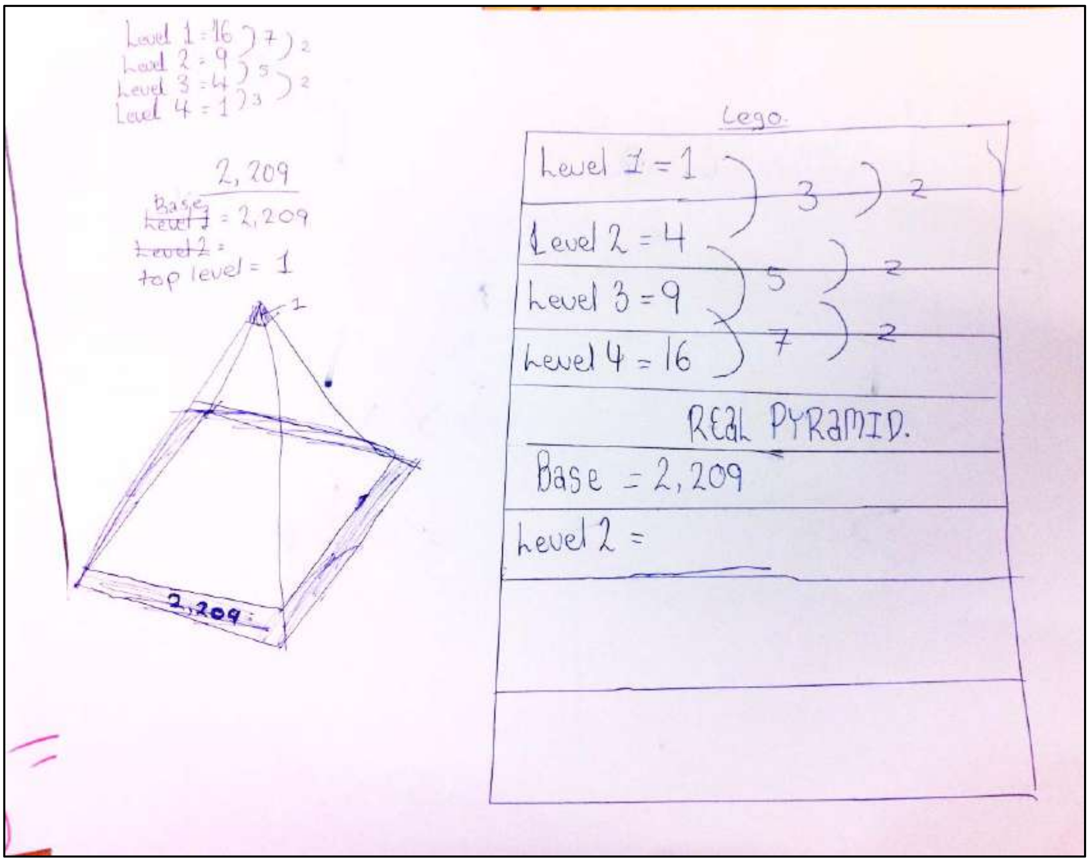


Figure 8 Sample of student work utilised for reflection

Where in their first post-lesson discussions teachers may have been wary of criticising any of the teaching approaches, as the meetings continued teachers' reflections focused more and more on students' learning and less on the teachers' actions. This reflection of students' thinking led to teachers anticipating more of students' mathematical responses and to seeing a lesson 'from the eyes of a student' in subsequent planning meetings. This changing perspective was an important element of teacher learning where members of the lesson study community began to develop or further hone their critical student lens (Fernandez et al., 2003) in noticing and interpreting students' mathematical thinking (Corcoran, 2011b; Jacobs et al., 2010). It is also worthy to note here that early career teachers who conducted research lessons welcomed feedback on their practice from their more experienced colleagues.

These post-lesson discussions proved to be important stages of lesson study where teachers were provided with opportunity to reflect on what they had observed in terms of interpreting students' strategies, recollecting student conversations, seeing the lesson through the eyes of a student, and developing their knowledge of sequencing learning trajectories for particular mathematical content.

Re-teaching a research lesson

In cycle 2 in Crannóg, teachers decided that their lesson on factorising quadratics had not achieved their intended learning objective and they concluded that they had included too much content in one research lesson. Teachers reviewed their lesson plan and a different teacher conducted the revised research lesson with a class within the same year group. In all other cycles in both schools teachers were satisfied that the lessons had met their objectives but re-teaching a research lesson remains a viable possibility for further developing research lessons.

Teacher Learning

In this section a number of findings around teacher learning and around the wider school context will be discussed.

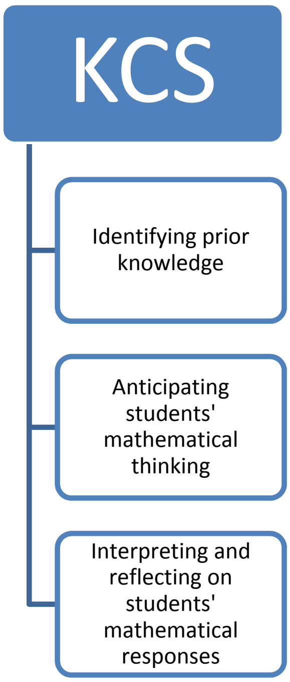
At the end of the research all teachers reported in their final interviews that they felt they had benefited, albeit in different ways, from participating in the research. All teachers felt they had an improved understanding of facilitating student group work, of holding whole class discussions, and of incorporating more communication within their mathematics classrooms. All but one teacher felt they had an improved knowledge of and understanding of the curriculum through engaging with curriculum documents throughout the planning of research lessons, that one teacher having felt that they already had a deep understanding of the curriculum prior to engaging in the research. A small number of teachers also felt their subject matter knowledge had improved.

The following section details findings of teacher learning in terms of PCK.

Developing Pedagogical Content Knowledge

Knowledge of Content and Students

A key finding of this research is the development of teachers' pedagogical content knowledge through their participation in iterative cycles of lesson study. Utilising a framework of mathematical knowledge for teaching (Ball et al., 2008), the following features of KCS emerged as developing through teachers' continued planning and reflection of research lessons:



- Identifying students' prior knowledge
- Anticipating students' mathematical thinking
- Interpreting and reflecting on students' mathematical responses

Identifying students' prior knowledge

What students learn in the classroom depends, to a large extent, on what they already know (Ball & Hyman, 2003; Holten & Thomas, 2001) and in mathematics students can be challenged by drawing on their prior knowledge and building on the assumptions they have already made (Baumert et al., 2010). Composing a lesson from the view point of a student and identifying their prior knowledge forms an important part of teachers' KCS and within the collective planning of the research lessons, it became more and more relevant to teachers to identify the prior knowledge that was required of students for a particular topic and to incorporate this within the lesson. This was an important exercise since discussions around this issue provided teachers with a definitive starting point from which to build students' mathematical thinking in particular topics.

Anticipating students' mathematical thinking

Through their classroom experiences teachers build on their knowledge of how students are likely to understand or have difficulties with a topic (Schoenfeld, 2011). An experienced teacher will know what issues are likely to be problematic and what misconceptions are common to students (Ball et al., 2008; Krauss et al., 2008). In this research through their observation and reflection of research lessons, teachers became more aware of anticipating and articulating students' mathematical thinking within subsequent research lessons and in their own teaching.

Interpreting and reflecting on students' mathematical responses

When students construct their own mathematical understanding, there are important learning moments for teachers to interpret and gain insight into students' thinking. Lesson study provides teachers with unique opportunities to observe students' activities within a co-constructed lesson and teachers can therefore notice, interpret and reflect on students' learning (Breen, McCluskey, Meehan, O'Donovan, & O'Shea, 2014; Jacobs et al., 2010; Mason, 2002). As part of participating in iterative cycles of lesson study teachers became more aware of and began explicitly reflecting on students' strategies and conversations around mathematical activities during their post-lesson discussions. This benefitted teachers in structuring subsequent research lessons and in further identifying student thinking in their own construction and conduction of classes outside of lesson study.

Knowledge of Content and Teaching

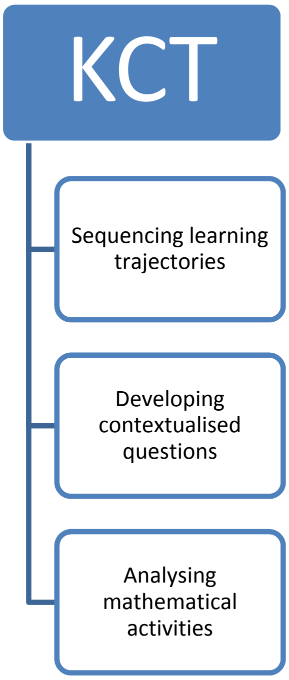
Utilising the same framework of mathematical knowledge for teaching (Deborah L. Ball et al., 2008), the following features of KCT emerged as developing through teachers' continued planning and reflection of research lessons:

- Sequencing learning trajectories
- Developing contextualised questions
- Analysing mathematical activities

Sequencing learning trajectories

Sequencing material for instruction requires teachers to make connections between the mathematics being taught and how the student can be best introduced to that topic. In order to make these connections the teacher has to be aware of the relative demands that different topics and tasks make of students (Rowland, 2007). In this research sequencing a learning trajectory over a series of lessons became a fundamental matter of planning for both lesson study communities engaged in this research and as their final output of each lesson study cycle, teachers had detailed series of lessons through which students' mathematical thinking would be developed.

Developing contextualised questions



In planning research lessons, teachers realised that traditional textbook questions did not always meet the learning objectives of the research lesson. Teachers wanted to design problems and activities that would be both relevant and contextualised for students (Schoenfeld, 1992) and developing these activities became a common element within the lesson study cycles in both schools. Teachers often sourced ideas from textbooks, PMDT curriculum materials, or education literature supplied to them by the facilitator and then modified these questions to ensure they held context for their own particular students. As an example, teachers in Doone developed a question around investigating quadratics that was both mathematically relevant and would be of interest for their transition year group of male students (Figure 9).

During the recent Ireland v England match in the Aviva Stadium as part of the Six Nations competition, Jonathan Sexton kicked a Garryowen before he went off with an injury. A Garryowen, also known as an "Up and Under", allows the attacking team to disrupt the defensive line, take the defense's pressure off themselves and put offensive pressure on their opponents. However, the kicking team risks losing possession of the ball, after which the opposing team may counter attack.

George Hook and Brent Pope, as well as being famous for their rugby commentary are also keen mathematicians who have calculated that the height of the ball above ground during the Garryowen can be described by

$$H = 25t - 4.9t^2$$

where H gives the height of the rugby ball above the ground at any time, t, in seconds.

a) How long is the ball in the air for?

b) What is the maximum height reached by the ball?

c) If Jonathan runs at a speed of 7m/s and is not tackled or hindered in any way. The ball will land 26m away. Will he catch the ball?

Figure 9 Contextualised question designed by teachers

Similarly in Crannóg teachers developed activities and tasks that incorporated learning objectives with engaging problems. In their first research lesson teachers devised an inquiry task introducing students to quadratic patterns (Figure 10).

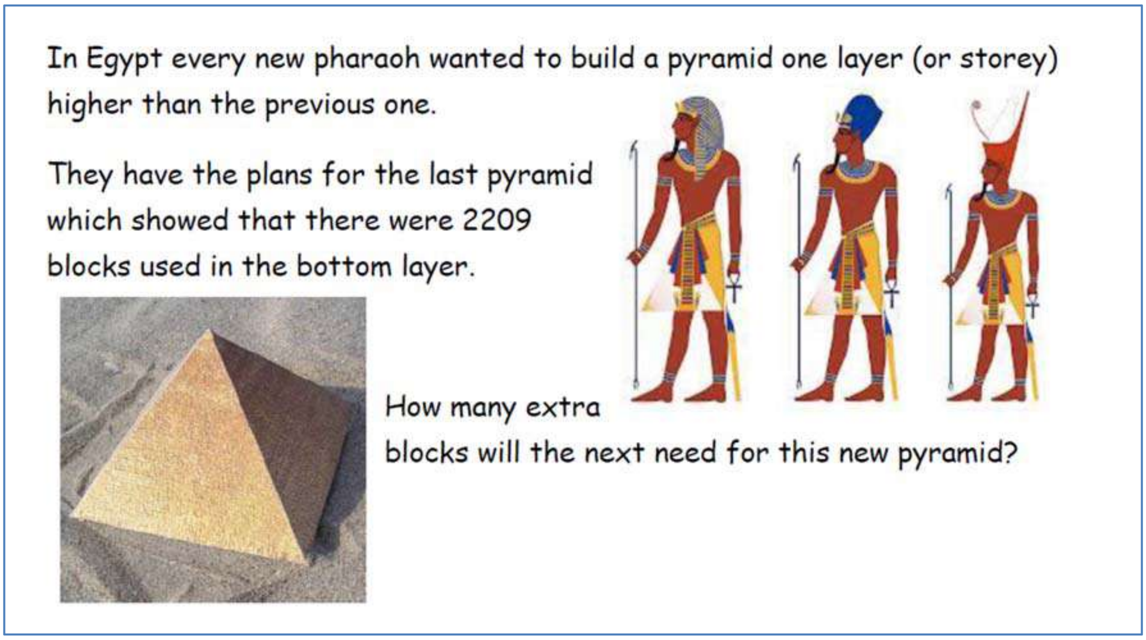


Figure 10 Contextualised student-inquiry activity

Analysing mathematical activities

Finally, teachers built on their KCT through analysing mathematics questions as a part of their planning of research lessons. Through attempting these tasks with their colleagues, teachers became aware of other solution strategies or aspects of questions that would otherwise have been unfamiliar to them (Ball et al., 2008; Borko, Jacobs, Eiteljorg, & Pittman, 2008). Furthermore, doing mathematics tasks prior to the research lesson encouraged teachers to anticipate students' mathematical responses, a feature of KCS.

In his second interview Dave reflected that in doing questions during the planning phase of lesson study he had benefitted in terms of becoming aware of different approaches to teaching that particular topic to students.

> *Dave*   *I have looked at more ways of solving quadratics in the last week than I have looked at in the last ten years. I have tried more ways of solving quadratics in the last week than I have looked at in the last ten years. I had two methods, the method I used at school for ten years and the method I met with [my colleague] Fiona and I knew there were others out there. Now I am just reading different types with algebra tiles and with area models – considering more of the curriculum.*

As with any professional development intervention, not all teachers gained in exactly similar ways (Jacobs et al., 2010) but from analysis of the qualitative data the research suggests that teachers built on their PCK through participation in iterative cycles of lesson study.

Changes to Classroom Practices

The most comprehensive change to classroom practice within research lessons was in incorporating more student group work and more communication of mathematical thinking through whole-class, pair, and group activities. In including these approaches to teaching within research lessons, teachers observed the benefit of engaging students in problem solving as a collaborative effort. The following is an excerpt from a post-lesson discussion which demonstrated teachers' surprise at the learning that occurred through students' working in groups:

Walter	*It was more exciting for them than I thought it would be*
Dave	*They came up with the questions themselves!*
Walter	*Their imagination of it, they just sort of plugged into that more…*
Fiona	*It worked. Those guys over there I was watching – they really helped each other. They were listening to each other!*
Gerald	*All that dynamic is great, to watch how they are hitting off each other and when they have their own ideas. There was an excitement in there about getting stuff worked out.*

Owen reported a dramatic change in his teaching from initially despising a "noisy classroom" and considering himself as "anti Project Maths" to consciously incorporating whole class discussions and including applications of mathematics within his teaching.

Following her conduction of the first research lesson, Lisa changed the layout of her classroom from one based on individual student work to that of groups of students. When her colleagues enquired about her changed classroom Lisa was extremely positive about how it impacted on students' learning which encouraged her colleagues to also incorporate student group work in their classrooms.

Owen	*Are you happy with yours?*
Kate	*You still have that layout?*
Lisa	*Yeah! Because they can all see each other's work 'you're not doing that right'. I don't have to go as far as the group. I would never go back.*

Owen *That's great...Will I change the tables in my classroom for the day [of the research lesson] to have them in group work?*

At the end of the year Lisa reflected that incorporating student communication of their mathematical thinking within research lessons led to a dramatic change in how participating teachers designed their classroom environment and activities.

Lisa *I think we've all looked at the way our classrooms are set up – to facilitate cooperative learning. All of our classrooms were the traditional single look-at-the-board and everybody has used cooperative learning practices as a result of the lessons that we designed. Very little of our work was done for the kid to do on their own.*

As well as these findings generated through a thematic analysis of the data, teachers also self-reported changes to their classroom practices as a result of their participation in lesson study. Both Walter and Eileen noted that from observing their colleagues, they were beginning to consciously extend the 'wait time' in allowing students to answer during their own lessons. Teachers were also consciously reducing their exposition time, instead incorporating more opportunity for students to engage with mathematical activities within the lesson.

Furthermore, teachers in both schools were less dependent on textbooks to guide their teaching. In Crannóg, teachers were already utilising textbooks as occasional question sources and through lesson study, designed their own module on introducing calculus to Transition Year students. In Doone, teachers became more and more proficient in designing their own questions and realised that the textbook strategy did not always define a sequence of learning that was relevant to their own students.

In-School Professional Development

Complementing In-service Workshops

In both schools, teachers felt this form of professional development had particular benefit in being located within the structures and cultures of their own schools where teachers collaborated with their colleagues while focusing on their own students. Furthermore, where teachers had attended professional development workshops outside of school and utilised curriculum materials, they had not had opportunity to implement resources or ideas from such in-service days. Lesson study provided teachers with opportunity to reflect on and incorporate teaching approaches and resources which they had met in workshops and could then modify and incorporate within their own teaching, relevant to their own students.

School Management

In both schools, the support of management was key in supporting teachers to participate in this form of professional development and in this research (Fullan, 2003). Principals, vice-principals, and other staff were supportive of their colleagues throughout the year and this impacted positively on teacher learning and on the development of teacher community in both schools. The support of principals and vice-principals in organising supervision for the purposes of observation or planning was a positive influence on teachers' engagement in this form of in-school professional development.

Voluntary Professional Development

Although participation in professional development should include the majority of education professionals and should occur regularly, participation in certain modules should occur on a voluntary basis (Erickson, Minnes Brandes, Mitchell, & Mitchell, 2005). In this research, all teachers reported in their final interviews that a vital element of their participation in this research was that they had all volunteered in the intervention. Teachers felt it important that if lesson study were to be incorporated as part of a suite of professional development models, it should remain the choice of the individual teacher if they wished to participate.

In their discussions in both schools, teachers felt that they had benefitted from participating in lesson study and wanted to continue with this form of professional development as a mathematics teacher community. This was a possibility in Crannog where teachers requested a common free period in the following academic year but was not possible to continue in Doone.

As a notable point, all teachers agreed that this form of professional development should be included in additional contract hours as a form of structured teacher collaboration in subject groups. A number of teachers expressed the wish for their participation in any professional development to be officially recognised and felt that a lack of acknowledgement of their engagement dis-incentivised teachers to continue with any form of professional development.

Scalability

While in this research there was not sufficient time to introduce teachers to lesson study prior to their engagement in the model, the introduction of teachers to a model of lesson study could be facilitated through research-based teacher education workshops mirroring Lewis et al.'s (2009) research. Such workshops would incorporate teacher participation in activities requiring facilitating of student group work and provide teachers with resources to share with their colleagues. In introducing such a model to schools, teachers should be provided with opportunity to volunteer to participate in the intervention which would necessarily adapt to the structures and cultures within individual schools and mathematics departments. A small number of well-trained facilitators would also be necessary as additional support for schools engaging in initial cycles of lesson study.

Sharing Resources

In both schools teachers were happy to share the resources created in lesson study since they were aware that their counterparts had given much of their free time in developing research lessons. It is perhaps an important finding of this research that teachers were generous in sharing the fruits of their collaboration with others who had proportionally invested time and ideas into developing such materials.

Limitations of the Research

In presenting this report it is necessary to recognise the limitations of the research.

One major limitation of this project is the sample population of the study. Only two schools were involved in this research and while both phases of the curriculum reform were represented, these two Mathematics departments cannot be stated as representative of all Mathematics departments that may exist around the country. In addition, these schools were both situated in Dublin and represented schools of large populations (over 500 students). Further research may be necessary to investigate how smaller or more rural schools with smaller populations of mathematics teachers might be impacted through participating in iterative cycles of lesson study.

In referencing the scalability of this model of professional development and referencing international research literature, I believe that lesson study can be introduced to post-primary schools on both regional and national levels. Dudley (2012) along with many other internationally based studies (C. Fernandez et al., 2003; Isoda & Katagiri, 2012; Murata et al., 2012) have demonstrated the scalability of this model in schools all over the world.

Conclusion

This research investigated lesson study as a model of school-based professional development for Irish post-primary mathematics teachers and was conducted as a double case study based in two schools. Lesson study was adopted as a model in order to structure the activities of teacher community while providing teachers with opportunity to investigate their own practices, through planning and reflection, at a time of curriculum reform.

Through planning multiple research lessons teachers collaborated in designing, conducting, observing, and reflecting on mathematics lessons that were based within the Project Maths initiative. Teachers became more aware of incorporating a critical student lens on their practice in incorporating students' prior knowledge, anticipating students' mathematical strategies, and interpreting students' classroom responses. In engaging with the processes of lesson study teachers became more comfortable in incorporating sociocultural practices within their teaching by planning and reflecting on whole class discussions, in facilitating student group work, and incorporating more communication of students' mathematical thinking. Furthermore, teachers designed their own mathematical activities and encouraged students to engage in problem solving practices during research lessons. Through engaging in iterative cycles of lesson study, all teachers of varying teaching experiences were facilitated in developing their pedagogical content knowledge.

While these schools were both Dublin based, they represented schools of differing cultures and ethos and provide evidence that lesson study may be introduced to Irish post-primary schools as a voluntary form of teacher professional development. The support of school management and teacher colleagues was incredibly important in promoting their colleagues' engagement with this research for the course of the academic year.

Lesson study, while seemingly simple to describe, holds much power in how it can engage teachers in investigating their own practices and in providing an environment within which mathematics teachers can be creative in their teaching. It also provides teachers with unique opportunities to access how students respond to and engage within mathematics lessons and, as identified in this research, participation in iterative cycles of lesson study can build teachers' pedagogical content knowledge.

This is a model of professional development which holds great potential as a structure within which teachers can enjoy, learn from, and reflect on their practices of teaching mathematics.

References

Ball, D. L., & Hyman, B. (2003). *Toward a Practice-Based Theory of Mathematical Knowledge for Teaching.* Paper presented at the 2002 Annual Meeting of the Canadian Mathematics Education Study Group, Edmonton.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*(5), 389-407.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., . . . Tsai, Y.-M. (2010). Teachers' Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress. *American Educational Research Journal, 47*(1), 133-180. doi: 10.3102/0002831209345157

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*(2), 417-436. doi: http://dx.doi.org/10.1016/j.tate.2006.11.012

Breen, S., McCluskey, A., Meehan, M., O'Donovan, J., & O'Shea, A. (2014). A year of engaging with the discipline of noticing: five mathematics lecturer's reflections. *Teaching in Higher Education, 19*(3), 289-300.

Cajkler, W., Wood, P., Norton, J., & Peddar, D. (2013). Lesson Study: towards a collaborative approach to learning in Initial Teacher Education? *Cambridge Journal of Education*. doi: DOI: 10.1080/0305764X.2013.834037

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., . . . Choi, Y. (2014). The Relationship Between Teachers' Mathematical Content and Pedagogical Knowledge, Teachers' Perceptions, and Student Achievement. *Journal for Research in Mathematics Education, 45*(4), 419-459. doi: 10.5951/jresematheduc.45.4.0419

Charalambos, C., & Philippou, G. (2010). Teachers' concerns and efficacy beliefs about implementing a mathematics curriculum reform: integrating two lines of inquiry. *Educational Studies in Mathematics, 75*(1), 1-21. doi: 10.1007/s10649-010-9238-5

Conway, P., & Sloane, F. (2005). *International trends in post-primary mathematics education*. National Council for Curriculum and Assessment.

Corcoran, D. (2011a). Learning from Lesson Study: Power Distribution in a Community of Practice. In L. Hart, A. S. Alston & A. Murata (Eds.), *Lesson Study Research and Practice in Mathematics Education* (pp. 251-268). New York: Springer.

Corcoran, D. (2011b). Lesson Study in practice: Staff ownership of mathematics teacher professional development. In T. Dooley, D. Corcoran & M. Ryan (Eds.), *Fourth Conference on Research in Mathematics Education MEI 4* (pp. 130-142). St. Patrick's College, Drumcondra, Dublin 9.

Darling-Hammond, L., Chung Wei, R., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional Learning in the Learning Profession: A status report on teacher development in the United States and Abroad. US: National Staff Development Council & Stanford University.

de Lima, J. Á. (2001). Forgetting about Friendship: Using Conflict in Teacher Communities as a Catalyst for School Change. *Journal of Educational Change, 2*(2), 97-122. doi: 10.1023/a:1017509325276

Dooner, A.-M., Mandzuk, D., & Clifton, R. A. (2008). Stages of collaboration and the realities of professional learning communities. *Teaching and Teacher Education, 24*(3), 564-574. doi: 10.1016/j.tate.2007.09.009

Erickson, G., Minnes Brandes, G., Mitchell, I., & Mitchell, J. (2005). Collaborative teacher learning: Findings from two professional development projects. *Teaching and Teacher Education, 21*(7), 787-798. doi: http://dx.doi.org/10.1016/j.tate.2005.05.018
Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US-Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*(2), 171-185.
Fernandez, C., & Chokshi, S. (2004). Challenges to Importing Japanese Lesson Study: Concerns, Misconceptions, and Nuances. *Phi Delta Kappa International, 85*(7), 520-525.
Fernández, M. L., & Robinson, M. (2006). Prospective teachers' perspectives on microteaching lesson study. [Article]. *Education, 127*(2), 203-215.
Fullan, M. (2003). *The Moral Imperative of School Leadership*. USA: Corwin Press, Inc
Grossman, P., Wineburg, S., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record, 103*(6), 942-1012.
Guskey, T. R. (2002). Professional Development and Teacher Change. *Teachers and Teaching: theory and practice, 8*(3), 381-392.
Hanley, U., & Torrance, H. (2011). Curriculum Innovation: Difference and Resemblance. *Mathematics Teacher Education and Development, 13*(2), 67-84.
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students. *Journal for Research in Mathematics Education, 39*(4), 372-400. doi: 10.2307/40539304
Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal, 42*(2), 371-406. doi: 10.2307/3699380
Holten, D., & Thomas, G. (2001). Mathematical Interactions and Their Influence on Learning. In D. Clarke (Ed.), *Perspectives on Practice and Meaning in Matheamtics and Science Classrooms*. Netherlands: Kluwer Academic Publishers.
Hord, S. M. (2004). Professional Learning Communities: An Overview. In S. M. Hord (Ed.), *Learning Together, Leading Together: Changing Schools Through Professional Learning Communities* (pp. 5-14). New York: Teachers College
Isoda, M., & Katagiri, S. (2012). *Mathematical Thinking: How to Develop it in the Classroom* (Vol. 1). Singapore: World Scientific Publishing.
Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.
Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology, 100*(3), 716-725. doi: 10.1037/0022-0663.100.3.716
Leavy, A. M., Hourigan, M., & McMahon, A. (2010). Facilitating inquiry based learning in mathematics teacher education *Third Science and Mathematics Education Conference*. Dublin City University
Lewis, C., & Hurd, J. (2011). *Lesson Study Step by Step: How Teacher Learning Communities Improve Instruction*. USA: Library of Congress Cataloging-in-Publication Data.
Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: a theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285-304. doi: 10.1007/s10857-009-9102-7

Lewis, C., Perry, R., & Murata, A. (2006). How Should Research Contribute to Instructional Improvement? The Case of Lesson Study. *Educational Researcher, 35*(3), 3-14.
Louis, K. S., & Marks, H. M. (1998). Does Professional Community Affect the Classroom? Teachers' Work and Student Experiences in Restructuring Schools. *American Journal of Education, 106*(4), 532 - 575.
Lyons, M., Lynch, K., Close, S., Sheerin, E., & Boland, P. (2003). *Inside Classrooms: The Teaching and Learning of Mathematics in Social Context*. Dublin: Insitute of Public Administration.
Mason, J. (2002). *Researching Your Own Practice: The Discipline of Noticing*. London: RoutledgeFalmer.
McLaughlin, M. W., & Talbert, J. E. (2006). *Building School-Based Teacher Learning Communities: Professional Strategies to Improve Student Achievement*. New York: Teachers College Press.
McLaughlin, T. H. (2004). Teaching as a Practice and a Community of Practice: The Limits of Commonality and the Demands of Diversity. In J. Dunne & P. Hogan (Eds.), *Education and Practice: Upholding the Integrity of Teaching and Learning* (Vol. 1, pp. 48-60). UK: Blackwell Publishing.
Murata, A., Bofferding, L., Pothen, B. E., Taylor, M. W., & Wischnia, S. (2012). Making Connections Among Student Learning, Content, and Teaching: Teacher Talk Paths in Elementary Mathematics Lesson Study. *Journal for Research in Mathematics Education, 43*(5), 616-650. doi: 10.5951/jresematheduc.43.5.0616
Murata, A., & Takahashi, A. (2002). *Vehicle to Connect Theory, Research, and Practice: How Teacher Thinking Changes in District-Level Lesson Study in Japan*. Paper presented at the North American Chapter of the International Group for the Pyschology of Mathematics Education, Athens.
Ni Shuilleabhain, A. (2014a). *Developing Mathematics Teachers' Pedagogical Content Knowledge through Lesson Study.* (Doctor of Philosophy Ph.D.), Trinity College Dublin, Trinity College Dublin Library.
Ni Shuilleabhain, A. (2014b). Lesson study and Project Maths: A Professional Development Intervention for Mathematics Teachers Engaging in a New Curriculum. In S. Pope (Ed.), *8th British Congress of Mathematics Education* (pp. 255-262). Nottingham University.
OECD. (2009). Creating Effective Teaching and Learning Environments: First Results from TALIS.
Oldham, E. (2001). The culture of mathematics education in the republic of Ireland: Keeping the faith? *Irish Educational Studies, 20*(1), 266-277. doi: 10.1080/0332331010200121
Remillard, J. T. (2005). Examining Key Concepts in Research on Teachers' Use of Mathematics Curricula. *Review of Educational Research, 75*(2), 211-246. doi: 10.2307/3516049
Remillard, J. T., & Bryans, M. B. (2004). Teachers' Orientations toward Mathematics Curriculum Materials: Implications for Teacher Learning. *Journal for Research in Mathematics Education, 35*(5), 352-388. doi: 10.2307/30034820
Ríordáin, M. N., & Hannigan, A. (2011). Who teaches mathematics at second level in Ireland? *Irish Educational Studies, 30*(3), 289-304. doi: 10.1080/03323315.2011.569117
Rowland, T. (2007). Developing Knowledge for Mathematics Teaching: A Theoretical Loop. In S. Close, D. Corcoran & T. Dooley (Eds.), *Second National Conference on Research in Mathematics Education: MEI2* (pp. 14-27). St. Patrick's College, Dublin.

Rowland, T. (2014). Mathematics Teacher Knowledge. In P. Andrews & T. Rowland (Eds.), *Masterclass in Mathematics Education* (pp. 87-100). London: Bloomsbury Publishing.

Schoenfeld, A. H. (1992). Learning to think Mathematically: Problem Solving, Metacognition, And Sense-Making in Mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMilan.

Schoenfeld, A. H. (2011). *How We Think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Takahashi, A., & Yoshida, M. (2004, May). Ideas for Establishing Lesson-Study Communities. *Teaching Children Mathematics,* 436-443.

van den Akker, J. (2003). Curriculum Perspectives: An Introduction. In J. v. d. Akker, W. Kuiper & U. Hameyer (Eds.), *Curriculum Landscapes an Trends* (pp. 1-10): Springer Netherlands.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244-276. doi: http://dx.doi.org/10.1016/j.tate.2006.11.005

Appendices

Appendix A: The Lesson Study Cycle

1. Study Curriculum and Formulate Goals

During the first meeting of the community of practice the following agenda is suggested:

A. Choose roles for the initial meeting: time keeper, note taker, facilitator etc.
B. Examine everyone's ideas about professional development and lesson study
C. Consider long-term goals and/or lesson goals
D. Build a timeline for the lesson study cycle
E. Review the curriculum and areas or topics which may be of interest
F. Review key decisions and/or insights made during the meeting
G. Agree upon assignments to be completed or followed up in the next meeting.

Facilitator: The facilitator's role is to aim to ensure that the preceding steps are met during the meeting or will agree with the group that they will be addressed at the next meeting. It is important that while an agenda is adhered to, the facilitator's role should be fluid and allow conversations to run freely.

Long term goals

In considering long-term goals teachers may ask themselves:

- What is a gap between the ideal and the actual that you would really like to work on as a teacher?
- Think about the students you serve. What qualities would you like these students to have in 5 or 10 years from now?
- What qualities do they have now and what could be improved upon?

Teachers should agree on the phrasing and terminology of their long term goal particular to their own set of students. This goal should remain a focus for all of the lesson study cycles throughout the year.

Examples:

For our students to find purpose in their mathematics and meaning that leads to their enjoyment and confidence in the subject.

Creating a culture which fosters independent thinking and fearlessness leading to autonomous learners.

2. Planning a Research Lesson

Planning Meetings

A. Choose roles for the planning meeting: note takers, lesson plan recorder, time keeper, facilitator, etc.
B. Decide on the class group and curriculum area on which to focus.
C. Select the lesson you will focus on.
D. Write a teaching and learning plan over a number of meetings (You may wish to use and/or modify resources already available to you).
E. Devise an observation strategy for collecting data.
F. Review key decisions and/or insights made during the meeting.
G. Agree upon assignments to be completed or followed up in the next meeting.

Teaching and Learning Plan

The teaching and learning plan should include:

- Long term goals
- Anticipated student thinking
- Data collection plan
- A sequence of learning
- The rationale for this chosen approach

In creating the teaching and learning plan teachers may wish to ask themselves:

(Note: These questions are only a guide)

1. What do students currently understand about this topic?

2. What do we want them to understand at the end of the unit or sequence of lessons?

3. What's the sequence of experiences (lessons) that will propel students toward the learning objective?

4. What will make the unit/sequence of lessons and each individual lesson motivating and meaningful to students?

5. Which lesson in the unit will be selected as the research lesson?
6. What will students need to know before this lesson?
7. What will they learn during this lesson?
8. What is the sequence of experience through which they will learn it?
9. How will students respond to the questions and activities in the lesson?
10. What problems and misconceptions will arise and how will teachers respond to them?
11. What evidence should we gather and discuss about student learning, motivation, and behavior?
12. What data collection forms are needed to do this?

Each member of the planning team should independently do the activity intended for students within the research lesson. Usually in a group of teachers, there will be a variety of strategies in attempting an activity and each teacher should have an opportunity to share how they approached the activity. It will be important to discuss the successes and difficulties the students will encounter and also the successful process(es) and outcomes for this task.

The point of anticipating student responses is not to design the activity so that student's won't struggle or so that misconceptions won't emerge, but rather to give teachers an opportunity to plan how they will respond and to think about what kind of struggles and misconceptions may be an intentional element of the lesson. It is important to discuss the instructional strategies and options that might facilitate the student learning as struggles and misconceptions emerge. Record the anticipated student responses and teacher responses in the lesson plan.

Approximate Time Guide	Student Learning Activities	Anticipated Student Responses	Anticipated Teacher Responses	Points to notice & evaluate

Figure 11 Suggested headings for a Teaching & Learning plan

Collecting Data

It will also be important for teachers to reflect on what data collected during the lesson will assess the learning and engagement of students.

NB: It may be useful for observing teachers to have both a copy of the lesson plan and seating map of the room with students' names during the lesson.

In designing and observation sheet teachers should ask:

1. What data will help you understand your students' progress on your lesson goals and long-term goals?
2. Would a prepared data collection form facilitate observation or should conversations between students be recorded?
3. What student work (if any) will be collected at the end of the lesson?
4. How will material that is presented on the board or in other locations be captured? (Photographs etc.)
5. What are the individual assignments of the lesson study team? Will one person transcribe the lesson and keep a timeline of lesson events? Will observers be assigned to observe specific students or groups?
6. If the lesson is to be video-recorded, who will be in charge of the video equipment during the lesson?

It is not necessary to write an answer for every question but it is important to discuss the questions within the meeting. You may wish to write your Lesson Plan as a mind map but it is important that there is a written record for you and your colleagues to return to for the reflection and following the Lesson Study cycle.

3. Conducting and Observing the Research Lesson

Teaching the Lesson

The teacher who will be instructing the lesson should follow the Teaching & Learning plan or may construct a condensed Lesson Flow for themselves. The teacher may introduce the class to the observing teachers but otherwise should not interact with his/her colleagues during the lesson. This teacher is conducting the lesson that all members of the group have designed.

Recommendations for observing teachers

- Minimise side conversations during the lesson

Remain in the classroom during the entire lesson to capture how the lesson is set up, its flow, and its conclusion

- Do not block the students' view of the blackboard or any area where the teacher is writing and posting materials or demonstrating an activity
- Minimise interactions with students. Refrain from teaching or assisting the students. Occasional interaction is permissible if done discreetly and with the purpose of understanding student thinking.

Observing teachers should choose specific behaviours or actions to focus on during the lessons. Teachers may wish to address specific research questions during the lesson and/or use observation forms.

Note: The observing teachers are explicitly focusing on students' engagement and learning during the lesson and not on their colleague's teaching.

Suggested Observations to Note

- Comments that come to your mind as you observe.
- Critical things that are happening in the classroom.
- Types of questions the students ask.
- Types of questions the teacher asks.
- Evidence of higher-order thinking.
- Evidence of confusion.
- Number of times students refer to and build on classmates' comments.
- Evidence of engagement.

4. Reflect on the Research Lesson

Reflection Meeting

A. Choose roles for the planning meeting: time keeper, note taker, facilitator etc.
B. The lesson teacher should first share their reflections on the lesson.
C. Observing teachers should share data and reflections from the lesson (each person should have an opportunity to share their opinions and provide evidence for their observations).
D. Use the data to highlight student learning, lesson design, and any broader issues in the teaching & learning.
E. Reflect on any changes that should/not be made to the Teaching & Learning template and decide whether or not to teach this adjusted plan or embark on a new cycle.
F. Record any changes to the Lesson Study plan and make note of new ideas that have resulted from the observation of the lesson.
G. Review key decisions and/or insights made during the lesson study meeting.
H. Agree upon assignments to be completed or followed up in the next meeting.

Note: If the group agrees to revise and re-teach the lesson, what changes affected the student learning? How was the learning affected?

Appendix B: Materials for use during lesson study

Long-term Goals and Objectives

Personal Objectives	
Teaching Objectives	
Classroom Objectives	
Objectives for Students	

Teaching-Learning Plan

Participating Teachers	
Instructor	
Date	
Intended Class & Year Group	

Lesson Title	
Research Theme/ **Subject Matter Goals/** **Lesson Goals**	
Lesson Rationale • Why focus on this topic? • What is difficult about learning or teaching this topic? • What is currently noticed about students learning this topic? • Why we have designed the lesson as shown.	
What do students already know about this topic? **How does students' understanding of this topic develop?**	
Where does this lesson relate to the curriculum?	
How will the learning be assessed?	

Approximate Time Guide	Teaching/ Learning Activities	Anticipated Student Responses	Anticipated Teacher Responses	Points to notice & evaluate

Lesson Flow for Instructing Teacher

Lesson Flow	Role of Teacher (s)

Observing the Research Lesson

Observers may wish to take some of the following actions:

1. Make notes on individual student comments and conversations, noting the names of students
2. Note situations in which students are collaborating or choosing not to collaborate
3. Look for examples of how students construct their understanding through their discussions and activities
4. Document the variety of methods that individual students use to solve problems, including errors.
5. Decide on pairs or groups of students on which to call on when students are asked to present their work.

The following are a number of options you may wish to follow during your observations. You and your colleagues may amend/use as you feel best impacts your own research on students' engagement.

Observer Questions	Notes
Was the goal clear? Did the supporting activities contribute effectively to achieving the goal?	
Was the flow of the lesson coherent and did it support students' learning of the concept?	
Were the activities and the materials helpful in achieving the goal of the lesson?	
Did the classroom discussions help promote student understanding?	
Was the content of the lesson appropriate for students' level of understanding?	
Did students apply their prior knowledge to understand the content of the lesson?	
Did the teacher's questions engage and facilitate student thinking?	
Were student ideas valued and incorporated into the lesson? Did the lesson summary refer to student theories or ideas?	
Was the lesson summary consistent with the lesson goal?	
How could the teacher reinforce what the students learned during the lesson?	

Recording Student Talk

Point in the lesson	Student Comments/Conversations	Significance

How Complex Instruction led to High and Equitable Achievement:
The Case of Railside School.

Jo Boaler, The University of Sussex.

Introduction.

This short paper will introduce you to the work of a group of equity-oriented teachers in an inner city school in California, who brought about amazing achievements in mathematics. The teachers used an approach called 'complex instruction', which is not well known in the UK, to bring about high achievements and great enjoyment of maths among students. I worked with a team of doctoral students to research the impact of the school's approach and we compared the students' learning experiences to students in other schools who worked differently. In this paper I will describe how the teachers worked and the results they achieved, in a related paper on this website (called 'Complex Instruction in England – the journey, the new schools, and initial results'.) I will describe my trip to Downing Street to show the approach to government ministers, which resulted in schools in England bravely taking on the innovative and unusual approach.

One of the most difficult challenges faced by teachers of maths is the wide range of students they teach. Even when taught in sets, maths classes often include students with low motivation and weak knowledge alongside others with advanced understanding and high motivation. Not surprisingly many teachers are supporters of ability grouping as it seems too hard to teach very mixed groups. In two different research studies I have conducted, in England and the US, I have followed students through secondary schools that teach and group students differently, investigating the impact of the different teaching and grouping methods upon achievement and enjoyment. In both studies the schools that used mixed ability approaches resulted in *extremely* impressive outcomes, including higher overall attainment and more equitable outcomes (Boaler, 2009). But in both cases the maths departments that achieved such goals used *particular methods* to make the mixed ability teaching effective. In this short paper I will describe the approach of "Railside school".

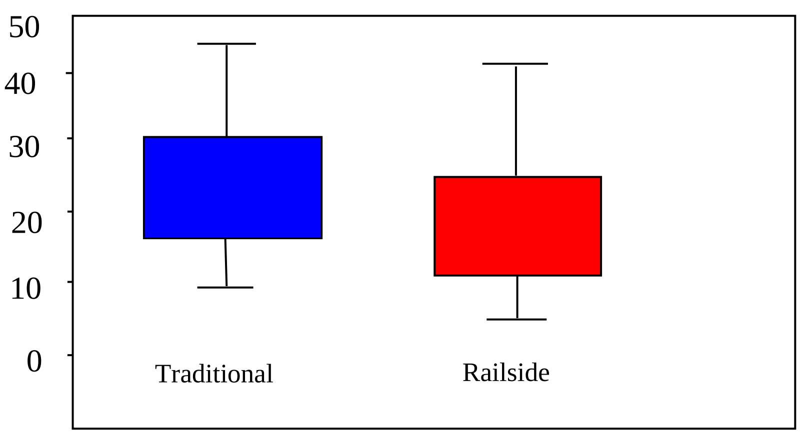
Railside School.

Railside High School is an inner-city school in California and lessons are frequently interrupted by the sound of trains passing right next to the playground. As with many inner-city schools the buildings look as though they are in need of some repair, but Railside is not like other inner-city schools. When I have taken visitors to the school and we have stepped inside the maths classrooms, they have been amazed to see all the students hard at work, engaged and excited about maths and advanced level classes filled to capacity. I first visited Railside a few years ago, when I was a professor at Stanford University, because I had learned that the teachers collaborated and planned teaching ideas together and I was interested to see their lessons. I saw enough in that visit to invite the school to be part of a longitudinal research project investigating the effectiveness of different mathematics approaches. Some four years later, after we had followed 700 students through three high schools, observing, interviewing and assessing students, we knew that Railside's approach was both highly unusual and highly successful.

The mathematics teachers at Railside originally taught using traditional methods, but the teachers were unhappy with the high failure rates among students and the students' lack of interest in maths The teachers decided to engage students in more group work and to follow a pedagogical approach designed to make group work equal, because they knew that group work often fails when students work unequally. They changed their classes from those taught in sets, to those taught in mixed ability groups and they designed a new curriculum together, that was based on what they called "group-worthy tasks". They moved from questions and exercises that gave students practice on mathematical methods such as 'factoring polynomials', or 'solving inequalities', and designed their curriculum around bigger mathematical ideas, with unifying themes such as "What is a linear function?" They also introduced an important theme of 'multiple representations' – this meant that they always asked students to communicate mathematical ideas in two or more ways, such as through words, diagrams, tables, symbols, objects, and graphs. They also asked students to discuss the different representations with each other and to move between the different representations, for example colour coding their diagrams and graphs to show where the same information was communicated (an example of one of their group-worthy, multiple representation tasks is given in the accompanying paper: 'An example of a group-worthy task in Railside school'.). When we interviewed students and asked them what they thought maths was, they did not tell us that it was a set of rules, as most students do, instead they told us that maths was a form of communication, or a language, as one student explained: "*Math is like kind of a language, because it has got a whole* *bunch of different meanings to it, and I think it is communicating. When you know the solution to a problem, I mean that is kind of like communicating with your friends*."

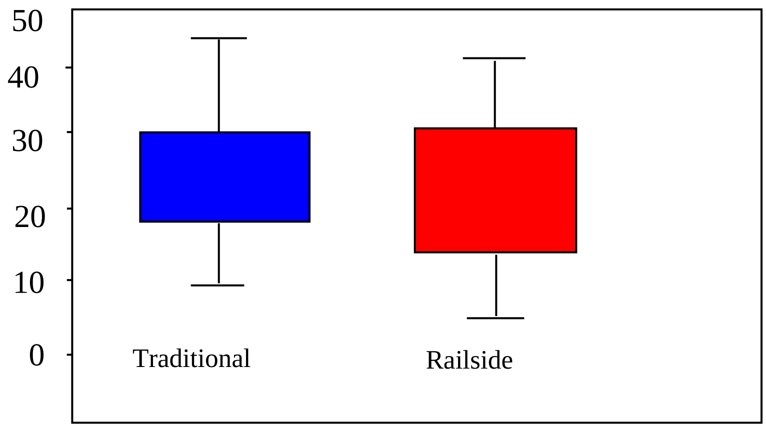
I was studying the learning of the Railside students as part of a larger, four-year study of three American high schools, during which we followed 700 students who were taught differently. The students at Railside worked on group tasks, in mixed ability classes, whereas the other students learned traditionally – practicing methods that a teacher demonstrated, through individual work. During the four-year study we collected a range of data, including approximately 600 hours of classroom observations, assessments given to the students each year, questionnaires and interviews. Railside school was more diverse than the other two schools, with more English language learners and higher levels of cultural diversity (approximately 38% of students were Latino/a, 23% African American, 20% White, 16% Asian or Pacific Islanders and 3% from other groups). We tested all of the students at the beginning of high school, before they had started working in different ways. At that time the Railside students were achieving at significantly lower levels than students at the other two schools, which is not atypical for students attending inner city schools, but within two years they were achieving at significantly higher levels. The following graphs show the achievement of students at the beginning of the study (year 1 pre-assessment) and at the end of year 1 (equivalent to Y9 in the UK) and the end of year 2 (equivalent to Y10):

Year 1 Pre-Assessment.



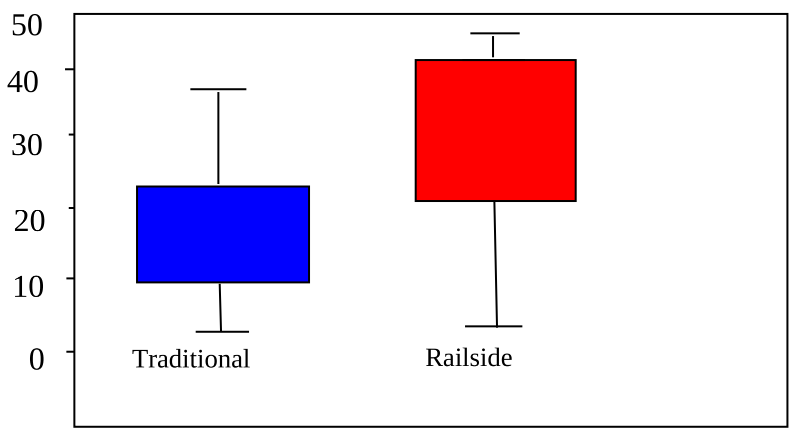
n=658, t = -9.141, p <0.001

End of Year 1



n=637, t = -2.040, p <0.04

End of Year 2



n=5128, t = -8.309, p <0.001

At the end of year 2 (similar to Y10 in the UK) the average achievement of the students taught traditionally was equivalent to a GCSE grade D, at Railside the average achievement was equivalent to a GCSE grade C. In other words, the Railside students learned in one year what it took the students taught traditionally to learn in more than two years. This result was achieved at Railside even though the students started the school at significantly lower levels, with severe gaps in their mathematical knowledge and understanding.

Students at Railside were also more positive about mathematics and took more courses. In year 4, *41% of seniors were enrolled in calculus*, compared with approximately 27% in the other two schools. The number of students taking advanced maths at Railside was incredibly high, for any school but particularly for a diverse, inner-city school where students arrived with low levels of mathematics knowledge. Importantly, inequities between students of different ethnic groups disappeared or were reduced in all cases at Railside whereas they remained at the other schools that employed ability grouping (for more detail see Boaler, 2009). Our statistical analysis showed that the students who were most advantaged by the mixed ability Railside approach were the high achievers, as they improved more than all other students. The high achievers at Railside also learned significantly more than the high achievers who went into top sets in the traditional schools. Many people are concerned about mixed ability teaching as they worry about the high achieving students who may be held back in mixed groups, but we found that the students were advantaged because they spent time explaining work, which helped their own understanding, and they were able to think more deeply about maths, rather than rushing through more and more work, as typically happens in top set classrooms.

The Railside classrooms were all organized in groups and students helped each other as they worked. The Railside teachers paid a lot of attention to the ways the groups worked together and they taught students to respect the contributions of other students, regardless of their prior attainment or their status with other students. One unfortunate but common side effect of some classroom approaches is that students develop beliefs about the inferiority or superiority of different students. In the other classes we studied, that were taught traditionally, students talked about other students as smart and dumb, quick and slow. At Railside the students did not talk in these ways. This did not mean that they thought all students were the same, but they came to appreciate the diversity of the class and the various attributes that different students offered. As Zane described to me: "*Everybody in there is at a different level. But what makes the class good is that everybody's at different levels so everybody's constantly teaching each other and helping each other out.*"

The teachers at Railside followed an approach called 'complex instruction' (http://cgi.stanford.edu/group/pci/cgi-bin/site.cgi), which is a method designed to make group work more effective and to promote equity in classrooms. They emphasized that all children were "smart" and had strengths in different areas and that everyone had something important to offer when working on maths. There were many reasons for the success of the Railside students. Importantly, the students were given opportunities to work on interesting problems that required them to think, and not just reproduce methods, and they were required to discuss mathematics with each other, increasing their interest and enjoyment. But there was another important aspect of the school's approach that is much more rare – the teachers enacted an expanded conception of mathematics and "smartness". The teachers at Railside knew that being good at mathematics involves many different ways of working, as mathematicians' accounts tell us. It involves asking questions, drawing pictures and graphs, rephrasing problems, justifying methods and representing ideas, in addition to calculating with procedures. Instead of just rewarding the correct use of procedures the teachers encouraged and rewarded all of these different ways of being mathematical.

In the remainder of this short paper, I will outline seven of the key methods that teachers used to bring about the incredible successes they achieved. I have already mentioned that the teachers used an approach called 'complex instruction', designed by Liz Cohen and Rachel Lotan (Cohen, 1994; Cohen & Lotan, 1997), to make group work equal. It is a pedagogical approach designed for use in any subject areas, and most schools use it in other subjects such as humanities. At Raiside the maths department was the only department using the approach at the time of our research, although now other departments have started to use the approach as the maths department had the best state test results in the school.

Some maths departments employ group work with limited success, particularly because groups do not always function well, with some students doing more of the work than others, and some students being excluded or choosing to opt out. Complex instruction (or CI) aims to counter social and academic status differences in classrooms, starting from the premise that status differences do not emerge because of particular students but because of group *interactions*. The approach includes a number of recommended practices that the mathematics department employed and refined for use in their subject area. In the next section I will review seven of the practices that the teachers employed and that our long term observations, interviews with students, and detailed analyses, showed to be important in the promotion of high achievement and equity. The first four (multidimensional classrooms, student roles, assigning competence, and student responsibility) are recommended in the complex instruction approach, the last three (high expectations, effort over ability, and clear expectations) were consistent with the approach and they were important to the high and equitable results that were achieved.

Complex Instruction.

(1) Multidimensionality

In many mathematics classrooms there is one practice that is valued above all others – that of executing procedures correctly and quickly. The narrowness by which success is judged means that some students rise to the top of classes, gaining good marks and teacher praise, while others sink to the bottom with most students knowing where they are in the hierarchy created. Such classrooms are uni-dimensional – the dimensions along which success is presented are singular. A central part of the complex instruction approach is what the authors refer to as *multiple ability treatment*. The idea is that when a more open set of task requirements is given, that values many different 'abilities,' then students will be more engaged and more successful. At Railside the teachers created multidimensional classes by asking the question – how does a mathematician work? Of course it is important to execute procedures but this is only one aspect of mathematical work and a mathematician also needs, for example, to ask good questions, connect methods, communicate ideas in different representations, use logic and reasoning, explain well and so on. What was unusual about Railside was that they did not only value all of these ways of being mathematical, but they assessed students on them. This meant that a student who may not have been the fastest or best at following and executing methods, could be very successful if they asked good questions, or saw problems in different ways, explaining them well to others. In analysing the success of the Railside approach, we concluded that *many more students were successful, because there were many more ways to be successful*. And this success transferred to state tests even though the tests were very narrow and did not assess the broad ways of working that students had learned, as the students had learned to feel good about themselves in maths classrooms and they had developed confidence in their own work.

The teachers created multi-dimensional classes by using what they referred to as *group-worthy problems* – open-ended problems that illustrated important mathematical concepts, allowed for multiple representations, and had several possible solution paths (Horn, 2005). The teachers had created the introductory algebra course themselves, adapting problems from different published curriculum to make them group-worthy. This enabled more students to contribute ideas and feel valued. When we interviewed the students from all three schools and asked them "What does it take to be successful in math class?" a stunning 94% of students from the traditional classes said the same thing - "you need to pay careful attention". At Railside the students answered the same question with many different practices such as: asking good questions, rephrasing problems, explaining well, being logical, justifying work, considering answers, and using manipulatives. The different dimensions that students believed to be an important part of mathematical work at Railside were valued in the teachers' interactions and the grading system.

The multidimensional nature of the classes at Railside was an extremely important part of the increased success of students. Students were aware of the different practices that were valued and all of them felt successful because they are able to excel at some of them (in contrast to students in many schools in England who believe that they are not good at maths). One of the messages that the teachers frequently gave was that "no-one is good at all of these ways of working, but everyone is good at some of them". The following comments given by students in interviews give an indication of the multidimensionality of classes –

> *Back in middle school the only thing you worked on was your math skills. But here you work socially and you also try to learn to help people and get help. Like you improve on your social skills, math skills and logic skills. (Janet, Y1)*

> *With math you have to interact with everybody and talk to them and answer their questions. You can't be just like "oh here's the book, look at the numbers and figure it out"*
> *Int: Why is that different for math?*
> *It's not just one way to do it (…) It's more interpretive. It's not just one answer. There's more than one way to get it. And then it's like: "Why does it work"? (Jasmine, Y1)*

It is rare to hear students describe mathematics as broader and more *interpretive* than other subjects. This breadth was important to the wide rates of success and participation achieved.

(2) Roles

When students were placed into groups they were also given a particular role to play, such as *facilitator, team captain, recorder/reporter* or *resource manager* (Cohen & Lotan, 1997). The idea behind this approach is that all students have important work to do in groups, without which the group cannot function. At Railside the teachers emphasized the different roles at frequent intervals, stopping, for example, at the start of class to remind facilitators to help people check their answers or show their work or to ask the group "What did you get for number 1?" Teachers also used the roles to distribute authority in the room differently – when they wanted to give new information to the class, for example, instead of talking to the whole class they would call the team captains out to the front and tell them the information the other students needed; they would then return to their groups and share the new information. Students changed roles at the end of each unit of work. The teachers reinforced the status of the different roles and the important part they played in the mathematical work that was being undertaken. The roles contributed to the complex interconnected system that operated in each classroom, a system in which everyone had something important to do and all students learned to rely upon each other.

(3) Assigning Competence

An interesting and subtle approach that is recommended within the complex instruction literature is that of *assigning competence*. This is a practice that involves teachers raising the status of students that may be of a lower status in a group, by, for example, praising something they have said or done that has mathematical value, and bringing it to the group's attention; asking a student to present an idea; or publicly praising a student's work in a whole class setting. This practice was one that I could not fully imagine until I saw it enacted. My first awareness of it came about when a quiet Eastern European boy muttered something in a group that was dominated by two happy and excited Latina girls. The teacher who was visiting the table immediately picked up on it saying "Good Ivan, that is important". Later when the girls offered a response to one of the teacher's questions he said, "Oh that is like Ivan's idea, you're building on that". He raised the status of Ivan's contribution, which would almost certainly have been lost without such an intervention. Ivan visibly straightened up and leaned forward as the teacher reminded the girls of his idea. Cohen (1994) recommends that if student feedback is to address status issues, it must be public, intellectual, specific and relevant to the group task (Cohen, 1994, p. 132). The public dimension is important as other students learn about the broad dimensions that are valued; the intellectual dimension ensures that the feedback is an aspect of mathematical work, and the specific dimension means that students know exactly what the teacher is praising.

Two of the mathematical practices that the teachers valued that seemed particularly important in the promotion of equity, were justification and reasoning. At Railside students were required to justify their answers, giving reasons for their methods, at almost all times. There are many good reasons for this – justification and reasoning are intrinsically mathematical practices (RAND, 2002; Martino & Maher, 1999) – but these practices also serve an interesting and particular role in the promotion of equity. The following boy was one of the lower achievers in the class, and it is interesting to hear him talk about the ways he was supported by the practices of justification and reasoning:

Most of them, they just like know what to do and everything. First you're like "why you put this?" and then like if I do my work and compare it to theirs. Theirs is like super different 'cos they know, like what to do. I will be like – let me copy, I will be like "why you did this? And then I'd be like: "I don't get it why you got that." And then like, sometimes the answer's just like, they be like "yeah, he's right and you're wrong" But like – why?" (Juan, Y2)

Juan made it clear that he was helped by the practice of justification and that he felt comfortable pushing other students to go beyond answers and explain *why* their answers were given. At Railside, the teachers carefully prioritized the message that each student had two important responsibilities – both to help someone who asked for help, but also to ask if they needed help. Both were important in the pursuit of equity, and justification and reasoning emerged as helpful practices in the learning of a wide range of students.

(4) Teaching Students to be Responsible for Each Other's Learning

A major part of the equitable results attained at Railside was the serious way in which teachers expected students to be responsible for each other's learning. Many schools employ group work which, by its nature, brings with it an element of student responsibility for others, but Railside teachers went beyond this to ensure that students took their responsibility to each other very seriously. One way in which teachers nurtured a feeling of responsibility was through the assessment system. For example, teachers occasionally graded the work of a group by rating the quality of the conversations groups had. In addition, the teachers occasionally gave group tests, which took several formats. In one version – (which often prompts a sharp intake of breath from teachers in England when I tell them about it!) – students work through a test individually, but the teachers only take in and mark one test paper from one student in each group, and that mark stood as the mark for all the students in the group. That is a very serious way that teachers communicate to students that they are responsible for each other as they learn! A third way in which responsibility was encouraged was through the practice of asking one student in a group to answer a follow-up question after a group had worked on something. If the student could not answer the question, the teacher would leave the group to further discussion before returning to ask the same student again. In the intervening time, it was the group's responsibility to help the student learn the maths they needed to answer the question.

The teaching strategy of asking one member of a group to give an answer and an explanation, without help from their group-mates, was a subtle practice that had major implications for the classroom environment. This practice meant that students were responsible to everyone in their group. In the following interview extract the students talk about this particular practice and the implications it held:

> *Int: Is learning math an individual or a social thing?*
> *G: It's like both, because if you get it, then you have to explain it to everyone else. And then sometimes you just might have a group problem and we all have to get it. So I guess both.*
> *B: I think both - because individually you have to know the stuff yourself so that you can help others in your group work and stuff like that. You have to know it so you can explain it to them. Because you never know which one of the four people she's going to pick. And it depends on that one person that she picks to get the right answer. (Gisella & Bianca, Y2)*

The students in the extract above made the explicit link between teachers asking any group member to answer a question, and being responsible for their group members. They also communicate the shared orientation that students at Railside developed, saying that the purpose in knowing individually is not to be better than others but so "you can help others in your group."

The four dimensions of complex instruction – of multidimensional classrooms, student roles, assigning competence, and shared responsibility, were enhanced by the 3 practices that the department chose to implement and that I will review briefly.

Three Practices Leading to High and Equitable Achievement.

(1) High Expectations

It was critical to the success of the students that teachers kept the demand of lessons high, both by providing complex problems and by following up with high-level questions. When students could not complete questions the teachers would leave groups to work through their understanding rather than providing them with small structured questions that led them to the correct answer. In interviews with the students, it became clear that they appreciated the high demands placed upon them. The students' appreciation was also demonstrated through questionnaires. For example, one of the questions started with the stem: "When I get stuck on a math problem, it is most helpful when my teacher…" This was followed by answers such as "tells me the answer" "leads me through the problem step by step" and "helps me without giving away the answer". Students could respond to each on a four-point scale (strongly agree, agree, disagree, strongly disagree). Almost half of the Railside students (47%) *strongly* agreed with the response: "Helps me *without* giving away the answer," compared with 27% of students in the 'traditional' classes at the other two schools. In interviews with the Railside students it became clear that one of the things they most appreciated about their teachers was the fact that they gave them hard work which they interpreted as the teachers *believing in them*.

(2) Effort Over Ability.

In addition to the actions in which teachers engaged, the teachers also gave frequent and strong messages to students about the nature of high achievement in mathematics. Unlike many teachers in England the Railside teachers did not believe in the idea of 'ability' and they, continually emphasised that mathematical success comes from hard work. I have already described the multidimensionality of classrooms and the fact that teachers took every opportunity to value something students could do, but they also kept reassuring students that they could achieve anything if they put in the effort. This message was heard by students and they communicated it to us in interviews. For example:

> *To be successful in math you really have to just like, put your mind to it and keep on trying – because math is all about trying. It's kind of a hard subject because it involves many things. (…) but as long as you keep on trying and don't give up then you know that you can do it. (Sara, Y1)*

In questionnaires, we offered the statement "Anyone can be really good at math if they try" 84% of Railside students agreed with this, compared with 52% of students in the traditional classes.

(3) Clear Expectations

The final aspect of the teachers' practice that I will highlight also relates to the expectations they offered the students. In addition to stressing the importance of effort the teachers were very clear about the particular ways of working in which students needed to engage. For example, the teachers would stop the students as they were working and talking and point out valuable ways in which they were working. In one videotaped example of this, Guillermo, the department co-chair, helped a boy named Arturo. Arturo was working on a problem about the number of pennies a person would be able to carry, considering their weight. He stopped and told the teacher that he was confused, so Guillermo told him to ask a specific question; as Arturo framed a question he realised what he needed to do and continued with his thinking. Arturo decided the answer to the question he was working on was "550 pennies" but then stopped himself saying "No, wait, that's not very much." At that point Guillermo interrupted him saying:

> Wait, hold on a second, two things just happened there. Number one is, when I said "what is the exact question?" you stopped to ask yourself the exact question and then suddenly you had ideas. That happens to a lot of students. If they're confused, the thing you have to do is say, "OK what am I trying to figure out? Like exactly", and, like, say it. So say it out loud or say it in your head but say it as a sentence. That's number one and number two, then you checked out the answer and you realized the answer wasn't reasonable and that is *excellent* because a lot of people would have just left it there and not said, "what, 500 pennies? That's not very much." (Guillermo, Math department co-chair)

Prior to the beginning of new work teachers set out the valued ways of working, encouraging students to, for example, pick "tricky" examples when writing a book (one of the projects they completed) as they would "show off" the mathematics that they knew; they also encouraged students individually as shown in the example above. The teachers communicated very clearly to students how they could achieve and what the teachers were looking for. This was also true of the teachers in the school in England that I studied (see Boaler, 2009) who also brought about high achievement and equitable outcomes.

Relational Equity

It would be hard to spend 4 years in the classrooms at Railside without noticing that the students were learning to treat each other in more respectful ways than is typically seen in schools and that ethnic cliques were less evident in the mathematics classrooms than they are in most schools. Further, such behavior did not just *happen* to take place in a mathematics classroom; it was fundamentally related to the students' conceptions of and work within mathematics. Thus, the work of students and teachers at Railside was equitable partly because they achieved more equitable outcomes on tests, with few achievement differences for students of different ethnic or gender groups, but also because they learned to act in more equitable ways in their classrooms. Because the teachers valued different ways of seeing and solving problems, students learned to appreciate the contributions of different students, from many different cultural groups and with many different characteristics and perspectives. The students reported that ethnic cliques did not develop in their school because of the ways they worked in their maths classes. It seemed to me that the students learned something extremely important, that would serve them and others well in their future interactions in society, which is not captured in conceptions of equity that deal only with test scores or treatment in schools. I propose that such behavior is a form of equity, and I have termed it *relational* equity (see also Boaler, 2008).

It is commonly believed that students will learn respect for different people and cultures if they have discussions about such issues or read about them in English or social studies classes. I propose that all subjects have something to contribute in the promotion of equity and that maths, often regarded as the most abstract subject removed from responsibilities of cultural or social awareness, has an important contribution to make. For the respectful relationships that Railside students developed across cultures and genders that they took into their lives were only made possible by a maths approach that valued different insights, methods and perspectives in the collective solving of particular problems.

Conclusion.

Railside school offers an important case of an inner-city, low-income, comprehensive school that brought about high and equitable achievement. Our four-year, longitudinal study, in which we monitored students at this and two other schools, revealed the importance of the approach that the school employed in supporting mixed ability teaching and providing high level learning opportunities for a wide range of students. Railside school is not a perfect place - the teachers would like to achieve more in terms of student achievement and the elimination of inequities, and they rarely feel satisfied with the achievements they have made to date, despite the vast amounts of time they spend planning and working. But research on urban schools, and the experiences of maths students in particular, tells us that the achievements at Railside are extremely unusual. Teachers who have heard about the achievements of Railside's maths department have asked for their curriculum so that they may use it, but while the curriculum plays a central role in what is achieved at the school, it is only one part of a complex, interconnected system. At the heart of this system is the work of the teachers, and the many different equitable practices in which they engage.

References

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal. 34 (2), 167-194.*

Boaler, J. (2009). *The Elephant in the Classroom: Teaching Students to Learn and Love Maths*. London: Souvenir Press.

Cohen, D. & Ball, D.L. (2001) Making change: instruction and its improvement, *Phi Delta Kappan*, (September), pp. 73-77.

Cohen, E. (1994). *Designing groupwork*. New York: Teachers College Press.

Cohen, E., & Lotan, R. (Eds.). (1997). *Working for equity in heterogeneous classrooms: sociological theory in practice*. New York: Teachers College Press.

Gutstein, E., Lipman, P., Hernandez, P., & de los Reyes, R. (1997). Culturally relevant mathematics teaching in a Mexican American context. *Journal for Research in Mathematic Education, 28*(6), 709-737.

Horn, I.S. (2005). Learning on the job: a situated account of teacher learning in high school mathematics departments. *Cognition & Instruction*,23(2).

Lee, C. D. (2001). Is October Brown Chinese? a cultural modeling activity system for underachieving students. *American Educational Research Journal, 38*(1), 97-141.

Martino, A.M. & Maher, C. (1999) Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior*, 18(1), pp. 53-78.

Oakes, J. (1985). *Keeping track. How schools structure inequality.* New Haven: Yale University Press.

RAND, M.S.P. (2002, October) Mathematical proficiency for all students: toward a strategic research and development program in mathematics education (DRU-2773-OERI)(Arlington, VA, RAND Education & Science and Technology Policy Institute.).

Schweder, R.A. (2003). Why do men barbeque? recipes for cultural psychology. Cambridge, MA: Harvard University Press, 74-133.

Additional Resources for Classroom Use.

http://www.sussex.ac.uk/education/profile205572.html

The website above includes 2 downloadable papers:

Boaler, J (*2008*). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal.*

Boaler, J & Staples, M. (*2008*). Creating Mathematical Futures through an Equitable Teaching Approach: The Case of Railside School. *Teachers' College Record.*

These are longer versions of this paper with more evidence and details on the approach described.

Complex Instruction:

Website:

http://cgi.stanford.edu/group/pci/cgi-bin/site.cgi

Cohen, E. (1986) *Designing groupwork: strategies for heterogeneous classrooms* (New York, Teacher's College Press).

Cohen, E. & Lotan, R. (Eds.) (1997) *Working for equity in heterogeneous classrooms: Sociological Theory in Practice* (New York, Teachers College Press).

"Opening Our Ideas": How a detracked mathematics approach promoted respect, responsibility, and high achievement.

Jo Boaler, Stanford University

In *Theory into Practice*, Winter 2006, Vol. 45, No. 1

Abstract

This article describes the ways in which the mathematics department of an urban, ethnically diverse school, brought about high and equitable mathematics achievement. The teachers employed heterogeneous grouping and *complex instruction*, an approach designed to counter status differences in classrooms. As part of this approach teachers encouraged multi-dimensional classrooms, valued the perspectives of different students, and encouraged students to be responsible for each another. The work of students and teachers at Railside was equitable partly because students achieved more equitable outcomes on tests, but also because students learned to act in more equitable ways in their classrooms. Students learned to appreciate the contributions of students from different cultural groups, genders and attainment levels, a behavior that I have termed *relational equity*. This article describes the teaching practices that enabled the department to bring about such important achievements.

"What makes the class good is that everybody's at different levels so everybody's constantly teaching each other and helping each other out." (Zane, Railside school)

Introduction.

One of the most difficult challenges facing teachers of mathematics, and other subjects, is the wide range of students they teach. Mathematics classes often include students with low motivation and weak knowledge alongside others with advanced understanding and high motivation. Not surprisingly many teachers support the practice of ability grouping so that they may narrow the range and teach more effectively. In two different research studies I have conducted, in England and the US, I have followed students through high schools, investigating the impact of different teaching and grouping methods upon learning. In both studies the schools that used mixed ability approaches resulted in higher overall attainment and more equitable outcomes (Boaler, 2002, 2004). But in both cases the mathematics departments that brought about higher and more equitable attainment employed particular methods to make the heterogeneous teaching effective. In this article I will describe the approach of Railside school, an urban high school in California. At Railside the students not only scored at high levels on tests, with differences in attainment between students of different cultural groups diminishing or disappearing while they were at the school, but the students learned to treat each other with respect. They learned to appreciate the contributions of students from different cultural groups, social classes, genders and attainment levels and develop extremely positive intellectual relations. I have termed this behavior relational equity (see also Boaler*, in press*), and this article will explain how it was achieved. It is commonly believed that students will learn respect for people from different cultures and circumstances if they learn through culturally relevant examples, or consider the history of different cultures. At Railside, the respectful relationships that students developed came about through a collaborative problem solving approach in which students worked together and learned to appreciate the different insights, methods, and perspectives that different students offered in the collective solving of problems.

Our study of Railside school was conducted as part of a larger, four-year study of three US high schools. At Railside, the department employed a mixed-ability reform-oriented approach, the other two mathematics departments employed tracking and traditional teaching methods. During the four-year study we collected a range of data, including approximately 600 hours of classroom observations, assessments given to the students each year, questionnaires and interviews. Railside school was more urban than the other two schools, with more English language learners and higher levels of cultural diversity (approximately 38% of students were Latino/a, 23% African American, 20% White, 16% Asian or Pacific Islanders. 3% were from other groups). On tests given to the students each year, the Railside students started at significantly lower levels than students at the other two schools but within two years they were achieving at significantly higher levels. Students at Railside were also more positive about mathematics and took more courses. In year 4, 41% of seniors were enrolled in calculus, compared with approximately 27% in the other two schools. Importantly, inequities between students of different ethnic groups disappeared or were reduced in all cases at Railside whereas they remained at the other schools that employed tracking (for more detail see Boaler, 2004).

Some mathematics departments employ group work with limited success, particularly because groups do not always function well, with some students doing more of the work than others, and some students being excluded or choosing to opt out. At Railside the teachers employed additional strategies to make group work successful. They adopted an approach called complex instruction, designed by Liz Cohen and Rachel Lotan (Cohen, 1994; Cohen & Lotan, 1997) for use in all subject areas. The approach aims to counter social and academic status differences in classrooms, starting from the premise that status differences do not emerge because of particular students but because of group *interactions*. The approach includes a number of recommended practices that the mathematics department employed and refined for use in their subject area. In the next section I will review seven of the practices that the teachers employed and that our long term observations, interviews with students, and detailed analyses, showed to be important in the promotion of equity. The first four (multidimensional classrooms, student roles, assigning competence, and student responsibility) are recommended in the complex instruction approach, the last three (high expectations, effort over ability, and learning practices) were consonant with the approach and they were important to the high and equitable results that were achieved.

Equitable Teaching Practices.

Multidimensionality

In many mathematics classrooms there is one practice that is valued above all others – that of executing procedures correctly and quickly. The narrowness by which success is judged means that some students rise to the top of classes, gaining good grades and teacher praise, while others sink to the bottom with most students knowing where they are in the hierarchy created. Such classrooms are uni-dimensional – the dimensions along which success is presented are singular. A central tenet of the complex instruction approach is what the authors refer to as *multiple ability treatment*. This approach is based upon the idea that expectations of success and failure can be modified by the provision of a more open set of task requirements that value many different abilities. Teachers should explain to students that no one student will be "good on all these abilities" and that each student will be "good on at least one" (Cohen & Lotan, 1977, p. 78).

At Railside the teachers created multidimensional classes by valuing many dimensions of mathematical work. This was achieved – in part – by giving students what the teachers referred to as *group-worthy problems* – – open-ended problems that illustrated important mathematical concepts, allowed for multiple representations, and had several possible solution paths (Horn, 2005). The teachers had created the algebra curriculum themselves, adapting problems from different curriculum to make them group-worthy. This enabled more students to contribute ideas and feel valued. When we interviewed the students and asked them "What does it take to be successful in mathematics class?" they offered many different practices such as: asking good questions, rephrasing problems, explaining well, being logical, justifying work, considering answers, and using manipulatives. When we asked students in the traditional classes in the other two schools in our study what they needed to do in order to be successful, they talked in much more narrow ways, saying that they needed to concentrate and pay careful attention. The different dimensions that students believed to be an important part of mathematical work at Railside were valued in the teachers' interactions and the grading system.

The multidimensional nature of the classes at Railside was an extremely important part of the increased success of students. Put simply, when there are many ways to be successful, many more students are successful. Students are aware of the different practices that are valued and they feel successful because they are able to excel at some of them. The following comments given by students in interviews give an indication of the multidimensionality of classes -

> *With math you have to interact with everybody and talk to them and answer their questions. You can't be just like "oh here's the book, look at the numbers and figure it out"*
>
> *Int: Why is that different for math?*

It's not just one way to do it (…) It's more interpretive. It's not just one answer. There's more than one way to get it. And then it's like: "Why does it work"? (Jasmine, Y1)

It is rare to hear students describe mathematics as more broad and more *interpretive* than other subjects. This breadth was important to the wide rates of success and participation achieved.

Roles

When students were placed into groups they were also given a particular role to play, such as *facilitator, team captain, recorder/reporter* or *resource manager* (Cohen & Lotan, 1997). The premise behind this approach is that all students have important work to do in groups, without which the group cannot function. At Railside the teachers emphasized the different roles at frequent intervals, stopping, for example, at the start of class to remind facilitators to help people check answers or show their work or to ask the group "What did you get for number 1?" Students changed roles at the end of each unit of work. The teachers reinforced the status of the different roles and the important part they played in the mathematical work that was being undertaken. The roles contributed to the complex interconnected system that operated in each classroom, a system in which everyone had something important to do and all students learned to rely upon each other.

Assigning Competence

An interesting and subtle approach that is recommended within the complex instruction literature is that of *assigning competence*. This is a practice that involves teachers raising the status of students that may be of a lower status in a group, by, for example, praising something they have said or done that has intellectual value, and bringing it to the group's attention; asking a student to present an idea; or publicly praising a student's work in a whole class setting. This practice was one that I could not fully imagine until I saw it enacted. My first awareness of it came about when a quiet Eastern European boy muttered something in a group that was dominated by two happy and excited Latina girls. The teacher who was visiting the table immediately picked up on it saying "Good Ivan, that is important". Later when the girls offered a response to one of the teacher's questions he said, "Oh that is like Ivan's idea, you're building on that". He raised the status of Ivan's contribution, which would almost certainly have been lost without such an intervention. Ivan visibly straightened up and leaned forward as the teacher reminded the girls of his idea. Cohen (1994) recommends that if student feedback is to address status issues, it must be public, intellectual, specific and relevant to the group task (Cohen, 1994, p. 132). The public dimension is important as other students learn about the broad dimensions that are valued; the intellectual dimension ensures that the feedback is an aspect of mathematical work, and the specific dimension means that students know exactly what the teacher is praising.

Teaching Students to be Responsible for Each Other's Learning

A major part of the equitable results attained at Railside was the serious way in which teachers expected students to be responsible for each other's learning. Many schools employ group work which, by its nature, brings with it an element of interdependence, but Railside teachers went beyond this to ensure that students took their responsibility to each other very seriously. One way in which teachers nurtured a feeling of responsibility was through the assessment system. For example, teachers occasionally graded the work of a group by rating the quality of the conversations groups had. In addition, the teachers occasionally gave group tests, which took several formats. In one version, students worked through a test together, but the teachers graded only one of the individual papers and that grade stood as the grade for all the students in the group. A third way in which responsibility was encouraged was through the practice of asking one student in a group to answer a follow-up question after a group had worked on something. If the student could not answer the question, the teacher would leave the group to further discussion before returning to ask the same student again. In the intervening time, it was the group's responsibility to help the student learn the mathematics they needed to answer the question.

The teaching strategy of asking one member of a group to give an answer and an explanation, without help from their group-mates, was a subtle practice that had major implications for the classroom environment. This practice meant that students were responsible to everyone in their group. In the following interview extract the students talk about this particular practice and the implications it held:

Int: Is learning math an individual or a social thing?
G: It's like both, because if you get it, then you have to explain it to everyone else. And then sometimes you just might have a group problem and we all have to get it. So I guess both.
B: I think both - because individually you have to know the stuff yourself so that you can help others in your group work and stuff like that. You have to know it so you can explain it to them. Because you never know which one of the four people she's going to pick. And it depends on that one person that she picks to get the right answer. (Gisella & Bianca, Y2)

The students in the extract above made the explicit link between teachers asking any group member to answer a question, and being responsible for their group members. They also communicate an interesting social orientation that becomes instantiated through the mathematics approach, saying that the purpose in knowing individually is not to be better than others but so "you can help others in your group."

Two of the practices that I have come to regard as being particularly important in the promotion of equity, and that are central to the responsibility students show for each other, are justification and reasoning. At Railside students were required to justify their answers, giving reasons for their methods, at almost all times. There are many good reasons for this – justification and reasoning are intrinsically mathematical practices (RAND, 2002; Martino & Maher, 1999) – but these practices also serve an interesting and particular role in the promotion of equity. The following boy was not one of the highest achievers in the class, and it is interesting to hear him talk about the ways he was supported by the practices of justification and reasoning:

> *Most of them, they just like know what to do and everything. First you're like "why you put this?" and then like if I do my work and compare it to theirs. Theirs is like super different 'cos they know, like what to do. I will be like – let me copy, I will be like "why you did this? And then I'd be like: "I don't get it why you got that." And then like, sometimes the answer's just like, they be like "yeah, he's right and you're wrong" But like – why?" (Juan, Y2)*

Juan made it clear that he was helped by the practice of justification and that he felt comfortable pushing other students to go beyond answers and explain *why* their answers were given. At Railside, the teachers carefully prioritized the message that each student had two important responsibilities – both to help someone who asked for help, but also to ask if they needed help. Both were important in the pursuit of equity, and justification and reasoning emerged as helpful practices in the learning of a wide range of students.

High Expectations

There were many other, related aspects of the teachers' approach that I can only briefly review in this short paper. For example, it was critical to the success of the students that teachers kept the demand of lessons intellectually high, both by providing complex problems and by following up with high-level questions. When students could not complete questions the teachers would leave groups to work through their understanding rather than providing them with small structured questions that led them to the correct answer. In interviews with the students, it became clear that they appreciated the high demands placed upon them. The students' appreciation was also demonstrated through questionnaires. For example, one of the questions started with the stem: "When I get stuck on a math problem, it is most helpful when my teacher…" This was followed by answers such as "tells me the answer" "leads me through the problem step by step" and "helps me without giving away the answer". Students could respond to each on a four-point scale (strongly agree, agree, disagree, strongly disagree). Almost half of the Railside students (47%) *strongly* agreed with the response: "Helps me *without* giving away the answer," compared with 27% of students in the 'traditional' classes at the other two schools.

Effort Over Ability.

In addition to the actions in which teachers engaged, the teachers also gave frequent and strong messages to students about the nature of high achievement in mathematics, continually emphasizing that it was a product of hard work and not of innate ability. I have already described the multidimensionality of classrooms and the fact that teachers took every opportunity to value something students could do, but they also kept reassuring students that they could achieve anything if they put in the effort. This message was heard by students and they communicated it to us in interviews, with absolute sincerity. For example:

> *To be successful in math you really have to just like, put your mind to it and keep on trying – because math is all about trying. It's kind of a hard subject because it involves many things. (…) but as long as you keep on trying and don't give up then you know that you can do it. (Sara, Y1)*

In the year 3 questionnaires, we offered the statement "Anyone can be really good at math if they try" 84% of Railside students agreed with this, compared with 52% of students in the traditional classes.

Learning Practices.

The final aspect of the teachers' practice that I will highlight also relates to the expectations they offered the students. In addition to stressing the importance of effort the teachers were very clear about the particular ways of working in which students needed to engage. Cohen and Ball (2001) describe ways of working that are needed for learning as *learning practices*. For example, the teachers would stop the students as they were working and talking and point out valuable ways in which they were working. In one videotaped example of this, Guillermo, the department co-chair, helped a boy named Arturo. Arturo said he was confused, so Guillermo told him to ask a specific question; as Arturo framed a question he realized what he needed to do and continued with his thinking. Arturo decided the answer to the question he was working on was "550 pennies" but then stopped himself saying "No, wait, that's not very much." At that point Guillermo interrupted him saying:

> Wait, hold on a second, two things just happened there. Number one is, when I said "what is the exact question?" you stopped to ask yourself the exact question and then suddenly you had ideas. That happens to a lot of students. If they're confused, the thing you have to do is say, "OK what am I trying to figure out? Like exactly", and, like, say it. So say it out loud or say it in your head but say it as a sentence. That's number one and number two, then you checked out the answer and you realized the answer wasn't reasonable and that is *excellent* because a lot of people would have just left it there and not said, "what, 500 pennies? That's not very much." (Guillermo, Math department co-chair)

Prior to the beginning of new work teachers set out the valued ways of working, encouraging students to, for example, pick "tricky" examples when writing a book (one of the projects they completed) as they would "show off" the mathematics that they knew; they also encouraged students individually as shown in the example above. The teachers communicated very clearly to students which learning practices would help them achieve. This was also true of the teachers in the school in England that I studied (Boaler, 1997, 2002) who also brought about more equitable outcomes.

Relational Equity

It would be hard to spend years in the classrooms at Railside without noticing that the students were learning to treat each other in more respectful ways than is typically seen in schools and that ethnic cliques were less evident in the mathematics classrooms than they are in most schools. Further, such behavior did not just *happen* to take place in a mathematics classroom; it was fundamentally related to the students' conceptions of and work within mathematics. Thus, the work of students and teachers at Railside was equitable partly because they achieved more equitable outcomes on tests, with few achievement differences aligned with cultural differences, but also because they learned to act in more equitable ways in their classrooms. Students learned to appreciate the contributions of different students, from many different cultural groups and with many different characteristics and perspectives. It seemed to me that the students learned something extremely important, that would serve them and others well in their future interactions in society, which is not captured in conceptions of equity that deal only with test scores or treatment in schools. I propose that such behavior is a form of equity, and I have termed it *relational* equity (see also Boaler, in press).

It is commonly believed that students will learn respect for different people and cultures if they have discussions about such issues or read diverse forms of literature in English or social studies classes. I propose that all subjects have something to contribute in the promotion of equity and that mathematics, often regarded as the most abstract subject removed from responsibilities of cultural or social awareness, has an important contribution to make. For the respectful relationships that Railside students developed across cultures and genders that they took into their lives were only made possible by a mathematics approach that valued different insights, methods and perspectives in the collective solving of particular problems.

Conclusion.

I have focused upon Railside school in this paper because it is an important case of an urban, low-income high school that brought about high and equitable achievement. Our four-year, longitudinal study, in which we monitored students at this and two other schools, revealed the importance of the approach that the school employed in supporting mixed ability teaching and providing high level learning opportunities for a wide range of students. Railside school is not a perfect place - the teachers would like to achieve more in terms of student achievement and the elimination of inequities, and they rarely feel satisfied with the achievements they have made to date, despite the vast amounts of time they spend planning and working. But research on urban schools, and the experiences of mathematics students in particular, tells us that the achievements at Railside are extremely unusual. In this paper, I have attempted to convey the work of the teachers in bringing about the reduction in inequalities as well as general high achievement. In doing so, I hope also to have given a sense of the complexity of the relational and equitable system that they have in place. Teachers who have heard about the achievements of Railside's math department have asked for their curriculum so that they may use it, but while the curriculum plays a part in what is achieved at the school, it is only one part of a complex, interconnected system. At the heart of this system is the work of the teachers, and the many different equitable practices in which they engage.

References

Boaler, J. (1997). Setting, social class and survival of the quickest. *British Educational Research Journal, 23*(5), 575-595.

Boaler, J. (2002). *Experiencing school mathematics: traditional and reform approaches to teaching and their impact on student learning*. (Revised and Expanded Edition ed.). Mahwah, NJ: Lawrence Erlbaum Association.

Boaler, J. (2004). *Promoting equity in mathematics classrooms - important teaching practices and their impact on student learning*. Paper presented at ICME, Copenhagen, Denmark.

Boaler, J. (in press). Promoting Relational Equity. *Educational Leadership*.

Cohen, D. & Ball, D.L. (2001) Making change: instruction and its improvement, *Phi Delta Kappan*, (September), pp. 73-77.

Cohen, E. (1994). *Designing groupwork*. New York: Teachers College Press.

Cohen, E., & Lotan, R. (Eds.). (1997). *Working for equity in heterogeneous classrooms: sociological theory in practice*. New York: Teachers College Press.

Gutstein, E., Lipman, P., Hernandez, P., & de los Reyes, R. (1997). Culturally relevant mathematics teaching in a Mexican American context. *Journal for Research in Mathematic Education, 28*(6), 709-737.

Horn, I.S. (2005). Learning on the job: a situated account of teacher
learning in high school mathematics departments. *Cognition & Instruction*,
23(2).

Lee, C. D. (2001). Is October Brown Chinese? a cultural modeling activity system for underachieving students. *American Educational Research Journal, 38*(1), 97-141.

Martino, A.M. & Maher, C. (1999) Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior*, 18(1), pp. 53-78.

Oakes, J. (1985). *Keeping track. How schools structure inequality*. New Haven: Yale University Press.

RAND, M.S.P. (2002, October) Mathematical proficiency for all students: toward a strategic research and development program in mathematics education (DRU-2773-OERI)(Arlington, VA, RAND Education & Science and Technology Policy Institute.).

Schweder, R.A. (2003). Why do men barbeque? recipes for cultural psychology. Cambridge, MA: Harvard University Press, 74-133.

Additional Resources for Classroom Use.

www.stanford.edu/~joboaler/

The website above includes a downloadable paper, entitled: 'Promoting Equity in Mathematics Classrooms – Important Teaching Practices and their impact on Student Learning' which is a longer version of this paper with more evidence and details on the approach described.

www.complexinstruction.org

More information on the complex instruction approach can be found at the website above.

Additional Resources for Classroom Use.

Effective Mathematics Teaching Approaches.

Website: http://www.stanford.edu/~joboaler/

Boaler, J. (2002) *Experiencing school mathematics: traditional and reform approaches to teaching and their impact on student learning.* (Mahwah, NJ, Lawrence Erlbaum Association).

Boaler, J. & Humphreys, C. (2005) *Connecting Mathematical ideas: middle school video cases to support teaching and learning* (Portsmouth, NH, Heinemann).

Boaler, J. (2004) Promoting equity in mathematics classrooms - Important teaching practices and their impact on student learning. *ICME* (Copenhagen, Denmark). (available for download at http://www.stanford.edu/~joboaler/)

Boaler, J. & Staples, M. (2005) Transforming students' lives through an equitable mathematics approach: the case of Railside school. (available for download at http://www.stanford.edu/~joboaler/)

Complex Instruction:

Website:

http://cgi.stanford.edu/group/pci/cgi-bin/site.cgi

Cohen, E. (1986) *Designing groupwork: strategies for heterogeneous classrooms* (New York, Teacher's College Press).

Cohen, E. & Lotan, R. (Eds.) (1997) *Working for equity in heterogeneous classrooms: Sociological Theory in Practice* (New York, Teachers College Press).

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Project Maths: Reviewing the project in the initial group of 24 schools – report on school visits

Contents

1. Introduction and context

Project Maths commenced in an initial group of 24 schools[1] in September 2008. Over 200 schools had applied to participate in the project and the initial group of 24 schools is reflective of the range of all post-primary schools. Changes to mathematics syllabuses and their assessment at both Junior Certificate and Leaving Certificate were phased in with these schools over a three-year period beginning in September 2008, with associated changes to the examinations commencing in 2010 (LC) and 2011 (JC).

Teachers of mathematics in the 24 schools have been supported through professional development workshops conducted by the Project Maths Development Team (PMDT) of Regional Development Officers (RDOs) and through complementary evening courses, with school-based support from the RDOs over the same period. The PMDT developed a range of teaching and learning support materials for teachers and students, which are published on their website (www.projectmaths.ie). The NCCA also developed student resources for the initial school group and these are now available to all students on the updated Project Maths pages of the NCCA website (www.ncca.ie/projectmaths).

A series of ten workshops, to which all maths teachers in the 24 schools were invited, focussed on the changed teaching and learning approaches advocated under Project Maths. Attendance at these workshops was consistently high – often in the 90%+ range. The workshops used specific topics from the different syllabus strands to illustrate a more investigative approach to teaching, learning and assessment and to emphasise the development of student problem-solving skills. The changed emphasis and approach to teaching and learning were also reflected in the examination papers for successive cohorts of students.

The complementary courses, including a series of summer courses by the National Centre for Excellence in Mathematics and Science Teaching and Leaning (NCE-MSTL), addressed areas of mathematics content where the need for support had been identified by teachers and/or the support team. Over 2,000 of almost 6,000 maths teachers nationally attended the complementary evening courses, which were held in local Education Centres. Each year, close to 100 of the approximately 230 teachers from the 24 initial schools, attended the summer courses held in 2009, 2010 and 2011.

[1] In preparation for its amalgamation in September 2011 with another school that was not one of the initial group of schools, Abbey Community College, Wicklow aligned itself with the national roll-out schedule from September 2010. Nonetheless, maths teachers from the former Abbey Community College were included in this review exercise.

Over the period of the project to date, the NCCA has had limited direct contact with the initial 24 schools. The main contact has been the RDO team, who visit the schools on a regular basis to provide individual and group support to maths teachers. Following the selection of the 24 schools in May 2008, regional meetings were held for maths teachers in these schools to outline the project and the planned programme of support. In December 2008 NCCA held a meeting of the 24 school principals to discuss progress and to identify particular issues that needed to be addressed. In May 2009 NCCA convened a meeting for principals/deputy principals and one or two maths teachers from each school, at which the feedback obtained through a teacher questionnaire was presented and discussed. This feedback resulted in adjustments to the initial syllabus drafts for the following year. In subsequent years, NCCA contact with the schools has mainly been in the form of regular Information Notes for teachers, and attendance by NCCA personnel at a selection of workshops in the different regions.

Now that the maths teachers in these schools have completed the set of ten workshops, and some changed syllabus strands have been through a full cycle at both Junior Certificate and Leaving Certificate, the opportunity was availed of by the NCCA to renew direct contact with the 24 schools in order to get teacher feedback on their experience of the process of change that Project Maths introduced and the impact it is having on their teaching practice.

This report presents the feedback that teachers gave in the series of school visits by NCCA personnel to the 24 schools in the period December 2011–January 2012.

2. Meetings with maths teachers

2.1 Arranging and conducting the meetings

The arrangements for the school visits were agreed with each school and the relevant RDO. The December Information Note (see appendix, page 26) to the schools outlined the purpose of the meetings with teachers and the main focus points:

- the impact of Project Maths on their practices as a maths teacher
- the impact of Project Maths on the school's maths department
- the impact of Project Maths on their students' experiences of maths.

By agreement with the schools, these meetings were limited to two class periods. While it was not possible to meet with all of the maths teachers in each of the schools, as many teachers as possible were included in the meetings that took place. In some schools, which had a small number of maths teachers, all of them attended the meeting. In a few schools, two meeting sessions were arranged so that disruption to class schedules was minimised. While principals were invited to attend the meetings, none did so. In a small number of schools the deputy principal, who was also a maths teacher, attended the meeting. In all, over 150 maths teachers attended the school-based meetings in the 24 schools.

Each of the review meetings was conducted by one of the NCCA's Project Maths personnel, who was accompanied by the RDO who normally works with that school. Teachers were assured of the confidentiality of the process; no one teacher or school would be identified or associated with particular comments. A series of question prompts was used to which the teachers were asked to respond and their responses were recorded on flip-charts. Teachers were free to amend or elaborate on the recorded responses, or to re-visit earlier responses in light of later discussion. They were informed that their responses would be collated and analysed, with feedback to be given to the schools at a general meeting later in the year. They were also informed that a report on the feedback from the teachers would be presented to the Council of the NCCA and would inform any further refinement of the syllabuses being finalised for national roll-out in September 2012, when all five strands would be in place for all schools.

2.2 Focus questions

The main areas of focus during the school visits were as follows.

1. **The impact of Project Maths on teaching, learning and assessment practice in mathematics classes**
   Teachers were asked to consider their classroom practices prior to the syllabus changes and at present, and also to consider what elements of practice they saw, and now see, as most valuable. They were asked to consider what forms of assessment they previously used, and now use, to assess student learning and progress in mathematics.

2. **The tools and resources that teachers find most beneficial**
   Having identified these resources, teachers were asked to elaborate on why they valued them.

3. **The impact of Project Maths on the school's maths department**
   As with individual teacher practices, they were asked to consider the functioning of the school's mathematics department (i) prior to the syllabus changes and (ii) at present and also to consider what aspects of the maths department and its role they saw, and now see, as most valuable to them as teachers.

4. **The impact of Project Maths on the student experience of maths**
   Teachers were asked, from their perspective, to identify the most significant change for their mathematics students as a result of Project Maths.

In addition, teachers were invited to give feedback on the syllabuses, focusing on strands 3, 4 and 5 (dealing with number, algebra and functions respectively). To facilitate written feedback on this item, a template was included as part of the Information Note sent to schools in advance of the visit (see Appendix, page 26).

3. Collating the responses, identifying themes

When all of the visits were completed, the sets of teacher responses were reviewed to identify emergent themes across the various focus points. The process of analysing and collating the feedback is set out below, together with the main themes to emerge from this feedback.

3.1 Thematic Analysis

The analysis of the feedback from teachers in the 24 schools was undertaken independently of the NCCA team involved in conducting the meetings. All responses gathered from all sessions in the 24 schools were reviewed through a process of data coding. First order coding was a descriptive level of coding and involved organising and categorising the views expressed by the teachers. Since, in many instances, teachers' responses related to more than one of the focus points, second-order coding was used to combine the descriptive first-order codes into meaningful super-ordinate codes. Finally, overarching themes emerging from the data were identified.

3.2 Overarching themes

Theme 1 – New roles

Across the board, there was recognition from teachers that participation in Project Maths calls for a change in the roles of the teacher and the student.

> *We taught in the same way we were taught at school, and now it's different.*
>
> *It was book led – all about ticking off chapters; rote learning, with the result tested in the exam.*
>
> *You just told them that's the rule and they have to learn it.*
>
> *Before, teaching was very much exam focused.*

Teachers recognised that the student now needs to be a more active learner, becoming involved in activity and discovery learning through new classroom practices such as group work, questioning and discussion. However, teachers reported struggling with this new role, which requires using a new skill set and a new set of classroom practices to enable learning for their students.

> *I am uncomfortable about this new role, there is an unknown.*
>
> *You are now a facilitator of learning as opposed to a giver of knowledge - I struggle with that.*

Teachers voiced fears that their own lack of confidence with the new approaches under Project Maths is picked up on by students.

I had more confidence – I knew the full story; the exam reflected teaching and there was predictability for both student and teacher.

Now, students have less confidence in that they don't know what is expected of them in the exam.

Some teachers described how students are gaining a different type of understanding in this new learning environment; this is especially true for the more junior classes. However, the feedback from the meetings indicates that, as exams approach, students and teachers value the old ways and there is a pressure to ignore their new role and to revert to previous exam preparation techniques rather than focusing on learning.

In the long run, it is a positive process and kids can see the relevance of maths to their lives. It was hard at the start, but once everything settles down I wouldn't go back.

It used to be very easy to prepare for the exams using repetition of practice and exam-style questions. We miss the comfort of past papers.

In sixth year maths you are pressurized and fall back into the old style of teaching under pressure; the reality is that there is a Leaving Cert that determines children's futures.

I am trying more to teach the maths, but at certain times of the year it's just the exams that I concentrate on.

Theme 2 – Supporting the changed approach; using resources

Teachers emphasised that they need support and resources to assist them in developing the skills and knowledge required for their new and different role. They are learning and developing and there is evidence of their being at different stages on this learning continuum; some have undergone major changes, some are still at the start of this development. As they learn, confidence grows.

> *I have so much learning to do (teacher doing HL after a gap). I'm still not in the comfort zone with strands 3, 4 and 5.*
>
> *My methods have totally changed since Project Maths came in.*
>
> *We're going a little slower. I find we're constantly changing because it might work with one group and not another. I find we need to use different methods for different groups; this is the way it should be, not making one method fit all.*
>
> *I have had to look at maths myself and it's made me improve my teaching practice; it's made me think outside the box.*

It is clear that teachers value and need support during this period of learning and development. However, their comments suggest that over-dependence on various forms of support may become an issue. As more resources become available there is a danger that teachers may lose sight of how intrinsic they are to the change.

> *There are so many (resources) but they're a bit all over the place. The problem is they are coming in dribs and drabs. (We need to) have resources together digitally in strands.*
>
> *All the resources are great but (students) still expect us to have all of the stuff in paper in front of them. The RDO is the saving of us.*
>
> *We struggle to be able to make up questions ourselves; we need the resources of past papers.*
>
> *I can't wait for the definitive book to be produced. I don't care how big it is, I want it.*

Teachers cited collaboration as being valuable to them. They reported that within schools and maths departments there is much more collaboration and support between colleagues than before. It was noted, however, that much of this 'team' work takes place in informal collaborations and that collaborations need to be planned for and supported by the school administration. The issue of time for planning was a recurrent theme and will be dealt with in more detail later in this report.

There is more interaction between teachers and the focus is on maths approaches.

Before, everyone was king of their own castle; now everyone depends on each other.

Sharing means we are learning how to change and it gives us different insights.

Project Maths emphasises student understanding of concepts. There is evidence that teachers need support in making connections within mathematics. They recognise this ability to connect as an advanced skill that only develops after a period of immersion in teaching with the revised syllabuses. It is apparent that being confident in maths supports this identification of connections and their efficient use.

I didn't tend to link topics, but I see that there are more connections between all the strands now.

(Identifying) linkages between strands is challenging, as we didn't really see the linkage at the beginning. We learnt it as we went through and this makes teaching it more difficult.

I am starting to make connections across strands and able to say 'do you remember that we did this at (a certain time)?'

More time is spent on linking different topics together now.

I've learned loads. I never thought of linking slope the way I do now.

I put greater emphasis on several approaches to solving a problem rather than on a singular approach.

Teachers reported that engaging with Project Maths had a positive impact on their teaching approaches in other subject areas. There was a concern from teachers whose first subject was not maths that they were not familiar enough with the maths to teach in the new way.

Business is my subject (but) I don't have the subtleties needed for maths. I have to think about it and engage lots with the material to get that confidence. I am a better business teacher now because of my experience with Project Maths.

There is evidence that the syllabus is now seen as a useful resource and some teachers are becoming less dependent on text books. However, they report that their students value the textbook and they feel under pressure to use the text books in class.

We read the syllabus (now). We have more understanding of the syllabus.

You have a copy of the syllabus in your back pocket now. Before, the book was the syllabus and you followed that.

Trouble is, they (students) respect the book, not worksheets.

They are used to having a book and perceive that it is not a real lesson without it. Having a textbook is a serious advantage – good kids are used to it from other subjects and weaker kids get structure from it.

Some comments point to the need to provide support in interpreting a syllabus now that it is written in terms of learning outcomes.

We had a syllabus that made sense, we didn't have these statements. I don't know the syllabus well and I'm not able to direct students to exams.

The syllabus is too vague; we may be over-teaching some topics and not emphasizing enough in other areas. The (text)book was easy – this isn't.

When asked about the resources they have used and valued teachers cite practical activities, resources and equipment that they have developed themselves or have been provided with through Project Maths. They also commented that having all the necessary materials and resources was challenging.

We need to use hands-on, open-ended activities.

The T& L plan is good – (doing) probability by making a game; statistics through doing their own surveys and presentations.

We did a school census. I used a newspaper article about statistics as a discussion.

(I'd) like to be in a position to have a set of maths equipment in the class; it's very useful to have and getting them is difficult. If we had these on a shelf it would be very helpful. Maths classes need resources like other practical subjects: dice, cards, spinners, geostrips, probability kits; this must be taken seriously.

In almost every school, teachers reported an increased use of IT and this too is valued.

IT is valued because, for example, geogebra allows visualization and very quickly aids understanding. It is good for constructions, graphs, and seeing what differentiation does.

(Students) are used to technology. It brings the maths home to them and they want to do it themselves.

The internet is great, but it is time consuming to get 'tailored material' – one size doesn't fit all and you have to rework it for different classes. It takes a while to gather a portfolio of resources.

Some teachers report that they find group work useful.

In group work you encourage students to learn themselves, making mistakes.

Highly motivated students benefit more from the approach of group work and open-ended activities.

Other teachers report on the challenges that they experience with group work.

In some classes discipline is a problem and this means you can't do group work and things like that.

There is not enough time for it (group work).

(The) effectiveness of group work depends on the size of the group, the ability/nature of group and the motivation of the group.

There's a higher percentage of demotivated kids in OL so I think group work is not suitable with these kids.

Teachers report that they are using questioning and discussion more frequently but, like group work, learning how to use this new learning methodology effectively is a challenge.

It's very difficult to get them to engage; they're not comfortable about sharing solution strategies; some are afraid to give their own opinions.

Sometimes I am met with silence and I end up answering it myself.

No time for this. I can't spend hours on a question and if you can't keep them on task it's disruptive and you'd lose a lot of them.

Questioning gives them an understanding of the vocabulary…..It's brilliant, you know from that that your message has got across.

Teachers had mixed opinions on the value of using open-ended tasks and activities to support learning. Some reported that using open-ended tasks is time consuming and there is evidence that, once again, teachers and students need time to learn how to use these effectively. Some see open-ended tasks as an add-on activity, not a core teaching methodology.

Open ended tasks and activities take time to prepare. It frustrates the hell out of them; depends on their ability ….. weak (students) can't break it into pieces.

(Open-ended tasks) perceived as a doss class by some kids who complain that teachers are not explaining it.

Better able kids can come up with strategies.

With problem solving there are lots of ways of skinning a cat – some of them only want one way of doing something.

Many teachers cite pressure from, and the dominating presence of, the terminal examination (Leaving Certificate) as inhibiting them from using the more student-centered methodologies such as open-ended tasks, discussion and group work.

Ultimately, (the) written exam is the thing, they have to do one – very bright kids answer questions very well in class but fall down in exams.

In class you can use questions to open up a problem but in the exam they need to be able to do this themselves.

I am trying more to teach the maths but at certain times of the year it is just the exam; we want to approach things in Project Maths style but we fail under pressure of exam structures.

Teachers report school issues such as classroom layout, maths-based classrooms and timetabling as being important to them in supporting the changes.

Theme 3 – Issues of assessment

It is clear from comments by teachers that there is still a heavy reliance on 'tests' as a way of assessing learning. There is evidence that teachers need help and support in developing new and trusted ways of assessing, adopting an approach which is reflective and focused on learning, assessing the extent to which learning has been achieved, and refining their teaching to reflect this.

When asked about the methods of assessing student learning that they value most, teachers reported that tests, exam questions and homework were the primary ways they were assessing learning.

> *Tests – assess each individual; they (students) have to be used to a written test.*
>
> *Exam questions and end of topic exams (are) most beneficial, and give practice for the final exams.*
>
> *(Tests) allow you to test a variety of concepts. You get to see where they are going wrong.*
>
> *The only way to check homework is by giving a test.*
>
> *I put more maths questions on tests to get them used to the unexpected.*
>
> *You know it's their own work and it is the way that they will be tested. All the resources are great but the homework and questions are the bread and butter.*
>
> *(Students) don't record enough when they're doing investigations.*
>
> *(Checking homework) gives a fair idea which ones (questions) were the problem, then open forum as to what was the problem and discuss to solve the problem.*
>
> *Swapping homework, marking each other's (work).*

However, even with tests, some teachers find it difficult to adopt a changed approach in marking student work.

> *I haven't a clue how to mark tests; I still use old scheme: +3, etc. Even with mocks I can't decide where marks should be awarded, or what constitutes 'appropriate information'.*
>
> *We need an inservice on marking students' work. I would like to see how a 'fair scheme' for marking works.*
>
> *If you are trying to be innovative, you can't give predictable homework.*

There is some evidence that alternative assessment methods such as project work, assignments, open discussion in class, questions, and examination of students' work are being used, but some teachers expressed the need for support on how to use these methods effectively with all their students.

> *(I use) discussion in open forum, getting them to describe the steps to answering the question.*
>
> *They can look at a question and just think "I can't do it". Some students could sit there doing nothing...They need prompts: Explain how you got there? Elaborate on…, Why did you start with that…?*
>
> *I got my students to make their own questions and give them to each other but I didn't know if that was worthwhile.*
>
> *Individual white boards are useful, you can quiz all students at the same time – it's efficient; they get feedback straight away.*
>
> *We use a folder system; students keep all their work and get graded on it.*
>
> *I listen in on their discussion, absorbing what they are answering.*
>
> *Sites such as ixl.com for online assessment (interactive assessment) give students feedback. A lot of students don't see this as homework. They know the teachers can log in and they are competitive.*

Teachers' comments about their experiences with different approaches to assessment reveal that in some cases the thinking about the purpose of assessment is beginning to change.

> *Formative assessment – it's different, gives you different insights and you can engage with them as you move forward.*
>
> *Extra work/ revision stuff and their attempts inform you.*
>
> *Use students' work to illustrate different solutions – that is useful*.
>
> *Students presenting work and explaining (their) strategy is great because it gives them confidence, they see different ways. If I just show them it gives preference to my thinking. (This way) shows them I am not an expert. I enjoy listening to their ways.*

Teachers have concerns about how well the new teaching methodologies and assessment are supporting the diverse learning needs of their students.

> *There is too much English on the paper. I worry about foreign students and those with weak language ability to interpret questions in an exam. The lack of help and resources for these students is becoming a bigger issue.*
>
> *Assessing the ability to display understanding of maths is an issue.*
>
> *Students with SEN – if they can't get info out of the question then they are at a disadvantage.*

Teachers have concerns about the exams. Currently the exam is impacting on the new teaching and many teachers feel under pressure to revert to old style 'drill and practice' teaching and abandon student-centered, inquiry-based methodologies. Teachers voiced concerns about the length, structure and format of the examination papers and wondered whether they adequately assessed the learning on Project Maths.

> *Exam papers are too long. There is a lack of structure in the (new) exam papers.*
>
> *(Before this) you could always say if you do this and this you'll get attempt marks, but now you can't.*
>
> *I feel sorry for them when we give them Project Maths questions; they don't get a sense of reward or achievement, they are fine working through the resources but then they can't equate what they are doing with exam questions*
>
> *Project Maths needs project-based assessment…there is a need to change the way it is examined.*
>
> *The exam is unpredictable; there's no (a), (b) and (c) parts anymore.*
>
> *Some of the exam questions are unfair, you might not be able to start a question whereas before you knew there were questions that everyone could do.*
>
> *I feel I'm engaging more kids in the class, they enjoy maths more, but they're still not doing well in tests. (LC) students can problem solve but can't do papers.*

Teachers cite fear and anxiety around exams as a feature of the student experience of Project Maths.

> *5th and 6th years are fearful, it's a negative experience.*
>
> *5th years who haven't been through Project Maths can't problem solve – they find it daunting. HL 5th years who engaged with Project Maths in JC are anxious about the LC because they got negative vibes from the last 6th years and are not as confident in the teacher's ability to deliver.*

Theme 4 – Time

This theme is constant throughout the Project Maths experience. Teachers made points about time to meet and plan. They mentioned time in relation to covering the course and using problem solving methodologies. They mentioned time being needed to use different kinds of assessment.

> *Involvement, discussion and activity learning are more time consuming than 'chalk and talk'. We don't have time to explore.*
>
> *You can't afford the time for the hands-on stuff even though the kids enjoy it and get it.*
>
> *In senior cycle I don't have as much time to show them things, whereas in JC you have more time to go through the explanations, investigations and discovery.*
>
> *We need time to teach for and to develop understanding.*

Teachers reported that time pressures inhibited student-centred approaches to learning.

> *Due to the length of the course I'm teaching new material in the old way – drill and practice – it's a time issue.*
>
> *In 6th year maths you are pressurized and fall back into the old style of teaching.*
>
> *Time pressure to get the course done reduces time for questions.*

It was also recognised that it takes time to become familiar and confident with the new syllabus and teaching methods, and teachers report that they find it difficult to know how much time to spend on each topic.

> *I need to be comfortable knowing how much time is available for a topic before I am willing to do the playful stuff.*
>
> *How much time to spend on certain topics is still an issue.*

There is a perception that the syllabus is long and time consuming and that it takes teachers longer to teach the same thing. There is evidence from some teachers' comments, however, that as teachers develop their familiarity with the connections between strands they can make more efficient and effective use of their time.

> *Strands 3 and 4 (take) too long to teach.*
>
> *The HL course is too long; every day you do something new and there's no time to go over stuff.*

The new course is longer, with more material, more depth; we are being asked to teach more to a greater level of understanding.

A small section on the syllabus may take a long time to cover.

Now I spend more time linking different topics together.

There is greater emphasis on several approaches to solving a problem rather than just one way.

Cross-linking, not going chapter by chapter, and looking for different representations takes more time.

There were reports in a small number of schools that teachers are teaching exam classes outside core school hours to cover the syllabus. It was acknowledged that this has lessened as the phasing of Project Maths progressed.

I need 240 hours to come at it (the syllabus) from different perspectives.

I came in after school 2 days a week in the first year. I didn't want to encourage panic and I knew this way we would cover the course.

I'm terrified of not getting the course covered; you can't get sick or you won't get it covered – that wasn't the case before.

Teachers reported that timetabling needs to support the maths learning needs. They also emphasised the pressure that exams exert on how time is used for learning.

35 minute periods are a constraint to new practices. Field work and practical work take more time.

5 class periods are proving inadequate for 5th and 6th years, given that 3 out of 5 classes are the final class of the day.

If maths is to become hands-on it should be treated as a science (a practical subject) with 24 students per class; the ability range in classes is huge.

I am concerned about the time element, it takes more time to get a topic covered and will that be reflected in the paper? I have to finish (the course) by Feb/March because of mock exams.

No revision time any more – 6th years are going into mocks to do material they haven't done since last year.

As noted already, teachers felt strongly about the need for time to be made available for planning and collaboration.

> *I spend more time preparing and thinking about methodology.*
>
> *Maths teachers are involved in other (subject) departments, so can't always meet formally.*
>
> *We have a class a week to meet…but I know we are only getting all this meeting time because we are a pilot school.*
>
> *We're not allowed to use the Croke Park hours, we've agreed on 1hour a term for subject planning, but this is inadequate.*
>
> *We underestimate what time different methodologies require and the amount of preparatory time required, e.g. group work is more than just putting people into groups and throwing stuff at them.*

Theme 5 – Issues of change

Teachers have views about the manner in which Project Maths was introduced simultaneously in first year and fifth year. Many of them felt that the exams were unfair to their students. For some there is still a sense that they don't have ownership of the change. The manner in which Project Maths issues are dealt with in the media impacts on teachers' perceptions.

> *There has been a lot of change in a relatively short time…Maybe it should have started only with first years.*
>
> *You're always having to justify (to students and parents) anything you do that's different as people don't like change; they will blame Project Maths for not doing well in maths.*
>
> *When the first cohort of students went through the exams there should have been more consultation.*
>
> *(We feel) the feedback we gave in June about the exam was ignored.*
>
> *We feel that we're being used as a test – every other school benefits – and that we're guinea pigs.*
>
> *Some of the kids have lost faith. The experience over the first couple of years was stressful for students and still is; they are afraid.*
>
> *The exam last year upset people, especially the students; they lost confidence. It has a lot to do with papers, poor publicity in the media.*

Theme 6 – Syllabus content issues

Teachers made a number of general points about strands 3, 4 and 5 along with other aspects of course content and its assessment. These are summarized in Table 1 below. Their comments suggest that teachers need support in understanding the aims of the syllabus, how to interpret learning outcomes and the purpose of assessment.

Table 1: Specific comments made by teachers about the syllabus

Strand 3	*Generalising a quadratic relationship from a pattern is very difficult, they cannot get the formula. This turned them off patterns.* *I'm happy with identifying a pattern as quadratic and continuing the pattern…but it's a step too far to generalize this.* *Manipulating equations is nightmarish stuff when they have to do procedures – a lot of procedure has gone – they need a balance.*
Strand 4	*Strand 4 is most enjoyable, and provides lots of linking. The different syllabus levels are appropriate.* *Connections - 5th years are actually making connections between different strands and previous work.* *LC - algebra is wider. The 3 nested columns leaves a lot to cover.* *Strands 3 & 4: patterns and algebra are better connected now.* *Standard question algebra style at JC-HL, formulas, etc.* *There is a huge jump from JC-HL algebra standard of the sample paper to that required to study at LC-HL. Problem solving in algebra is a problem…kids are used to algebra as patterns.*
Strand 5	*Differentiation is too short; product, quotient and chain rules are gone, differentiation from first principles is gone, this was always a comfort to the student; they could do it.* *Calculus is a complete change; it's lovely because now it's more applicable, trig the same. Before it was all rules, rules, rules.* *I am disappointed by the amount of integration on LC-HL.*
Foundation	*FL syllabus is needed at JC.*

level	*It's not all about OL and HL and 3rd level; we must think of FL – they are aground completely, with no questions directed for them.* *There is a serious lack of understanding for FL – it needs to be more tightened/specific. It's vague - what depth for FL?*
Common Introductory Course (CIC) in First Year	*Not sure whether the CIC material is working.* *The CIC is so broad and, for good students, it's not challenging enough. I don't know whether they gain anything.*
Comments on exams	*Paper 2 is too long.* *It's difficult to judge the syllabus without a selection of exam papers to see how it is examined.* *The a, b, c structure for questions was better.* *There needs to be a hint to help students identify the differentiation question.* *Top students are unnerved – for some questions they require life experiences beyond their years.* *Title the question – the words make it difficult to know which section this is. There is a need to read and comprehend.* *There is a serious language issue – students ask 'Is this an English test or a Maths test?'*

Note: Teachers comments on specific learning outcomes and other queries on the syllabus were brought to the attention of the relevant course committee. Some additional clarifications are being made in the syllabuses to be issued in September.

4. Learning from the experiences of the 24 schools

The teacher feedback above indicates that the teaching of maths in these 24 schools is changing, albeit at a slow pace, as a result of Project Maths.

For many teachers there has been a change in their role, teaching practices and methods as they have moved away from teacher led and didactic approaches to more student-centered and active methodologies. Many teachers now see themselves as facilitators of learning rather than givers of knowledge. This change has not been easy and many teachers have described a loss of confidence when compared with their familiarity with the previous syllabus and exam. They have also described being very challenged by the increasing time demands of the new syllabus. Using active learning methods, characterised by a higher level of student involvement, classroom discussion and practical work, has proved very time consuming so far and many teachers have reported that covering the whole syllabus is challenging. Some teachers reported that they taught extra classes outside core maths hours to complete the syllabus.

Learning approaches such as group work, classroom discussion and questioning are being used by more teachers. All teachers report that these methods are more time consuming than ‘chalk and talk’ and ‘drill and practice’ methods, and many report that they are challenging to use as not all students are yet comfortable with them. Teachers have indicated a need for support to enable them and their students to develop the skills to use these methods efficiently and effectively. Not all teachers are convinced that these teaching practices offer additional learning benefits over the ‘chalk and talk’ and ‘drill and practice’ approaches that they have relied on in the past.

While embracing new approaches in their teaching, many teachers still focus on the examinations, particularly in sixth year, and want more exam-focused questions and sample papers to use in exam preparation. The practice of striving to finish the full syllabus in time for early mock exams adds additional time pressures. Some teachers have reported that they have reverted to ‘chalk and talk’ teaching methods under these time pressures in sixth year. From teachers’ comments, there would appear to be some danger that, with increased availability of exam-oriented resources, they may revert to old practices and not fully embrace their new role.

Teachers report that increasingly they are using the syllabus as a guide, whereas previously they had used the text book and past exam papers to guide their teaching. However, not all teachers report being comfortable with the language and level of detail provided by the syllabus. If the syllabus is to be a useful guide, then teachers need to be able to read and understand it and it has to be more than simply a list which is ticked off when a topic has been ‘covered’. Reading and understanding a syllabus and using it to design learning activities that ’fit’ the group of students in a class appears to be an important competency for teachers, and an area in which teachers expressed the need for support. This is also a challenge for the NCCA in syllabus design and development.

Collaboration among maths teachers in each of the schools has increased. While most of the schools did have maths department meetings prior to Project Maths, these tended to be more focused on timetabling, exams and sequencing issues. These meetings are now increasingly focused on collaboration around challenging aspects of teaching the syllabus using the new approaches. Teachers report having more meetings and, in particular, more informal meetings, which are often between two or three colleagues and focused on maths. This collegial support has been found to be very valuable by all.

Teachers believe that the student learning experience has changed. They report that students are now engaged in greater discussion, collaboration and activity within their maths classrooms although, as has been reported above, this often changes under the pressure of the exam year. Teachers also report that not all students are comfortable with this new type of learning and it appears that younger students and those not in exam years are most comfortable with the new methods, whilst exam year students are disconcerted by the absence of past exam papers and want teaching geared to answering exam questions. On the other hand, teachers do report that there has been an increase in understanding maths concepts among students. Students who performed best in the previous ‘chalk and talk’ and ‘drill and practice’ learning environment seem to be more challenged by the move to discovery learning and those who were less able in that environment are now performing better.

Another area where teachers need new skills, and support in developing these skills, is the area of assessment. It appears that the majority of teachers used, and many still continue to use, tests as their only assessment tool. The experience of some teachers who have attempted to explore other assessment methods is that it can be challenging. There is some evidence of the insight that teachers have gained into students thinking and learning through changed classroom practices, such as listening as students explain how they solved a problem, or group discussions on different ways of answering a question. These insights can change teachers’ perceptions of student understanding and learning, and also change their perceptions of the efficacy of the newer teaching and learning practices.

It appears that the changes introduced by Project Maths propel both teachers and students on a new learning continuum. Not all teachers are at the same point or proceeding at the same pace along this continuum. From the meetings held in the schools, it would appear that experienced and fully qualified maths teachers who are teaching maths full-time have found the experience of introducing Project Maths less challenging than their colleagues who teach a range of subjects. These teachers have often been able to provide peer assistance to their colleagues and have been an internal source of support within maths departments, which has facilitated collaboration between teachers.

A range of concerns have been voiced about the changed Leaving Certificate exam. Some teachers are worried that it doesn't reflect the type of learning that Project Maths promotes and that there should be a move towards an additional assessment component, such as project work. Others have focused more on issues of exam performance and perceive that students who would previously have got an A1 in maths are now not achieving this high grade and they consider that this is a problem with the exam rather than a reflection of student learning and understanding. Other exam-related concerns include issues of reward for effort through the year, in that a lot of time may be spent teaching a concept that then doesn't feature specifically on the exam.

The experience of these 24 schools has demonstrated that teaching using the approaches in Project Maths is only the starting point in changing the culture of maths teaching and learning within a school. The new syllabus is only one element in this transition. Ongoing supports for teachers, a collaborative maths department, organised and accessible resources, a timetable that supports a discursive learning environment, a classroom infrastructure that supports this type of learning, an assessment methodology that reflects the syllabus learning outcomes, and methodologies and external leadership and support from the educational establishment all have an important role to play. Indeed the experience of the 24 schools to date demonstrates the synergies between these.

Across Europe, it is recognised that professional development opportunities can play a key role in equipping all teachers with the necessary skills to adapt their teaching to changes and developments in mathematics education (*Mathematics Education in Europe: Common Challenges and National Policies;* a report of the Eurydice network, 2011). The report acknowledges the specific reforms in Ireland which target mathematics teachers – one of only two countries where such reforms have been introduced.

5. Next steps

The feedback from the schools will be discussed with the Project Maths Development Team, with a view to planning future support for these schools as they complete the full cycle of changed examinations.

The maths committees were kept informed of emerging issues in relation to the syllabuses and their assessment. From September 2012, all schools nationally will engage with the same syllabuses. This will see the final section of the 'retained' syllabus being replaced by strand 5 (functions). In light of the fact that some topics in this section of the LC Maths syllabus will no longer be included, the committee decided not to make any additional adjustment to the length of the syllabus at this stage. As students come through to senior cycle having experienced the revised syllabus and new approaches in the junior cycle, future consideration of syllabus length will be informed by ongoing feedback from the initial schools.

A seminar involving the principals/deputy principals and two maths teachers from each of 24 schools will take place in April, at which the overall findings of the feedback from the school visits will be presented and discussed.

Appendix

Project Maths
Information Note #14
December 2011

Please ensure that all maths teachers in your school receive this bulletin.

This Information Note provides details about the upcoming Project Maths review meeting in schools.

Project Maths school-based review meeting

One of the NCCA Project Maths team together with your RDO will be visiting your school in the next two weeks. The purpose of this visit is to get feedback from you on your experiences of the Project Maths initiative. On the day of the visit we will be meeting with maths teachers for two class periods; the meeting will be informal and we intend to cover the same ground with all schools. The main focus of the session will be:

- The impact of Project Maths on your practices as a maths teacher
- The impact of Project Maths on your school maths department
- The impact of Project Maths on your students' experience of maths.

Feedback on strands 3,4 and 5

We will also be looking for your feedback as initial participants in the project on strands 3, 4 and 5 so that the committees can consider this before they finalise the syllabus for national roll-out in September 2012. In order to direct your thoughts, and in the event that we are under pressure with time on the day, we would ask that you note any comments on strands 3.4 and 5 in advance of the meeting on the sheet provided. We will take this away with us for discussion with the committees.

NCCA contact details

Contact details for NCCA staff working on Project Maths are set out below.

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Strands 3,4 and 5

The four main headings under which we would like to get your feedback are:

- The syllabus topics and learning outcomes

- The appropriateness of the different syllabus levels

- The progression from JC to LC

- The connections between the strands

Reviewing Statistics

Throughout your study of strand 1 you will have considered all aspects of a statistical approach:

- asking a question that results in data that varies
- displaying this data in a way that allows you to see patterns in the variation
- analysing the patterns in the data
- drawing conclusions from that data.

You may even have had an opportunity to get a glimpse of what it is like to become a statistical detective; attempting to account for unexpected variability you observe in a particular set of data.

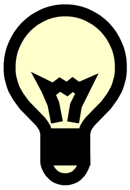
As you review for the final examination in June, it is important that you can connect each element of your study and consider the BIG IDEA of the strand so that you will be able to use the elements appropriately to help you solve problems that you may not have seen before.

The following is an extract from Strand 1 of the syllabus; it summarises what you should be able to do when you finish studying this strand.

> It is envisaged that throughout the statistics course students will be involved in identifying problems that can be explored by the use of appropriate data, designing investigations, collecting data, exploring and using patterns and relationships in data, solving problems, and communicating findings. This strand also involves interpreting statistical information, evaluating data-based arguments, and dealing with uncertainty and variation.

You may decide to form a study group with your friends or you may prefer to work alone; either way as you work through this review document you will consider issues such as framing a question in order to obtain meaningful ***reliable*** data, selecting a sample in order to avoid ***bias, displaying*** your data in a way that will allow you to see patterns in the variation and ***drawing conclusions*** from your data.

Asking the Question

Think 

Do you use a computer?

How did you answer the question?
What were you thinking when you answered it?

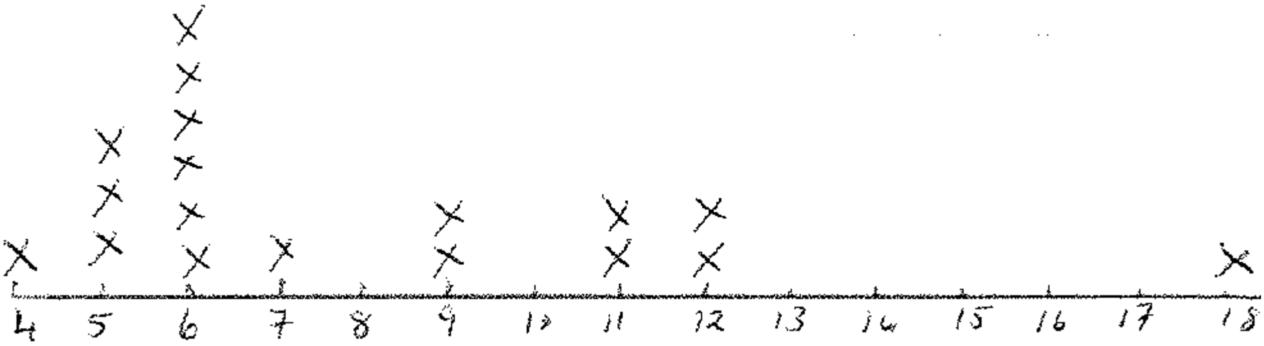
A university sports outlet was considering shutting down their campus shop and becoming an on-line store in an effort to reduce costs. A group of students was surveyed and asked that same question:

Do you use a computer?

Sophie answered **Yes** because she thought the question meant had she ever used a computer.
Joe answered **No** because he thought the question was asking whether he used one regularly.
Andrew answered **No** because he played games on the computer and didn't think this counted as "using" one.
Do you think the results of this survey are ***reliable?***

How could you rephrase the question so that it is less ambiguous and more likely to provide useful answers?

A group of students interested in finding the typical family size for their class obtained the data displayed in this line plot



What question do you think they asked in order to elicit this data?

What issues would they have needed to consider when framing the question?

Displaying the data and drawing conclusions from it

Use fractions or percentages to describe the data.

Can you see any clumps or areas where a large proportion of the data falls?

Are there any unusual family sizes? [18 is an unusual value in this set.]

What do you think is the typical family size of this group ? Why?

If you were asked to predict the family size of someone from this group what value would you give? Why?

How certain would you be? Can you lower this to a smaller range? How **confident** would you be now?

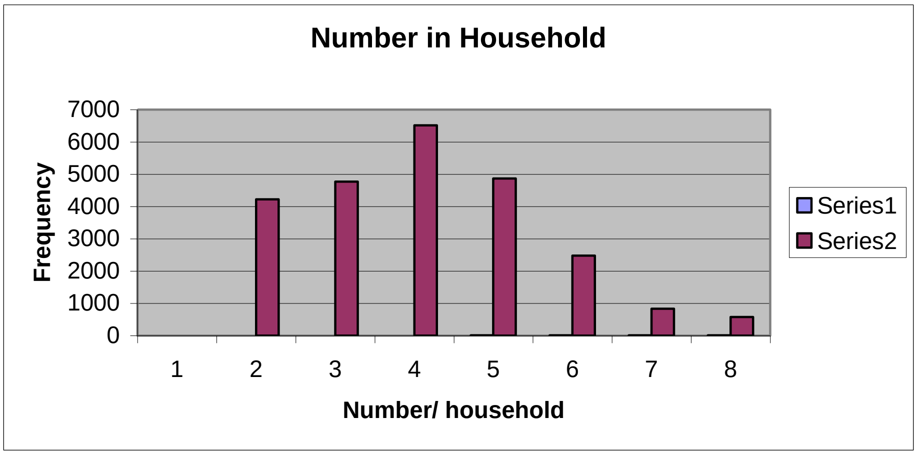
Calculate the mean family size for this group and identify the median family size. Which is a more reasonable estimate of typicality?

You could do a similar survey of your class, display the data in a line plot and compare the two data sets.

Or you could visit

http://beyond2020.cso.ie/Census/TableViewer/tableView.aspx?ReportId=109241 and retrieve some data from the area in which you live, use Excel to display the data and compare it to the sample above.

Household size	Frequency
2	4218
3	4773
4	6512
5	4870
6	2478
7	833
8	572

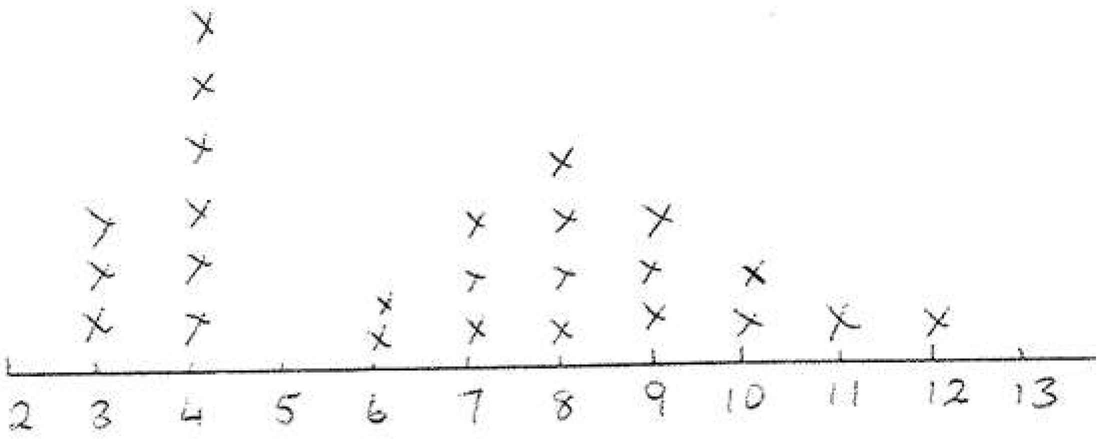


This bar chart was drawn with data from Carlow.

Compare this data with that from the sample set above. What is the range of this data set? What is the range from the sample data set?

Is there any evidence to suggest that the sample was from Carlow? Explain.

The following data set was gathered from a TY class who were interested in finding out what was the typical amount of money spent by their parents on the lotto each week.



Use fractions or percentages to describe the data.

Can you see any clumps or areas where a large proportion of the data falls?

Are there any unusual amounts?

What do you think is the typical amount spent on the Lotto each week by this group ? Why?

If you were asked to predict the amount spent on the Lotto each week by someone from this group what value would you give? Why?

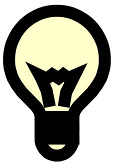
How certain would you be? Can you lower this to a smaller range? How **confident** would you be now?

Return to the value you think is the typical amount spent on the Lotto each week by this group.

Calculate the mean amount spent on the Lotto by this group and identify the median amount spent on the Lotto each week. Which is a more reasonable estimate of typicality?

The data below was obtained by students trying to find out how good they are at judging a minute.

18	25	26	30	40	41
45	45	47	52	52	56
67	68	74	79	109	

Think 

Will a line plot be a meaningful display for this data? Try it out and see?

Now display the data in a **stem and leaf plot.** Which display is more meaningful? Why?

What is the range of the data? Where is all the data concentrated? Calculate the percentage of data in this region. How good were people at guessing a minute?

Predict how long do you think people in this group think a minute is? How **confident** are you of this answer?

Now consider **Q4** on the **2010 FL** mock paper, which requires you to use those skills you've been practicing.

Question 4 **(25 marks)**

Sam wanted to see how well the people in his class could judge how long one minute is.
He asked each student to say *'Start'* and then to say *'Stop'* when they thought a minute had gone by.
Sam used a stop-watch to time each of them.
He recorded the results in the table below.

Times in seconds					
63	56	86	52	75	65
57	59	64	55	89	54
39	67	82	70	68	57
66	72	33	42	52	79
60	59				

(a) Display the data in a stem and leaf plot.

(b) How many people did Sam time?

(c) What is the median time?

Extension to OL

Compare Sam's set of data with the previous set.

- Use two **stem and leaf plots** or **back-to-back stem and leaf plots.**
- Use % and fractions.
- Mention areas where the data is clumped.
- State the range of both sets.

Which group do you think was better at estimating a minute?

Justify your answer.

Q9A a) and **b)** on the **2010 OL** mock paper

Oxygen levels in a polluted river were measured at randomly selected locations before and after a clean-up. These results are given in the table.

Before (mg/l)				After (mg/l)			
20	25	20	9	26	10	10	9
23	23	10	11	11	15	11	11
2	10	11	5	3	8	11	4
11				13			

(a) Construct a back-to-back stem-and-leaf plot of the above data.

(b) State **one difference** and **one similarity** between the distributions of the measurements before and after cleanup.

Difference:

Similarity:

Q7 a) and **b)** on the **2010 HL** mock paper

Some research was carried out into the participation of girls and boys in sport. The researchers selected a simple random sample of fifty male and fifty female teenagers enrolled in GAA clubs in the greater Cork area. They asked the teenagers the question: *How many sports do you play?*

The data collected were as follows:

Boys	Girls
0, 4, 5, 1, 4, 1, 3, 3, 3, 1,	3, 3, 3, 1, 1, 3, 3, 1, 3, 3,
1, 2, 2, 2, 5, 3, 3, 4, 1, 2,	2, 2, 4, 4, 4, 5, 5, 2, 2, 3,
2, 2, 2, 3, 3, 3, 4, 5, 1, 1,	3, 3, 4, 1, 6, 2, 3, 3, 3, 4,
1, 1, 1, 2, 2, 2, 2, 2, 3, 3,	4, 5, 3, 4, 3, 3, 3, 4, 4, 3,
3, 3, 3, 3, 3, 3, 3, 3, 3, 3	1, 1, 3, 2, 1, 3, 1, 3, 1, 3

(a) Display the data in a way that gives a picture of each distribution.

(b) State **one difference** and **one similarity** between the distributions of the two samples.

Difference:

Similarity:

(c) Do you think that there is evidence that there are differences between the two populations? Explain your answer.

Note: you are not required to conduct a formal hypothesis test.

Answer: ______________________________

Justification:

Q 7 OL 2010 mock paper

(a) A teacher asked the students in her class to estimate the height of the church opposite the school in metres.

The stem-and-leaf diagram shows all the results:

3	5 9
4	0 2 6 8 8
5	3 3 5 7 7 9
6	0 5 5 5
7	4 8
8	2 7

Key: 3 | 5 represents 35 m

(i) How many students were in the class?

(ii) Describe the **shape** of the distribution of the data.

(iii) What was the median estimate?

(iv) Explain the answer to part **(iii)** to someone who does not know what the word "median" means.

Stop and think

Under what conditions would a **line plot** be a meaningful representation?

Under what conditions would a **stem and leaf plot** be a more meaningful representation?

Try using statistics to solve this problem.

PROBLEM: Climbing helmets are made in a variety of styles and sizes.

The manager of You Climb Safely must decide what styles of helmet to keep in stock and how many helmets of each size to order. A standard fit helmet is offered in 10 sizes. When you order helmets you must order 1000. How many of each helmet size should the manager order?

In order to get an idea of how head sizes are **distributed** the manager decided to measure the head circumferences of a group of people.

Think: what is the **population** of interest? Can he measure the circumferences of the heads of the whole population? How will he choose a **sample**?

The manager chose a **Simple Random Sample** of climbers from clubs around the country and recorded their head circumference and gender ion the table overleaf.

Is this a suitable sample? Why or Why not?

Gender	Head Circumference (mm)
F	522
M	580
M	552
F	531
M	563
F	546
F	545
M	545
M	545
M	568
F	560
M	613
F	555
F	573
M	585
F	584
M	600
M	595
M	593
F	590
M	594
F	564
F	536
M	586
F	540
M	585
M	550
M	565
F	600
F	590
F	551
M	590
M	580
F	577

Is a line plot a good representation of this data?

Display the data in a stem and leaf plot.

Describe the data…..Are there any clumps or areas where the data is concentrated? Are some head sizes more common than others?

Use your representation to answer the original question: ***how many helmets of each size should the manager order?***

Begin by counting the number of leaves on each stem.

Look at the first stem…52 ..How many leaves are there on stem 52? What fraction of the total is this? What % of the total number of head circumference measurements does stem 52 represent?

How many helmets size 520cm- 530cm should the manager order?

Continue working like this until you have decided how many helmets of each size the manager should order.

Return to your representation…Do you think there are **gender effects?** Try representing the male and female data in ***back to back stem plots*** Compare the two sets of data; is there any evidence to suggest that there are differences in the sizes of heads of men and women?

If there are gender effects will this affect the number of helmets the manager should order? Or are helmets unisex?

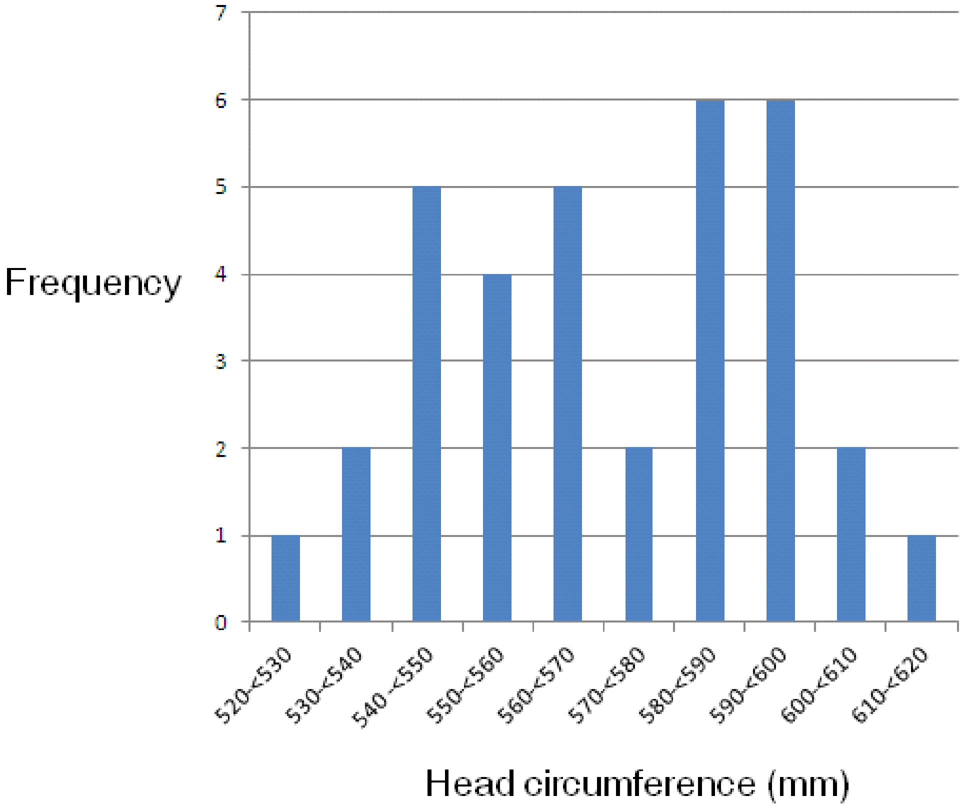
So far you have looked at **line plots** and **stem and leaf plots.** Both are very useful representations for allowing you to see patterns in the variation of your data. A histogram is another useful representation and it is especially useful when dealing with lots of data.

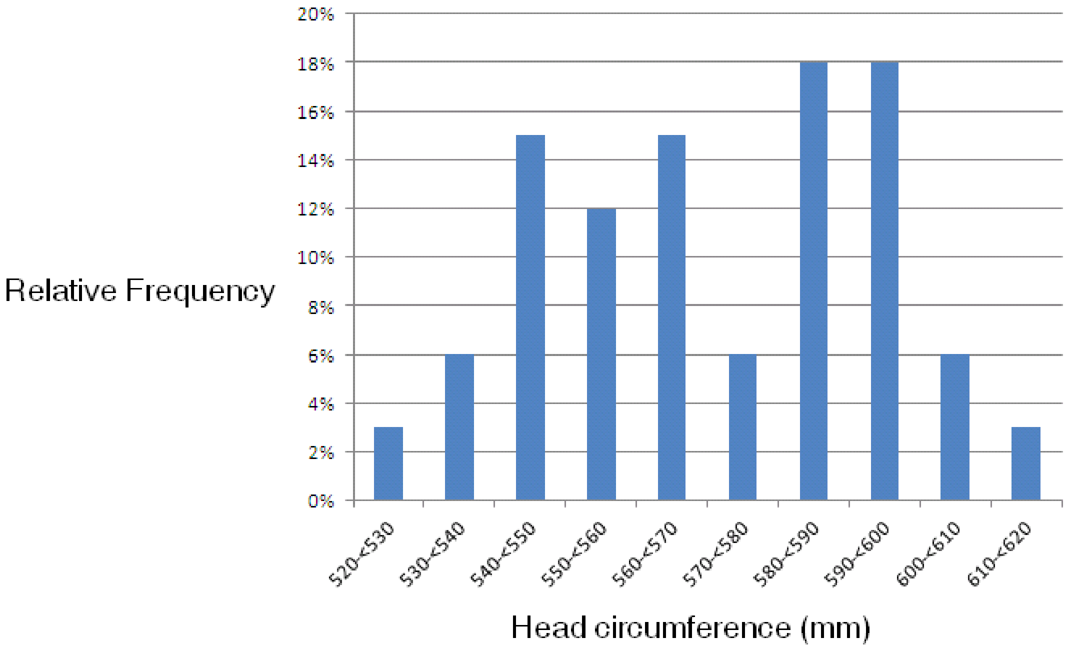
Consider the following:

The frequency and relative frequency for each stem was calculated.

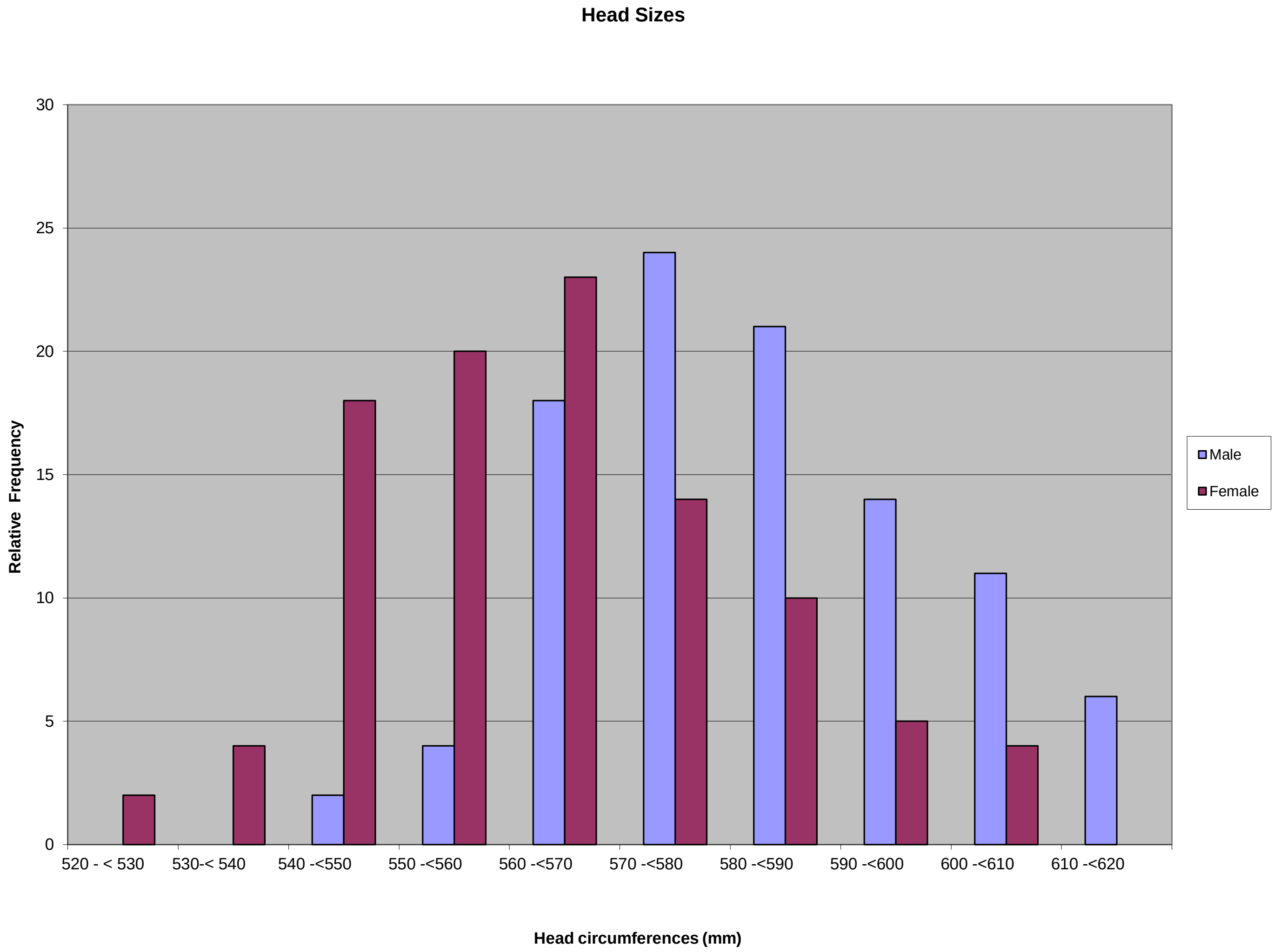
Stem	Leaves		Frequency	Relative Frequency
52	2	520 - < 530	1	1/34 = 3%
53	1 6	530 - < 540	2	2/34 = 6%
54	0 5 5 5 6	540 - <550	5	5/34 = 15%
55	0 1 2 5	550 - <560	4	4/34 = 12%
56	0 3 4 5 8	560 - <570	5	5/34 = 15%
57	3 7	570 - <580	2	2/34 = 6%
58	0 0 4 5 5 6	580 - <590	6	6/34 = 18%
59	0 0 0 3 4 5	590 - <600	6	6/34 = 18%
60	0 0	600 - <610	2	2/34 = 6%
61	3	610 - <620	1	1/34 = 3%

Using Excel we can draw a histogram. The diagrams below show two representations. Examine the axes. When would it be more suitable to use relative frequency as opposed to frequency ?





Look at the following histogram showing the distribution of head sizes for a different group of males and females. Compare the distributions. Is there any evidence to suggest that there are differences in the head sizes of men and women?



Why do you think the relative frequency is used for this histogram? Does it matter that the actual numbers of males and females in this sample are not given?

Consider the following piece of research.

In the late 1990s a **study** was undertaken in the South Island of New Zealand to explore iron levels in babies and toddlers (age 6-24 months). The participants were selected ***randomly*** from Christchurch, Dunedin and Invercargill (South Island Urban).

The iron, fibre, calcium and vitamin C intake per day was collected over three non-consecutive days and the ***iron (ferritin)*** levels in the blood were measured. Information such as whether the child was being breastfed, fed with formula milk or cows milk, as well as things like gender, ethnicity, maternal education, income level of household, if there were smoker(s) in the household and marital status of the mother were also obtained.

What **type** of study was this? Explain your choice.

Suggest how the participants may have been selected.

Having explored the literature, a number of factors were suggested that could have had an effect on the levels of iron or ferritin in the blood. One of these was gender – boys are at higher risk of having reduced levels of iron or ferritin in their blood.

Some sample data from the study is shown in the tables oopposite.

Ferritin Levels	Sex
22.80	1.00
8.00	1.00
	1.00
	1.00
16.80	1.00
	1.00
22.40	1.00
	1.00
17.30	1.00
	1.00
48.20	1.00
27.80	1.00
12.40	1.00
	1.00
5.30	1.00
	1.00
22.20	1.00
39.10	1.00
16.20	1.00
	1.00
18.70	1.00
38.00	1.00
	1.00
8.60	1.00
13.50	1.00
13.00	1.00
5.20	1.00
19.40	1.00
6.21	1.00
	1.00

Key
0 = Boy
1= Girl

Ferritin levels	Sex
27.20	0.00
	0.00
44.30	0.00
25.00	0.00
17.20	0.00
12.60	0.00
	0.00
9.90	0.00
23.70	0.00
12.75	0.00
	0.00
8.50	0.00
2.00	0.00
32.40	0.00
53.10	0.00
27.50	0.00
16.40	0.00
8.20	0.00
4.80	0.00
16.70	0.00
14.40	0.00
19.50	0.00
20.20	0.00
11.50	0.00
32.70	0.00
	0.00
12.70	0.00
6.30	0.00
10.40	0.00
8.00	0.00

Display this data in a way that will allow you to see patterns in the variation and compare the two **distributions.**

Describe and compare the **shape** of both **sample distributions.**

Describe and compare the **spread** of both **sample distributions.**

Use the data to answer the question “Do the iron levels of South Island urban boys tend to be lower than the iron levels of South Island urban girls?”

Explain why you have made this conclusion.

What aspects of this question did you find confusing?

Was the data difficult to display? What caused this difficulty?

What type of display did you decide to use? How did you make your choice?

How did you deal with the missing data when you displayed your data?

Did the missing data have any impact on the conclusions you made about the study?

Now see how well you have understood these concepts by answering the examination style questions. Make a note of the parts of the questions you find difficult or confusing. You may like to discuss these areas with your friends or your teacher.

Further examination questions can be found on the State Examinations Commission website (www.examinations.ie), including Sample Papers and the June 2010 and 2011 examinations.

STRAND 1 REVIEW

Working through these questions will help you assess your understanding of the learning outcomes listed here:

Level	All
Learning outcomes	– list outcomes of an experiment – apply the fundamental principle of counting – apply the principle that in the case of equally likely outcomes the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, urns with coloured objects, playing cards, etc.) – recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – use stem and leaf plots and histograms (equal intervals) to display data – evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others – use a variety of summary statistics to describe the data: central tendency – mean, median, mode; variability – range – evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others – recognise that correlation is a value from -1 to +1 and that it measures the extent of the linear relationship between two variables

Q. A group of people was asked "***What is your blood type***?" Here is the data they gave.

Type A	**Type B**	**Type O**	**Type AB**
50	65	70	15

If a person from this group is selected at random, what is the probability that this person has type **O** blood?

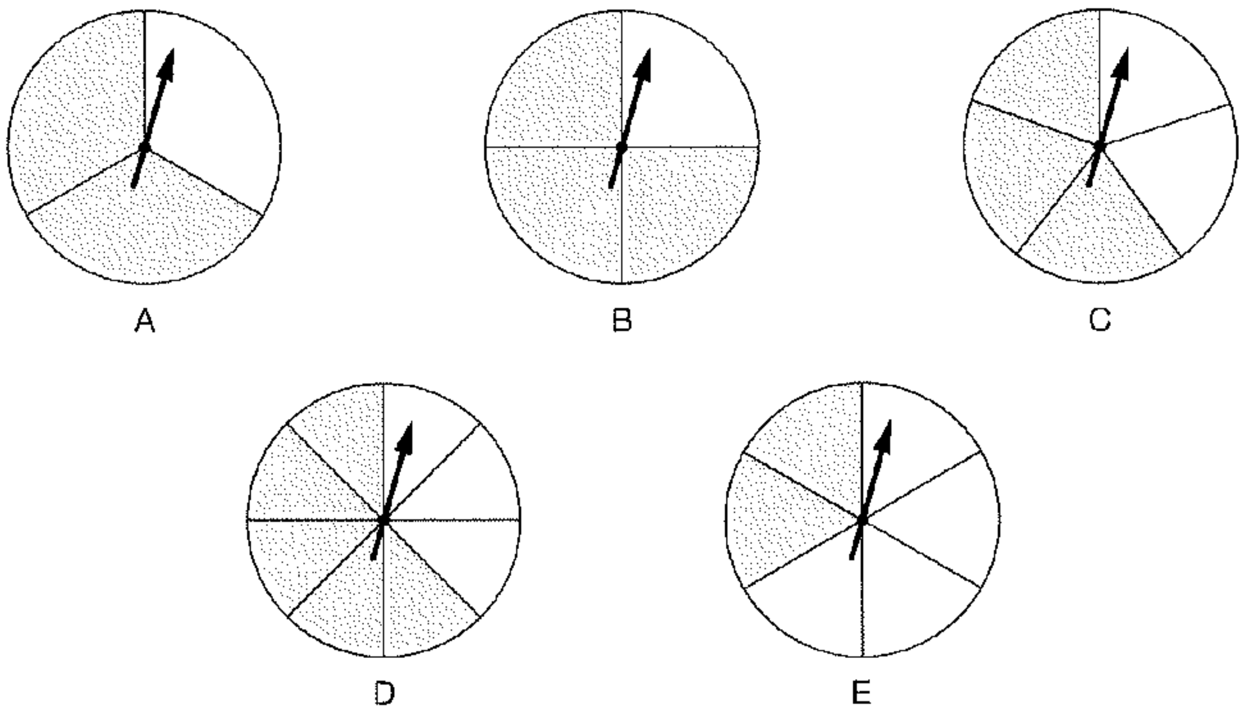
How many people answered the question?

How many people have type O blood?

Remember probability is always a number between **0** and **1.** This means it is a **fraction.** You should write fractions in their **lowest terms.**

Q. Five fair spinners are shown below.

Each spinner is divided into equal sectors, which are coloured either grey or white.



a) Identify the spinner for which the probability of spinning grey is ¾.

b) For two of the spinners, the probability of spinning grey is **more than 60%** but **less than 70%.** Which two spinners are these?

a) If the probability is ¾ what does this mean?

What does the 3 represent? What does the 4 represent?

Can you write ¾ in a different way?

Is 6/8 the same as ¾? Why? Why not?

If a student said the probability of spinning grey was 6/8 what might the spinner look like?

Would the student be correct in saying the probability of spinning grey was 6/8? Why? Why not?

b) Represent 60% and 70% as fractions.

Now work out the probabilities of spinning grey on each spinner.

Can you answer the question now?

Q Two coins are tossed. Complete the diagram to show all the **possible outcomes**.

a) What is the probability of getting 2 heads?

	H	T
T		
H		

b) Jennifer tossed the two coins 50 times and got a head and a tail 28 times.

Is there reason for Jennifer to think that one of the coins is not fair?

Explain.

c) Describe an experiment that would allow Jennifer to determine whether or not the coin was fair.

a) Can you make sense of the diagram? Does it help you to keep track of all the **possible outcomes**? How many possible outcomes are there?

b) Is it **more likely** that you get two heads than two tails? Why? Why not? Is it **more likely** that you get a head and a tail? Why? Why not? If you tossed the coins four times how many times would you expect to get a head and a tail? Why?

c) What would Jennifer have to do? How many times should she throw the dice – twice? 3 times? 100 times?

Remember the learning outcome

Students should be able to

- recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability

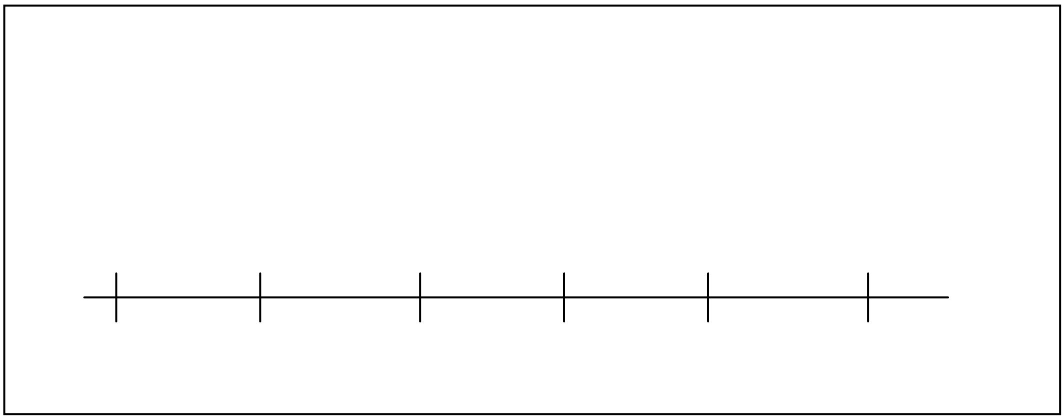
Try this experiment yourself

Q. The table below is a record of the number of texts sent by a group of students in one month.

No of texts sent	0 - 50	50 - 100	100 - 150	150 - 200	200 - 250
Number of students	10	15	25	18	8

a) How many students are in the group?

Illustrate the data on a histogram.



b) Using the table and /or histogram to help you estimate, complete this sentence:

On average these students send about ___________ texts each month.

c) Sarah is in the group and she sends 210 texts every month. Describe in one sentence Sarah's text sending by comparison to the others in the group.

What is unusual about the way the data is displayed in the table?

If John sends 100 texts in a month where in the table would you enter his data?

What does a **histogram** look like?

What do you think is the **typical** amount of texts sent by students in this group? Does your histogram help you decide on what is the typical amount of texts sent by students in this group?

Q A teacher asked 21 students to estimate the height of a building in metres.

The stem-and-leaf diagram shows all 21 results

6 | 5 represents 6.5 m

6	5	9				
7	0	2	6	8	8	
8	3	3	5	7	7	9
9	0	5	5	5		
10	4	8				
11	2	7				

a) What is the range of the estimated values?

b) What was the median estimated height?

c) The height of the building was 9.2m .How many people overestimated the height?

What other information can you get form the **stem and leaf plot**?

Is there any **evidence** to suggest that the group are good at estimating building height?

Q. Carol opened a new sandwich bar. She offers a lunch special consisting of a sandwich and a drink for €5.

The different choices available are shown below

Type of bread	**Filling**	**Drinks**
Brown	Salad	Tea
White	Egg	Coffee
Wrap	Meat	Hot Chocolate
Panini		Cold drink

All of the different combinations are possible. For example, you can order a salad sandwich on brown bread and a coffee.

How many different lunch specials are possible?

Think of a way to organise your thoughts. Can you write out all the possible combinations? Can you see a pattern as you write out all the combinations? Can you **generalise** this pattern that will help you to find out how many combinations there are without writing them all out?

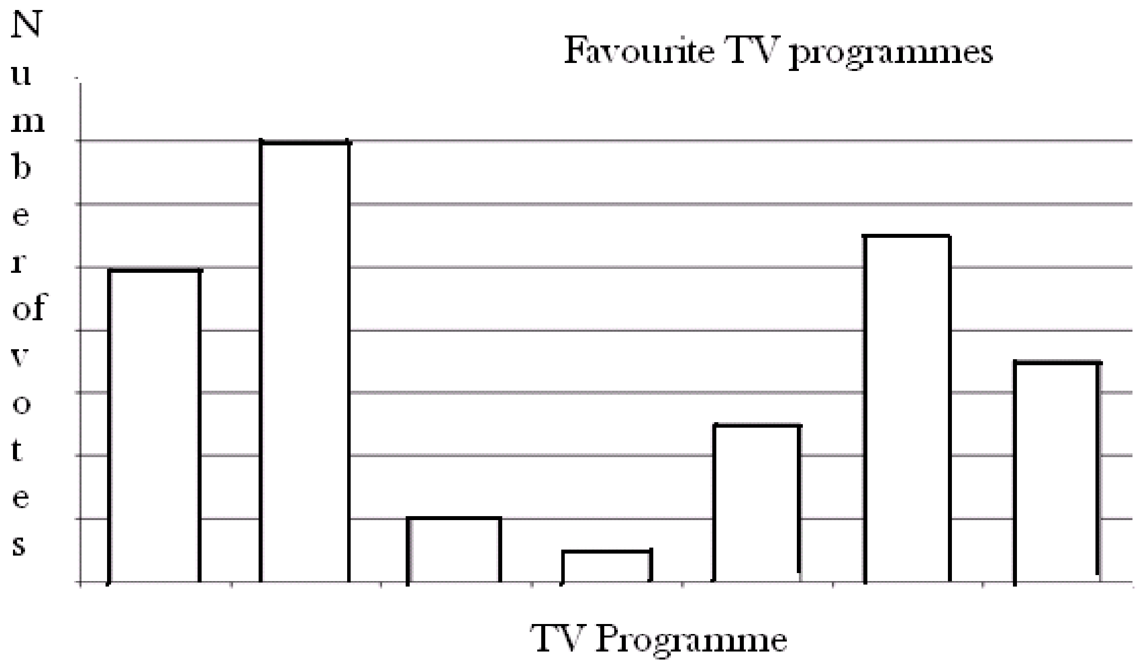
Q. The lists of test results for two maths classes were posted on the college notice board. You do not know which of the lists is for your class.

List 1	List 2
75	92
80	85
83	87
46	91
35	85
27	81
95	89
84	88
65	87
76	88
15	90
100	92
23	87
20	6
15	0

- Display the data from each list in stem and leaf plots.
- Give one reason why you would hope that list 1 is for your class and one reason why you would hope that list 2 is for your class
- Which list represents the better results? Give a reason for your answer.

Think about what mark you would be hoping to get in the test. Is it **likely** that you would get this mark if your class results were on list 1? List 2?
What is the **typical** mark on list 1? On list 2?
Would you like to get 100? How likely is it that you would get 100 if your class results were on list 1? On list 2? What does it mean to have the **better results**? Is there any evidence that list 1 has **better** results than list 2? Is there any evidence that list 2 has **better** results than list 1?

Q.



Clues

- Coronation Street was the most popular TV show
- Twice as many liked Coronation Street as Eastenders
- Fair City got 4 votes less than Coronation Street
- Casualty was the second most popular TV show
- Primetime got 4 votes more than Frontline
- 5 voted for Primetime
- Some people voted for Desperate Housewives

Use the information above to complete the frequency table

TV Programme	No of Votes

Place Coronation street first then Casualty

The bar representing Eastenders must be half the size of the bar representing Coronation Street. Why is this?

Can you locate the bar representing Eastenders?

How will you decide which are the bars representing Primetime and Frontline?

What about the bars representing Desperate Housewives and Fair City?

Task

In 1999 a university librarian put a number of measures in place to try to stop students "stealing" books from the library.

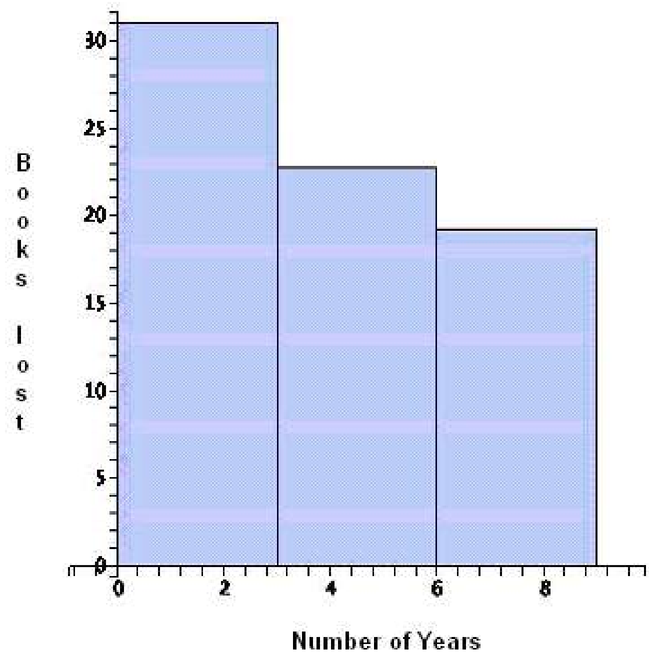
To see how effective these measures were she recorded the number of non-returned books over the next number of years.

The data is recorded below

Year	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**
No of non-returned books	9	10	9	14	6	3	4	5	10

When asked to report to the budget committee on book loss she wrote:

Whilst the drain on resources due to lost books is significant, the histogram below shows that over the last nine years the number of books lost to the library is steadily decreasing, which suggests that the measures implemented to combat this practice are working.



The finance officer was not convinced that the measures were working.

Plot the same data in a histogram but, instead of using three year intervals like the librarian did, divide the data into nine intervals, one for each of the last nine years.

Now, use your histogram to write two statements about the trend.
Does your histogram support the librarian's view that the measures are working, or does it lend more support to the doubts of the finance officer?
Explain your reasoning.

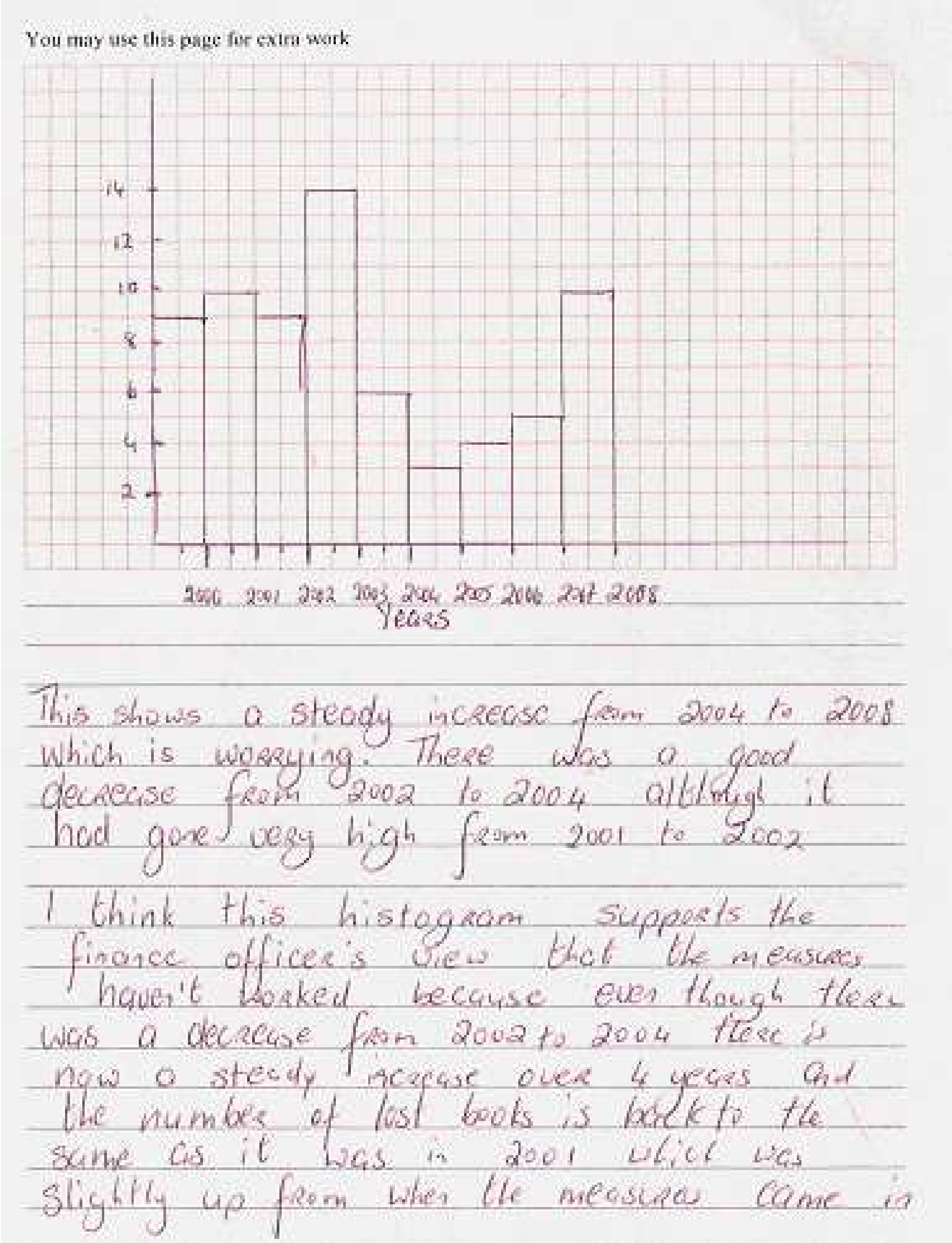
Note to student

This question highlights the fact that the choice of interval length can **reveal** certain trends or **hide** others.

Look at the plot that is given. How many intervals are there? How many years are in each interval? What can you **conclude** about the number of lost books? Would you say that the measures taken to discourage non-return are working?

Now divide the data into nine intervals and plot the histogram. Is there a difference in the **trend**?

Examine the student work below. Compare this with your work



Remember:

Histograms can be cleverly designed to hide or highlight certain trends. Remember this when you are interpreting histograms.

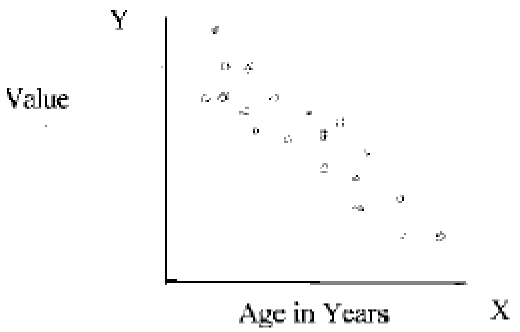
Which type would give you more detail? Which type would give you less detail?

Why might you want to highlight or hide certain trends?

Learning outcome	This partial question gives you the opportunity to provide evidence that you can – recognise that correlation is a value from -1 to +1 and that it measures the extent of the linear relationship between two variables

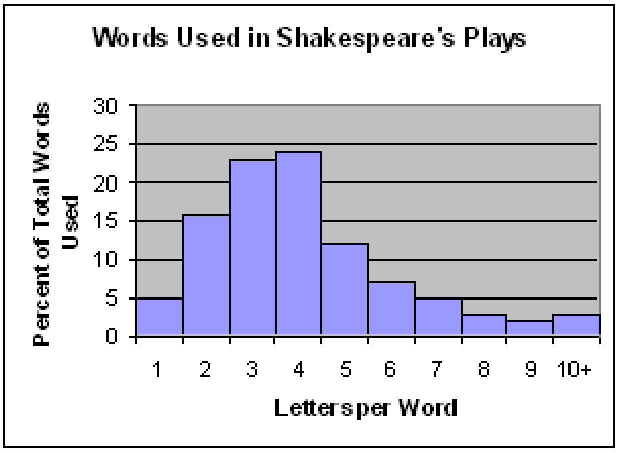
The points in the scatter plot below represent the ages of cars and their values.
Based on this scatter plot, which of the following is a reasonable conclusion?

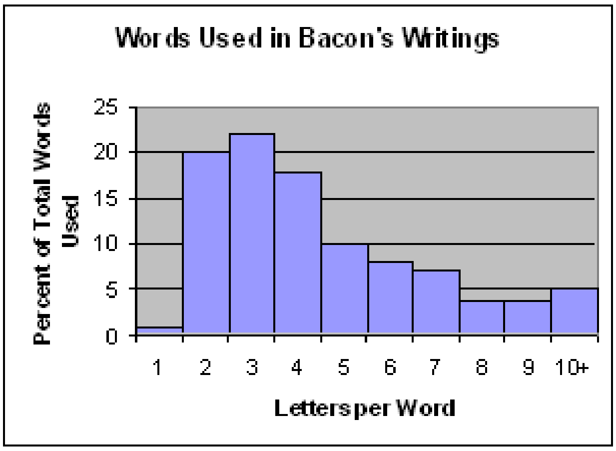
(i) Age and value have a correlation coefficient that is equal to zero.
(ii) Age and value have a correlation coefficient that is greater than 0.5.
(iii) Age and value have a correlation coefficient that is less than zero.
(iv) Age and value have a correlation coefficient that is between zero and 0.5.



Is the correlation positive or negative? How do you know?
Is there a strong or weak correlation? How do you know?
What numbers represent a strong correlation?
What numbers represent a weak correlation?
Sketch graphs that could represent each of the other answer options.

Q. Some scholars think William Shakespeare was really just a **pen name** for Sir Francis Bacon. (A pen name is a 'fake' name used by another person when writing.) In order to determine if this was true, a researcher counted the letters in every word of Shakespeare's plays and Bacon's writing. The results are recorded in the histograms below.





Based on these histograms, do you think that there is any **evidence** to suggest that William Shakespeare was really just a pen name for Sir Francis Bacon? Explain.

There is a lot of information in these histograms that you could use to support either argument. Yes, William Shakespeare was a pen name for Sir Francis Bacon; or no, William Shakespeare was not a pen name for Sir Francis Bacon. Might there be another explanation?

Are the **distributions** similar? Describe each **distribution.** Use fractions and percentages.

What percentage of Shakespeare's words have 4 letters per word or less?

What % of Bacon's words have 4 letters per word or less?

What percentage of Shakespeare's words have 5 letters per word or less?

What % of Bacon's words have 5 letters per word or less?

Q. The data shows the head circumferences for a group of men and women.

(a) Display the data in a way that will allow you to compare the distributions of head circumferences for both men and women.

(b) Is there any evidence to suggest that men have larger heads than women? Explain your reasoning.

Gender	Head Circumference
F	522
M	580
M	552
F	531
M	563
F	546
F	545
M	545
M	545
M	568
F	560
M	613
F	555
F	573
M	577
F	584
M	600
M	595
M	593
F	590
M	594
F	564
F	536
M	586

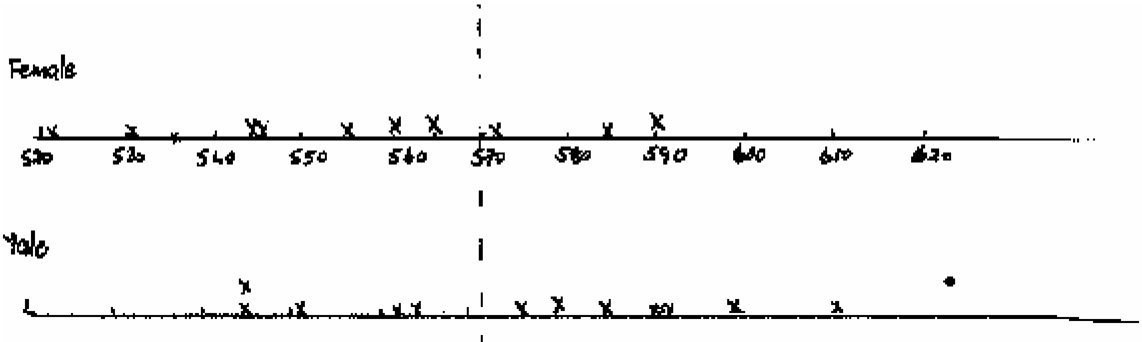
There are a lot of ways to display this data; a line plot, a back to back stem and leaf plot, or a histogram. Eyeball the data and think about how you would display it.

- What features are you looking at in the data?
- How are you deciding which display is most appropriate?

Once displayed you will be able to **comment** on the **distributions** and **draw conclusions** about the relative sizes of the heads of men and women.

Have a go at this and then examine the following examples of how other students displayed the data and drew conclusions from it.

Student work



Only 27% of female heads are above 570cm while 62% of male heads are larger than 570cm
That is evidence.

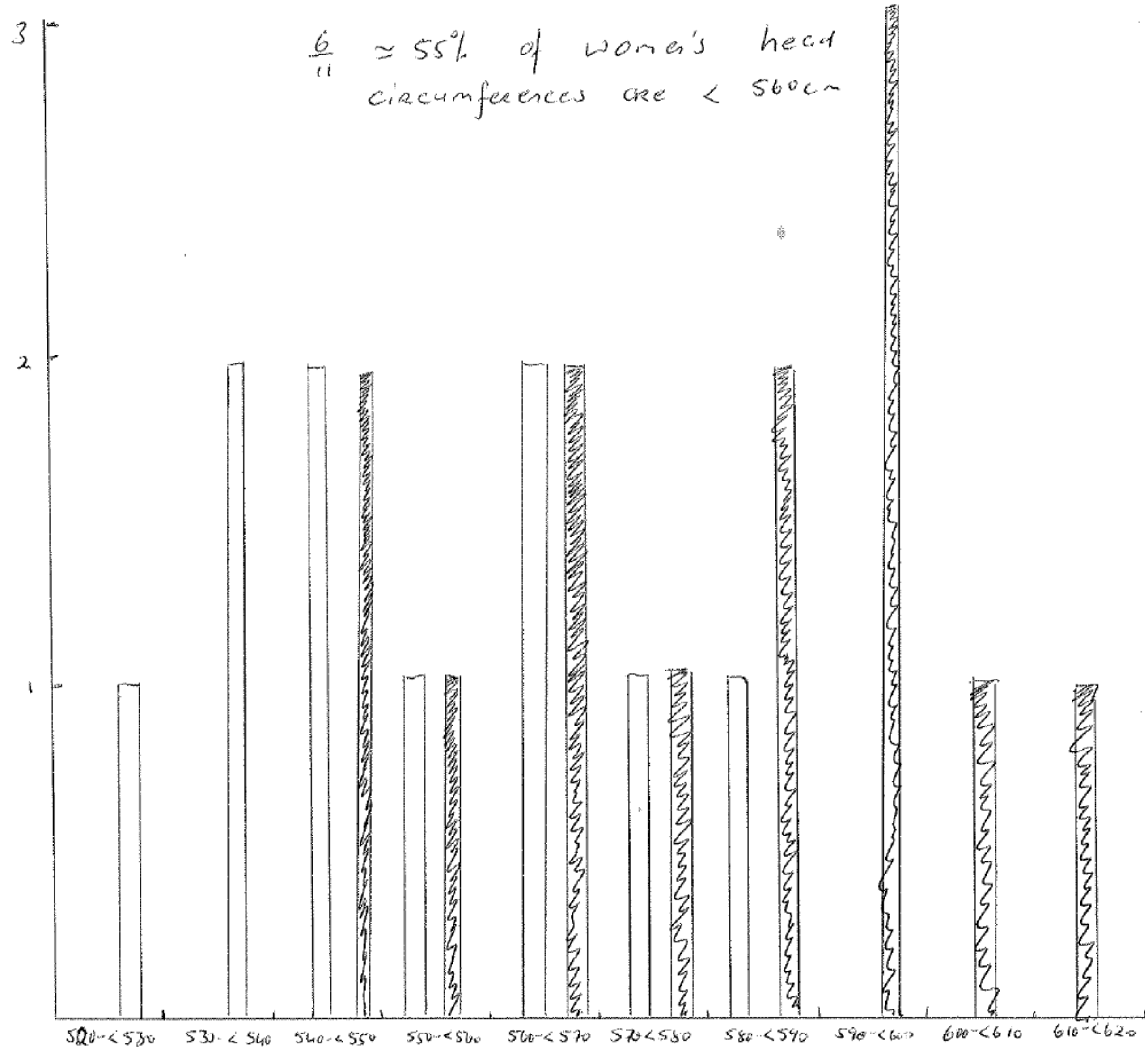
Male		Female
	50	
	51	
	52	2
	53	1 6
5 5	54	6 5
2	55	5
8 3	56	0 4
7	57	3
6 0	58	4
4 3 5	59	0
0	60	
3	61	

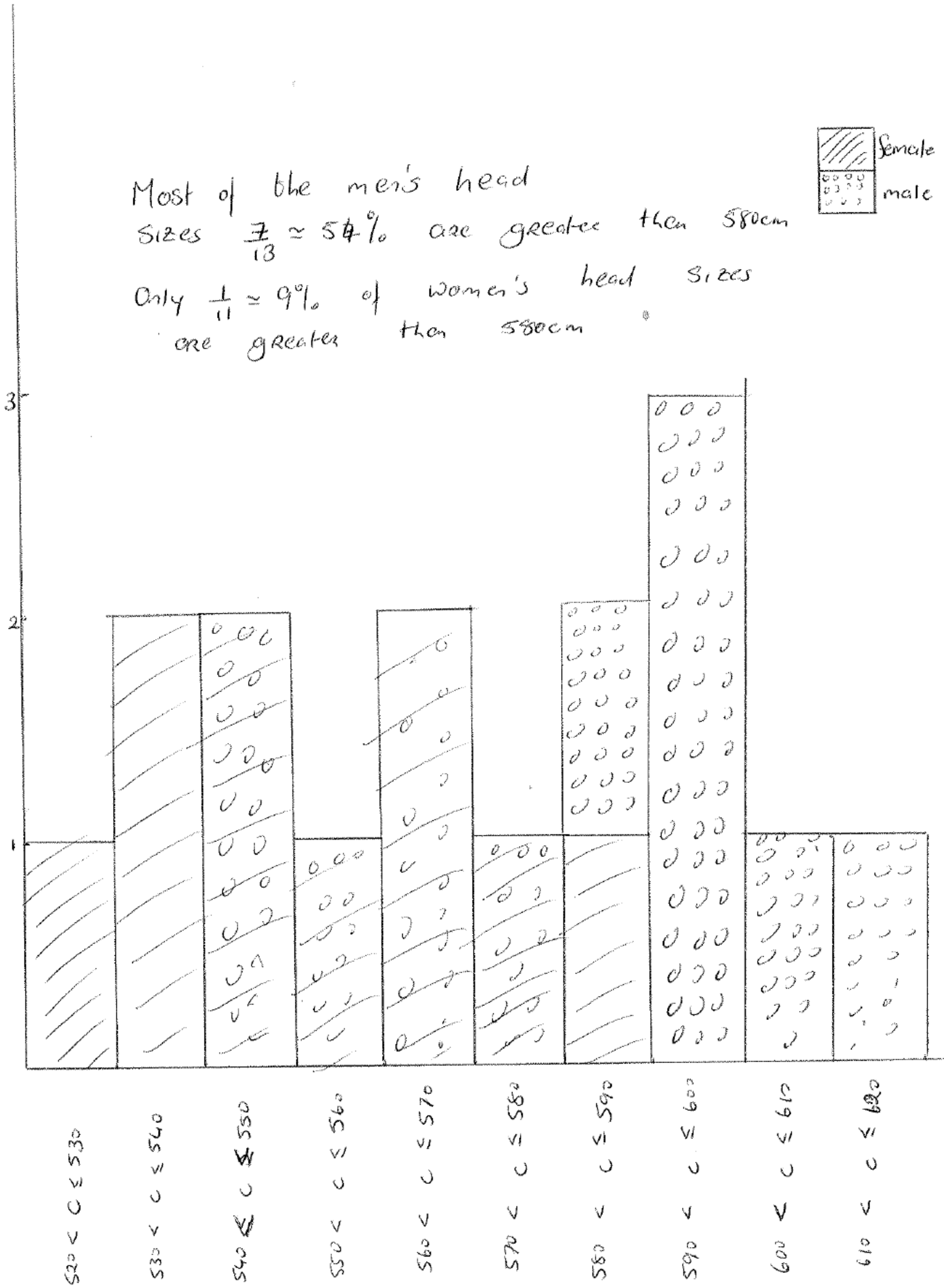
Range of female is 590 - 522 = 68
Range of male is 613 - 545 = 68

50% of the female head sizes lie between 522cm and 555cm while only 23% of the males lie in that Range
54% of male head sizes are between 580 and 613cm and only 18% of female head sizes are in this Range so yes there is evidence to support statement men have bigger heads then women.

	Male	Female
$520 < C \leq 530$	0	1
$530 < C \leq 540$	0	2
$540 < C \leq 550$	2	2
$550 < C \leq 560$	1	1
$560 < C \leq 570$	2	2
$570 < C \leq 580$	1	1
$580 < C \leq 590$	2	1
$590 < C \leq 600$	3	0
$600 < C \leq 610$	1	0
$610 < C \leq 620$	1	0

$\frac{7}{13} \simeq 54\%$ of men's head circumference is > 580cm




Most of the men's head sizes 7/13 ≈ 54% are greater then 580cm
Only 1/11 ≈ 9% of women's head sizes are greater then 580cm
female
male
1
2
3
520 < C ≤ 530
530 < C ≤ 540
540 < C ≤ 550
550 < C ≤ 560
560 < C ≤ 570
570 < C ≤ 580
580 < C ≤ 590
590 < C ≤ 600
600 < C ≤ 610
610 < C ≤ 620

Q. The 5^{th} year and 6^{th} year students in a local school were asked about the number of hours per week they spent playing on a games console. The results are shown below.

Number of hours spent playing on a games console	**Number of 5^{th} year students**	**Number of 6th year students**
1		
2	1	1
3	2	3
4	1	1
5	1	2
6	5	2
7		3
8		
9	1	3
10		1
11		3
12		2
13	3	3
14	1	1
15	4	
16	4	3
17	2	1
18	4	2
19	4	4
20	3	2
21	2	
22	3	
23	1	
24		
25	1	4

Display the data in a way that allows you to **comment** on the **shape of the distributions**. Is there any **evidence** to suggest that 6th year students spend longer playing a games console than 5th year students?

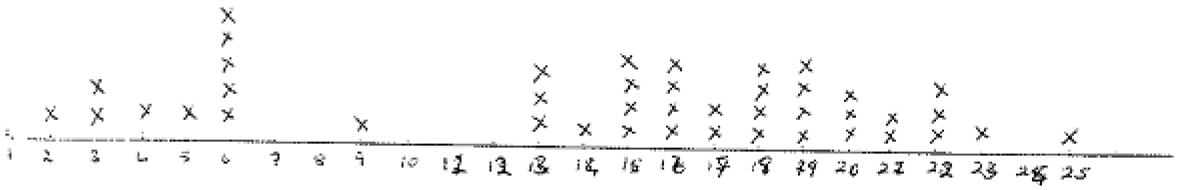
Note to Students

There are many ways you may choose to answer this question.
The data could be displayed in line plots, a back to back stem and leaf plot, or a histogram.
Once displayed you will be able to **comment** on the **distributions** and **draw conclusions** about the relative times spent by 5th and 6th year students on games consoles.

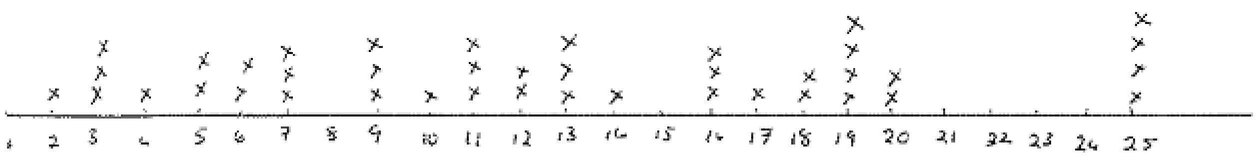
Have a go at this and then examine how student A below displayed the data and drew conclusions from it.
Now try to use a back to back stem and leaf plot and a histogram. Evaluate each display.

Student A

5TH Year



6th year

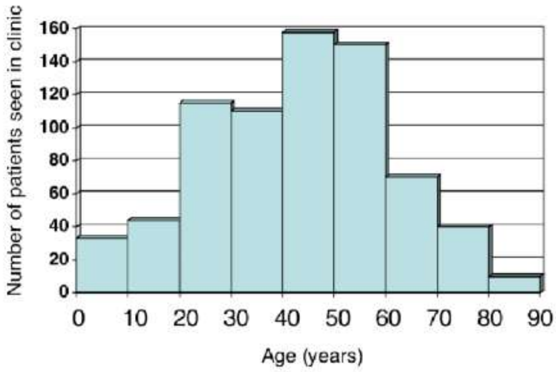


In the 5th year data there are two clusters: between 2 and 6 hours per week and 13-23 hours per week. 10 out of 43 or almost 25% of students play the console over the range of the first cluster. 31 of 43 or 72% are in the second cluster.
Only 1 out of 43 students uses the games console between 7 and 12 hours per week.

The data from 6th years are more evenly spread than the 5th year data there are no real clusters. 16/41 or 39% of students play the games console between 16 and 25 hours per week while ½ of the students play between 2 and 12 hours per week.
The range is the same as the 5th years.

The fact that 24 of 43 or approximately 56% of 5th year students play the console between 16 and 25 hours per week whilst only 39% of the 6^{th} year students play for this length of time indicates that there Is no evidence to suggest that 6^{th} year students spend longer playing a games console than 5^{th} year students. In fact the evidence shows the opposite.

Q. The ages of the patients seen by a group of doctors in a clinic over the last month are shown in the histogram below.

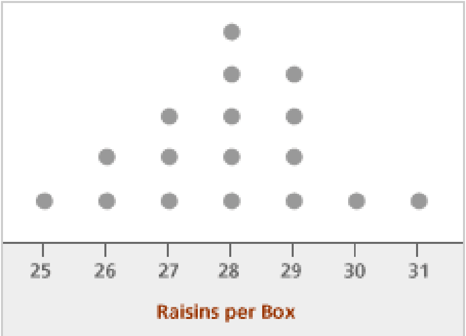


The clinic is about to begin a Swine Flu vaccination programme and must order the drugs they need from the HSE.

If $^1/_3$ of the 40-90 year olds, ½ of the 20-40 year olds, $^1/_5$ of the 10-20 year olds and all the 0-10 year olds who attended the clinic last month are likely to attend for vaccination, what is the minimum number of vaccinations that the clinic should order from the HSE?
Show your workings.

Use the histogram to decide how many of each age group visited the clinic over the last month.
Try to organise your work into a table
How many 40-90 year olds attended the clinic in the last month? What fraction of these is likely to attend for vaccination? How many of these are likely to attend for vaccination?
What about the 20-40 year olds? What fraction of these attended the clinic in the last month? How many of these are likely to attend for vaccination?

Q. Students were investigating the number of raisins contained in individual boxes of Sun-Maid raisins. They recorded their results in the diagram shown.



(a) If the students choose a box at random from all the boxes they surveyed what is the probability that the box contains 29 raisins?

(b) Four boxes were found after the students had completed the line plot above.
Jack, Sarah, Amy and Kevin were each given a box and asked to count the contents.
Jack said his contained 28 raisins. Sarah said hers contained 28 raisins also.
Another student said: "I bet Amy's contains 28 raisins also.
Kevin said "Wait, Amy; don't reveal the contents of your box yet."

He and Amy whispered together and then Kevin said "I will tell you that if the contents of our two boxes are added to the data the mean number of raisins per box will be 28.

Give one possible value each for the number of raisins in Kevin's and Amy's boxes.
Justify your choice.

Is it possible that the student won the bet?
Explain your reasoning.

LC-HL only

The next day the students looked up the statistics from the Sun-Maid company.
The mean number of raisins per box is published as 25 and the standard deviation as 5.
If a normal distribution applies, what is the probability that a box chosen at random from the production line contains 29 or more raisins?

Compare this probability with that obtained from the class sample.

- How many boxes contained 29 raisins? How many boxes were there altogether?
- When the 4 extra boxes were found were you surprised that two of them contained 28 raisins? Why or why not? Is it possible that the final two boxes contained 28 each also? Why do you think this? How likely is it that both boxes contain 28 raisins? Is it possible they contain 24 raisins or 25 raisins each? Why, or why not?
- If the mean of the 21 boxes is 28 what does this mean? How many raisins are there altogether? How many are in the 19 boxes? How do you know? How many must be in the other two boxes if the mean stays at 28? How do you know? Is it possible that only one of the other two boxes contains 28 raisins? Justify your thinking.
- If the published data states that the mean number of raisins per box is 25 with a standard deviation of 5, would 29 raisins be unusual? How many standard deviations from the mean is 29? Is it a whole number of standard deviations? Can you use the tables to find the probability of obtaining 29 raisins? Compare this with the probability of getting a box with 29 in the sample described above.

Question LCHL: Descriptive Statistics

To enter a particular college course, candidates must complete an aptitude test. In 2010 the mean score was 490 with a standard deviation of 100. The distribution of the scores on the aptitude test is a normal distribution.

(a) What percentage of candidates scored between 390 and 590 on this aptitude test?

(b) One student scored 795 on this test. How does this student's score compare to the rest of the scores?

(c) The college admits only students who were among the highest 16% of the scores on this test. What score would a student need on this test to be qualified for admission to this college? Explain your answer.

(d) Alice is preparing to sit the aptitude test in 2011. She heard that a score of over 650 would guarantee her a place on the course. She knew 20 people who were going to take the test. Based on the mean and standard deviation in 2010, approximately how many of the people Alice knew were likely to get a score of above 650 and secure a place on the course? Justify your answer.

LCOL: Descriptive Statistics.

David noticed that, when he drank a bottle of sports drink before going out for a run one day, his performance time improved. He set about doing an experiment to see whether drinking the sports drink increases performance when running.

He recorded the times of people in his running club to complete a 5km run without drinking the sports drink and then on another day he recorded the time it took the same people to complete 5km having taken the sports drink.

He recorded the information in a back-to-back stem and leaf plot:

Without taking the sports drink.		Having taken the sports drink.
5	20	3 4
1 1 1	21	3 4 7
8 8 4 3 2	22	
	23	1 2 2
	24	0
	25	8
	26	1
	27	
	28	2 3 6 6 7 7
	29	2 4 4 5 5 5 8 9 9
	30	1 3 4 5 6 7 8 8 9
5	31	
6 4 4 3 0 0	32	1 1 4 9
9 9 6 5 4 4 3 3 2 1	33	3 3 3 2
7 7 5 5 6 6 6 1 0	34	5
8 8 8 3 3	35	0 0
7 3 2	36	1
1	37	2
	38	3 5
2 2	39	
4 4 2 0	40	

Key: 32 | 1 means 32.1 minutes

(i) Based on the diagrams approximate the median speed without drinking the sports drink and the median speed having taken the sports drink. What does this information tell you?

(ii) Compare the distributions of each of the data sets above.

(iii) Is there evidence from the diagram to suggest that taking the sports drink improves performance? Justify your conclusions.

(iv) Make an argument, based on the two data sets, that taking the sports drink does not improve performance.

(v) After completing the experiment, David wondered how accurate his study was. He realised that he had not specified how much of the sports drink the runners should take. He asked 20 of the runners approximately how many millilitres of sports drink they had taken and recorded ithis alongside their time. The results are as follows:

Time (mins)	Sports drink (ml)
20.3	250
21.7	100
21.8	120
24	80
28.6	300
29.4	130
29.5	300
29.9	280
32.1	300
32.1	100
33.2	80
35	220
38.3	180
20.6	100
29.2	200
29.8	250
36.1	80
29.9	120
30.9	240
30.1	280

Display the data in a way that allows you to examine the relationship between the two data sets.

(vi) Is there evidence to suggest that there is a relationship between the time taken to complete 5km and the amount of sports drink taken before the race?

(vii) The correlation coefficient for data in part **(v)** above is one of the following.

Circle the correct correlation coefficient, based on your graph.

A -0.82 **B** 0.13 **C** 0.95 **D** 0.6

JCOL: Descriptive Statistics

A group of students were asked "*Do you get worried about your exams*?" They were asked to circle one of following to answer the question: Never, Rarely, Sometimes, Frequently.

The data below shows the answers from a sample of boys and girls.

Boys	Girls
Frequently	Never
Never	Sometimes
Never	Sometimes
Sometimes	Rarely
Sometimes	Never
Rarely	Frequently
Sometimes	Frequently
Sometimes	Never
Frequently	Sometimes
Never	Rarely
Sometimes	Frequently
Rarely	Rarely
Rarely	Sometimes
Frequently	Frequently
Never	Frequently
Rarely	Frequently
Rarely	Rarely
Frequently	Frequently
Never	Frequently
Frequently	Frequently
Never	Sometimes
Sometimes	Sometimes
Never	Sometimes
Frequently	Never
Rarely	Rarely
Sometimes	Frequently
Rarely	Frequently
Never	Never
Sometimes	Never
Rarely	Frequently

(a) How many students were in each sample?

(b) Display the data in a way which allows you to compare the two samples.

(c) Compare the two sets based on your display.

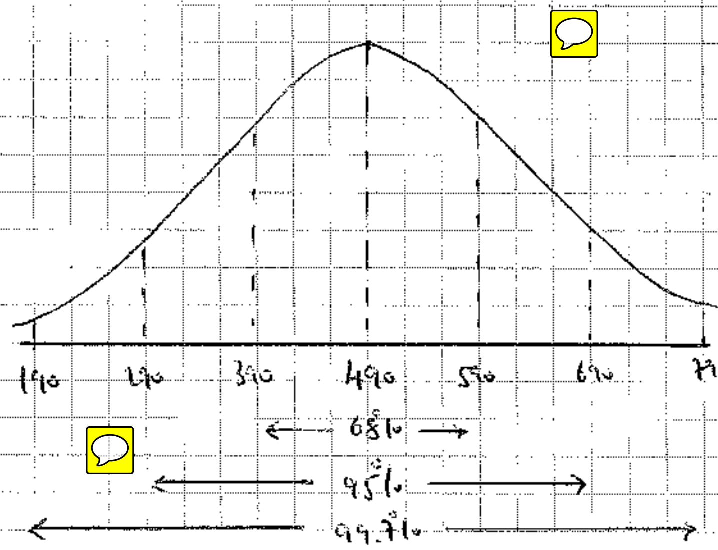
Note:
When looking at the solution set, click on the "sticky note" for additional comments.

Question LCHL: Descriptive Statistics

To enter a particular college course, candidates must complete an aptitude test. In 2010 the mean score was 490 with a standard deviation of 100. The distribution of the scores on the aptitude test is a normal distribution.

(a) What percentage of candidates scored between 390 and 590 on this aptitude test?

390 → 590 ⇒ 1 s dev each side of the mean.
∴ 68% of scores in this region



(b) One student scored 795 on this test. How does this student's score compare to the rest of the scores?

190 → 790 ⇒ 3 s.d. ⇒ 99.7% So 0.3% outside this region ∴ 795 ⇒ top 0.15%

(c) The college admits only students who were among the highest 16% of the scores on this test. What score would a student need on this test to be qualified for admission to this college? Explain your answer.

Highest 16% ⇒ 32% on the "tails".

∴ 68% Region ⇒ 1 standard deviation from mean

Thus highest 16% is above 590 on test.

The College will accept students who score higher than 590

(d) Alice is preparing to sit the aptitude test in 2011. She heard that a score of over 650 would guarantee her a place on the course. She knew 20 people who were going to take the test. Based on the mean and standard deviation in 2010, approximately how many of the people Alice knew were likely to get a score of above 650 and secure a place on the course? Justify your answer.

$\frac{650 - 490}{100} = 1.6$ Z score of 1.6 ⇒ 0.9452

$1 - 0.9452 = 0.0548$

$20 \times 0.0548 = 1.096$

Thus approximately 1 person out of the 20 is likely to score above 650 and secure a place on the course.

LCOL: Descriptive Statistics.

David noticed that, when he drank a bottle of sports drink before going out for a run one day, his performance time improved. He set about doing an experiment to see whether drinking the sports drink increases performance when running.

He recorded the times of people in his running club to complete a 5km run without drinking the sports drink and then on another day he recorded the time it took the same people to complete 5km having taken the sports drink.

He recorded the information in a back-to-back stem and leaf plot:

Without taking the sports drink.		**Having taken the sports drink.**
5	20	3 4
1 1 1	21	3 4 7
8 8 4 3 2	22	
	23	1 2 2
	24	0
	25	8
	26	1
	27	
	28	2 3 6 6 7 7
	29	2 4 4 5 5 5 8 9 9
	30	1 3 4 5 6 7 8 8 9
5	31	
6 4 4 3 0 0	32	1 1 4 9
9 9 6 5 4 4 3 3 2 1	33	3 3 3 2
7 7 5 5 6 6 6 1 0	34	5
8 8 8 3 3	35	0 0
7 3 2	36	1
1	37	2
	38	3 5
2 2	39	
4 4 2 0	40	

Key: 32 | 1 means 32.1 minutes

(i) Based on the diagrams, approximate the median speed without taking the sports drink and the median speed having taken the sports drink. What does this information tell you?

Median times: No sports drink → 33.9 mins
with sports drink → 29.9 mins

Median speeds: No sports drink → $\frac{5}{33.9} = 0.147$ km/min

with sports drink → $\frac{5}{29.9} = 0.167$ km/min

So the median speed increased when they ran 5 km after taking the sports drink.

(ii) Compare the distributions of each of the data sets above.

The range of times without the sports drink is 20.5 – 40.4 mins
The range of times after taking the sports drink is 20.3 – 38.5 mins

The distribution of times without the sports drink is more skewed than the times after taking the sports drink, which is more symmetrical

For the data without the sports drink, the data are clustered around 32 – 36 minutes, whereas for the data where the runners had taken the sports drink the times are clustered around 28 – 30 minutes.

(iii) Is there evidence from the diagram to suggest that taking the sports drink improves performance? Justify your conclusions.

There is evidence to suggest that performance improves after taking the sports drink. The range of times is smaller after taking the drink before running the 5 km. Without the drink only 20% of the runners took less than 32 mins to run the 5 km. after taking the drink, 70% of the runners completed the 5 km in less than 32 mins.

(iv) Make an argument, based on the two data sets, that taking the sports drink does not improve performance.

18% of the runners took less than 23 minutes to run the 5 km without taking the sports drink. Only 10% of the runners ran the 5 km, after taking the drink, in less than 23 minutes

(v) After completing the experiment, David wondered how accurate his study was. He realised that he had not specified how much of the sports drink the runners should take. He asked 20 of the runners approximately how many millilitres of sports drink they had taken and recorded this alongside their time. The results are as follows:

Time (mins)	Sports drink (ml)
20.3	250
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21.8	120
24	80
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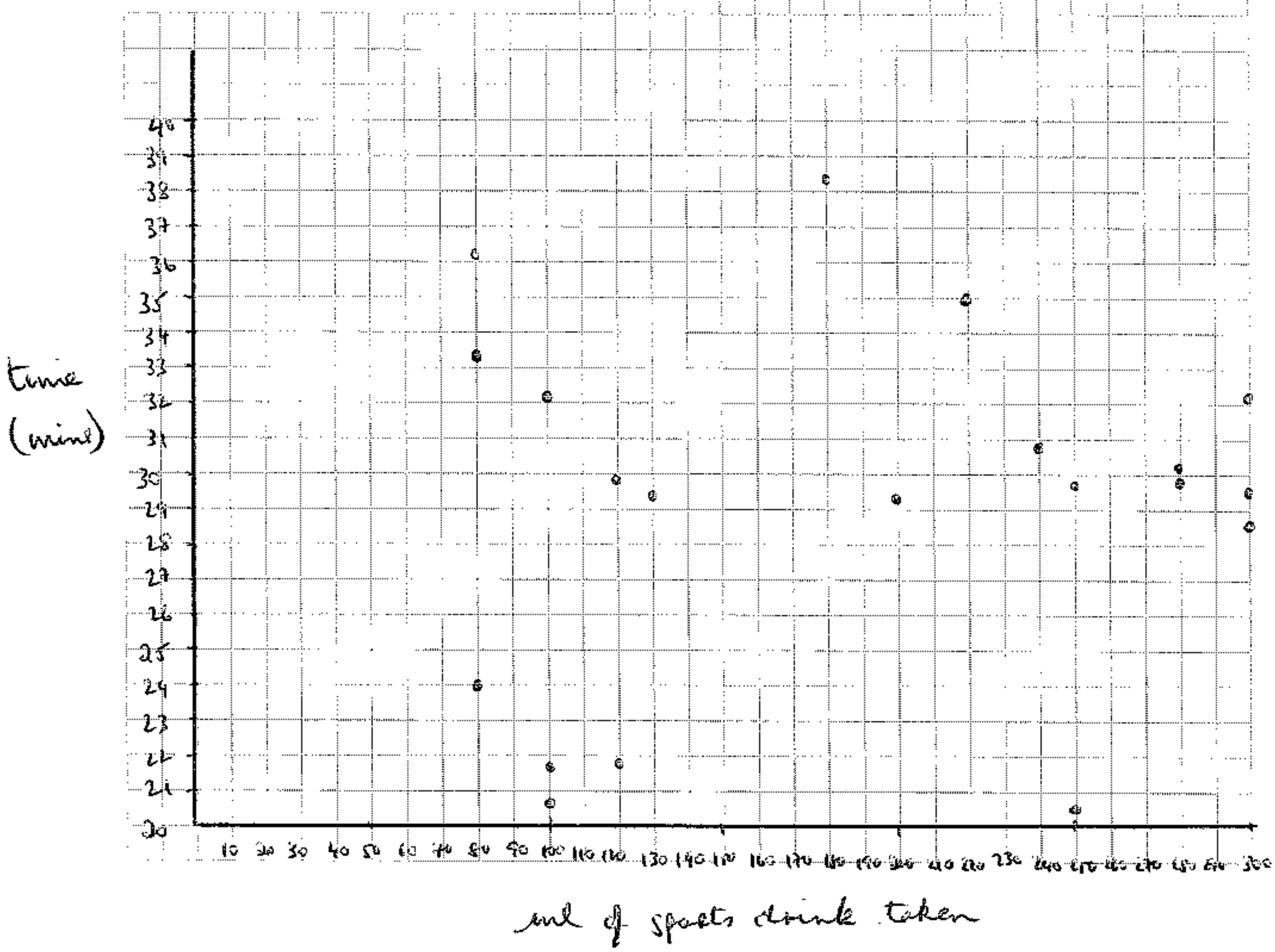
29.4	130
29.5	300
29.9	280
32.1	300
32.1	100
33.2	80
35	220
38.3	180
20.6	100
29.2	200
29.8	250
36.1	80
29.9	120
30.9	240
30.1	280

Display the data in a way that allows you to examine the relationship between the two data sets.



(vi) Is there evidence to suggest that there is a relationship between the time taken to complete 5km and the amount of sports drink taken before the race?

No, the scatterplot doesnt show a noticeable relationship between the time taken to run the 5 km and the amount of sports drink taken before the run.

(vii) The correlation coefficient for data in part **(v)** above is one of the following. Circle the correct correlation coefficient, based on your graph.

A -0.82 **B** 0.13 **C** 0.95 **D** 0.6

JCOL: Descriptive Statistics

A group of students were asked "*Do you get worried about your exams*?" They were asked to circle one of following to answer the question: Never, Rarely, Sometimes, Frequently.

The data below shows the answers from a sample of boys and girls.

Boys	Girls
Frequently	Never
Never	Sometimes
Never	Sometimes
Sometimes	Rarely
Sometimes	Never
Rarely	Frequently
Sometimes	Frequently
Sometimes	Never
Frequently	Sometimes
Never	Rarely
Sometimes	Frequently
Rarely	Rarely
Rarely	Sometimes
Frequently	Frequently
Never	Frequently
Rarely	Frequently
Rarely	Rarely
Frequently	Frequently
Never	Frequently
Frequently	Frequently
Never	Sometimes
Sometimes	Sometimes
Never	Sometimes
Frequently	Never
Rarely	Rarely
Sometimes	Frequently
Rarely	Frequently
Never	Never
Sometimes	Never
Rarely	Frequently

(a) How many students were in each sample?

30

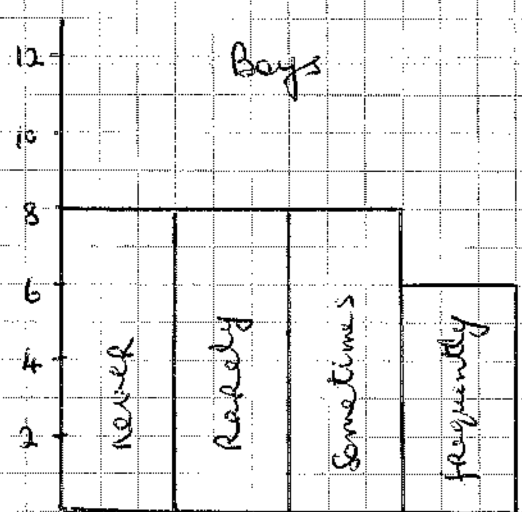
(b) Display the data in a way which allows you to compare the two samples.

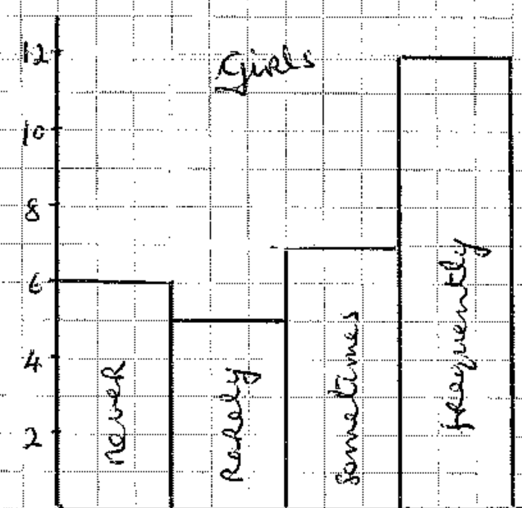
Boys

frequently	6
Sometimes	8
Rarely	8
never	8

Girls

Frequently	12
Sometimes	7
Rarely	5
never	6





(c) Compare the two sets based on your display.

More girls than boys said they worried frequently (twice as many).

More boys than girls said they rarely or never worry about their exams (16 compared to 11.

NCCA
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Project Maths
Tionscadal Mata

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Mathematics Resources for Students

Leaving Certificate – Strand 1

Statistics and Probability

INTRODUCTION

This material is designed to supplement the work you do in class and is intended to be kept in an A4 folder. Activities are included to help you gain an understanding of the mathematical concepts and these are followed by questions that assess your understanding of those concepts. While there are spaces provided in some activities/questions for you to complete your work, you will also need to use your copybook/A4 pad or graph paper. Remember to organise your folder so that it will be useful to you when you revise for tests and examinations. As you add pages to your folder, you might consider dating or coding them in a way that associates them with the different topics or syllabus sections. Organising your work in this way will help you become personally effective. Being personally effective is one of the five key skills identified by the NCCA as central to learning (www.ncca.ie/keyskills). These key skills are important for all students to achieve their full potential, both during their time in school and into the future.

As you work through the material in this booklet and with your teacher in class, you will be given opportunities to develop the other key skills. You will frequently work in pairs or groups, which involves organising your time effectively and communicating your ideas to the group or class. You will justify your solutions to problems and develop your critical and creative skills as you solve those problems. As you complete the activities you will be required to process and interpret information presented in a variety of ways. You will be expected to apply the knowledge gained to draw conclusions and make decisions based on your analysis. The sequence in which the sections/topics are presented here is not significant. You may be studying these in a different order, or dipping in and out of various sections over the course of your study and/or revision.

The questions included in this booklet provide you with plenty of opportunities to develop communication skills and to promote mathematical discourse. When your teachers mark your work they will gain insights into your learning and will be able to advise you on what you need to do next.

The material in the booklet is suitable for both Junior Certificate and Leaving Certificate, since at Leaving Certificate you consolidate and build on the concepts you learned at Junior Certificate and, occasionally, you may need to explore some of the activities or exercises which were used to introduce these concepts. Through completing the activities and questions contained in this booklet, you will develop a set of tools that will help you become a more effective learner and these tools can be used across the curriculum. Solving problems of this nature should also improve your confidence in doing mathematics, thus helping you to develop a positive attitude towards mathematics and to appreciate its role in your life.

The mathematics syllabus documents can be accessed at www.ncca.ie and you will find other relevant material on www.projectmaths.ie.

PROBABILITY 1

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to
- decide whether an everyday event is likely or unlikely to happen
- recognise that probability is a measure on a scale of 0 - 1 of how likely an event is to occur.
- connect with set theory; discuss experiments, outcomes, sample spaces
- use the language of probability to discuss events, including those with equally likely outcomes

INTRODUCTION

The activities described below and the questions that follow give you the opportunity to reinforce your understanding of the basic concepts of probability. The activities are designed to build on previous experiences where you estimated the likelihood of an event occurring. Some of the activities will be done in class under the direction of your teacher; others can be done at home.

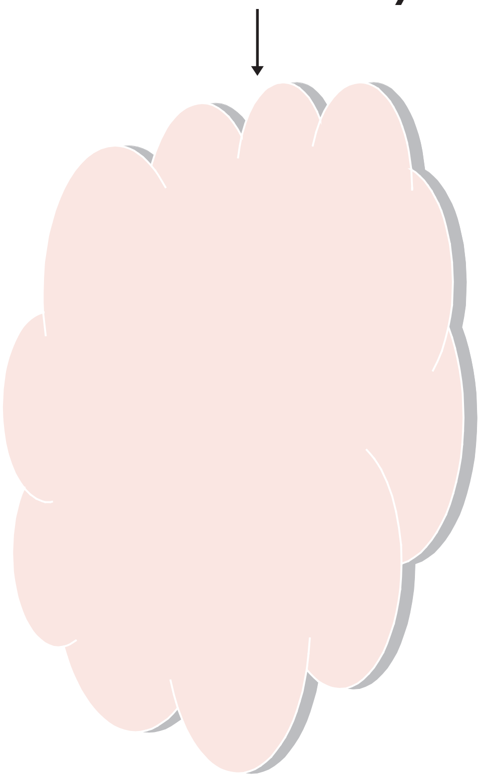
Activity 1.1

A probability describes mathematically how likely it is that something will happen. We can talk about the probability it will rain tomorrow or the probability that Ireland will win the World Cup.

Consider the probability of the following events
- It will snow on St Patrick's day
- It will rain tomorrow
- Munster will win the Heineken Cup
- It is your teacher's birthday tomorrow
- You will obtain a 7 when rolling a die
- You will eat something later today
- It will get dark later today

Words you may decide to use: certain, impossible, likely, very likely

Student Activity

Certain not to happen	Area of Uncertainty	Certain to happen
1.		1.
2.		2.
3.		3.

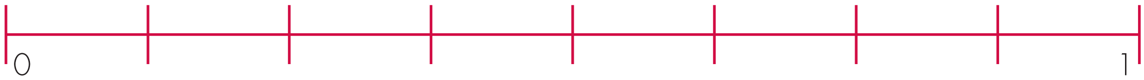
Phrases used to describe uncertainty

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________

Use the table provided or mark your work page out in a similar way and place each of the events in the approriate section.Note the phrases you used to describe uncertainty.

Activity 1.2

The Probability Scale



Extremely unlikely	50/50	3/8	1 in 4 chance
Probability of getting an odd number when rolling a die	87.5%	Extremely likely	1/2
	0.125	3/4	Impossible
1/4	Certain	75%	1
Equally likely	0.25	0	

1. Place the above phrases, numbers and percentages at the correct position on the probability scale.
2. Find and write down instances from TV, radio, or in the newspaper which illustrate how probability affects people's lives.

Questions

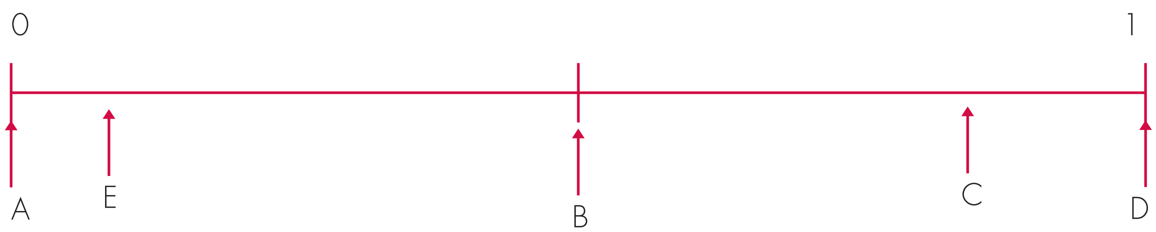
Q.1 For each event below, estimate the probability that it will happen and mark this on a probability scale.

- It will snow in Ireland on August 16th
- Your maths teacher will give you homework this week
- Your will eat fish later today
- You will go to bed before midnight tonight
- You will go to school tomorrow

Q. 2 Use one of the words certain, likely, unlikely, impossible to describe each of the events below. Give a reason for each of your answers.

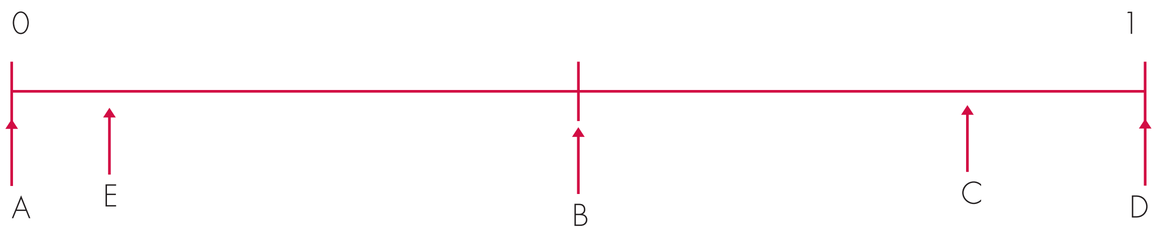
- You are more than 4 years old
- You will arrive on time to school tomorrow
- You will miss the school bus tomorrow
- Your county will win the Championship this year.

Q. 3 The probability line shows the probability of 5 events A, B, C, D and E



a. Which event is certain to occur?
b. Which event is unlikely but possible to occur?
c. Which event is impossible?
d. Which event is likely but not certain to occur?
e. Which event has a 50:50 chance of occurring?

Q. 4 The events A, B, C, D have probabilities as shown on this probability line;



i. Which event is the **most likely** to take place?

ii. Which event is the **most unlikely** to take place?

iii. Which event is **more likely than not** to take place?

Q. 5 When you toss an unbiased coin the probability of getting a head is ½, because you have an equal (or even) chance of getting a head or tail. Name two other events that have a probability of ½.

Q. 6 The 'events' A, B, C, D are listed below;

A: You will live to be 70 years old
B: You will live to be 80 years old
C: You will live to be 100 years old
D: You will live to be 110 years old

Make an estimate of the probability of each event, and place it on a probability scale.

Q. 7 Sarah and Alex are exploring probability and Sarah has these cards:

2	3	9	5	6	7	8	9
△	△	■	△	■	■	△	△

Alex takes a card without looking. Sarah says

> On Alex's card ■ is more likely than △

i. Explain why Sarah is wrong.

ii. Here are some words and phrases that can be associated with probability:

impossible	not likely	certain	likely

Choose a word or a phrase to fill in the gaps below.

It is .... that the number on Alex's card will be smaller than 10.

It is .... that the number on Alex's card will be an odd number.

Sarah mixes up the cards and places them face down on the table.
Then she turns the first card over, like this:

Alex is going to turn the next card over

iii. Complete the sentence:

On the next card, ______ is less likely than ______

The number on the next card could be higher than 5 or lower than 5

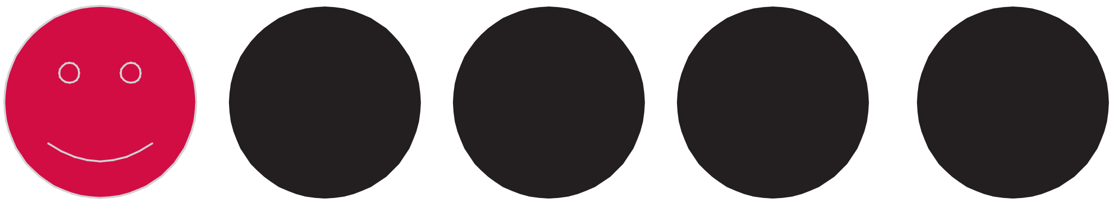
iv. Which is more likely? Tick the correct box below.

☐ Higher than 5 ☐ Lower than 5 ☐ Cannot tell

Explain your answer.

Q. 8 Lisa has some black counters and some red counters.

The counters are all the same size.
She puts 4 black counters and 1 red counter in a bag.



a. Lisa is going to take one counter out of the bag without looking.

She says:

> There are two colours, so it is just as likely that I will get a black counter as a red counter.

i. Explain why Lisa is wrong. What is the probability that the counter she takes out is black?

ii. How many more red counters should Lisa put in the bag to make it just as likely that she will get a black counter as a red counter?

b. Jack has a different bag with 8 counters in it. It is more likely that Jack will take a black counter than a red counter from his bag.

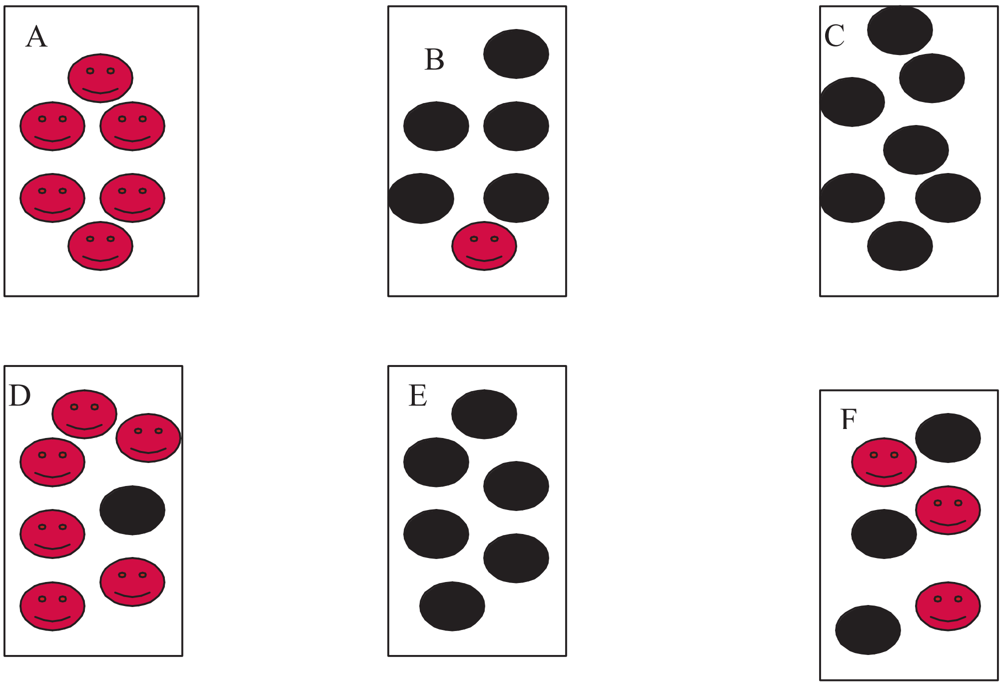
iii. How many black counters might there be in Jack's bag? Suggest a number and explain why this is a possible answer.

c. Jack wants the probability of taking a black counter from his bag to be the same as the probability Lisa had at the start of taking a black counter from her bag, so he needs to put extra counters into his bag.

iv. Assuming Jack had the number of black counters you have suggested at (iii) above, how many extra black counters and how many extra red counters (if necessary) should Jack put in his bag?

Explain your reasoning.

Q. 9 (a) Josh has some boxes containing red and black counters.



He is going to take a counter from each box without looking.

a. Match boxes (using the letters A-F) to the statements below. Explain your reasoning each time.

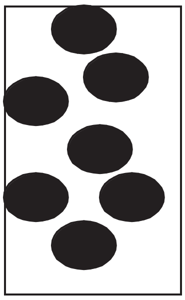
It is **impossible** that Josh will take a black counter from box........because

It is **equally likely** that Josh will take a black or red counter from box........because

It is **likely** that Josh will take a red counter from box............because

It is **certain** that Josh will take a black counter from box...........because

Josh selects box C which has 7 black counters in it



He wants to make it **more likely** that he will take a red counter than a black counter out of the box.

How many red counters must he put into the box? Explain your answer.

b. In another box, there are 30 counters which are either red or black in colour.
It is **equally likely** that Josh will take a red counter or a black counter from the box.
How many red counters and how many black counters are there in the box?

c. Extension question
There are 40 counters in a box which are either red or black in colour.
There is a **75% chance** that Josh will take a red counter from the box.
How many black counters are in the box? Explain your answer.

PROBABILITY 2

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- estimate probabilities from experimental data; appreciate that if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability
- associate the probability of an event with its long run relative frequency

INTRODUCTION

The activities described below and the questions that follow give you the opportunity to reinforce your understanding of the basic concepts of probability. You begin by rolling two coins and progress to playing a game involving rolling two dice. You will use a sample space to list all the possible outcomes and begin to consider the concept of expected value as you investigate the idea of fairness in relation to the game.

Activity 2.1

Toss two coins simultaneously about 30 times and record all the outcomes.
Do you notice any outcomes coming up over and over again?
Do some of these come up more frequently than others?
Use the grid below to show the 4 possible outcomes (the sample space) of heads (H) and tails (T).

		Coin 1	
		H	**T**
Coin 2	**H**		
	T		

Use the sample space to calculate the probability of each outcome occurring (i.e. the theoretical probability).
From the results you obtained in the 30 tosses, construct a table showing the number of times each outcome occurred and its relative frequency. Compare these to the theoretical probability.

Outcome	Tally	Relative Frequency

Activity 2.2

Working in pairs, roll a die 30 times (i.e. 30 trials) and enter your results into a table similar to the one outlined below

Number which appears on die (outcome of trial)	**How many times did this happen?** (Use tally marks to help you count.)	**Total** (frequency)
1		
2		

As you complete your own table compare it with that of another group.
Are there any similarities?

Your teacher may ask you to complete a Master sheet showing the results of all the groups in the class (a total of N trials).

Outcome of trial	**Frequency (group results)**	**Total of frequencies**	**Relative frequency)** $\frac{\textit{Total of frequencies}}{\textit{sample size (N)}}$	**% of total scores Rel. Freq × 100**	**Probability**
1	E.g. 5+6+5+…				
2					
3					
4					
5					
6					
		SUM			

The sum of all the relative frequencies is

The sum of all the percentages is

The sum of all the probabilities is

Conclusion:

What does your experiment tell you about the chance or probability of getting each number on the die you used?

Your die can be described as being unbiased. Can you explain why?

Activity 2.3

a. Each student tosses a coin 30 times and records their results for every 10 tosses.

No of tosses	No of Heads	Relative frequency
10		
10		
10		

b. What does the table you completed in (a) tell you about the probability of getting a head?

c. Now put all the results for the class together and obtain a new estimate of the probability of getting a head.

d. Is your new estimate closer to ½ than the estimate in (a)?

Activity 2.4

This is a game for two players, A and B. They take turns to roll two dice and add the two numbers shown on each toss. The winner is determined as follows:

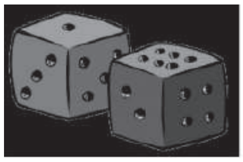
A wins if the sum of the numbers on the dice (i.e. outcome) is 2, 3, 4, 10, 11 or 12.

B wins if the sum of the numbers on the dice is 5, 6, 7, 8, 9.

Before you begin predict which player is most likely to win.

I think player ______ will win because ______________________

Play the game until one player reaches the bottom of the game sheet.



GAME SHEET

	A wins	**A wins**	**A wins**	B wins	B wins	B wins	B wins	B wins	**A wins**	**A wins**	**A wins**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

Record the number of times each player wins in the table below. The relative frequency is the **total no. of wins divided by the total no. of games.**

	Total (frequency)	**Relative frequency**
Player A wins		
Player B wins		
Totals		

As a class exercise construct a Master Tally sheet and record the results of the whole class

	Total (frequency)	**Relative frequency**
Player A wins		
Player B wins		
Totals		

Does your predicted result agree with your actual result? Think about why this happens.
Complete the table below showing all the possible outcomes for throwing two dice.

	1	**2**	**3**	**4**	**5**	**6**
1	(1,1)					
2						
3			(3,4)			
4					(4,6)	
5						
6						

In the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes.

Construct a table to show the probability of each outcome above, with the probability = $\frac{\text{no of outcomes in the event}}{\text{no of outcomes in the sample space}}$

Sum of two dice	Frequency	Probability
2	1	1/36
3	2	2/36

Look back at the rules of the game.

Original Rules: Player A wins when the sum is 2, 3, 4, 10, 11 or 12.

Player B wins when the sum is 5, 6, 7, 8 or 9.

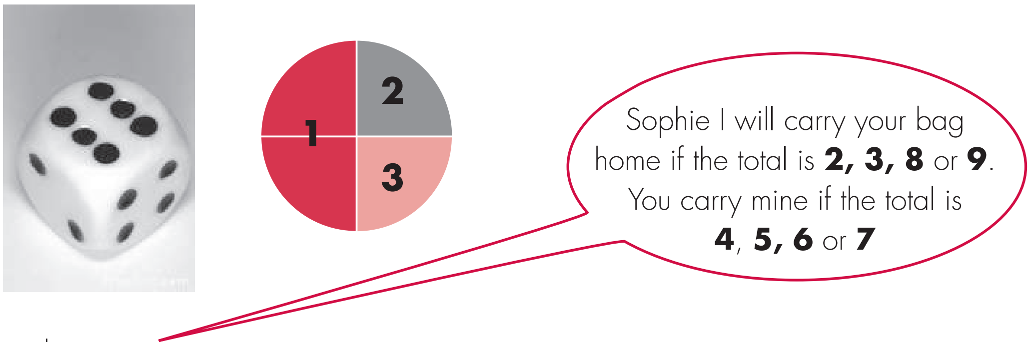
For how many outcomes will player A win?_______

For how many outcomes will player B win?_______

Does the game seem fair? If not, suggest a change to the rules which would make it fairer.

Create a mind map or a graphic organiser (http://www.action.ncca.ie) that will help you remember how to calculate the relative frequency of an event occurring.

Q. 1 Sophie and Andrew are playing a game with a fair, six-sided die and the spinner shown. They throw the die and spin the spinner simultaneously and note the total



Andrew said

Create a sample space showing the possible outcomes and use it to help Sophie decide whether or not she should play the game. Justify your advice to Sophie.

Q. 2 What is the probability of getting a head and a 6 when you simultaneously toss a fair coin and roll a fair, six-sided die?

How would this probability change if the die was replaced with:

a. A four-segment spinner (segments of equal area) numbered 1, 6, 6, 5?

or

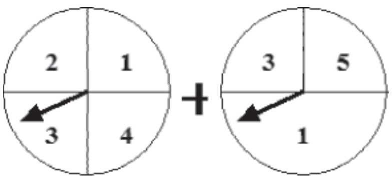
b. A suit of spades from a deck of playing cards (and 1 card is chosen at random from the suit)?

Q. 3 A spinner has four unequal sections, red, black, pink and grey.

The probability that the spinner will land on red is 0.1 [P(red) =0.1]
The probability that the spinner will land on black is 0.2 [P(black) =0.2]
The probability that the spinner will land on pink is the same as the probability that it will land on grey.

Calculate the probability that the spinner will land on grey. Justify your answer.

Q. 4 Design a spinner that simulates in 1 spin the sum of the outcome from spinning the two spinners below. Explain and justify your design.



Q. 5 A calculator can be used to generate random digits. Sandra generates 100 random digits with her calculator. She lists the results in the table below.

0	𝍸 II	**5**	𝍸 𝍸
1	𝍸 III	**6**	𝍸 𝍸 IIII
2	𝍸 I	**7**	𝍸 𝍸 II
3	𝍸 𝍸 II	**8**	𝍸 𝍸 I
4	𝍸 𝍸 II	**9**	𝍸 𝍸 IIII

Based on Sandra's results, estimate the probability that the calculator produces:
a) 9, b) 2, c) a digit that is a multiple of 3, d) a digit that is prime.

Q. 6 Four students each threw 3 fair dice.

They recorded the results in the table below.

Name	Number of throws	All different numbers	Exactly 2 numbers the same	All 3 numbers the same
Jane	50	36	12	2
Paul	150	92	45	13
Tom	40	18	20	2
Patti	120	64	52	4

a. Which student's data are **most likely** to give the best estimate of the probability of getting

All numbers the same ☐ Exactly 2 numbers the same ☐ All 3 numbers the same ☐

Explain your answer.

b. This table shows the students' results collected together:

Number of throws	All different	Exactly 2 numbers the same	All 3 numbers the same
360	210	129	21

Use these data to estimate the **probability** of throwing numbers that are **all different**.

c. The theoretical probability of each result is shown below:

	All Different	2 the same	All the same
Probability	5/9	5/12	1/36

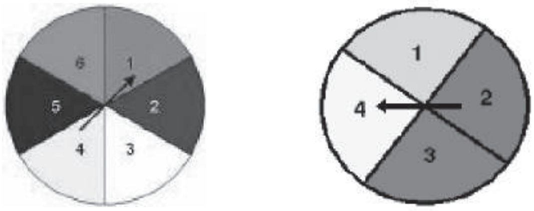
Use these probabilities to calculate, for 360 throws, **how many times** you would theoretically expect to get each result. Complete the table below.

Number of throws	All different	2 the same	All the same
360			

d. Give a reason why the students' results are not the same as the theoretical results.

Think: How would this question be different if coins, spinners or playing cards were used?

Q. 7 Pierce and Bernie were investigating results obtained with the pair of spinners shown.



They used a table to record the total of the two spinners for 240 trials. Their results are given in one of the three tables A, B and C below.

Table A

Sum	Frequency	Relative frequency
2	10	1/24
3	20	1/12
4	30	1/8
5	30	1/8
6	60	1/4
7	40	1/6
8	20	1/12
9	20	1/12
10	10	1/24
Total	240	1

Table B

Sum	Frequency	Relative frequency
2	12	12/240
3	12	12/240
4	27	27/240
5	27	27/240
6	35	35/240
7	45	45/240
8	24	24/240
9	18	18/240
10	40	40/240
Total	240	

Table C

Sum	Frequency	Relative frequency
2	11	
3	19	
4	32	
5	30	
6	29	
7	28	
8	17	
9	14	
10	60	
Total	240	

Complete the relative frequency column in table C.
Use your results to decide which, if any, of these three tables might represent the results found by Pierce and Bernie. Explain your reasoning.

Q. 8 A spinner with 3 equal segments numbered 1, 2 and 3 is spun once.

i. Give the sample space of this experiment.
ii. What is the probability that the spinner stops on number 2?
iii. What is the probability that the spinner stops on a number greater than or equal to 2?

Q. 9 Pierce and Bernie were investigating the results given by the spinner shown, by spinning it 60 times and recording the results.

Their results are given in one of the three tables below, A, B and C

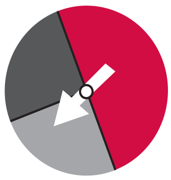


Table A

result	tally	count
red	卌 卌 卌 卌 卌 I	21
grey	卌 卌 卌 IIII	19
black	卌 卌 卌 卌	20

Table B

result	tally	count
red	卌 卌 卌 卌 卌 卌 卌 卌 卌 II	47
grey	卌 I	6
black	卌 II	7

Table C

result	tally	count
red	卌 卌 卌 卌 卌 卌 II	32
grey	卌 卌 卌	15
black	卌 卌 III	13

a. Which of the three tables above is most likely to be like the one that Pierce and Bernie made? Explain how you made your decision.

b. For each of the other two tables, draw a diagram of a spinner that is likely to produce results like those shown in each table.

The following questions represent an application of the concept of probability. They are most suited for LCHL. They require you to have an understanding of the fact that probability is a quantity that gives a measure on a scale of 0-1 (0-100%) of how likely an event is to occur. They also require you to display an understanding that, if an experiment is repeated, there will be different outcomes and increasing the number of times an experiment is repeated generally leads to better estimates of probability. Discussing such probability scenarios with your peers in class allows you to make sense of probability and its application to real life phenomena. When explaining your answers you are displaying your competency in the key skill of communication.

Q. 10 Ten patients with HIV are each given a new treatment which the specialist says has a 30% chance of completely curing them.

What can each patient deduce from this?

Q. 11 The weather forecast for tomorrow states there is a 25% chance of rain in Leinster.

What exactly does this statement mean?

PROBABILITY 3

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- apply the principle that in the case of equally likely outcomes the probability is given by the number of outcomes of interest divided by the total number of outcomes
- use binary/counting methods to solve problems involving successive random events where only two possible outcomes apply to each event
- discuss basic rules of probability (AND/ OR, mutually exclusive) through the use of Venn Diagrams
- find the probability of intersection of two independent events

HL learners will

- extend your understanding of the basic rules of probability (AND/ OR, mutually exclusive) through the use of formula (addition rule):

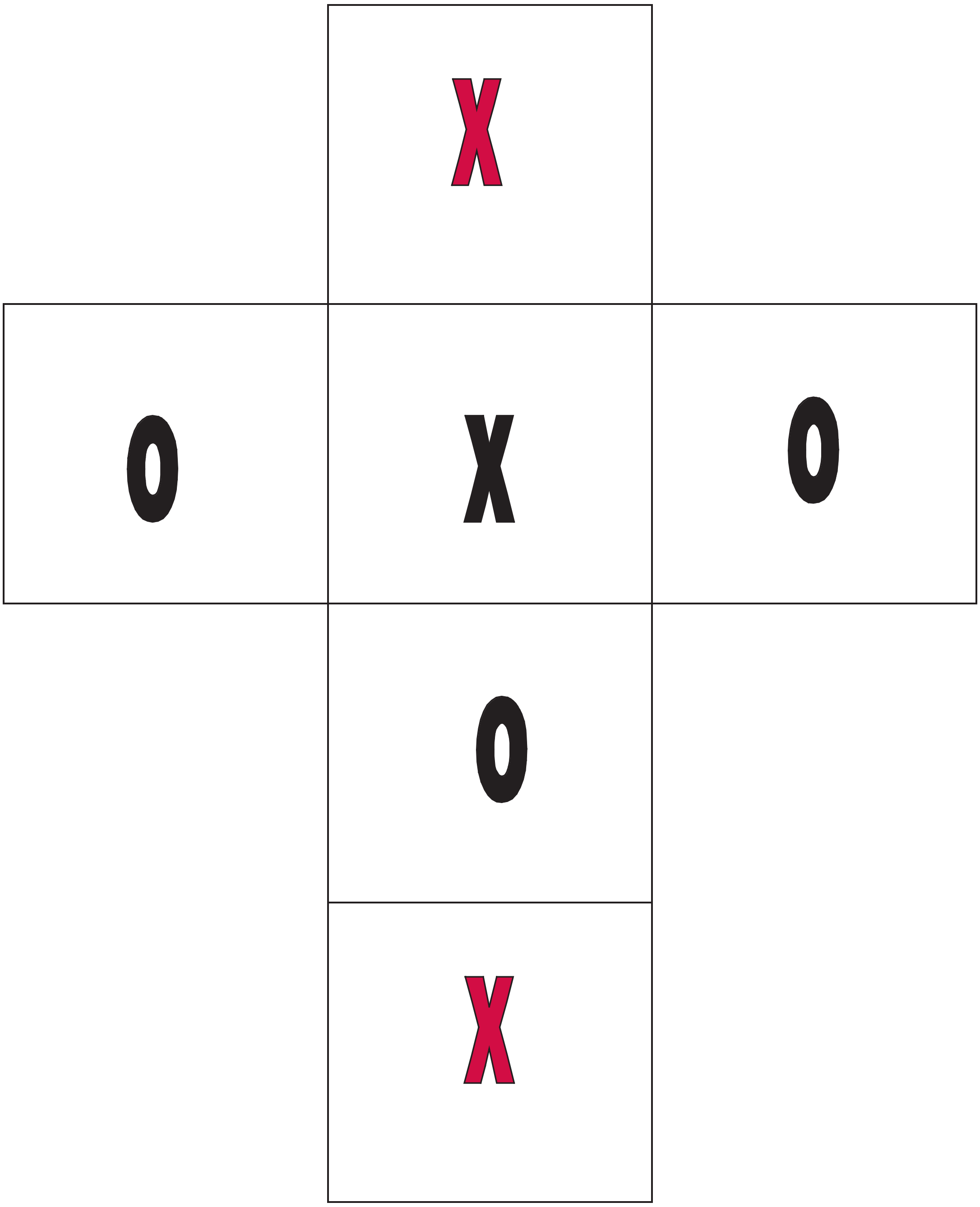
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

INTRODUCTION

The activities described below are intended for JC HL, LC OL and LC HL students.
The questions that follow the activities allow you to construct an understanding of outcome spaces and of the 'and' and 'or' rules of probability and use them to solve and compose problems.

Activity 3.1

Fold the net shown overleaf to form a cube.
Roll the cube; there are 6 possible outcomes.
Complete the boxes to show the 6 possible outcomes



a.

i. What is the probability of getting a cross?

ii. What is the probability of getting a circle?

iii. What is the probability that you will get a red symbol?

iv. What is the probability that you will get a black symbol?

Explain how you arrived at your answer.

b. Let's take the probability of getting a cross.

i. What will happen to this probability if I say the cross has to be black?

ii. What will happen to this probability if I say the cross has to be red?

How do these answers relate to your answer in (a) (i)?

c. Now look at the probability of getting a circle.

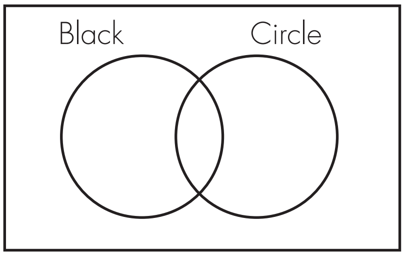
i. What will happen to the probability if I say the circle has to be red?

ii. What will happen to the probability if I say the circle has to be black?

If two events cannot possibly happen together we say they are ***Mutually Exclusive***

d. What events are mutually exclusive in the example above?

Represent this situation in the Venn diagram



We can write the questions in mathematical language

P (Closed ∩ Cross) means the probability of the symbol being closed AND a cross

Write mathematical sentences for the following and give the answer in each case

i. The probability of the symbol being black AND a circle

ii. The probability of the symbol being red AND a circle

iii. The probability of the symbol being red AND a cross

e. What about OR?

Think about this question: What is the probability of the symbol being black OR a circle? Can you see this in the Venn diagram above?

> [HL] Use the Venn diagram to help you write an equation for P(Black ∩ Circle) in terms of the P(Black) and P(Circle)

f. Now think about the probability of the symbol being black OR a cross P(Black U Cross)

Can you see this in the Venn diagram?

> HL Use the Venn diagram to help you write an equation for P (Black U Cross) in terms of the P(Black) and P(Cross)

Activity 3.2

a. Suppose you were asked to randomly choose one number from the numbers 1 to 10.

i. What is the probability that the number you choose would be greater than 5?

ii. What is the probability that the number you choose would be even?

iii. What is the probability that the number you choose would be greater than 5 AND even?

iv. What is the probability that the number you choose would be greater than 5 OR even?

b. Write the numbers 1 to 10 on ten pieces of paper. Now fold the pieces of paper so that you can no longer see the numbers on each piece. Working in pairs, place the pieces of paper in a hat or a bag and ask a classmate to choose a piece of paper from the hat/bag. Get your classmate to record the number they chose in a table similar to this.

Number	1	2	3	4	5	6	7	8	9	10
Frequency										

Place the piece of paper back into the bag/hat and stir the pieces of paper around. Now repeat this experiment until you have done it 20 times and combine the results from the other pairs in the class.

i. How many times was a number greater than 5 chosen?
ii. How many times was an even number chosen?
iii. How many times was the number chosen greater than 5 AND even.
iv. How many times was the number chosen greater than 5 OR even.

Use these answers to compute relative frequencies for each of the events listed above and compare these relative frequencies with the probabilities you obtained earlier. Are they the same?

You may like to look back at **Probability 2** to remind yourself how relative frequencies are calculated.

What do you expect would happen to the relative frequencies if you were to repeat this experiment one million times?

The following questions provide you with the opportunity to

- connect with JC set theory to discuss experiments, outcomes, sample spaces
- discuss basic rules of probability (AND/ OR, mutually exclusive) through the use of Venn diagrams

You will need to recall your work on the binary operations of addition and subtraction of fractions

Q. 1 In a class 1/2 of the pupils represent the school at winter sports and 1/3 represent the school at summer sports and 1/10 at both. Draw a Venn diagram to represent this. If a pupil is chosen at random, what is the probability that someone who represents the school at sport will be selected?

Q. 2 In a certain street 1/5 of the houses have no newspaper delivered, 1/2 have a national paper delivered and 1/3 have a local paper delivered. Draw a Venn diagram to represent this information and use it to find the probability that a house chosen at random has both papers delivered.

Q. 3 Consider the following possible events when two dice, one red and one green, are rolled.

A: The total is 3
B: The red is a multiple of 2
C: The total is ≤ 9
D: The red is a multiple of 3
E: The total is ≥ 11
F: The total is ≥ 10

a. Which of the following pairs are mutually exclusive? Discuss your choice with a classmate.

(i) A,D (ii) C, E (iii) A ,B (iv) C,F (v) B,D (vi) A,E

b. Write a definition explaining what is meant by saying that two events are mutually exclusive.

Q. 4 The probability that Josh will be in the school football team is 0.6, the probability that he will be in the hurling team is 0.5, and the probability that he will be in both the hurling and football teams is 0.3. His father says he will buy him a bicycle if he makes any of the school sport teams and he plays only those two sports at school. What is the chance that he gets a bicycle from his dad?

Note to HL learners: Revisit the questions above and try answering them with the formula.

Activity 3.3

Consider the following game

Players roll 2 four-segment spinners, which have equal segments numbered 1, 2, 3 and 4 . Player 1 wins if the sum of the spinner numbers is 3, 4, or 5,; player 2 wins if the sum is 2, 6, 7, or 8.

a. Predict whether player 1 or player 2 has the greater chance of winning. Play the game a few times to check your prediction. Now use the table below to help you decide in a more mathematical way. Write a sentence explaining why you think the game is, or is not, fair.

	1	2	3	4
1				
2				
3				
4				

b. Now consider this game

Players roll 3 four-segment spinners, which have equal segments numbered 1, 2, 3, and 4. Player 1 wins if the sum of the spinner numbers is 3, 4, 5, 6 or 12; Player 2 wins if the sum is 7, 8, 9, 10 or 11.

Is this game fair?

Can you represent the possible outcomes in the same way? It is difficult because there is an extra dimension – the 3rd spinner.

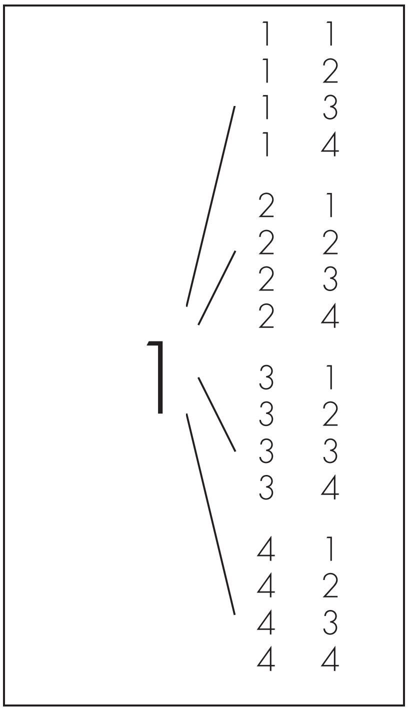
Consider all the possibilities when the first spinner shows a 1.

This is only ¼ the total number of outcomes and the process of completing the rest gets very repetitive.

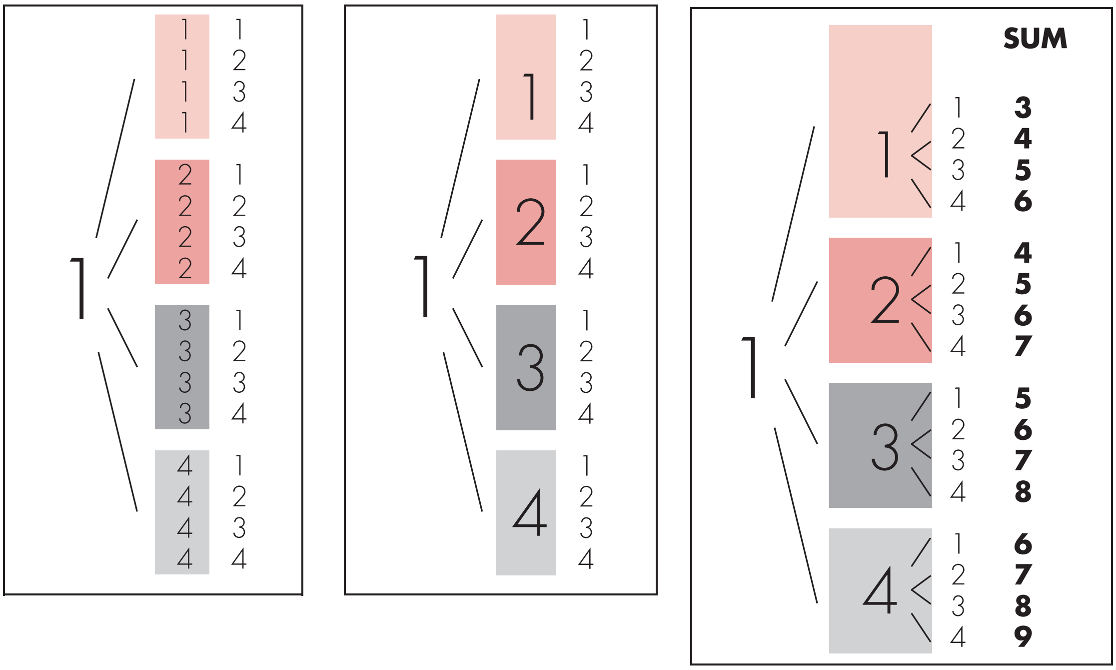
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
1	2	2
1	2	3
1	2	4
1	3	1
1	3	2
1	3	3
1	3	4
1	4	1
1	4	2
1	4	3
1	4	4

We could get rid of the repetitions by replacing the first column of 1's with 1 big 1.

1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
1	2	2
1	2	3
1	2	4
1	3	1
1	3	2
1	3	3
1	3	4
1	4	1
1	4	2
1	4	3
1	4	4



Can you get rid of any more repetitions?



Can you see a pattern forming ?
This is called a tree diagram; can you see why? Can you see how the required outcome (sum of the three spinners) is calculated for each 'branch' of the 'tree'?

i. Draw tree diagrams showing the possible outcomes when the first spinner shows 2, 3, and 4.
ii. How many possible outcomes are there? Now use your diagrams to decide if the game is fair (see the rules at the start).

This is how one student explained why tree diagrams are very useful when counting outcomes such as in this question:

Well, tree diagrams are useful for counting the total number of outcomes. There are four 'trunks' (for the possible numbers on the first spinner), and each has four 'branches' (for the possible numbers on the second spinner), and each has four 'twigs' (for the possible numbers on the third spinner). An outcome is formed as we go from a trunk to a branch to a twig. There are as many outcomes as there are twigs: 4x4x4=64.

c. Draw a tree diagram showing the number of possible outcomes when three coins are tossed

Could you have answered this question without drawing the tree diagram?

Explain

i. Use your tree diagram to answer the following

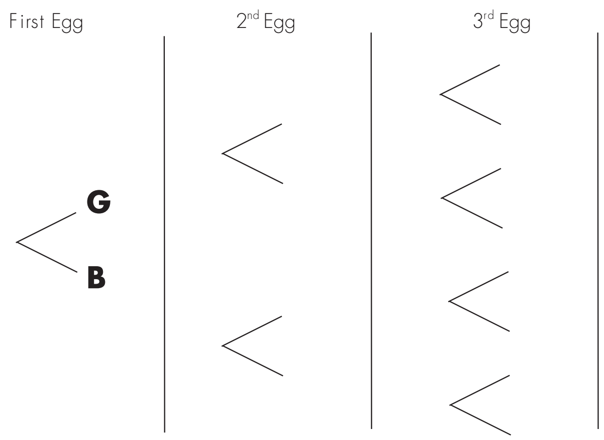
P (All tails) =

P (All heads) =

ii. Predict the number of possible out comes when two coins are tossed and 1 die is rolled. Check your prediction by drawing a tree diagram.

Q. 3 There are a dozen eggs in a box and 3 of them are 'bad'. 3 eggs are chosen at random from the box.

a. Complete the probability tree diagram below, showing good (G) and bad (B) eggs.



b. Work out the probability that

i. all three eggs are 'good'

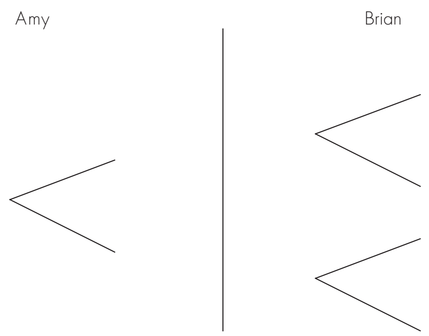
ii. 1 egg is 'bad'

iii. 2 eggs are 'bad'

iv. all three eggs are 'bad'

Q. 4 Jessica is taking part in a quiz. She is unsure of the answer to a question and needs to ask her team-mates, Amy and Brian. The probability that Amy will get it right is 0.7.The probability that Brian will get it right is 0.4.

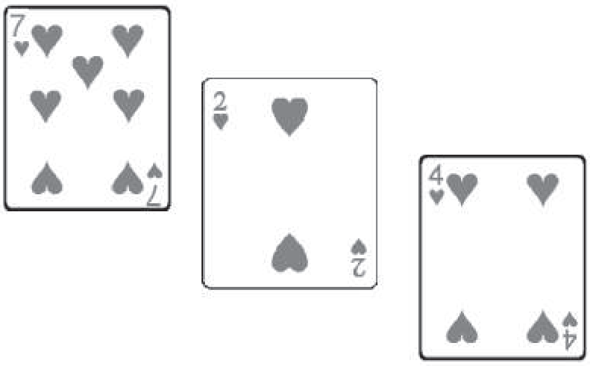
a. Complete the probability tree diagram below.



i. What is the probability that at least one of her two friends will give her the correct answer?

ii. What is the probability that neither of them will give her the correct answer?

Q. 4 John and Sophie each have three cards numbered 2, 4 and 7. They each select one of their own cards. They then add together the numbers on the four remaining cards. What is the probability that their answer is an even number? Explain how you arrived at your answer.



Note to LC-HL learners: Revisit the question and answer it without the aid of a tree diagram.

Q. 5 Suppose that every child that is born has an equal chance of being born a boy or a girl.

i. Write out the sample space for the situation where a mother has two children.

ii. What is the probability that a randomly chosen mother of two children would have two girls?

iii. What is the probability that this mother of two children would have two boys?

iv. What is the probability that this mother of two children would have one boy and one girl?

Q. 6 Consider the situation where a die is rolled once; list the outcomes in the sample space.

The following events are defined:

A: a number greater than 2 but less than 5 is obtained.

B: an odd number is obtained.

i. Give the Venn diagram that depicts this situation.

ii. Find the following probabilities:

(a) P(A) (b) P(B) (c) $P(A \cap B)$ (d) $P(A \cup B)$ (e) $P(A^C)$

iii. Give a full account of what you understand by P(A) and $P(A \cup B)$.

Q. 7 A company that makes baby food tins finds the probability of producing a tin with a scratch on it to be 0.04, the probability of producing a tin with a mark on it is 0.07, and the probability of producing a tin with a scratch and a mark on it is 0.03.

a. Define events

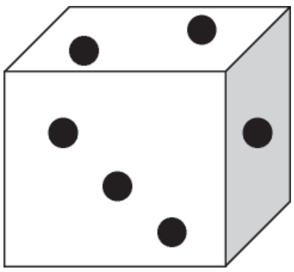
M: Tin produced with mark on it
S: Tin produced with scratch on it

i. Are these events mutually exclusive?

ii. Find the probability of producing a tin with a mark OR a scratch on it.

iii. How many tins, out of every 1000 produced, are expected to be scratched OR have a mark on them?

Q. 8 Judith is playing a game in which she moves a marker along a circular track. She uses a coin and a six-sided die to decide the steps she takes at each move.



She tosses the coin to see whether she moves forwards (if a head is shown) or backwards (if a tail is shown) and then rolls the die once to get the number of steps to be moved.

a. Complete a sample space to show all possible outcomes.

b. What is the probability that Judith will have to move

i. backwards fewer than 3 steps?

ii. forwards more than 4 steps?

Show how you arrived at your answer in each case by showing the steps involved.

Q. 9 A spinner with three equal segments numbered 1, 2 and 3 is spun twice and the number obtained each time is noted.

i. Draw a tree diagram to show all possible outcomes of the experiment.

ii. Write out the sample space of this experiment.

iii. Find the probability that the spinner stops on an odd number both times it is spun.

iv. Find the probability that the spinner stops on an odd number the first time it is spun and an even number the second time it is spun

Q. 10 For each of the pairs of events, A and B, determine whether the events are mutually exclusive and/or independent.

Event A	Event B	Mutually Exclusive	Independent
First born child is male	Second born child is female		
Coin displays head upon landing	Coin displays tail upon landing		
Die shows even number when tossed	Die shows number > 3 when tossed		
Card drawn is black	Card drawn is 3 of Clubs		
Ball chosen from bag containing different coloured balls	Second ball chosen from same bag without replacement of first ball		
First-born child has red hair	Second-born child has red hair		
Student will get A in mathematics exam	Student will get A in Irish exam		
Father is Bald	Eldest son will become bald		
Person abuses drugs	Person is left handed		
Person smokes cigarettes	Person dies of lung cancer		
Parent is a twin	Child is a twin		
Traffic Jam occurs on M50 between time T and T+1hour	Traffic jam occurs on the M50 between the time T+1hour and T+2hours		
Flooding occurs on July 2nd this year in Galway	Flooding occurs on July 2nd this year in Mayo		
Football team wins FAI cup	Football team wins League of Ireland Championship		
Player wins GAA hurling All Ireland medal	Player wins GAA camogie All Ireland medal		
Country possesses nuclear technology	Country is member of European Union		
A fair coin is tossed and 99 times in a row the coin displays heads	The 100th fair coin toss displays a head		

PROBABILITY 4

SYLLABUS TOPIC: COUNTING AND CONCEPTS OF PROBABILITY

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- list outcomes of an experiment
- apply the fundamental principle of counting
- count the arrangements of n distinct objects (n!)
- count the number of ways of selecting r objects from n distinct objects
- count the number of ways of arranging r objects from n distinct objects

INTRODUCTION

The activities are designed to help you build up an understanding of the concepts of arranging and combining numbers. Some of the activities will be done in class under the direction of your teacher; others can be done at home. Your answers to the questions will give you and your teacher an indication of how well you have understood the concepts.

Activity 4.1

Choose 7 classmates, friends or family members and ask them to stand in a row facing you. Write the numbers 1 to 7 on seven separate pieces of paper. One way to hand out the seven numbers to the seven people is to give the person on the left of the row the number 1 and the next person the number 2 and so on until the person on the right end of the row receives the number 7.

Alternatively you could hand out the numbers in decreasing order from left to right so that the person on the left receives the number 7 and the person on the right end receives the number 1.

This gives us two different **assignments** of numbers.

i. How many different assignments of the seven numbers are possible?

Make a table to help organise your work. Can you see any patterns in your work?

Place the seven pieces of paper in a bag and moving along the row of seven people ask each person in turn to choose one piece of paper without looking at the pieces of paper.

ii. What is the probability that the numbers were assigned in perfectly increasing order from 1 to 7 from the left of the row to the right of the row?

Ask everyone to show you the number that they got.

iii. Did the numbers line up perfectly in an increasing order from left to right?

Now choose the three tallest of the seven people and ask them to stand to one side in a new row.
Take the three pieces of paper from the three people.

iv. Are the numbers 1, 2 and 3 on the three pieces of paper?

v. Is it possible that the three pieces of paper could have the numbers 1,2 and 3?

vi. Is it possible they could have had the numbers 4,5 and 6?

vii. Is it possible that all three pieces of paper could have odd numbers?

viii. Is it possible all three could have even numbers?

ix. In total how many different sets of numbers could be on the pieces of paper that the group of three people have?

Ask each of the three people to choose one of the three pieces of paper that you took from them. (Make the three people close their eyes when they are choosing, so that their choices are random.)

x. Did the three people choose the same numbers that they had originally chosen?

xi. How many ways could the three people choose from among the three numbers?

Q. 1 Considering the experiment you have just performed, answer the following questions.

xii. How many ways can three people choose three numbers from the numbers 1-7 if all you are interested in is the set of three numbers that are chosen and you are not interested in who received which number? (LCOL)

xiii. How many ways can three people choose three numbers from seven numbers taking into account that you now do care about who receives which number? (LCHL)

Q. 2 Supposing now that instead of the numbers 1-7 we had the numbers 1-42,

i. how many ways could we choose three numbers from the forty two numbers assuming that the order of the three numbers is irrelevant?

ii. how many ways could we choose six numbers from the forty two numbers assuming the ordering of the numbers is irrelevant?

Activity 4.2

a. Get a pack of playing cards and remove the two jokers from the pack. How many cards are left in this pack? Choose a card at random from the pack and record on a piece of paper what the card is then replace the card in the pack and shuffle the cards. Repeat this until you have selected 100 cards in total.

i. Compute the relative frequency of cards that were chosen which were red cards.

ii. Compute the relative frequency of cards that were chosen which were hearts.

iii. Compute the relative frequency of cards that were chosen which were both red and hearts.

You may find it helpful to create a table in which to record your results. Look back at previous tables you have used.

b. You are now going to calculate some theoretical probabilities. Each time we calculate a probability we must first determine the sample space.

Consider choosing 1 card from the above pack of cards.
What is the sample space for this experiment? How many elements are in the sample space?
Now consider some different events that we could choose from this sample space.

i. How many ways can you choose a heart from the pack of cards?

ii. How many ways can you choose a red card from the pack of cards?

Dividing the number of elements in each event by the number of elements in the sample space compute, the probability of

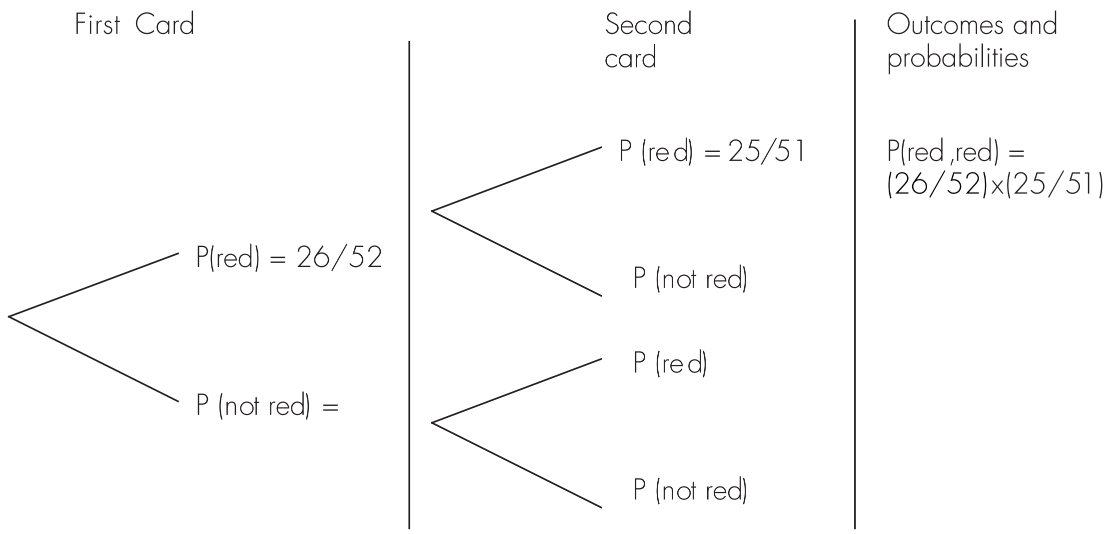
iii. choosing a heart from the pack of cards

iv. choosing a red card from the pack of cards

Activity 4.2A

So far you have chosen only 1 element from the sample space. Now you will progress to more challenging examples where you are required to choose more than 1 element from the sample space

Complete the tree diagram and use it to find the probability of choosing two red cards from a pack of 52 cards. Now realising that cards that are not red are black, what is the probability of choosing two black cards?



What is the probability of choosing one red card followed by one black card?
What is the probability of choosing one black card followed by one red card?
Hence, by adding probabilities, calculate the probability of choosing one red and one black card from a pack of 52 cards.

Activity 4.2B

You can approach this question a different way.
Consider choosing 2 cards from the above pack of cards. Think about the sample space for this experiment. How many elements are in the sample space? (In other words, how many ways can you choose 2 objects from a set of 52 objects? In mathematical language this is written as ${}^{52}C_2$.)
Now consider some different events that we could choose from this sample space.

> If you are having difficulty answering this question try to simplify it by using smaller amounts of cards e.g. how many ways can you choose 2 cards from a pack of 3 different cards (${}^{3}C_2$) or how many ways can you choose two letters from A, B, C (${}^{3}C_2$)?. In this way you may see some patterns in your work that will help when you use larger numbers.

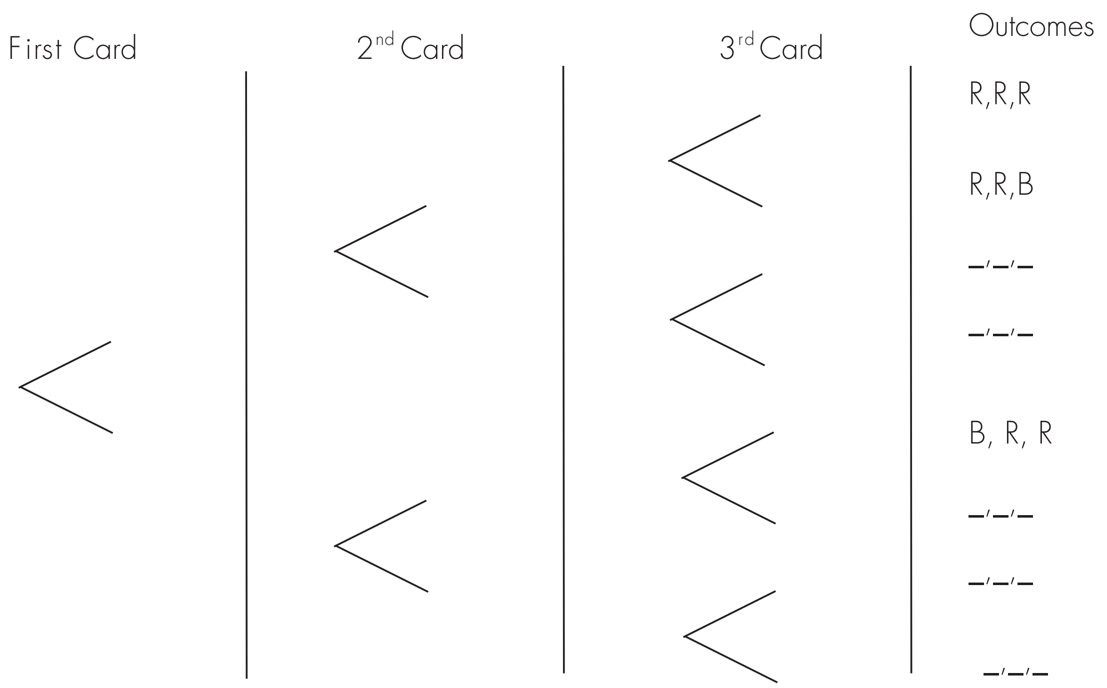
i. How many ways can you choose two red cards from the pack of cards? (When answering this question first consider how many red cards there are and then consider how many ways to choose two of these)
ii. Dividing the number of elements in this event by the number of elements in the sample space compute the probability of choosing two red cards from the pack of cards

> You have now answered the same question in two different ways; compare this answer to the answer that you obtained earlier.
>
> Show that the answers you got from these two methods are actually the same answer (you will need to review the definition of combinations and remember how to divide fractions).
>
> Which approach to answering this question did you find the most helpful?

Now, by counting the number of elements in the events and in the sample space as above what is the probability of choosing two black cards? Compare this answer with the one you got using the tree-diagram approach.

Activity 4.2C

We have seen questions involving choosing one and then two cards from a pack of cards. Now imagine choosing three cards from a pack of 52 cards. Using the tree diagram approach find the probability of choosing three red cards from a pack of 52 cards.



Now consider answering this question by counting the number of elements in the sample space and counting the number of elements in the event (i.e. that there are three red cards). Firstly, how many ways can you choose three cards from a pack of 52 cards? This is the size of your sample space. Now how many ways can you choose three red cards from 26 red cards? This is the number of elements in your event. Now divide the number of elements in the event by the number of elements in the sample space; what is your answer? This is the probability of choosing three red cards from the pack of 52 cards.

Show that the answers you got using the tree diagram method and the sample space method are identical.

Now consider choosing two red cards and one black card from the pack. How many ways can you do this?

Look back at your work from the previous section.
What is the probability of getting a black card first followed by two red cards?
What is the probability of getting a red card first followed by a black and then another red?
What is the probability of getting two red cards followed by a black card?
Think about the sample space for this experiment. How many elements are in the sample space? (In other words, how many ways can you choose 3 objects from a set of 52 objects? In mathematical language this is written as ${}^{52}C_3$.)
Now consider some different events that we could choose from this sample space.

Compare the answers you obtain from both approaches.

Q. 1 Consider choosing two cards from a pack of 52 playing cards.

i. How many ways can you choose two cards from the pack of cards?
ii. How many ways can you choose two Queens from the pack of cards?
iii. What is the probability of getting two Queens?
iv. What is the probability of getting one Queen and one King?
v. How many ways can you choose five cards from a pack of cards?
vi. What is the probability that you get four Kings and 1 queen in a hand of five cards?
vii. What is the probability that you get three Kings and Two Queens in a hand of five cards?

Construct a mind map or graphic organiser to help you remember how you
- count the arrangements of n distinct objects (n!)
- count the number of ways of selecting r objects from n distinct objects
- count the number of ways of arranging r objects from n distinct objects

Activity 4.3

Find out how the National Lottery game of Lotto is played.

a. How many different sets of six balls could possibly be chosen in the Lotto game?

> Can you see any similarity between how you approach this question and the work you have done previously. In activity 4.2 you saw two approaches to answering these questions How many branches would you have in your tree diagram? Is it wise to attempt to write out all the combinations? Is the tree diagram going to be too complicated for this example?

b. Suppose that you wrote down six numbers from among the set of numbers 1 to 45. What is the size of the sample space? (How many ways can you choose 6 numbers from 45 different numbers?)

c. What is the probability that the numbers you wrote down would be the exact same numbers chosen in next Saturday's Lotto game?

d. How many ways could you choose a set of five numbers from the six numbers that are chosen in the Lotto game next Saturday?

> This is complicated. Think how it relates to the card situation in activity 4.2. Consider that the red cards are the winning 6 numbers chosen in the draw. The black cards are the numbers not chosen in the draw. This gives you a pack of 45 cards with 6 of them red and 39 of them black.

e. How many numbers will NOT be chosen in the Lotto game on Saturday?

f. How many ways could you choose one number from among this set of numbers which will NOT be chosen in the Lotto on Saturday?

g. Combining the answers to the last three questions, can you establish how many ways EXACTLY five of the six numbers that you wrote down match with five of the numbers that will be chosen on Saturday's Lotto draw?

Suppose the Lotto game was structured so that you chose 6 numbers from the numbers 1-36. Would that be an easier game to win?

Can you calculate the probability of matching all 6 numbers in a Lotto game that contains 36 numbers? What about one that contains 39 numbers and one that contains 42 numbers?

Would it be good for people playing to decrease the number of numbers in the Lotto game from 45 to 36? Can you predict what would happen to the number of winners if this were to happen?

PROBABILITY 5

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- calculate expected value and understand that this does not need to be one of the outcomes
- recognise the role of expected value in decision making with a focus on fair games
- change the rules of a game to make it 'fair'
- invent games for your peers
- engage in discussions about why car insurance premiums are more costly for some

INTRODUCTION

The activities described below and the questions that follow allow you to construct an understanding of expected value and how the expected value can be used to inform decision making. You will recall your work on deciding whether a game is fair or not and extend this to understand how calculating the expected value facilitates these types of decisions. The activities are grounded in context so that you will not only appreciate how the mathematical concepts relate to real life situations; but also appreciate the usefulness of mathematics in gaming, the insurance and financial industry.

Prior knowledge

You will need to be able to use sample spaces and tree diagrams to calculate the probability of events occurring.

Note: Sometimes it is desirable to know the mean or average outcome of an experiment; in the long run this is called the **expected value**.

Read and reflect on the following definition

In probability theory the expected value of a discrete random variable is the sum of the probability of each possible outcome of the experiment multiplied by the outcome value. Thus, it represents the average amount one 'expects' as the outcome of the random trial when identical odds are repeated many times. Note that the value itself may not be expected in the general sense - the 'expected value' itself may be unlikely, or even impossible.

This has been summarised below

Expected Value calculations essentially involve taking all the possible outcomes, weighting the more likely outcomes, and coming to a conclusion. It is an average of all the likely outcomes.

We will consider some examples from real life where it is desirable to know the expected value. The most easily understood examples are from gaming. Consider the following

Ann and Barry each have a pile of sweets and they play a game:
Barry rolls a die. If he gets a 6, Ann gives him four sweets; if he gets a 1, he gives Ann two sweets; if he gets a 2, 3 or 4, he gives Ann one sweet; if he gets a 5, there's no exchange of sweets.

Is this a fair game, or does it favour one player over the other?
If it's not fair, how can it be changed to make it fair?
Make up some similar games that are fair /unfair

Think:

What constitutes a 'win' for Barry?

What constitutes a 'loss' for Barry?

Complete the **Before** section of the prediction sheet below

Before		**Statement**	**After**	
Agree	**Disagree**		**Agree**	**Disagree**
		1. Playing this game requires skill.		
		2. Barry will usually win when he plays this game.		
		3. Every time Barry plays this game, he will have the same chance of losing.		
		4. This is a fair game.		

What is the probability that Barry will win?

What is the probability that Barry will lose?

What is the 'value' for each outcome? For example, if he throws a 6 he will get four sweets (i.e. he is 'up' 4) so the value would be +4.

Complete the following table by listing all the possible outcomes, calculating the corresponding probabilities and inserting the 'value' for each outcome

Possible Outcome	Probability	Value

Look back at the definition try to work out how to calculate the expected value from the table.

......the expected value is the sum of the product of probability times the value....

So what is the expected value in this game?
What does this mean?
Is the game fair?
Who does it favour?
If the expected value was positive who would it favour?
If the game is fair what should the expected value be?

Refer back to the table; complete the **After** section. This will give you an indication of how well you have grasped the underlying concepts.

Would the game be fair if, when Barry throws a 5, Ann gives him one sweet?
Justify your reasoning

Reflect on how you calculated the expected value.

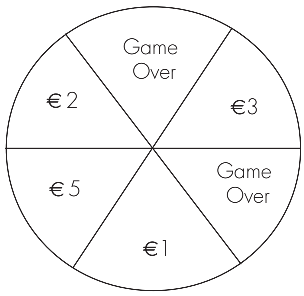
Create a mind map or a graphic organiser that will help you remember how to calculate the expected value (http://www.action.ncca.ie).

Q. 1 Spin and Win

Instructions

- one game costs €2 to play
- spin and win the amount shown
- Tick Agree or Disagree in the Before column beside each statement before you start to play the Spin and Win game.
- Revisit your choices at the end of the investigation.

Before		Statement	After	
Agree	Disagree		Agree	Disagree
		1. Playing this game requires skill.		
		2. You will usually win when you play this game.		
		3. Every time you play this game, you will have the same chance of losing.		
		4. This is a fair game.		

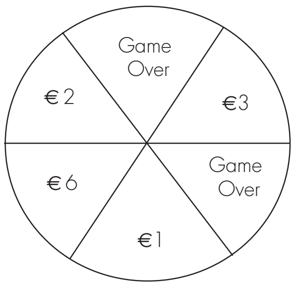


Think: How can what you have learned in the previous example help you make a decision about whether or not this game is fair?

Look back at how you assigned values in the previous example.
What is the difference in this game? Did Barry or Ann have to pay any sweets to play the game?
Do you have to pay to play *Spin and Win*? Will this affect the 'value' you assign to a particular outcome?
Is this a fair game? Justify your answer

Now return and complete the **After** column of your table.

b. Consider the following statements. What effect, if any, will they have on your decision about whether or not the game is fair? Justify your answer.
   i. The cost of the game is increased to €3
   ii. The following spinner was used



Return again to your graphic organiser and adjust it to take into consideration what you have learned from playing ***Spin and Win***

Q. 2 You and a friend are playing the following game:

Two dice are rolled. If the total showing is a prime number, you pay your friend €6. Otherwise, your friend pays you €2.

i. What is the expected value of the game to you?

Remember: ***expected value is a weighted average of the values of the outcomes. List all the outcomes; multiply the value of each outcome by the probability of that outcome, then add.***

ii. If you played the game 40 times, what are your expected winnings?

iii. After playing the game a while, you begin to think the rules aren't fair and you decide to change the game. How much (instead of €6) should you pay your friend when you lose so that your expected winnings are exactly €0?

Q. 3 A roulette wheel has 38 numbers; 18 black, 18 red and 2 green

a. Game1: *Bet on a Number*

Bet €1 on a number; if you are right you win €35 (and you get your €1 back). The outcome is +35 because you are 'up' €35. If you are wrong, you lose €1. What is your outcome?
Is this a fair game?

b. Game 2: *Bet On Black*

Bet €1 on a black number
If right win €1 (and get your €1 stake back). If wrong, you lose €1.
What is the expected value? Is this a fair game?

The following example is more challenging, since it doesn't refer to gaming.

Q. 4 Consider two family planning strategies.

(a) have children until you have a girl or (b) stop after two boys.
Would either strategy upset the male /female mix in the population? What do you think? Make a prediction.

If you calculate the **expected value** or mean number of boys for a large population of families and compare this with the **expected value** or mean number of girls for a large population of families you will be able to answer this question.
Discuss in your group why this information will help you decide if either strategy would upset the male/female mix.

Use probability trees to help find the probability of each outcome. Think about the possible outcomes
Girl first (what happens now?); Boy, Girl (what happens now?); Boy, Boy (what happens now?)
What's the average number of boys per family? Complete the probability column in the table below.

Now look at the definition of expected value again and see if you can calculate the expected value or the average number of boys per family. Can you complete the final column in the table?

Boys	Probability	Value × Probability
0		
1		
2		

So, the Expected Value, or average number of boys per family =
Hence, the mean number of boys per family for a large population of families =

Now in a similar way calculate the average number of girls per family.
Is there any difference in these values?

Decide whether either strategy will affect the mix of boys and girls in the population?
Was your prediction correct?

Thoughts
Think about how bookmakers and casinos ensure that they make a profit – by making sure that the expected return for the player is always lower than the stake. Players will of course occasionally win, but the house wins in the long run (average over all players and all games).

Insurance is mathematically identical to gambling against a casino. For example, when you pay your life insurance premium each month, you're betting that you're going to die this month and the insurance company is betting that you won't! In game-theory terms, insurance is a bet that does not favour the player, which leads to the question about why you would have it, and this could lead on to the idea of 'risk'. There's more to decision-making than just looking at the expected value. The bottom line here is that when sums of money or other consequences are very large, their true value is not really in proportion to their actual face value. For example, one might judge the need for their family to get half a million euro if they die this year to be more than a thousand times more valuable to their lives than the €500 loss they suffer by buying the insurance. The insurance company deals with sums of money on this scale as being in proportion to their face value, but to the individual it's the impact on your life that counts. (This kind of issue can also provide interesting discussion on how the optimal strategy for 'Who wants to be a millionaire?' may be different for people in different financial circumstances.)

Think about expected value and become an informed citizen.

Why are car insurance premiums higher for some categories of driver?
What are the advantages/disadvantages of community rating in the health insurance market?
Why is a standard retirement age of 65 in an era of increasing longevity a pensions/actuarial nightmare?

PROBABILITY 6

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY; CONDITIONAL PROBABILITY

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- solve problems involving conditional probability in a systematic way
- extend your understanding of the basic rules of probability (AND/ OR, mutually exclusive) through the use of formulae, in particular $P(A \cap B) = P(A) \times P(B|A)$

You will also examine specific situations that will help you to

- appreciate contexts where $P(A | B) \neq P(B | A)$

INTRODUCTION

The activities described below and the questions that follow give you the opportunity to reinforce your understanding of the basic concepts of probability and to construct an understanding of the concept of conditional probability.

Activity 6.1

Consider a dice game with two six-sided dice. The sum of the dice decides who wins. What is the probability that the sum of the numbers on the dice is seven?
Use the table below to help make your decision.

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

How many possible outcomes are there when you roll the 2 dice?

Consider two events

Event A = {The sum of the numbers the dice is 7 or 9}.
Event B = {The second die shows 2 or 3}

Compute P(A) and P(B) (the probabilities of events A and B)

Use the table below to help work out P(A)
Colour the table to show all the possible outcomes for event A
Now find P(B). Use a different colour and show all the possibilities of event B

Are there any outcomes that are common to both?
If so, how many?

Now use your shaded table to find P(AIB) ie the probability of the sum of the numbers the dice shows is 7 or 9 and the 2nd die shows 2 or3

Use this result to write a general equation for P(AIB) in terms of

i. possible outcomes,and
ii. probabilities

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Activity 6.2

For this experiment you will need ten girls and ten boys and three rooms. If you do not have access to ten boys and ten girls write the word 'boy' on ten pieces of paper, the word 'girl' on ten other pieces of paper; then use three paper bags to represent the three rooms.

a. Place all ten boys and ten girls in Room 1. Now choose four girls and six boys and put them in a Room 2. Leave the room and place a blindfold over your eyes and re-enter Room 2. Walk around until you encounter a person. Ask that person to leave Room 2 and go to Room 3. Continue walking around while wearing your blindfold until you bump into a second person. Ask this person to leave the room and go to Room 3.

Go to Room 3 and determine whether there are two boys, two girls or one boy and one girl in Room3.

Now start again from the beginning and repeat this experiment 20 times. Each time record the gender of the two people who are in Room 3.

Compute the relative frequency for each of the events:

- A Room 3 contains two girls
- B Room 3 contains two boys
- C Room 3 contains a boy who was chosen first and then a girl who was the second person chosen
- D Room 3 contains a girl who was chosen first and also a boy who was the second person chosen

Now let's calculate the theoretical probabilities of each of these events. Recall that in Room 2 there are four girls and six boys.

i. What is the probability that the first person chosen from Room 2 is a boy?

ii. Suppose that the first person chosen was a boy, how many boys and how many girls are left in Room 2?

iii. Now what is the probability that the second person chosen from this remaining set of girls and boys in Room 2 is a boy?

iv. Combining these two probabilities appropriately what is the probability that the two people chosen to enter Room 3 were both boys?

v. Using the same logic, compute the probabilities of events A, C and D. Are the probabilities of C and D the same?

vi. What is the probability that Room 3 at the end of the experiment contains one boy and one girl? (You do not know the order in which the people were chosen to enter Room 3.)

b. Put all of the twenty people back in Room 1 and this time place five boys and five girls in Room 2. Now wearing your blindfold enter Room 2 and choose one person at random. Ask this person to enter Room 3 and write on a blackboard in Room 3 whether they are male or female and then return to Room 2. Now repeat this process again.

Compute the probability of the following events:

- **A.** The first person to have been chosen was a girl.
- **B.** The first person to have been chosen was a boy.
- **C.** The second person to have been chosen was a girl.
- **D.** The second person to have been chosen was a boy.
- **E.** The first person chosen was a boy and the second person chosen was a girl.
- **F.** The first person chosen was a girl and the second person chosen was a boy.
- **G.** The first person chosen was a boy and the second person chosen was a boy.
- **H.** The first person chosen was a girl and the second person chosen was a girl.

i. Examining these probabilities determine whether the events A and C are independent. What about A and D?

ii. How do the probabilities for events E, F, G and H compare? Does it matter in which order the boys and girls were chosen?

c. Suppose the event X was that a boy is chosen from Room 2 and event Y is that a girl is chosen.

iii. What is $P(X|Y)$ – the probability that a boy is chosen given that a girl was already chosen?

iv. What is $P(Y|X)$ – the probability that a girl is chosen given that a boy was already chosen?

d. Suppose a third experiment was performed in which we again start with 5 boys and 5 girls in Room 2 but this time when someone is chosen to enter Room 3 they stay in Room 3 as they did in our first experiment.

i. Under these rules, what are the probabilities of the events A, B, C, D, E, F, G and H?

ii. Are A and C independent in this experiment?

iii. What about A and D?

iv. How do the probabilities for events E, F, G and H compare now?

v. What is $P(X|Y)$ – the probability that a boy is chosen given that a girl was already chosen?

vi. What is $P(Y|X)$ – the probability that a girl is chosen given that a boy was already chosen?

Let Z be the number of boys who could be in Room 2 at the end of the last experiment.

Think: What values can Z possibly take?

Looking at the probabilities you have previously calculated, assign a probability to each possible value of Z. Now compute E(Z), the expected number of boys in Room 3. Compute E(Z) for the two experiments described earlier in this example.

Q. 1 You may remember doing the first part of this question before. Now you will extend the concept to consider conditional probability.

Suppose every child that is born has an equal chance of being born a boy or a girl.

i. Write out the sample space for the experiment where a mother has two children.
ii. What is the probability that a randomly chosen mother of two children would have two girls?
iii. What is the probability that this mother of two children would have two boys?
iv. What is the probability that this mother of two children would have one boy and one girl?

Suppose you meet someone who tells you they have two children and that one of their children is a girl.

v. What is the probability that this person has two daughters?
vi. Is it the same as it was in the previous situation of the randomly chosen mother?

Consider the sample space that you wrote out in the case of the randomly chosen mother.

vii. Do all of the elements of this sample space still apply to the example we are now considering?
viii. Write out the sample space for this example, i.e. where the person tells you that one of their children is a girl.
ix. Now, considering this sample space again, try to answer the question 'What is the probability that this person has two daughters?'

Extension

What if this person tells you that they have two children and that one of their children is a *girl named Florida*? Does this extra piece of information make any difference? If you have access to the internet, you might look up this well known puzzle/problem, which is not as straightforward as it might first appear.

Q. 2 100 glasses of mineral are placed on a table. One of these glasses of mineral contains a new brand of cola called Green Cow Cola and the other 99 are a well known brand of cola.

a. A student is asked to go into the room and choose one of the glasses at random not knowing in advance which glass contains the new brand of cola. The student considers themselves a connoisseur of cola and is able to correctly identify colas 95% of the time. This means that 95% of the time if the student is given any brand of cola in a glass and is asked to identify it they will assign the correct name to that cola.

i. What is the probability that they will choose the glass of Green Cow Cola?

ii. What is the probability they will not choose the glass of Green Cow Cola?

Let A be the event that a glass contains Green Cow Cola.

iii. What, in ordinary English, is the event A^C, the complement to A?

Let B be the event that the student says that a glass contains Green Cow Cola after they have tasted the cola.

iv. What is the event B^C?

v. What is the $P(B|A)$ – the probability that the student says the glass contains Green Cow Cola given that the glass actually contains Green Cow Cola?

vi. What is $P(B|A^C)$?

vii. What is $P(B^C|A)$?

viii. What is $P(B^C|A^C)$?

ix. In ordinary English describe what $P(A|B)$. Do you think it is the same as $P(B|A)$?

x. What is $P(A \cap B)$ in English? What is $P(B \cap A)$ in English? How do these two probabilities compare?

b. Write down formulae to express each of $P(A|B)$ and $P(B|A)$ in terms of P(A), B(B) and $P(A \cap B)$.

Considering these formulae and making an appropriate substitution from one formula into the other can you see how to express $P(A|B)$ in terms of $P(B|A)$?
Looking at the new expression you have created, once again try to answer the question: 'Is $P(A|B)$ the same as $P(B|A)$?'
Utilising the expression you have created can you compute a value for $P(A|B)$?

> If you are having difficulty answering this question it may help to consider smaller numbers; in this way you will organise your work in a systematic way and you may see patterns that will help you answer questions with larger numbers. It may also help to draw a diagram to represent the situation

Activity 6.3

A disease called Stats-itis is known to affect one in every 100,000 people in the world. People who have this disease have a natural proficiency for Statistics and many of them eventually become employed as Statisticians.
There is a test that has been developed which can test teenagers to determine if they suffer from Stats-itis. While this test is not completely accurate it does have an accuracy that is comparable to tests used for many other well known diseases. If the test is performed on someone who suffers from Stats-itis there is a 0.95 probability that the test will correctly

indicate that the person has the disease. If the test is performed on someone who does not have the disease then the test will correctly identify that the person does not have the disease 95% of the time.

Suppose that you get tested for this disease and the test says that you have the disease. Recalling the 95% accuracy of the test, does your intuition tell you that it is pretty certain that you do indeed have the disease?

Let's now try and consider this question from an in-depth mathematical point of view and see how the true answers we receive compare with our intuition.

a. Let A be the event that the test indicates that someone has Stats-itis. Let B be the event that someone actually has Stats-itis.

   i. What is $P(A|B)$?
   ii. What is $P(B)$?
   iii. Using these two answers compute a value for $P(A \cap B)$ and for $P(B \cap A)$.
   iv. Draw a Venn-diagram for this scenario and identify the sets A, A^C, B and B^C.
   v. The following equation is missing one part. Can you fill in the missing part?

      $P(A) = P(A \cap B)$ + [missing part]

   vi. Can you express $P(B|A)$ in terms of the probabilities of the events A, B and 'the intersection of A and B'?
   vii. Express $P(A \cap B)$ in terms of $P(A|B)$.
   viii. Can you find similar expressions for $P(A \cap B^C)$, $P(B \cap A^C)$ and $P(A^C \cap B^C)$?
   ix. By combining several of the results you have already computed, express $P(B|A)$ in terms of $P(A|B)$.

b. Suppose you do not know whether you have Stats-itis.

   i. What is the probability that the test says you have Stats-itis given that you do have Stats-itis?
   ii. What is the probability that you have Stats-itis given that the test says you do have Stats-itis?

Suppose that you went to a doctor and the doctor performed this test on you and told you that you had the disease. Should you be concerned?

c. You ask the doctor how sure she is that you have the disease and the doctor says that the test is 95% accurate. The doctor seems to be indicating that she is 95% sure that you have the disease.

   i. Is this an accurate reflection of the true probability that you have the disease given that the test has come back positive?

   ii. How does the probability that you have the disease before you took the test compare with the probability that you have the disease given that you have taken a test and it has given a positive result indicating that you have the disease?

Consider this example in detail and discuss it with your friends and/or family. Does the answer you received for the 'probability that you actually have the disease given that the test says you have the disease' agree with your intuition and that of your friends or family?

Q. 2 Consider the following pairs of events:

1. A: A creature has a Tail.
   B: A creature is an Elephant.

2. A: A creature has wings.
   B: A creature can fly.

3. A: You are someone who plays a game of chance.
   B: You are someone who wins a game of chance.

For each pair of events answer the following questions:

i. What in words is $P(A|B)$?

ii. What in words is $P(B|A)$?

iii. Is $P(A|B) = P(B|A)$?

Q. 3 Consider the following events:

A: A person wears a black hoodie and blue jeans.
B: A person commits a crime.

John was arrested for stealing a woman's handbag. At the trial it is pointed out that a Garda arrested John 20 minutes after the crime took place 100 metres from the scene of the crime and that John was wearing blue jeans and a black hoodie.

i. Is $P(A|B) = P(B|A)$?
ii. How is this relevant to this court case?
iii. Is $P(A)$ a useful piece of information in establishing the innocence or guilt of John?
iv. Would it make any difference if this crime took place in the centre of Dublin compared to if the crime took place in a village with a population of 50 people?
v. Can you see any similarity between this question and the previous question about Elephants?

Evidence is then presented that the DNA matches an individual named Tom. The prosecution does not feel the need to present any other evidence relating Tom to the crime as they have the DNA match.

vi. Does your intuition tell you that this evidence is sufficient to establish with certainty that Tom committed the crime?
vii. By considering the total population of the world to be 6.75 billion, what is the number of individuals whose DNA would match the DNA from the crime scene?
viii. What is the probability that someone other than Tom could be the individual whose DNA was left at the crime scene?

PROBABILITY 7

SYLLABUS TOPIC: OUTCOMES OF RANDOM PROCESSES

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- apply the principle that, in the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, urns with coloured objects , playing cards, etc.)
- solve problems involving experiments whose outcome is random and can have two possibilities (labelled 'success' or 'failure'), such as tossing $n \leq 3$ coins or rolling $n \leq 3$ dice
- toss n coins or roll n dice and count number of 'successes' and calculate the probability of this occurring
- toss a coin or roll a die until the kth 'success' and calculate the probability of this occurring

Activity 7.1

Consider tossing three coins. Each coin has one face with a Head and one with a Tail. Coins never land on their side but always land face up.

Drawing a tree diagram will help you organise your thoughts and answer the following questions.

a. What is the probability that the first coin shows a Head?

b. What is the probability that all three coins show Heads?

Consider the event that two of the three coins were Heads and one was a Tail. Is it certain that the first two coins were Heads?
Is it possible that the last two coins were Heads?

c. What other possibilities are there?

If you are having difficulty answering this part of the question, write the numbers 1, 2 and 3 on three pieces of paper and place them in a bag. Draw three empty boxes in a horizontal row on a piece of paper. Number the boxes 1, 2 and 3 from left to right. Choose two pieces of paper from the bag and write the word Head in the boxes with the numbers on the two pieces of paper that you drew from the bag. Write the word Tail in the remaining box.

Place the pieces of paper back in the bag and repeat this experiment many times.
How many DIFFERENT possibilities are there for the arrangements of the two Heads and one Tail?
How many ways can you choose two numbers from the set of numbers 1, 2 and 3?

Activity 7.2

Imagine now that you are tossing four coins and consider the event that two of the coins show up as Heads and two as Tails.

i. Can you list all the possible arrangements of two Heads and two Tails among the four coins?
ii. How many different possibilities are there?
iii. How many ways can you select two objects from among four objects?

Q. 1 Suppose you were to toss one hundred coins and twenty three were to land as heads. It could be that the first twenty three coins were heads and the last seventy seven were tails. There could be other possibilities as well. How many ways can you select twenty three numbers from among the set of numbers 1 to 100? So how many different possible ways could you get twenty three heads among the one hundred coins?

Activity 7.3

a. You perform an experiment in which you will toss six coins, each time getting a Head or a Tail.

i. What is the probability that an individual coin toss results in a Head?
ii. What is the probability that an individual coin toss results in a Tail?

You may find it useful to draw tree diagrams to work out the probability for the next few questions.

iii. Is the event that you get a Head on the first coin independent of the event that you get a Tail on the second toss?
iv. What is the probability that the first two tosses reveal a Head on the first coin followed by a Tail on the second coin?
v. What is the probability that the first three tosses give exactly the pattern (Head, Tail, Tail)?
vi. What is the probability that the first coin shows a Head and all of the subsequent five coins show Tails?
vii. How many ways can you select one object from 6 objects?

viii. How many other arrangements of the coins would give one Head and five Tails?

ix. What is the probability that the coins land with the first five coins showing Tails and the last coin showing a Head?

x. What is the probability that the coins land with the second coin showing a Head and all the other coins showing Tails?

xi. By adding up the individual probabilities of each possible way that you can get one Head and five Tails, compute the probability that you would get exactly one Head when you toss six coins.

b. Now imagine repeating this experiment where you toss six coins, but suppose this time you get two Heads and four Tails. One way this could happen is that the first and third coins were Heads and the other four were Tails. What is the probability of getting exactly this arrangement? In total how many different ways can six coins display two Heads and four Tails? Are all of these different arrangements equally likely or is any one more likely than the others? What is the probability of each of these arrangements? Combining the information you have for the number of ways that you can get two Heads and four Tails with the probability of each of the possible arrangements, write down an expression for the probability of getting exactly 2 Heads when six coins are tossed.

   a. How many ways could you get six Heads among the six coins?

   b. How many ways could you get three Heads among the six coins?

Activity 7.4

a. Get a 2 Euro coin with a Harp on one side. Toss this coin and observe whether it lands with the Harp facing upwards. If it does, stop tossing the coin. If it does not, and instead you get a map of Europe displayed then toss the coin again repeatedly until the coin first displays a Harp. Count the number of tosses that were required until the first Harp appeared. Repeat this experiment 100 times (each class group could complete the experiment a number of times and compile the class results) each time noting the number of tosses until the first Harp appeared.

Construct a table in which to record your results

b. Draw a histogram of the relative frequencies of N where N is the 'Number of tosses required to get the first Harp'.

   i. Is there any pattern to the histogram?

   ii. What is the ratio of the relative frequency for N=3 versus the relative frequency for N=2?

c. You will now attempt to construct a probability distribution for this experiment.

What is the largest possible value that N can take? What is the smallest value that N can take?

If N=1 what happened when you tossed the coin?
If N=3 what happened each time you tossed the coin?

iii. What is the probability that you get a 'Europe' on the first toss of the coin?

iv. What is the probability that you get a Harp on the second toss of the coin?

v. Are the events 'Europe on first toss' and 'Harp on second toss' mutually exclusive and/ or independent?

vi. What is the probability that you get a Europe on the first toss of the coin AND you get a Harp on the second toss of the coin?

vii. What is the probability that N=2?

viii. What is the probability that N=4?

If N=17, how many Europes have occurred?
If N=17, how many Harps have occurred?

ix. What is the probability that you get 25 Europes displayed one after another?

x. What is the probability that N=26?

Activity 7.5

Roll a standard six-faced cuboid die. (A die which has six faces displaying differing numbers of dots on each side from one dot to six dots.)

i. How many different possible numbers of dots can be displayed on the topmost side when the die lands?

ii. What is the sample space for this experiment?

iii. What is the probability that you get a 1 on the first roll? (The die falls with one dot facing upwards.)

iv. If you got a 6 on the first roll does that make it more or less likely that you will get another 6 on the second roll?

v. What is the probability that you will get a 6 on the second roll given that you have already gotten a 6 on the first roll?

vi. What is the probability that you get a 6 on the first roll and get a 6 on the second roll?

vii. What is the probability that you will not get a 6 on the first roll?

Now we will play a game in which you must roll the die as many times as is necessary until you get a 6. Record, N, the number of rolls required until you get a 6.

viii. What is the probability that you will not get a 6 on the first two rolls and you will get a 6 on the third roll?

ix. If N=67 what do you know about the 27th roll?

x. If N=34 what do you know about the 34th roll?

The table below describes the probability distribution of N. Fill in the missing values in the table.

N	Pattern of Rolls	Probability of N
1		1/6
2		
3	(Not 6, Not 6, 6)	
4		
5		
6		
7		
8		
9		
10		
11		
12		

Q.2 You have 10 keys on your key ring. All the keys look exactly alike. You arrive at your front door at midnight tonight. You try one of the keys in the door and if it opens you enter and place your keys back in your pocket. If however the door does not open you take the keys out of the door and they happen to fall on the ground. You pick up the keys not being able to identify which key you had previously used you try any one of the 10 keys in the door. You repeat this entire procedure, dropping the keys after each unsuccessful entry, until you eventually gain entry to your house.

i. What is the probability that you gain entry on the first attempt?

ii. What is the probability that it takes you three attempts to gain entry?

Each attempt at entry takes a total of 30 seconds to complete.

iii. What is the probability that you gain entry in 2 minutes exactly?

iv. What is the probability that you gain entry in a time not exceeding 3 minutes?

v. What is the maximum number of attempts it could take to enter your house?

vi. What is the probability that it takes at least 3 attempts to enter your house?

vii. If every night you and everyone else in the world were to repeat this procedure until the end of the world, and each time a record of the number of attempts that it took to enter houses was taken. What would be the average number of attempts?

Activity 7.6

Roll a fair die until the third 1 appears. Record the outcome of each roll on a piece of paper, classifying any roll which does not lead to a 1 as a failure and any roll leading to a 1 as a success. Now repeat this experiment 99 more times. For each of the experiments count the number of successes that occurred and the number of failures. Group the 100 experiments according to N, where N is the total number of rolls required to achieve the third success.

i. What is the minimum value for N in the 100 rolls?

ii. What was the largest value for N in the 100 rolls?

iii. What's the smallest possible theoretical value for N?

iv. What is the largest possible theoretical value for N?

v. For each value of N in your 100 experiments, write down a relative frequency for N.

You have now constructed a relative frequency table for N from the sample of experiments that you conducted. Can you construct a theoretical probability distribution for this experiment? For each value of N you will need to compute the probability of N occurring.

vi. What do you know about each experiment?

vii. What happens on the last roll?

viii. How many successes are achieved in the preceding N–1 rolls?

ix. What different possibilities are there for the way that those successes were arranged among the N–1 rolls?

x. What is the probability of getting a success on any individual roll?

xi. What is the probability of getting a failure on any individual roll?

xii. What is the probability of getting 3 successes AND N–3 failures?

xiii. How many different ways can 3 successes and N–3 failures occur in a set of N rolls, remembering that the game always ends when the third 1 is obtained?

Q.3 Ireland has made the final of the FIFA World Cup. The game has ended scoreless after extra time has been played and you are the only player on the Irish team willing or able to take penalty kicks. As the goalkeeper of the opposing team also plays for the same team as you in Serie A you have practised taking penalties against him many times and you know that you are successful exactly one quarter of all times you take a penalty against him.

i. What is the probability that you score on all of the first five kicks?

ii. What is the probability that your first successful kick occurs on the 4th attempt?

Suppose that under new rules the opposing team gets to take five penalties before you take any and the Irish goalkeeper saves all but one of the first 5 penalties that the opposing team takes. So this means you can win the World Cup if you score two penalties in the first 5 attempts.

iii. What is the probability that you will win the World Cup for Ireland with your first two kicks?

iv. If your winning kick occurs on the 3rd kick what are the possible occurrences on the 1st and 2nd kicks that you took?

v. What is the probability of each of these occurrences which led to you winning the World cup for Ireland on the 3rd kick?

vi. Are each of these possibilities that lead to you winning on the 3rd kick mutually exclusive or not?

vii. Considering the different things that might happen on your 1st and 2nd kicks, what is the overall probability that you will win the World cup on your 3rd kick?

viii. If you need two successful kicks to win, what is the probability that you win on the 5th and final kick?

ix. What is the probability that you win on the 4th kick?

As soon as you have scored two goals Ireland has won and you do not need to take any more penalties. What is the probability that you can win the World Cup for Ireland with the five kicks that are available to you?

The following activities can be conducted by groups of two people, one carrying out the activity and the other recording the results obtained. A minimum of ten groups is required.

Activity 7.7

Each of ten people should roll a fair die one hundred times, with the outcome of each roll recorded. Count the number of occurrences of each outcome for each individual and construct a relative frequency table and relative frequency histogram for this experiment.

a. What do you notice about the relative frequency of each of the outcomes 1, 2, 3, 4, 5 and 6?

Now combine the results of the experiments for all of the people, so that instead of having data for one hundred rolls you have data for at least one thousand rolls.

Again construct a relative frequency histogram for the experiment. What class intervals should you use for this histogram?

b. What do you now notice about the relative frequency of each of the outcomes 1, 2, 3, 4, 5 and 6?

Now using a calculator or a spreadsheet, compute an average (mean) of the outcomes that occurred on the first roll of the die for each person. Record this number and repeat this averaging process for each of the one hundred rolls of the die.

Now construct a relative frequency histogram using these one hundred averages.

c. What do you notice about the shape of this histogram? Is it the same as you had before you took averages?

Activity 7.8

Each of ten people should toss a coin until they get the first head, recording N, the number of tosses required until the first head was achieved. This is repeated a further ninety nine times by each person, to give a total of 1000 results. Construct a relative frequency histogram for N.

a. What does the histogram look like?

b. Are all of the bars of roughly equal height as they were when you rolled a die one hundred times?

Now combine all of the data from each person until you have results from at least one thousand tosses. Construct a relative frequency histogram for N using all of the data.

c. Has the shape of the histogram changed a lot from the ones constructed by each individual person?

Now average the values of N that were achieved in the first attempt at the game by each of the people participating. Remember you should have at least ten people playing this game so you should be averaging at least ten numbers. Now average the values of N from each person for the second game and repeat this averaging for each of the one hundred games until you have one hundred averaged values.

Construct a relative frequency histogram for these averages using class intervals of width 1.

d. What shape does this histogram have?

e. Is it similar to the histogram which was constructed prior to the averaging process?

f. Compare this histogram to that which was constructed by averaging the outcomes from the die rolling experiment. Do you notice any similarity?

Activity 7.9

Each of ten people should take a deck of cards and remove all of the picture cards and the jokers so that what remains are cards with denominations 1 to 10.

Each person shuffles their smaller pack of cards and randomly chooses one card from the deck, recording the card chosen on a piece of paper. The card is replaced in the deck and the cards are shuffled. Another card is chosen at random and its face value recorded; it is then replaced and the deck shuffled as before. This drawing of cards is repeated until one hundred cards have been drawn.

a. Construct a relative frequency histogram for this experiment using the data for each individual person.

b. Comment on the shape of this histogram.

c. Now amalgamate all of the data from all of the people playing the game and construct another relative frequency histogram.

d. Comment on the shape of this histogram.

Average the face values of the cards drawn by all the players on their first draw. Repeat this averaging process for each of the remaining ninety nine draws.

Using these one hundred averages construct a new relative frequency histogram with class interval widths equal to 1.

e. Comment on the shape of this histogram. Is it similar in shape to any other histograms you have constructed?

Activity 7.10

Open a blank worksheet in Excel and type =10*RAND() into cell A1.

Go to the Help Menu in Excel and search for help on the function RAND(). Having discovered what RAND() does can you explain =10*RAND() ?

Selecting the bottom right hand corner of Cell A1 with your mouse drag the formula down until it is entered into cells A1 to A400.

Select Column A with your mouse and paste a copy of it in each of columns B, C, D, E, F, G, H, I , J and K.

You will notice that the entries in each of the copies of Column A are not exactly the same. This is because the function RAND() updates itself each time it is copied.

Using the Help Menu in Excel search for help on the function COUNTIF.

In cell M1 enter =COUNTIF(A1:A400, '<=0.5')-COUNTIF(A1:A400,'<=0')

In cell M2 enter =COUNTIF(A1:A400, '<=1')-COUNTIF(A1:A400,'<=0.5')

Enter similar formulae, changing as appropriate for each of the cells M3 to M20 until you have entered =COUNTIF(A1:A400, '<=10')-COUNTIF(A1:A400,'<=9.5') in M20

a. Using the values in M1 to M20 construct a relative frequency histogram for the data in A1 to A400

What is the shape of the histogram?

In Cell L1 enter =AVERAGE(A1:J1)

By selecting the bottom right hand corner of Cell L1 drag this formula until it is entered in Cells L1 to L400

In cell N1 enter =COUNTIF(L1:L400, '<=0.5')-COUNTIF(L1:L400,'<=0')

In cell N2 enter =COUNTIF(L1:L400, '<=1')-COUNTIF(L1:L400,'<=0.5')

Enter similar formulae, changing as appropriate for each of the cells N3 to N20 until you have entered =COUNTIF(L1:L400, '<=10')-COUNTIF(L1:L400,'<=9.5') in N20

Using the values in N1 to N20 construct a relative frequency histogram for the data in L1 to L400

b. What is the shape of the histogram?
c. Is it similar to the histogram for the entries in column A?
d. Is it similar to any other histograms that you have produced?

Q.4 Continuous Distributions

A machine is created to select real numbers between 0 and 10. This machine is constructed to select all numbers in an equally likely manner so that no number is more likely to be chosen than any other number. The machine operates only once and produces one real number between 0 and 10.

a. What is the probability that it chooses the number 1.0786578?
b. What is the probability that it chooses the number 4.0?
c. What is the probability that it chooses a number which is greater than 0 and less than 5?
d. What is the probability that it chooses a number which is greater than or equal to 0 and less than or equal to 5?
e. If Z represents the number chosen by the machine, what is $P(1<Z<3)$?
f. What is $P(Z=2)$?

Q.5 Continuous Distributions

The author of this question has chosen a real number between 0 and 100 and has written this number down to its full accuracy of all of its decimal places on an extremely large piece of paper. The author will offer a substantial prize of one hundred million euro to anyone who guesses this number to all decimal places. After you have completed your Leaving Certificate you may enter this competition as many times as you like. Entry to the competition is entirely free.

a. Would it make sense to spend four years of your life attempting to guess this number rather than working or pursuing further study?
b. How many integers are there between 0 and 100?
c. If the author of this question had written an integer value between 0 and 100 down on a piece of paper and you were asked to guess an integer value between 0 and 100 what is the probability that you could guess the same value as the author chose?
d. If the author instead chose a real number between 0 and 1 would this game be easier to win?
e. How many possible real numbers are there between 0 and 1?
f. What is the probability that you could correctly guess the exact real number between 0 and 1 that the author wrote down?

Activity 7.11

Figure 1 contains a picture of the density function for a Standard Normal Random Variable Z.

a. The area under the curve between any two values, Z_1 and Z_2, on the x axis measures what quantity for the random variable Z?

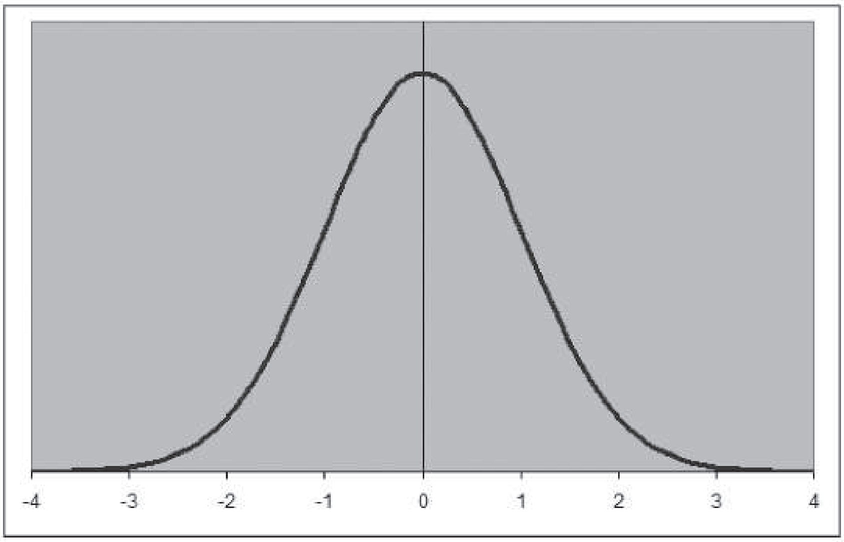


Figure 1 – The Standard Normal Density Function

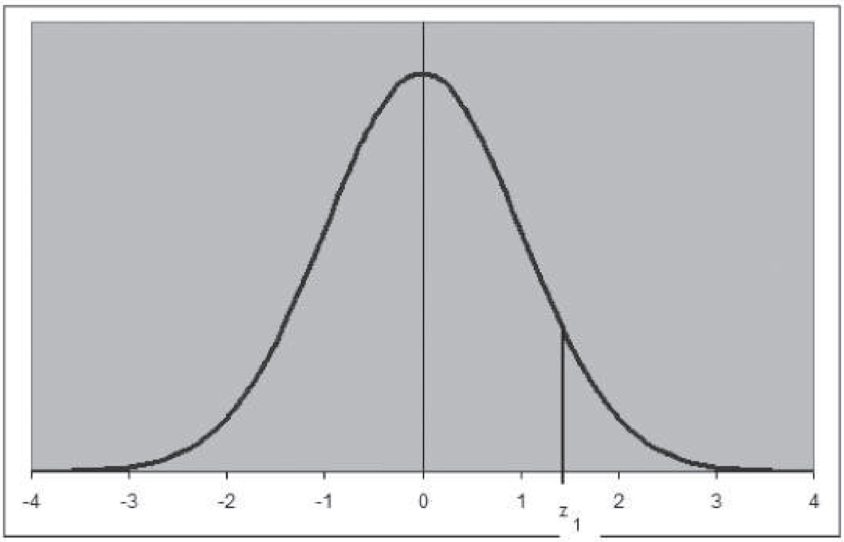


Figure 2: The Standard Normal Density Function

A number Z_1 is entered above on the horizontal axis of graph of the Standard Normal Density Function. Using a red pen enter the number '$-Z_1$' on the horizontal axis of this graph.

a. Shade in the region representing $P(Z > Z_1)$. Shade in the region representing $P(Z < -Z_1)$.

i. The area under the entire curve represents $P(-\infty < Z < +\infty)$. Considering this, what is the numerical value of the area under the entire curve?

ii. What is the relationship between $P(Z > Z_1)$ and $P(Z \leq Z_1)$?

iii. What is $P(Z > 0)$? What is $P(Z < 0)$? What is $P(Z \leq 0)$? What is $P(Z=0)$?

b. By referring to the Standard Normal Tables in your set of Mathematical Tables answer the following questions:

(i) What is $P(Z<1)$? (ii) What is $P(Z<2)$? (iii) What is $P(Z>1)$?
(iv) What is $P(0<Z<1)$? (v) What is $P(-1<Z<0)$? (vi) What is $P(-1<Z<1)$?
(vii) What is $P(-2<Z<2)$? (viii) What is $P(-3<Z<3)$?

In Figures 1 and 2 the density function is centred on the value 0. This is because the mean of Z is 0.

c. Draw a new graph, Figure3, which is identical in shape to that in Figure 1 but which is centred on 5 instead of on 0. Instead of Z we will name the variable whose density function is displayed in this new graph as W.

(i) What is the mean of W? (ii) What is E(W), the expected value of W?

Looking at the new graph that you have drawn, answer the following questions:
a) What is $P(-\infty < W < +\infty)$? b) What is $P(W>5)$? c) What is $P(W<5)$?
d) What is $P(W=5)$? e) What is the relationship between W and Z?

d. Suppose a new variable X is created where X=2Z. Answer the following questions.

(i) What is the mean of X? (ii) How does the mean of X compare with the mean of Z?
(iii) What is $P(-\infty < X < +\infty)$? (iv) What is $P(X>0)$? (v) What is $P(X<0)$? (vi) What is $P(X=0)$?

By referring to the Standard Normal Tables in your set of Mathematical Tables, and by remembering the relationship between X and Z answer the following questions:

(vii) What is $P(X<2)$? (viii) What is $P(X<4)$? (ix) What is $P(X>2)$? (x) What is $P(0<X<2)$?
(xi) What is $P(-2<X<0)$? (xii) What is $P(-2<X<2)$? (xiii) What is $P(-4<X<4)$?
(xiv) What is $P(-6<X<6)$?

e. Recall that the mean of a random variable determines where the density function is centred, and that the standard deviation of a random variable determines the width of the density function.

i. Compare $P(0<Z<1)$ with $P(0<X<2)$. The standard deviation of Z is 1, what is the standard deviation of X?

ii. Suppose H is a new random variable with mean 0 and standard deviation 3, what is $P(0<H<3)$?

iii. Suppose G is a random variable with mean equal to -2 and standard deviation equal to 1, draw a graph of the density function of G. a) What is $P(G> 0)$? b) What is $P(G>-2)$?

Q.6 Normal Distribution

X is a random variable with mean equal to 3 and standard deviation equal to 2.
(i) What is $P(X>4)$? (ii) What is $P(X>0)$? (iii) What is $P(X<-2)$? (iv) What is $P(0<X<3)$?

Q.7 Normal Distribution

Suppose the scores on the Leaving Certificate Mathematics Exam turn out to be normally distributed with a mean of 60% and a standard deviation of 15%.

i. What is the probability that a randomly selected student scores 75% or above?
ii. What is the probability that a randomly selected student scores 30% or below?

Q.8 Normal Distribution

i. The mean percentage achieved by students on a Statistics exam is 60%. The standard deviation of the exam marks is 10%. What is the probability that a randomly selected student scores a percentage above 80%? What is the probability that a randomly selected student scores a percentage below 45%? What is the probability that a randomly selected student scores a percentage between 50% and 75%?
ii. Suppose you were sitting this exam and you are offered a prize for getting a mark which was greater than 90% of all other students sitting the exam. What percentage would you need to get on the exam to win the prize?

Q.9 The heights of Irish women are normally distributed with mean 5 feet 6 inches and standard deviation 3 inches. Using this information, approximately what proportion of Irish women are taller than 6 feet? (Note there are twelve inches in one foot).

STATISTICS 1

SYLLABUS TOPIC: REPRESENTING DATA GRAPHICALLY AND NUMERICALLY

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- explore concepts that relate to ways of describing data, such as the shape of a distribution, what's typical in the data, measures of centre (mode, median, mean), and range or variability in the data
- use a variety of summary statistics to analyse the data: central tendency; mean, median, mode
- select appropriate graphical or numerical methods to describe the sample (univariate data only)
- evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others
- use pie charts , bar charts, line plots, histograms (equal intervals), stem and leaf plots to display data
- use back to back stem and leaf plots to compare data sets

There are links with Strand 3 (Number) where you will investigate models such as accumulating groups of equal size to make sense of the operation of multiplication.

INTRODUCTION

Being able to see a data set as a whole and so being able to use summary statistics such as averages to describe the 'big picture' or the overall shape of the data is an important learning intention of strand 1.

The activities described below allow you to investigate how the mean is constructed and the relationship of the mean to the data set it represents. You will also explore the different ways the median and mean represent the data - the median as a middle point in the data, and the mean as a 'point of balance' or the 'fair share' value of the data. Using two different representations of the mean gives you a chance to view the relationship between the mean and the data set through different models and so construct a firm understanding of the mathematical concept.

Prior learning

The idea that a set of data can be viewed and described as a unit is one of the key ideas about data that develops across primary school and is built on at second level. Initially, you looked at each individual piece of data. Gradually, you began to move away from a focus on individual pieces of data to looking at larger parts of the data. You learned to make general statements about the group of things or phenomena that the data represent, such as 'most people in our class have 1 or 2 siblings, and the range is from no siblings to 6 siblings.' Now you are ready to move away from making general statements and begin to make summary statements that describe the whole data set.

Activity 1.1

There are 5 bags of sweets, each of a different brand. All bags are the same size. The average price for a bag is €1.43

a. What could the individual prices of the 5 bags be? Think of at least two different sets of prices.

b. If both of your sets of prices included €1.43 as a price for at least one of the bags, price the five bags without using €1.43 as one of the prices.

c. Did you use €1.43 as the median? If so, what is the mean for your sets of prices? If you didn't use €1.43 as the median, what is the median for your sets of prices? Are the mean and median the same or different?

Discuss one of your lists of five prices with your group. How did you decide on your list of prices? How do you know what the average is in each example?
Note to each small group: Make sure you consider some lists that do not include a value of €1.43 as one of the prices.

d. There are seven bags of beads. Five of the bags have the following numbers of beads in them: 5, 7, 8, 9, and 12. Now work through parts (i), (ii) and (iii) with your group.

   i. Make a representation of the five bags by using small objects such as cubes, counters, marbles, etc. Make another representation of the five bags on a line plot.

   ii. Now use your representation to figure out how many beads could be in the other two bags so that 8 is the mean number of beads for all seven bags. Try to figure this out without adding up the beads in the five bags. Find at least two different sets of numbers for the two bags that will solve this problem.

   iii. Revise your two representations – counters and line plot – so that they show all 7 pieces of data. Can you 'see' the average in your representation?

e. What is the least number of beads there could be in one of the additional bags? What is the greatest number?

f. What numbers of beads could be in the two other bags if the mean number of beads was 7? What if the mean number was 10?

Q1. A teacher had some cards with groups of numbers displayed on them, as shown below

1, 7, -8, 0,	0, 0, 0	-2, 8, -6, 7, 11
0, 11, 8, 0, 13	-5, -4, -3, -2, -1 0, 1, 2, 3, 4, 5	2, 3, 4, 5, 6, 7, 8 9, 10

John was asked to calculate the mean of the numbers on each card and to put the cards that had a **mean of zero** into a box.

a. Circle the cards that John should put into the box.

The teacher has another card and tells the students that the mean of the numbers on this card is also zero.

b. Tick the correct box for each statement about this extra card.

Statement	Must be true	Could be true	Cannot be true
All of the numbers are *zero*			
Some of the numbers are *zero*			
There are as many negative numbers as positive numbers			
The sum of ***all*** the numbers is zero			
All of the numbers are positive numbers			
Some of the numbers are positive numbers			

Q.2 3 girls and 5 boys received text messages

The mean number of messages received by the 3 girls was **31**.

The mean number of messages received by the 5 boys was **27**.

Decide whether the following statements are true (T) or false (F), and justify your answer in each case:

i. The person who received the most messages must have been a girl.

ii. The mean number of messages for the 8 people was 29.

Q.3 Three girls and five boys were studying climate change in various countries around the world. They were examining the maximum daily temperatures in these areas

The mean daily temp of the locations studied by the 3 girls was 31° C
The mean daily temp of the locations studied by the 5 boys was 27 ° C

Decide whether the following statements are True or False, and justify your answer in each case.

i. The person who encountered the max daily temperature must have been a girl.
ii. The person who encountered the min daily temperature must have been a boy.
iii. The mean max daily temperature encountered by the 8 people was 29° C. .

Q.4 Sophie has six cards, each of which has a positive whole number printed on it. Four of the cards each have the number 9 on it.

a. Without knowing the numbers on the other two cards, can you give the value of the
   i. median
   ii. mode
   iii. range

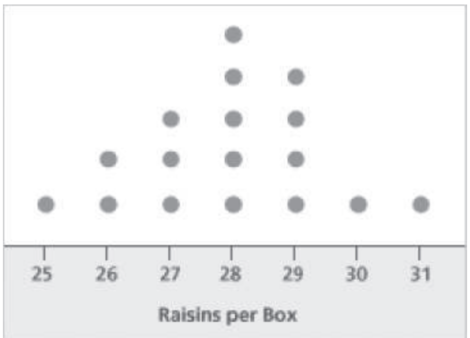
Explain your reasoning.

b. You are told that the six cards have a mean of 9. Give some possible whole numbers that could be on the other two cards. Which of your answers would give the greatest range? Why?

If the six cards have a mean of 9 and a range of 6 how many answers can you now find for the numbers on the remaining two cards?

Q.5 Students were investigating the number of raisins contained in individual mini-boxes of Sun-Maid raisins.

They recorded their results in the diagram shown.



a. Use the diagram to answer the following:

  i. How many boxes of raisins did they survey?

  ii. What was the modal number of raisins per box?

  iii. What is the median number of raisins per box? Explain how you found this answer.

b. If the students chose a box at random from all the boxes they surveyed what is the probability that the box contained 29 raisins?

Having done this activity, the students are asked to write down the answer they would give to the question: 'How many raisins are in a mini-box of Sun-Maid raisins?' Here are some of the answers they wrote down:

A 'There could be any number of raisins in a box.'
B 'There are about 28 raisins in a box.'
C 'There are almost always 28 raisins in a box.'
D 'You can be fairly sure there are 27, 28 or 29 raisins in a box.'
E 'Probably 28'.

c. Which of the answers above do you think is the best answer to the question? Explain why you think it's the best.

d. Which of the answers above do you think is the worst answer? Explain why you think it's the worst.

Extension (LC-OL)

Two extra boxes were found after the students had completed the diagram above. When the contents of these two boxes were added to the data, the mean number of raisins per box became 28.

e. Give one possible value each for the number of raisins in the two extra boxes.

Explain how you decided on these two numbers.

Extension (LC- HL)

The students wonder whether their sample was typical. On searching the internet, they read about a much larger-scale experiment on mini-boxes of Sun Maid raisins which indicates that the true mean is 31 raisins, with a standard deviation of 4.5. Some are surprised that the mean of the sample examined by the class was so much lower; others are not surprised, saying that this kind of variation between different samples is to be expected.

f. Calculate the probability that a random sample of 19 boxes will have a mean of less than or equal to 28 raisins per box.

g. What does this probability tell you about whether or not the sample examined by the class should be regarded as surprising?

Activity 1.2

A good part of one's day is spent travelling from one place to another. How much time do you spend travelling to school? How much time do your classmates spend travelling to school?

Carry out a survey to find out how everyone in your class travels to school, and how long the journey takes, on a given day. Your survey should enable you to answer a series of questions.

Deciding to walk or to go by car may depend on the distance, but, after choosing the method of transportation, does everybody spend about the same amount of time travelling to school?
Do those who take the bus to school spend less time than others?
Does the time it takes to get to school depend on where you live?

To better understand the situation, consider the 'time travelling to school' variable. Analyse the data you collect based on the method of transportation used.
Do you think this situation varies from one region in Ireland to another?

Time to get to school

Enter the class data in a table, such as the one below, grouping them in intervals of ten minutes, for example. First write down the numbers as you collect them. Then put them in ascending order to create a stem and leaf plot, where the tens are the 'stems' and the units are the 'leaves'. For example, a time of 15 minutes is recorded by placing a '5' in the Units column in the row which corresponds to the '1' in the Tens column.

Time to get to school Raw data	
Tens	Units
0	
1	
2	
3	…
…	…

Now, try to get an overview.

1. Look at all the ordered data. Half the class takes less than how many minutes to get to school? This number is called the median; it's the central value that divides the list of ordered data into two equal sections.
2. What is the average time that students in your class spend travelling to school?
3. Which row contains the most data? In your opinion, what does this mean?
4. What is the shortest time? What is the longest? What is the difference between them?
5. What can you say about the time that students in your class spend to get to school?

To get a better picture of the situation, it would help to add a column to your table that shows the number of students.

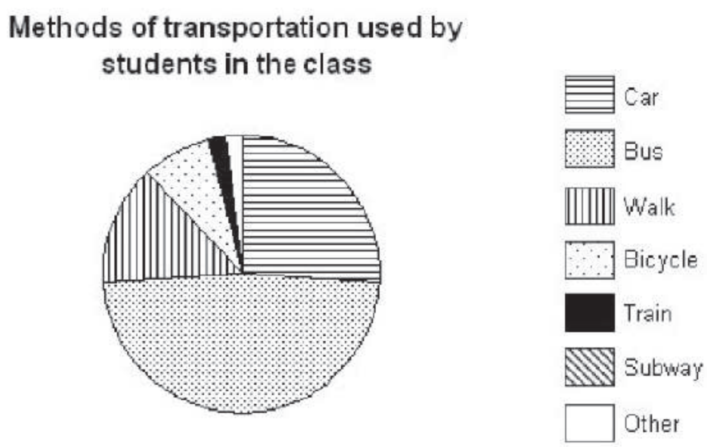
Time to get to school Raw data		
Tens	Units	No of Students
0		
1		
2		
3	…	
…	…	
	Total	

6. Now, can you create a graph that shows how much time the students in your class spend travelling to school? As you can see, everybody does not spend the same amount of time travelling to school.

You can now examine whether this time changes with the method of transportation.

Time spent by method of transportation

First, group together the students who use the same method of transportation. You can quickly determine the distribution of students by transportation method by creating a pie chart with a spreadsheet program. Your chart might look something like this:



From your chart, what are the most popular methods of transportation?
Approximately what fraction of the students in your class walk to school?

Now, for each method of transportation:

a. sort the time spent getting to school, from the shortest to the longest time.

b. determine the total time spent, which lets you calculate the average.

c. find the number of minutes or less that the faster half of the students spent travelling to school. This is the median or the value of the middle item of the ordered data.

d. add the minimum and maximum amount of time spent travelling to school.

Create a descriptive table that will look like this:

Method of Transportation	Time to get to school (mins)	No	Total Time	Average Time	Median	Min	Max
Car	5, 12, 12, 2, 32,	5	83	83/5	12	5	32

You can now examine the time by method of transportation.
Do you notice any significant differences?
Which method of transportation takes the longest?
Which method of transportation shows the biggest difference between the shortest time (minimum) and the longest time (maximum)? What might explain this?
Can you describe the overall situation for your class and present your point of view? What type of transportation do you think we should encourage? Under what conditions? Why?
Finally, use the data you have obtained to create a graph that properly conveys the information about your class that you feel is important.

Comparing your class to a sample of Irish students

Do you think the situation of your class resembles that of most Irish students?
Obtain a sample of 50 students from your school. Then do the same analysis that you did for your own class.
Is the time spent getting to school approximately the same for both groups? If not, how does it vary?
To help you better compare the data, create two tables side-by-side for each group.

Time to get to school				
Raw data for the school			Raw data for our class	
Students	Units	Tens	Units	Students
		0		
		1		
		2		
		3	…	…
		4	…	…
		5	…	…
50	Total	TOTAL	Total	

'A picture is worth a thousand words' and can certainly make it easier to read all these numbers. Create appropriate graphs to easily compare the time spent getting to school for both groups.

You can also compare the methods of transportation used.
For each group: create a pie chart to illustrate the distribution of students for the different methods of transportation used to get to school.
Use a descriptive table to examine the time spent by method of transportation used.
Do you arrive at the same observations for both groups? Are there any significant differences? If yes, what are they? Can you explain the differences taking into account the characteristics of your region?
Create a visual representation that properly illustrates and conveys your main conclusions.

STATISTICS 2

SYLLABUS TOPIC: FINDING, COLLECTING AND ORGANISING DATA

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- clarify the problem at hand
- formulate one (or more) questions that can be answered with data
- explore different ways of collecting data
- design a plan and collect data on the basis of above knowledge
- generate data, or source data from other sources including the internet
- discuss different types of studies: sample surveys, observational studies and designed experiments
- select a sample (Simple Random Sample)
- recognise the importance of representativeness so as to avoid biased samples
- design a plan and collect data on basis of above knowledge.

The activities described below and the questions that follow give you the opportunity to construct an understanding of the concept of finding, collecting and organising data in a statistical investigation. By carrying out a complete data investigation, from formulating a question through drawing conclusions from your data, you will gain an understanding of data analysis as a tool for learning about the world.

The activities are designed to build on your previous experiences with data, and to introduce you to the ideas you will work on as you progress through statistics in Strand 1.

During these activities you will work with categorical data, noticing how these data can be organised in different ways to give different views of the data.
As a result you should be able to

- gather data from a group
- classify the data
- write sentences that describe the 'Big Picture' of the data
- appreciate how the purpose of the research will affect how the data is gathered
- understand that the way data is represented can illuminate different aspects of the data.

Activity 2.1: A data Investigation

With what well-known person would you like to meet?

1. You will be working in groups on a data investigation. The first step is for each student to decide on his/her own how they would answer the survey question. Each student will need to write their answer a number of times on separate pieces of paper so that they can give their individual answers to each group, including their own.
2. Each group collects answers from everyone; make sure your group has a full class set of data that you can discuss.
3. Before you look at the data spend a few minutes discussing what might be interesting about them.
4. As a group sort the class data into three piles according to what they have in common. This is called classifying your data.
5. Choose one of your ideas for sorting and arrange your cards on a large piece of paper to show that classification
6. Write a sentence or two on your display that tells what you notice about the data
7. Post your display on the wall. If you finish before other groups, discuss issues about data that arose while you did this activity.
8. Can you represent this data in a chart?

Key Words: **Category, Data**

As you work through this activity reflect with your group on

- What issues came up for you as you tried to represent these data?
- What does the data tell you about the group?
- What questions arise for you while looking at this data? How might you modify the survey in order to address these?
- Did everyone interpret the original question in the same way?
- What were you thinking when you made your own decision?

Consider the following question:

How many countries have you visited?

Elect a scribe to sketch a line plot with reasonable intervals on the board. Collect data on the line plot by marking an X for the value of each person's response. (Note: a line plot is a graph for numerical data that is similar to a bar chart. It is one of the plots in common use in statistics.) Try to form statements that describe the data. What can they say for the class as a whole about the number of countries that they have visited?

Activity 2.2

1. Note: You have 30 mins to complete this assignment and post a representation of your data for others to see. That means you will need to decide on a question and collect your data efficiently. You may need to design a data collection sheet. Think about how you will make sure you get a response from every person. After 15 mins you should be ready to start making a data representation. Your representation need not be decorative or elaborative. Focus on how well it communicates information about your data.
2. Select a question that will result in numerical data
3. Collect data from everyone in the class.
4. Create a line plot for your data
5. Write three to five sentences on your display that describe your data
6. When your display is complete, discuss issues that arose in your group as you defined your question
7. What further questions might you want to pursue based on these initial data?

Sample data collection sheet

Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name

STATISTICS 3

SYLLABUS TOPIC: REPRESENTING DATA GRAPHICALLY AND NUMERICALLY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- explore the distribution of data, including concepts of symmetry and skewness
- interpret a histogram in terms of distribution of data
- recognise standard deviation as a measure of variability
- make decisions based on the empirical rule

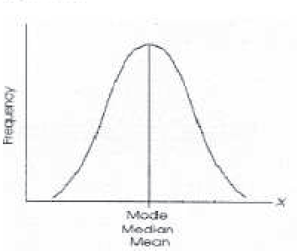
INTRODUCTION

The following activities and questions will enable you to construct an understanding of distributions and how the shape of the distribution relates to the mean, mode and median of the data. You will also construct an understanding of the concept of variability and how standard deviation relates to the representation of the data in a histogram. You will derive the empirical rule and make decisions based on it.

Activity 3.1

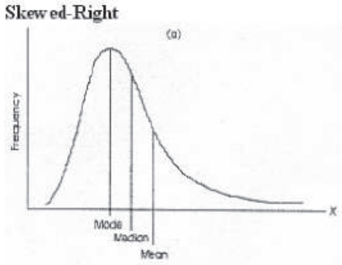
Examine the following distributions; note how the mean, median and mode compare in each situation and the shape characteristics of each distribution. A normal distribution is an example of a symmetric distribution.

Symmetric



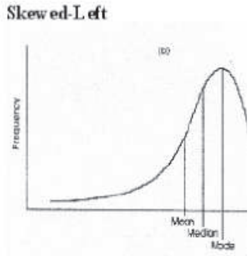
Mean= Median= Mode

Skewed-Right



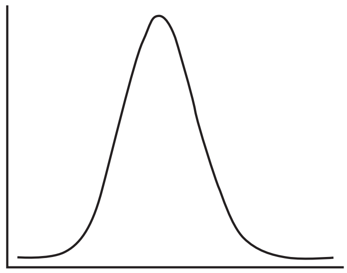
Mean> Median> Mode

Skewed-Left

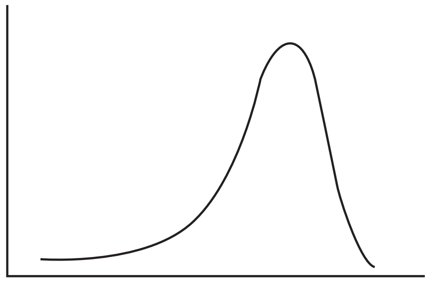


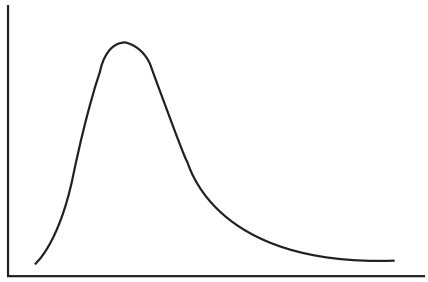
Mean< Median< Mode

Q.1 Examine the distributions sketched below. Label them as symmetric/normal, skewed right, or skewed left . Develop a list of situations in which the data gathered would produce each of the 3 different distributions.



……………………………………..





Q.2 Using the mean, median and mode, sketch the shape of the frequency histogram with the following characteristics

a. mean: 7.5 median: 6 mode: 5.7

b. mean: 6 median: 6 mode: 10, 12

c. mean: 7.5 median: 8.5 mode: 9

d. mean: 7.5 median: 7.5 mode: 7.5

Q.3 A pair of dice is rolled numerous times. The sum of the dice, as well as the frequency, is recorded. Calculate the mean, median and mode. Use these results to identify the shape of this distribution.

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	2	3	5	7	9	11	8	7	4	2	1

Q.4 The table shows information about the age of a sample of internet Facebook users

Age (t years)	Frequency
$10 < t \leq 15$	24
$15 < t \leq 20$	37
$20 < t \leq 25$	42
$25 < t \leq 30$	65
$30 < t \leq 35$	24
$35 < t \leq 40$	17

Draw a histogram of the data and calculate the mean and median age of the Facebook users.
How are the data distributed? Explain your reasoning

Q.5 A dataset consists of the prices of all new cars sold in Ireland last year. For this dataset which do you think would be higher, the mean or the median, or are they about equal?

Q.6 Draw a rough histogram of a dataset that is skewed to the right.

A dataset consists of the salaries in a company employing 100 factory workers and two highly paid executives. For this dataset which is higher, the mean or median or are they about equal?

Would the range or quartiles be more heavily influenced by outliers?

Q.7 The following data set represents the ages, to the nearest year, of 27 university students in a statistics class.

17 21 23 19 27 18 20 21 28 31
18 21 24 30 25 19 22 27 35 18
29 22 20 30 28 21 23

e. Determine the mean, median and mode for the data set

f. Do you think the data is normally distributed? give a reason for your answer

g. Determine the standard deviation of the data.

Q.8 A dataset consists of the ages when students in your class will get married. If you were to draw a histogram representing the relative frequency of marriage at different ages for the students in your class what would the histogram look like?

i. Would it be symmetric, skewed to the right or skewed to the left?
ii. For this dataset which do you think would be higher, the mean or median, or are they about equal?
iii. 'Most Irish people have more than the mean number of legs for human beings.' Explain how this can be, using the concepts of mean, median and mode.
iv. Explain how most Irish houses cost less than the average house price in Ireland.

Q.9 In a data set would the range, the standard deviation or the inter-quartile range be more heavily influenced by outliers?

A dataset consists of percentage scores obtained by students on an examination. The median of the students marks is 50%, the lower quartile (ie 25th percentile) is 25% and the mean of the marks is 70%. Would you expect the upper-quartile of the data set (ie the 75th percentile) to be greater or less than 75%?

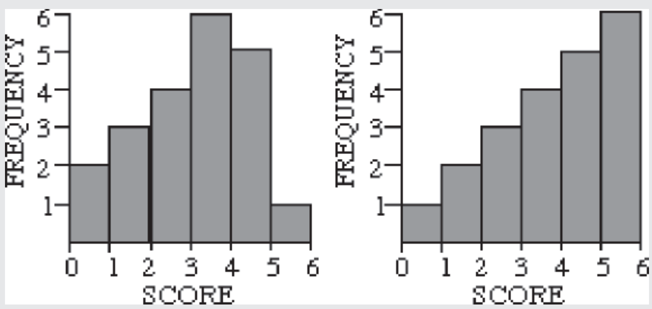
What Makes the Standard Deviation larger or smaller?
Study the 15 pairs of graphs that follow. The mean for each graph (μ) is given just above each histogram.

For each pair of graphs presented below:

1. Indicate whether one of the graphs has a larger standard deviation than the other or if the two graphs have the same standard deviation.
2. Identify the characteristics of the graphs that make the standard deviation larger or smaller.

1.

A $\mu = 2.57$ B $\mu = 3.33$



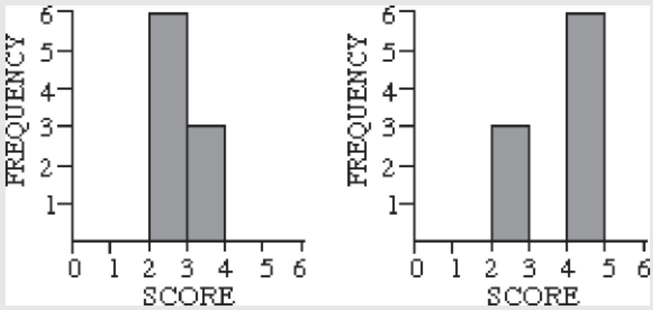
A has a larger standard deviation than B ☐

B has a larger standard deviation than A ☐

Both graphs have the same standard deviation ☐

2.

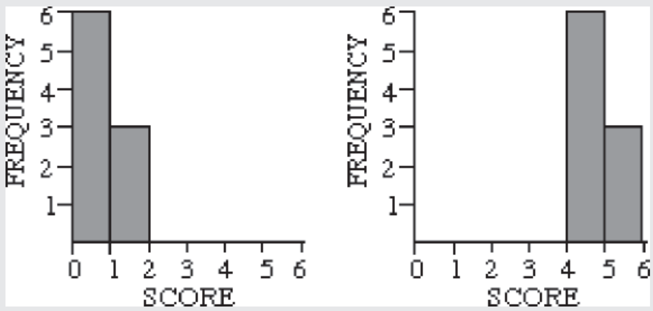
A μ = 2.57 B μ = 3.33



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

3.

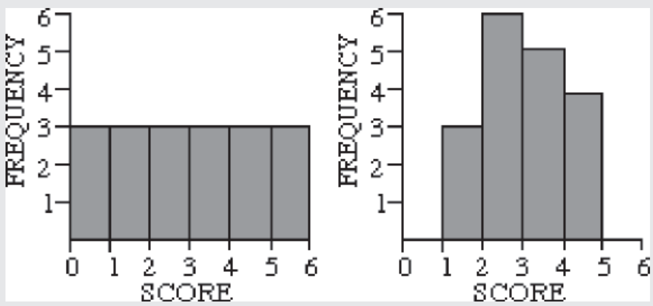
A μ = .33 B μ = 4.33



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

4.

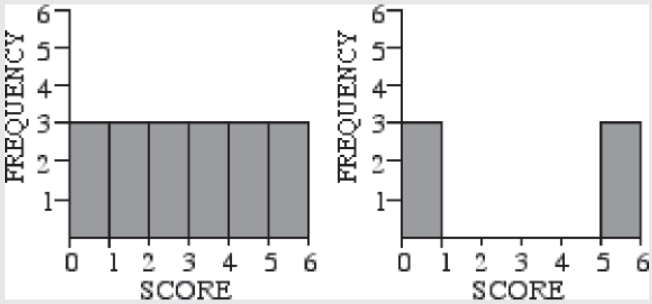
A μ = 2.50 B μ = 2.56



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

5.

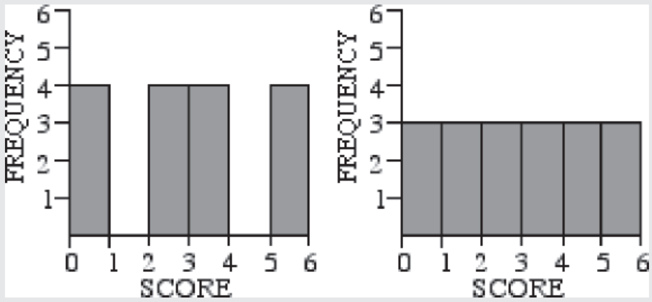
A $\mu = 2.50$ B $\mu = 2.50$



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

6.

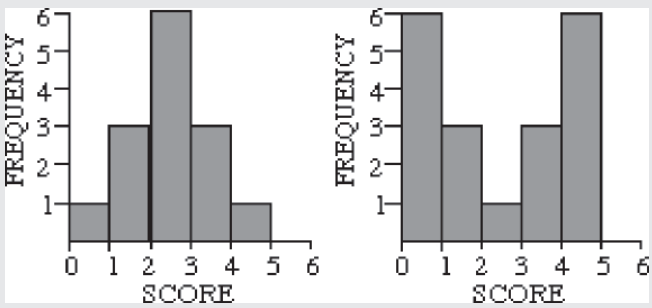
A $\mu = 2.50$ B $\mu = 2.50$



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

7.

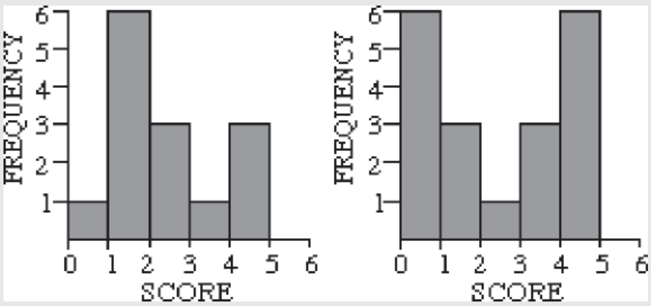
A $\mu = 2.00$ B $\mu = 2.00$



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

8.

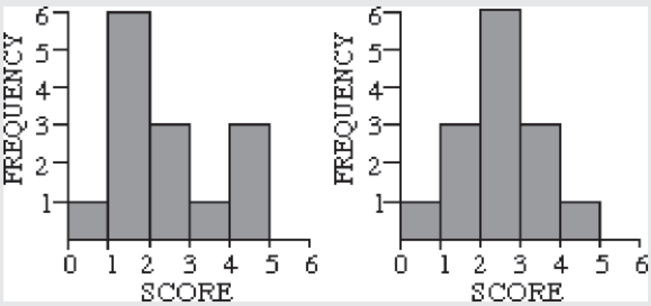
A $\mu = 1.93$ B $\mu = 2.00$



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

9.

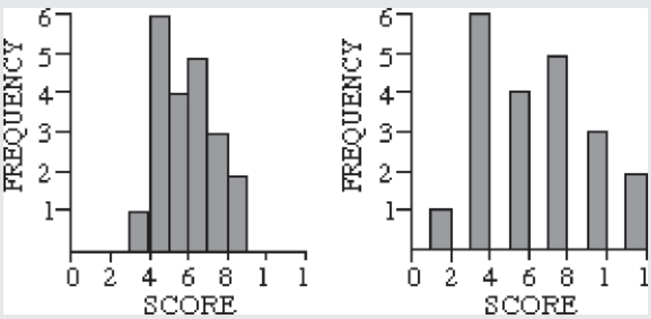
A $\mu = 1.93$ B $\mu = 2.00$



- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

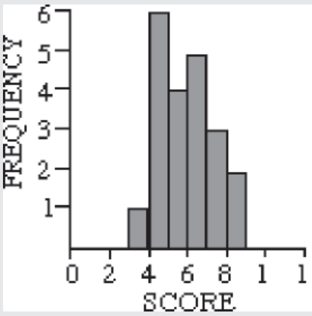
10.

A $\mu = 5.43$ B $\mu = 5.86$

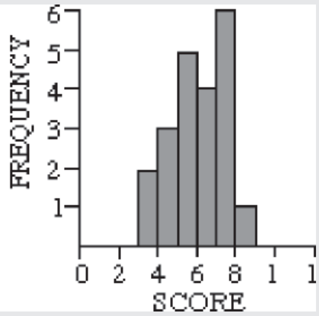


- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

11. A $\mu = 5.43$

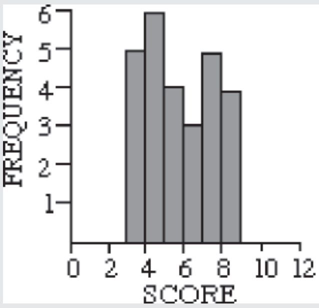


B $\mu = 5.57$

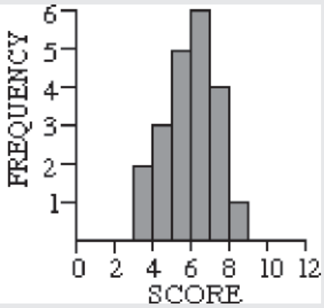


- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

12. A $\mu = 5.33$

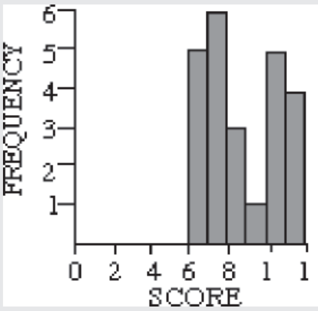


B $\mu = 5.48$

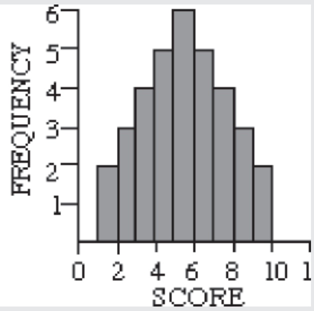


- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

13. A $\mu = 8.33$



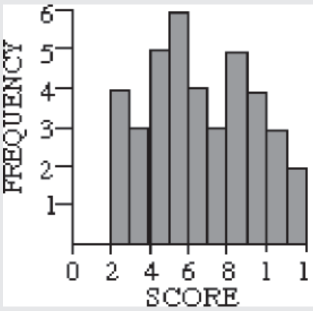
B $\mu = 5.00$



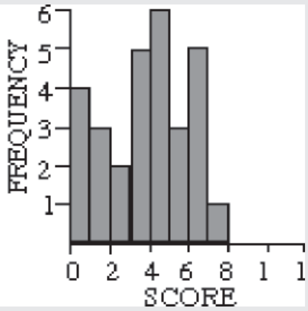
- A has a larger standard deviation than B ☐
- B has a larger standard deviation than A ☐
- Both graphs have the same standard deviation ☐

14.

A $\mu = 6.15$



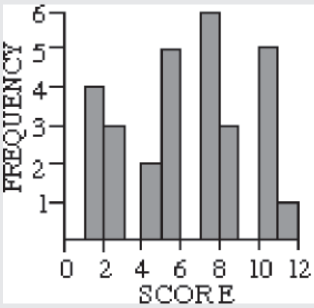
B $\mu = 3.38$



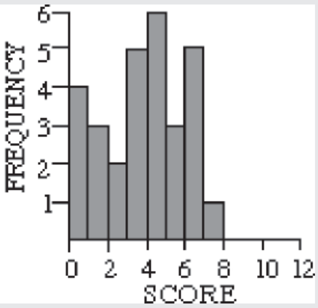
A has a larger standard deviation than B	☐
B has a larger standard deviation than A	☐
Both graphs have the same standard deviation	☐

15.

A $\mu = 5.86$



B $\mu = 3.38$



A has a larger standard deviation than B	☐
B has a larger standard deviation than A	☐
Both graphs have the same standard deviation	☐

Activity 3.2

The office manager of a small office wants to get an idea of the number of phone calls made by the people working in the office during a typical day in one week in June.

The number of calls on each day of the (5-day) week is recorded. They are as follows:
Monday:15; Tuesday: 23; Wednesday: 19; Thursday: 31; Friday: 22

1. Calculate the mean number of phone calls made

2. Calculate the standard deviation (correct to 1 decimal place).

3. Calculate 1 Standard Deviation from the mean:

$\bar{x} \pm \mathbf{s}$ = ...or ....

The interval of values is ($\bar{x} - \mathbf{s}$; $\bar{x} + \mathbf{s}$) = ...

4. On how many days is the number of calls within one Standard Deviation of the mean?

Number of days =
Percentage of days =
Therefore the phone calls on ....% of the days lies within 1 Standard Deviation of the mean.

Activity 3.3

The pilot study for a Census@School project gave the following data for the heights of 7 067 students from Grade 3 to Grade 11.

Height Less than (m)	Total number of students
$0 \leq h < 1.06$	8
$1.06 \leq h < 1.21$	111
$1.21 \leq h \leq 1.36$	1,114
$1.36 \leq h \leq 1.52$	2,218
$1.52 \leq h \leq 1.67$	2,413
$1.67 \leq h \leq 1.83$	1,105
$1.83 \leq h \leq 1.98$	108

The mean x = 1.52 cm and the Standard Deviation s = 15 cm

Draw a histogram of this data .Does the distribution look symmetric?

a.

i. Calculate $\bar{x} + s$

ii. Calculate $\bar{x} - s$

iii. What is the total number of students whose heights are less than $\bar{x} + s$?

iv. What is the total number of students whose heights are less than $\bar{x} - s$?

v. Calculate the number of students whose heights are within 1 standard deviation of the mean

vi. Write this number as a percentage of the total number of students

b.

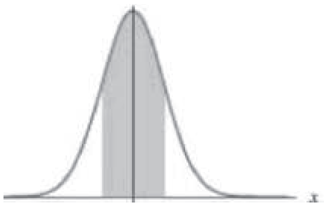
i. Calculate the number of students whose heights are within 2 standard deviations of the mean i.e. within the interval ($\bar{x} - 2s$; $\bar{x} + 2s$)

ii. Write this as a percentage

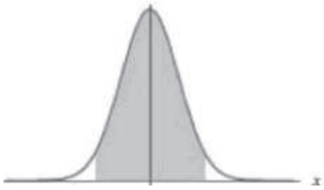
iii. Calculate the number of students whose heights are within 3 standard deviations of the mean i.e. within the interval ($\bar{x}$ -3s ; $\bar{x}$ +3s)

iv. Write this as a percentage.

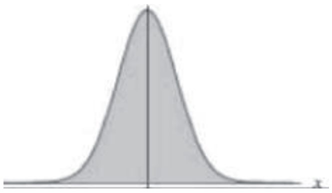
In the activities above you worked out the percentage of data items within 1, 2 and 3 standard deviations of the mean. Summarise your findings below. [Your findings are usually referred to as the empirical rule]



……% of the population falls within 1 standard deviation of the mean.



…….% of the population falls within 2 standard deviations of the mean.

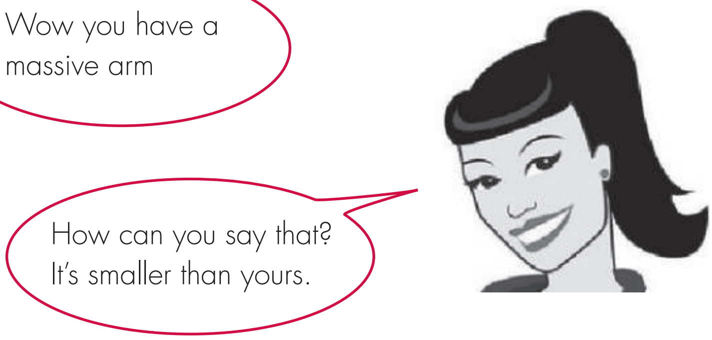


……% of the population falls within 3 standard deviations of the mean.

Q.10 For data that is symmetrically distributed and bell shaped (similar to a normal distribution), approximately what proportion of observations lies within one standard deviation of the mean according to the empirical rule? What proportion lies within two standard deviations of the mean? What proportion lies within three standard deviations of the mean?

Activity 3.4

Sophie and Jack were discussing proportions in an Art class. Sophie measured the length of her arm and found it to be 783mm long. Jack measured the length of his arm and found it to be 789mm long.



Later in the day their mathematics teacher suggested they use mathematics to see if there was any truth in Jack's statement. Sophie and Jack trawled the internet for data on arm lengths and found the results of a survey, which are summarised in the table below, recording the lengths of the arms of 500 females over the age of 16 and 500 males over the age of 16 measured from the shoulder to the finger tip with the arm outstretched.

Arm length (mm)	No of females	No of males	No of adults
$620 \le x \le 640$	3	0	
$640 \le x \le 660$	11	0	
$660 \le x \le 680$	41	0	
$680 \le x \le 700$	92	0	
$700 \le x \le 720$	132	2	
$720 \le x \le 740$	120	9	
$740 \le x \le 760$	69	27	
$760 \le x \le 780$	25	71	
$780 \le x \le 800$	6	114	
$800 \le x \le 820$	1	122	
$820 \le x \le 840$	0	89	
$840 \le x \le 860$	0	46	
$860 \le x \le 880$	0	15	
$880 \le x \le 900$	0	4	
$900 \le x \le 920$	0	1	
Total	500	500	

Work in groups of three to complete the following tasks.

1. Choose a data set to work with; female, male or adult.
2. Work with your set of grouped data and calculate
   a. The mean
   b. The median
   c. The standard deviation
3. Draw a histogram to illustrate your data
4. Check the following:
   a. Do the mean and median of your set of data have approximately the same value?
   b. Does approximately 99.7% of the data lie within three standard deviations of the mean?.
5. Compare the histograms and comment on similarities and differences. Which set of data, if any, is normally distributed?

Extension for HL

Refer to Jack's statement. Is Sophie's arm length unusual? Explain your answer.
What about Sophie's reply? Is Jack's arm length unusual? Explain you answer.

Is it reasonable to conclude from the data that men have longer arms than women?

Q.11 A recent survey of Irish school-going teenagers reported their 'Attitude-Toward-Authority' scores to have mean 107 and standard deviation 14 among the males, and 115 and 13, respectively, among the females. A score which is higher than 90 indicates pro-authority feelings.

1. Relative to his/her own group, who is more pro-authority: a male teenager with a score of 120 or a female with a score of 125?
2. Assuming that the scores are normally distributed, what proportion of the male teenagers can be considered pro-authority?
3. In a group of 250 female teenagers, how many do you expect to be pro-authority?
4. Ninety percent of female teenagers have their scores between 93.55 and what other score?
5. A teenager is considered rebellious if he is in the first percentile among his peers. Suppose Luke's score is 79; can we consider him rebellious?
6. Where does the Empirical rule say that 95% of the observations lie in a distribution which is approximately bell shaped?

Q.12 Record the heights in centimetres and weights in kilograms of all of the people in your mathematics class. Compute the mean and standard deviation of these heights and weights. How many heights in your data set were within one standard deviation of the mean height (i.e. between the mean minus one standard deviation and the mean plus one standard deviation)? How many were further than three standard deviations from the mean? Answer the same two questions for the weights.

Q.13 100 students sit a statistics examination and the marks scored by the students are found to be approximately normally distributed. The mean mark scored by students is 50% and the standard deviation of the students' marks is 5%. Two students scored 99%. These students were called out of class by the principal, who had a degree in statistics, and accused of cheating on the examination. Why did the principal feel that the students had cheated?

Q.14 Using the webpage http://uk.finance.yahoo.com/, or another source search, for historical share prices for any company listed there. From this webpage it is possible to download share price information into a spreadsheet. Download the daily share prices of any company for the last year.

a. For these share prices compute the mean, the median and the mode.

b. Compute the interquartile range, the range and the standard deviation.

c. Do any prices appear to lie more than one and half interquartile ranges above the upper quartile? Do any share prices lie more than three interquartile ranges above the upper quartile?

d. Do any share prices lie more than four standard deviations above the mean?

e. Construct a relative frequency histogram for the daily share prices. Do the share prices seem to be normally distributed?

STATISTICS 3

SYLLABUS TOPIC: FINDING, COLLECTING AND ORGANISING DATA

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- clarify the problem at hand
- formulate one (or more) questions that can be answered with data
- explore different ways of collecting data
- design a plan and collect data on the basis of above knowledge
- generate data, or source data from other sources including the internet

Students working at OL will be given the opportunity to

- discuss different types of studies: sample surveys, observational studies and designed experiments
- select a sample (Simple Random Sample)
- recognise the importance of representativeness so as to avoid biased samples
- design a plan and collect data on basis of above knowledge

Whilst those working at HL will learn to

- recognise the importance of randomisation and the role of the control group in studies
- recognise biases, limitations and ethical issues of each type of study
- select a sample (stratified, cluster, quota etc. – no formulae required, just definitions of these)
- design a plan and collect data on basis of above knowledge

Activity 3.1

There are many situations where it is impossible or impractical to gather information on all the items in a survey.
List some such situations.

Consider the following

Frank has to find out about the breakfast eating habits of students in his school. The school is co-educational. There are 150 students in Transition Year (TY) and about 200 students in each of the other year groups.

Frank decides it is impractical to interview every student in the school, so he decides to gather the information from a sample of 100–200 students. Here are some ways for him to choose his sample

A. Frank can get to school early and use the first 100 students who arrive

a. Is any group of students likely to be under-represented?

b. Are equal numbers of boys and girls likely to be included in this sample?

c. Is age group likely to be represented in the sample in the same proportion as they are in the school?

d. Do you think a student's early arrival has any relationship with the breakfast they eat?

e. From the experiences of your own school, can you think of any reasons why this would not be a representative sample?

B. Frank can take one class from each year group as his sample.

a. If the classes are banded (i.e. grouped according to ability), is this likely to affect their breakfast eating habits?

b. Does the inclusion of a complete TY class reflect the proportion of these students in the whole school?

c. Are the age groups in the sample represented in the same proportion as they occur in the school?

d. Can you suggest ways of improving the selection of the sample to make it as representative as possible?

This kind of sample (identifying groups with different characteristics and taking a sample of each group) is called a stratified sample.

C. Frank can draw his sample from the main school register. The students are listed in this register with the boys first in alphabetical order followed by girls, also in alphabetical order.

Frank can take the first 50 boys and the first 50 girls as his sample. Alternatively, Frank can take every fifth student on the register as his sample. (This method of sampling is an attempt at random selection.)

a. Can you suggest reasons why either method may not give a sample that represents the whole school population?

b. Devise a way of selecting about 100 students from your own school so that they reflect, as nearly as possible, known characteristics of the whole set of students.

Construct a mind map or graphic organiser (see www.ncca.ie) to help you remember how to choose a representative sample from a population.

Q.1 Are men or women more adept at remembering where they leave misplaced items (like car keys)? According to University College Dublin researchers, women show greater competence in actually finding these objects. Approximately 300 men and women from Dublin participated in a study in which each person placed 20 common objects in a 12-room 'virtual' house represented on a computer screen. Thirty minutes later, the subjects were asked to recall where they put each of the objects. For each object, a recall variable was measured as 'yes' or 'no.'

a. Identify the population of interest to the researcher.

b. Identify the sample.

c. Does the study involve descriptive or inferential statistics? Explain.

d. Are the variables measured in the study quantitative or qualitative?

Q.2 Which of the following statements is correct regarding observational studies?

a. A researcher can observe but not control the explanatory variables.

b. A researcher can define but not observe the explanatory variables.

c. A researcher can minimise but not eliminate the explanatory variables.

d. A researcher can control but not observe the explanatory variables.

Q.3 Suppose you wanted to find out whether there should be fewer days in the school year. Select twenty people in your school and ask them whether schools should be closed for all of the months of May, June, July and August. What proportion of people agreed that the school should be closed?

If the Government were to hold a referendum tomorrow to decide whether to close schools for four months during the summer do you think that the same result would occur in the referendum vote as you got in your survey? Can you identify some reasons why your survey might disagree with the referendum vote?

Q.4 Many surveys today are conducted using the internet. A person, while browsing a particular website, is asked to participate in a brief survey. Do you think the results of such surveys would give the same answers as a survey of the entire population?

Q.5 Another method that survey companies have of selecting people is to hire people to go onto the street with a clipboard, stop people, and ask them questions. Do you think that the people who participate in these surveys are representative of the entire population? Can you identify any people who may not participate in a street survey like this? Would it make a difference if the survey was conducted on a Thursday in April at 2.00pm or the following Saturday at 3.00pm? Would students from secondary or primary schools be more likely to be included in one of these surveys? What about people who are at work?

Q.6 Some companies use computers to randomly dial phone numbers and, if someone answers the phone, a company employee asks the individual who answered the phone if they would participate in a survey. Do you think that a survey which is conducted like this gives an accurate reflection of the opinions of the entire population?

Q.7 A crèche needs to paint its building but has only two colours of paint available, pink and blue. The owners of the crèche will make the decision on which of the two colours to use by asking ten children which colour they should paint the building.

What do you think the answer will be if the ten children who are asked are all girls?
Do you think a different answer would be given if the ten children are boys?
Suppose there are fifty children in the crèche. Can you recommend a way to identify what proportion of girls and boys should be in the group of ten that are surveyed?

Q.8 Suppose that the government of Ruritania was concerned that mobile phones caused brain tumours. In order to determine if this is true they decided in 1980 to conduct a study in which they required one thousand people to speak on their mobile phones for ten hours every day. The study was conducted over twenty nine years and at the end of the study the number of people who developed brain tumours was measured in the group. This information was then be used to decide if mobile phones should be banned in Ruritania.

Comment on the design of this study. Do you think that the results of this study are valid? Do you think the study is ethical? How would you recommend a way to decide if mobile phones cause brain tumours?

Q.9 A company claims that it has developed a new drug which is completely harmless and has the ability to cure obesity. The company wants to market this new drug and is asked to prove that the drug works. The company says that it has conducted a 'clinical trial' of the drug in which one hundred people participated. There were two groups of fifty people in the trial. The treatment group was given the drug to take daily over a two year period and the control group did not take the drug. The company was able to show that the group who took the drug had an average weight of 50kg and the group who did not take the drug had an average weight of 80kg.

Based on this information do you think the drug works?

Suppose that you knew that the drug company had selected the fifty people for the treatment group from among the members of a gym and they chose the control group from those who ate everyday in a nearby fast food restaurant. Does this information change your opinion about the efficacy of the drug?

Suppose instead that the members of the groups were evenly balanced as to members of the gym and those who ate in the fast food restaurant. But it was noted that all of the members of the treatment group happened to be called Pauline and all the members of the control group were called Paul. Do you believe the drug was effective in reducing weight if this was the case?

When a company is selecting a control group and a treatment group to test a new drug, what differences or similarities should there be between the two groups?

In Sweden, many studies are conducted using pairs of twins. Why is this done? How would you recommend designing a study in which you had fifty pairs of twins available to you?

Q.10 Suppose that a tobacco company claims that they have discovered that one of their brands of cigarette does not cause cancer. The cigarette has been on the market for the last forty years. How should the claim that it does not cause cancer be tested? What is the best study design to use in this situation? What ethical issues may there be in designing this type of study? Is it possible to design a study to test this claim which does not pose any ethical problems?

Q.11 A teacher wants to determine which of her thirty students is best at solving a particular type of mathematical problem. To answer this question she has constructed thirty different examples of this problem and she assigns one problem to each student. Ten of the students must answer the problem on the blackboard in front of the entire class. Ten of the students must answer the problem in class while the teacher moves around behind their desks and looks over their shoulders as they solve the problem. The last ten students are allowed to answer the question in class at a time when the other twenty students are reading books and the teacher is at the top of the class also reading.

a. Do you think there will be a difference in the average performance of the three groups? Which group do you think will perform best?

b. Do you think that participants in a study perform differently depending on whether they are being observed or not?

c. Suppose that during one half of a double maths class period a teacher is present in class but during the following period the teacher is called away and asks the students to work on their own. During which period do you think students will work hardest?

Q.12 A survey is conducted in a school in the centre of Dublin to determine what students do in their time outside school. Do you think that the results of this survey would match the results of the same survey which was conducted in a rural school in County Roscommon?

Q,13 An electronic manufacturer wants to sell a new MP3 player with one hundred songs preloaded on to it. They decide that by preloading one hundred of the most popular songs they will have a best selling product when it is released onto the market. The company is based in France and they survey thirty thousand French people to determine which one hundred songs should be loaded onto the MP3 player. They then release the player in every country in the world.

Do you think that this MP3 player will be popular in Ireland? In the USA? In France? In Belgium?

Q.14 The European Union (EU) wants to standardise timekeeping and establish one time-zone for the entire EU. Before bringing forward legislation the EU decides it should gauge public opinion by carrying out a survey of EU citizens.

The population of the EU is approximately five hundred million people, so the EU decides that fifty thousand people will be surveyed.

a. How should the EU choose the fifty thousand people to participate in the survey?

b. One person suggests that to save money the survey could easily be conducted by stopping people outside the EU offices in Luxembourg and asking their opinion. Is this an appropriate method?

c. Another person indicates that the sample should be chosen as a simple random sample, so that the EU should place the names and addresses of every person in the EU over the age of 18 into a computer and then randomly select fifty thousand people who would be written to and surveyed. Is this simple random sampling approach one that you would recommend?

d. How would the principle of stratified random sampling be applied to this survey? Would it be preferable to simple random sampling?

Q.15 A company owns fifty shops, each of which is the same size and shape and stocks the same selection of products. Each shop contains ten aisles and each aisle contains different brands of one single product. For example, aisle number 1 in each shop contains 40 inch LCD televisions and aisle 2 contains packets of tea.

The company manager wants to conduct a survey of the stores to determine which brands sell best but he does not want to survey each item in every store. Instead, he wants to select a sample of items. The manager has heard that cluster sampling and stratified sampling are good techniques to use but he does not understand the difference between the two methods.

Explain how each method could be applied to select a sample of products from the company's shops. Discuss the relative merits of each sampling method in this context.

Q.16 You are asked to conduct a survey of students in your school to decide if everyone should wear a new school uniform that has a red and black striped design. You have a picture of the new uniform and you must choose a sample of fifty students to show the picture to. Design four different plans to select the students for your sample, one for each of the following different sampling strategies:

(a) Cluster Sampling, (b) Simple Random Sampling,
(c) Stratified Random Sampling, (d) Quota Sampling.

Q.17 Construct a sampling plan to decide what proportion of sweets in packets of M and Ms are yellow. Indicate why the sampling scheme you have chosen is the best possible scheme and will yield the most accurate prediction.

Q.18 Suppose that RTE wants to know how viewers feel about a new soap opera they are broadcasting. They decide that after the show they will show two phone numbers for people to call one for people who like the show and one for people who don't like the show. Will this result be biased? Explain in one sentence.

Q.19 Suppose your school principal wants to know how students feel about a policy on banning students being dropped by car to school. The principal wants to make all students travel by foot, on bicycle or by public transport. The principal doesn't want to ask all students so s/he must rely on a sample. Which of the following sampling methods would you recommend that the principal choose?

a. Walk through the school building and pick a classroom at random, choose the students inside as the sample.

b. Place a list of all students in the school on a table and randomly choose 40 students.

c. Stand outside the school in the morning time and stop every 5th student who arrives by car.

d. Randomly pick 5 students from each class in the school.

e. Stop one school bus outside the school in the morning and question all the students on board the bus prior to their disembarkation.

Did you recommend more than one method? Why did you prefer the methods that you chose? What quality were you looking for in the sample you chose? Could you recommend a better sampling scheme than any of the methods listed above?

Q.20 In order to survey the opinions of its passengers an airline made a list of all its flights and randomly selected 25 flights. All the passengers on those flights were asked to fill out a survey. What kind of sampling procedure was used in this case?

Q.21 Pre-election polls are often conducted by asking the opinions of a few thousand adults nationwide and using the results to infer the opinions of all the adults in the nation. Explain what the sample is and what the population for such polls is.

Q.22 In order to find out how people connected to your school feel about the re-introduction of undergraduate fees for third level education. Your principal considered three different categories of people: teachers in your school, students in your school and parents of students in your school. A random sample from each group was surveyed. What kind of sampling procedure was used in this case?

Q.23 An experiment is conducted to measure the effectiveness of step aerobics as a means of weight loss. What would be the explanatory variable and the outcome variable in this experiment?

List two reasons why it might be preferable to conduct a sample survey rather than a census.

Q.24 In a designed experiment the sample chosen for the study is often split into a 'treatment group' and a 'control group'. What differences or similarities exist between these two groups?

In a case-control study the sample chosen for the study is split into a 'case group' and a 'control group'. How do these two groups differ from the 'treatment group' and 'control group' used in a designed experiment?

A study wishes to examine whether there is a link between baldness and susceptibility to having a heart attack for males. Recommend how a case control study could be used to answer this question. Why is it more appropriate to use 'case and control' rather than 'treatment and control' as the categories for dividing the sample group in this study?

STATISTICS 4

SYLLABUS TOPIC: ANALYSING, INTERPRETING AND DRAWING INFERENCES FROM DATA

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- recognise how sampling variability influences the use of sample information to make statements about the population.
- develop appropriate tools to describe variability drawing inferences about the population from the sample.
- interpret the analysis
- relate the interpretation to the original question

Students working at OL will be able to

- discuss different types of studies: sample surveys, observational studies and designed experiments
- select a sample (Simple Random Sample)
- recognise the importance of representativeness so as to avoid biased samples
- design a plan and collect data on basis of above knowledge

INTRODUCTION

Throughout the activities you will analyse and interpret the data from a statistical investigation and draw inferences and conclusions based on your analysis of the data.

Q.1 In a school there are two 6th year Mathematics teachers called Holly and Luke. Over the last ten years the mean mark achieved by Holly's students on the Leaving Certificate Mathematics Exam was 60%, with a standard deviation of 2%. During the same period Luke's students also have a mean mark of 60% but the standard deviation of these students' marks is 15%.

Students in this school have a choice of teacher when entering 6th year. Which teacher should a student who is good at mathematics choose? Which teacher should a student who is weak at mathematics choose?

Q.2 Indoor radon concentrations in Ireland, X, are measured and it is found that the logs of these concentrations, Y=log(X), follow an approximate normal distribution. When 20,000 radon values were considered it was found that the mean indoor radon concentration was 89 Becquerels per cubic metre (Bm^{-3}). Using the information provided in this question would 200 Bm^{-3} or 20 Bm^{-3} be a more likely value for the standard deviation of these radon concentrations?

Suppose that a house is found with an indoor radon concentration of 40,000 Bm^{-3}. Taking the mean and standard deviation above, and taking into account the normality of the Y values, do you think that there is anything unusual about this house? Would you consider a measurement of 8,000 Bm^{-3} to be an outlier for this data set?

Q.3 Two hundred athletes enter a race. One athlete wins the race with a time that is very much faster than any of the other athletes. The athlete is accused of using performance enhancing drugs. At a subsequent court hearing the judge becomes frustrated with the cases put forward by the defence and the prosecution. The judge calls you as an expert witness to use your knowledge of statistics to try to establish the truth. What do you do? How would you analyse the data to establish the innocence or guilt of the accused athlete?

Q.4 One thousand people are selected using simple random sampling to take part in a survey. The participants are blindfolded and are asked to taste two different brands of crisps and to indicate which of the two brands, A or B, they prefer. What is the margin of error for this survey? If 60% of people in the sample indicated a preference for Brand A, what can you say about the proportion of individuals in the population that preferred brand B?

How many individuals should participate in a survey to achieve a margin of error of 2%?

Q.5 The Blue Party have been the party in power for the last 4 years. The leader of the Blue Party, knows that he will have to face the electorate at some time during the next year. He can call an election today or wait for up to 12 months before he must call an election. He wants to maximise the chances that his Blue Party is re-elected to government, so he asks a polling company to conduct an opinion poll of the electorate. The company is instructed not to allow the knowledge of this poll to become known so it decides to survey just 100 people rather than the usual 1000. The survey indicates that 56% of those polled say they will vote for the Blue Party. The sample was chosen using simple random sampling to be representative of the entire electorate. Should the leader of the Blue Party call an election on the basis of the survey results?

Q.6 A national newspaper publishes the results of an opinion poll on the 22nd of May indicating that 45% would vote for the Democratic People's Party. The following week a different national newspaper reports that a new poll shows 42% of people would now vote for the Democratic People's Party. This newspaper prints the headline 'Significant decline in support for Democratic People's Party'. If a simple random sample of 1000 people were surveyed in each of the two polls, what is your opinion of the newspaper's headline?

Activity 4.1

Do taller people have longer arms than shorter people?

Select 20 students from your school and line them up from the shortest to the tallest. Now hand a piece of paper with the letter A to each of the 10 shortest people and a B to the 10 tallest people. Now ask everyone to hold their arms out horizontally and measure the length of each person's arms from the tip of the middle finger on their left hand to the tip of the middle finger on their right hand, measuring across their backs.

Write the length that you have measured on the piece of paper you previously gave to each person. Gather up each of the pieces of paper and order them from the one with the smallest 'arm length' to the one with the longest 'arm length'. Now starting with the first piece of paper write down whether it contained an A or a B, continue writing down As or Bs for each piece of paper until you have a sequence of As and Bs.

Now count the number of As before the first B and add it to the number of Bs after the last A to get a value for the 'Tukey Test Statistic'.

Now making use of the Tukey Quick Test Tables answer the original question: 'Do taller people have longer arms than shorter people?'

Q.7 A company believes it has invented a new drink which will make people better able to understand statistics. The company recruits twenty students studying for a Statistics degree to take part in this experiment. The company does not see the point in using control groups and treatment groups and so it gives its new drink to all the students and measures the students' ability to answer statistical questions. The students all perform well so the company believes it has proven that its drink can improve peoples' statistical understanding.

Comment on the validity of this study.

How would you design a different study to test the effectiveness of the new drink making use of the concept of control groups and treatment groups?
Would this study have any benefits over the study that was conducted by the company? If so, what are the benefits?

Q.8 It is believed that randomisation plays an important role in designing studies to test the effectiveness of a new product. Why is randomisation important?

Consider the case of a company which has developed a shampoo that it claims will make men irresistible to women. To test this claim the company's female scientists recruit forty male test subjects. The female scientists choose some of the male subjects because they consider the males to be particularly handsome, other males are chosen by the female scientists because they consider these males to be particularly unappealing.

The test will involve assigning twenty male test subjects to a treatment group who will wash their hair with the new shampoo every day for a week. The other twenty males will wash their hair with an ordinary shampoo chosen by the scientists. At the end of the week both groups will be sent to a reality television show where a judging panel of five celebrity females will rate each male as to their desirability.

a. How should the forty males be assigned to the treatment and control groups to test the effectiveness of this new product?
b. Why would randomisation play an important role in this assignment?
c. If randomisation were not employed in this study what problems could arise in determining the effectiveness of the shampoo?
d. Suppose that twenty of the males were initially classified as handsome and twenty were classified as unappealing by the female scientists. One of the scientists recommends that in this situation it might be possible to assign the males in a non-random manner to the control and treatment groups. The scientist says that in this particular situation she can produce a study design which is better than the study design that uses randomisation. Is she right? Why? What do you believe is her improved non-random assignment method?

Q.9 Do men or women have more friends?

To answer this question a researcher asked 10 men and 10 women to list the names of their friends whom they had spent time with during the past year. The researcher then counted the numbers of friends for each woman and each man. The results are given below:

Men: 5, 9, 10, 4, 13, 22, 21, 19, 28, 17
Women: 35, 21, 10, 18, 9, 18, 23, 42, 33, 29

Apply the Tukey Quick Test to answer the original question: 'Do men or women have more friends?'

Q.10 A student was given two lists of numbers by his teacher and asked to perform the Tukey Quick Test to decide which of the lists contains the larger numbers.
The two lists of numbers were:

List A: 10, 12, 40, 56, 23, 34, 43, 19, 21,
List B: 8, 10, 34, 36, 23, 36, 39, 17, 9.

The student refused to perform the Tukey Quick Test, saying instead that it was obvious that the numbers in list A were larger than the numbers in List B and that there was no point in performing the Tukey Quick Test.

The teacher was dismayed at the student's refusal and reported him to the principal, where the student still steadfastly refused to conduct the Tukey test, insisting that it was obvious that the numbers in List A were larger than those in List B. The student was expelled from school for subordination. What do you think of the student's position? If you were a statistically literate judge at the subsequent court case, whose side would you rule in favour of, the school or the student? Why?

Q.11 What is the point of performing a Hypothesis test? In making decisions based on samples of data, why can't we just calculate means or medians for each sample and compare the means or medians to reach a conclusion?

STATISTICS 5

SYLLABUS TOPIC: REPRESENTING DATA GRAPHICALLY AND NUMERICALLY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- use scatterplots to determine the relationship between variables (OL)
- recognise that correlation is a value from -1 to +1 and that it measures the extent of linear relationship between two variables (OL)
- match correlation coefficient values to appropriate scatter plots (OL)
- draw the line of best fit by eye (HL)
- make predictions based on the line of best fit (HL)

Activity 5.1

Consider the following:

The Proprietor of Mike's Machines Garage was anxious to encourage his customers to regularly change the oil in their vehicles. He surveyed his customers over a period of time and recorded the data in the table below.

Oil Changes per Year	3	5	2	3	1	4	6	4	3	2	0	10	7
Cost of Repairs (€)	300	300	500	400	700	250	100	400	450	650	600	0	150

a. Represent this data in a graph.

Think: How should the axes be labeled? Should you include units on the axes? What scale should you use on each axis?

b. Visualise a straight line as a representation of the data. Draw a line that seems to 'fit' the plotted points. [A line that 'fits' the points should have the same characteristics as the set of points; it should actually summarise the data. A line drawn this way is called a line of best fit by eye.]

c. Look at the points you plotted and the line of best fit. Do you think there is a relationship?

If so:

- Is it a positive or negative relationship? (Look at how the line 'slopes' to help you decide.)
- Is it a strong or weak relationship? (Look at how close the points are to the line of best fit to help you decide.)

Is there a correlation between the cost of repairs and the number of oil changes per year? If so, describe this correlation.
Do you think the number of oil changes is the only factor which causes the need for engine repairs?
Think of other factors that might affect the need for engine repairs.

It is important to understand that a relationship doesn't mean a cause.

d. Would this data convince Mike's customers that they should change their oil regularly? Give a reason for your answer.

e. According to Mike's data what is the optimum number of oil changes per year?

f. By choosing some points on the line you have drawn that best fits the data, calculate the equation of this 'line of best fit'.

g. Use your graph to predict how much a customer is likely to spend on repairs if they change the oil 5 times a year.

h. Comment on your answer to g) above; is it good value to change the oil 5 times in the year.

i. Examine the slope of the graph; you can do this by counting units.

It is an important learning outcome that you can interpret the slope as 'rate of change'.
What does a 'unit' represent for the number of oil changes per year?
What does a 'unit' represent for the cost of engine repairs?

Complete the sentence below, describing the change in the cost of repairs with the number of oil changes.

A rate of change (or slope) indicates that, for each additional oil change per year, the cost of engine repairs will tend to

Q.1 The chart below is used by veterinary surgeons to decide the dose of a certain drug to be used in fighting bacterial infection in the joints of small animals.

Animal's Weight (kg)	Usual dosage (mg)	Max dosage (mg)
0.9	9	13.5
1.4	14	21
1.8	18	27
2.3	23	34.5
2.7	27	40.5
3.2	32	48
3.6	36	54
4.1	41	61.5
4.5	45	67.5
5.0	50	75
5.4	54	81
5.9	59	88.5
6.4	64	96
7.7	77	115

1. Use graph paper to plot the data (animal weight, usual dosage) and draw a line of best fit.
2. Plot (animal weight, maximum dosage) on the same axes. Draw a line of best fit.
3. Find the slope for each line. What do they mean, and how do they compare?
4. By choosing two points on your lines of best fit write an equation for each of the two lines.
5. Are the two lines parallel? Why or why not?

Note: The data is real data obtained from the drug company website.

Q.2 In BMX dirt-bike racing, jumping high or 'getting air' depends on many factors: the rider's skill, the angle of the jump, and the weight of the bike. Here are data about the maximum height for various bike weights.

Weight (kg)	Height (cm)
8.6	26.3
8.8	26.2
9.1	26
9.3	25.9
9.5	25.7
9.8	25
10.4	24.9
10.2	24.9
10.7	24.6
10.9	24.4

1. Use graph paper to plot the data (weight, height). If the data show a linear relationship, draw a line of best fit.
2. Explain the relationship between bike weight and jump height by completing the following sentence:

   As the weight of the bike increases

3. Use two points on the line of best fit to find the slope or rate of change. What does this mean?
4. Predict the maximum height for a bike that weighs 9.8kg if all other factors are held constant.

Note: Data obtained from BMX magazine article.

Q.3 Use the data provided to help you decide whether or not there is a relationship between the Total Fat content of fast food and the number of Total Calories in the food.

Sandwich	Total Fat (g)	Total Calories
Hamburger	9	260
Cheeseburger	13	320
Quarter Pounder	21	420
Quarter Pounder with Cheese	30	530
Big Mac	31	560
Sandwich Special	31	550
Sandwich Special with Bacon	34	590
Crispy Chicken	25	500
Fish Fillet	28	560
Grilled Chicken	20	440
Grilled Chicken Light	5	300

Q.4 A student was investigating whether students who study more watch less television.

The results obtained are displayed below.

Number of hours per week spent watching television	Number of hours per week spent studying
33	10
28	10
27	12
22	11
28	13
25	15
21	17
23	20
19	20
18	20
21	25
17	25
12	25
7	28
4	30
11	30
22	30
10	32
8	32

Draw a suitable graph and use it to decide whether or not there is a correlation between the time per week spent studying and the time per week spent watching television. Describe the correlation, if there is one.

The students who gathered this information said there is a correlation between the two variables. They said the equation of the line of best fit is; $y = x + 40$.
Look at your graph and explain how you can tell that this equation is wrong.
Draw a line of best fit.
Choose any two points on this line and use them to find the equation of the line of best fit.

A student who was absent on the day of the investigation reported that he spends 6 hours per week watching TV. Use the information the students gathered to predict how many hours that student spends studying /week.

Is it possible that your predicted answer is incorrect? Explain your thinking.

Q.5 (a) Which implies a stronger linear relationship, a correlation coefficient of +0.2 or –0.5? (b) Is the correlation between the following pairs of variables likely to be strong, moderate, or weak? Is it likely to be positive or negative?

a. Daily rainfall in Sydney and Dublin
b. The number of hours that students spend studying during three weeks prior to their Leaving Certificate Mathematics Exam and their percentage score on the exam
c. Daily rainfall in Drogheda and Dundalk
d. Engine size of a car and its petrol consumption
e. Weight of a randomly selected woman and the amount of food she eats
f. The average daily temperature for different towns in Europe and the average cost of heating homes in the same towns
g. Number of students in University College Dublin who drive cars to college each day and number of free car parking spaces in the university each day

Q.6 Is there a relationship between a student's height and their shoe size? Select twenty students from your school and make a note of the height (X) and shoe size (Y) for each student.

Draw a scatterplot of the data. Does there seem to be a relationship between the two variables?
Compute the correlation coefficient for this set of data and use this value to answer the question as to whether there is a relationship between a student's height and their shoe size.

When you select students to participate in this study should you choose students who are all of similar heights? If your school had both girls and boys would it make sense to have both in your study or should you limit your group to just males or females?

Q.7 The shelf life of packaged food depends on many factors. No one likes soggy cereal so it is clear that moisture content is important in determining the shelf life of cereal. Statistics students with part-time jobs in supermarkets conducted an experiment on one particular brand of cereal. They recorded time on shelf X (days), and moisture content Y (percentage). The table below shows the data they collected.

X	0	3	6	8	10	13	16	20	24	27	30	34	37	41
Y	4.1	4.3	4.4	4.9	2.8	3.0	3.1	3.2	3.4	3.4	3.5	3.1	3.8	4.0

i. Construct a scatterplot for this data. From the scatterplot does it seem that there is a relationship between the length of time that a package spends on the shelf and the moisture content of the cereal package?

ii. Compute a correlation coefficient for this data set and interpret the value of the correlation coefficient in the context of this data.

Q.8

a. University researchers conducted a study looking at children under the age of 12 and they found that there was a strong positive correlation between the numbers of fillings in children's teeth and the children's vocabulary. Does this mean that eating more sweets would increase a child's vocabulary? Explain.

b. These researchers also conducted a study where they examined each country in the world and they found that there was a strong positive correlation between the number of storks in a country and the number of babies born in that country. When a newspaper discovered this information they had a front page headline which read 'Researchers have shown that storks really are responsible for bringing babies'. Explain the error made by the journalist. Can you identify a possible explanation for the researchers' result?

c. The same researchers also found a strong positive correlation between the sales of ice-cream in Ireland and the number of people who drowned in Ireland for each week of the year. Does this mean that consuming ice-cream increases the likelihood that someone will drown? Suggest an explanation for the result the researchers found.

Q.9 The following information was obtained from the manager of a local Water Department for predicting the weekly consumption of water in litres from the size of household:

Household Size	2	7	9	4	12	6	9	3	3	2
Water Used	650	1200	1300	430	1400	900	1800	640	793	925

a. Without performing any computations, predict what the correlation coefficient would be for this set of data.

b. Which variable should be labelled X and which should be labelled Y in a scatter plot of this data?

c. Construct a scatter plot of this data and determine whether there is a relationship between household size and water consumption.

d. By looking at the scatter plot decide whether you believe your previous estimate of the correlation coefficient. If necessary make a new prediction based on the scatter plot.

e. Compute the correlation coefficient for this data set. Does the computed correlation coefficient match with your previous two predictions?

f. Interpret the value of the correlation coefficient that you computed. Suppose that, instead of measuring the water consumption in litres, the engineers had measured the water consumption in gallons. If you were to convert the values in the table above from litres to gallons what effect would this have on the correlation coefficient?

Q.10 Which of the following is not a property of correlation?

a. A negative correlation indicates that the variables increase together.

b. A correlation will be between −1 and +1.

c. Correlations are not affected by changes in units of measurement.

d. A correlation of zero indicates that there is no linear relationship between the two variables

NCCA
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Junior Certificate – Strand 1

Statistics and Probability

INTRODUCTION

This material is designed to supplement the work you do in class and is intended to be kept in an A4 folder. Activities are included to help you gain an understanding of the mathematical concepts and these are followed by questions that assess your understanding of those concepts. While there are spaces provided in some activities/questions for you to complete your work, you will also need to use your copybook/A4 pad or graph paper. Remember to organise your folder so that it will be useful to you when you revise for tests and examinations. As you add pages to your folder, you might consider dating or coding them in a way that associates them with the different topics or syllabus sections. Organising your work in this way will help you become personally effective. Being personally effective is one of the five key skills identified by the NCCA as central to learning (www.ncca.ie/keyskills). These key skills are important for all students to achieve their full potential, both during their time in school and into the future.

As you work through the material in this booklet and with your teacher in class, you will be given opportunities to develop the other key skills. You will frequently work in pairs or groups, which involves organising your time effectively and communicating your ideas to the group or class. You will justify your solutions to problems and develop your critical and creative skills as you solve those problems. As you complete the activities you will be required to process and interpret information presented in a variety of ways. You will be expected to apply the knowledge gained to draw conclusions and make decisions based on your analysis. The sequence in which the sections/topics are presented here is not significant. You may be studying these in a different order, or dipping in and out of various sections over the course of your study and/or revision.

The questions included in this booklet provide you with plenty of opportunities to develop communication skills and to promote mathematical discourse. When your teachers mark your work they will gain insights into your learning and will be able to advise you on what you need to do next.

The material in the booklet is suitable for Junior Certificate. It builds on the concepts learned in primary school and continues the investigative and experimental approach to learning about data, data handling, and probability (chance). Through completing the activities and questions contained in this booklet, you will develop a set of tools that will help you become a more effective learner and these tools can be used across the curriculum. Solving problems of this nature should also improve your confidence in doing mathematics, thus helping you to develop a positive attitude towards mathematics and to appreciate its role in your life.

The mathematics syllabus documents can be accessed at www.ncca.ie and you will find other relevant material on www.projectmaths.ie.

PROBABILITY 1

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- decide whether an everyday event is likely or unlikely to happen
- recognise that probability is a measure on a scale of 0 - 1 of how likely an event is to occur.
- connect with set theory; discuss experiments, outcomes, sample spaces
- use the language of probability to discuss events, including those with equally likely outcomes

INTRODUCTION

The activities described below and the questions that follow give you the opportunity to reinforce your understanding of the basic concepts of probability. The activities are designed to build on previous experiences where you estimated the likelihood of an event occurring. Some of the activities will be done in class under the direction of your teacher; others can be done at home.

Activity 1.1

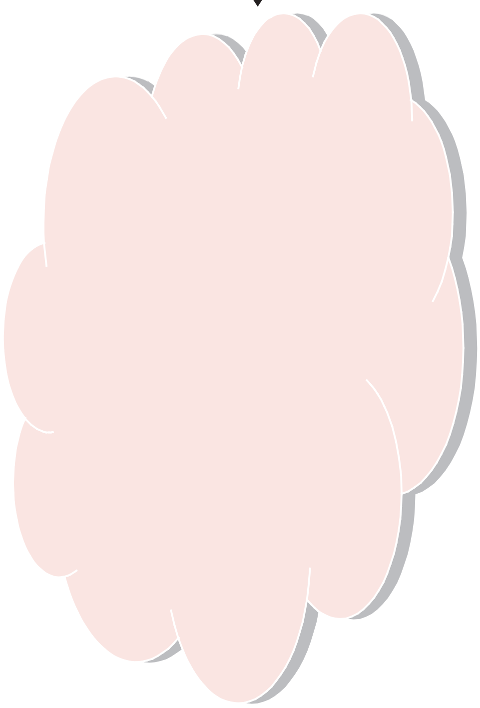
A probability describes mathematically how likely it is that something will happen. We can talk about the probability it will rain tomorrow or the probability that Ireland will win the World Cup.

Consider the probability of the following events

- It will snow on St Patrick's day
- It will rain tomorrow
- Munster will win the Heineken Cup
- It is your teacher's birthday tomorrow
- You will obtain a 7 when rolling a die
- You will eat something later today
- It will get dark later today

Words you may decide to use: certain, impossible, likely, very likely

Student Activity

Certain not to happen	Area of Uncertainty	Certain to happen
1.		1.
2.		2.
3.		3.

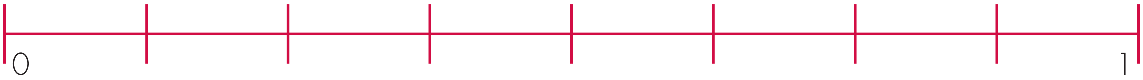
Phrases used to describe uncertainty

1.
2.
3.
4.
5.

Use the table provided or mark your work page out in a similar way and place each of the events in the approriate section.Note the phrases you used to describe uncertainty.

Activity 1.2

The Probability Scale



Extremely unlikely	50/50	3/8	1 in 4 chance
Probability of getting an odd number when rolling a die	87.5%	Extremely likely	1/2
	0.125	3/4	Impossible
1/4	Certain	75%	1
Equally likely	0.25	0	

1. Place the above phrases, numbers and percentages at the correct position on the probability scale.
2. Find and write down instances from TV, radio, or in the newspaper which illustrate how probability affects people's lives.

Questions

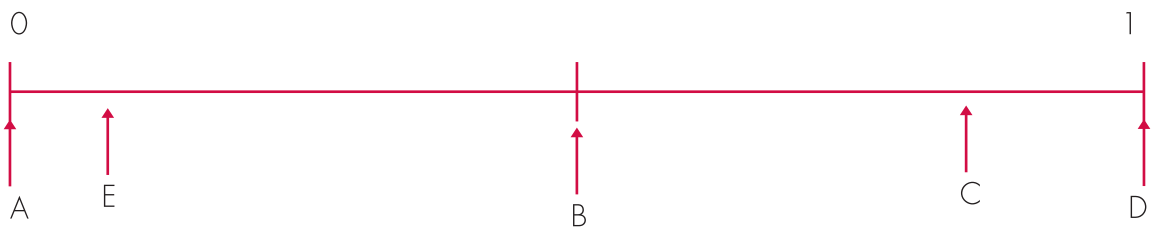
Q.1 For each event below, estimate the probability that it will happen and mark this on a probability scale.

- It will snow in Ireland on August 16th
- Your maths teacher will give you homework this week
- Your will eat fish later today
- You will go to bed before midnight tonight
- You will go to school tomorrow

Q. 2 Use one of the words certain, likely, unlikely, impossible to describe each of the events below. Give a reason for each of your answers.

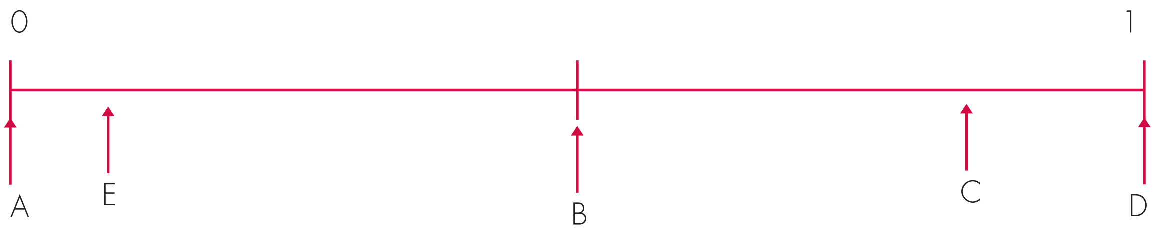
- You are more than 4 years old
- You will arrive on time to school tomorrow
- You will miss the school bus tomorrow
- Your county will win the Championship this year.

Q. 3 The probability line shows the probability of 5 events A, B, C, D and E



a. Which event is certain to occur?
b. Which event is unlikely but possible to occur?
c. Which event is impossible?
d. Which event is likely but not certain to occur?
e. Which event has a 50:50 chance of occurring?

Q. 4 The events A, B, C, D have probabilities as shown on this probability line;



i. Which event is the **most likely** to take place?

ii. Which event is the **most unlikely** to take place?

iii. Which event is **more likely than not** to take place?

Q. 5 When you toss an unbiased coin the probability of getting a head is ½, because you have an equal (or even) chance of getting a head or tail. Name two other events that have a probability of ½.

Q. 6 The 'events' A, B, C, D are listed below;

A: You will live to be 70 years old
B: You will live to be 80 years old
C: You will live to be 100 years old
D: You will live to be 110 years old

Make an estimate of the probability of each event, and place it on a probability scale.

Q. 7 Sarah and Alex are exploring probability and Sarah has these cards:

2	3	9	5	6	7	8	9
△	△	■	△	■	■	△	△

Alex takes a card without looking. Sarah says

> On Alex's card ■ is more likely than △

i. Explain why Sarah is wrong.

ii. Here are some words and phrases that can be associated with probability:

impossible	not likely	certain	likely

Choose a word or a phrase to fill in the gaps below.

It is .... that the number on Alex's card will be smaller than 10.

It is .... that the number on Alex's card will be an odd number.

Sarah mixes up the cards and places them face down on the table.
Then she turns the first card over, like this:

Alex is going to turn the next card over

iii. Complete the sentence:

On the next card, __________ is less likely than __________

The number on the next card could be higher than 5 or lower than 5

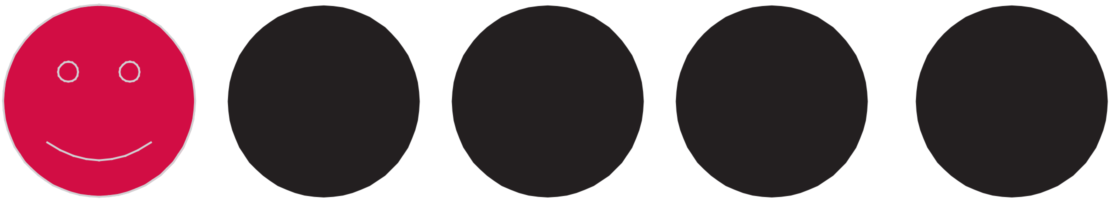
iv. Which is more likely? Tick the correct box below.

☐ Higher than 5 ☐ Lower than 5 ☐ Cannot tell

Explain your answer.

Q. 8 Lisa has some black counters and some red counters.

The counters are all the same size.
She puts 4 black counters and 1 red counter in a bag.



a. Lisa is going to take one counter out of the bag without looking.

She says:

> There are two colours, so it is just as likely that I will get a black counter as a red counter.

i. Explain why Lisa is wrong. What is the probability that the counter she takes out is black?

ii. How many more red counters should Lisa put in the bag to make it just as likely that she will get a black counter as a red counter?

b. Jack has a different bag with 8 counters in it. It is more likely that Jack will take a black counter than a red counter from his bag.

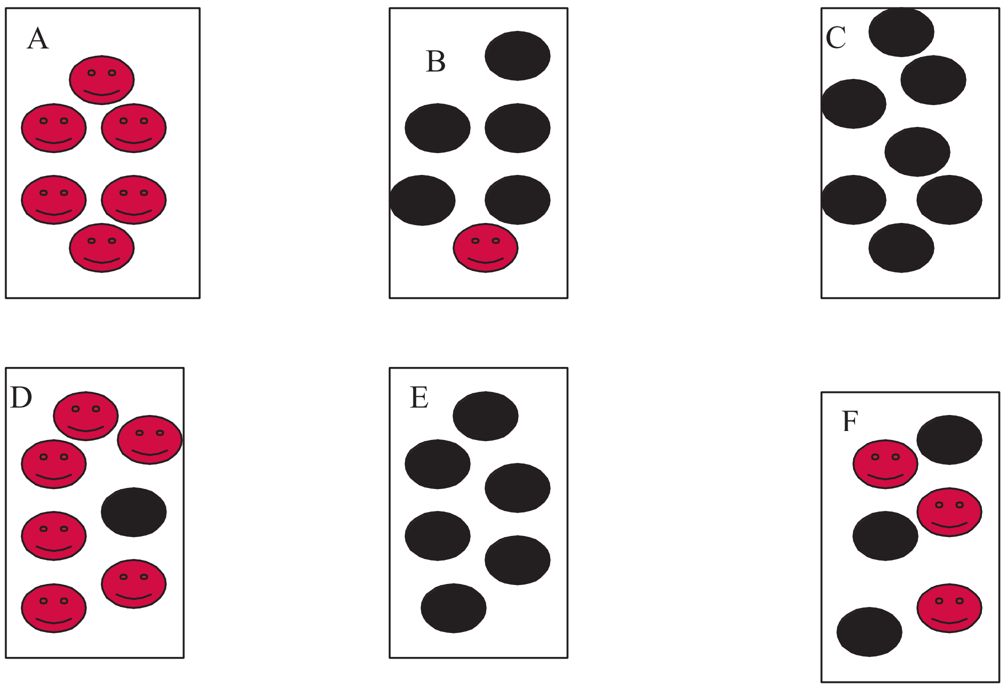
iii. How many black counters might there be in Jack's bag? Suggest a number and explain why this is a possible answer.

c. Jack wants the probability of taking a black counter from his bag to be the same as the probability Lisa had at the start of taking a black counter from her bag, so he needs to put extra counters into his bag.

iv. Assuming Jack had the number of black counters you have suggested at (iii) above, how many extra black counters and how many extra red counters (if necessary) should Jack put in his bag?

Explain your reasoning.

Q. 9 (a) Josh has some boxes containing red and black counters.



He is going to take a counter from each box without looking.

a. Match boxes (using the letters A-F) to the statements below. Explain your reasoning each time.

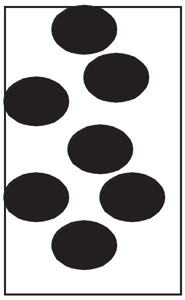
It is **impossible** that Josh will take a black counter from box........because

It is **equally likely** that Josh will take a black or red counter from box........because

It is **likely** that Josh will take a red counter from box............because

It is **certain** that Josh will take a black counter from box...........because

Josh selects box C which has 7 black counters in it



He wants to make it **more likely** that he will take a red counter than a black counter out of the box.

How many red counters must he put into the box? Explain your answer.

b. In another box, there are 30 counters which are either red or black in colour.
It is **equally likely** that Josh will take a red counter or a black counter from the box.
How many red counters and how many black counters are there in the box?

c. Extension question
There are 40 counters in a box which are either red or black in colour.
There is a **75% chance** that Josh will take a red counter from the box.
How many black counters are in the box? Explain your answer.

PROBABILITY 2

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- estimate probabilities from experimental data; appreciate that if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability
- associate the probability of an event with its long run relative frequency

INTRODUCTION

The activities described below and the questions that follow give you the opportunity to reinforce your understanding of the basic concepts of probability. You begin by rolling two coins and progress to playing a game involving rolling two dice. You will use a sample space to list all the possible outcomes and begin to consider the concept of expected value as you investigate the idea of fairness in relation to the game.

Activity 2.1

Toss two coins simultaneously about 30 times and record all the outcomes.
Do you notice any outcomes coming up over and over again?
Do some of these come up more frequently than others?
Use the grid below to show the 4 possible outcomes (the sample space) of heads (H) and tails (T).

		Coin 1	
		H	**T**
Coin 2	**H**		
	T		

Use the sample space to calculate the probability of each outcome occurring (i.e. the theoretical probability).
From the results you obtained in the 30 tosses, construct a table showing the number of times each outcome occurred and its relative frequency. Compare these to the theoretical probability.

Outcome	Tally	Relative Frequency

Activity 2.2

Working in pairs, roll a die 30 times (i.e. 30 trials) and enter your results into a table similar to the one outlined below

Number which appears on die (outcome of trial)	**How many times did this happen?** (Use tally marks to help you count.)	**Total** (frequency)
1		
2		

As you complete your own table compare it with that of another group.
Are there any similarities?

Your teacher may ask you to complete a Master sheet showing the results of all the groups in the class (a total of N trials).

Outcome of trial	**Frequency (group results)**	**Total of frequencies**	**Relative frequency)** $\frac{\textit{Total of frequencies}}{\textit{sample size (N)}}$	**% of total scores Rel. Freq × 100**	**Probability**
1	E.g. 5+6+5+…				
2					
3					
4					
5					
6					
		SUM			

The sum of all the relative frequencies is

The sum of all the percentages is

The sum of all the probabilities is

Conclusion:

What does your experiment tell you about the chance or probability of getting each number on the die you used?

Your die can be described as being unbiased. Can you explain why?

Activity 2.3

a. Each student tosses a coin 30 times and records their results for every 10 tosses.

No of tosses	No of Heads	Relative frequency
10		
10		
10		

b. What does the table you completed in (a) tell you about the probability of getting a head?

c. Now put all the results for the class together and obtain a new estimate of the probability of getting a head.

d. Is your new estimate closer to ½ than the estimate in (a)?

Activity 2.4

This is a game for two players, A and B. They take turns to roll two dice and add the two numbers shown on each toss. The winner is determined as follows:

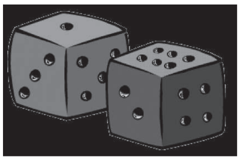
A wins if the sum of the numbers on the dice (i.e. outcome) is 2, 3, 4, 10, 11 or 12.

B wins if the sum of the numbers on the dice is 5, 6, 7, 8, 9.

Before you begin predict which player is most likely to win.

I think player ______ will win because ______

Play the game until one player reaches the bottom of the game sheet.



GAME SHEET

	A wins	**A wins**	**A wins**	B wins	B wins	B wins	B wins	B wins	**A wins**	**A wins**	**A wins**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

Record the number of times each player wins in the table below. The relative frequency is the **total no. of wins divided by the total no. of games.**

	Total (frequency)	Relative frequency
Player A wins		
Player B wins		
Totals		

As a class exercise construct a Master Tally sheet and record the results of the whole class

	Total (frequency)	Relative frequency
Player A wins		
Player B wins		
Totals		

Does your predicted result agree with your actual result? Think about why this happens. Complete the table below showing all the possible outcomes for throwing two dice.

	1	2	3	4	5	6
1	(1,1)					
2						
3			(3,4)			
4					(4,6)	
5						
6						

In the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes.

Construct a table to show the probability of each outcome above,
with the probability = $\dfrac{\text{no of outcomes in the event}}{\text{no of outcomes in the sample space}}$

Sum of two dice	Frequency	Probability
2	1	1/36
3	2	2/36

Look back at the rules of the game.

Original Rules: Player A wins when the sum is 2, 3, 4, 10, 11 or 12.

Player B wins when the sum is 5, 6, 7, 8 or 9.

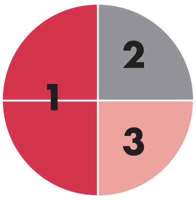
For how many outcomes will player A win?_______

For how many outcomes will player B win?_______

Does the game seem fair? If not, suggest a change to the rules which would make it fairer.

Create a mind map or a graphic organiser (http://www.action.ncca.ie) that will help you remember how to calculate the relative frequency of an event occurring.

Q. 1 Sophie and Andrew are playing a game with a fair, six-sided die and the spinner shown. They throw the die and spin the spinner simultaneously and note the total



Sophie I will carry your bag home if the total is **2, 3, 8** or **9**. You carry mine if the total is **4**, **5, 6** or **7**

Andrew said

Create a sample space showing the possible outcomes and use it to help Sophie decide whether or not she should play the game. Justify your advice to Sophie.

Q. 2 What is the probability of getting a head and a 6 when you simultaneously toss a fair coin and roll a fair, six-sided die?

How would this probability change if the die was replaced with:

a. A four-segment spinner (segments of equal area) numbered 1, 6, 6, 5?

or

b. A suit of spades from a deck of playing cards (and 1 card is chosen at random from the suit)?

Q. 3 A spinner has four unequal sections, red, black, pink and grey.

The probability that the spinner will land on red is 0.1 [P(red) =0.1]
The probability that the spinner will land on black is 0.2 [P(black) =0.2]
The probability that the spinner will land on pink is the same as the probability that it will land on grey.

Calculate the probability that the spinner will land on grey. Justify your answer.

Q. 4 A calculator can be used to generate random digits. Sandra generates 100 random digits with her calculator. She lists the results in the table below.

0	𝍸 II	**5**	𝍸 𝍸
1	𝍸 III	**6**	𝍸 𝍸 IIII
2	𝍸 I	**7**	𝍸 𝍸 II
3	𝍸 𝍸 II	**8**	𝍸 𝍸 I
4	𝍸 𝍸 II	**9**	𝍸 𝍸 IIII

Based on Sandra's results, estimate the probability that the calculator produces:
a) 9, b) 2, c) a digit that is a multiple of 3, d) a digit that is prime.

Q. 5 Four students each threw 3 fair dice.

They recorded the results in the table below.

Name	Number of throws	All different numbers	Exactly 2 numbers the same	All 3 numbers the same
Jane	50	36	12	2
Paul	150	92	45	13
Tom	40	18	20	2
Patti	120	64	52	4

a. Which student's data are **most likely** to give the best estimate of the probability of getting

All numbers the same ☐ Exactly 2 numbers the same ☐ All 3 numbers the same ☐

Explain your answer.

b. This table shows the students' results collected together:

Number of throws	All different	Exactly 2 numbers the same	All 3 numbers the same
360	210	129	21

Use these data to estimate the **probability** of throwing numbers that are **all different**.

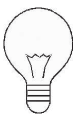
c. The theoretical probability of each result is shown below:

	All Different	2 the same	All the same
Probability	5/9	5/12	1/36

Use these probabilities to calculate, for 360 throws, **how many times** you would theoretically expect to get each result. Complete the table below.

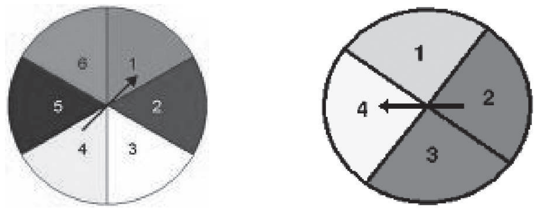
Number of throws	All different	2 the same	All the same
360			

d. Give a reason why the students' results are not the same as the theoretical results.



Think: How would this question be different if coins, spinners or playing cards were used?

Q. 6 Pierce and Bernie were investigating results obtained with the pair of spinners shown.



They used a table to record the total of the two spinners for 240 trials. Their results are given in one of the three tables A, B and C below.

Table A

Sum	Frequency	Relative frequency
2	10	1/24
3	20	1/12
4	30	1/8
5	30	1/8
6	60	1/4
7	40	1/6
8	20	1/12
9	20	1/12
10	10	1/24
Total	240	1

Table B

Sum	Frequency	Relative frequency
2	12	12/240
3	12	12/240
4	27	27/240
5	27	27/240
6	35	35/240
7	45	45/240
8	24	24/240
9	18	18/240
10	40	40/240
Total	240	

Table C

Sum	Frequency	Relative frequency
2	11	
3	19	
4	32	
5	30	
6	29	
7	28	
8	17	
9	14	
10	60	
Total	240	

Complete the relative frequency column in table C.
Use your results to decide which, if any, of these three tables might represent the results found by Pierce and Bernie. Explain your reasoning.

Q. 7 A spinner with 3 equal segments numbered 1, 2 and 3 is spun once.

i. Give the sample space of this experiment.
ii. What is the probability that the spinner stops on number 2?
iii. What is the probability that the spinner stops on a number greater than or equal to 2?

Q. 8 Pierce and Bernie were investigating the results given by the spinner shown, by spinning it 60 times and recording the results.

Their results are given in one of the three tables below, A, B and C

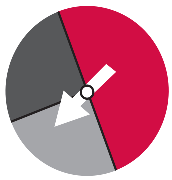


Table A

result	tally	count
red	卌 卌 卌 卌 卌 |	21
grey	卌 卌 卌 ||||	19
black	卌 卌 卌 卌	20

Table B

result	tally	count
red	卌 卌 卌 卌 卌 卌 卌 卌 卌 ||	47
grey	卌 |	6
black	卌 ||	7

Table C

result	tally	count
red	卌 卌 卌 卌 卌 卌 ||	32
grey	卌 卌 卌	15
black	卌 卌 |||	13

a. Which of the three tables above is most likely to be like the one that Pierce and Bernie made? Explain how you made your decision.

b. For each of the other two tables, draw a diagram of a spinner that is likely to produce results like those shown in each table.

PROBABILITY 3

SYLLABUS TOPIC: CONCEPTS OF PROBABILITY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- apply the principle that in the case of equally likely outcomes the probability is given by the number of outcomes of interest divided by the total number of outcomes
- use binary/counting methods to solve problems involving successive random events where only two possible outcomes apply to each event

Activity 3.3

Consider the following game

Players roll 2 four-segment spinners, which have equal segments numbered 1, 2, 3 and 4 . Player 1 wins if the sum of the spinner numbers is 3, 4, or 5,; player 2 wins if the sum is 2, 6, 7, or 8.

a. Predict whether player 1 or player 2 has the greater chance of winning. Play the game a few times to check your prediction. Now use the table below to help you decide in a more mathematical way. Write a sentence explaining why you think the game is, or is not, fair.

	1	2	3	4
1				
2				
3				
4				

b. Now consider this game

Players roll 3 four-segment spinners, which have equal segments numbered 1, 2, 3, and 4. Player 1 wins if the sum of the spinner numbers is 3, 4, 5, 6 or 12; Player 2 wins if the sum is 7, 8, 9, 10 or 11.

Is this game fair?

Can you represent the possible outcomes in the same way? It is difficult because there is an extra dimension – the 3rd spinner.

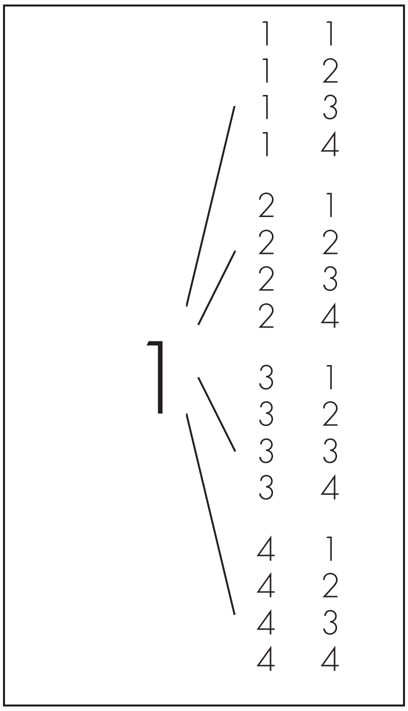
Consider all the possibilities when the first spinner shows a 1.

This is only ¼ the total number of outcomes and the process of completing the rest gets very repetitive.

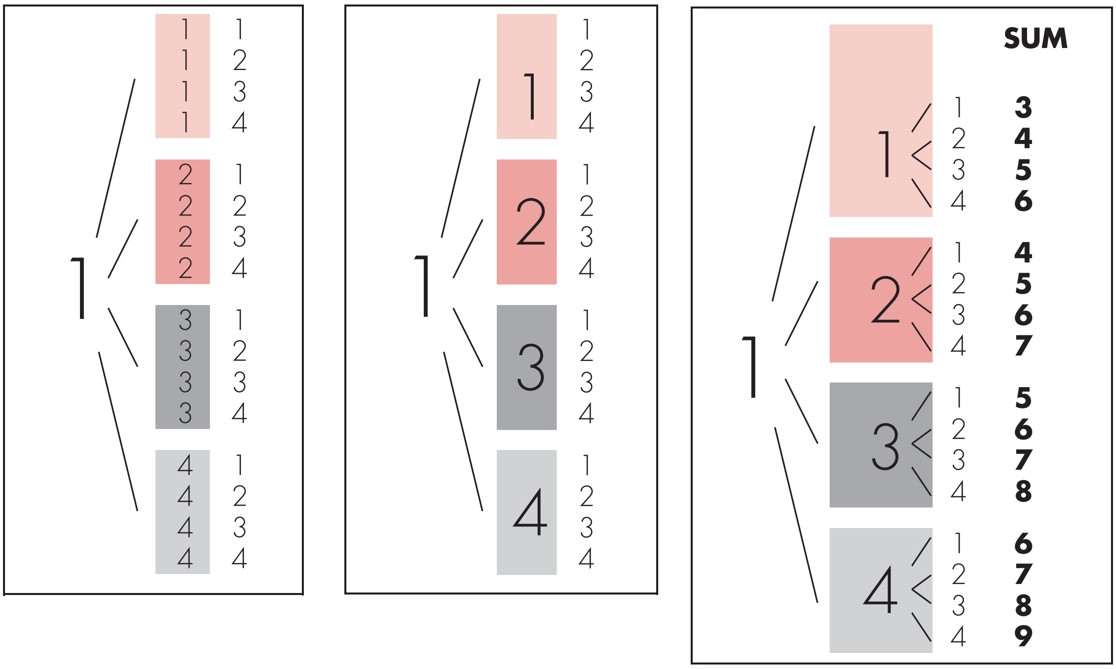
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
1	2	2
1	2	3
1	2	4
1	3	1
1	3	2
1	3	3
1	3	4
1	4	1
1	4	2
1	4	3
1	4	4

We could get rid of the repetitions by replacing the first column of 1's with 1 big 1.

1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
1	2	2
1	2	3
1	2	4
1	3	1
1	3	2
1	3	3
1	3	4
1	4	1
1	4	2
1	4	3
1	4	4



Can you get rid of any more repetitions?



Can you see a pattern forming?
This is called a tree diagram; can you see why? Can you see how the required outcome (sum of the three spinners) is calculated for each 'branch' of the 'tree'?

i. Draw tree diagrams showing the possible outcomes when the first spinner shows 2, 3, and 4.
ii. How many possible outcomes are there? Now use your diagrams to decide if the game is fair (see the rules at the start).

This is how one student explained why tree diagrams are very useful when counting outcomes such as in this question:

> *Well, tree diagrams are useful for counting the total number of outcomes. There are four 'trunks' (for the possible numbers on the first spinner), and each has four 'branches' (for the possible numbers on the second spinner), and each has four 'twigs' (for the possible numbers on the third spinner). An outcome is formed as we go from a trunk to a branch to a twig. There are as many outcomes as there are twigs: 4x4x4=64.*

c. Draw a tree diagram showing the number of possible outcomes when three coins are tossed

Could you have answered this question without drawing the tree diagram?

Explain

i. Use your tree diagram to answer the following

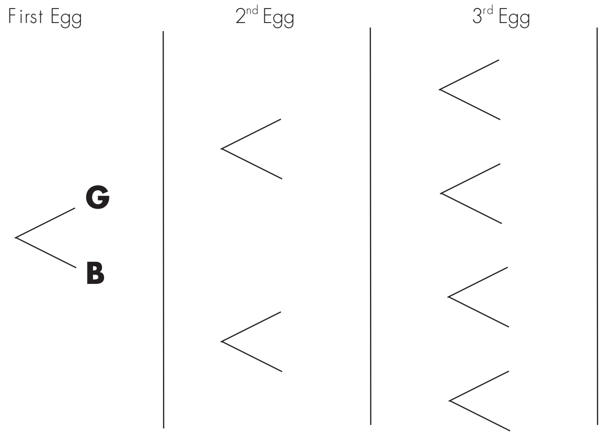
P (All tails) =

P (All heads) =

ii. Predict the number of possible out comes when two coins are tossed and 1 die is rolled. Check your prediction by drawing a tree diagram.

Q. 1 There are a dozen eggs in a box and 3 of them are 'bad'. 3 eggs are chosen at random from the box.

a. Complete the probability tree diagram below, showing good (G) and bad (B) eggs.



b. Work out the probability that

i. all three eggs are 'good'

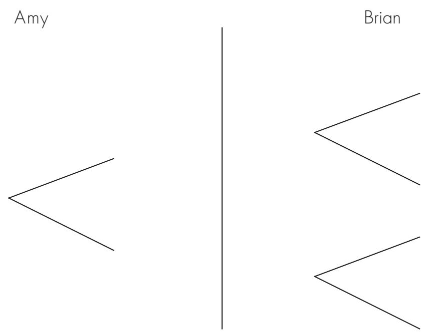
ii. 1 egg is 'bad'

iii. 2 eggs are 'bad'

iv. all three eggs are 'bad'

Q. 2 Jessica is taking part in a quiz. She is unsure of the answer to a question and needs to ask her team-mates, Amy and Brian. The probability that Amy will get it right is 0.7.The probability that Brian will get it right is 0.4.

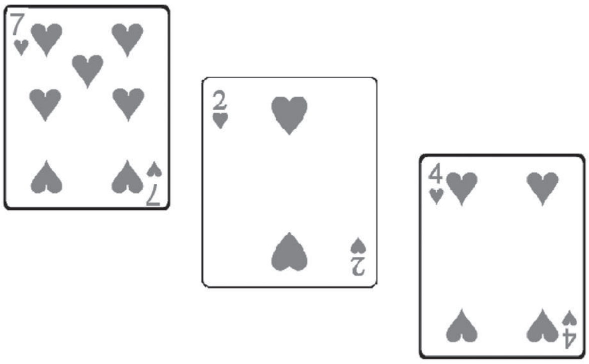
a. Complete the probability tree diagram below.



i. What is the probability that at least one of her two friends will give her the correct answer?

ii. What is the probability that neither of them will give her the correct answer?

Q. 3 John and Sophie each have three cards numbered 2, 4 and 7. They each select one of their own cards. They then add together the numbers on the four remaining cards. What is the probability that their answer is an even number? Explain how you arrived at your answer.



Q. 4 Suppose that every child that is born has an equal chance of being born a boy or a girl.

i. Write out the sample space for the situation where a mother has two children.

ii. What is the probability that a randomly chosen mother of two children would have two girls?

iii. What is the probability that this mother of two children would have two boys?

iv. What is the probability that this mother of two children would have one boy and one girl?

STATISTICS 1

SYLLABUS TOPIC: REPRESENTING DATA GRAPHICALLY AND NUMERICALLY

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- explore concepts that relate to ways of describing data, such as the shape of a distribution, what's typical in the data, measures of centre (mode, median, mean), and range or variability in the data
- use a variety of summary statistics to analyse the data: central tendency; mean, median, mode
- select appropriate graphical or numerical methods to describe the sample (univariate data only)
- evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others
- use pie charts , bar charts, line plots, histograms (equal intervals), stem and leaf plots to display data
- use back to back stem and leaf plots to compare data sets

There are links with Strand 3 (Number) where you will investigate models such as accumulating groups of equal size to make sense of the operation of multiplication.

INTRODUCTION

Being able to see a data set as a whole and so being able to use summary statistics such as averages to describe the 'big picture' or the overall shape of the data is an important learning intention of strand 1.

The activities described below allow you to investigate how the mean is constructed and the relationship of the mean to the data set it represents. You will also explore the different ways the median and mean represent the data - the median as a middle point in the data, and the mean as a 'point of balance' or the 'fair share' value of the data. Using two different representations of the mean gives you a chance to view the relationship between the mean and the data set through different models and so construct a firm understanding of the mathematical concept.

Prior learning

The idea that a set of data can be viewed and described as a unit is one of the key ideas about data that develops across primary school and is built on at second level. Initially, you looked at each individual piece of data. Gradually, you began to move away from a focus on individual pieces of data to looking at larger parts of the data. You learned to make general statements about the group of things or phenomena that the data represent, such as 'most people in our class have 1 or 2 siblings, and the range is from no siblings to 6 siblings.' Now you are ready to move away from making general statements and begin to make summary statements that describe the whole data set.

Activity 1.1

There are 5 bags of sweets, each of a different brand. All bags are the same size. The average price for a bag is €1.43

a. What could the individual prices of the 5 bags be? Think of at least two different sets of prices.

b. If both of your sets of prices included €1.43 as a price for at least one of the bags, price the five bags without using €1.43 as one of the prices.

c. Did you use €1.43 as the median? If so, what is the mean for your sets of prices? If you didn't use €1.43 as the median, what is the median for your sets of prices? Are the mean and median the same or different?

Discuss one of your lists of five prices with your group. How did you decide on your list of prices? How do you know what the average is in each example?
Note to each small group: Make sure you consider some lists that do not include a value of €1.43 as one of the prices.

d. There are seven bags of beads. Five of the bags have the following numbers of beads in them: 5, 7, 8, 9, and 12. Now work through parts (i), (ii) and (iii) with your group.

   i. Make a representation of the five bags by using small objects such as cubes, counters, marbles, etc. Make another representation of the five bags on a line plot.

   ii. Now use your representation to figure out how many beads could be in the other two bags so that 8 is the mean number of beads for all seven bags. Try to figure this out without adding up the beads in the five bags. Find at least two different sets of numbers for the two bags that will solve this problem.

   iii. Revise your two representations – counters and line plot – so that they show all 7 pieces of data. Can you 'see' the average in your representation?

e. What is the least number of beads there could be in one of the additional bags? What is the greatest number?

f. What numbers of beads could be in the two other bags if the mean number of beads was 7? What if the mean number was 10?

Q1. A teacher had some cards with groups of numbers displayed on them, as shown below

1, 7, -8, 0,	0, 0, 0	-2,8,-6, 7, 11
0, 11, 8, 0, 13	-5,-4,-3,-2,-1 0,1,2,3,4,5	2,3,4,5,6,7,8 9,10

John was asked to calculate the mean of the numbers on each card and to put the cards that had a **mean of zero** into a box.

a. Circle the cards that John should put into the box.

The teacher has another card and tells the students that the mean of the numbers on this card is also zero.

b. Tick the correct box for each statement about this extra card.

Statement	Must be true	Could be true	Cannot be true
All of the numbers are *zero*			
Some of the numbers are *zero*			
There are as many negative numbers as positive numbers			
The sum of ***all*** the numbers is zero			
All of the numbers are positive numbers			
Some of the numbers are positive numbers			

Q.2 3 girls and 5 boys received text messages

The mean number of messages received by the 3 girls was **31**.

The mean number of messages received by the 5 boys was **27**.

Decide whether the following statements are true (T) or false (F), and justify your answer in each case:

i. The person who received the most messages must have been a girl.

ii. The mean number of messages for the 8 people was 29.

Q.3 Three girls and five boys were studying climate change in various countries around the world. They were examining the maximum daily temperatures in these areas

The mean daily temp of the locations studied by the 3 girls was 31° C
The mean daily temp of the locations studied by the 5 boys was 27 ° C

Decide whether the following statements are True or False, and justify your answer in each case.

i. The person who encountered the max daily temperature must have been a girl.
ii. The person who encountered the min daily temperature must have been a boy.
iii. The mean max daily temperature encountered by the 8 people was 29° C. .

Q.4 Sophie has six cards, each of which has a positive whole number printed on it. Four of the cards each have the number 9 on it.

a. Without knowing the numbers on the other two cards, can you give the value of the
   i. median
   ii. mode
   iii. range

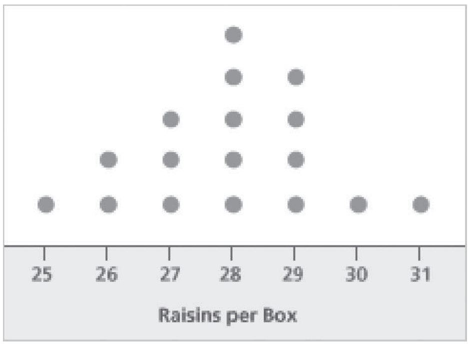
Explain your reasoning.

b. You are told that the six cards have a mean of 9. Give some possible whole numbers that could be on the other two cards. Which of your answers would give the greatest range? Why?

If the six cards have a mean of 9 and a range of 6 how many answers can you now find for the numbers on the remaining two cards?

Q.5 Students were investigating the number of raisins contained in individual mini-boxes of Sun-Maid raisins.

They recorded their results in the diagram shown.



a. Use the diagram to answer the following:
   i. How many boxes of raisins did they survey?
   ii. What was the modal number of raisins per box?
   iii. What is the median number of raisins per box? Explain how you found this answer.

b. If the students chose a box at random from all the boxes they surveyed what is the probability that the box contained 29 raisins?

Having done this activity, the students are asked to write down the answer they would give to the question: 'How many raisins are in a mini-box of Sun-Maid raisins?' Here are some of the answers they wrote down:

A	'There could be any number of raisins in a box.'
B	'There are about 28 raisins in a box.'
C	'There are almost always 28 raisins in a box.'
D	'You can be fairly sure there are 27, 28 or 29 raisins in a box.'
E	'Probably 28'.

c. Which of the answers above do you think is the best answer to the question? Explain why you think it's the best.

d. Which of the answers above do you think is the worst answer? Explain why you think it's the worst.

Activity 1.2

A good part of one's day is spent travelling from one place to another. How much time do you spend travelling to school? How much time do your classmates spend travelling to school?

Carry out a survey to find out how everyone in your class travels to school, and how long the journey takes, on a given day. Your survey should enable you to answer a series of questions.

Deciding to walk or to go by car may depend on the distance, but, after choosing the method of transportation, does everybody spend about the same amount of time travelling to school?
Do those who take the bus to school spend less time than others?
Does the time it takes to get to school depend on where you live?

To better understand the situation, consider the 'time travelling to school' variable. Analyse the data you collect based on the method of transportation used.
Do you think this situation varies from one region in Ireland to another?

Time to get to school

Enter the class data in a table, such as the one below, grouping them in intervals of ten minutes, for example. First write down the numbers as you collect them. Then put them in ascending order to create a stem and leaf plot, where the tens are the 'stems' and the units are the 'leaves'. For example, a time of 15 minutes is recorded by placing a '5' in the Units column in the row which corresponds to the '1' in the Tens column.

Time to get to school Raw data	
Tens	Units
0	
1	
2	
3	…
…	…

Now, try to get an overview.

1. Look at all the ordered data. Half the class takes less than how many minutes to get to school? This number is called the median; it's the central value that divides the list of ordered data into two equal sections.
2. What is the average time that students in your class spend travelling to school?
3. Which row contains the most data? In your opinion, what does this mean?
4. What is the shortest time? What is the longest? What is the difference between them?
5. What can you say about the time that students in your class spend to get to school?

To get a better picture of the situation, it would help to add a column to your table that shows the number of students.

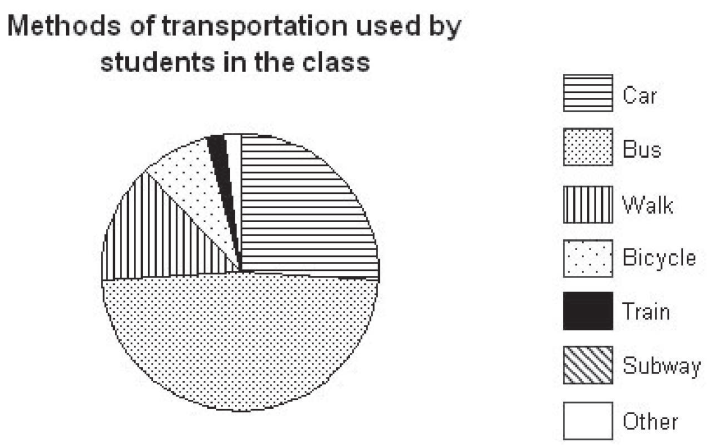
Time to get to school Raw data		
Tens	Units	No of Students
0		
1		
2		
3	...	
...	...	
	Total	

6. Now, can you create a graph that shows how much time the students in your class spend travelling to school? As you can see, everybody does not spend the same amount of time travelling to school.

You can now examine whether this time changes with the method of transportation.

Time spent by method of transportation

First, group together the students who use the same method of transportation. You can quickly determine the distribution of students by transportation method by creating a pie chart with a spreadsheet program. Your chart might look something like this:



From your chart, what are the most popular methods of transportation?
Approximately what fraction of the students in your class walk to school.

Now, for each method of transportation:

a. sort the time spent getting to school, from the shortest to the longest time.

b. determine the total time spent, which lets you calculate the average.

c. find the number of minutes or less that the faster half of the students spent travelling to school. This is the median or the value of the middle item of the ordered data.

d. add the minimum and maximum amount of time spent travelling to school.

Create a descriptive table that will look like this:

Method of Transportation	Time to get to school (mins)	No	Total Time	Average Time	Median	Min	Max
Car	5, 12, 12, 2, 32,	5	83	83/5	12	5	32

You can now examine the time by method of transportation.
Do you notice any significant differences?
Which method of transportation takes the longest?
Which method of transportation shows the biggest difference between the shortest time (minimum) and the longest time (maximum)? What might explain this?
Can you describe the overall situation for your class and present your point of view? What type of transportation do you think we should encourage? Under what conditions? Why?
Finally, use the data you have obtained to create a graph that properly conveys the information about your class that you feel is important.

Comparing your class to a sample of Irish students

Do you think the situation of your class resembles that of most Irish students?
Obtain a sample of 50 students from your school. Then do the same analysis that you did for your own class.
Is the time spent getting to school approximately the same for both groups? If not, how does it vary?
To help you better compare the data, create two tables side-by-side for each group.

Time to get to school				
Raw data for the school			Raw data for our class	
Students	**Units**	**Tens**	**Units**	**Students**
		0		
		1		
		2		
		3	…	…
		4	…	…
		5	…	…
50	Total	TOTAL	Total	

'A picture is worth a thousand words' and can certainly make it easier to read all these numbers. Create appropriate graphs to easily compare the time spent getting to school for both groups.

You can also compare the methods of transportation used.
For each group: create a pie chart to illustrate the distribution of students for the different methods of transportation used to get to school.
Use a descriptive table to examine the time spent by method of transportation used.
Do you arrive at the same observations for both groups? Are there any significant differences? If yes, what are they? Can you explain the differences taking into account the characteristics of your region?
Create a visual representation that properly illustrates and conveys your main conclusions.

STATISTICS 2

SYLLABUS TOPIC: FINDING, COLLECTING AND ORGANISING DATA

LEARNING OUTCOMES

As a result of completing the activities in this section you will be able to

- clarify the problem at hand
- formulate one (or more) questions that can be answered with data
- explore different ways of collecting data
- design a plan and collect data on the basis of above knowledge
- generate data, or source data from other sources including the internet
- discuss different types of studies: sample surveys, observational studies and designed experiments
- select a sample (Simple Random Sample)
- recognise the importance of representativeness so as to avoid biased samples
- design a plan and collect data on basis of above knowledge.

The activities described below and the questions that follow give you the opportunity to construct an understanding of the concept of finding, collecting and organising data in a statistical investigation. By carrying out a complete data investigation, from formulating a question through drawing conclusions from your data, you will gain an understanding of data analysis as a tool for learning about the world.

The activities are designed to build on your previous experiences with data, and to introduce you to the ideas you will work on as you progress through statistics in Strand 1.

During these activities you will work with categorical data, noticing how these data can be organised in different ways to give different views of the data.
As a result you should be able to

- gather data from a group
- classify the data
- write sentences that describe the 'Big Picture' of the data
- appreciate how the purpose of the research will affect how the data is gathered
- understand that the way data is represented can illuminate different aspects of the data.

Activity 2.1: A data Investigation

With what well-known person would you like to meet?

1. You will be working in groups on a data investigation. The first step is for each student to decide on his/her own how they would answer the survey question. Each student will need to write their answer a number of times on separate pieces of paper so that they can give their individual answers to each group, including their own.
2. Each group collects answers from everyone; make sure your group has a full class set of data that you can discuss.
3. Before you look at the data spend a few minutes discussing what might be interesting about them.
4. As a group sort the class data into three piles according to what they have in common. This is called classifying your data.
5. Choose one of your ideas for sorting and arrange your cards on a large piece of paper to show that classification
6. Write a sentence or two on your display that tells what you notice about the data
7. Post your display on the wall. If you finish before other groups, discuss issues about data that arose while you did this activity.
8. Can you represent this data in a chart?

Key Words: **Category, Data**

As you work through this activity reflect with your group on

- What issues came up for you as you tried to represent these data?
- What does the data tell you about the group?
- What questions arise for you while looking at this data? How might you modify the survey in order to address these?
- Did everyone interpret the original question in the same way?
- What were you thinking when you made your own decision?

Consider the following question

How many countries have you visited?

Elect a scribe to sketch a line plot with reasonable intervals on the board. Collect data on the line plot by marking an X for the value of each person's response. (Note: a line plot is a graph for numerical data that is similar to a bar chart. It is one of the plots in common use in statistics.) Try to form statements that describe the data. What can they say for the class as a whole about the number of countries that they have visited?

Activity 2.2

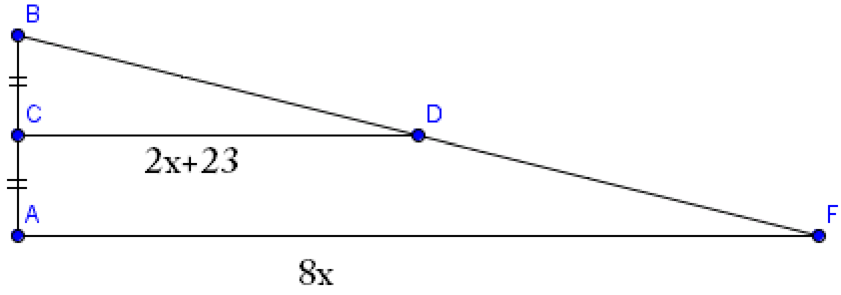
1. Note: You have 30 mins to complete this assignment and post a representation of your data for others to see. That means you will need to decide on a question and collect your data efficiently. You may need to design a data collection sheet. Think about how you will make sure you get a response from every person. After 15 mins you should be ready to start making a data representation. Your representation need not be decorative or elaborate. Focus on how well it communicates information about your data.
2. Select a question that will result in numerical data
3. Collect data from everyone in the class.
4. Create a line plot for your data
5. Write three to five sentences on your display that describe your data
6. When your display is complete, discuss issues that arose in your group as you defined your question
7. What further questions might you want to pursue based on these initial data?

Sample data collection sheet

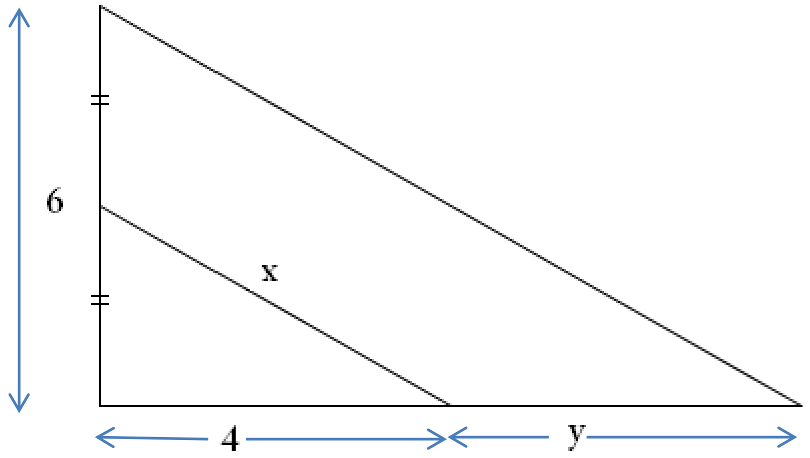
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name
Name	Name	Name	Name	Name

Q1 (LCOL) In the diagram, CD is parallel to AF and equal lengths are marked.

Find the value of x.

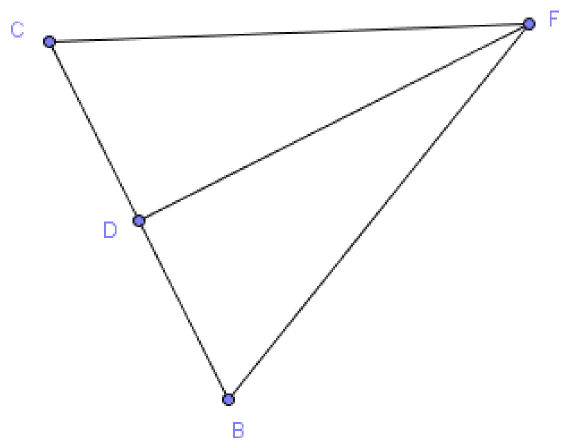


Q2 (JCHL) If the sloped lines are parallel, find the value of x and the value of y.

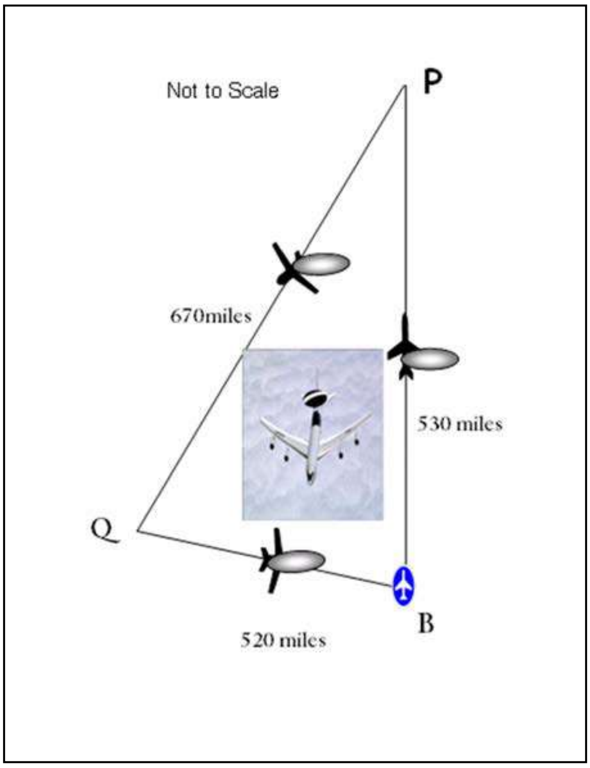


Q3 (JCHL) In triangle FCB |CD| = |DB| and |∠FDC|= |∠FDB| = 90°

Explain why the triangles FDC and FDB are congruent.



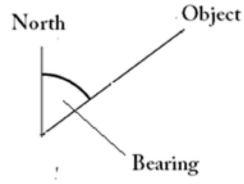
Q4 (LCOL)



An aircraft takes off from Baldonnel **(B**) on a navigation exercise. It flies 530 miles directly North to a point **(P)** as shown. It then turns and flies directly to a point **(Q),** 670 miles away**.** Finally it flies directly back to base, a distance of 520 miles.

a) Calculate the angle QPB.

b) If the bearing is defined as the clockwise angle measured from the North direction, calculate the bearing of Q from P.

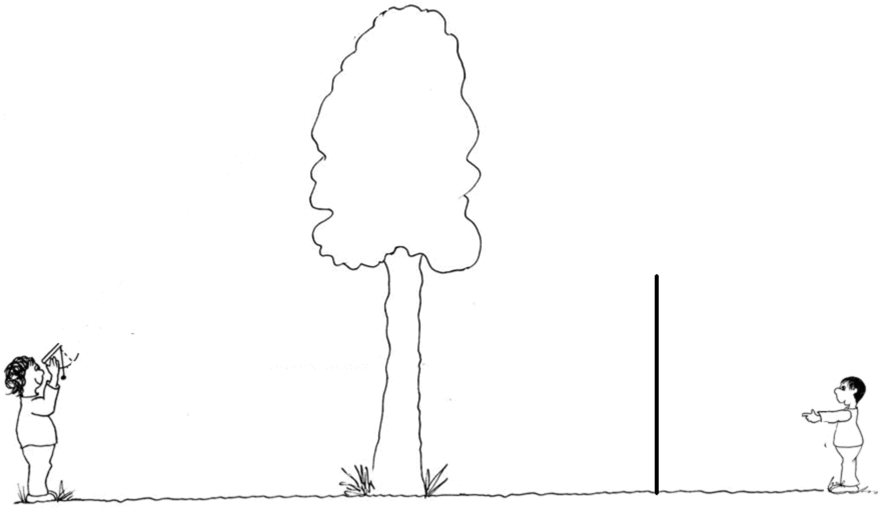


Q5 (JCOL). Jane and Stephen want to estimate the height of a tall tree which is vertical and stands on horizontal ground.

Jane has a **clinometer** and Stephen has a 100m measuring tape and a large **stake**.

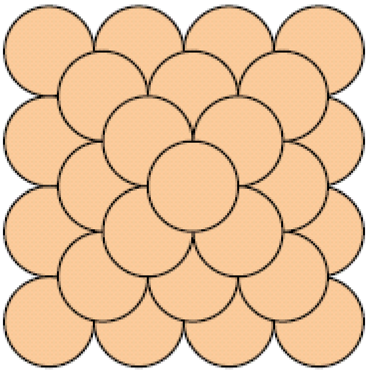
Explain, using diagrams and your own reasonable measurements, how each of them can make an estimate of the tree's height.

Account for any inaccuracies that might occur and suggest how you could minimise these inaccuracies.

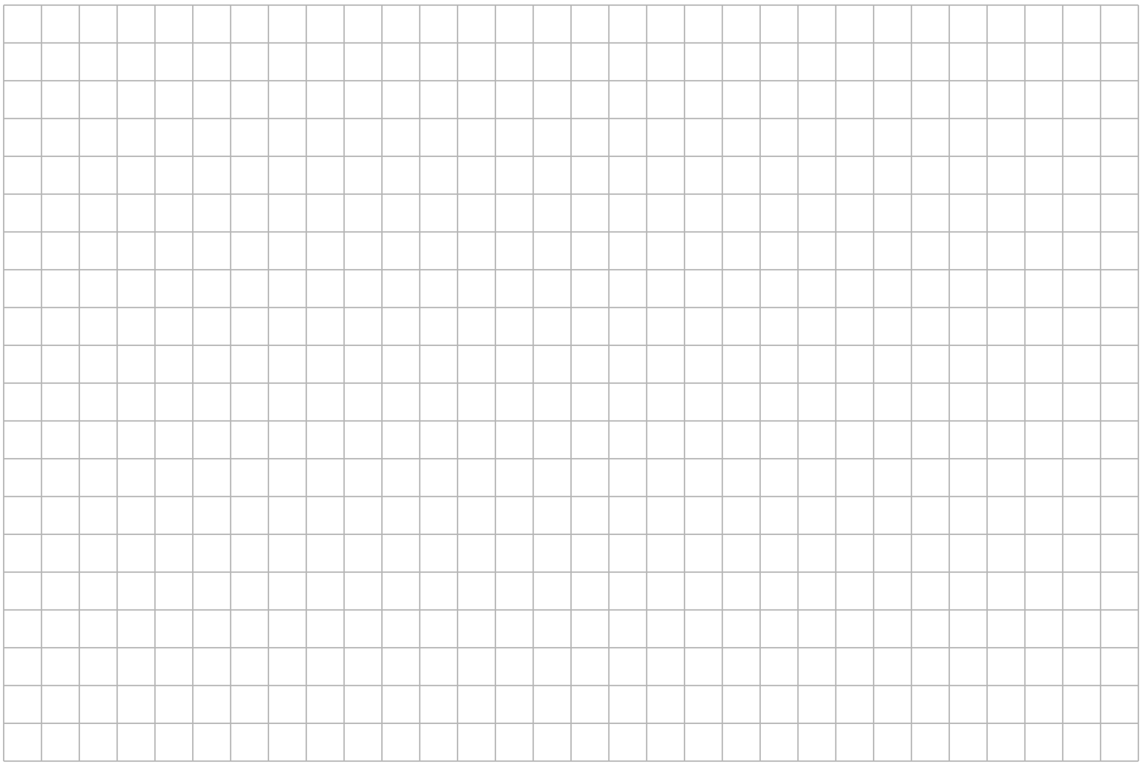


Q6 (LCHL) Joan was asked to design a box for 30 chocolates. Each chocolate is cylindrical with diameter **1.5 cm** and height **1 cm**. She decided the box should be made from card and in the shape of a **square-based pyramid**.

Inside the box the chocolates would be stacked in 4 layers and would look like this when viewed from above.



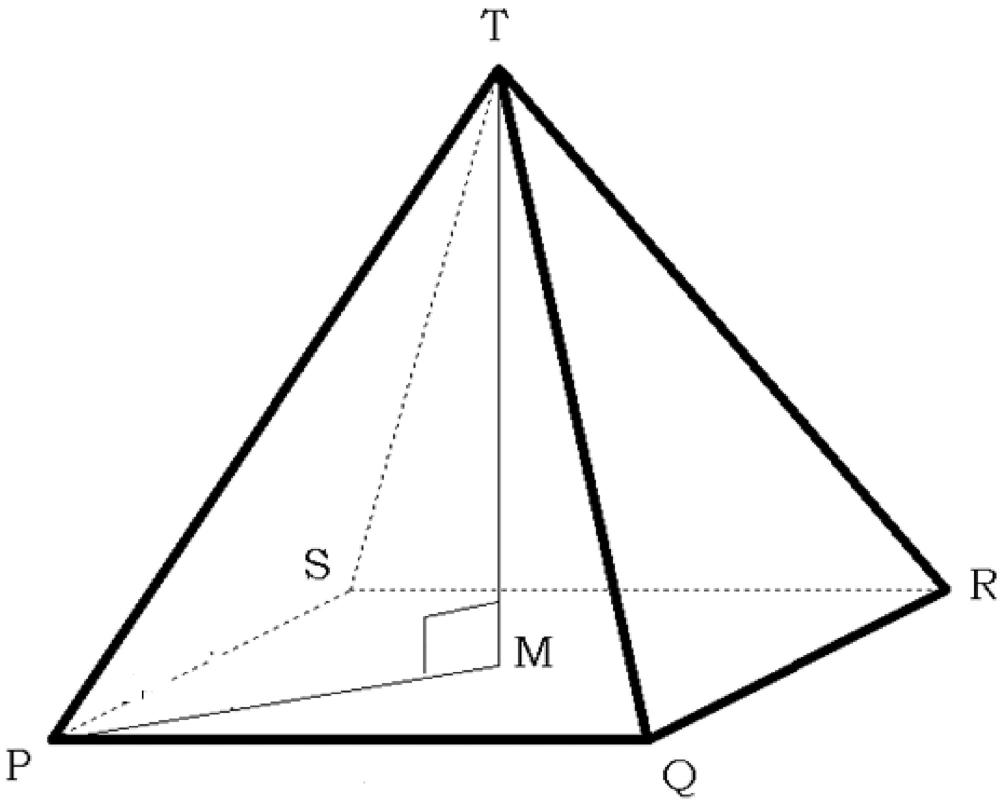
By sketching a net of the box, without including any joining flaps, calculate how much card the design will need. Show all measurements on your sketch.

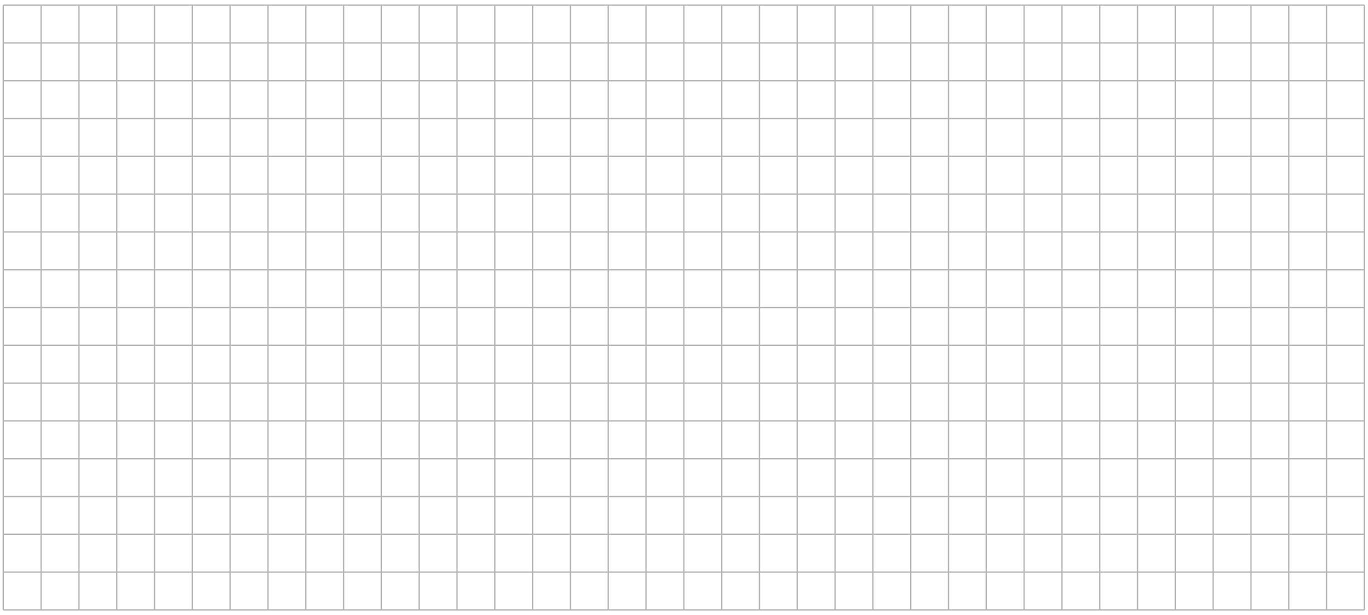


Sarah claims that it would need less card if the 30 chocolates were stacked in a closed rectangular box that would hold two layers, each 5 chocolates long by 3 chocolates wide. By calculating the surface area of such a box, decide whether or not the claim is accurate.

Q7 (LCHL) Show that the length of steel tubing required to make a sculpture in the shape of a square-based pyramid, as illustrated below, is given by the equation:

Length of tubing = $4\sqrt{h^2 + \frac{x^2}{2}} + 4x$, where x = base length and h = vertical height.





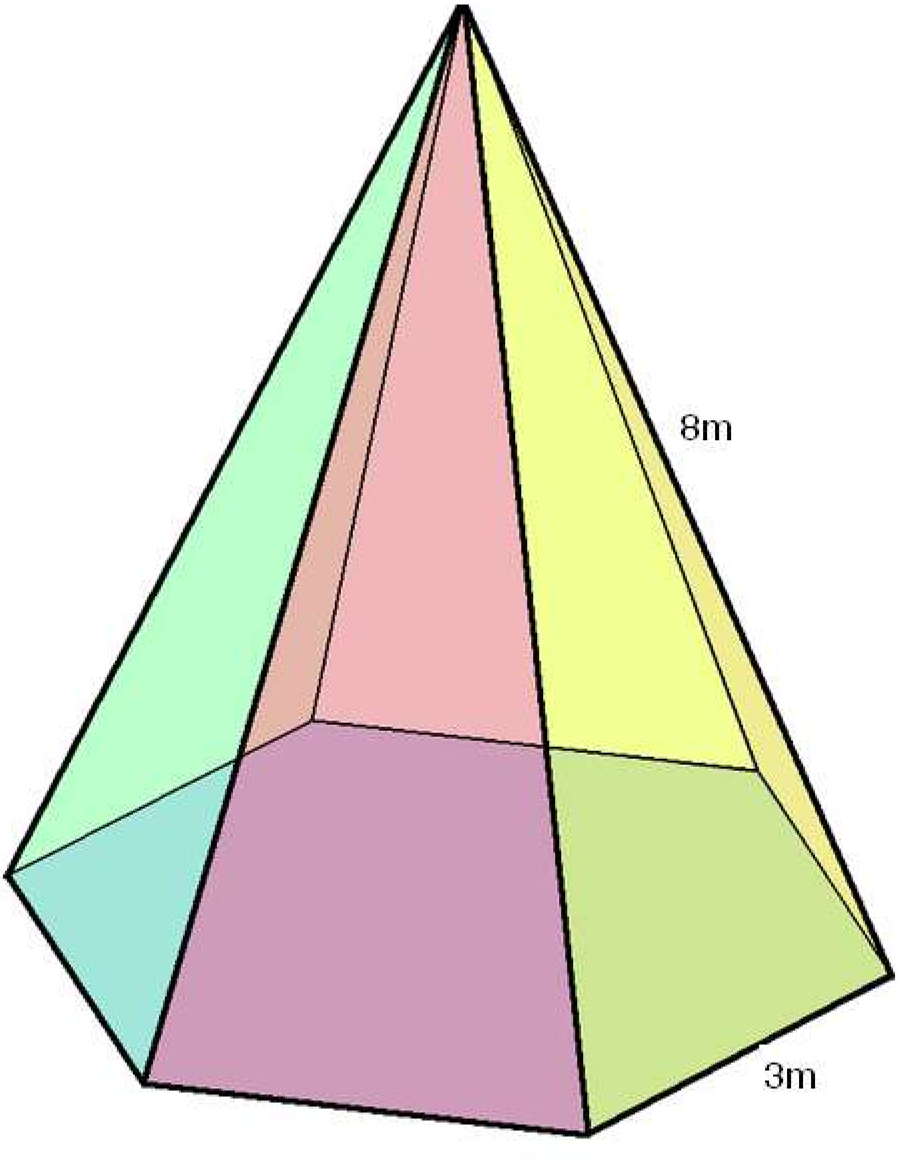
Q8 (LCHL) The diagram shows the structure of a climbing frame. The structure is in the shape of a pyramid on a hexagonal (6 equal sides) base.

The length of each sloping edge is 8m and the pyramid's base is a regular hexagon with sides of length 3m as shown in the diagram.

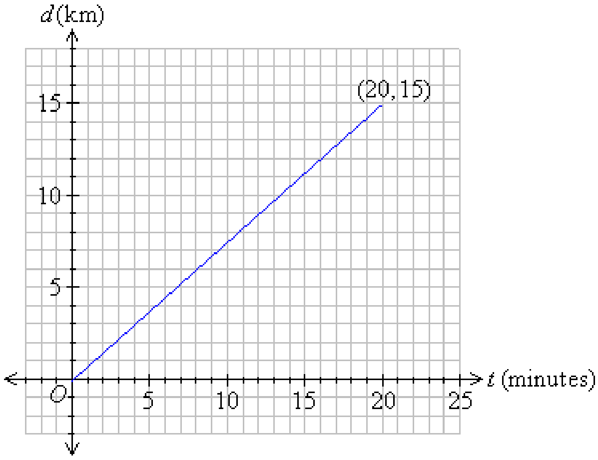
The regulations state that the frame cannot exceed 7.5m in height.

Will these dimensions comply with regulations?

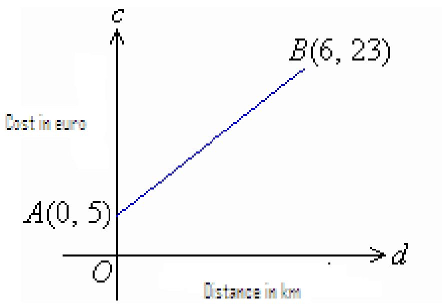
Support your answer with calculations.



Q9 (LCFL) (a) A cyclist travels for 20 minutes at a constant speed and covers a distance of 15 km, as shown in the diagram. Find the slope of the line and describe its meaning.



(b) The cost of transporting documents by courier can be represented by the following straight line



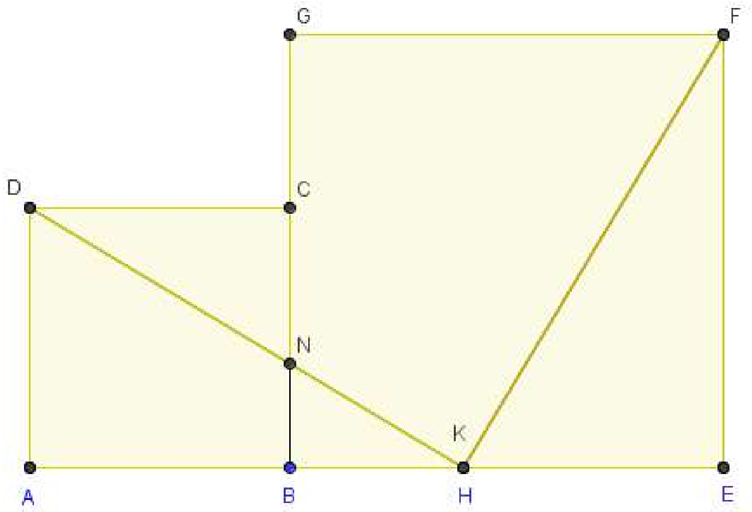
(i) What does each point represent?

(ii) Calculate the slope. What does this represent?

Q10 (JCHL) – Geometry

The diagram (Fig. 1) shows two square tiles, ABCD and BEFG placed alongside each other. The point H is chosen along the side BE so that $|HE| = |AB|$.

Fig. 1



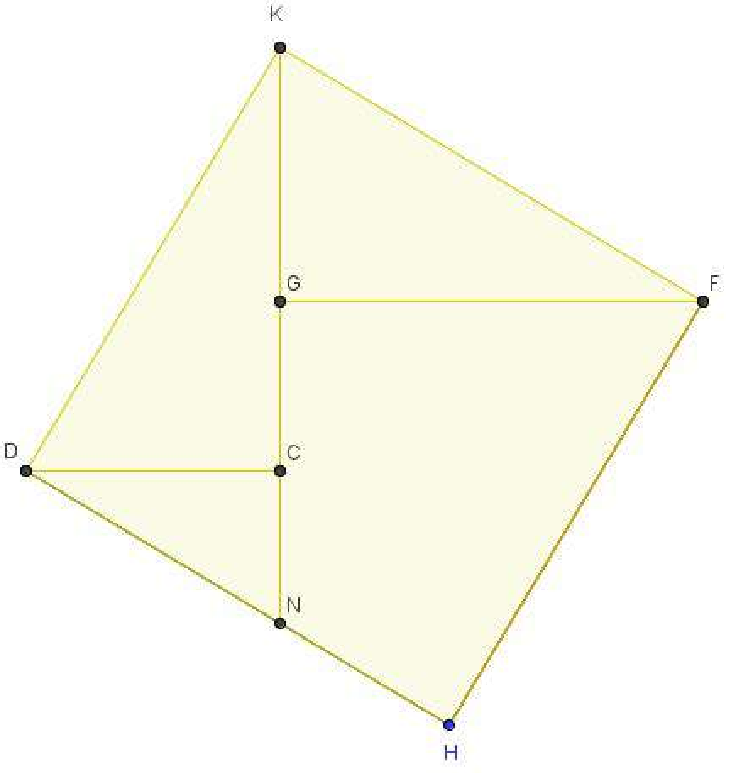
(i) Prove that the triangles DAH and HEF are congruent.

(ii) Prove that $\angle$ DHF is a right angle

The square tiles are cut along the lines DH and HF as shown and the pieces are moved so that ∆HEF lies in the position DCK and ∆DAH lies in the position KGF (see Fig. 2).

(iii) Prove that the new figure formed, DHFK, is a square.

Fig. 2

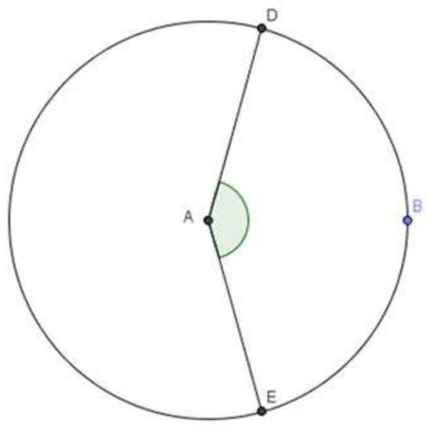


Q11 (LCOL)

Fig. 1 shows a circle with centre A.

(a) If the $|\angle DAE| = 150°$ and $|AD| = 12$ cm, find the length of each arc.

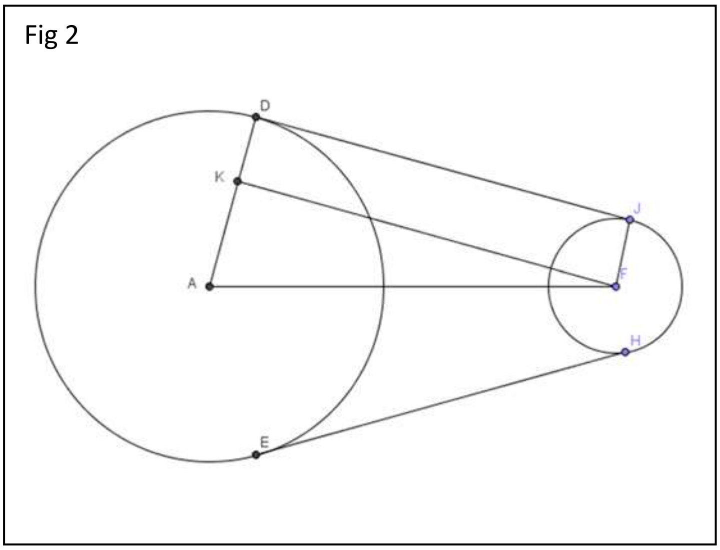
Fig 1



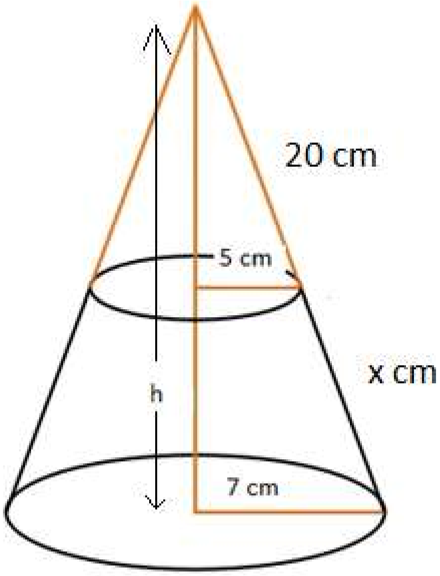
(b) Fig. 2 shows a belt-driven pulley system with pulleys of radii 12 cm and 5 cm respectively. The centres of the pulleys are 24 cm apart.

(i) Find the measure of the angle DAF to the nearest degree.

(ii) Find the total length of the belt needed for this pulley system.



Q12 (LCHL) Geometry/Trigonometry



The figure to the left shows a cone from which a lampshade is to be made.

The smaller cone with base radius 5 cm is cut off the top to form the lampshade.

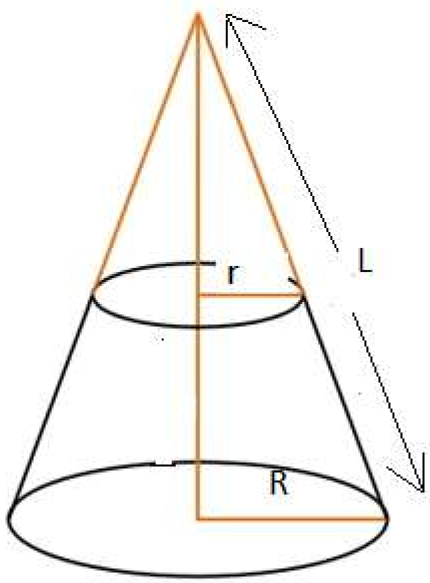
(i) Calculate the slant height of the lampshade (marked x cm in the diagram).

(ii) Hence calculate the value of h, correct to one decimal place.

(iii) Calculate the surface area of the lampshade correct to two decimal places.

(iv) By letting r = the base radius of the small cone, R= base radius of the large cone and L = the slant length of the large cone, show that the curved surface area of the lampshade is given by

$$A = \frac{\pi L(R^2 - r^2)}{R}$$

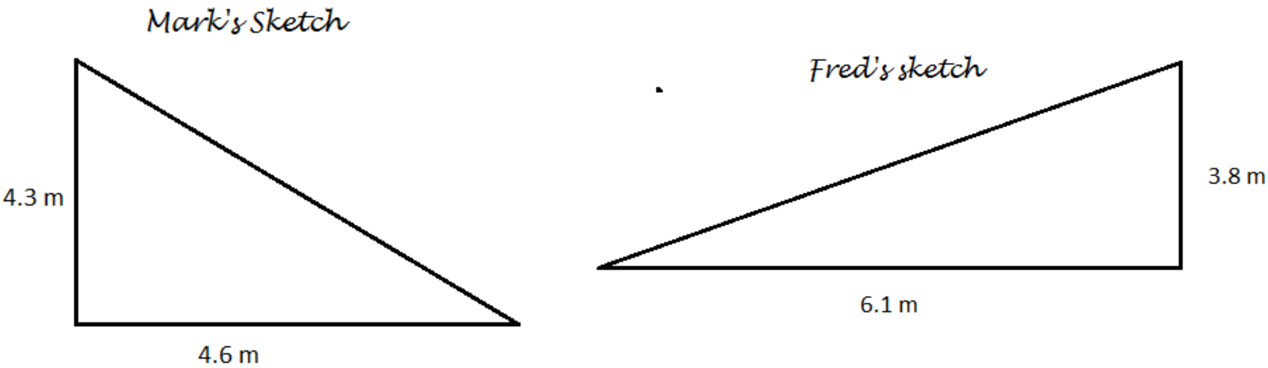


Q13 (JCHL) Trigonometry

Mark and Fred are designing a skateboard ramp. In *Skate Monthly*, they read the following advice

"to make a good skateboarding ramp, you need to find the balance between being too steep and too low. If it's too low, all you end up doing is getting a few inches off the ground, wiping out and looking silly. If it's too steep, you get halfway up, come back down, fall and look even sillier. It's best to keep the ramp angle with the ground between 30 and 45 degrees".

Here are Mark's and Fred's sketches:



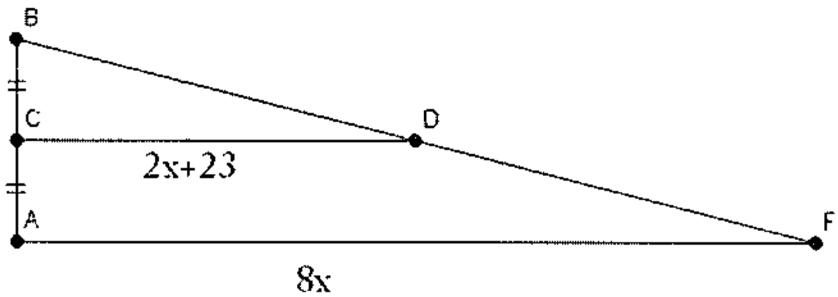
(i) Use mathematics to decide which ramp is steeper (that is, has the greater slope).

(ii) Which ramp would ensure that the skater travels a greater distance on the ramp?

(iii) Does the angle which each ramp makes with the ground comply with the advice about angles given in *Skate Monthly*? Use mathematics to justify your conclusion**.**

Q1 (LCOL) In the diagram, CD is parallel to AF and equal lengths are marked.

Find the value of x.



CD ∥ AF ⇒ ∠C = ∠A

∠B is common to △ABF and △CBD

} Remaining angles are equal

Thus △ABF + △CBD are similar

⇒ corresponding sides are proportional

$|BC| = \frac{1}{2}|BA| \Rightarrow |CD| = \frac{1}{2}|AF|$

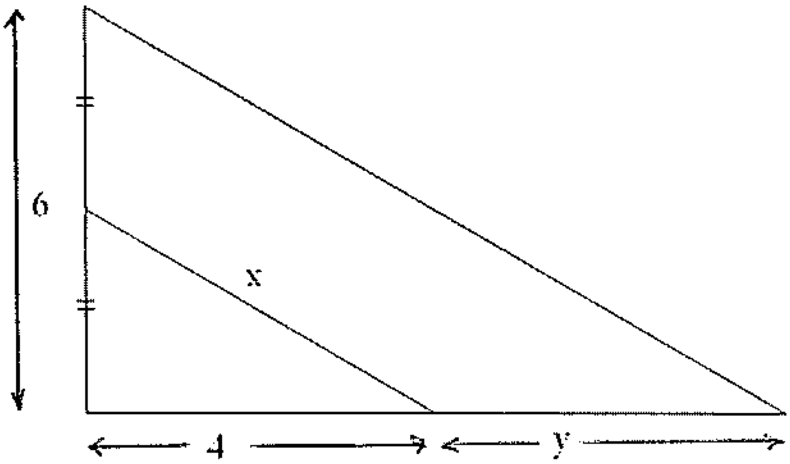
$\therefore \quad 2x + 23 = 4x$

$23 = 2x$

$11.5 = x$

Ans $x = 11.5$

Q2 (JCHL) If the sloped lines are parallel, find the value of x and the value of y.



$$x^2 = 4^2 + 3^2$$
$$= 16 + 9$$
$$= 25$$

$x = 5$

Sloped lines parallel ⇒ Δs are similar

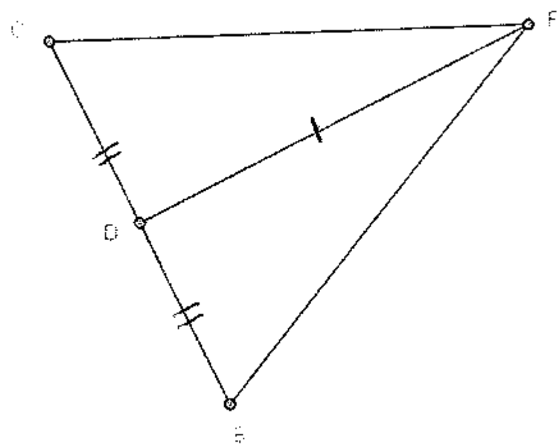
$$\therefore \frac{6}{3} = \frac{4+y}{4}$$

$$24 = 12 + 3y$$
$$12 = 3y$$
$$4 = y$$

OR Sloped lines parallel, so sides are divided in the same proportion. Because 6 is divided equally, y must also be 4 in length.

Q3 (JCHL) In triangle FCB $|CD| = |DB|$ and $|\angle FDC| = |\angle FDB| = 90°$

Explain why the triangles FDC and FDB are congruent.



In $\triangle FDC$ and $\triangle FDB$

$|CD| = |DB|$

$|\angle FDC| = |\angle FDB|$

$|FD| = |FD|$

So (SAS) they are congruent

OR

In $\triangle FDC$

$|FC|^2 = |CD|^2 + |FD|^2$ (Pythagoras)

In $\triangle FDB$

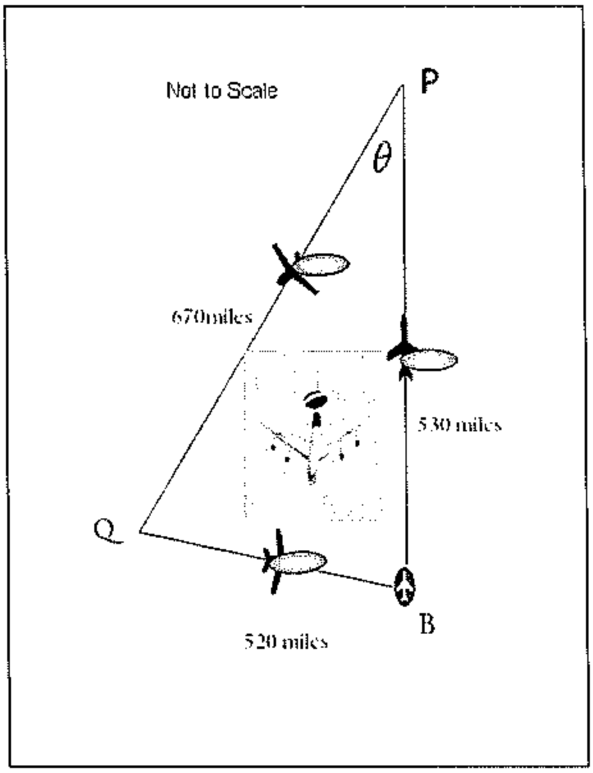
$|FB|^2 = |DB|^2 + |FD|^2$ (Pythagoras)

$= |CD|^2 + |FD|^2$ $(|CD| = |DB|)$

$\therefore |FB|^2 = |FC|^2$

Hence the 3 corresponding sides in each $\triangle$ are equal. $\Rightarrow$ congruent

Q4 (LCOL)



An aircraft takes off from Baldonnel **(B)** on a navigation exercise. It flies 530 miles directly North to a point **(P)** as shown. It then turns and flies directly to a point **(Q)**, 670 miles away. Finally it flies directly back to base, a distance of 520 miles.

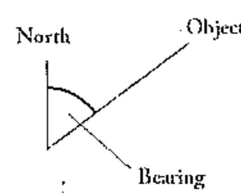
a) Calculate the angle QPB.

$$520^2 = 530^2 + 670^2 - 2(530)(670)\cos\theta$$

$$\cos\theta = \frac{530^2 + 670^2 - 520^2}{2(530)(670)} = 0.64686$$

$$\Rightarrow \theta = 49.7^\circ$$

b) If the bearing is defined as the clockwise angle measured from the North direction, calculate the bearing of Q from P.



$$49.7^\circ + 180^\circ = 229.7^\circ$$

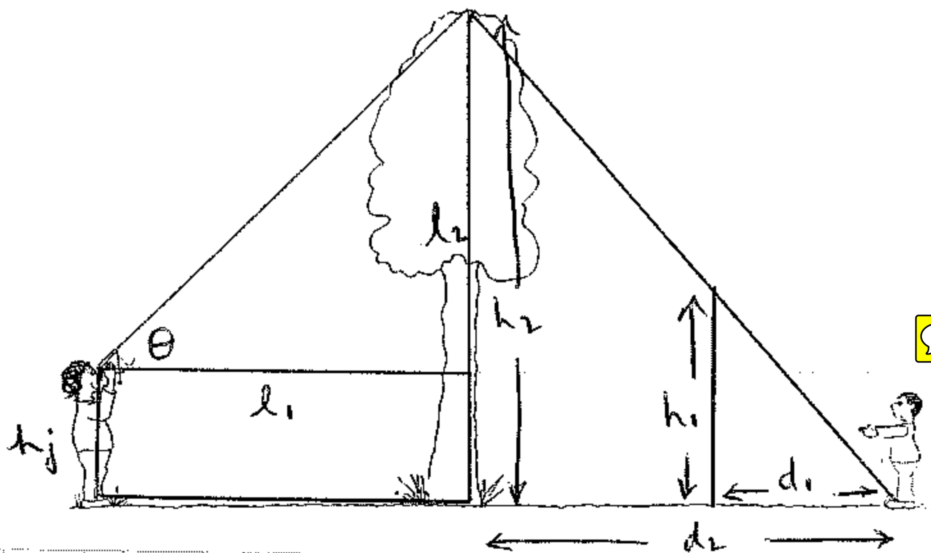
$\Rightarrow$ Bearing is 229.7° from P

Q5 (JCOL) Jane and Stephen want to estimate the height of a tall tree which is vertical and stands on horizontal ground.

Jane has a **clinometer** and Stephen has a 100m measuring tape and a large **stake**.

Explain, using diagrams and your own reasonable measurements, how each of them can make an estimate of the tree's height.

Account for any inaccuracies that might occur and suggest how you could minimise these inaccuracies.



Jane measures the angle of elevation θ and the length l_1 to the tree.

She calculates l_2 from

$$\tan\theta = l_2/l_1$$

and then calculates the tree height by adding her own height (h_j)

$$h_2 = l_2 + h_j$$

Stephen measures distances d_1, d_2 and the height of the stake h_1 (when the top of the stake coincides with the top of the tree when viewed from a point on the ground where his feet are).

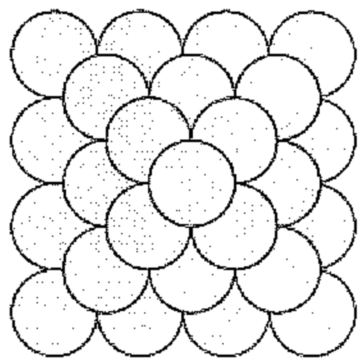
Then

$$\frac{d_1}{h_1} = \frac{d_2}{h_2}$$

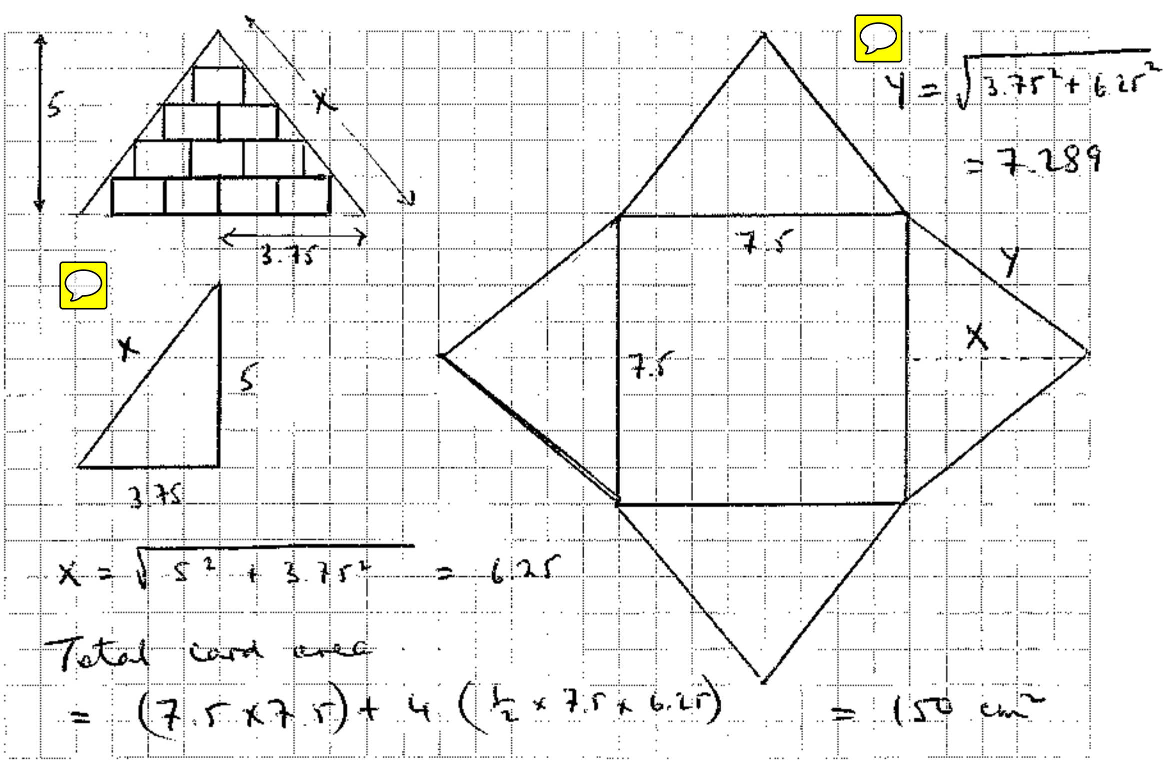
$$\Rightarrow h_2 = \frac{d_2 \times h_1}{d_1}$$

Q6 (LCHL) Joan was asked to design a box for 30 chocolates. Each chocolate is cylindrical with diameter **1.5 cm** and height **1 cm**. She decided the box should be made from card and in the shape of a **square-based pyramid**.

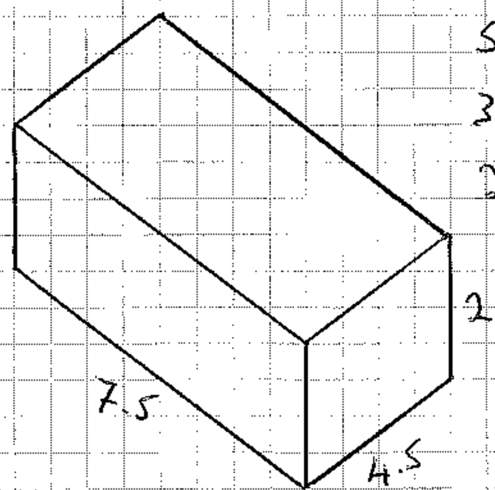
Inside the box the chocolates would be stacked in 4 layers and would look like this when viewed from above.



By sketching a net of the box, without including any joining flaps, calculate how much card the design will need. Show all measurements on your sketch.



Sarah claims that it would need less card if the 30 chocolates were stacked in a closed rectangular box that would hold two layers, each 5 chocolates long by 3 chocolates wide. By calculating the surface area of such a box, decide whether or not the claim is accurate.



5 chocs long ⇒ 7.5 cm

3 chocs wide ⇒ 4.5 cm

2 chocs high ⇒ 2 cm

$(7.5 \times 4.5) \times 2 = 67.5$

$(7.5 \times 2) \times 2 = 30$

$(4.5 \times 2) \times 2 = 18$

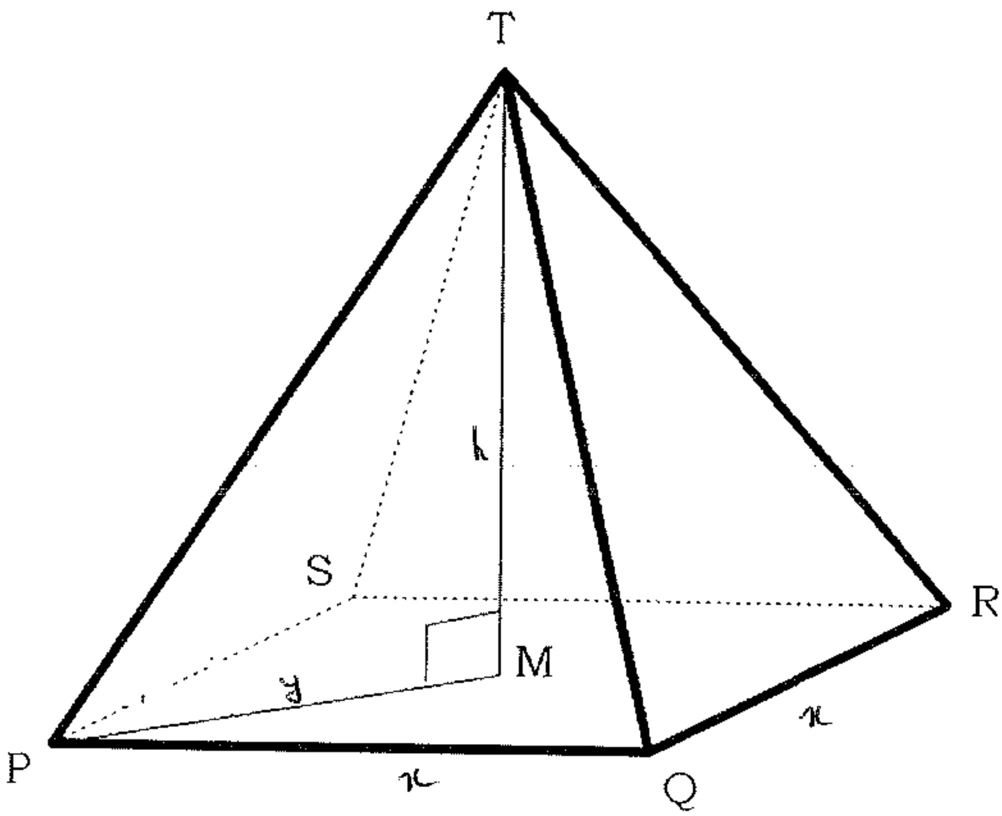
115.5

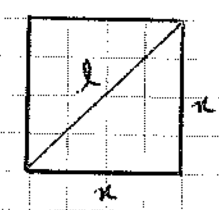
Since $115.5\ cm^2 < 150\ cm^2$

Sarah's claim is accurate

Q7 (LCHL) Show that the length of steel tubing required to make a sculpture in the shape of a square-based pyramid, as illustrated below, is given by the equation:

Length of tubing = $4\sqrt{h^2 + \frac{x^2}{2}} + 4x$, where x = base length and h = vertical height





$l^2 = x^2 + x^2$

$l = x\sqrt{2}$

$\therefore y = \frac{1}{2}x\sqrt{2}$

$PT^2 = y^2 + h^2$

$= \frac{1}{4}x^2(2) + h^2$

$PT^2 = \frac{1}{2}x^2 + h^2$

$\Rightarrow PT = \sqrt{\frac{1}{2}x^2 + h^2}$

Total length $= (4 \times PT) + 4x$

$= 4\sqrt{\frac{1}{2}x^2 + h^2} + 4x$

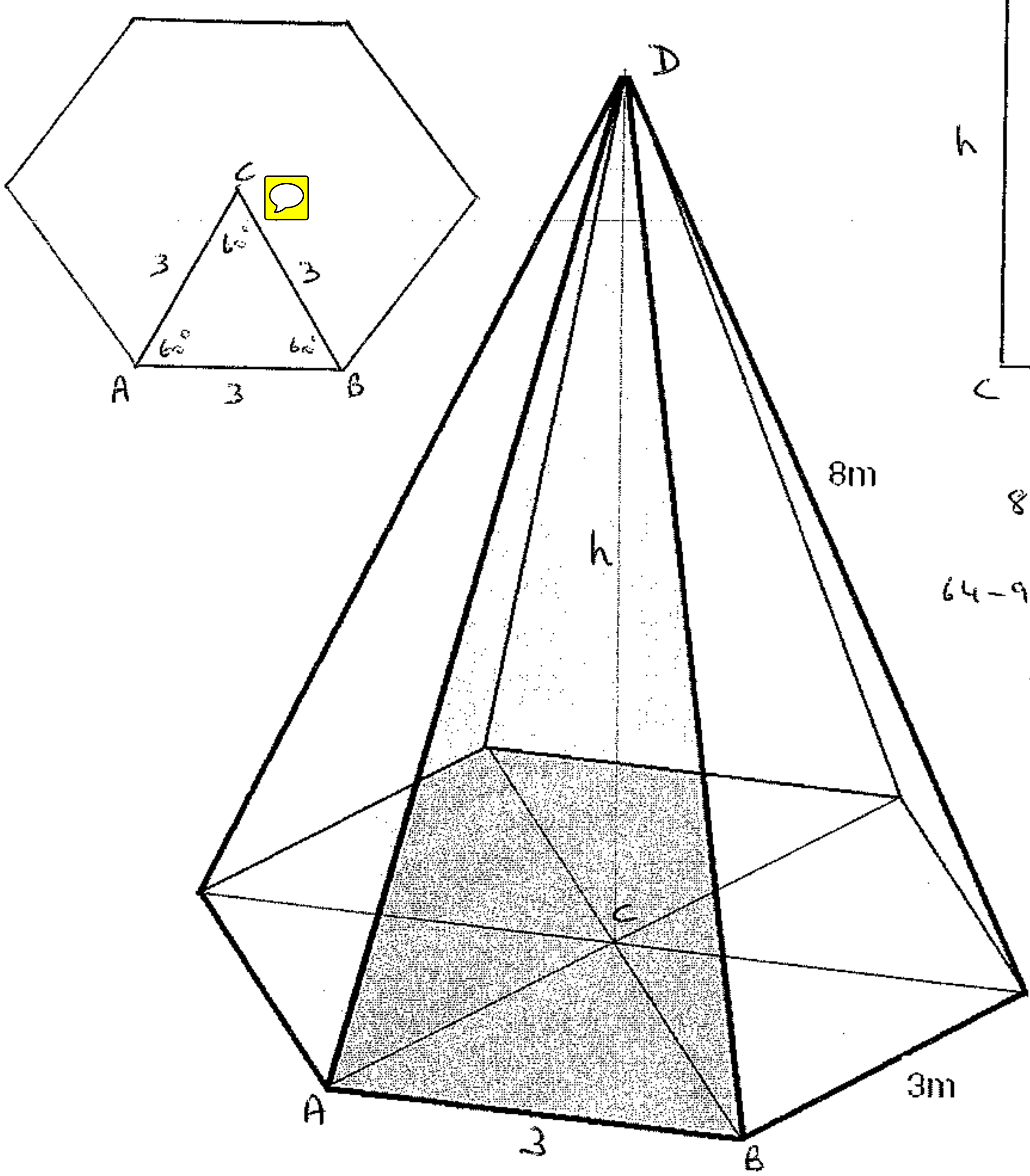
Q8 (LCHL) The diagram shows the structure of a climbing frame. The structure is in the shape of a pyramid on a hexagonal (6 equal sides) base.

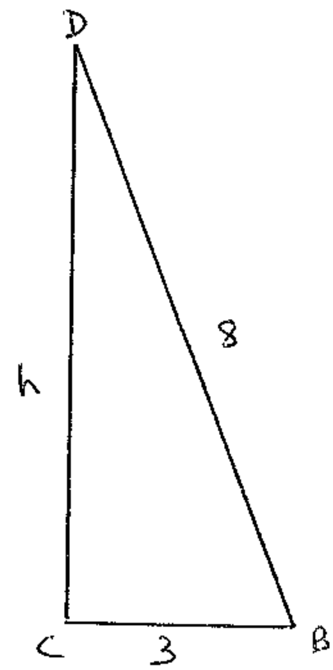
The length of each sloping edge is 8m and the pyramid's base is a regular hexagon with sides of length 3m as shown in the diagram.

The regulations state that the frame cannot exceed 7.5m in height.

Will these dimensions comply with regulations?

Support your answer with calculations.





$8^2 = h^2 + 3^2$

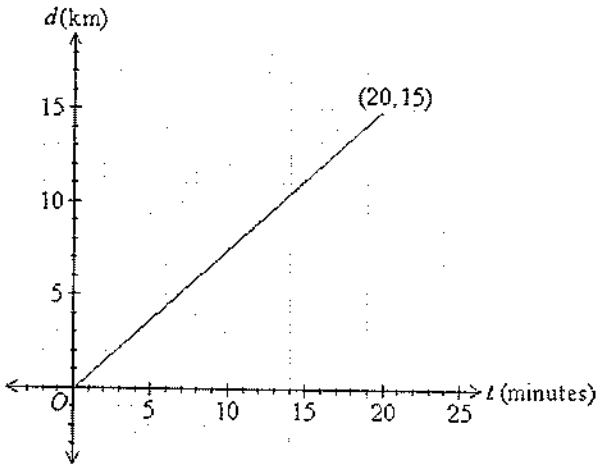
$64 - 9 = h^2$

$h = \sqrt{55}$

$= 7.416$

Since $7.416 < 7.5$ it complies with Regulations

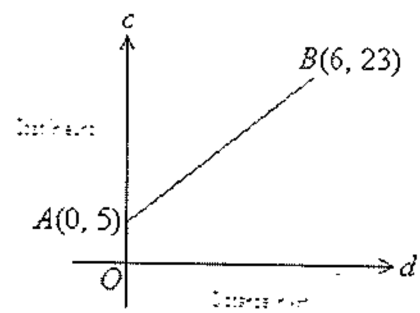
Q9 (LCFL) (a) A cyclist travels for 20 minutes at a constant speed and covers a distance of 15 km, as shown in the diagram. Find the slope of the line and describe its meaning.



$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15-0}{20-0} = 0.75$$

It means that the cyclist covers 0.75 km per minute.

(b) The cost of transporting documents by courier can be represented by the following straight line



(i) What does each point represent?

A is the flat or standing charge for any distance = €5

B shows that it costs €23 for a 6 km journey

(ii) Calculate the slope. What does this represent?

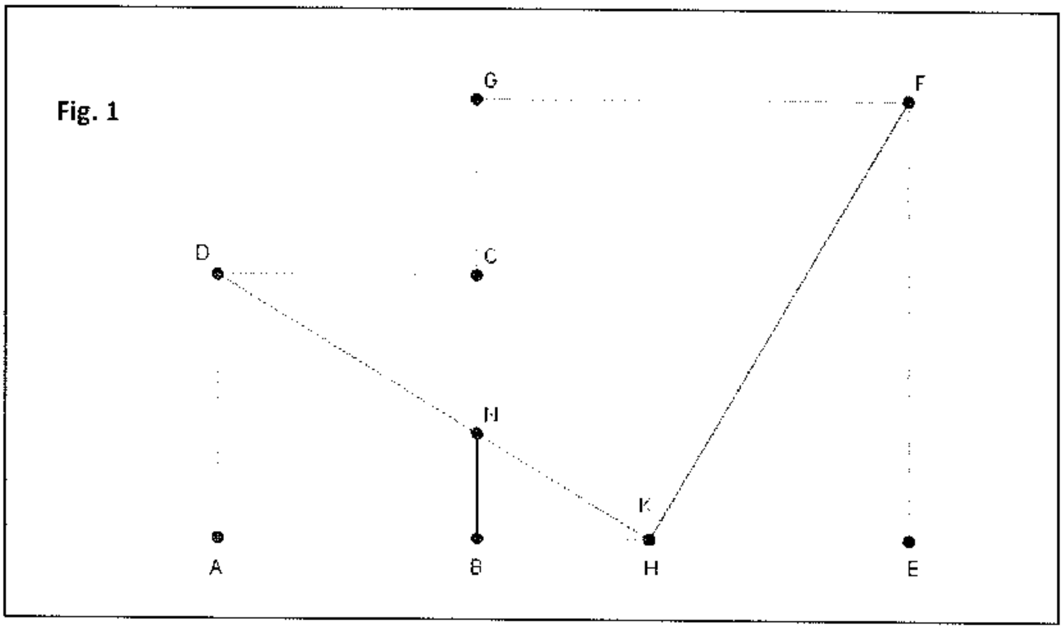
$$\text{slope} = \frac{23-5}{6-0} = 3$$

This means it costs €3 for every kilometre on top of the standing charge.

Total cost is €5 plus €3 per km.

Q10 (JCHL) – Geometry

The diagram (Fig. 1) shows two square tiles, ABCD and BEFG placed alongside each other. The point H is chosen along the side BE so that $|HE| = |AB|$.



(i) Prove that the triangles DAH and HEF are congruent.

$|DA| = |HE|$ (both $= |AB|$)

$|AH| = |AB| + |BH| = |HE| + |BH| = |EF|$ (sides of the square)

$\angle A = \angle E = 90°$

SAS $\Rightarrow$ $\triangle DAH$, $\triangle HEF$ are congruent

(ii) Prove that $\angle$ DHF is a right angle

$|\angle AHD| = |\angle HFE|$ from the $\triangle$s DAH and HEF

$|\angle FHE| + |\angle HFE| = 90°$ since $\angle E = 90°$

$\therefore |\angle FHE| + |\angle AHD| = 90°$

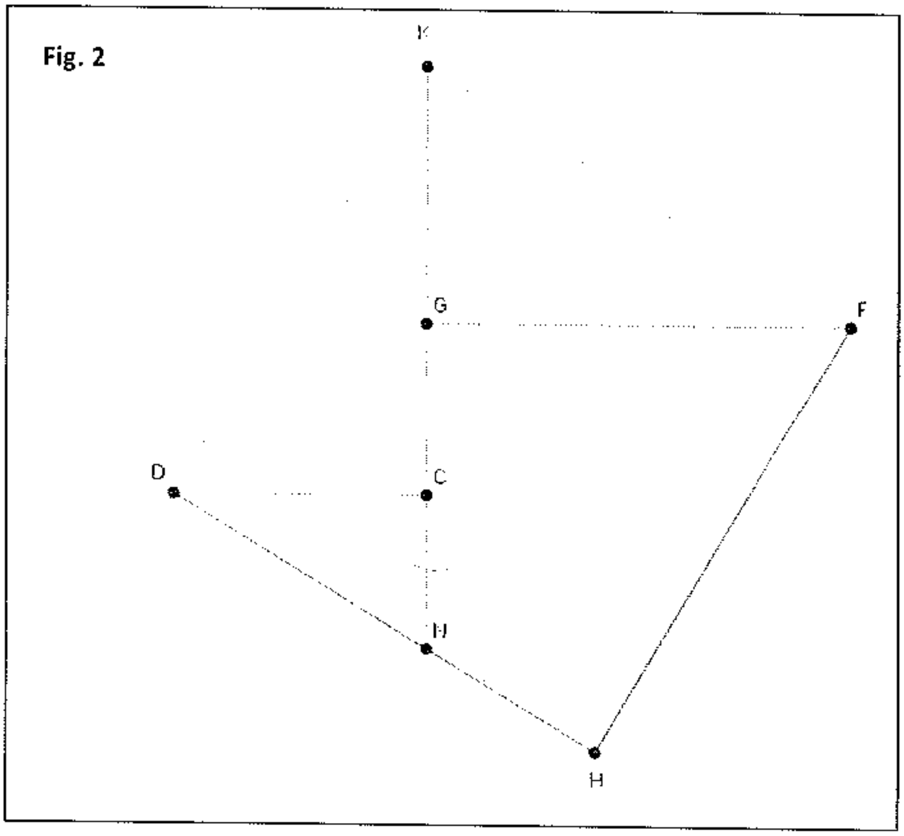
Since AHE is a straight line $\rightarrow$ Remaining angle $= 90°$

$\therefore |\angle DHF| = 90°$

The square tiles are cut along the lines DH and HF as shown and the pieces are moved so that ΔHEF lies in the position DCK and ΔDAH lies in the position KGF (see Fig. 2).

(iii) Prove that the new figure formed, DHFK, is a square.

Fig. 2



$|DH| = |HF| = |FK| = |KD|$ (sides of the congruent triangles)

the new side KD came from FH

and side KF " " DH

The angle DHF is 90° from proof (ii) above

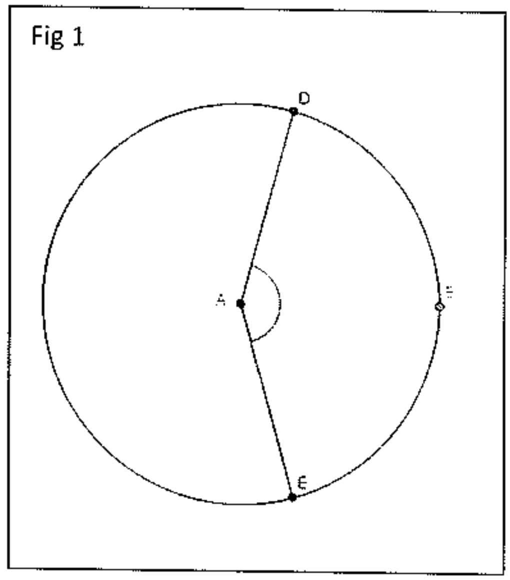
Hence DHFK has 4 equal sides and is right angled.

So it is a square.

Q11 (LCOL)

Fig. 1 shows a circle with centre A.

(a) If the $|\angle DAE| = 150°$ and $|AD| = 12$ cm, find the length of each arc.



Small arc

$2\pi(12) \times \frac{150}{360}$

$= 10\pi$ cm

large arc

$2\pi(12) \times \frac{210}{360}$

$= 14\pi$

(b) Fig. 2 shows a belt-driven pulley system with pulleys of radii 12 cm and 5 cm respectively. The centres of the pulleys are 24 cm apart.

(i) Find the measure of the angle DAF to the nearest degree.

(ii) Find the total length of the belt needed for this pulley system.

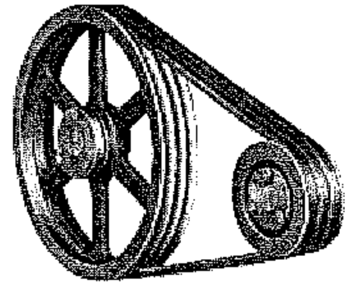
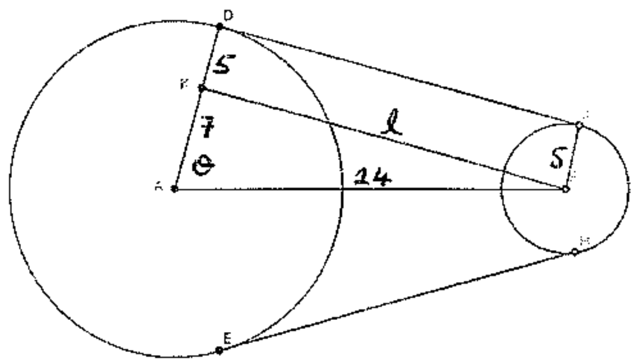


Fig 2



$l^2 + 7^2 = 24^2$

$l^2 = 24^2 - 7^2 = 527$

$l = 22.96$

$\Rightarrow |DJ| = 22.96$ cm

$360 - 146 = 214$

$\cos\theta = \frac{7}{24} \Rightarrow \theta = 73^\circ \longrightarrow 2\theta = 146^\circ$

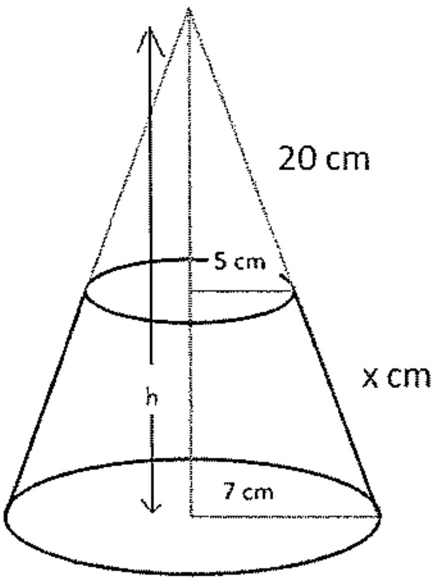
large pulley; large arc $\rightarrow 2\pi(12) \times \frac{214}{360} = 44.82$ cm

small pulley, small arc $\rightarrow 2\pi(5) \times \frac{146}{360} = 12.74$

$\therefore$ belt length $= (2 \times 22.96) + 44.82 + 12.74$

$= 103.48$ cm

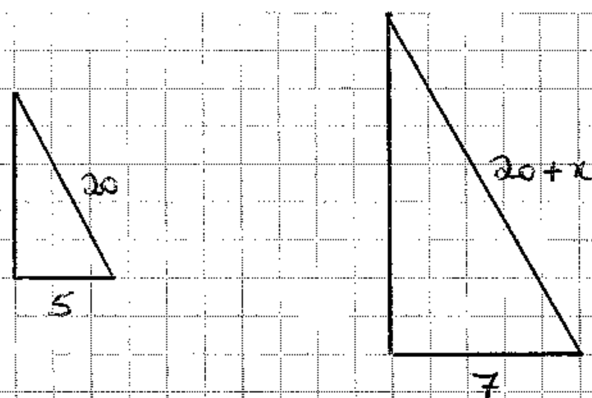
Q12 (LCHL) Geometry/Trigonometry



The figure to the left shows a cone from which a lampshade is to be made.

The smaller cone with base radius 5 cm is cut off the top to form the lampshade.

(i) Calculate the slant height of the lampshade (marked x cm in the diagram).



$\Rightarrow$ Similar triangles

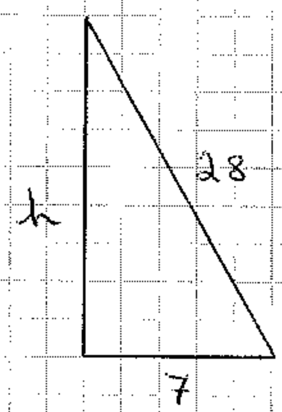
$$\frac{5}{7} = \frac{20}{20+x}$$

$$100 + 5x = 140$$

$$5x = 40$$

$$x = 8 \text{ cm}$$

(ii) Hence calculate the value of h, correct to one decimal place.



$28^2 = h^2 + 7^2$

$28^2 - 7^2 = h^2$

$27.11 = h$

$h = 27.1$ cm

(iii) Calculate the surface area of the lampshade correct to two decimal places.

Large cone: area $= \pi r l$

$= \pi(7)28$

$= 196\pi$

Small cone:

area $= \pi(5)(20)$

$= 100\pi$

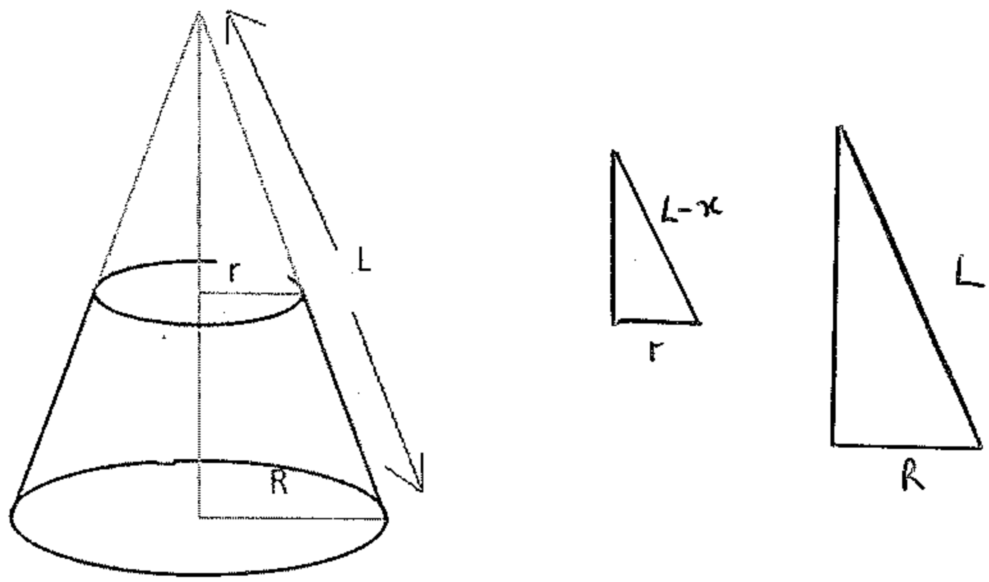
lampshade area $= 196\pi - 100\pi$

$= 96\pi$

$= 301.5928$

Ans 301.59 cm^2

(iv) By letting r = the base radius of the small cone, R= base radius of the large cone and L = the slant length of the large cone, show that the curved surface area of the lampshade is given by $A = \dfrac{\pi L(R^2 - r^2)}{R}$



$$\frac{r}{R} = \frac{L-x}{L}$$

$$rL = RL - Rx$$

$$Rx = RL - rL$$

$$x = \frac{L(R-r)}{R}$$

∴ slant height of small cone

$$= L - \frac{L(R-r)}{R}$$

$$= \frac{RL - RL + rL}{R} = \frac{rL}{R}$$

Large cone area $= \pi RL$

Small cone area $= \pi r\left(\frac{rL}{R}\right) = \frac{\pi r^2 L}{R}$

∴ area of lampshade $= \pi RL - \frac{\pi r^2 L}{R}$

$$= \frac{\pi R^2 L - \pi r^2 L}{R}$$

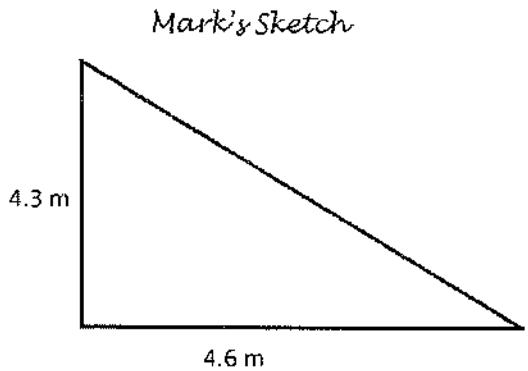
$$= \frac{\pi L(R^2 - r^2)}{R}$$

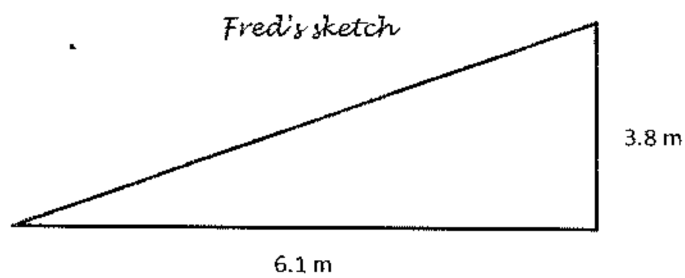
Q13 (JCHL) Trigonometry

Mark and Fred are designing a skateboard ramp. In *Skate Monthly*, they read the following advice

"to make a good skateboarding ramp, you need to find the balance between being too steep and too low. If it's too low, all you end up doing is getting a few inches off the ground, wiping out and looking silly. If it's too steep, you get halfway up, come back down, fall and look even sillier. It's best to keep the ramp angle with the ground between 30 and 45 degrees".

Here are Mark's and Fred's sketches:





(i) Use mathematics to decide which ramp is steeper (that is, has the greater slope).

Slopes: Mark Fred

$= \frac{4.3}{4.6}$ $= \frac{3.8}{6.1}$

$= 0.935$ $= 0.623$

$\therefore$ Mark's ramp is steeper than Fred's

(ii) Which ramp would ensure that the skater travels a greater distance on the ramp?

Mark $l^2 = 4.3^2 + 4.6^2$

$= 39.65$

$\Rightarrow l = 6.3$ m

Fred $l^2 = 3.8^2 + 6.1^2$

$= 51.65$

$\Rightarrow l = 7.2$ m

$\therefore$ Fred's ramp will have a greater distance on the Ramp.

(iii) Does the angle which each ramp makes with the ground comply with the advice about angles given in *Skate Monthly*? Use mathematics to justify your conclusion.

Mark $\tan^{-1}(0.935) = 43^\circ$

Fred $\tan^{-1}(0.623) = 32^\circ$

Both angles are between 30° and 45°

So they comply with the advice given.

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Project Maths
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Mathematics Resources for Students

Leaving Certificate – Strand 2

Geometry and Trigonometry

INTRODUCTION

This booklet is designed to supplement the work you have done in Leaving Cert geometry with your teacher. There are activities included for use as homework or in school. The activities will help you to understand more about the concepts you are learning in geometry. Some of the activities have spaces for you to fill in answers, while others will require you to use drawing instruments and paper of your own. You may not need or be able to complete all of the activities; your teacher will direct you to activities and/or questions that are suitable.

You should note that Foundation level material is a subset of Ordinary level. Students at OL can expect to be tested on material from the Foundation level course, but at a greater degree of difficulty. The same will apply to Higher level students, who can be tested on syllabus material from a lower level, but at a greater degree of difficulty. The sequence in which the sections/ topics are presented here is not significant. You may be studying these in a different order, or dipping in and out of various sections over the course of your study and/or revision.

In the first section (synthetic geometry), it is important you understand the approach taken. Although only Higher level students are required to reproduce the proof of some theorems, all students are expected to follow the logic and deduction used in the proofs. This type of understanding is required when solving problems such as those from activity 5 onwards.

Each activity or question you complete should be kept in a folder for reference and used for revision at a later date. A modified version of this booklet and syllabus documents are available at www.ncca.ie/projectmaths and other related materials at www.projectmaths.ie.

GEOMETRY 1

SYLLABUS TOPIC: SYNTHETIC GEOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- complete a number of constructions
- use the listed terms related to logic and deductive reasoning: theorem, proof, axiom, corollary, converse, implies;
- investigate theorems and solve problems.

HL learners will

- extend their understanding of geometry through the use of formal proof for certain theorems.

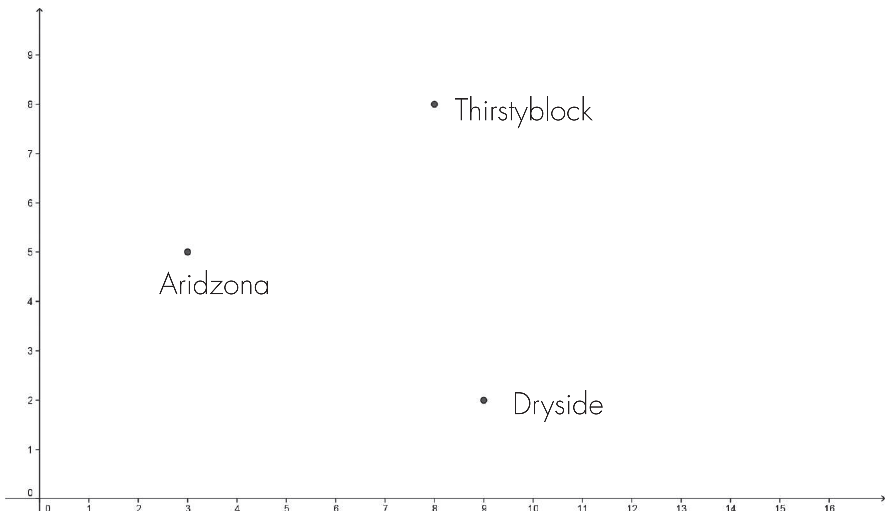
INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in geometry as well as to solve problems using these concepts and their applications.

Activity 1.1 LCFL

In a developing country an engineer is given the problem of digging a new well in a community where there are three villages. The ideal situation would be to have a well in each village, but this is not possible due to cutbacks. He has a map with the towns on it and he wants to minimise the walking to be done by the people who will carry the water.

Map



i. Where is the best place to dig the well?

ii. How would you find the nearest point to the three villages?

iii. Is there any other point that is also nearest, or is it unique?

iv. What would happen to this point if the Thirstyblock was more to the left or right?

v. What do we call this 'nearest' point?

The 'nearest' point for the three villages seems always to be inside the triangle. Could it ever be outside the triangle? Try drawing a triangle with a circumcentre outside of the triangle.

Activity 1.2 LCOL

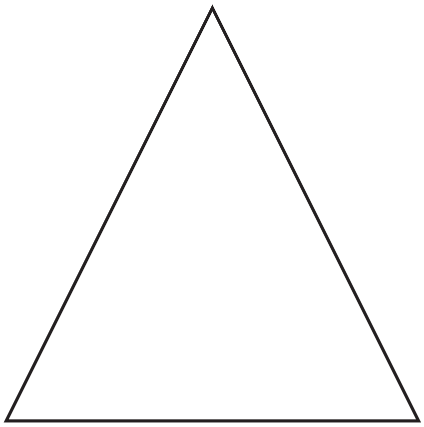
An architect wants to design the biggest circular clock-face that will fit onto the wall of a train station. The diagram of an isosceles triangle represents the top section of the gable of the building. She needs to find where the centre of the clock will be located in this triangle.

i. Mark what you think are the two sides that are equal in length.

ii. Write a sentence which describes an isosceles triangle.

iii. Find the centre of the face of the clock on the diagram, showing how you found it by including all lines of construction. The clock can touch the edges of the triangle but not go outside of it.

iv. There are other types of triangles. List these and draw a diagram of each one that shows its properties.



Gable end of the train station onto which the clock will fit.

Activity 1.3 LCFL and LCOL

This activity could be very long and boring if you tried to define all terms in geometry. The selection here is to make you think about how we define things in geometry. We always need to be as clear as we can when trying to solve problems or prove that things are true when using geometry. Your teacher will guide you through the differences and the uses of terms in geometry.

State what you understand by each of the following terms, writing one sentence in each case.

(i) angle	(ii) definition	(iii) theorem	(iv) axiom
(v) corollary	(vi) converse	(vii) geometrical proof	

Make out a list of 6 other terms you use in geometry.

Q. 1 LCOL

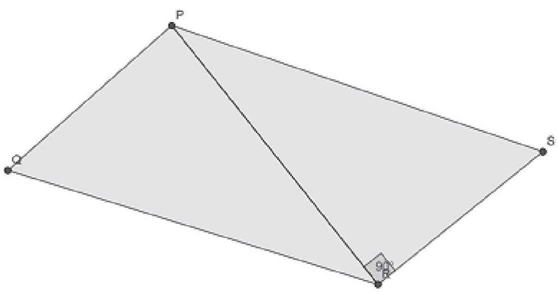
PQRS is a parallelogram, as shown.

i. Name four pairs of equal angles.

Calculate the length of [RS] if
|PR|=12, |PS|= 13 and
|∠PRS|= 90°.

ii. Find the area of the parallelogram.

iii. If |∠RPS|= 25°, show that
|∠PQR|= 65°

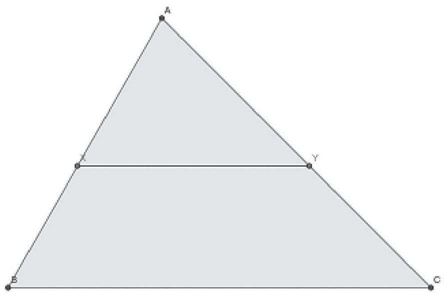


Q. 2 LCOL

In the diagram XY|| BC.

i. If |AX|:|XB|= 4:3 and |XY|= 8, write down|BC|.

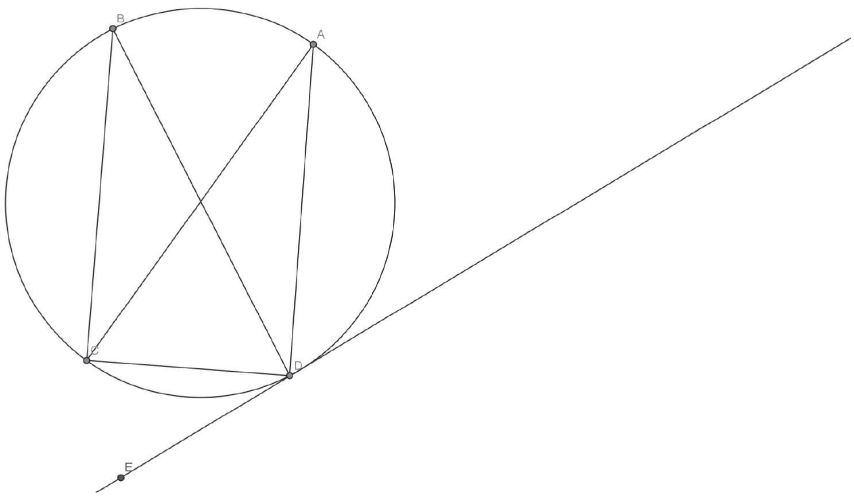
ii. If |AY|= 7 find |AC| and |YC|



Q. 3 LCOL

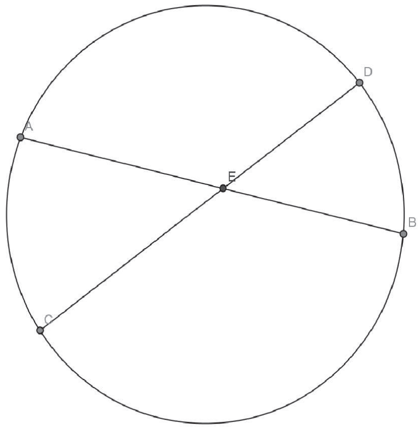
Each tangent is perpendicular to the radius that goes to the point of contact (Theorem 20).

DE is a tangent to the circle at D and DB is a diameter. Show that |∠CDE|=|∠CAD|



Q. 4 LCHL

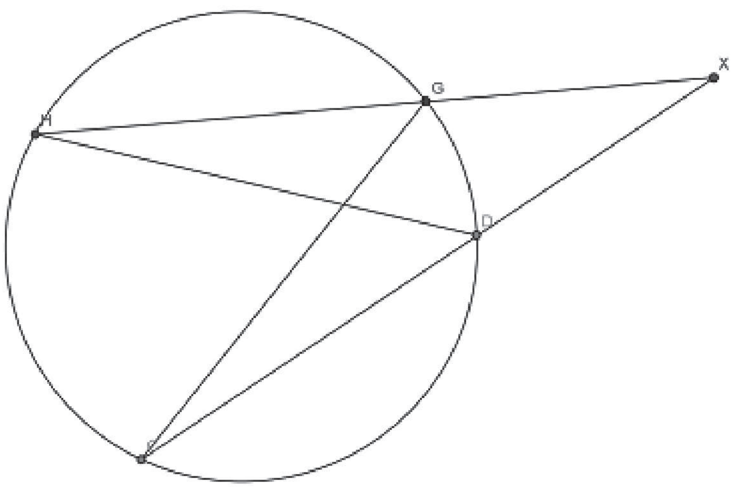
The diagram below shows a circle with two intersecting chords.
Prove that the triangles ACE and DBE are equiangular.



What theorem(s) are we using to prove this?

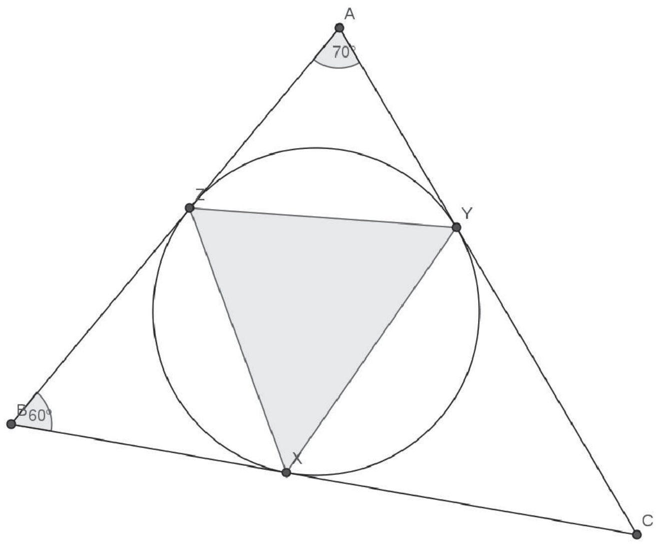
Q. 5 LCHL

The diagram below shows a circle with two chords CD and HG which intersect outside the circle at the point X. Prove that the triangles HDX and CGX are equiangular.



Q. 6 LCHL

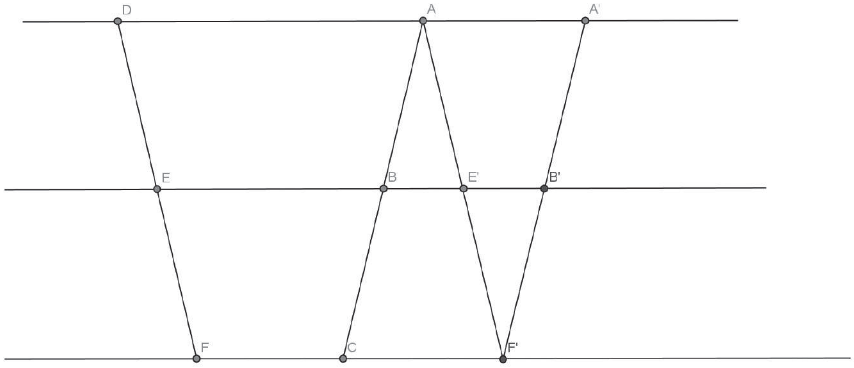
The diagram below shows the incircle of a triangle ABC. Given the measures of the angles of triangle ABC as in the diagram, find the measures of the angles of the triangle XYZ



Activity 1.4 LCHL

At higher level you are expected to prove certain theorems. The proof of theorem 11 is given here. The theorem is stated and a diagram with proof is given. You may have done a slightly different proof with your teacher, but that should not prevent you from following the logic and deductive reasoning of the proof given here.

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*



Proof. [This uses opposite sides of a parallelogram, AAS, and the axiom of parallels.]

Suppose AD||BE||CF and |AB|= |BC|. We wish to show that |DE|=|EF|.

Draw AE′ ||DE, cutting EB at E′ and FC at F′.

Draw F′B′ ||AB, cutting EB at B′ Write the justification in the spaces below

|B′F′| = |BC|

= |AB|

|∠BAE′| = |∠E′F′B′|

|∠AE′B| = |∠F′E′B′|

∴ ΔABE′ is congruent to ΔF′B′E

∴|AE′| = |E′ F′|

But |AE′|= |DE| and |E′ F′|= |EF|

∴|DE|= |EF|

GEOMETRY 2

SYLLABUS TOPIC: CO-ORDINATE GEOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- co-ordinate the plane
- calculate distance, slope, and the equations of lines; find the point of intersection of two lines
- use the equations of lines to solve problems
- explore the properties of circles and perform calculations using the equations of circles and lines

HL learners will

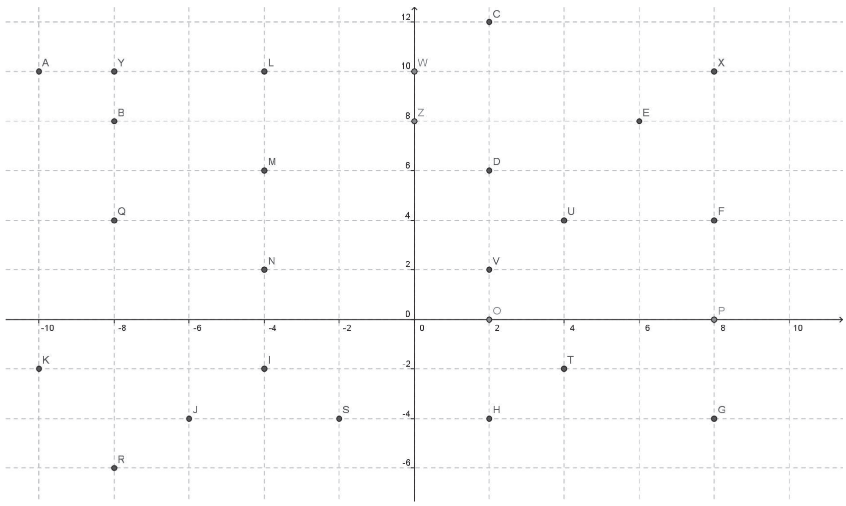
- further extend their understanding of co-ordinate geometry through solving problems involving lines and circles

INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in co-ordinate geometry as well as solve problems using these concepts and their applications. You will investigate various properties of lines and circles and perform calculations and solve problems based on distance, slope and the equations of lines and circles.

Activity 2.1 LCFL

If you and your friends in school have a grid with each letter of the alphabet on it you can compose messages to each other that cannot be read by anyone who does not have the grid. Use the grid below to answer the questions following.



i. What is your first name?

ii. Who is your favourite band or singer?

iii. What is your favourite TV programme

Here are my answers to those questions; can your figure them out?

i. (4,-2), (6,8), (-8, -6), (-8, -6), (-4,-2)

ii. (2,12), (2,0), (-4,10), (2,6)   (8,0), (-4,10), (-10,10), (-8,10)

iii. (8,-4), (-8,-6), (6,8), (-8,10), (-2,-4) (-10,10),(-4,2), (-10,10), (4,-2), (2,0) (-4,6), (-8,10)

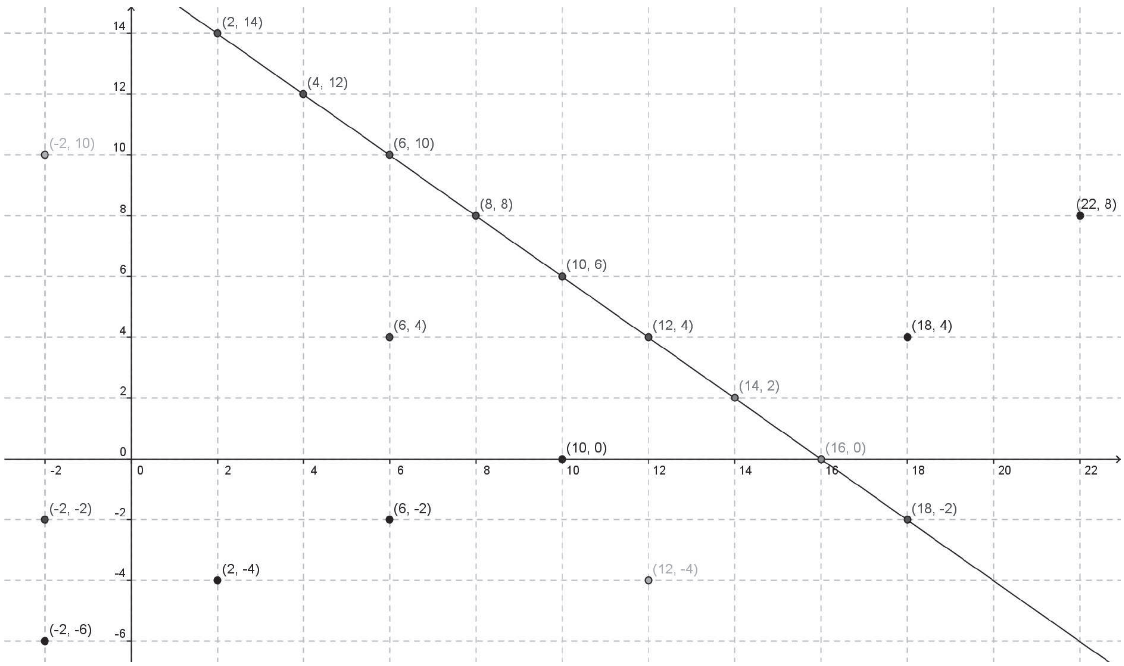
Q. 1 LCFL

Make up five questions that you can contribute in class tomorrow that could be answered in codes by other students.

Q. 2 LCFL

Write a message of one sentence in code that you can give to another student to decrypt in class tomorrow. The sentence should contain at least eight words and should be written with spaces for the words and brackets for each pair of co-ordinates.

Activity 2.2 LCFL and LCOL



There are a lot of points marked on the Cartesian plane shown above. One of the sets of points is joined to make a straight line. If they are on the same line they must have a common link!

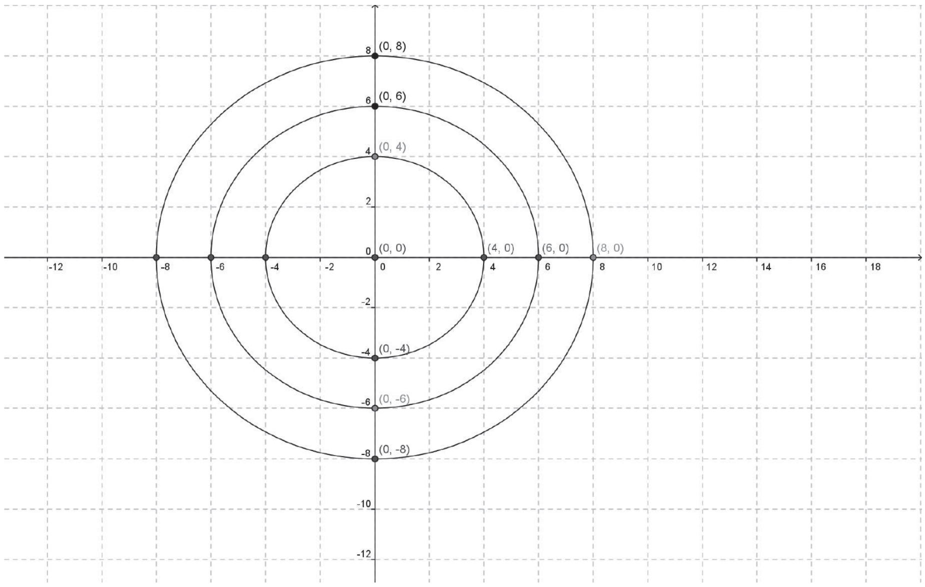
There is another set of points which appear to form a straight line. Join them to check if they all belong to this line.

The lines appear to be going upwards or downwards at constant rates. What do we call this? If the rate of rising/falling is constant, we should be able to measure this at different places and it should remain the same. Pick two points and find the slope of the line, then pick two different points on the same line and show that the slope is the same.

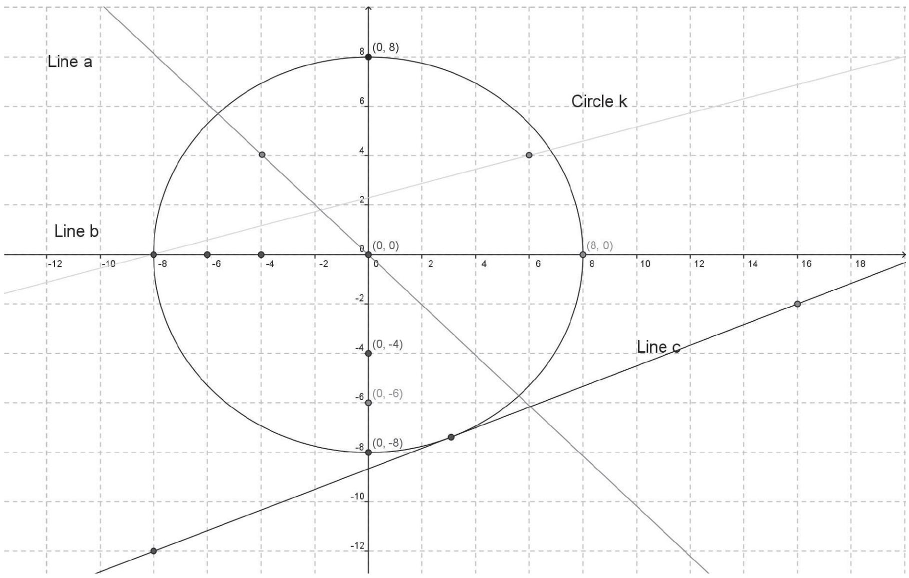
We can see where the line cuts the X-axis (16,0). Where does the line cut the Y-axis? Do we have a name for that point?

The point (14, 2) is a significant point. What do we call it? If we did not have the point given to us, could we have found it by drawing the two lines? Is there another way to find that point without drawing? Is there a connection with algebra here?

Activity 2.3 LCOL



The diagram shows three circles which are centred at the origin. Write the co-ordinates of the three points not already marked on the diagram, where the circles cut the x-axis.
Write down the equations of the three circles. How is the difference between the circles shown in their equations?



In the diagram above we see how lines can cut through a circle in different ways. Line a cuts through the circle k in two places and contains the centre (0,0). Line b intersects circle k at two points but does not contain (0,0), and line c touches circle k at one point only.

Line a is called

Line b is called

Line c is called

Consider the circle k which has the equation $x^2 + y^2 = 8^2$.

The task now is to find the points of intersection between the three lines and this circle.

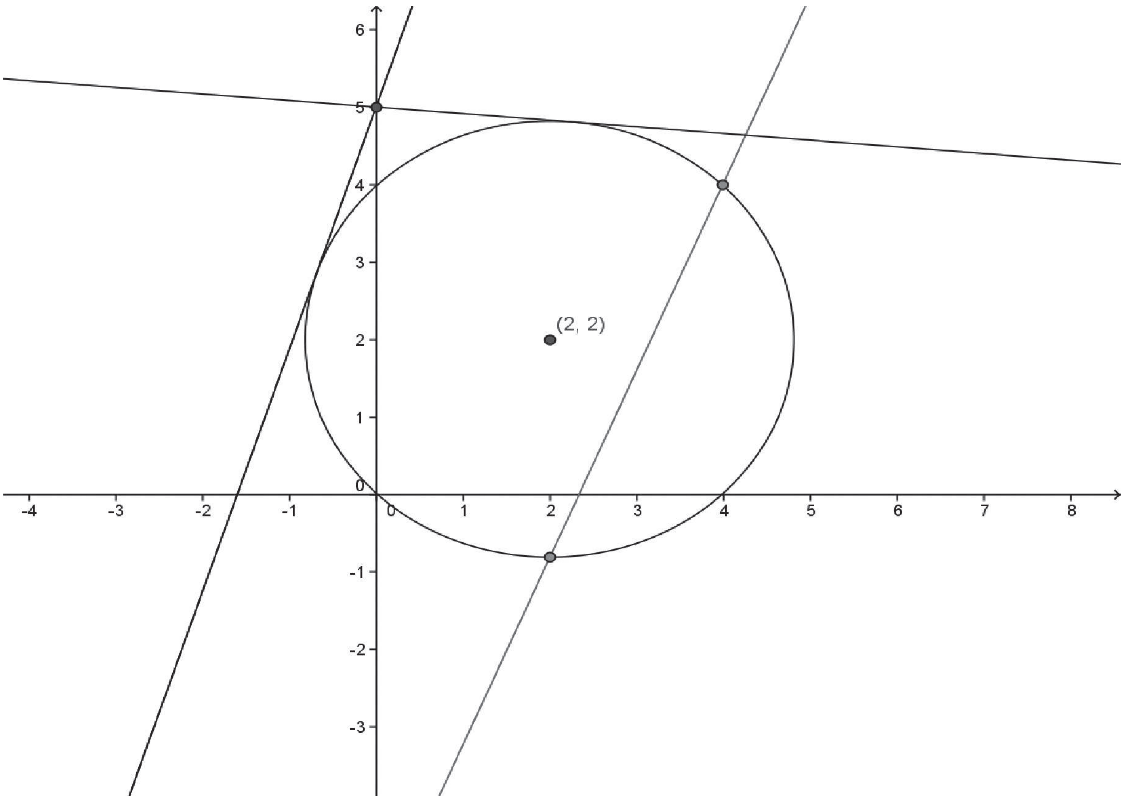
i. Read off the values of the points of intersection from the diagram.
Is this an accurate way to find the points?

ii. Could we do this activity using algebra, and be more confident of our answers?
It involves the equations of the three lines so we need to find them first.

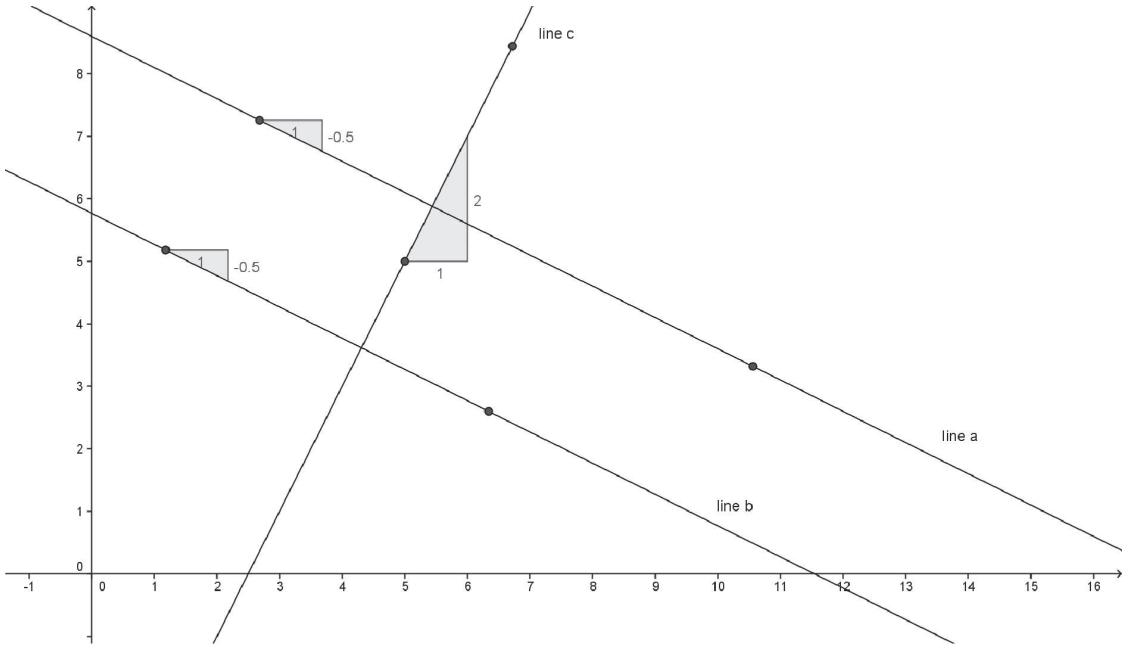
iii. Find the three equations and then use them and the equation of the circle to get more accurate results for the points of intersection between the lines and the circle.

For further consideration

iv. Repeat the exercise above for the situation when the centre of the circle is not at (0,0) but at (2,2) as shown in the diagram below.



Activity 2.4 LCOL



a. The diagram above has the slopes of three lines shown. It also allows us to see the relationship between parallel and perpendicular lines.

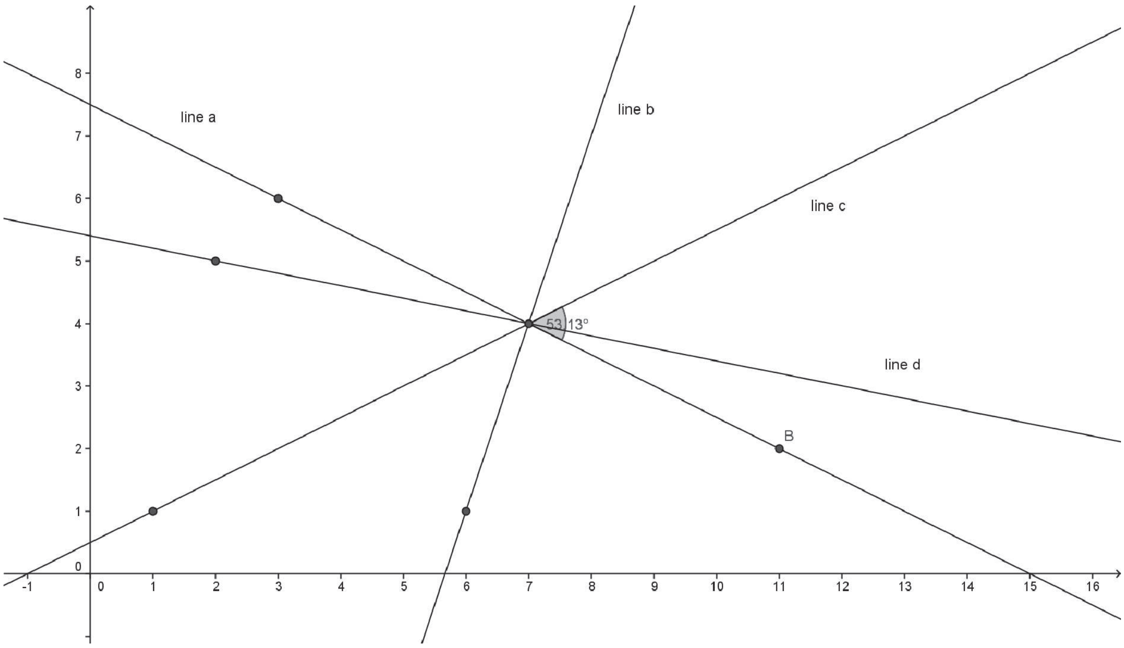
   i. Try using Geogebra to illustrate the relationships yourself. Plot two lines by using random points and manoeuvre these lines to make them either parallel or perpendicular. How can you be sure they are parallel or perpendicular? (Hint: use what you have learned about slope.)

   ii. Summarise the relations for any pair of parallel or perpendicular lines using words and symbols.

a. The following section is aimed at Higher level students

   i. Using the diagram above, or your own window in Geogebra, pick a point on a line that is parallel to another line and find the distance from it to the parallel line. Now pick another point on the first line; what distance should that point be from the parallel line?

   ii. Using pencil and ruler, or a dynamic geometry worksheet, draw a series of lines that intersect with each other at the same point.



In the diagram above, the acute angle between line a and line c is marked. You can see how this angle will vary when the point B on line a is rotated towards line c. It is easy to visualise this as line a will coincide with line d in this rotation at some point, and we can see that the angle between line d and line c is smaller than 53.13°.

iii. Is there a relationship between the angle between lines and the slopes of the lines? Write down any observations you have made as the angle between the lines gets bigger or smaller.

iv. How is this relationship stated?

v. Find the angle between line a and line c.

vi. Why is the angle either + or -?

For further exploration at Higher level
You have learned that another way of representing the equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where the centre is $(-g, -f)$ and $r = \sqrt{(g^2 + f^2 - c)}$

Q. 1 H is a circle $x^2 + y^2 + 8x - 10y + 32 = 0$

Write down the co-ordinates of the centre of the circle, the length of the radius, and make an accurate drawing of the circle.

Q. 2 The circles $(x+2)^2 + (y-1)^2 = 8$ and $(x-5)^2 + (y+2)^2 = 50$ intersect at two points.

Use the theorem of Pythagoras to show the tangents to both circles at the point of intersection are perpendicular.

Q. 3 Find the point of contact between the line $3x - 4y + 13 = 0$ and the circle $x^2 + y^2 + 6y - 16 = 0$.

Q. 4 Find the length of the tangents from $p(-4,0)$ to the circle $x^2 + y^2 - 4x - 8y - 30 = 0$

Find the equations of the tangents to the circle from P.
The line joining the centre of the circle to the point of tangency T cuts the x-axis inside the circle at Q. Find the co-ordinates of T and the $|QT|$.

GEOMETRY 3

SYLLABUS TOPIC: TRIGONOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- solve right angle triangle problems, use the theorem of Pythagoras and the trigonometry ratios of sin, cos and tan

and you will also

- extend your knowledge of the trigonometric ratios
- solve problems using rules; find areas of given shapes

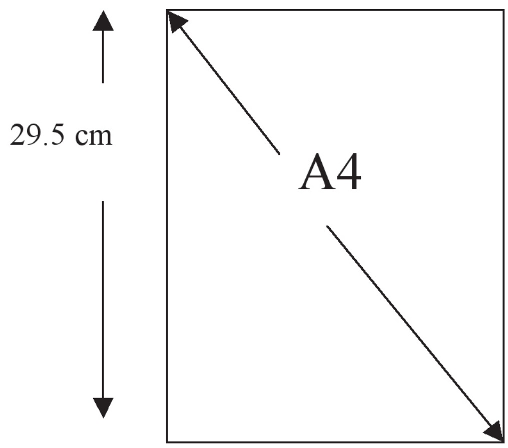
HL learners will be able to

- graph trigonometric functions
- solve trigonometric equations, including use of radians
- apply trigonometric formulae

INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in trigonometry as well as solve problems using these concepts and their applications.

Activity 3.1 — LCOL



The sheet is 20.8 cm wide and 29.5 cm long. The diagonal length is not so easy to measure accurately with a standard 30 cm ruler as it is longer than this.

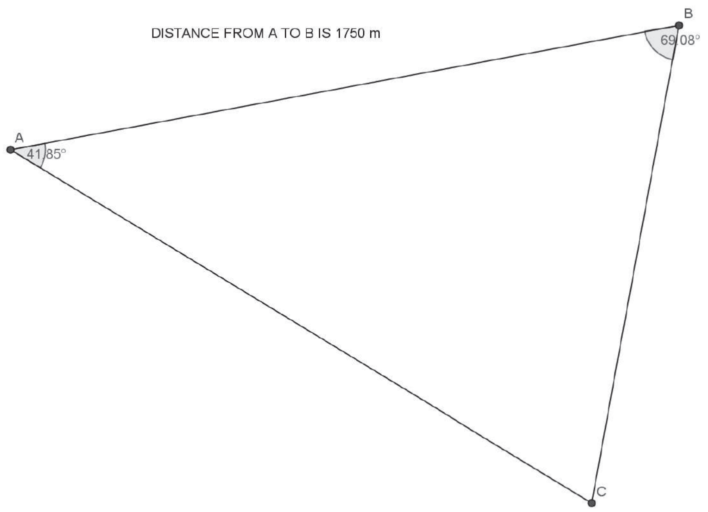
The theorem of Pythagoras gives us a way of calculating this distance to a desired degree of accuracy.

i. Find the length of the diagonal to
   - one decimal place
   - two decimal places
   - three decimal places

ii. Now measure the diagonal with a piece of string, metre stick or a tape measure and compare this with the values calculated at (i).

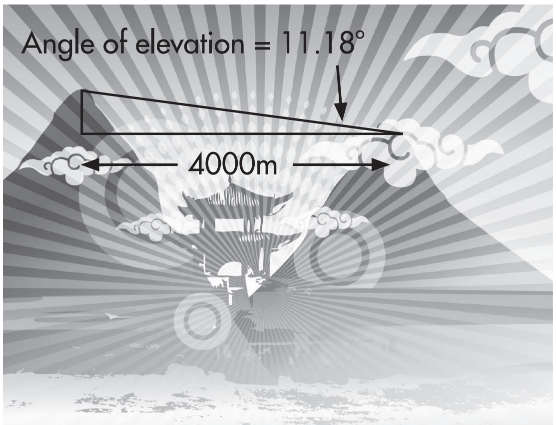
Q.1 LCOL

In the triangle shown below a cartographer (mapmaker) has made measurements from A to two places B and C on the same level plain. The angle at A is 41.85° and the angle at B is 69.09°. The distance from A to B is 1750 m. Find the distance from A to C and the distance from B to C.



Q.2 LCOL

The climber on top of the mountain on the right wants to know how much higher the mountain on the left is. The angle of elevation from right to left is 11.18° and the horizontal distance between the peaks is 4000m. Find the difference in height between the peaks.

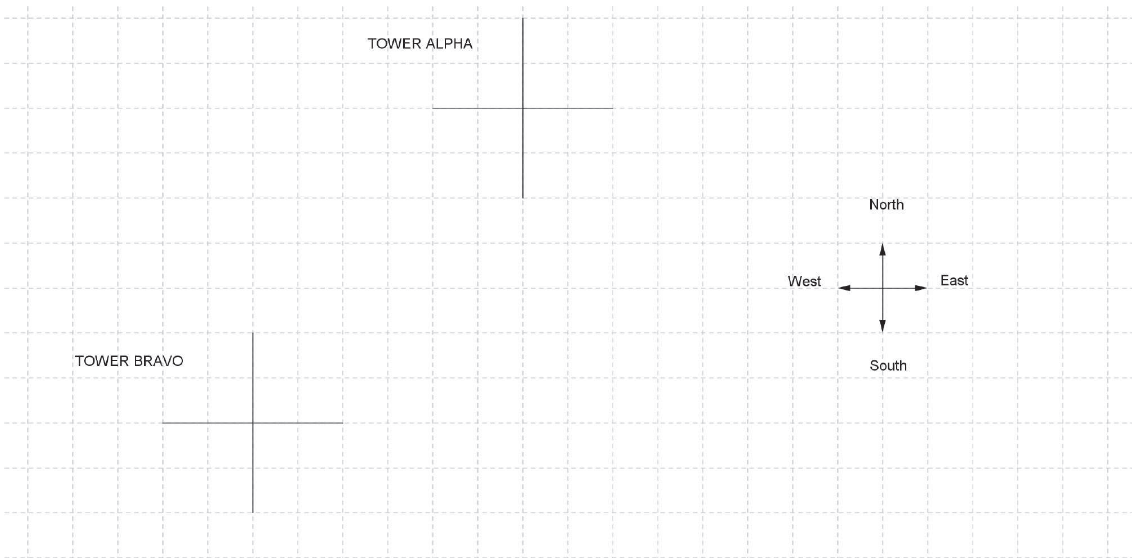


Q.3 LCOL

In fighting crime, police forces in recent years have been able to pinpoint the location of mobile phones at the times calls are made from them. By drawing as accurately as you can on the grid below, show the location of the phone, given the following data:

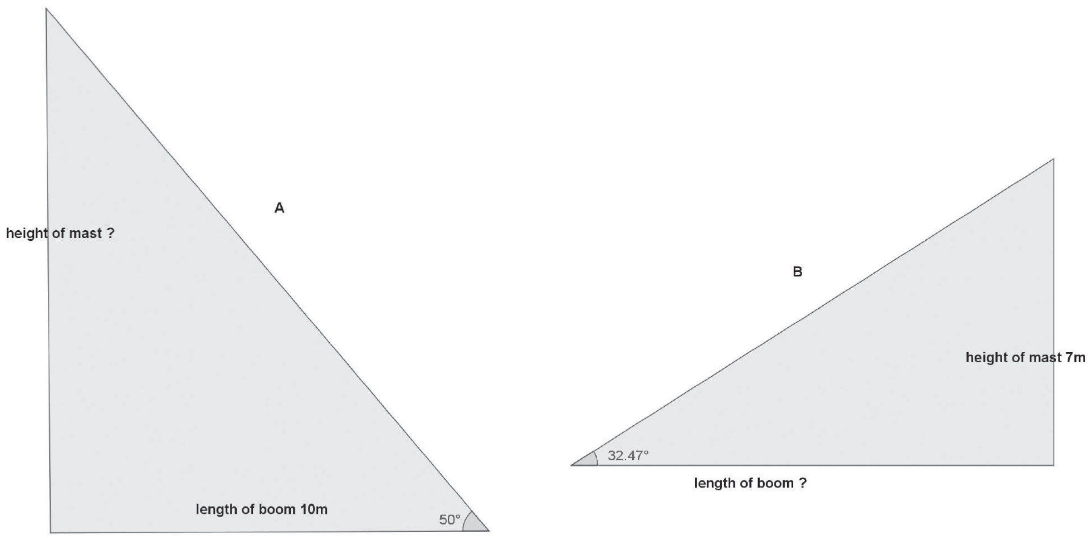
i. from tower Alpha, the phone is 25° south of east

ii. from tower Bravo the phone is 30° north of east.



Q.4 LCHL

The sails on the boats that were in the recent 'round the world yacht race' that visited Galway are where the power for the boats comes from. The bigger they are, the more wind they catch and the faster the boat can go. However, the sail designers are restricted by the length of the boom (bottom) of the sail and the height of the mast. Our designer has two models to work from. The Captain has told him that a height of 9 m is the tallest the mast can be, and a boom of length 10 m is the longest that is permitted.



i. Work out the height of the mast in design A.
ii. Work out the length of the boom in design B.
iii. Work out the area of each triangle to find which one has the bigger area?
iv. There is a problem with design A; explain why.
v. There is a problem with design B; explain why.
vi. Restrict the mast height in design A to what is permitted and recalculate the area.
vii. Restrict the boom size in design B to what is permitted and recalculate the area.

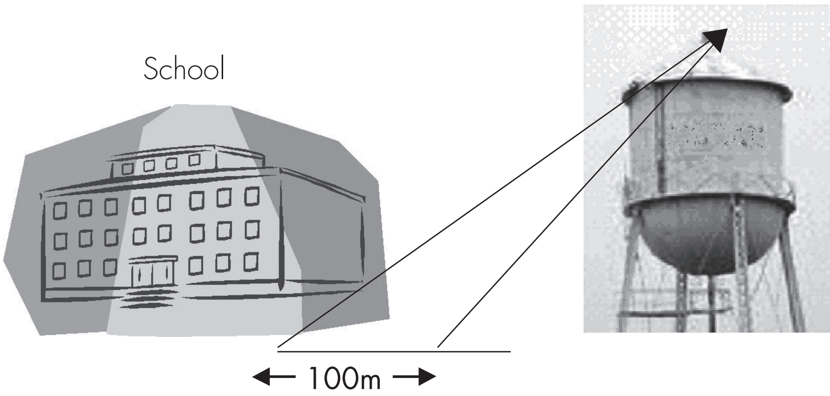
Which design now gives the greater sail area?

Activity 3.2 LCOL and LCHL

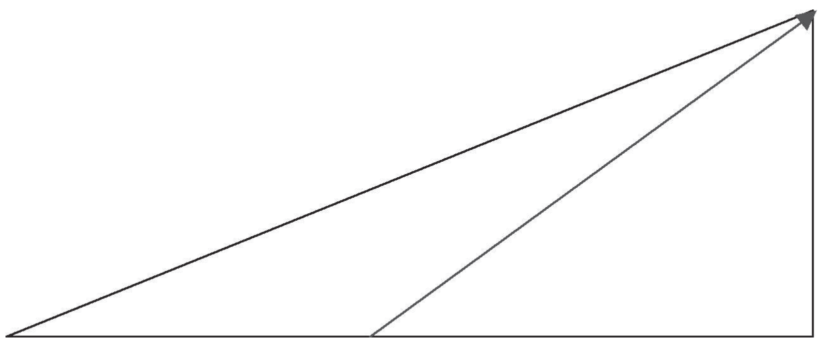
How do you measure the heights of trees without climbing them, or distances between planets? Making measurements can be difficult in space or in environments that are too small or too hostile. Trigonometry helps us to make measurements and use them to find more difficult measurements easily. You may have tried some of these in school.

A compass clinometer is a useful tool to find 'dip and strike' for geologists in the field. They use this information to track underground rock formations that they cannot see but are trying to map. The information is useful for mining companies, farmers, planners and archaeologists. A clinometer is also useful for approximating heights.

Students are given a challenge by their teacher to find the height of the local water tower in the town without leaving the school grounds. The students made two measurements of angles using the clinometer. The first angle, measured at the side of the school was 25° and the second angle, at a spot 100 m away from the school, was 35°.



We can start to solve this problem by drawing a triangle that shows the measurements made by the students (as shown above). This drawing can be extended to include the height of the tower as a vertical to the horizontal ground. We have made a right-angled triangle and that is generally useful when solving triangles.



i. Draw separate diagrams to help you solve the problem in stages.

ii. Write out the rule that you are going to use to solve the problem.

iii. What level of accuracy would be suitable for the answer to this problem?

iv. Work out the height of the tower.

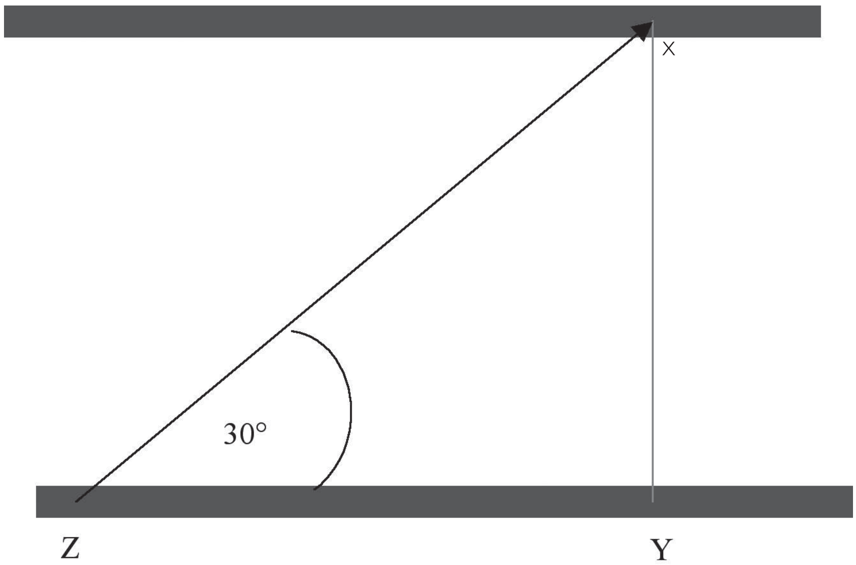
For further consideration

v. Work out how far away the tower is from the position of the first measurement.

vi. If the sun is behind the tower when viewed from the school, and its angle of elevation is 40°, how long is the shadow? Would the shadow reach the school?

Q.1 LCOL and LCHL

Jo wants to swim across the river from X and can go to either Y or Z. She can swim directly across to Y at a rate of 25m per minute or, with the help of the current, she can swim to Z at 35 m per minute. If the distance from X to Z is 200 m, find which route is quicker.



Activity 3.3 LCHL

Graphs of trig functions are used to model situations in real life involving populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals. The best way to see these graphs in action is to draw them or see them drawn. There are a number of websites that are free to use, where these graphs can be drawn dynamically. If you have access to the web, check out the following sites to see graphs in action:

http://www.analyzemath.com/unitcircle/unitcircle.

http://www.intmath.com/Trigonometric-graphs/1_Graphs-sine-cosine-amplitude.php.

N.B. There may be links to other sites and/or pages that may require payment. It is not intended or required that you should pay for any information or display of graphs in action.

Activity 3.4 LCHL

The derivation and rewriting of trigonometric formulae has long been a skill required in trigonometry. The derivations of the following are required learning.

1. $\cos^2 A + \sin^2 A = 1$
2. sine formula: $\dfrac{a}{SinA} = \dfrac{b}{SinB} = \dfrac{c}{SinC}$
3. cosine formula: $a^2 = b^2 + c^2 - 2bcCosA$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
5. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
6. $\cos 2A = \cos^2 A - \sin^2 A$
7. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
8. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Prepare and learn each of these derivations and test yourself by picking a number at random and deriving the proof for that formula as practice.

The formulae are used along with the others from appendix 2 to solve problems. The questions below were asked on past exam papers, and you can test yourself with these.

Q.1 show that $\frac{\sin 2A}{1 + \cos 2A} = \tan A$ 2008

Q.2 show that $(\cos A + \sin A)^2 = 1 + \sin 2A$ 2007

Q.3 using $\cos 2A = \cos^2 A - \sin^2 A$, or otherwise 2005
prove $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

Q.4 Prove that $\cos 2A = \cos^2 A - \sin^2 A$ 2004
Deduce that $\cos 2A = 2\cos^2 A - 1$

GEOMETRY 4

SYLLABUS TOPIC: TRANSFORMATIONS

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- draw scale figures
- solve problems involving lengths, scale factors and centres of enlargement
- solve problems involving area and scale factor

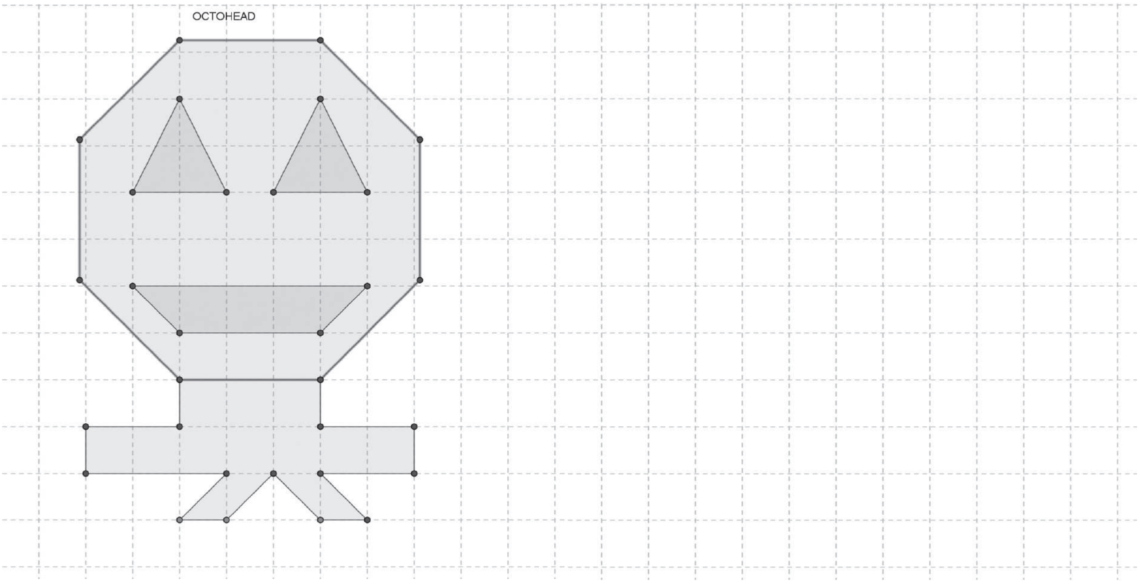
INTRODUCTION

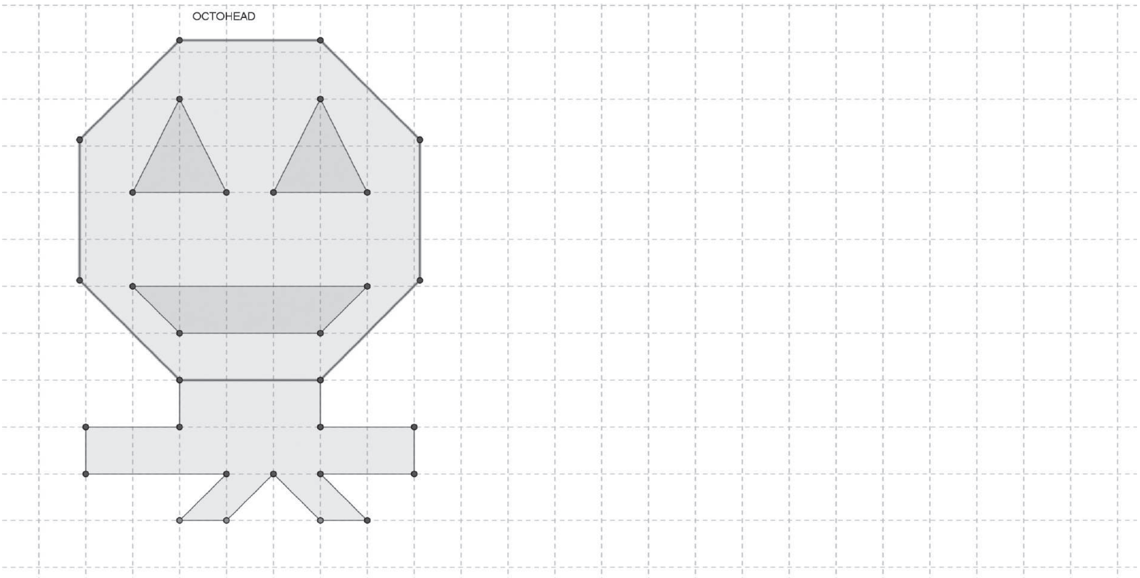
The activities described below and the questions that follow allow you to deepen your understanding of concepts in transformations as well as solve problems using these concepts and their applications. Knowledge of Junior Certificate geometry is assumed.

Activity 4.1 LCFL

For generations, artists, sculptors, architects and cartoonists have used scale diagrams to complete their work. Those of you who studied Technical Graphics may find this work familiar and easy to complete, but it presents an opportunity to understand the maths and geometry that underpin such work.

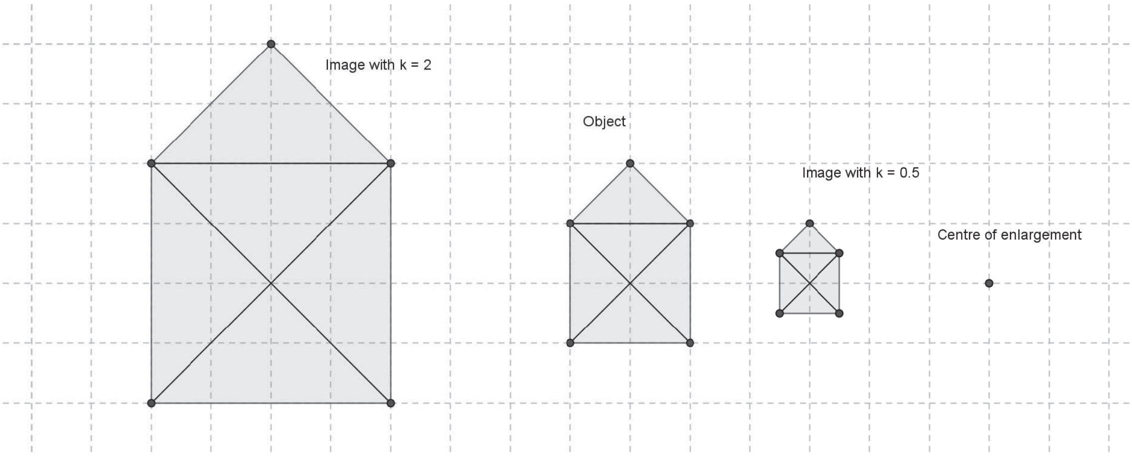
Using the right-hand side of the grid below, copy the image of the character shown at the left-hand side. In the first instance make the image twice as big; in the second case make it half the size.





Activity 4.2 LCFL and LCOL

When making a scale diagram we start with an object and make an image of it that is either smaller or larger, depending on the scale factor (k). Below is a well known diagram that you can use to test your skill. Can you complete the diagram without lifting the pencil from the page or going over the same line twice? In the diagram, two images are shown, one which is twice as big (k=2) and the other which is half the original size (k= 0.5). This is easy to do if you use graph paper, a pencil and a ruler. The use of dynamic geometry software can also give you accurate images very quickly, but doing the drawing manually is the best way to start.

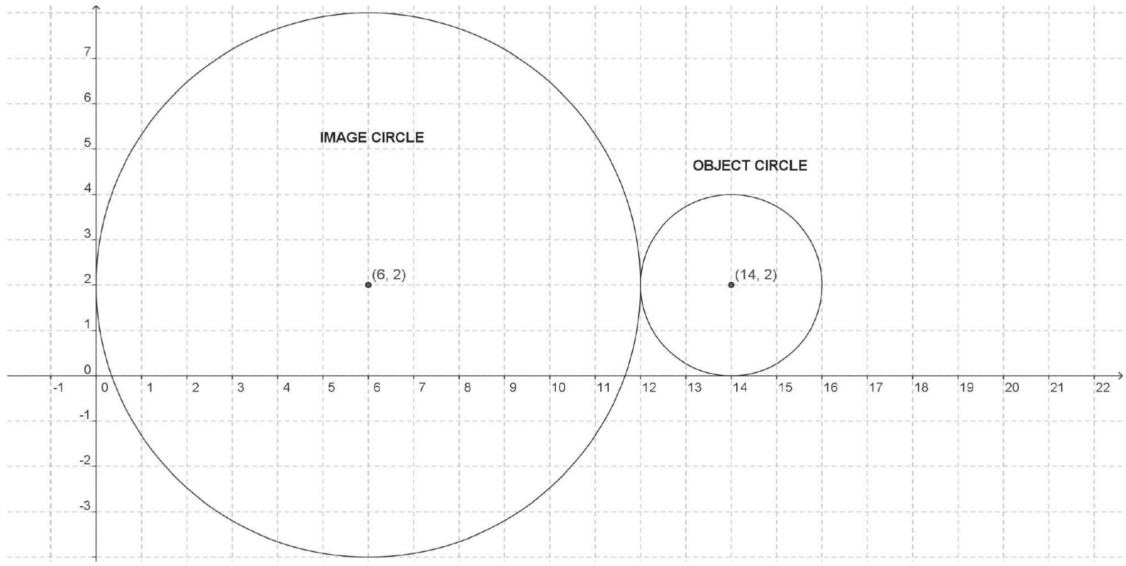


i. Draw in the rays that connect the marked points back to the centre of enlargement on your diagram.

ii. Make measurements on the corresponding sides of the objects and the images. What do you notice?

iii. Measure the distance from any point on the larger image to the centre of enlargement. Compare this distance to the corresponding measurement for the smaller image. What do you notice?

Q1 Try to find the centre of enlargement in the diagram below if the scale factor is k = 3.



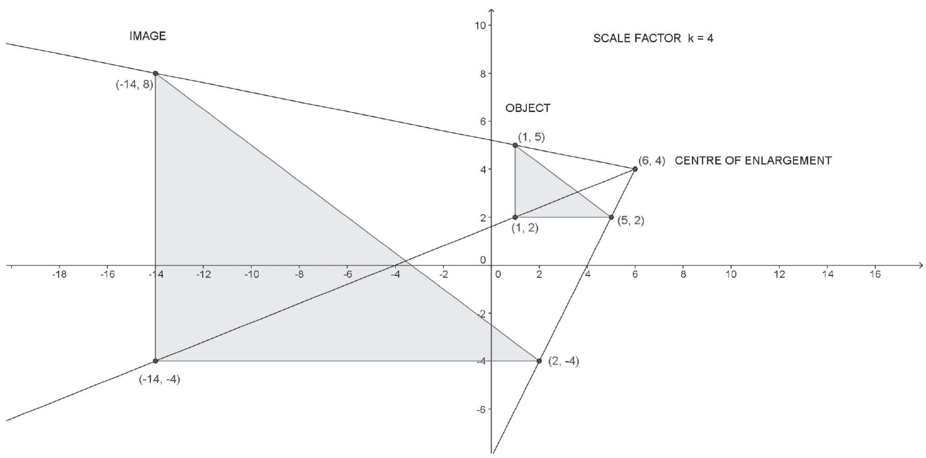
What is the most important point of reference in a circle when finding its centre of enlargement? Do co-ordinates make it easier to find the centre of enlargement?

How would you find the centre of enlargement without the co-ordinates?

Activity 4.3 LCOL

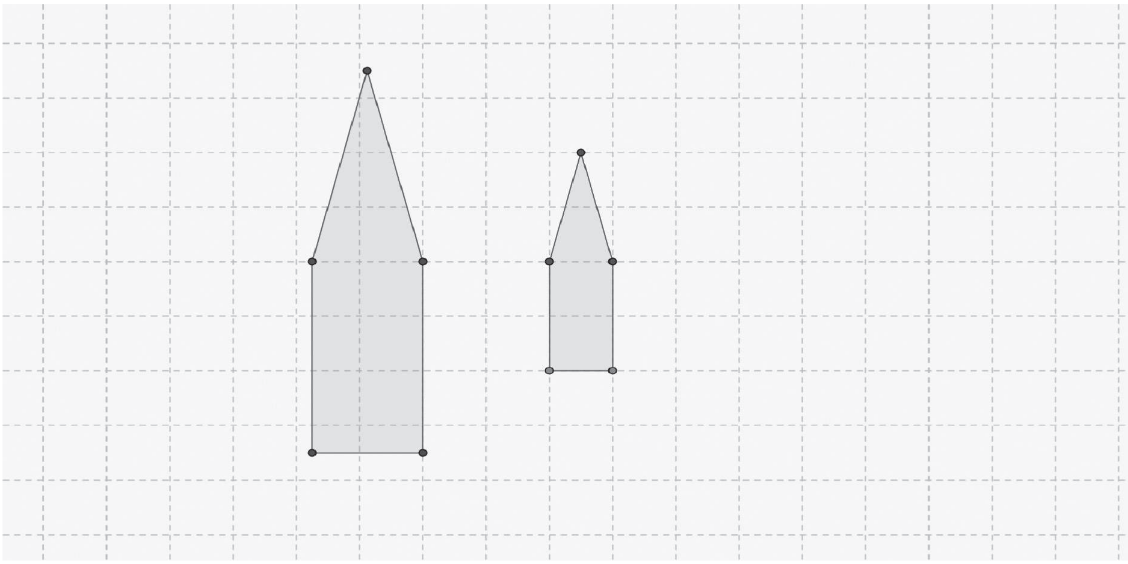
The area of an image is closely related to the scale factor and the object.
Examine the triangles below. The scale factor is 4.

i. Calculate the area of the object triangle.
ii. Calculate the area of the image triangle.
iii. Divide the area of the image by the area of the object.
iv. Compare your answer to the scale factor. What is the relationship between them?



Activity 4.4 LCOL and LCHL

Mark the centre of enlargement on the diagram which shows the object building enlarged by a factor of $k = 1\frac{3}{4}$.



Q.1 LCOL

If the length of the side of an object is 10 cm before it is enlarged and 15 cm afterwards, what is the scale factor (k) by which it has been enlarged?

Q.2 LCOL

In a design illustration, a rectangular housing unit measures 2 cm × 3 cm × 4 cm. The dimensions of the real housing unit will need to be bigger by a factor of 3.5. Calculate the volume of the unit in the design illustration and then calculate the volume of the real unit. What is the ratio of two volumes? How is this related to the scale factor?

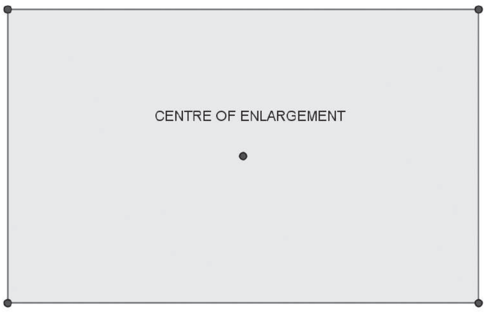
Q.3 LCOL

A toy manufacturer has a prototype toy boat with a mast that is 40 cm high. The mast is too tall to fit in the boxes already made. By what factor (k) must the mast be reduced if it can be only 30 cm tall?

Q.4 LCOL and LCHL

A TV company wants to replace their current plasma screen with a newer, bigger one. The old one is drawn below, with the centre of enlargement shown. They want to increase the dimensions of the screen by 20%, so the scale factor is 1.2. Draw the image of the new screen on the diagram below.

Plasmatron electronics – new developments



i. If the original screen measures 40 cm × 25 cm, find the area of the screen.

ii. Find the area of the new, enlarged screen.

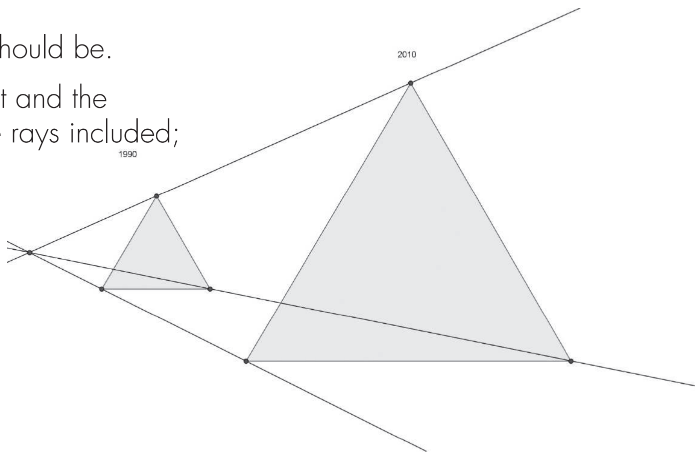
iii. Divide the new area by the old area, and then find the square root of the answer. Correct your answer to one decimal place. What have you discovered?

Q.5 LCOL and LCHL

Triadtechnics have been using the same logo for 20 years, but then decided to change the corporate logo to show that they have grown a lot in size in the 20 years. Their logo is an equilateral triangle and the company has grown by 200% in the 20 years. The new image of the triangle is to illustrate this growth.

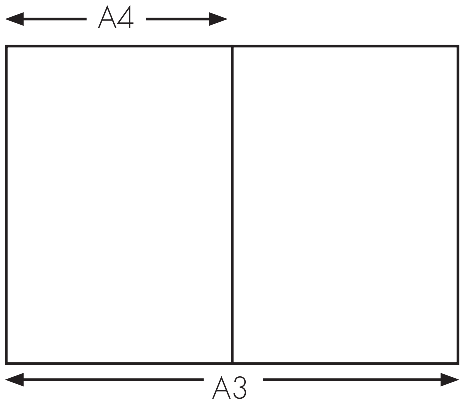
i. Work out what the scale factor should be.

ii. On graph paper draw the object and the image for the new logo, with the rays included; a sample diagram is given



Q. 6 LCHL

An A3 sheet of paper is as large as two A4 sheets placed side-by-side, as shown below.



If I have a map that is A3 in size and want to reduce it so that it fits on an A4 sheet, the photocopier selects 70% as the scale factor, not 50%. Based on what you have learned already in this section, explain why the scale factor should be (approximately) 70% .

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Mathematics Resources for Students

Junior Certificate – Strand 2

Geometry and Trigonometry

INTRODUCTION

This booklet is designed to supplement the work you have done in Junior Cert geometry with your teacher. There are activities included for use as homework or in school. The activities will help you to understand more about the concepts you are learning in geometry. Some of the activities have spaces for you to fill in, while others will require you to use drawing instruments and paper of your own. You may not need or be able to complete all activities; your teacher will direct you to activities and/or questions that are suitable.

You should note that Ordinary level material is a subset of Higher level and that HL students can expect to be tested on material from the Ordinary level course, but at a greater degree of difficulty. The sequence in which the sections/topics are presented here is not significant. You may be studying these in a different order, or dipping in and out of various sections over the course of your study and/or revision.

In the first topic (synthetic geometry) it is important that you understand the approach taken. Although only HL students are required to present the proof of some theorems, all students are expected to follow the logic and deduction used in these theorems. This type of understanding is required when solving problems such as those given in the section headed 'other questions to consider'.

Each activity or question you complete should be kept in a folder for reference and revision at a later date. A modified version of this booklet and syllabus documents are available at www.ncca.ie/projectmaths and other related materials at www.projectmaths.ie

GEOMETRY 1

SYLLABUS TOPIC: SYNTHETIC GEOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- complete a number of constructions
- use the following terms related to logic and deductive reasoning: theorem, proof, axiom, corollary, converse, implies
- investigate theorems and solve problems.

HL learners will

- extend their understanding of geometry through the use of formal proof for certain theorems.

INTRODUCTION

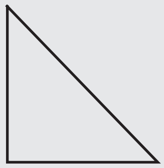
The activities described below and the questions that follow allow you to deepen your understanding of concepts in geometry as well as solve problems using these concepts and their applications.

Activity 1.1 JCFL

The following activity is to help you understand the properties of different triangles. Revise the ideas of acute angle, obtuse angle and right angle. You may want to discuss with someone else the difference between 'could be true' and 'could never be true' before you start.

Read each statement about triangles.

Decide if the statement could ever be true, and tick the correct column in the table. Then draw a diagram of the triangle in the box provided. The first one has been done for you.

Statement	Could be true	Could never be true	Diagram
Triangles can have one right angle and two acute angles	✓		
Triangles can have two right angles.			
Triangles can have one obtuse angle and two acute angles			
Triangles can have two obtuse angles and one acute angle			

Activity 1.2 OL and HL

Geometry has a language all of its own. You are not required to learn all of the vocabulary associated with it but you do need an understanding of the different terms. Fill in the spaces in the activity items below to assess your understanding of what the terms mean. Your teacher will guide you through the differences and the uses of terms in geometry.

What do you understand by the word **line**? Write your answer in one sentence.

What do you understand by the word **triangle**? Write your answer in one sentence.

What do you understand by the word **angle**? Write your answer in one sentence.

What do you understand by the word **definition**? Write your answer in one sentence.

What do you understand by the word **theorem**? Write your answer in one sentence.

What do you understand by the word **axiom**? Write your answer in one sentence.

What do you understand by the word **corollary**? Write your answer in one sentence.

What do you understand by the phrase **geometrical proof**? Write your answer in one sentence.

What do you understand by the word **converse**? Write your answer in one sentence.

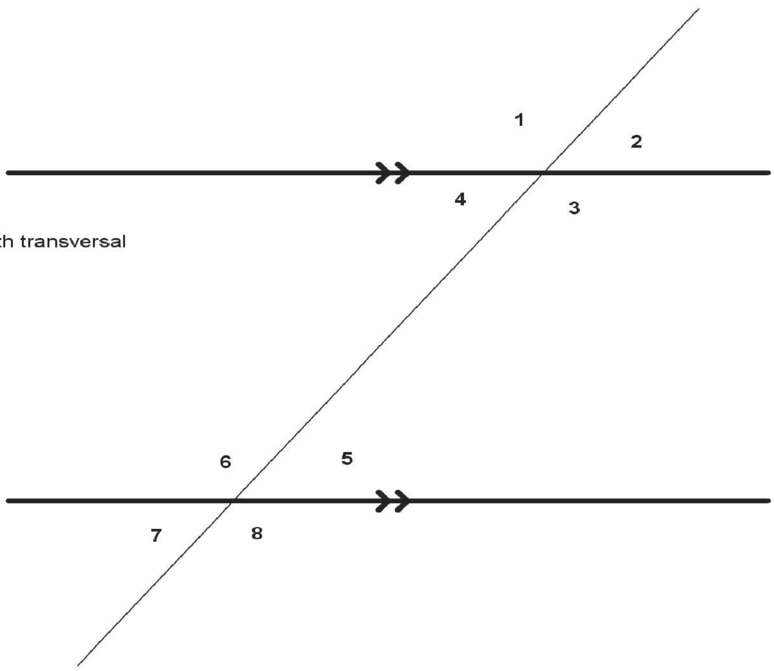
Make out a list of any terms in geometry where you are not sure of their meaning.

1.
2.
3.
4.
5.
6.

Activity 1.3 OL and HL

In geometry we often have to find angles or the lengths of shapes and we can use known 'facts' to establish these or to prove that particular geometrical statements are true. The use of deductive reasoning is important and our ability to piece the clues together makes it easier to do this. This reasoning comes from our ability to build on what we know to be true in order to discover something new.

When examining a pair of parallel lines with a transversal cutting across them, we can see lots of different angles which are equal and we can classify these in different ways.



Parallel lines with transversal

Using the numbers shown on the diagram, list pairs of angles which are equal in measure, using one of the following terms to justify each pair: vertically opposite angles, alternate angles, corresponding angles, supplementary angles.

Use the notation | ∠7 | to mean the number of degrees in angle 7.

a. ____ and ____ are equal because they are ____________

b. ____ and ____ are equal because they are ____________

c. ____ and ____ are equal because they are ____________

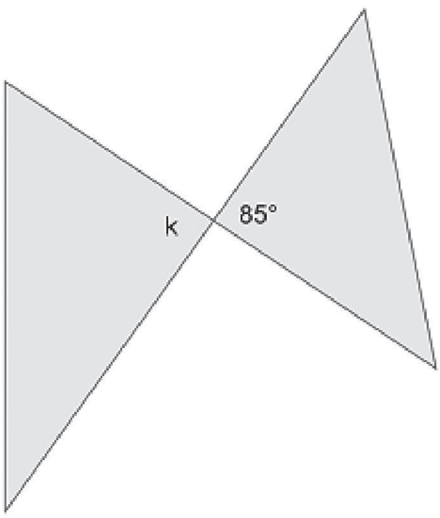
d. ____ and ____ are equal because they are ____________

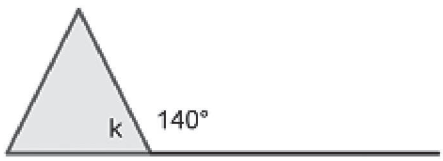
e. ____ and ____ are equal because they are ____________

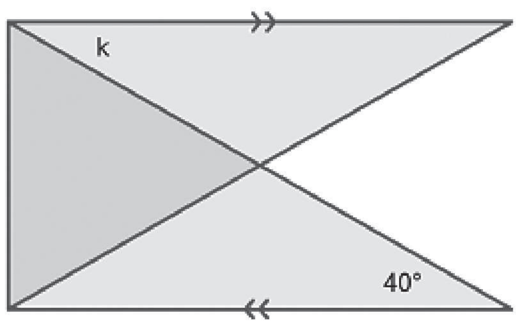
f. ____ and ____ are equal because they are ____________

Are there more than six pairs?

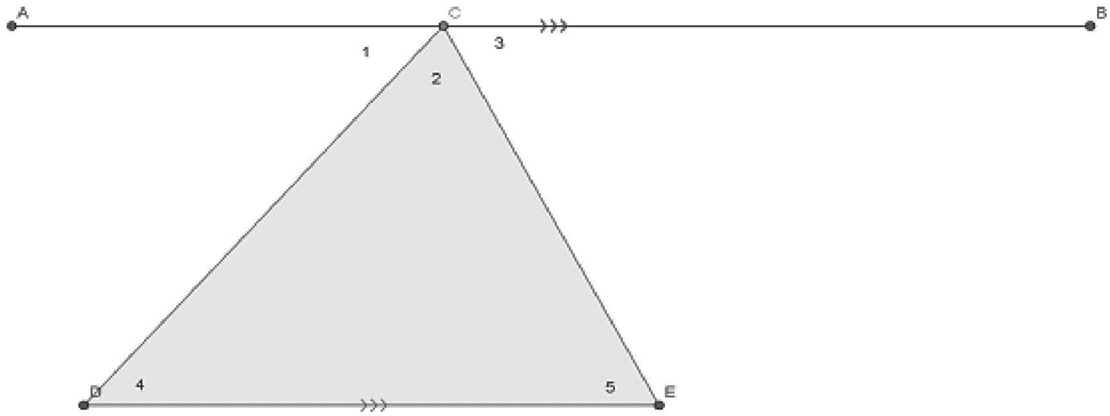
Q. 1 Based on what you have learned above, can you now find the value of the angle k in each of the following diagrams?







Q. 2 Now consider the following diagram and answer the questions below it.



a. What is the sum of the angles in a straight line?

b. Name three angles in the diagram above that make a straight line.

c. Name two pairs of alternate angles.

For HL

d. Can you draw any conclusions from the sum of the angles in the triangle in the diagram?

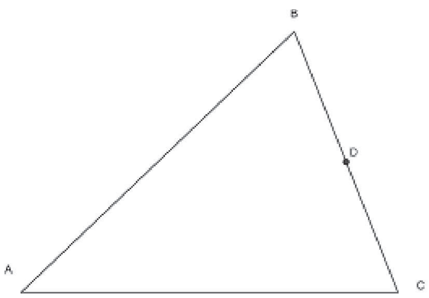
e. If the point C was in a different place on the line segment [AB], would it make any difference?

From this example, what, if anything, can you say generally about all triangles?
Could you show that this is true?

Other questions to consider

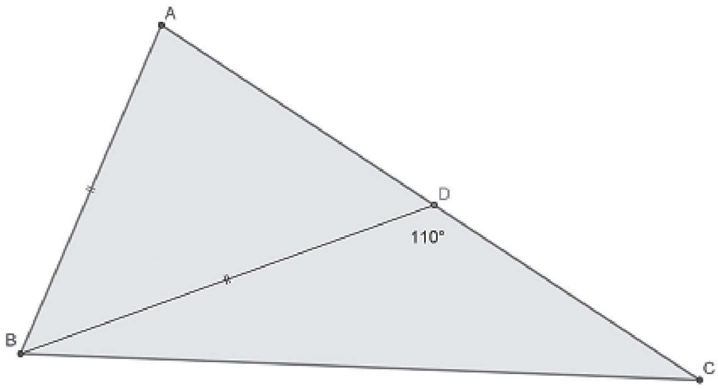
Q. 3 OL and HL

In the diagram |AB|= |AC|, |BD|= |DC|. Show that D is equidistant from AB and AC.



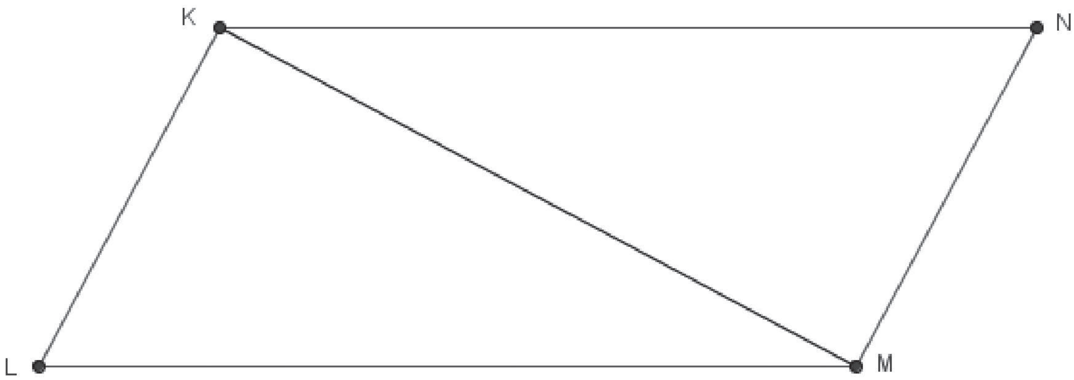
Q. 4 OL and HL

If |BA |= |BD| and |DB |=|DC |, Find the value of | $\angle$ ABC|.|



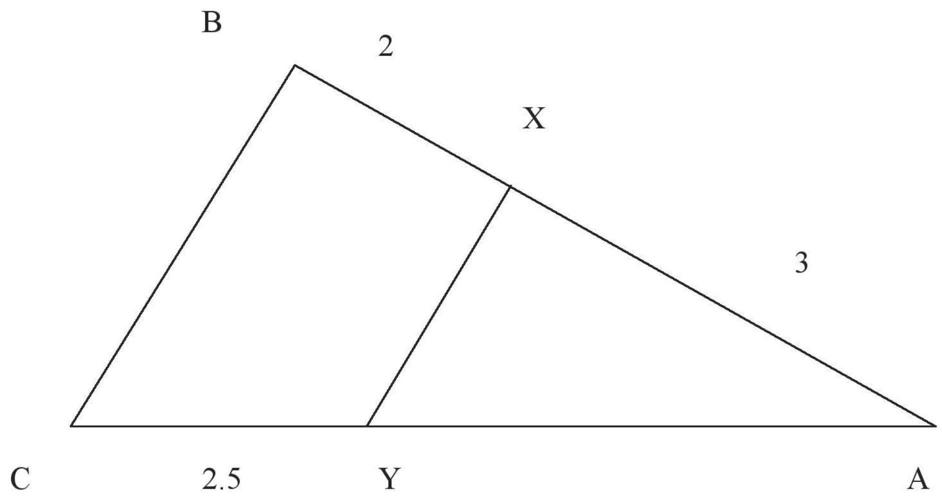
Q. 5 OL and HL

In the diagram KLMN is a parallelogram and KM is perpendicular to MN. If |KM|= 7.5 cm and |LM|= 8.5 cm, find the area of the parallelogram.



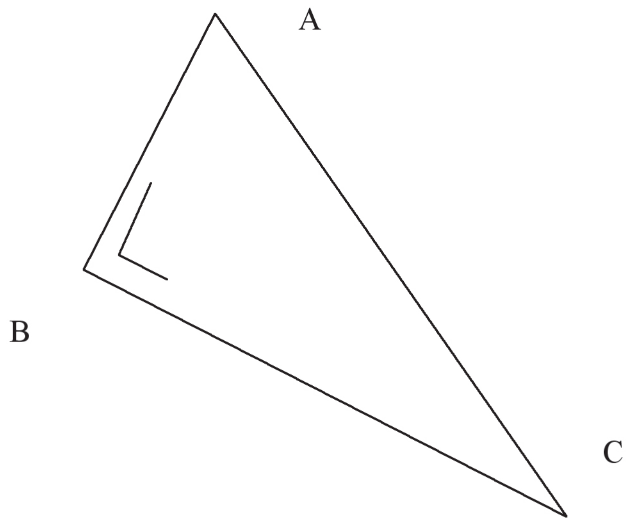
Q. 6 HL

In Δ ABC, XY || BC. Find |AY|



Q. 7 OL and HL

In ΔABC, |∠ ABC|= 90°, |AB|= 7 − a number, and |AC|= 8 + the same number. Find |BC|^2.

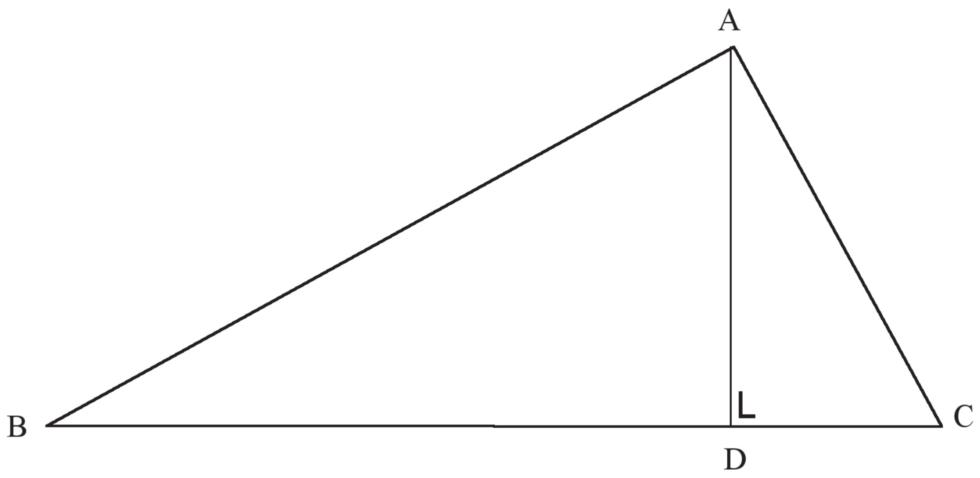


Q. 8 OL and HL

In the diagram |∠BAC|= 90°= |∠ADC |

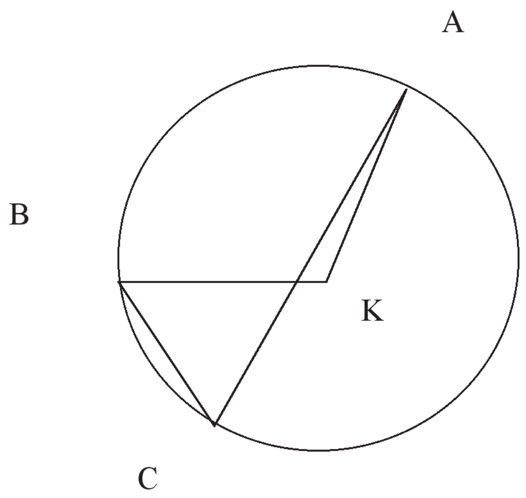
Show that ΔABD and ΔABC are equiangular and that ΔADC and ΔABC are equiangular.

From this, can you show that $|AB|^2 + |AC|^2 = |BC|^2$



Q. 9 HL

The centre of the circle is K. |∠AKB|= 100°. Find |∠ACB|.



Activity 1.4

You are required to carry out a number of constructions and the best way to learn is by doing. If you are studying Technical Graphics, ask your teacher whether there are different ways of doing a given construction that might be easier for you to remember. Trying to learn them off for a test is more difficult than learning them by completing exercises like the ones that follow.

Q. 1 Divide the shape below into four equal parts using only a compass, ruler and pencil.

Q. 2 The following six constructions involve drawing triangles. Try to construct them, but note that not all of them are possible. If it is not possible to construct the triangle, briefly explain why. Also, note if more than one solution is possible.

i. A triangle with sides of length 3 cm, 6 cm and 12 cm.

ii. A triangle with sides of 10 cm each. What kind of a triangle is this? Using this triangle, can you find a way of making two triangles which have a right angle, a side of 10 cm and a side of 5 cm? What do you notice about these two triangles?

iii. A triangle with one side of 4 cm and two angles of 50° each.

iv. A triangle with angles of 55°, 65°and 65°.

v. A right-angled triangle which has two sides the same length. Label the triangle and measure the angles with a protractor. Record their values on the diagram.

vi. A right angled triangle with one side twice as long as the other. Label the triangle and measure the angles with a protractor. Measure the third side as accurately as you can. Record all the measurements on the diagram.

GEOMETRY 2

SYLLABUS TOPIC: TRANSFORMATION GEOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- locate axes and centres of symmetry
- recognise images, points and objects under transformations

INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in transformation geometry as well as solve problems using these concepts and their applications.

Translation in geometry is about movement of a point or object. The point or object can change its position or an object can change the direction in which it is facing. When we have located the object in a new position we call it the image, because it is like the original.

When we describe movement in the plane we can say

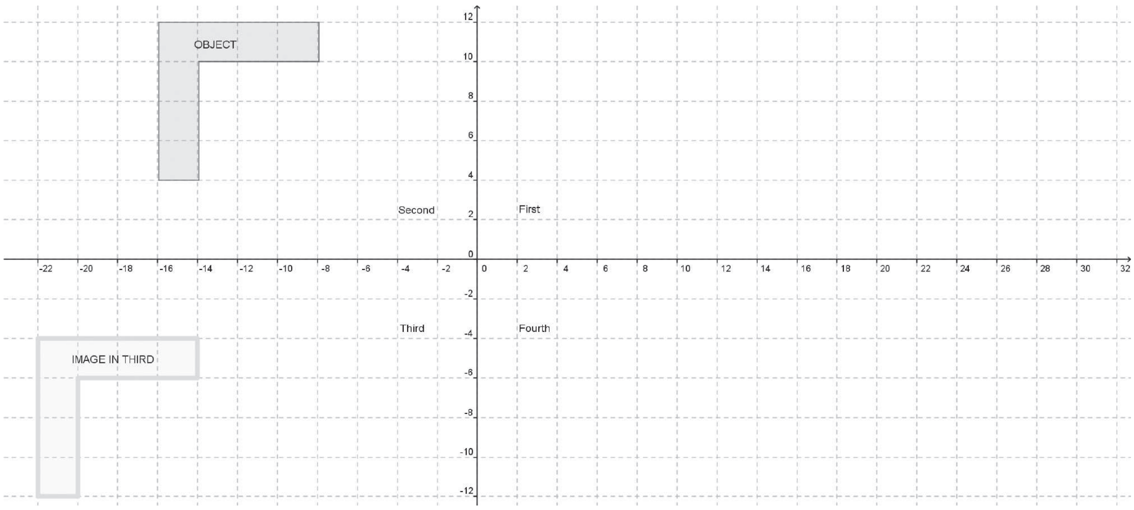
to the left or right, up or down;

to the north, south, east or west;

or, later, we can use co-ordinates to describe the location of the image of an object or point.

Activity 2.1

The object shown in the second quadrant of the plane has been translated into the third quadrant. Your task is to translate it, into the first quadrant and then into the fourth quadrant. After doing so, consider the questions that follow.



i. Does it matter where the image is in each quadrant?

ii. Have the lengths of the sides of the object changed?

iii. Has the area of the shape changed?

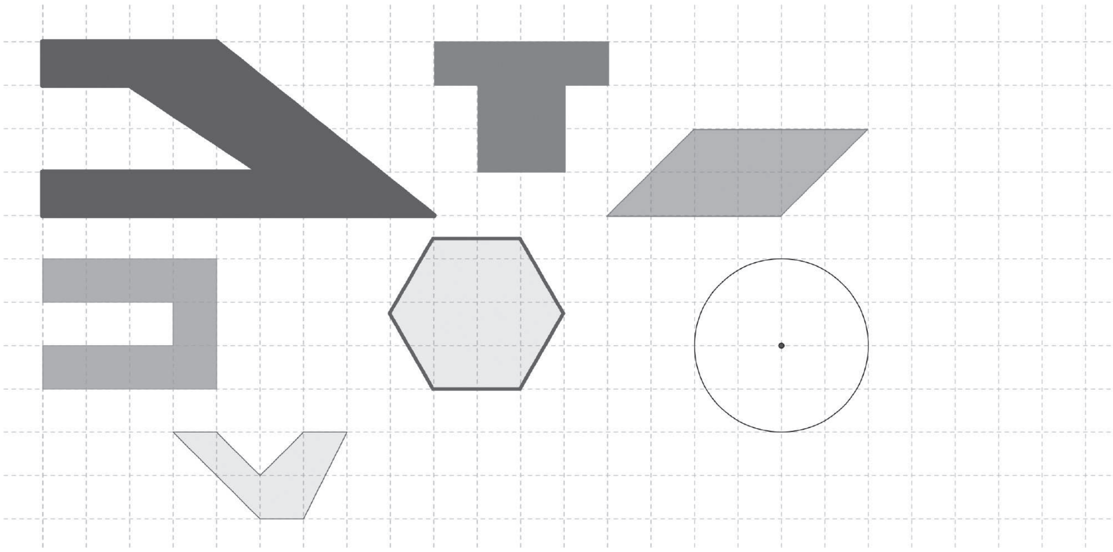
iv. Is there any difference between the image and the object?

v. Comment on the positions of the images?

vi. What can you conclude about the operation of translation on an object?

Activity 2.2

If we can fold one side of a shape exactly onto another it has a line of symmetry. Look at each of the shapes below and draw in the line of symmetry, if it has one. Some shapes may have more than one line of symmetry.



Which of the above are regular polygons? Can you draw any conclusions about the axes of symmetry of these?

Symmetry can occur in the natural world around us. Reflections of ourselves in the mirror or reflections of the sky and landscape in the water on a calm day are usually one of the first ways we experience symmetry.

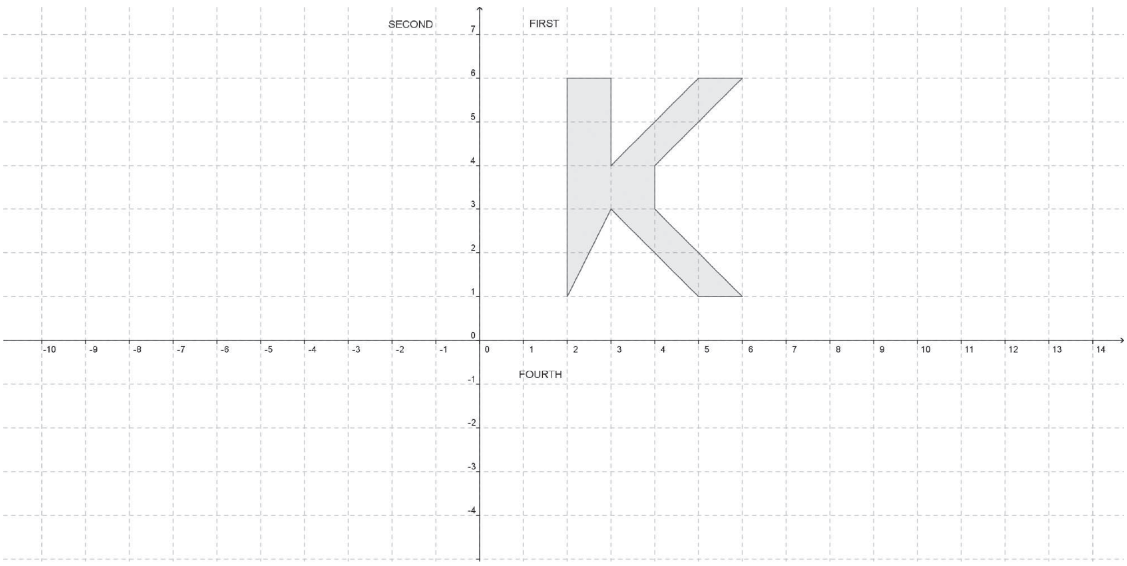
There are some things about symmetry that we should be aware of from looking at our own reflections in the mirror. The closer we stand to the mirror the closer the image appears to be. As you raise your right hand and wave, the image raises its left hand and waves. If you step to the left away from the mirror, the image steps away to its right. Try this yourself in front of a mirror and, if you study science, you can find out more about light, mirrors and images.

There are creatures in the insect world, such as butterflies and beetles, which have symmetrical bodies. You can see some of these at http://www.misterteacher.com/symmetry.html. Snakes may have symmetrical patterns on their skins and flowers can also show symmetry. Some sea creatures such as starfish and ammonites are also well known for their symmetrical and spiral shells.

Axial symmetry

Axial symmetry, or reflection in a line, is another type of transformation. Imagine the axis of symmetry as a line along which you can fold the plane.

If you print off a copy of this diagram you can carry out the activity by folding the sheet of paper along the axes to see where the images will appear. The object is in the first quadrant; find the images of the object in the second and fourth quadrants by reflection in the Y and X axes respectively.



What do you notice about the pointed part of the K as it is reflected into different quadrants?

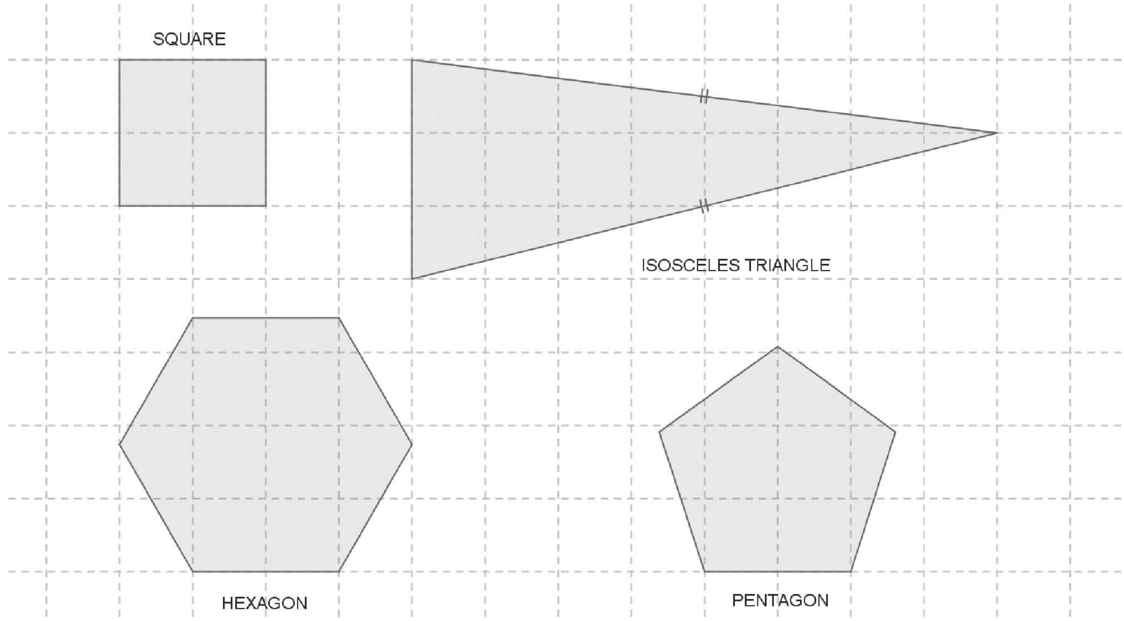
Is the image of K always facing in the same direction?

Describe the image in each quadrant using one short sentence. Focus on what is different from the original object.

Is it possible to find the image of the K in the third quadrant using axial symmetry?

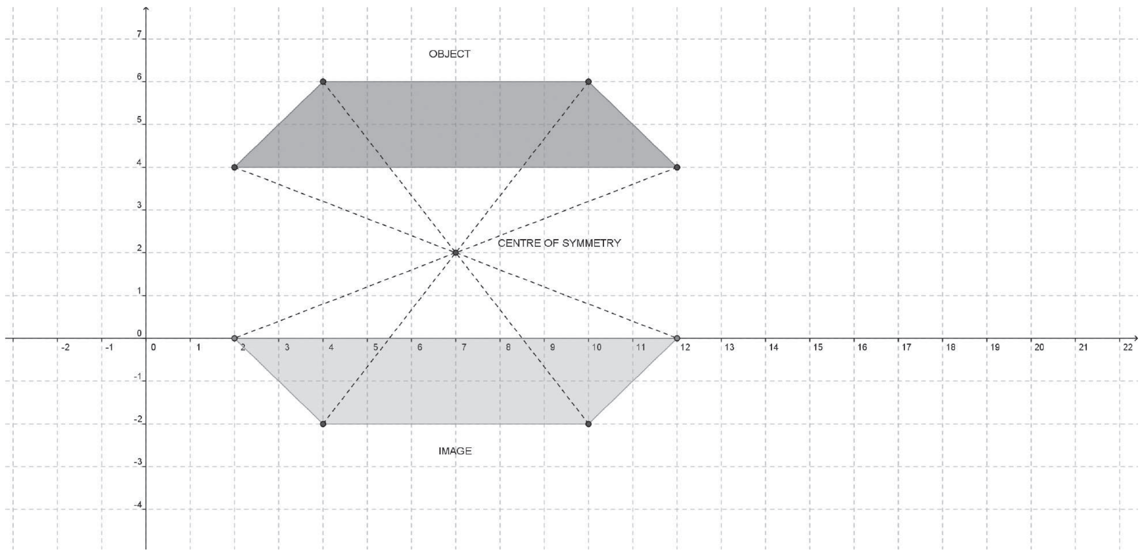
Activity 2.3

Central symmetry or reflection in a point is another type of transformation. Find the centre of symmetry of the following objects, if they have one.

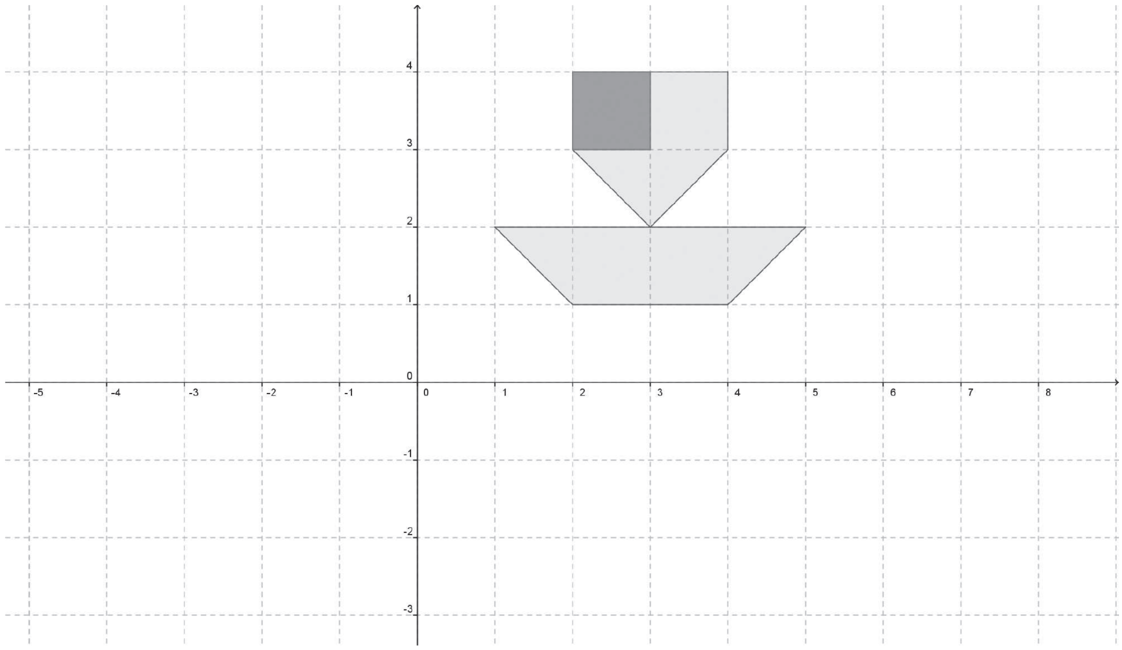


What can you say about the image and the object when the centre of symmetry is inside the shape?

When reflection takes place through a point outside of the object, as seen from the diagram below, we get a new image and we say that the image has been formed by central symmetry in the point (or by reflection in the point). Note that the point is the centre of symmetry for the combined shape (the object and its image).



Q. 1 Draw the image of the shape in the diagram by central symmetry through the origin (0,0) and then answer the questions that follow.



i. What happened to the object when it was reflected in the origin (0,0)? Explain this in one sentence.

ii. How would you describe the orientation of the image after reflection?

iii. If you reflected the image back through the origin, would you get the original shape?

Central symmetry can be done through any point on the plane. Pick a point which will reflect the object above

i. into the fourth quadrant

ii. into the second quadrant.

Briefly describe the image in each case.

GEOMETRY 3

SYLLABUS TOPIC: CO-ORDINATE GEOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- read and plot points
- find midpoint, distance, slope, and points of intersection of lines with the axes

At Higher level

- find the point of intersection of two lines
- find the slopes of parallel and perpendicular lines

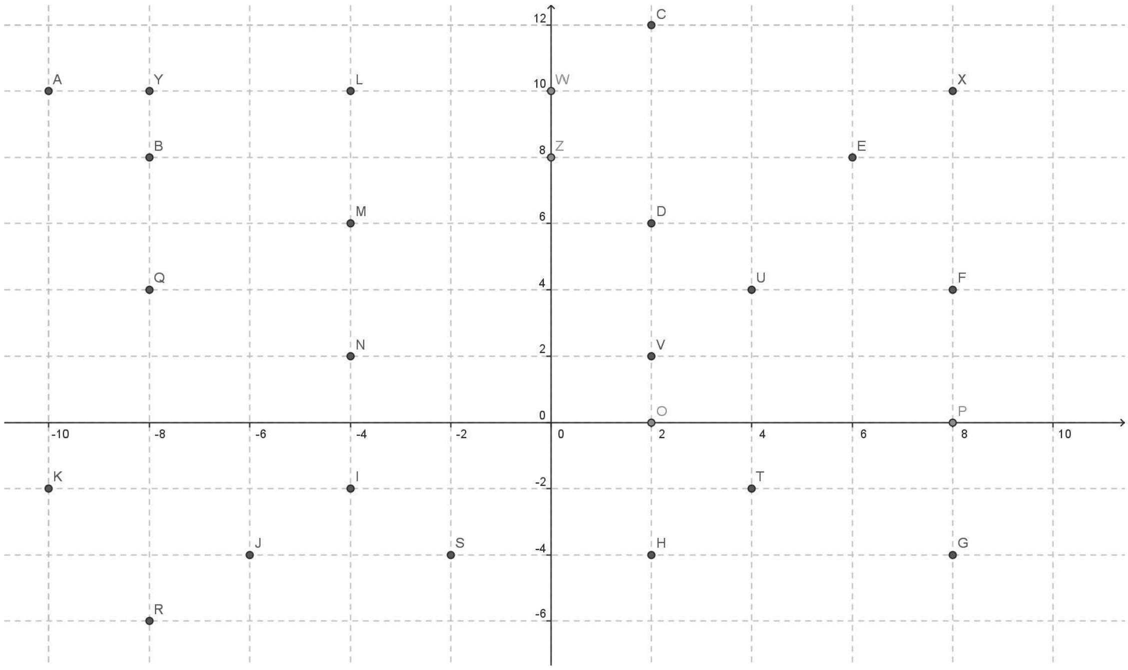
INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in co-ordinate geometry as well as solve problems using these concepts and their applications.

The plotting of points has allowed us to find locations on maps for generations. A global positioning system (GPS) has replaced 6 figure grid references for specifying locations these days. Our mobile phones have this function and many cars have in-built or mobile 'sat-nav' technology to help us navigate our way around big cities or to unfamiliar places. When we get there we can store the location as a 'way point' and find it again easily next time.

Activity 3.1

If you and your friends in school have a grid with each letter of the alphabet on it you can write secret messages to each other. You need to have the grid to compose and to read the messages. Use the grid below to answer the questions that follow.



i. What is your first name?

ii. Who is your favourite band or singer?

iii. What is your favourite TV programme

Here are my answers to those questions; can your figure them out?

i. (4,-2), (6,8), (-8, -6), (-8, -6), (-4,-2)

ii. (2,12), (2,0), (-4,10), (2,6)   (8,0), (-4,10), (-10,10), (-8,10)

iii. (8,-4), (-8,-6), (6,8), (-8,10), (-2,-4)   (-10,10),(-4,2), (-10,10), (4,-2), (2,0) (-4,6), (-8,10)

Q. 1 Make up five questions that could be answered in codes by other students.

Q. 2 Write a message of one sentence in code that you can give to another student to work out. The sentence should be written with pairs of co-ordinates in brackets for each letter and spaces between each of the words.

Extension activity

It is easy for anyone to read a message using the above grid.
Make your own grid with the letters in different places. Consider the following.

1. Can you confuse someone trying to break the code by having the same letter in two locations on your grid?
2. It is possible to include numbers or words in your grid?
3. Could you put the 'txt words' that you use on your mobile onto a grid?

For further exploration

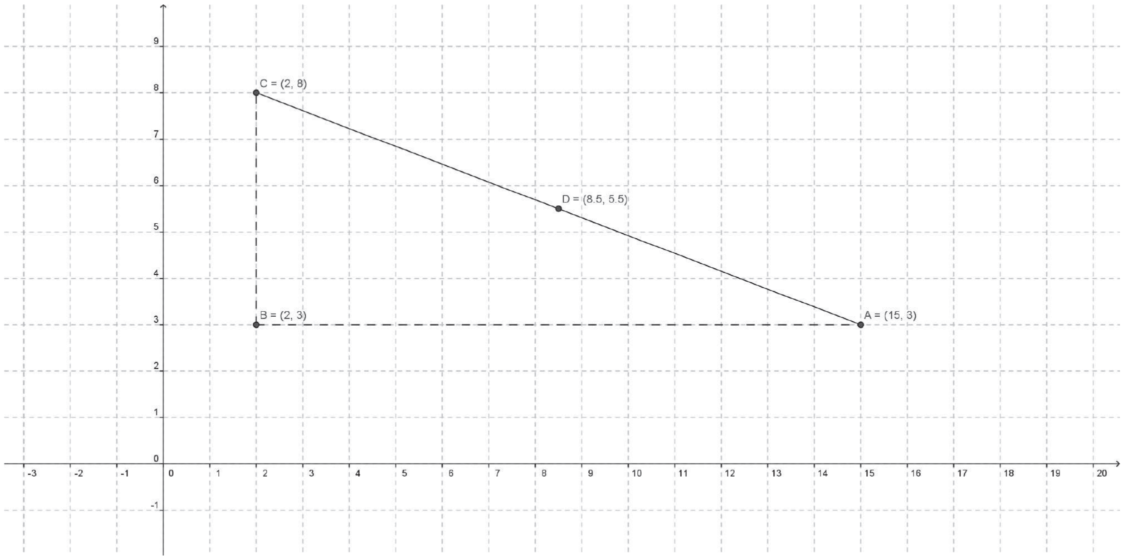
Rene Descartes was the mathematician famed for first devising the co-ordinated plane. Find out more about him and his life in your school library or on the internet if you have access (check out www.projectmaths.ie for information).

There are some famous codes that were used in the past to communicate or to encrypt messages. ENIGMA was one of these, and a movie was made about this code.

If you have access to the internet, search for information about Public Key Encryption (PKE).

Activity 3.2

You are familiar with finding the average or mean of a set of numbers. We add them and divide by the number of numbers. Using this idea, we can find the midpoint of a line segment in co-ordinate geometry.



In the example above we have $\frac{2+15}{2}=8.5$, and $\frac{8+3}{2}=5.5$

This gives us the result (8.5, 5.5). This seems to correspond with the midpoint of the line segment in our example.

Q. 1 Find the midpoint of the line segments formed by the following pairs of points; remember to be careful with the minus signs.

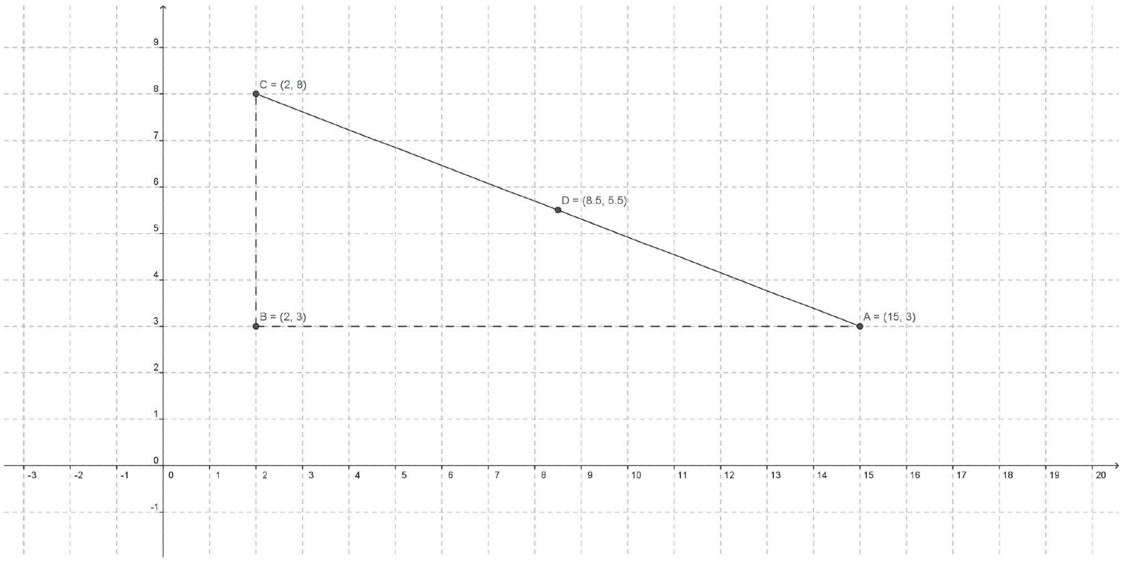
[E(1,1) and F(7,7)], [G(1,2) and H(3,6)], [J(4,7) and K(11,16)],
[P(0,4) and Q(−2,2)], [R(2,1) and S(4,3)], [T(2, −3) and U(− 2,5)]

The midpoint should be the average of the two end points. So, add the x co-ordinates and divide by two. Then repeat for the y co-ordinates. This can be summarised by the formula for the midpoint that you may be familiar with:

$$\text{midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Activity 3.3

The distance between two points is a length. We can physically measure a length with a ruler or tape measure, but it is not always possible to do this. So, we need to find a way of measuring in geometry without using instruments. The engineers who built the pyramids at Giza in Egypt figured out a way of finding lengths so they would not make errors; they based it on right angles. What became called the theorem of Pythagoras was known to the Egyptians long before Pythagoras was born. We can see how it helps us to devise a method for finding the length of a line segment, using the diagram from Activity 2 above.



There is a right angle at B (2,3). The length of the line segment [AB] can be found by subtracting the x values (15-2 = 13). The length of the line segment [BC] can be found by subtracting the y values (8-3 = 5). Since the angle at B is a right angle, the line segment [CA] is the hypotenuse. So, using what we know about the theorem of Pythagoras, we can say $13^2 + 8^2 = [CA]^2$.

i. Find the value of [CA] to the nearest whole number?
ii. Find the value of [CA] correct to the first decimal place?
iii. Find the value of [CA] correct to two decimal places?
iv. Using the co-ordinates P(-1,8), Q(-1,4) and R(2,4), draw another right angled triangle on the grid above. Find the length of the hypotenuse using the same method as before.

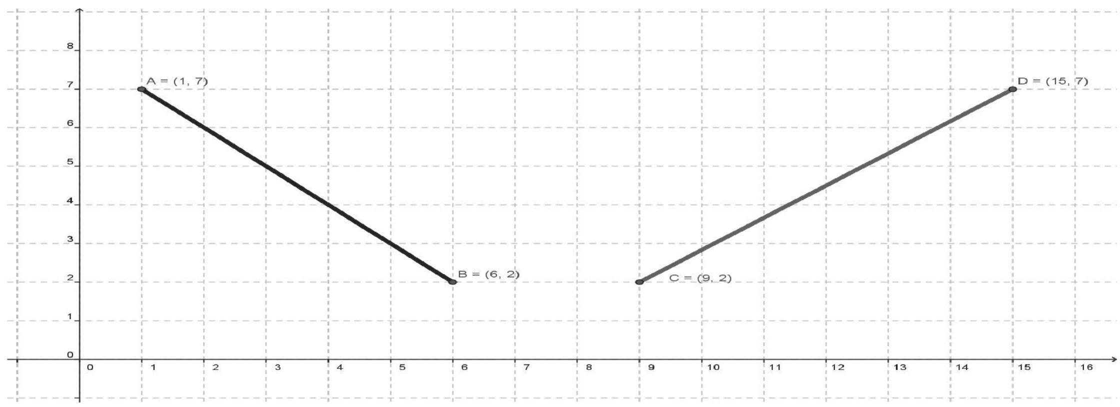
If this method works for these two triangles, will it work for all right angled triangles when we have the coordinates of the three vertices?
If it does work for all right angled triangles, can we work out a general rule or formula to calculate the length of the hypotenuse?

Activity 3.4

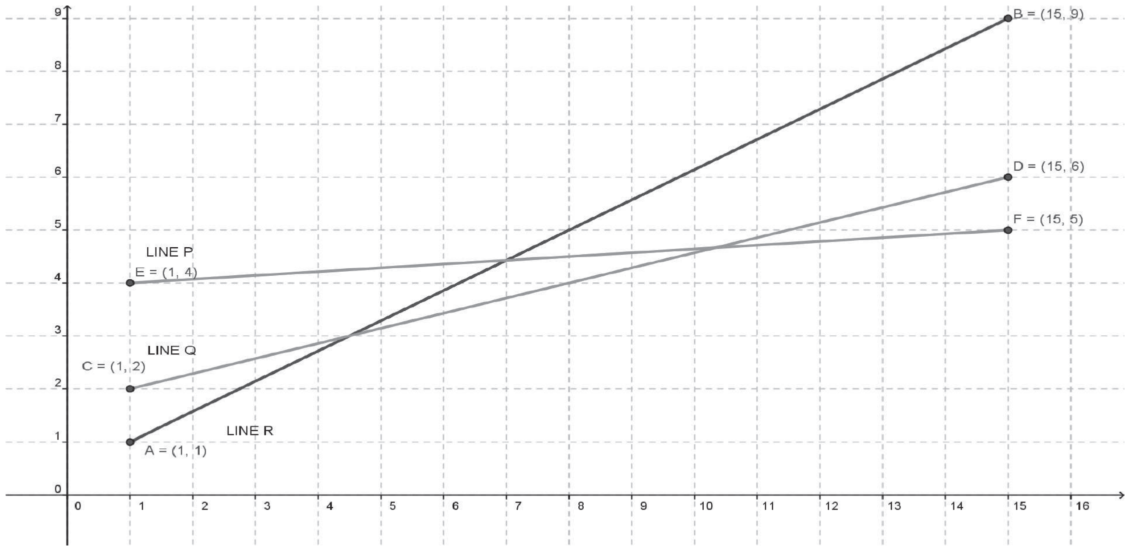
The slope or gradient of a line is the amount by which it goes up or down. In geography, where you have to do a cross-section on an ordinance survey map, you are making a profile of the slopes on the map – its topography. The slope of a hill tells us a lot about how suitable it can be for a road or for climbing. The slope of a road is often shown as a percentage or ratio on a road sign.

So, how do we explain how steep or shallow a slope is? We measure how much it goes up or down as we travel along it from left to right.

One of the following line segments has a positive slope and one has a negative slope. Write underneath which one is which and write a short explanation for your answer.



The slopes of the lines P, Q and R below are of one type. Are they positive or negative? How would you explain the difference between the slopes of these three lines?



Let's look at how the slope of line R changes between points A(1,1) and B(15,9). The line rises up from 1 to 9 in height and it goes forward from 1 to 15 along the horizontal. If we compare the change in height to the change in horizontal distance, we get an idea of how steep the slope is.

Change in height: 9–1 = 8; , change in horizontal: 15–1 = 14.

The comparison can be written down as a fraction or ratio: $\frac{8}{14} = \frac{4}{7}$

i. Compare the slopes of the other two lines P and Q. What can you conclude about the size of the slope and the size of the fraction?

ii. Is the slope of each line constant?

iii. Would it matter if we took different points on the lines to find the slope of?

iv. If this method works for calculating any slope, can we summarise it into a rule?

v. Slope is a property of a line; what does this mean?

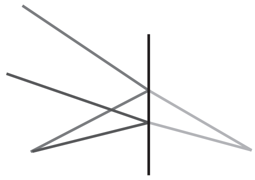
vi. It is easy to draw a line when you have two points on it. Draw the line which contains the points G(3,6) and H(9,–1). Work out the slope of this line. This line is different from other lines. What makes it different?

vii. We can use a point on a line and its slope to get its equation. How would you describe the equation of a line in your own words?

viii. We have seen above that it is possible to find the slope of a line when you have two points.Is it possible to find the equation of a line using the same two points?

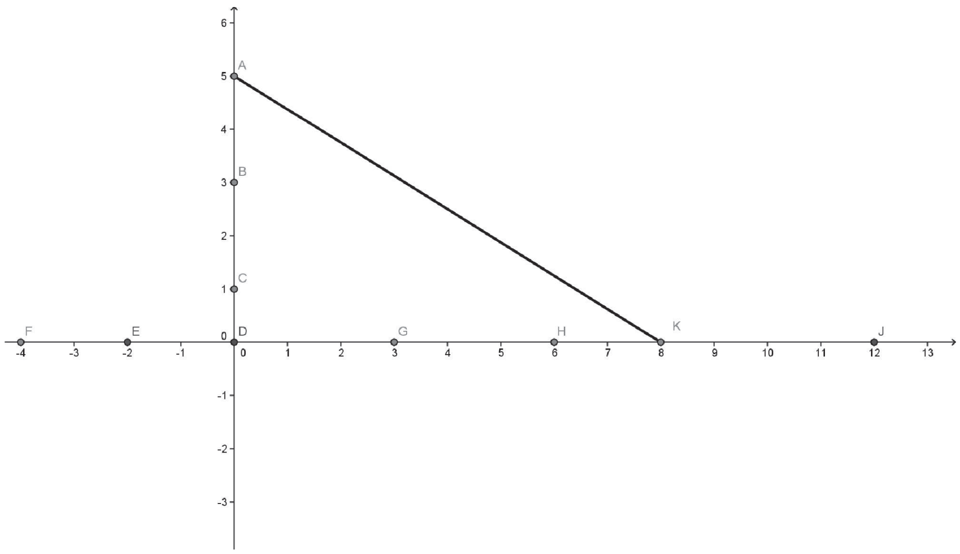
ix. If you were told about the slope of a line and were given a point on a line could you draw a picture of it?

Activity 3.5



Lines are long, and we draw parts of them on diagrams. These 'parts of lines' are called line segments. The more interesting parts of lines are when they come in contact with other lines or points or shapes. Then they have common elements or a shared location.

If we begin by looking at the X and Y axes and plotting the points along them, we can see that there is a common co-ordinate for points on the X-axis and also a common co-ordinate for points on the Y-axis. Give the co-ordinates of the points labelled on the diagram and write down the common co-ordinate for each axis.



Points on x-axis

What is the common coordinate for points on the x-axis?

Points on y-axis

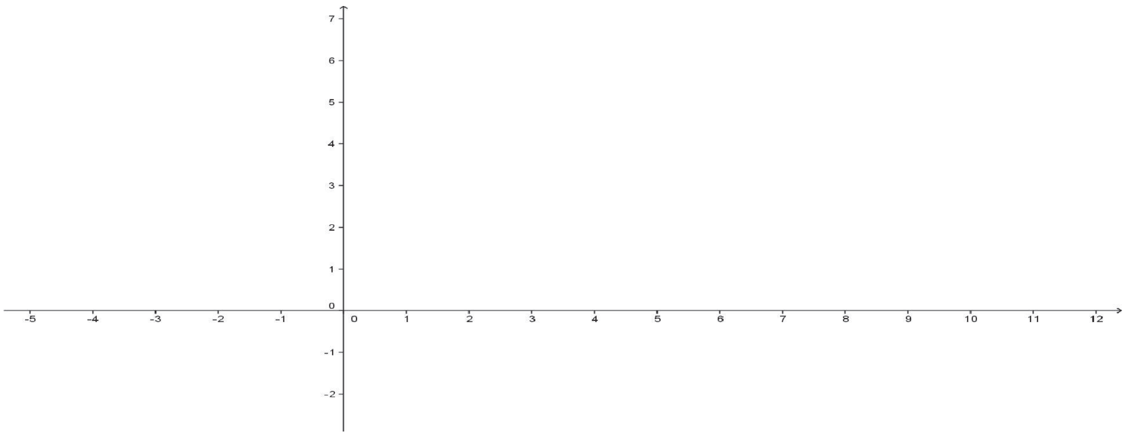
What is the common coordinate for points on the y-axis?

Find the equation of the line linking points A and K.

Activity 3.6

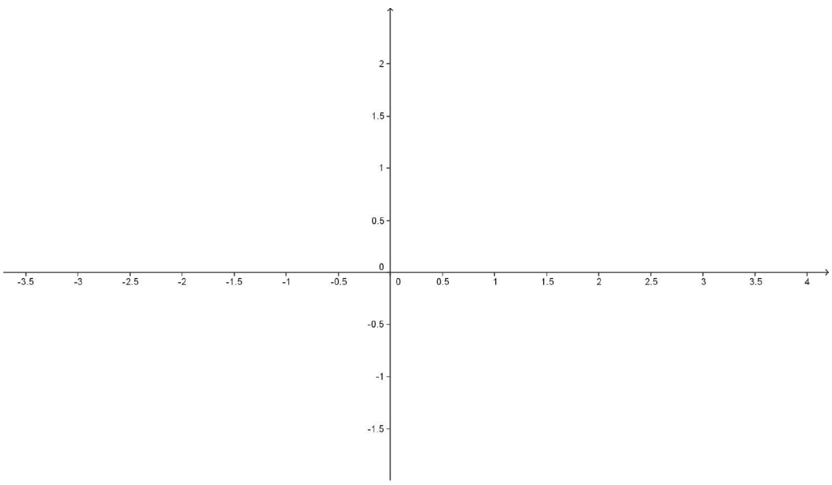
In algebra you may have studied simultaneous equations. If so, you have been able to find a unique value of x and one for y that satisfies two equations. If you were to draw a graph of each of those equations what would they look like?

$3x + 2y = 8$ and $2x - y = 3$ can be used as an example. Solve these simultaneous equations in the normal fashion to find the x and y values that satisfy each equation. Now consider what each of these equations represents when it is drawn as a graph. Plot both of the equations as lines on the axes provided below. Notice the co-ordinates of the point of intersection



Q.1 Now try to find the solution to a similar problem, this time plotting the lines first and then solving the equations using algebra.

$2x + 3y = -2$ and $3x + 7y = -6$ are the equations of two lines. Plot these lines on the grid below.

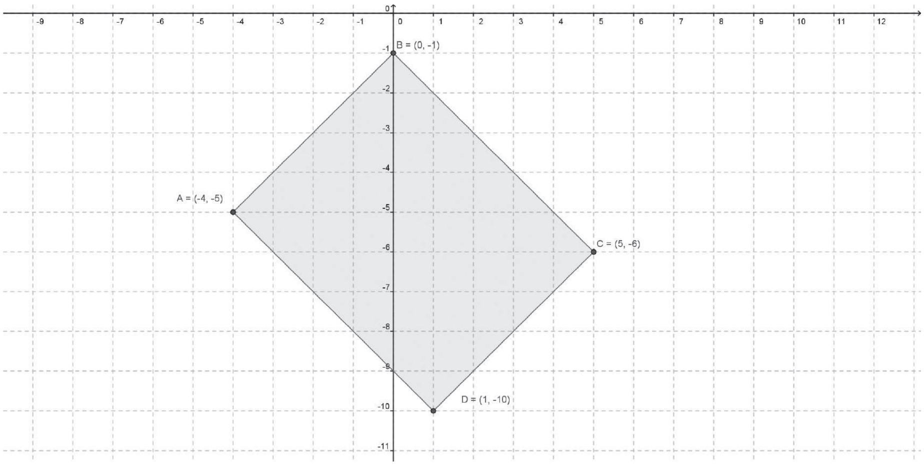


Although both methods give you an answer, comment on the two answers that you got. What conclusion did you reach about the accuracy of each method?

Activity 3.7

Describe parallel lines in one sentence.

What common property would you identify from the opposite sides [AB] and [CD] of the shape shown in the diagram below. How can you show that the property is the same in both sets of lines? Will this always be true for parallel lines?

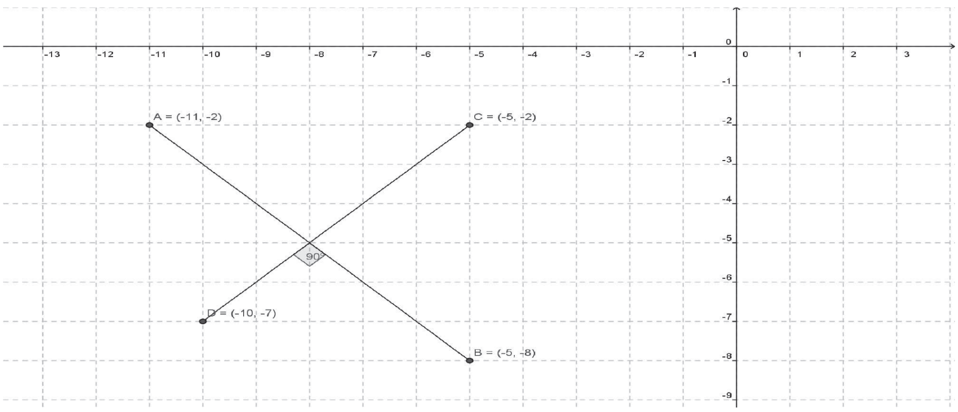


Generalise your findings into a rule that will apply to all parallel lines.

Other lines

Draw the line x = 4 and the line y = 4. Investigate the slopes of these lines. Write a couple of sentences to summarise your findings.

Q. 1 In the diagram below the lines are at right angles. Find the slope of each line and compare them. Write down any ideas you have about the slopes of perpendicular lines.



GEOMETRY 4

SYLLABUS TOPIC: TRIGONOMETRY

LEARNING OUTCOME

As a result of completing the activities in this section you will be able to

- apply the result of the theorem of Pythagoras
- use trigonometric ratios to solve problems(0° to 90°)

Higher level students will be able to

- solve problems involving surds
- solve problems involving right angled triangles
- manipulate measures in DMS and decimal forms

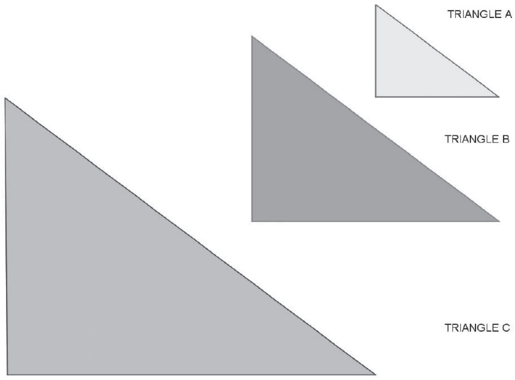
INTRODUCTION

The activities described below and the questions that follow allow you to deepen your understanding of concepts in trigonometry as well as solve problems using these concepts and their applications.

Activity 4.1

It is claimed that the ancient Egyptians used a rope with twelve equally spaced beads on it to check if their right angles were correct. Tie twelve equally spaced beads onto a length of wool or cord as illustrated below and see if you can form a right angle the way they did in ancient times.

Are all of the following triangles right angled? Triangle A has sides of 3, 4 and 5. Triangle B has sides of 6, 8 and 10. Triangle C has sides of 9, 12 and 15.



Activity 4.2

The amount of turning between the two rays (or arms) of an angle tells us how big the angle is. Using the table below, which has some values included already, look at how the values of sine, cosine and tangent change as the angle gets bigger. Give all answers correct to four decimal places.

Size of the angle in degrees	Sine (sin)	Cosine (cos)	Tangent (tan)
10°	0.1736	0.9848	0.1763
20°			
30°			
40°			
45°	0.7071	0.7071	1
50°			
60°			
70°			
80°			
90°	1	0	Error?

Q. 1 Write a single sentence as an answer to each of the following questions.

i. What have you noticed about the values of sin as the angle got bigger?

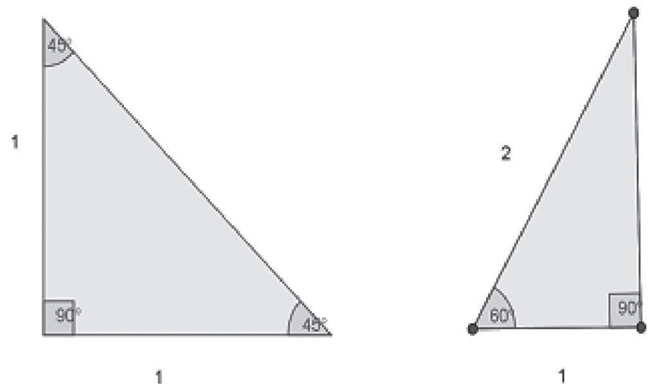
ii. What have you noticed about the values of cos as the angle got bigger?

iii. What have you noticed about the values of tan as the angle got bigger?

iv. Can you give a reason why a calculator displays 'ERROR' when you try to get the tan of 90°?

Activity 4.3

You have studied surds already in strand 3; they are numbers that can only be expressed exactly using the root sign. Surds occur quite often when we are trying to solve problems in triangles. Some of the best known surds are √2 , √3 and √5 and they occur in familiar triangles.



Find the missing sides and angles in the two triangles above.

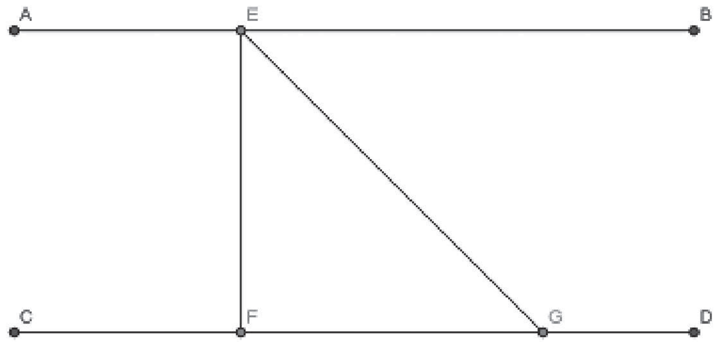
Complete the following table

Angle A	30°	60°	90°
cos A			
sin A			
tan A			

Activity 4.4

The work of surveyors, planners and engineers often involves solving real-life problems. Measurements can be made using instruments, and angles or distances easily found using trigonometry. In the diagram below the width of a river is being calculated. In order to make this calculation decide what information you require. Below the diagram is a series of measurements that were made and certain conditions are given. Is each piece of information necessary to solve the problem? Explain why. Is there another way of solving the problem without needing all of the conditions given?

Q. 1



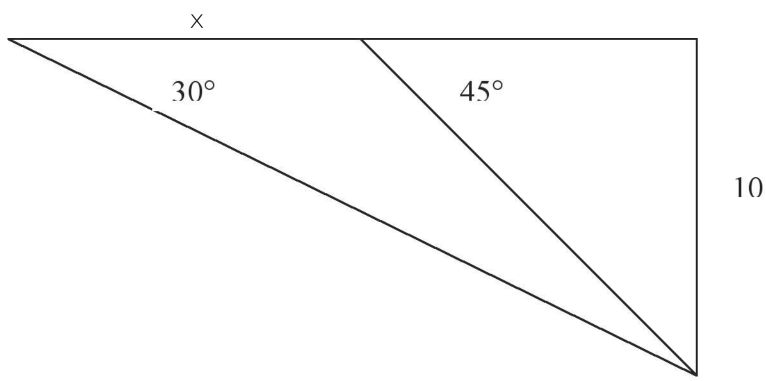
[AB] and [CD] represent river banks that are parallel.
[EF] makes a right angle with [CD]; |FG| is 100m and |∠EGF| is measured to be 52°

i. Find the width of the river.

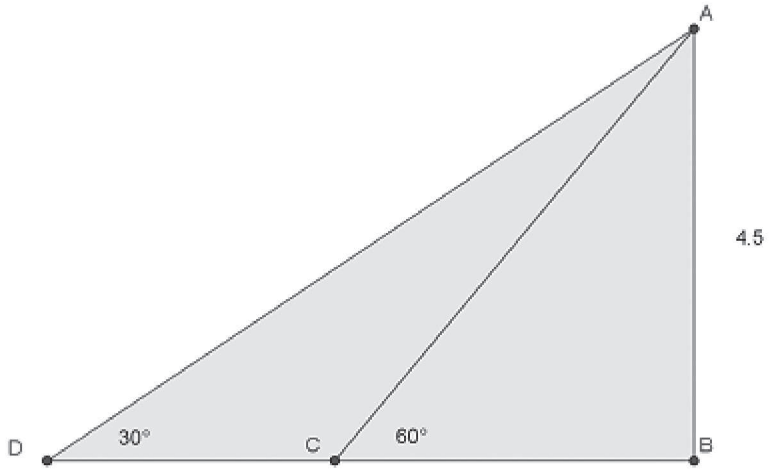
ii. Can you suggest another way to find the width of the river, without directly measuring it?

Q. 2 Try solving the following triangles

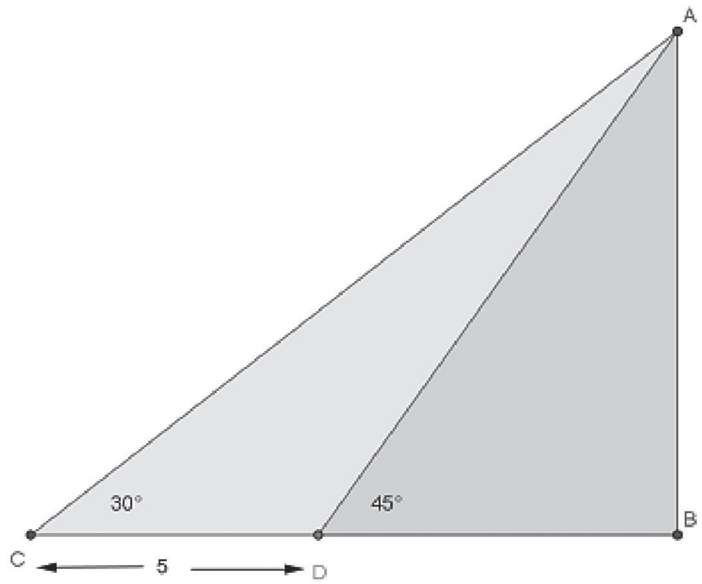
i. From the diagram find the length of x.



ii. From the diagram, find the value of |DC| in surd form.



iii. Find the lengths of [AD] and [AC] in surd form.

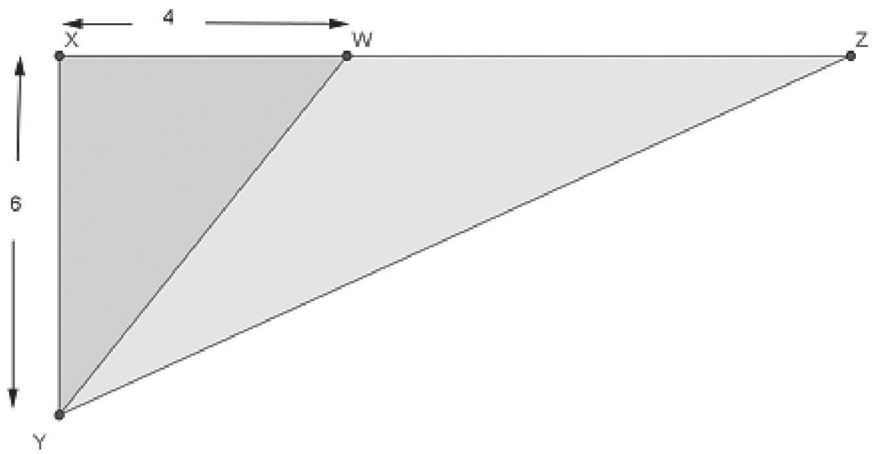


iv. Construct an angle B such that cos B = $\frac{\sqrt{3}}{5}$

Activity 4.5

If there are 360 degrees in a circle, 60 minutes in a degree, and 60 seconds in a minute how many seconds are there in a circle? [N.B. the usual notation is 360°, 60′ and 60″ respectively.]

In the right angled triangle XYZ, [YW] bisects ∠XYZ. |XW|= 4 and |XY|= 6. Calculate|∠XYW| (in degrees and minutes as well as in decimal form) and the length of [WZ].



Leaving Certificate

Strand 3 and 4 Tasks

This set of tasks promotes understanding and gives you an opportunity to provide evidence of your learning in Strand 3 (Number) and Strand 4 (Algebra).

It is broken up into sections

Section A: Working with number and Applied measure

Section B: Complex numbers, solving equations

Section C: Generating arithmetic expressions, geometric expressions, sequences and series.

Remember, if you are following the HL syllabus you need to be able to display evidence of understanding of the learning outcomes on the three syllabus levels; FL, OL and HL. In the same way, if you are following the OL syllabus you need to be able to display evidence of understanding of the learning outcomes on the two syllabus levels; FL and OL. Sometimes you will see similar tasks at two different levels, if you are taking the higher of the two levels it is useful for you to see how the same learning outcomes can be assessed at different levels. There is more help provided in the lower of the two levels, this help in a question is called **scaffolding**. If you are following HL expect to be given very little help in the question, students following HL are expected to display a sophisticated problem solving ability. When you look at scaffolded FL or OL tasks think about how the task might look if the scaffolding was removed altogether, this is likely to be the way it will be presented to you.

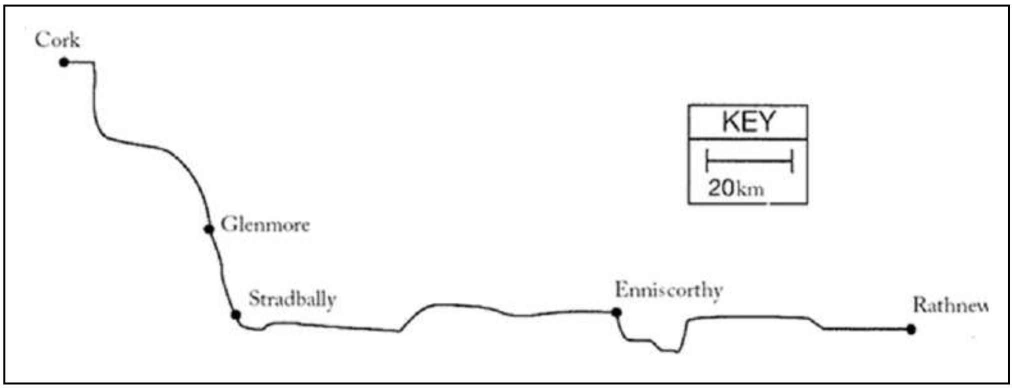
Examples of student work are included for a selection of the tasks. Try the tasks yourself before you look at other students' work. We invite you to **Compare, Examine, Discuss and Evaluate** the solution strategies provided.

Section A

TASK 1 and 2	Exploration, investigation and discussion Strand 3
Level	LCFL/OL
Learning outcome	This material provides you with the opportunity to display evidence that you can – interpret scaled diagrams – solve problems that involve calculating averages, speed, distance and time

Task 1 LCFL

Sean and Amy are travelling from Cork to Rathnew; the route they are taking is shown below.



(a) Using the key, complete the table showing the distances between the towns.

Stage of Journey	Distance travelled
Cork to Glenmore	
Glenmore to Stradbally	
Stradbally to Enniscorthy	
Enniscorthy to Rathnew	

(b) Sean and Amy will drive at an average speed of 60km/hr. They will stop for lunch along the way for half an hour. How long will it take for Sean and Amy to complete the journey?

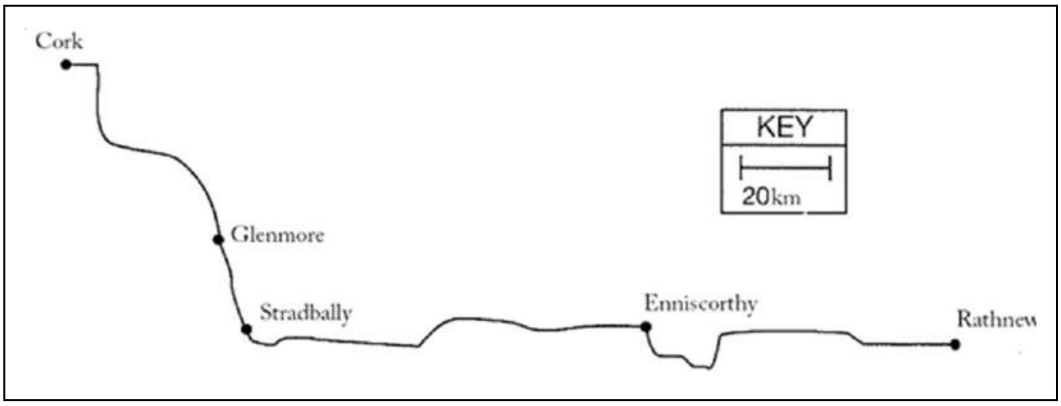
Explain your thinking.

(c) Sketch a distance time graph showing their journey. Mark each of the 5 towns on the graph.

One of the challenges of this question is using the key to estimate the distances between towns. If you estimate the distance badly how will this affect your answer to part (b)? How will if affect your graph in part (c)? If you underestimate the distance what effect would this have on your answers to parts (b) and (c)? If you overestimated the distance what effect would this have on your answers to parts (b) and (c)? What **precautions** could you take to make sure your estimate is as accurate as possible? That means how can you make sure that you don’t make any mistakes when you are estimating the distance.

Task 2 LCOL

Sean and Amy are travelling from Cork to Rathnew; the route they are taking is shown below.

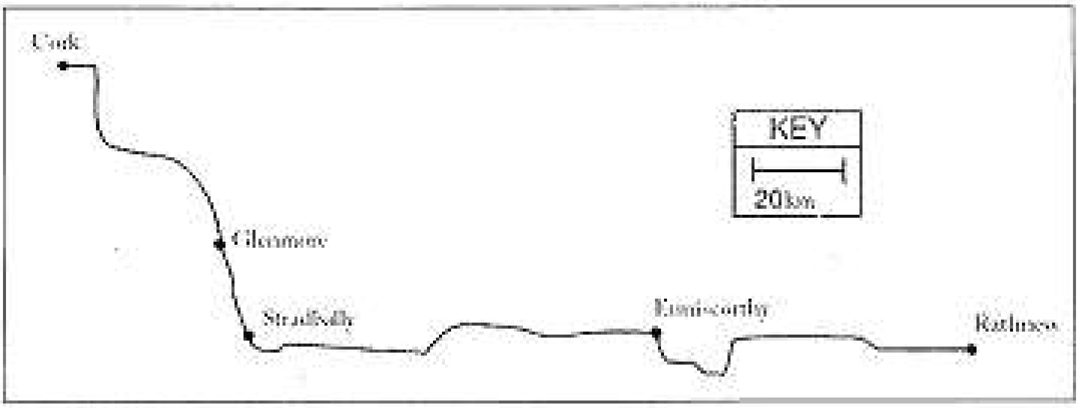


(a) Sean and Amy will drive at an average speed of 60km/hr. They will stop for lunch along the way for half an hour. How long will it take for Sean and Amy to complete the journey?
Explain your thinking.

(b) Sketch a distance time graph showing their journey. Mark each of the 5 towns on the graph.

Compare, Examine, Discuss and Evaluate

Sean and Amy are travelling from Cork to Rathnew the route they are taking is shown below



a) Sean and Amy will drive at an average speed of 60km/hr. They will stop for lunch along the way for half an hour.
How long will it take for Sean and Amy to complete the journey?.
Explain your thinking

b) Sketch a distance time graph showing their journey. Mark each of the 5 towns on the graph.

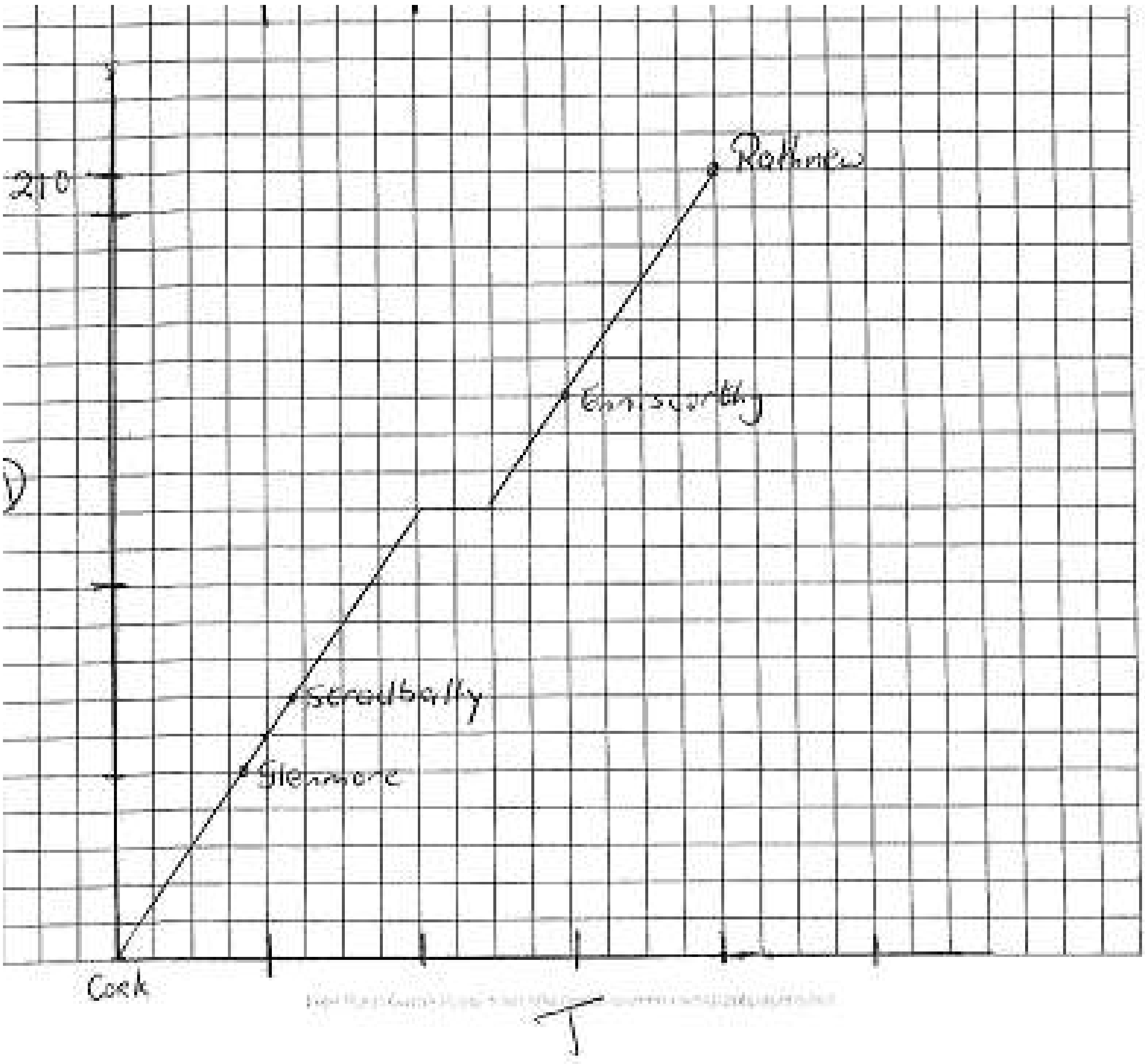
a) I worked out how far it is from Cork to Glenmore to Stradbally to Eniscorthy to Rathnew

Cork to Glenmore 50km
Glenmore to Stradbally 20km
Stradbally to Eniscorthy 80km
Eniscorthy to Rathnew 60km

Total Distance = 210km

$T = \frac{D}{S} = \frac{210}{60} = 3.5$ hrs and .5 hrs for Lunch

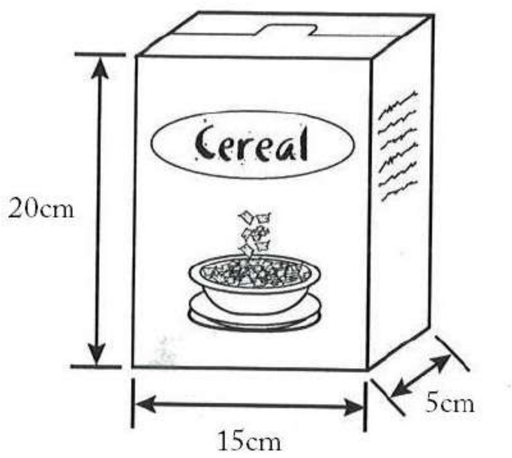
4hrs

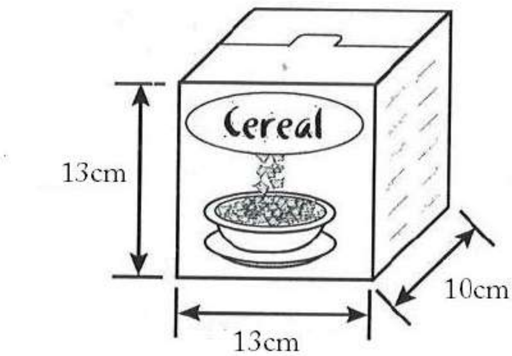

210
D
Rathnew
Enniscorthy
Stradbally
Glenmore
Cork
T

Tasks 3 - 18	Exploration, investigation and discussion Strand 3
Level	LCFL/OL/HL
Learning outcome	This material provides you with the opportunity to display evidence that you can – investigate the nets of rectangular solids – select and use suitable strategies to find length of the perimeter and the area of the following plane figures: disc, triangle, rectangle, square, parallelogram, trapezium, and figures made from combinations of these – select and use suitable strategies to find surface area and volume of the following solid figures: rectangular block, cylinder, right cone, right prism and sphere – perform calculations involving surface area and volume of the following solid figures: rectangular block, cylinder, right cone, triangular-based prism (right angle, isosceles and equilateral), sphere, hemisphere, and solids made from combinations of these.

Task 3: LCFL

A company introducing a new cereal wants to use one of the boxes shown.





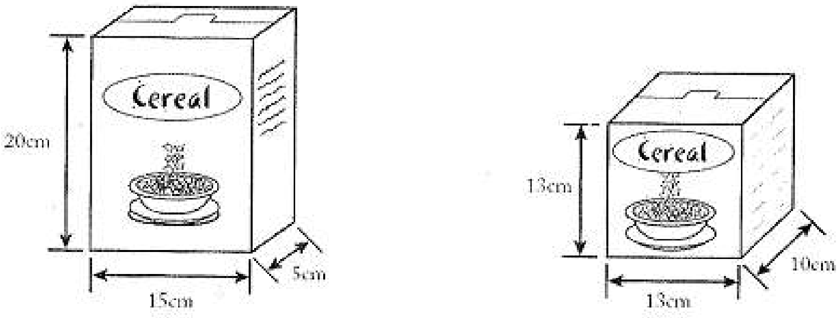
(a) Which box will hold the greater volume of cereal?

(b) Draw nets of each box (ignore the tabs for joining the sides). Clearly label the dimensions and use these to help you calculate the area of cardboard needed to make each box.

Compare, Examine, Discuss and Evaluate

LCFL

A company introducing a new cereal wants to use one of the boxes shown.



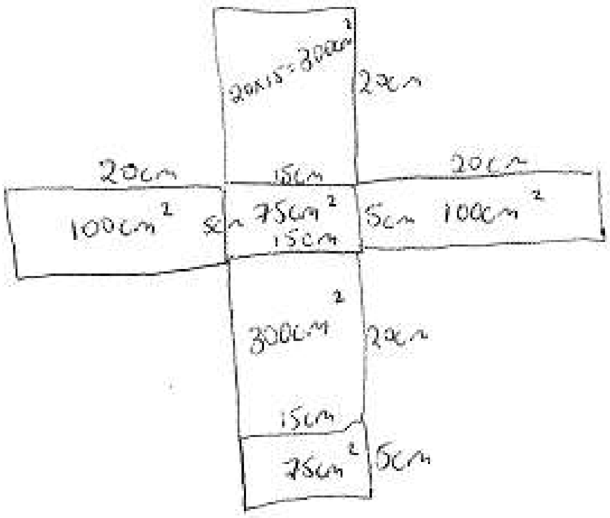
a) Which box will hold the greatest volume of cereal?

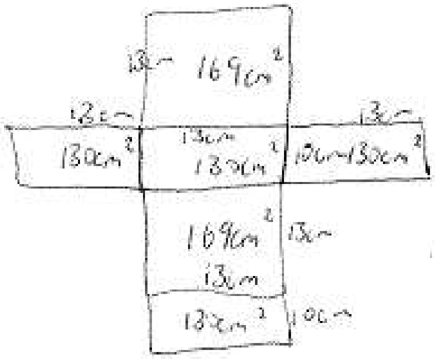
V= lxwxh
= 20x15x5
= 1500cm³

V = lxwxh
= 13x13x10
= 1690cm³

The one that is 13cm X 13cm X 10cm

b) Draw nets of each box. Clearly label the dimensions and use these to help you calculate the are of cardboard needed to make each box.



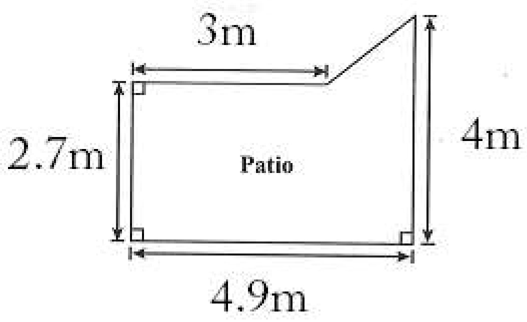


Area = 300+100+75+100+300+75
= 950 cm²

Area = 520 + 338
= 858 cm²

Task 4 LCFL

The Baker family patio is shown below.



Mrs Baker wants to pave the patio and fence it off from the rest of the garden.

Calculate:

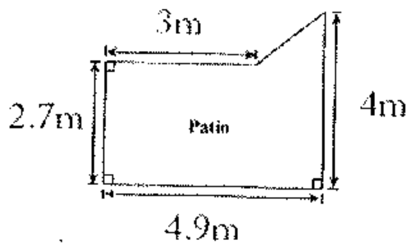
(a) the area that is to be paved

(b) the length of fencing required

Compare, Examine, Discuss and Evaluate

LCFL

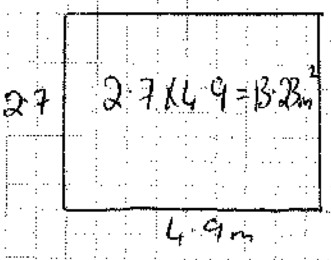
The Baker family patio is shown below.

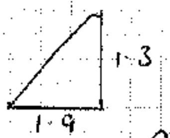


Mrs Baker wants to pave the patio and fence it off from the rest of the garden.

Calculate

a) the area that is to be paved.





$4.9 - 3 = 1.9$

$4 - 2.7 = 1.3$

$A = \frac{1}{2}(1.9 \times 1.3)$

$= \frac{1}{2}(2.47)$

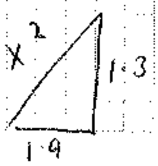
$= 1.235\,m^2$

$Area = 13.23 + 1.235 = 14.465\,m^2$

b) the length of fencing required

$2.7m + 4.9m + 4m + Xm + 3m$

$= 14.6m + 2.3m$



$x^2 = 1.3^2 + 1.9^2$

$= 1.69 + 3.61$

$= 5.30$

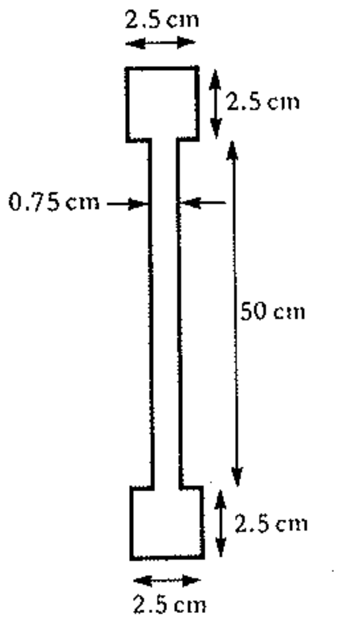
$X = \sqrt{5.3}$

$= 2.3m$

$= 16.9m$

Task 5 LCOL

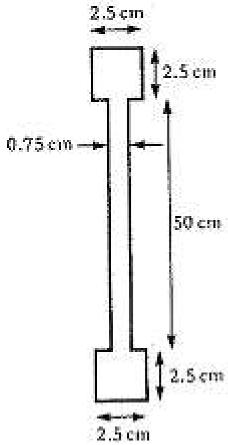
The cross-section through a domestic central heating radiator is shown below. The centre section is 50cm high and 0.75cm thick; at the top and bottom there are squares of side 2.5cm. Calculate the area of cross-section and hence find the volume of water in litres, inside a radiator 2m long.



Compare, Examine, Discuss and Evaluate

LCOL

The cross- section through a domestic central heating radiator is shown below. The centre section is 50cm high and 0.75cm thick; at the top and bottom there are squares of side 2.5cm. Calculate the area of cross- section and hence find the volume of water in litres, inside a radiator 2m long.



Area $= (2.5 \times 2.5) + (50 \times .75) + (2.5 \times 2.5)$

$= 6.25 + 37.5 + 6.25$

$= 50 cm^2$

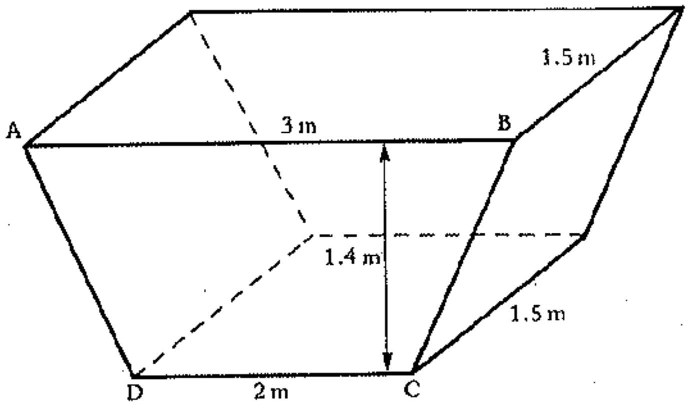
Volume $= 50 \times 200$

$= 10,000 cm^3$

$= 10$ litres

Task 6 LCOL

The diagram shows a builder's skip. The skip has a **rectangular** base measuring 2m by 1.5m, a **rectangular** open top measuring 3m by 1.5m, and is 1.4m deep. The vertical sides are **trapeziums** and the sloping sides are **rectangles.**



(a) Find the area of the vertical side ABCD

(b) Find the volume of waste in the skip when it is filled level with the top

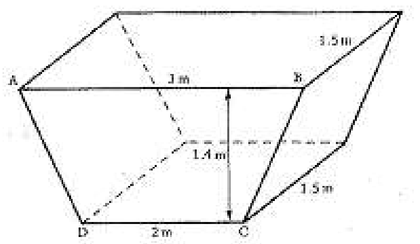
(c) By heaping it above the top a further 20% of waste material may be carried in the skip. How much does the skip hold when the waste is carried in this way?

(d) If $1m^3$ of waste has a mass of 600kg, find the mass of waste in the skip when it is filled level with the top. Give your answer in tonnes.

Compare, Examine, Discuss and Evaluate

LCOL

The diagram shows a builder's skip. The skip has a **rectangular** base measuring 2m by 1.5m, a **rectangular** open top measuring 3m by 1.5m, and is 1.4m deep. The vertical sides are **trapeziums** and the sloping sides are **rectangles**.



a) Find the area of the vertical side ABCD

Area = $\frac{1}{2}(2+3) \times 1.4$

= $3.5\,m^2$

b) Find the volume of waste in the skip when it is filled level with the top

Volume = Area × 1.5

= 3.5 × 1.5

= $5.25\,m^3$

By heaping it above the top a further 20% of waste material may be carried in the skip. How much does the skip hold when the waste is carried in this way?

$5.25 + \frac{20}{100} \times 5.25$

$5.25 + 1.05$

$6.30\,m^3$

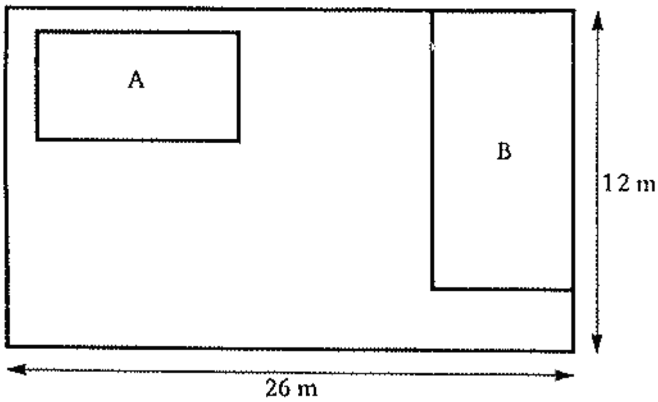
c) If $1m^3$ of waste has a mass of 600kg, find the mass of waste in the skip when it is filled level with the top. Give your answer in tonnes.

6.30 × 600 = 3,780 kg

= 3.78 tonnes

Task 7 LCFL

Ben Carey is moving into his new house. A plan of the rectangular garden is shown in the diagram. He wants to have a swimming pool 1.5m deep measuring 8m by 4m at A. He also wants to have a workshop at B, measuring 10m by 6m, for which the ground must be excavated to a depth of 50cm in order to lay the foundations.



(a) Find the total area of the plot before any soil is removed.

(b) Find the volume of soil removed in order to build the pool.

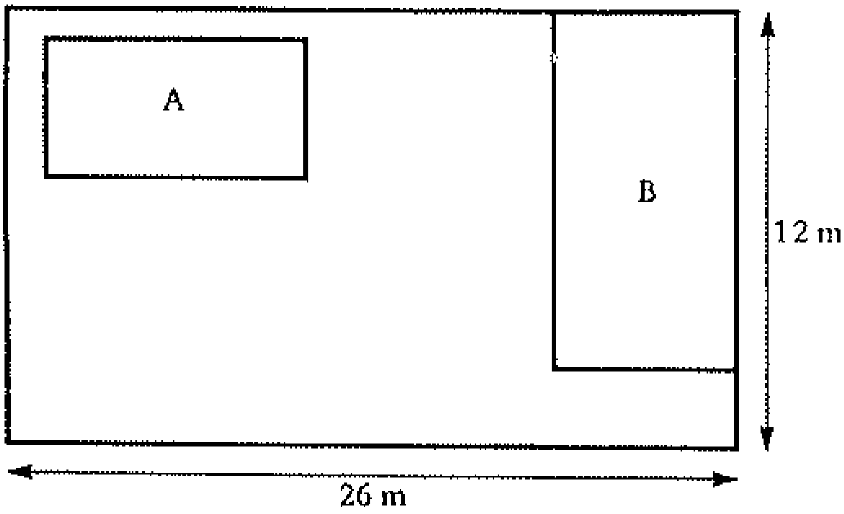
(c) Find the volume of soil excavated to lay the foundation of the workshop.

(d) Find the ground area of A and B.

(e) Find the area of ground that is untouched

Task 8 LCOL

Ben Carey is moving into his new house. A plan of the rectangular garden is shown in the diagram. He wants to have a swimming pool 1.5m deep measuring 8m by 4m at A. He also wants to have a workshop at B, measuring 10m by 6m, for which the ground must excavated to a depth of 50cm in order to lay the foundations.

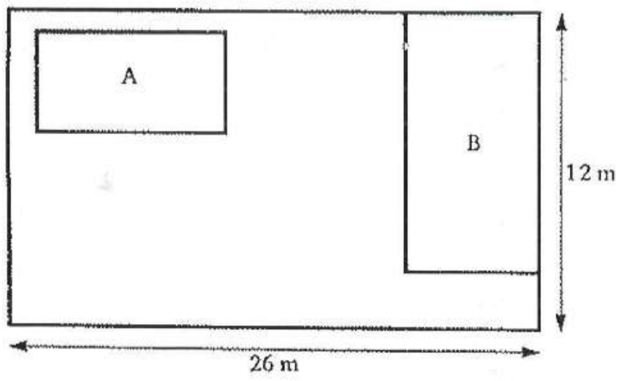


The earth that is excavated from A and B is now laid evenly over the remainder of the plot. By how much will the level of this area rise? Show how you worked out your answer.

Compare, Examine, Discuss and Evaluate

LCOL

Ben Carey is moving into his new house. A plan of the rectangular garden is shown in the diagram. He wants to have a swimming pool 1.5m deep measuring 8m by 4m at A. He also wants to have a workshop at B, measuring 10m by 6m, for which the ground must excavated to a depth of 50cm in order to lay the foundations.



The Earth that is excavated from A and B is now laid evenly over the remainder of the plot. By how much will the level of this area rise?. Show how you worked out your amswer.

Area of garden = 12×26
= 312m²

Vol of Earth from A = 8×4×1·5
= 48m³

Area of Swimming pool = 8×4
= 32m²

Vol of Eart from B = 10×6×0·5
= 30cm³

Area of Workshop = 10×6
= 60m²

Vol of Earth = 78cm³

Area Left = 312 − (92)
= 230m²

$V = (l \times w) \times h$

$78 = 230 \times x$

$x = \frac{78}{230} = \cdot 339\text{m}$

= 33·9cm Deep

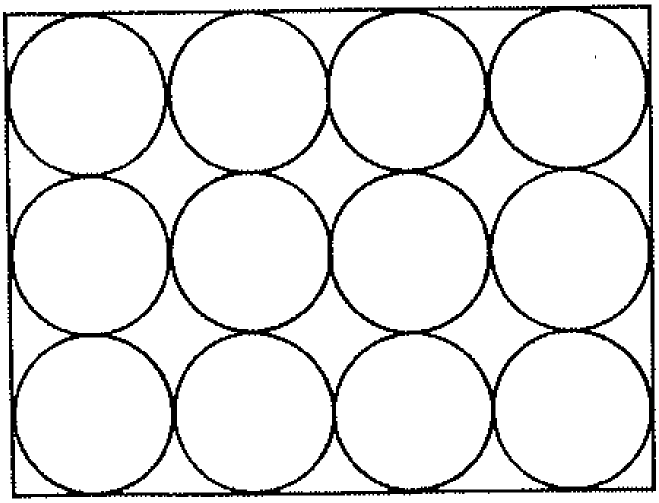
Task 9 : LCFL

Freshfruit is a company that manufactures canned fruit.

The fruit is packed into cylindrical cans **10cm** high with a base **6cm** wide**.**

For shipping purposes the cans are packed into open-ended, rectangular cardboard boxes.

The diagram below shows one possible arrangement, when the cans are packed 12 at a time.



Find

(a) The length and width of the box

(b) The capacity of the box

(c) The volume of one of the cans

(d) The amount of space in the box that is unused

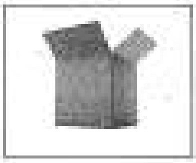
(e) The area of card used to make the box (ignore overlaps)

Compare, Examine, Discuss and Evaluate

LCFL

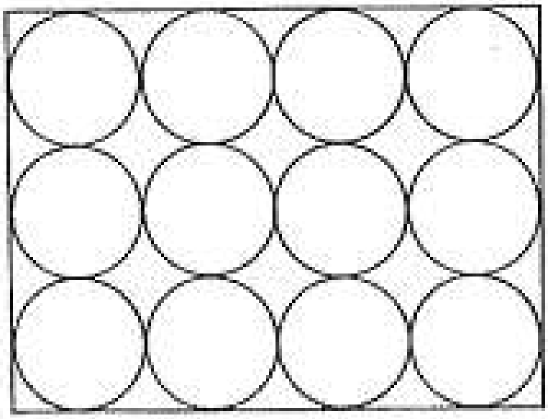
Freshfruit is a company that manufactures canned fruit.

The fruit is packed into cylindrical cans **10cm** high with a base of **6cm.**



For shipping purposes the cans are packed into open ended rectangular cardboard boxes.

The diagram shows one possible arrangement when the tins are packed 12 at a time



Find

a) The length and width of the box

$L = 4 \times 6 = 24cm$

$W = 3 \times 6 = 18cm$

b) The capacity of the box

$V = L \times W \times H$

$= 24 \times 18 \times 10$

$= 4320cm^3$

Task 10: LCOL

Freshfruit is a company that manufactures canned fruit.

The fruit is packed into cylindrical cans **10cm** high with a base of **6cm.**

For shipping purposes the cans are packed into open-ended, rectangular cardboard boxes.

(a) If the cans are to be packed into boxes in one layer, 12 at a time, what are the dimensions of the box? How much spare space is in the box?

(b) If cardboard costs €2.50/m^2 what is the cost of one packaging box?

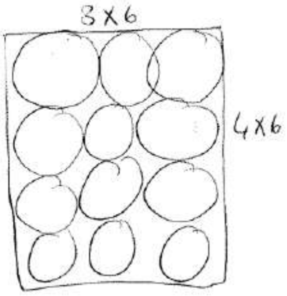
An employee suggests that to cut costs they should pack the cans in two layers with 6 cans on the base, arranged 2X3, and six cans placed on top of these, since less cardboard would be needed to make the boxes.

Do you agree with this suggestion? Explain why or why not? Illustrate your answer with a labelled diagram.

Compare, Examine, Discuss and Evaluate

For shipping purposes the cans are packed into open ended rectangular cardboard boxes.

If the cans are to be packed into boxes in one layer 12 at a time what are the dimensions of the box? How much spare space is in the box?



Length = 24 cm
Width = 18 cm
Height = 10 cm

$V = 24 \times 18 \times 10$
$= 4320 cm^3$

$V = 6(\pi R^2 h)$
$12(\pi(3)^2(10))$
$= 12282.7 cm^3$
$= 3392.6 cm^3$

Spare Space = 4320 − 3392.4 = 927.6 cm³

If cardboard costs €2.50/m² what is the cost of one packaging box?

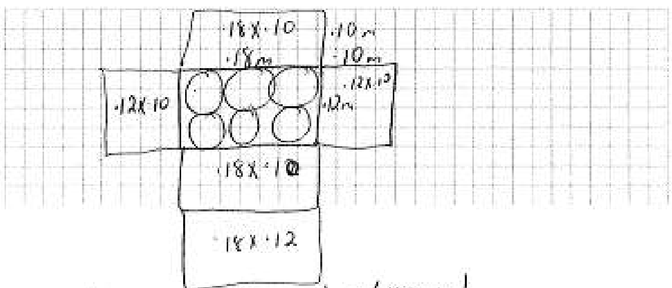
$A = 2(240) + 2(432) + 2(180)$
$= 1704 cm^2$
$= .1704 m^2$

Cost = 2.50 x .1704 = 42.6c

An employee suggests that to cut costs they should pack the cans in two layers with 6 cans on the base, arranged 2X3, and six cans placed on top of these as less cardboard would be needed to make the boxes.

Do you agree with this suggestion? Explain why or why not? Illustrate your answer with a labelled diagram.

Give reasons for your choice



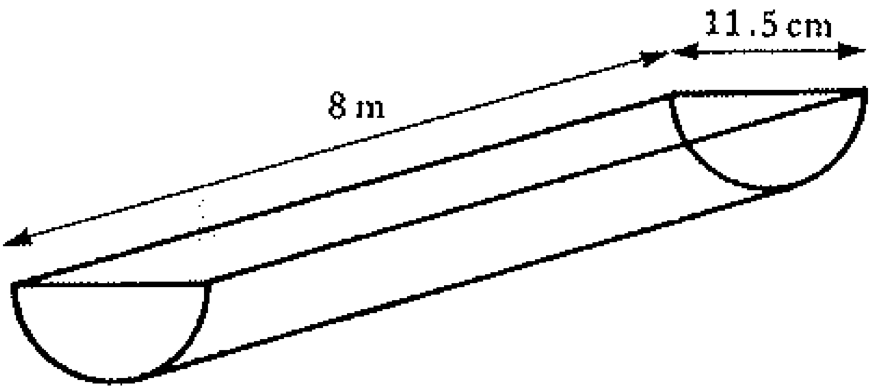
$A = 2(.18 \times .12) + 2(.12 \times .10) + 2(.18 \times .10)$
$= .0432 + .024 + .036$
$= .1032 m^2$

Yes I do cos if I arrange them like this I need less cardboard in fact .0672m² less which saves me .168c / box It all adds up...

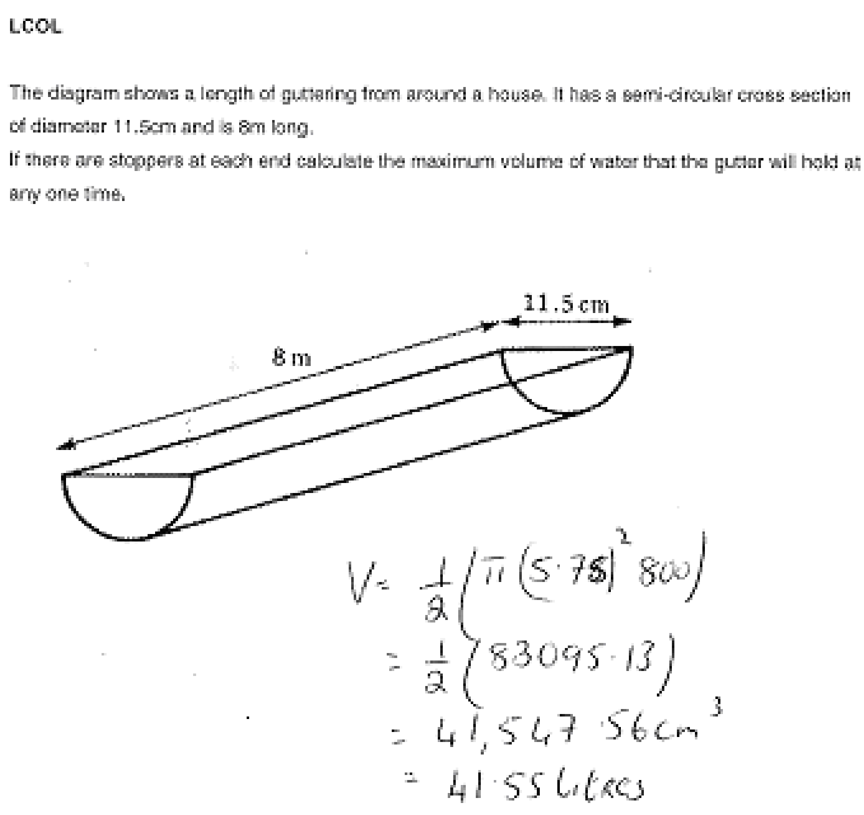
Task 11 LCOL

The diagram shows a length of guttering from around a house. It has a semi-circular cross section of diameter 11.5cm and is 8m long.

If there are stoppers at each end, calculate the maximum volume of water in litres that the gutter will hold at any one time.



Compare, Examine, Discuss and Evaluate



LCOL

The diagram shows a length of guttering from around a house. It has a semi-circular cross section of diameter 11.5cm and is 8m long.

If there are stoppers at each end calculate the maximum volume of water that the gutter will hold at any one time.

$V = \frac{1}{2}\left(\pi (5.75)^2\ 800\right)$

$= \frac{1}{2}\left(83095.13\right)$

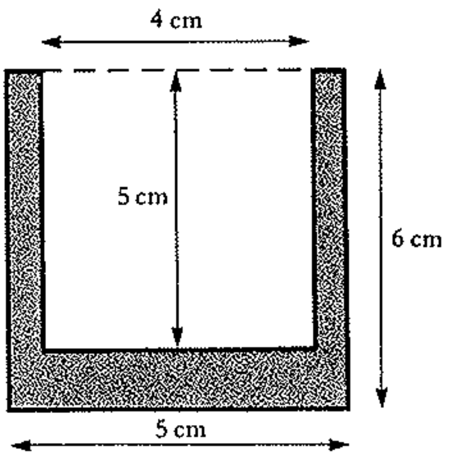
$= 41,547.56\text{cm}^3$

$= 41.55$ Litres

Task 12 LCFL

The diagram shows the vertical cross-section through a machine part.

It shows a solid metal cylinder of diameter 5cm and height 6cm from which a cylinder of diameter 4cm and depth 5cm has been removed.



Calculate

(a) the volume for the cylinder before the hole is bored

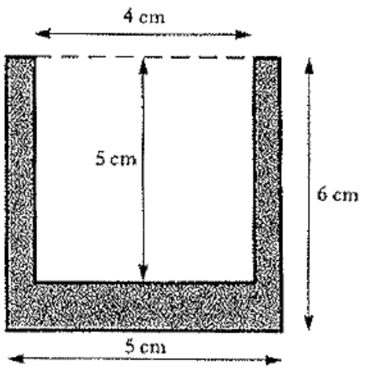
(b) the volume of the metal removed

(c) the mass of the finished machine part if the mass of 1cm^3 of the metal is 8.3g.

Compare, Examine, Discuss and Evaluate

LCFL

The diagram shows the vertical cross section through a machine part.
It shows a solid metal cylinder of diameter 5cm and height 6cmfrom which a cylinder of diameter 4cm and depth 5cm has been removed.



a) Calculate the volume for the cylinder before the hole is bored

$$V = \pi R^2 h$$
$$= \pi (2.5)^2 6 = 117.8 cm^3$$

b) The volume of the metal removed.

$$V = \pi R^2 h$$
$$= \pi (2)^2 (5)$$
$$= 62.83 cm^3$$

c) The mass of the finished machine part of the mass of 1cm^3 of the metal is 8.3g

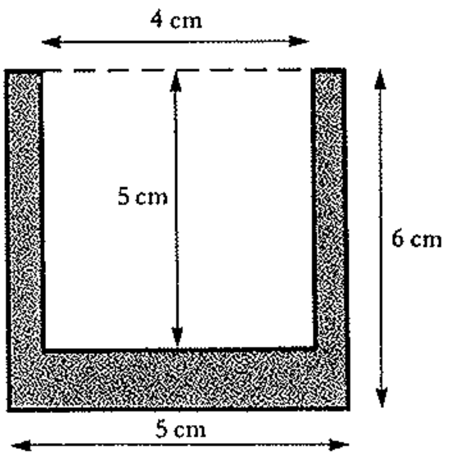
$$V = 117.8 - 62.83$$
$$= 54.97 cm^3$$

$$Mass = 54.97 \times 8.3$$
$$= 456.24 g$$

Task 13 LCOL

The diagram shows the vertical cross-section through a machine part.

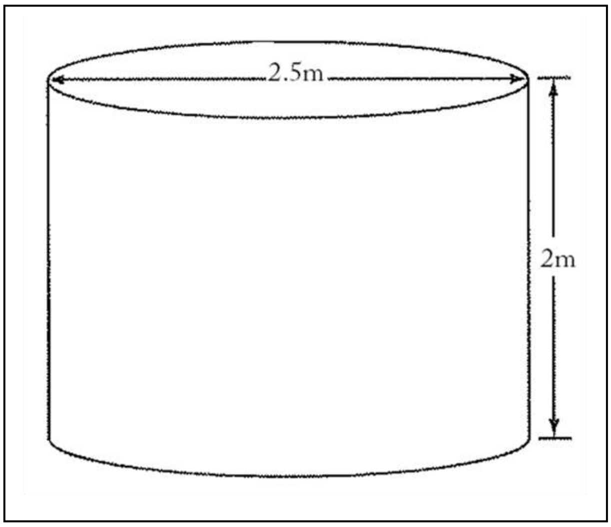
It shows a solid metal cylinder of diameter 5cm and height 6cm from which a cylinder of diameter 4cm and depth 5cm has been removed.



Calculate the mass of the finished machine part if the mass of $1cm^3$ of the metal is 8.3g.

Task 14: LCOL

The diagram shows a water-storage tank proposed for the Byrne family home. It is in the shape of a cylinder of diameter 2.5m and height 2m.



The Byrnes want get a new tank to increase the volume of water they can store.

They can either increase the diameter by **1m** or increase the height by **1m**.

What is the difference, in litres, in the volume obtained by increasing the diameter by **1m** and the volume obtained by increasing the height by **1m**?

Show your calculations of the two volumes or give a complete explanation with numerical support to justify your answer.

The Byrnes decide to choose the tank that maximises the volume of water they can store. They want to buy insulation for their new water storage tank. Insulation comes in rolls 33m long and 2m wide.

How many layers of insulation can they wrap around the new tank if they use a complete roll? Explain your thinking

Compare, Examine, Discuss and Evaluate

Increase Diameter by 1m

$V = \pi\left(\frac{3.5}{2}\right)^2 2$

$= \pi(3.06)2$

$= 19.24m^3$

Increase height by 1m

$V = \pi\left(\frac{2.5}{2}\right)^2 3$

$V = \pi(1.56)3$

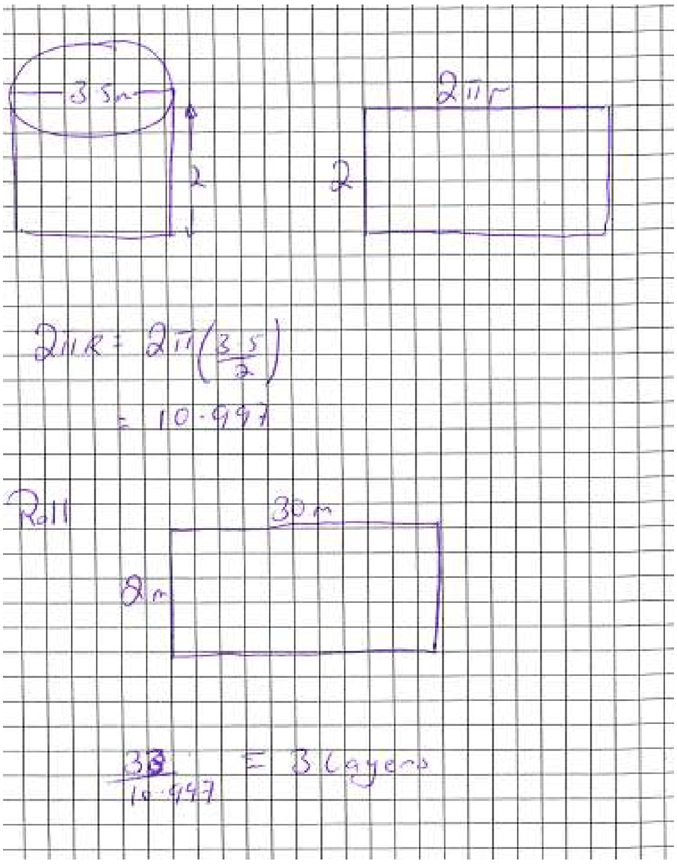
$V = 14.73m^3$

Diff $= 19.24 - 14.73$

$= 4.51m^3$

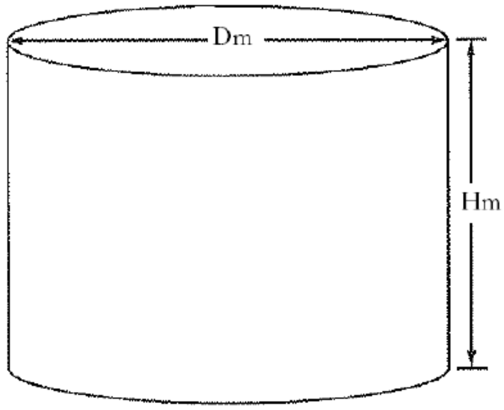
$= 4510000cm^3$

$= 4,510$ Litres


3.5m
2
2πr
2
2πR = 2π(3.5/2)
= 10.997
Roll
30m
8m
33/10.997 = 3 Layers

Task 15: LCHL

The diagram shows a water-storage tank proposed for the Byrne family home.



The Byrnes want to get a new tank to increase the volume of water they can store.

They can either increase the diameter by **1m** or increase the height by **1m**.

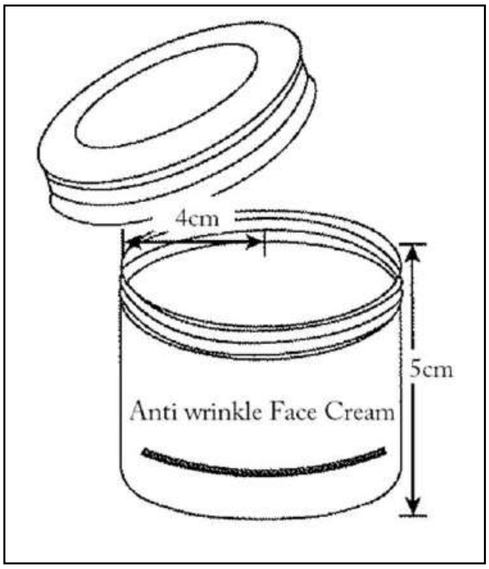
What is the difference, in litres, in the volume obtained by increasing the diameter by **1m** and the volume obtained by increasing the height by **1m**?

Task 16: LCOL

A cosmetic company manufacturing face cream wants to use larger jars.

The new jar is to have exactly twice the volume of the current jar.

Is it best to double the height of the jar or double the radius of the jar?



Explain your thinking.

Compare, Examine, Discuss and Evaluate

$Vol = 2\pi R^2 h$

$= 2(\pi)(4)^2(5)$

$= 502.65 cm^3$

Double height

$V = 2\pi(4)^2 10$

$= 1005.31 cm^3$

Double radius

$V = 2\pi(8)^2 5$

$= 2010.62$

$\frac{\text{Double height}}{\text{Vol}} = \frac{1005.31}{502.65} = 2$

$\frac{\text{Double Radius}}{\text{Vol}} = \frac{2010.62}{502.65} = 4$

I think they should double the height cos doubling the radius increases the vol by 4 not 2.

Task 17: LCHL

A cosmetic company manufacturing face cream wants to use larger jars.

The new jar is to have exactly twice the volume of the current jar.

Is it best to double the height of the jar or double the radius of the jar?

Explain your thinking.

Compare, Examine, Discuss and Evaluate

$V = \pi r^2 h$

Double r $\quad V = \pi (2r)^2 h$

$= 4\pi r^2 h$

$= 4V$

Double h : $\quad V = \pi r^2 (2h)$

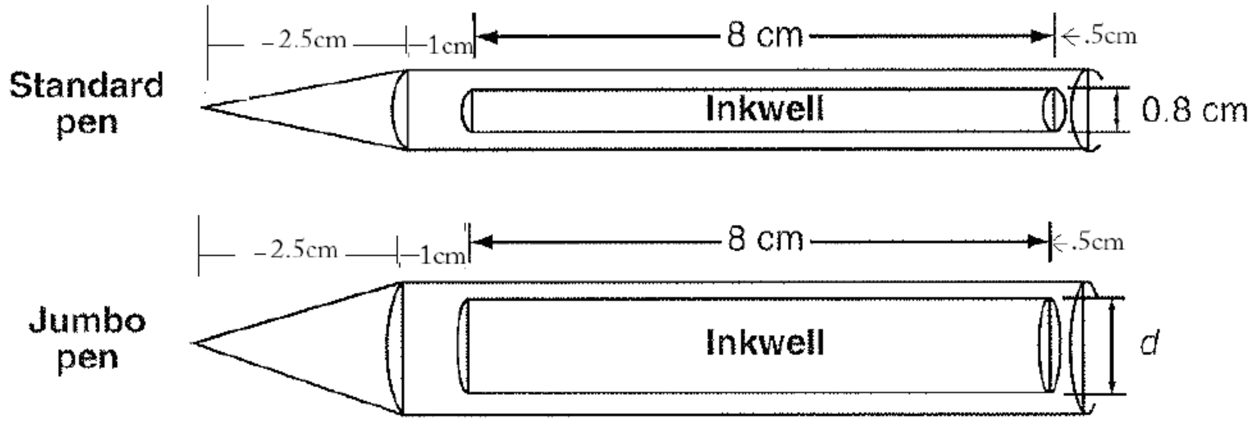
$= 2\pi r^2 h$

$= 2V$

So best to double height.

Task 18 LCHL

A company makes two sizes of pens similar to the ones shown below.



The **Jumbo pen** holds three times as much ink as the **Standard pen** and the company claims that it lasts three times longer.

A paper label surrounds the inkwell of each pen. What is the area of each label?

Compare, Examine, Discuss and Evaluate

Area of label = length x width

$= 8 \times 2\pi R$

$= 16\pi R$

Standard Pen Area = $16(\pi)(0.4) = 20.11cm$

Jumbo Pen Area = $16(\pi)\left(\frac{d}{2}\right) = 16(\pi)\frac{1.39}{2} = 39.93cm^2$

Vol of Standard Pen = $\pi R^2 h$

$= \pi(.4)^2 8$

$= 4.02cm^3$

Vol of Jumbo Pen = $3(4.02) = \pi R^2 h$

$12.06 = \pi R^2 8$

$R^2 = 0.48$

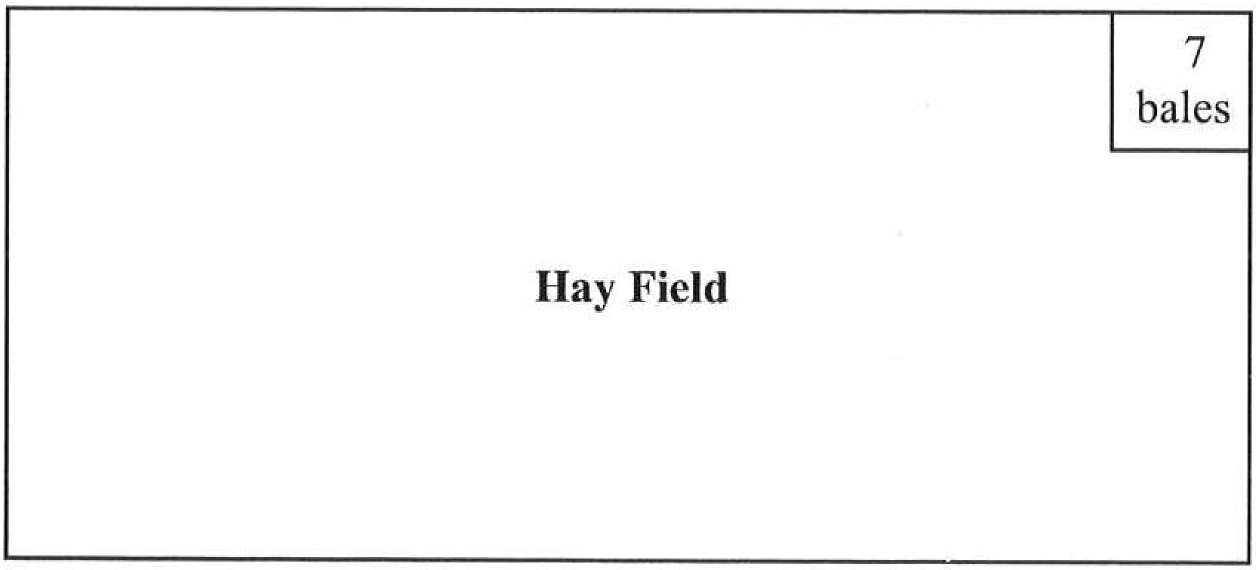
$R = 0.69$

$d = 2 \times R$

$= 1.39cm$

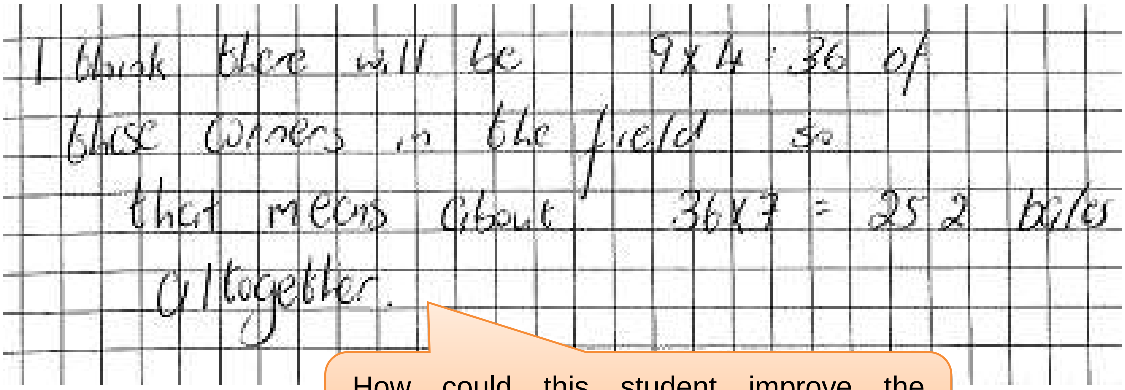
Task 19	Exploration, investigation and discussion	Strand 3
Level	**LCFL**	
Learning outcome	This material provides you with the opportunity to display evidence that you can – estimate the world around you	

A farmer is baling hay; he got 7 bales from the corner of the field shown.



Estimate the number of bales in the whole field. Explain your thinking.

Compare, Examine, Discuss and Evaluate



How could this student improve the accuracy of their estimate?

Task 20	Exploration , investigation and discussion Strand 3
Level	**LCFL**
Learning outcome	This material provides you with the opportunity to display evidence that you can – develop decimals as special equivalent fractions strengthening the connection between these numbers and fraction and place value understanding

Students were sorting glassware in their science laboratory. The measures of the thickness of the glass are shown in the table below

Glass item	Glass thickness (cm)
A	1/8
B	0.25
C	3/16
D	0.5

Sort the glassware in order from the thinnest to the thickest. Explain your working.

Section B

Task 1	Exploration , investigation and discussion Strand 4
Level	LCHL
Learning outcome	This question gives you the opportunity to display **evidence** that you can – work with complex numbers in rectangular and polar form to solve quadratic and other equations including those in the form $z^n = a$, where $n \in \mathbf{Z}$, and $z = r \cos\theta + i\sin\theta$ – use De Moivre's Theorem – use applications such as n^{th} roots of unity, $n \in \mathbf{N}$

Task 1 LCHL

With w denoting either of the two complex cube roots of unity, find

$$\frac{2w+1}{5+3w+w^2} + \frac{2w^2+1}{5+w+3w^2}$$

giving your answer as a fraction a/b, where a, b are integers with no factor in common.

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how they solved the problem: Complex 1

Well, when I looked at this first I thought… right.. what I have here is two fractions so I should get the common denominator and simplify them so that was what I was going to do and then I looked at them and I thought this is going to be awful because I have two quadratic equations as the denominators and I'm going to have powers of 4 and all sorts....so I looked again and I thought how can the fact that w is one of the two complex cube roots of unity help me here?....

I worked out the cube roots of unity are 1, $-\frac{1}{2}$ minus root 3 over 2i, and $-\frac{1}{2}$plus root 3 over 2i, that is 1, w and w^2.

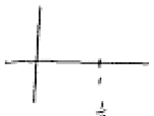
I know w cubed = 1 and $1 + w + w^2 = 0$ because the three roots of unity add to 1

So that really helped me solve the problem.

$z = 1 + 0i$

In Polar form

$z = 1(\cos 2\pi n + i\sin 2\pi n)$



$z^{\frac{1}{3}} = (1+0i)^{\frac{1}{3}} = \cos\frac{2\pi n}{3} + i\sin\frac{2\pi n}{3} \qquad n = 0, 1, 2$

$n = 0 \quad z^{\frac{1}{3}} = \cos 0 + i\sin 0 = 1$

$n = 1 \quad z^{\frac{1}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$n = 2 \quad z^{\frac{1}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

So there are 3 cubed roots of unity

$1, \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$1 \qquad \omega \qquad \omega^2$

So $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$

$$\frac{2\omega+1}{5+3\omega+\omega^2} + \frac{2\omega^2+1}{5+\omega+3\omega^2}$$

$$\frac{2\omega+1}{(\omega^2+\omega+1)+2\omega+4} + \frac{2\omega^2+2\omega+2-2\omega-1}{3\omega^2+3\omega+3-2\omega+2}$$

$$\frac{2\omega+1}{2\omega+4} + \frac{2(\omega^2+\omega+1)-2\omega-1}{3(\omega^2+\omega+1)-2\omega+2}$$

$$\frac{2\omega+1(2-2\omega) + (2\omega+1)(2\omega+4)}{(2\omega+4)(2-2\omega)}$$

$$\frac{4\omega-4\omega^2+2-2\omega-[4\omega^2+8\omega+2\omega+4]}{4\omega-4\omega^2+8-8\omega}$$

$$\frac{-4\omega^2+2\omega+2-4\omega^2-8\omega-2\omega-4}{-4\omega^2-4\omega+8}$$

$$\frac{-8\omega^2-8\omega-2}{-4\omega^2-4\omega+8}$$

$$\frac{-8\omega^2-8\omega-8+6}{-4\omega^2-4\omega-4+12}$$

$$\frac{-8(\omega^2+\omega+1)+6}{-4(\omega^2+\omega+1)+12} = \frac{6}{12} = \frac{1}{2}$$

Task 2	Exploration, investigation and discussion	Strand 4
Level	**LCHL**	
Learning outcome	This question gives you the opportunity to display **evidence** that you can – use the factor theorem for polynomials	

It is important that you are able to interpret this theorem both geometrically and algebraically.

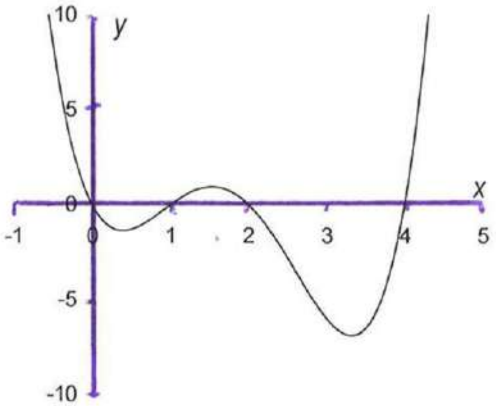
The **geometric** interpretation: *if f(x) is a polynomial whose graph crosses the x-axis at x = a, then (x-a) is a factor of f(x).*

The **algebraic** interpretation: *if f(x) is a polynomial and f(a) = 0, then (x-a) is a factor of f(x)*

Try the tasks below

Task 2:

The graph of the polynomial $y = x\,(x–1)\,(x–2)\,(x–4)$ is shown below.



Argue that the algebraic formula given in the form of factors allows you to see right away where the graph is above the x-axis, where it is below the x-axis and where it crosses the x-axis.

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how they completed the task 2

Well, first I thought where does the graph cross the x-axis? I know from the algebraic equation that it crosses at the points where $y = 0$. In the algebraic formula y is given as a product of factors, this is nice and handy for me because I know this will happen exactly when one of the factors is zero. Now the first factor is zero only at $x = 0$, and the second factor is zero only at $x = 1$, and the third and fourth factors are zero only at $x = 2$ and $x = 4$. So I can say that the graph crosses the x-axis at the points $x = 0$, 1, 2 and 4, and nowhere else. And that's exactly what I see in the diagram.

Now I need to know where y is positive and where it's negative.

Since y is a product, I can determine this if I know the signs of all the factors. It can be tedious working out the sign of each of the factors every time I have a new point but I can do the whole real line in one continuous swoop.

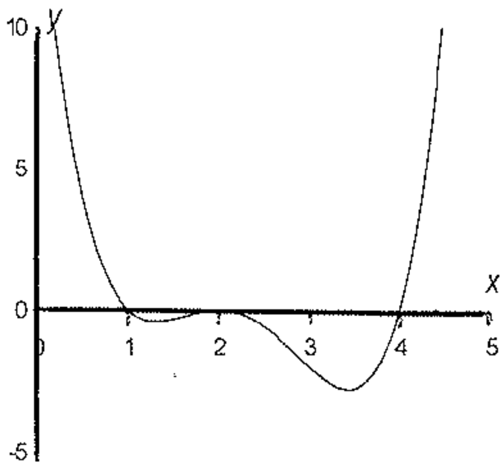
I am going to start with the value of x that is above 4 and let it decrease—sort of going from right to left along the x-axis.

- At the beginning (x above 4) all the terms are positive so y must be POSITIVE.
- As x passes below 4, the (x–4) factor changes sign, but nothing happens to the other factors, so the whole expression changes sign and y becomes NEGATIVE.
- As x passes below 2, the (x–2) factor changes sign, but nothing happens to the other factors, so the whole expression changes sign again and y becomes POSITIVE.
- As x passes below 1, the (x–1) factor changes sign, but nothing happens to the other factors, so the whole expression changes sign once more and y becomes NEGATIVE.
- Finally as x passes below 0, the (x) factor changes sign, but nothing happens to the other factors, so the whole expression changes sign yet again and y becomes POSITIVE.

If I was going to summarise my method I would say that I passed through the zeros one by one and recorded not the sign of y, but the change in the sign and I used that to update the sign.

Note: Sometimes the graph does not cross the x axis at a zero or a root, i.e. when y = 0, but rather it hits the axis and bounces back on the same side of the axis.

Example 2B: At x = 2 y= 0 so 2 is a “zero” or “root” of the equation but the graph doesn’t cross the x axis. It hits the axis and bounces back down again.



At the other zeros or roots 1 and 4, the graph of the polynomial crosses the x axis.
So what is the difference?

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how they explained the difference: Example 2B
I'm just looking at the graph and I am thinking.. OK..I see 3 roots or zeros: 1, 2 and 4. I prefer to call these zeros because they are x values that make y zero. Starting at the biggest one 4 I think: OK, if 4 is a root of the polynomial or a zero; Then (x-4) must be a factor of the polynomial ..when x is above 4 the factor is positive and so are the values of y and that is why the graph is above the x axis and Positive…then as x falls below 4 ..that's when I 'm going back from 4 towards zero the factor (x-4) will become negative and the product of factors which is y will be negative and the graph goes negative below the x axis. Then x gets to 2 and hits the axis because it is a zero. This means that (x-2) is a factor and as x goes below 2 the factor (x-2) becomes negative and y should become positive but it doesn't; it stays negative. So the only way that can be is if the factor is not (x-2) but $(x-2)^2$, then when x drops below 2 the factor (x-2) becomes negative but the factor $(x-2)^2$ becomes positive, and there is no change in the sign of y. It stays negative so the graph bounces back down again. Then when x gets to 1 y is zero so (x-1) is a factor and when x drops below 1 the factor (x-1) becomes negative and the sign of y changes again to be positive.

So it's like if you see the graph hitting the x axis but not crossing it, it means one of the factors must be squared (i.e. of degree 2)…I think it would probably be a bit like that if it hit and bounced back and sort of did a loop without hitting again it would mean the factor was cubed or degree 3.
I think I could write out the polynomial now. It would be $f(x) = (x-1)(x-2)^2(x-4)$.
Now if I was given the polynomial factorised I could sketch the graph.

Task 3:

Below are the algebraic equations of a number of polynomials in factored form. In each case make a rough sketch of the graph and explain the basis of your decisions. Do not plot points; simply identify the roots or zeros and observe whether the graph crosses or does not cross the axis at each of these.

Note: without plotting points, you won't really know how "high" the graph gets between the zeros, but that is not the focus of this exercise.

(a) $y = x(x-1)(x-2)(x-3)(x-5)$

(b) $y = (x+1)(x-1)(x-3)(x-4)^2$

(c) $y = x(x-1)^2(x-2)^2$

(d) $y = (x-1)(x-2)^2(x-3)(x-4)^3$

(e) $y = x^2(x^2-1)(x-4)^2$

(f) $y = (x^2-x)(x^2-1)(x+1)$

(g) $y = x(x-1)(x-2)^2(x-3)^3(x-4)^4$

NCCA

Task 4 LCHL

Verify that $x = a$ is always a solution of the equation:

$$x^4 - 2ax^3 + (2a^2-1)x^2 - a(a^2-1)x = 0$$

and use this fact to find all the roots of the equation in terms of a.

Compare, Examine, Discuss and Evaluate

LCHL

Verify that $x=a$ is always a solution of the equation:

$$x^4 - 2ax^3 + (2a^2-1)x^2 - a(a^2-1)x = 0$$

and use this fact to find all the roots of the equation in terms of a.

$$a^4 - 2a(a)^3 + (2a^2-1)(a)^2 - a(a^2-1)(a) = 0$$

$$a^4 - 2a^4 + 2a^4 - a^2 - a^4 + a^2 = 0$$

$$0 = 0$$

$$x\left(x^3 - 2ax^2 + (2a^2-1)x - a(a^2-1)\right) = 0$$

$x = 0$ $\qquad x^3 - 2ax^2 + (2a^2-1)x - a(a^2-1) = 0$

$x = a$ is a solution then $(x-a)$ is a factor

$$\begin{array}{r|l} & x^2 - ax + a^2 - 1 \\ x-a & x^3 - 2ax^2 + (2a^2-1)x - a(a^2-1) \\ & x^3 - x^2a \\ \hline & -ax^2 + (2a^2-1)x - a(a^2-1) \\ & -ax^2 - a^2x \\ \hline & (a^2-1)x - a(a^2-1) \\ & (a^2-1)x - a(a^2-1) \\ \hline & 0 \end{array}$$

$$x^2 - ax + (a^2-1)$$

$$x = \frac{+a \pm \sqrt{a^2 - 4a^2 + 4}}{2}$$

$$x = \frac{a + \sqrt{4-3a^2}}{2}, \quad \frac{a - \sqrt{4-3a^2}}{2}$$

So Roots are

$$x = 0, \quad x = a, \quad x = \frac{a + \sqrt{4-3a^2}}{2}, \quad x = \frac{a - \sqrt{4-3a^2}}{2}$$

Task 5 LCHL

Explain where the graph of the polynomial y= x(x-1)(x-2)(x-4)

- crosses the x axis
- Is above the x axis
- Is below the x axis

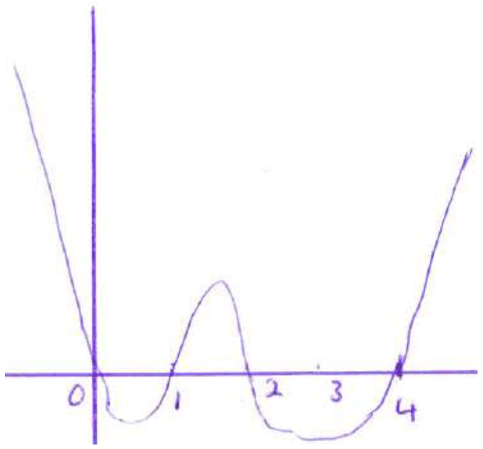
Sketch the graph of the polynomial.

Compare, Examine, Discuss and Evaluate

It crosses the x axis at x=0, 1, 2 and 4

because when
$x=0 \quad y=0(-1)(-2)(-4)=0$
$x=1 \quad y=1(0)(-1)(-3)=0$
$x=2 \quad y=2(1)(0)(-2)=0$
$x=4 \quad y=4(3)(2)(0)=0$

It is above the x axis when $x>4$
because the product of all the factors is positive so y is positive.
as x goes below 4 (x-4) factor becomes negative so product of factors is neg and the graph drops below x axis and hits axis again at x=2 then as x goes below 2 (x-2) factor becomes negative and the product becomes positive so y is positive and graph goes above the x axis again until x=1 and as x goes below 1 (x-1) factor becomes neg and so does the product so graph goes below x-axis again till x=0 below 0 x is neg so product will be +ve and graph will go above the x axis


0
1
2
3
4

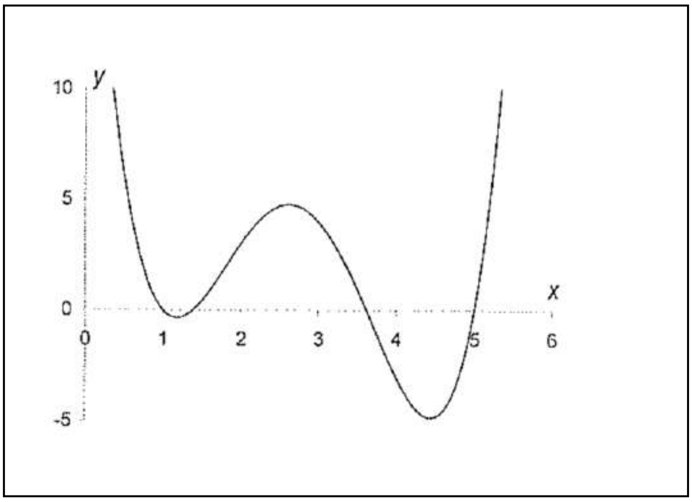
Task 6: LCHL

The graph of the polynomial is shown

$$y= x^4 -11x^3 +40x^2-55x+25$$

It cuts the x axis at x=1 and x=5 and two other places.

Find these two other intersections.



Compare, Examine, Discuss and Evaluate

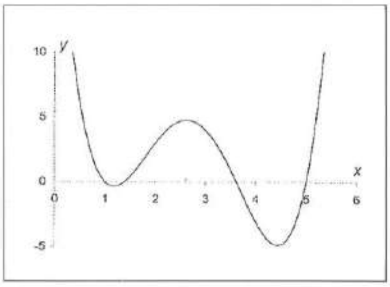
LCHL

The graph of the polynomial is shown

$$y= x^4 -11x^3 +40x^2-55x+25$$

It cuts the x axis at x=1 and x=5 and two other places.

Find these two other intersections.



So $(x-1)$ and $(x-5)$ are factors

$x-5 \overline{) x^4-11x^3+40x^2-55x+25}$ = $x^3-6x^2+10x-5$

x^4-5x^3

$-6x^3+40x^2-55x+25$

$-6x^3+30x^2$

$10x^2-55x+25$

$10x^2-50x$

$-5x+25$

$-5x+25$

0

$x-1 \overline{) x^3-6x^2+10x-5}$ = x^2-5x+5

x^3-x^2

$-5x^2+10x-5$

$-5x^2+5x$

$5x-5$

$5x-5$

0

x^2-5x+5

$x = \frac{5 \pm \sqrt{25-20}}{2}$

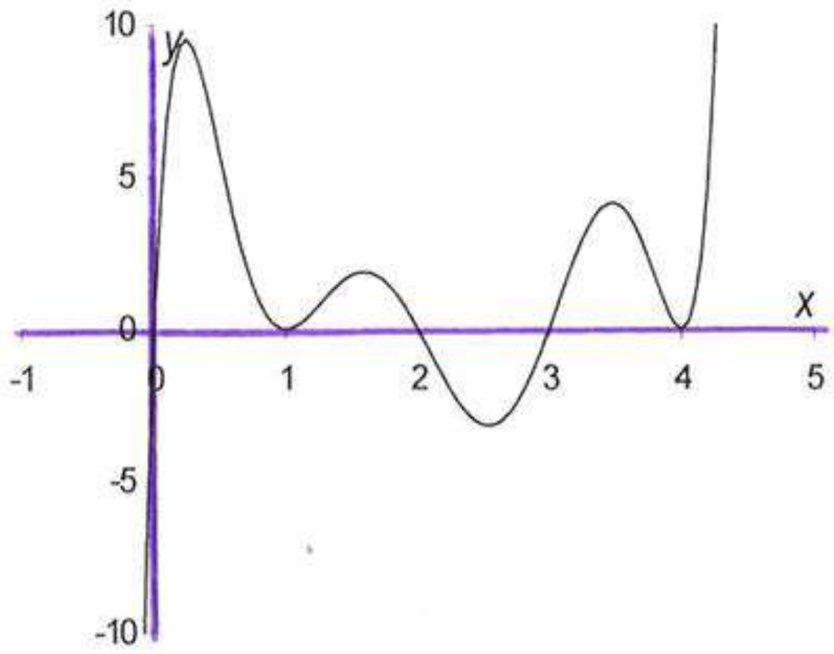
$= \frac{5 \pm \sqrt{5}}{2}$

Intersections

$x=1 \quad x=5 \quad x=1.38 \quad x=3.62$

Task 7 LCHL

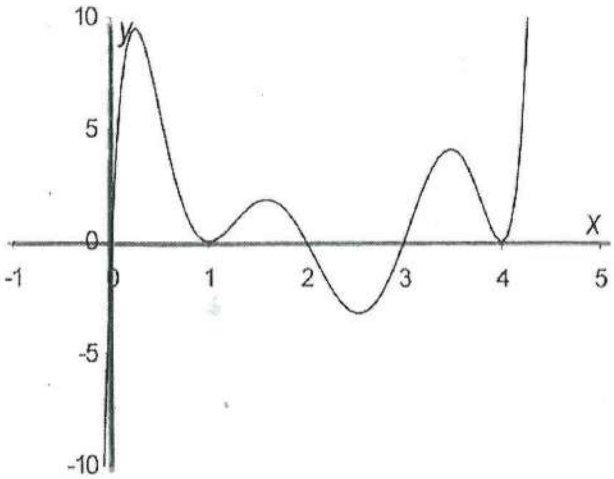
What is the algebraic equation of the polynomial whose graph is shown below? Explain your thinking.



Compare, Examine, Discuss and Evaluate

LCHL

What is the algebraic equation of the polynomial whose graph is shown below? Explain your thinking.



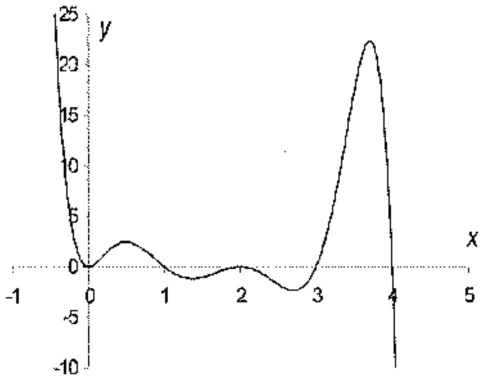
I can see from the graph that the roots or zeros of the polynomial are 0, 1, 2, 3 and 4 because this is where $y=0$ so that means x $(x-1)$ $(x-2)$ $(x-3)$ and $(x-4)$ are factors of the polynomial.

At $x=4$ the graph just hits the axis and doesn't cross it so this means the graph doesn't go from +ve to -ve it should if the factor is $(x-4)$ since this would be neg if $x<4$. So $(x-4)^2$ must be a factor and so must $(x-1)^2$ be a factor for the same reason that the graph just hits the x axis and doesn't cross it as x drops below 1.

So polynomial is $y = x(x-1)^2(x-2)(x-3)(x-4)^2$

Task 8 LCHL

The graph shown is that of a polynomial of degree 7. Find its equation and justify your thinking.

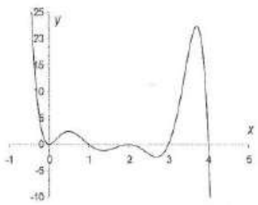


Check the vertical scale of your equation by using a couple of suitable values of x: x=0.5 or x=3.5

Compare, Examine, Discuss and Evaluate

LCHL

The graph shown is that of a polynomial of degree 7.Find it's equation and justify your thinking.



Check the vertical scale of your equation by plugging in a couple of suitable values of x ; x=0.5 or x=3.5

Well the zeros or roots are 0, 1, 2, 3 and 4

The graph crosses the x axis at 4, 3 and 1

So $(x-4)(x-3)(x-1)$ are all factors

Graph doesn't cross at $x=0$ or $x=2$ so x^2 and $(x-2)^2$ are factors

So $y = x^2(x-1)(x-2)^2(x-3)(x-4)$

But when $x > 4$ this y would be +ve but it's not its neg so then it must be

$$y = -x^2(x-1)(x-2)^2(x-3)(x-4)$$

$x = .5$

$y = -(.5)^2(-.5)(-1.5)^2(-2.5)(-3.5)$

$= 2.46$

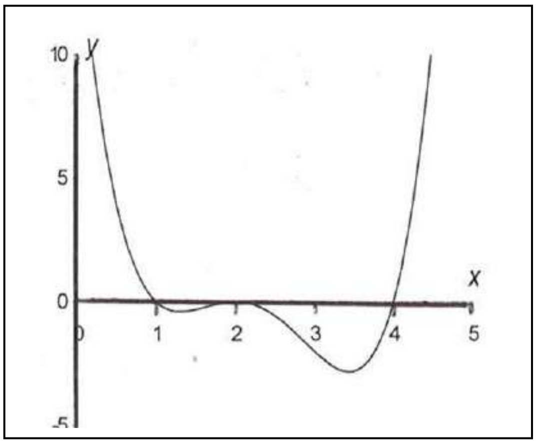
$x = 3.5$

$y = -(3.5)^2(2.5)(1.5)^2(.5)(-.5)$

$= 17.22$

Task 9 LCHL

The graph of the polynomial $y = x^4 - 9x^3 + 28x^2 - 36x + 16$ is shown below

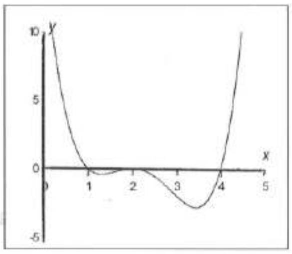


Write the polynomial as a product of its factors. Explain your thinking

Compare, Examine, Discuss and Evaluate

LCHL

The graph of the polynomial $y = x^4 - 9x^3 + 28x^2 - 36x + 16$ is shown below



Write the polynomial as a product of it's factors

Roots are $x = 1, 2, 4$

factors $(x-1)(x-2)(x-4)$

Since ~~equatio~~ the graph doesn't cut the x-axis at 2 $(x-2)^2$ is a factor

$y = (x-1)(x-2)^2(x-4)$

I will check this

$y = (x^2 - 5x + 4)(x^2 - 4x + 4)$

$= x^4 - 4x^3 + 4x^2 - 5x^3 + 20x^2 - 20x + 4x^2 - 16x + 16$

$= x^4 - 9x^3 + 28x^2 - 36x + 16$

Section C

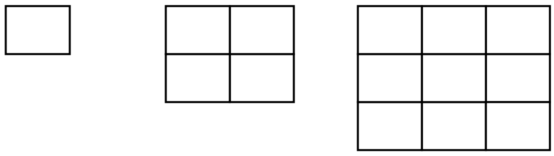
Exploration	Exploration, investigation and discussion. Strand 3, 4
Level	JCHL/ LCFL/OL
Learning outcome	Seeing patterns, making generalisations and giving explanations for them promotes a way of thinking that underpins school algebra. This activity provides you with the opportunity to develop your algebraic reasoning by engaging in problems that encourage you to – use tables to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence – use tables, diagrams and graphs as tools for representing and analysing patterns and relations – develop and use their own generalising strategies and ideas and consider those of others Throughout the activity you will be examining non-constant rates of change and LC learners will have the opportunity to make explicit the link between **linear functions** and **arithmetic sequences** and **exponential functions** and **geometric sequences**. These activities provide an ideal segue to the investigation of **series** and **logarithms.** The slope presentation will also help you as you work through these problems.

Prior knowledge: This activity introduces you to non-linear relations and provides you with the opportunity to further examine relationships between the representation, the table and the graph. It is essential that you have engaged with linear relationships before embarking on this work where you look in detail at the concept of "rate of change" and what this means in the different representations. Try the slope presentation now, it will help you reinforce your understanding of linear relationships.

Note to Students: The focus of the first four questions is on making tables and graphs and not formulas. There is an opportunity to work on formulas when you get to problem 5.

Exploration 1

Growing Squares: Extend this table for the number of tiles needed to make squares with side lengths 1,2,3….up to 10. Make observations about the values in the table. What would a graph for this situation look like? Will it be linear? How do you know? Make a graph to check your prediction.



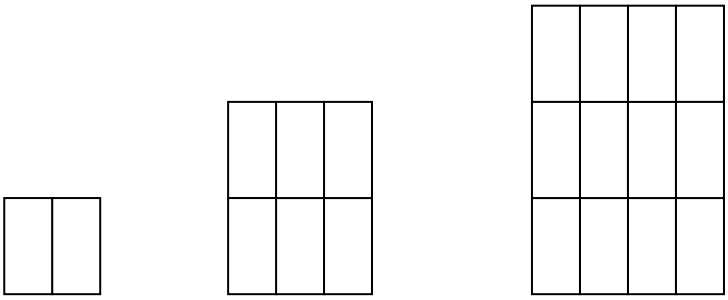
Side Length	Number of Tiles
1	1
2	4
3	9
4	16
5	25

Exploration 2

Growing Rectangles: Look at the pattern of growing rectangles below. Continue the table up to rectangles with a height of 10. Use drawings if you need to. Make observations about the values in the table

What would a graph for this table look like? Will it be linear? How do you know?

Make a graph to check your prediction.



Height	No of Tiles
1	2
2	6
3	12
4	?

Exploration 3

Pocket money story

I ask my Dad for pocket money. All I want, I say is for you to give me pocket money for this month. Give me 2 cent on the 1st day of the month, double that for the 2nd day, and double that for the 3rd day and so on. On the first day I will get 2 cents; on the 2nd day 4 cents; on the 3rd day 8 cents, etc .That is all I want.

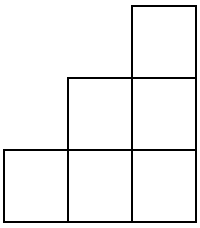
Make a table to show how much money I will get each day for the first 10 days of the month. Make observations about the values in the table. What would a graph look like? Would it be linear? How do you know? Make a graph to check your prediction.

Exploration 4

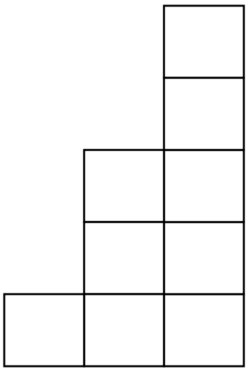
Staircase Towers: Look at the staircases below. Make a table expressing the relationship between the total number of tiles and the number of towers. Make observations about the values in the table. What would a graph look like? Would it be linear? How do you know?

Make a graph to check your prediction.

Number of Towers	Number of Tiles
1	1
2	3
3	6



No of Towers	No of Tiles
1	1
2	4
3	9



Exploration 5

In the problems in **exploration 1 – 4** you have looked at tables and graphs for the situations. Now work to express the relationships by finding an expression for the 50th or 100th case and then by finding an expression for the nth case.

As you work through the questions, think about the properties of linear relationships compared with non-linear ones.

- How are the situations different?
- How are the tables different?
- How are the graphs different?
- How are the generalised expressions different?

A distinguishing feature of quadratic and exponential relations is the way the changes vary. These are key ***ideas to arrive at.***

In a group discussion, compare the tables and graphs for ***growing rectangles***, ***staircase towers*** and the ***pocket money problem***.

Think about the following

- what is the same about the graphs and tables?
- what is different about the graphs and tables?
- how is the non-linear nature of the graph related to the tables and the context?

Focus on how the numbers change in each column; for example, for ***growing rectangles*** and ***staircase towers.***

Here are some student comments

"*the numbers grow in a particular way in the right column, but it's different from the left. They go up in 1 in the left and in **growing rectangles** they go up by 4 and 6 and 8, it's not the same but in a pattern, 2 extra each time. In **staircase towers** they go up in 2 and 3 and 4 – again, a pattern, 1 extra each time*."

Compare the increase in the right column for ***growing rectangles*** and the first ***staircase towers.*** This is a very important observation***;*** the **<u>change of the change</u>** increase by 1 each time in the first staircase towers and the **<u>change of the change</u>** increase by 2 each time in growing rectangles.

The rate of change of the changes therefore for ***growing rectangles*** is twice as fast as the rate of change of the changes for ***staircase towers***. Now focus on how this observation manifests itself in the graphs.

Once the connection between the changes of the changes has been made, analyse the tables for each of the relations and see how each is manifested in the table.

Refer to the concept of slope presentation.

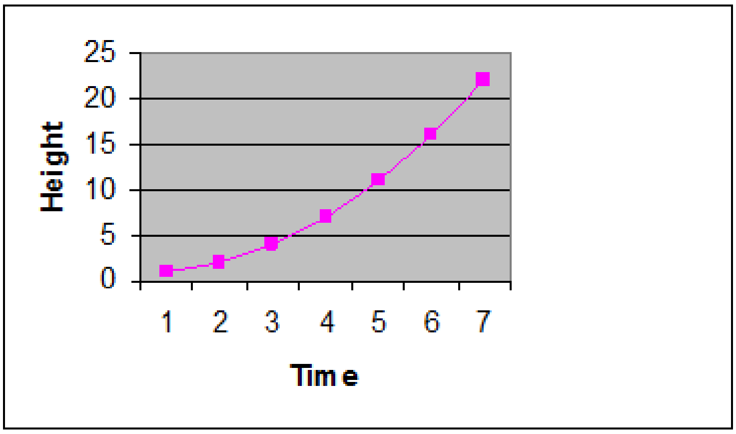
Exploration 6

Consider the case of a fantasy animal called Walkasaurus. The following table shows how the Walkasaurus's height changes with time.

Age (Years)	Height (cm)
0	1
1	2
2	4
3	7
4	11
5	16
6	22

Would a graph of this table be linear? How can you tell?

Now look at the graph.

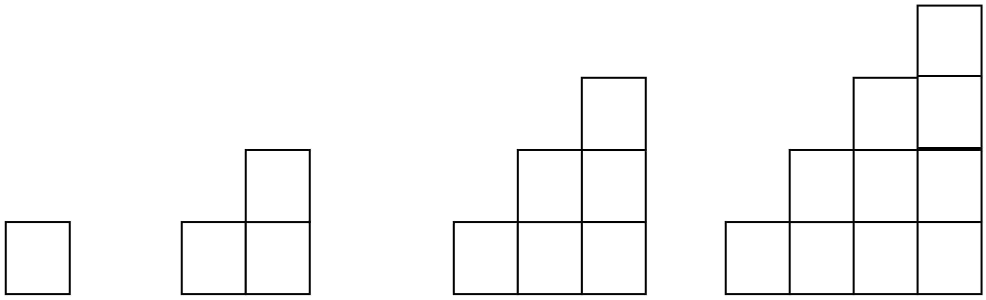


Explain why the graph looks like it does. Which of the problems in the previous *Growing Patterns* assignments is this similar to? How is it similar and how is it different?

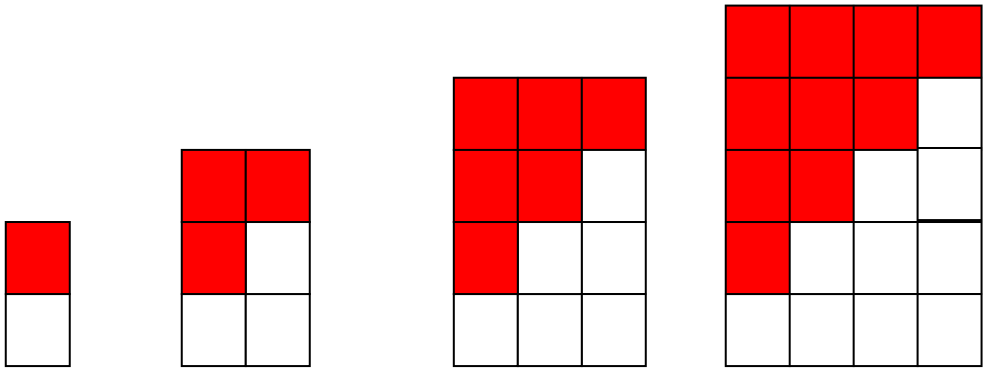
Note:

The contexts for the problems in this section have been carefully chosen to give relationships that can be compared. These comparisons can be used to help find the formula that define the relations.

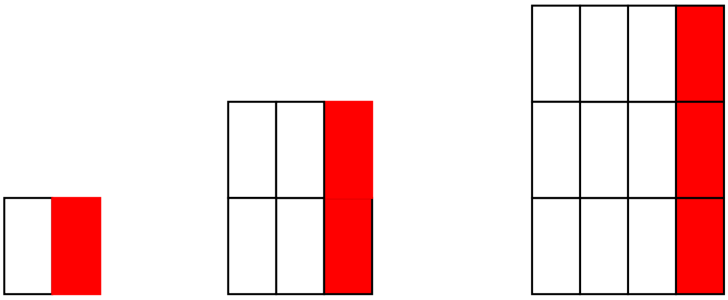
Consider looking for the formula for the staircase tower function.



Look at the following representation. It is double the original.



See that this is very similar to the growing rectangles situation.



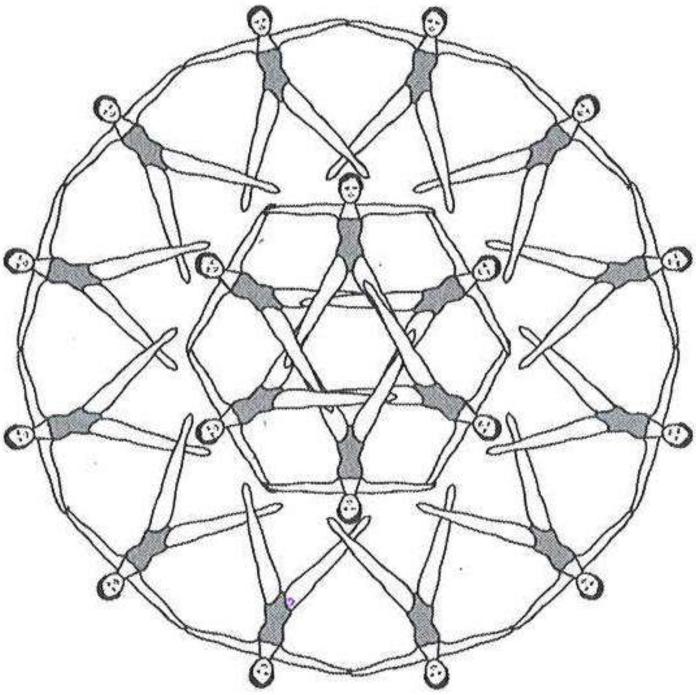
So, a connection can be made between the formulae.

Similarities can be seen between the table for Walkasaurus and that in the staircase tower problem in so far as the changes are similar. An important difference however is the fact that Walkasaurus' height begins at year 0 whereas the staircase tower table begins at 1. Try this!

Graphing these two on the same axis will make it easy for you to see that the curve is the same, just that one starts higher than the other. The formula for Walkasaurus therefore will be similar to that of the staircase tower but will need to be adjusted upwards by 1. Try this!

Task 1: LCFL

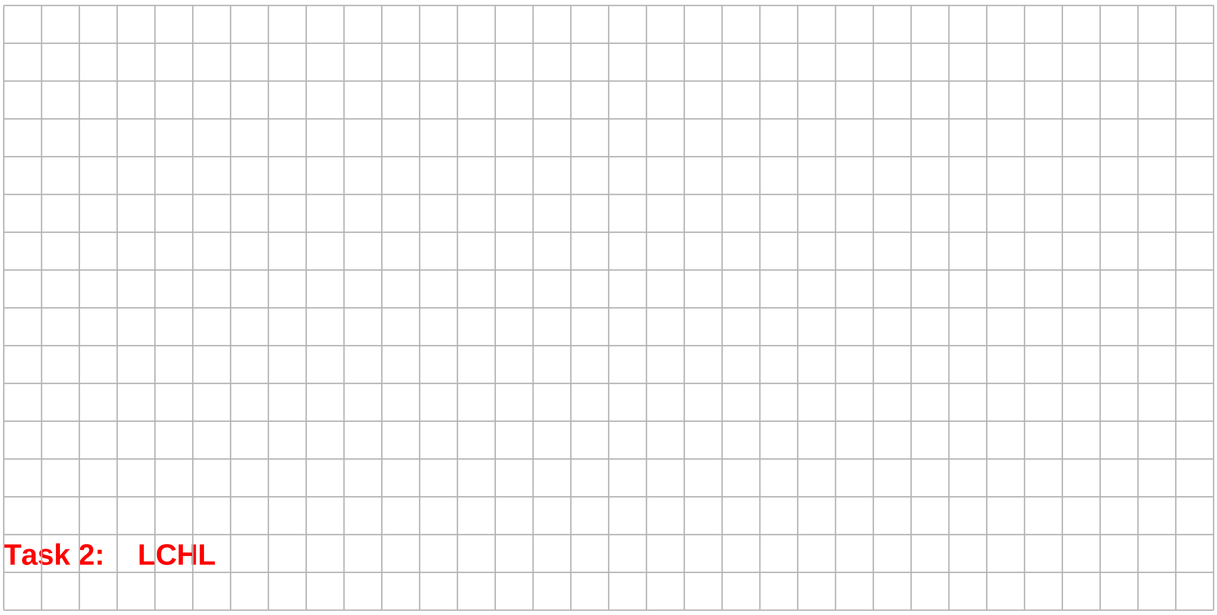
Sophie is planning for a synchronised swimming show. The diagram shows the pattern she wants to create**.**



How many more swimmers would she need if she were to add an extra ring to the pattern? Explain your thinking.

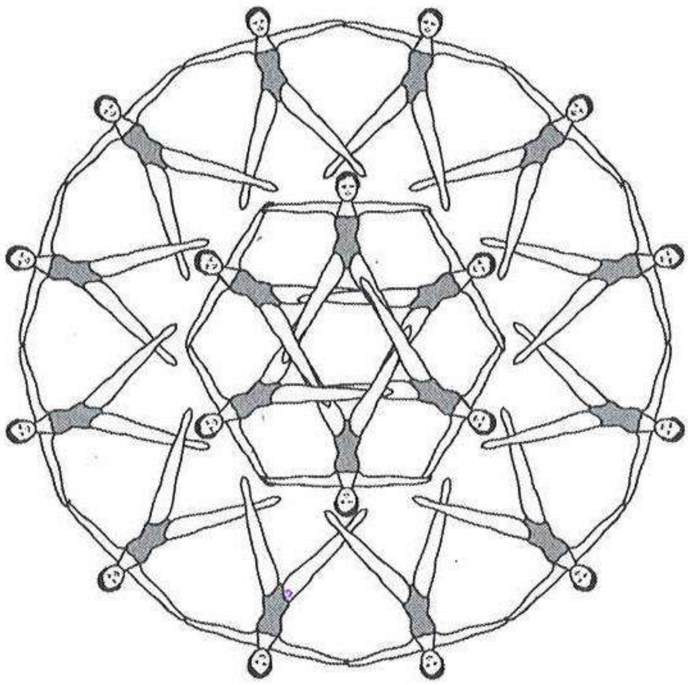
If she were to add a 4th ring to the pattern how many swimmers would be in this ring?

Explain the pattern in words.



Task 2: LCHL

Sophie is planning for a synchronised swimming show. The diagram shows the pattern she wants to create.



The number of swimmers in each ring forms a sequence. Is this sequence arithmetic or geometric? Explain your thinking.

How many swimmers would be in the 5^{th} ring?

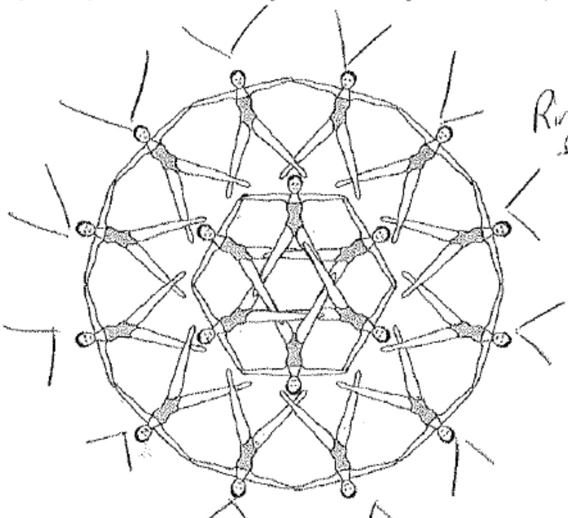
Generalise the pattern in algebraic form.

Sophie needs 6 rings in her pattern. How many swimmers does she need ?

Compare, Examine, Discuss and Evaluate

Task: LCHL

Sophie is planning for a synchronised swimming show. The diagram shows the pattern she wants to create.



Ring No	Amount of Swimmers		
1	6	6	6
2	12	6x2	6×2^1
3	24	6x4	6×2^2
4	48	6x8	6×2^3
5	96	6x16	6×2^4
n			$6\times2^{n-1}$

The number of swimmers in each ring forms a sequence. Is this sequence arithmetic or geometric? Explain your thinking.

Geometric because it doesn't increase by a constant amount 1st Ring has 6 2nd Ring has 12 next Ring will have 24 for every 1 you get 2 more swimmers

How many swimmers would be in the 5th ring? 96

Generalise the pattern in algebraic form.

No of swimmers in the nth Ring is $6(2)^{n-1}$

Sophie needs 6 rings in her pattern. How many swimmer does she need ?

$6 + 6(2) + 6(2)^2 + 6(2)^3 + 6(2)^4 + 6(2)^5$

$6 + 12 + 24 + 48 + 96 + 192$

278

11

What does this student understand? Evaluate the method. Are there any limitations? How would you know how many swimmers would be need for a pattern with 68 rings?

Task 3: LCOL

In a lecture theatre there are 28 seats in the first row, 29 seats in the 2nd row, 30 seats in the third row, and so on.

How many seats are in row 10 of the lecture theatre?

The pattern of seats forms a sequence. Is this an arithmetic sequence or a geometric one ? Explain your thinking.

If there are 70 seats in the last row of the lecture theatre, how many rows are in the theatre?

How many seats are there in total in the first 20 rows?

Compare, Examine, Discuss and Evaluate

Task: LCOL

In a lecture theatre there are 28 seats in the first row, 29 seats in the 2nd row, 30 seats in the third row, and so on.

How many seats are in row 10 of the lecture theatre?

Row No	Amount of Seats
1	28
2	29
3	30
4	31
5	32

+1 +1 +1 +1

28, 29, 30, 31, 32 . . . 70

0 1 2 3

No of Seats = 28 + 1(Row No)

The pattern of seats forms a sequence. Is this an arithmetic sequence or a geometric one ? Explain your thinking.

Arithmetic because it goes up by 1 each time

There are 70 seats in the last row of the lecture theatre. How many rows are there in the lecture theatre?

How many seats are there in total in the first 20 rows?

70 – 28 = 42

70, 69, 68, 67

28 + 1(No of Rows)

28 + 1(Row No)

12

421

28 + 1(42)

28 + 1(41)

So 70 in Row No 42 means 43 Rows cos I started at zero

28 + 29 + 30 + 31 + 32 = 2(60) + 30

33 + 34 + 35 + 36 + 37 = 2(70) + 35

38 + 39 + 40 + 41 + 42 = 2(80) + 40

43 + 44 + 45 + 46 + 47 = 2(90) + 45

120 + 140 + 160 + 180 + 150

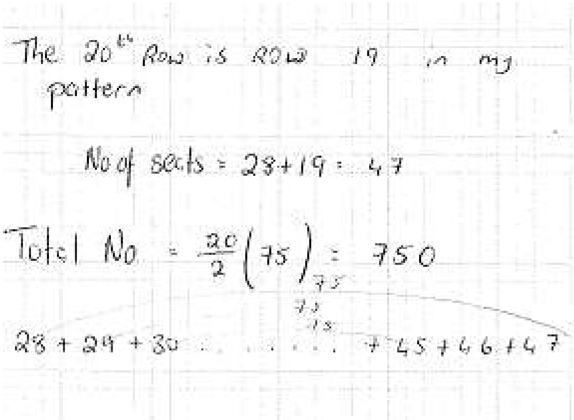
500 + 250

750

What does this student mean by the statement …"I started at zero."?

Examine this strategy can you extend it for larger numbers? Can you generalise it for any numbers?

Examine the work of this student as they answer the question '*How many seats are there in the first 20 rows of the lecture theatre?'*


The 20^{th} Row is Row 19 in my pattern

No of seats = 28+19 = 47

Total No $= \frac{20}{2}(75) = 750$

28 + 29 + 30 . . . . + 45 + 46 + 47

Task 4: LCOL

Jason examined the total in his savings account over a number of months and recorded the amounts in the table below.

Month	Amount (€)
January	20
February	40
March	80
April	160

(a) Predict how much will be in his account in a) June, b) August? Explain your prediction.

(b) Write a general expression describing Jason's pattern of savings.

(c) If Jason continues to save in this way when will he have €10,000 in his account?

Compare, Examine, Discuss and Evaluate

1	20
2	40
3	80
4	160

Differences: +20, +40, +80; differences of the differences: +20, +40

The relationship is exponential because the difference of the differences is not constant

The terms form a geometric sequence

I predict that for June which is the 6^{th} month there will be $2\times(2\times160)$ = €640 in his account because the amount in his account on the 6^{th} month will be double what was there on the 5^{th} month which will be double what was there on the 4^{th} month

August is the 8^{th} month so there will be $2\times(2\times640)$ = €2560 in the account

1	20		
2	40	20 × 2	20×2
3	80	20 × 4	20×2^2
4	160	20 × 8	20×2^3
⋮			
n			$20\times(2)^n$

$20(2)^n = 10{,}000$

$2^n = 500$

$2^9 = 512$ too big

$2^8 = 256$ too small

between August and September

I'd say closer to September

This student was confused. The generalised expression is incorrect. Can you see why? Correct the student's mistake.

Task 5: LCOL

You begin a biology experiment with 10 amoeba in a Petri dish and record the number of amoeba every minute in the table below

Time (mins)	No of amoeba
0	10
1	20
2	40
3	80

(a) How many amoeba will there be after 5 mins? Explain how you arrived at your answer.

(b) What type of relationship is there between the time in minutes and the number of amoeba in the Petri dish? Is it linear? How do you know? Explain your thinking.

(c) The terms produce a sequence; is this sequence **arithmetic** or **geometric**? How do you know?

(d) Write a general expression showing how the number of amoeba varies with time.

(e) How many amoeba will be in the dish after 1 hour?

Compare, Examine, Discuss and Evaluate

Time	No of amoeba
0	10
1	20
2	40
3	80

Differences: +10, +20, +40; differences of differences: +10, +20

I see a pattern It is double each time so after 5 mins the number will be

$2(2 \times 80) = 320$

b) The relationship is not linear because the difference is not constant each time it goes up by different amounts each time It is exponential because the difference of the difference is not constant.

c) geometric because the relationship is exponential and because each term is got by multiplying the previous term by a constant amount

d)

Time	No of amoeba		
0	10	10	10
1	20	10×2	10×2
2	40	10×4	10×2^2
3	80	10×8	10×2^3
:			:
n			$10(2)^n$

No of amoeba $= 10(2^t)$

e) No of amoeba $= 10(2^{60})$

=

Task 6: LCOL

Look at a calendar for this month. Examine the column that represents all the Thursdays in this month.

(a) What are the dates?

(b) What kind of sequence do these numbers represent? Explain how you know.

(c) If it is arithmetic, what is d, the common difference? If geometric, what is r, the common ratio?

(d) If that sequence continued, what would be the 100th term?

Task 7 LCOL Suppose that a, b, c, d….represent an arithmetic sequence. For each of the sequences below, indicate if it is arithmetic, geometric, or neither. Explain your reasoning in each case.

1) a + 2, b + 2, c + 2, d + 2…..
2) 2a, 2b, 2c, 2d…..
3) a^2, b^2, c^2, d^2……

Task 8 LCHL

Find x to make the sequence 10, 30, 2x + 8

a) arithmetic

b) geometric

Task 9 LCOL

Suppose a litre of petrol cost €1.20 in January, and the price goes up by 3% every month throughout the year.

(a) Find the cost of a litre of petrol, rounded to the nearest cent, for each month of the year.

(b) Is this sequence arithmetic, geometric, or neither?

(c) If it keeps going at this rate, how many months will it take to reach €10 per litre?

(d) How about €1000 per litre?

Task 10 LCOL

In an arithmetic sequence, each term is the previous term plus a constant. In a geometric sequence, each term is the previous term times a constant. Is it possible to have a sequence which is both arithmetic and geometric? Can you give an example?

Task 11 LCOL

A ball is dropped from a height of 3.0 m vertically on a horizontal surface. After each bounce it rises to 75% of its previous height.

(a) What height does the ball reach *after 6 bounces*?

(b) The terms form a sequence; what type of sequence is it? Explain your reasoning.

(c) Write an expression showing how the height of the ball varies with the number of bounces.

(d) After how many bounces does the ball reach a height of only 7 cm?

Compare, Examine, Discuss and Evaluate

No of bounces	Height
0	300
1	225
2	168.75
3	126.56
4	94.92
5	71.19
6	53.39

This is an exponential relationship the terms form a geometric sequence with common ratio (.75) because each term is found by multiplying the one before by (.75)

How accurate is this method? Thinking about the accuracy of different methods is very important, when evaluating different methods you should always consider this.

0	300
1	$300(.75)$
2	$300(.75)(.75)$
3	$300(.75)(.75)(.75)$
.	
.	
.	
n	$300(.75)^n$

Height $= 300(.75)^n$

n = no. of bounces

$7 = 300(.75)^n$

$\frac{7}{300} = (.75)^n$

$.023 = (.75)^n$

$(.75)^{13} = .023$

So after 13 bounces.

Task 12: LCOL

Sarah was investigating savings options

Option 1: Invest €1000 at 11% per year for a number of years.

Option 2: Deposit €1000 and add €100 each year.

Investigate how the money begins to grow for each option by filling in the table.

Time in years (n)	0	1	2	3	4	5	6
Amount (A) Option 1	€1000						
Amount (A) Option 2	€1000						

Sarah said "*the terms in option 1 produce a geometric sequence whilst the terms in option 2 produce an arithmetic sequence"*

(a) Examine these terms to see if Sarah is correct. Explain your reasoning

(b) Write an expression for the value of the *nth* term of each sequence (the general term)?

(c) Which investment option would you advise Sarah to go with? Explain your reasoning

Compare, Examine, Discuss and Evaluate

Option A

Time	Amount	Amount
0	1000	1000
1	1110	$1000(1{\cdot}11)$
2	1232·1	$1000(1{\cdot}11)(1{\cdot}11)$
3	1367·63	$1000(1{\cdot}11)(1{\cdot}11)(1{\cdot}11)$

$1110 = 1000 + {\cdot}11(1000)$

$= 1000(1 + {\cdot}11)$

$= 1000(1{\cdot}11)$

After t years

Amount $= 1000(1{\cdot}11)^t$

$1232{\cdot}1 = 1110 + {\cdot}11(1110)$

$= 1110(1 + {\cdot}11)$

$= 1110(1{\cdot}11)$

$= 1000(1{\cdot}11)(1{\cdot}11)$

terms make a geometric sequence with common ratio 1·11

$1367{\cdot}63 = 1232{\cdot}1 + {\cdot}11(1232{\cdot}1)$

$= 1232{\cdot}1(1 + {\cdot}11)$

$= 1232{\cdot}1(1{\cdot}11)$

$= 1000(1{\cdot}11)(1{\cdot}11)(1{\cdot}11$

Option B

Thomas

Time	Amount	
0	1000	
		+ 100
1	1100	
		+ 100
2	1200	

n $1000 + n \times 100$

Amount = 1000 + 100t

- 1000 → What you Start with
- 100 → Slope or how much it goes up by
- t → time in years

This is a linear Relationship because a constant is added each time.

The terms form an arithmetic sequence with Common difference = 100

Task 13: JCHL

(a) If a cent is dropped from a height of 45m, its height changes over time according to the formula: h=45 -4.9t^2, where t is measured in seconds. Use the mathematical tools (numerical analysis, tables and graphs) to determine how long it will take for the cent to hit the ground. What does the graph of height Vs time look like? What connections do you see between the graph and the table?

(b) Suppose you wanted the cent to land after exactly 4 seconds. From what height would you need to drop it? Explain how you figured this out.

(c) Suppose you dropped the cent from the top of the Eiffel Tower (300m). How long would it take to hit the ground? What does the graph of height Vs time look like? What connections do you see between the graph and the table?

(d) Suppose a machine tosses the coin vertically into the air so that the instant it leaves the machine it is travelling at 30m/s. The formula for height after t seconds is given by h=30t -4.9 t^2 +4.Why might that make sense? How long will it take to hit the ground? What does the graph of height Vs time look like? What connections do you see among the graph, the table, the situation, and the formula?

There is opportunity here to discuss the need for a more accurate approach to answering the question.

Would a graph have given a more accurate result? What about the equation?

Compare, Examine, Discuss and Evaluate

$h = 45 - 4.9t^2$

t	h
0	45
1	40.1
2	20.99
3	.9
4	-33.4

It will hit the ground between 3 and 4 secs after it starts to fall

The table shows the height falling each time it doesn't fall by a constant amount so the graph won't be linear it will be curved downwards because it is losing height but not at a constant rate

If it is to hit the ground after 4 secs Then

The height must be 0

$0 = x - 4.9(16)$

$x = 78.4m$

If you drop it from 300m Then $h = 300 - 4.9(t^2)$

Could you have worked this out from the table?

t	h
0	300
1	295
2	280
3	256
4	222
5	178
6	124
7	60
8	−13·6

It will hit
the ground
between 7 and 8
secs after it is
dropped.

Compare, Examine, Discuss and Evaluate

$h = \{30t\} - \{4.9t^2\} + \{4\}$

When the coin is tossed it goes up and gravity pulls it down too.

How far it goes up

How far it comes down

How high it is at the start

Altogether this is the height.

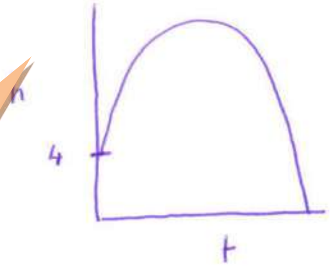
t	h
0	4
1	29.1
2	44.4
3	49.9
4	45.6
5	31.5
6	7.6
7	-26.1

increasing (t = 0 to 4)

decreasing (t = 5 to 7)

The height increase then decreases and hits the ground between 6 and 7 secs after it is tossed

The graph will be like



Note What would a more accurate approach to answering the question involve?

Would a graph have given a more accurate result? What about the equation?

Task 14: LCOL

A radioactive substance has a ***half-life*** of 20 minutes. That means after 20 minutes only half the substance remains. If a substance starts with 1000 grams, complete the table to show the amount of substance remaining at each of the times shown.

Time (minutes)	Half-Lives	Substance remaining (grams)
0	0	1000
20	1	
40		
60		
80		
100		

(a) What type of sequence do the terms produce? How do you know?

(b) Write an equation showing how many grams there are remaining after n half-lives.

(c) After n half-lives, how many minutes have gone by?

(d) Write an equation showing after t minutes how many half-lives have gone by.

Now put it all together. After t minutes, how many grams are there?

(e) Test that equation to see if it gives you the same result you found above after 100 minutes.

Predict what the graph will look like. Explain your thinking

Check your prediction

(f) How much substance will be left after 70 minutes?

(g) How much substance will be left after two hours?

(h) How long will it be before only one gram of the original substance remains?

Compare, Examine, Discuss and Evaluate

LCOL A radioactive substance has a ***half-life*** of 20 minutes. That means after 20 minutes only half the substance remains. If a substance starts with 1000 grams, complete the table to show the amount of substance remaining at each of the times shown.

Time (minutes)	Half-Lives	Substance remaining (grams)
0	0	1000
20	1	$\frac{1}{2}(1000) = 500$
40	2	$\frac{1}{2}\frac{1}{2}(1000) = \frac{1}{4}(1000) = 250$
60	3	$\frac{1}{2}\frac{1}{2}\frac{1}{2}(1000) = \frac{1}{8}(1000) = 125$
80	4	$\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}(1000) = \frac{1}{16}(1000) = 62.5$
100	5	$\frac{1}{32}(1000) = 31.25$

i) What type of sequence do the terms produce? How do you know?

Geometric because it doesn't go down by a constant amount

j) Write an equation to showing how many grams there are remaining after n half-lives.

$$\frac{1}{2^n}(1000)$$

k) After n half-lives, how many minutes have gone by?

$n \times 20$

l) Write an equation showing after t minutes (for instance, after 60 minutes, or 80 minutes) how many half-lives have gone by.

after 60 minutes $\frac{60}{20} = 3$ half lives
" 80 minutes $\frac{80}{20} = 4$ half lives
" t minutes $\frac{t}{20}$ half lives

Now put it all together. After t minutes, how many grams are there?

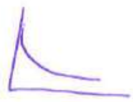
$\frac{1}{2^{\frac{t}{20}}}(1000)$ grams

m) Test that equation to see if it gives you the same result you gave above after 100 minutes.

after 100 mins $t = 100$

$\frac{1}{2^{\frac{100}{20}}}(1000) = \frac{1}{2^5}(1000) = \frac{1}{32}(1000)$
$= 31.25g$

Predict what the graph will look like. Explain your thinking

Curved downwards like this 

Check your prediction

n) How much substance will be left after 70 minutes?

o) How much substance will be left after two hours?

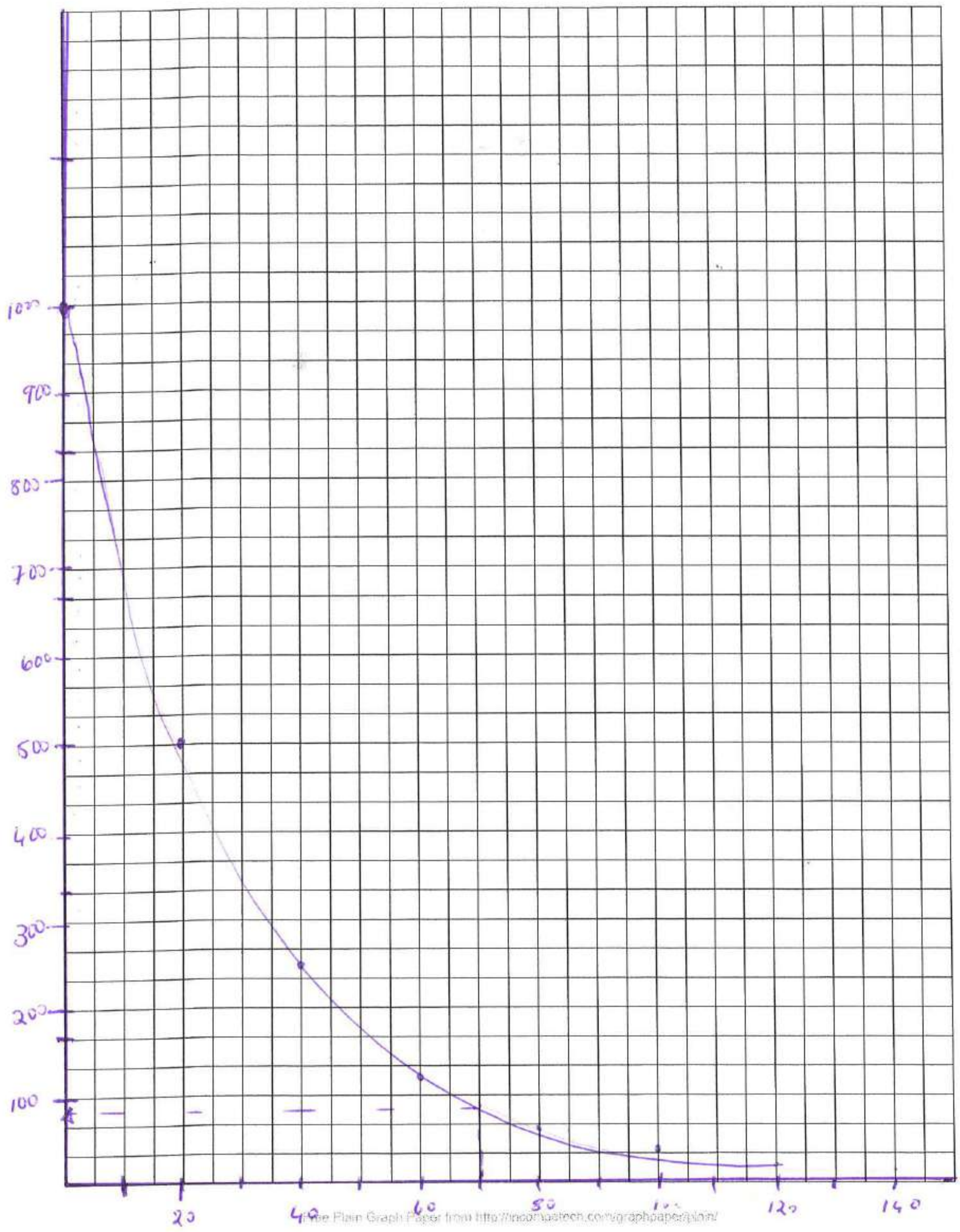
p) How long will it be before only one gram of the original substance remains?

after 70mins $t = 70$

$\frac{1}{2^{\frac{70}{20}}}(1000) =$

afte 2hrs $t = 120 = 6 \frac{1}{2}$ lifes

$\frac{1}{2^{\frac{120}{20}}}(1000) = \frac{1}{2^6}(1000) = 15.63g$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Free Plain Graph Paper from http://incompetech.com/graphpaper/plain/

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how they solved the radioactivity problem: Radioactivity 1

Well……when I started doing this question I just started filling in the table and I could see immediately that the relationship was not going to be linear because it was going down by different amounts each time ….first 500 then 250 …it was getting smaller each time. So I knew the terms couldn't form an arithmetic sequence,but they would form a geometric one.

Then I started looking at the grams remaining…. first one was half of a thousand …then half of half of a thousand and then half of half of half of a thousand and so on in a pattern that was a quarter of a thousand, one eight of a thousand and so on .It was easy really to see the relationship between the number of half lives and the grams remaining…..1 over 2 to the power of n times a thousand where n is the number of half lives

Then the question sort of made it easy for me to see the connection between the time and the grams remaining ....you just have to work out how many half lives have gone by in a given time….sometimes it's easy because it's a whole number of half lifes like in the table but other times it's not so easy ..like for 70 mins it's 3.5 half lives because that's 70 ÷ 20 = 3.5…..two hours was easy because that's just 120 ÷ 20 = 6 half lives…I had no problem predicting the graph because I knew it was an exponential relationship and that means the graph curves with the slope changing rapidly..I knew this time it would curve down because it was getting less and less. …If you draw the graph it's probably easier to find the grams remaining from that although it's probably not very accurate because it's hard to draw curves accurately on a graph….you just have to use a calculator ..once you get into exponential relationships the maths gets hard…it's all like powers and sometimes they're straightforward and sometimes they're not.

The last part is hard because it's like…you're given the grams and you have to find n. It's like we don't know how to do this we need to learn new maths. Because one over 2 to the n is equal to one over a thousand which means two to the n is equal to a thousand but like I can't do this because a thousand isn't two to the anything…..do you see what I mean? Like if you said two to the n is equal to 32 I would know that n is 5 because 32 is two to the 5..but a thousand isn't two to the anything…I can only do this by trying out numbers …so like two to the 8 is 256..(too small) two to the 9 is 512 (too small) ..two to the ten is one thousand and twenty four.(too big). So I know that n must be between 9 and 10; probably closer to 10. I could try two to the 9.6 that's 776.04 (too small); two to the 9.7 equals 831.75 (too small); two to the 9.9 equals 955.42 (too small); two to the 9.98 equals 1009.9 (too big); ..two to the 9.97 equals1.002 (too big); two to the 9.96 equals 995.99

(too small); so it must be between 9.96 and 9.97. This takes a long time and isn't really accurate We probably have to learn new maths to find a way to solve these types of questions.

The student who featured in the audio file has spoken about the need…." to find a way to solve these types of questions .." Logarithms were invented to facilitate difficult calculations; can logs help in this situation?

Well firstly, think about why the student thought the problem was difficult.

It was because, instead of going from time to amount, it asked her to go from amount to time. The question required her to invert the exponential function. This is called an ***inverse function***.

Exponential equations can be interpreted as questions.

$\sqrt{36}$ asks the question: What squared equals 36? The answer, of course, is 6.

Logarithmic functions are the inverse of exponential functions and they ask similar questions.

$\text{Log}_2 8 = x$ asks the question…. Two to what power equals 8 ?…The answer is 3.

The following question set in context gives you an indication of the usefulness of logs.

Task 15 LCHL

Sound is a wave in the air; the loudness of the sound is related to the intensity of the wave.

Type of sound	Intensity
Whisper	100
Background noise in a quiet rural area	1000
Normal conversation	1,000,000
Rock concert	1,000,000,000,000

Note: Try to place these points on a number line, and label them. Did you find this difficult? Why? It is because the range is huge, the function grows so quickly. Logarithmic scales are used when working with a function that, by itself, grows too quickly.

Sound volume is usually not measured in intensity, but in loudness, which is given by the formula:

$$L = 10 \log_{10} I$$

where L is the loudness (measured in decibels), and I is the intensity.

(a) What is the loudness, in decibels, of a whisper?

(b) What is the loudness, in decibels, of a rock concert?

(c) Now draw the number line again, labeling all the sounds; but this time, graph loudness instead of intensity.

(d) The quietest sound a human being can hear is intensity 1. What is the loudness of that sound?

(e) Sound intensity can never be negative, but it can be less than 1. What is the loudness of such inaudible sounds?

(f) If sound A is twenty decibels higher than sound B, how much more intense is it?

From Sequences to series

A series is essentially the sum of all the terms of a sequence. So we can have arithmetic series and geometric ones.

Note: A difficulty in dealing with series is getting to grips with the notation ($T_{n,}$, S_n, a, r, d); spend some time finding out about using notation. You can get help in textbooks, on the internet or by asking your teacher.
Now use the notation to describe each finite series below:

(a) 6 + 7 + 8 + 9 + 10
(b) -6 - 7 - 8 - 9 -10
(c) 6 + 8 + 10 + 12 + 14
(d) 6 + 12 + 24 + 48
(e) 6 - 7 + 8 - 9 + 10
(f) All the even numbers between 50 and 100.

Sequences can be finite or infinite. So too can series.

Applications of series

In economics, **geometric series** are used to represent the present value of an annuity (a sum of money to be paid in regular intervals).

For example, suppose that you expect to receive a payment of €100 once per year for an indefinitely long time. Receiving €100 a year from now is worth less to you than an immediate €100, because you cannot invest the money until you receive it. In particular, the present value of a €100 one year in the future is €100 / (1 +yearly interest rate)

Similarly, a payment of €100 two years in the future has a present value of €100 / (1 + yearly interest rate)^2 – squared because it would have received the yearly interest twice. Therefore, the present value of receiving €100 per year for an indefinitely long time can be expressed as an infinite series because the payment is being made for an indefinite length of time.

$$\frac{100}{1+I} + \frac{100}{(1+I)^2} + \frac{100}{(1+I)^3} + \frac{100}{(1+I)^4} + \cdots$$

Investigate this.
Have a look at the transcript from the audio file “Investment 1”

This is a geometric series with common ratio 1 / (1 + *I*). The sum is

$$\frac{a}{1-r} = \frac{100/(1+I)}{1-1/(1+I)} = \frac{100}{I}.$$

For example, if the yearly interest rate is 10% (***I*** = 0.10), then the entire annuity has a present value of €1000.

This sort of calculation is used to compute the **APR** of a loan (such as a **mortgage** loan). It can also be used to estimate the present value of expected **stock dividends**, or the **terminal value** of a **security**. You may find financial terminology intimidating use Google to help you make sense of it.

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how they solved the investment problem: Investment 1

Well…what we did was to look at €100 and what it would become 3, 4 or 5 years from now if we invested it at, say, 5% interest.

So we made a table…it's always useful to do this because you can see patterns easily.

Time	Amount
0	100
1	100+5/100(100)= 100(1+.05)
2	100(1+.05) +.05(100(1+.05) =100(1+.05)(1+.05)
3	100(1+.05)(1+.05)(1+.05)

So I see the pattern it is an exponential relationship and the terms form a geometric sequence with common ratio (1+.05). That means to find the amount for each year you multiply the previous year by (1+.05).

So then we started to think about the €100 we are going to get each year.

The present value of €100 one year in the future is 100/ (1+.05)

Then the present value of €100 two years in the future is $100/(1+.05)^2$

This payment goes on indefinitely so the present value forms an infinite series with common ratio

1/(1+.05)… we can get the sum of this series by using the formula from the tables.

a/1-r = 100/(1+.05) divided by 1 -1/(1+.05)…2000 is what the whole annuity is worth now….. that is its **present value.**

We can generalise this for any value and any interest rate.

LCHL

You have won a lottery that pays €1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Note: This annuity is different from the last example because there is a finite (20 years) sequence of payments which form a **finite** series. The reasoning will be the same as the above except the sum of the finite series is found using the formula

$$\frac{a(r^n - 1)}{r - 1}$$

The "Formulae and tables" book states this formula in a different way. Are they both the same? Why?

Look on this questions as a situation where two people have won the same lottery; John and Susan.

John, a young student is happy with his €1000 monthly payment, but Susan, a slightly older lady wants to have the entire amount now. Your job is to determine how much Susan should get.

Work out how much Susan would get if she invested this lump sum at 8% per annum compounded monthly. You should find it interesting that in 20 years her lump sum should be worth €568, 999.07. Have a look through the next transcript from the audio file.

It's a good idea to look for other contexts; look in the papers, go online to banking websites and create your own questions.

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how she solved the investment problem: Investment 2 (John and Susan)

Well; let's say Susan accepts x euro, then that x euro deposited at 8% for 20 years should yield the same amount as the €1,000 monthly payments for 20 years. So what we are doing really is comparing the future values for both Susan and John, and these future values have to be equal.

Now since Susan is getting a lump sum of x euro, its future value is given by

$$x(1+(\sqrt[12]{1.08}-1))^{240}$$

$$x(\sqrt[12]{1.08})^{240}$$

and John is getting a sequence of payments, or an annuity, of €1,000 per month; its future value is given by

$1000 + 1000(1 + \sqrt[12]{1.08} - 1) + 1000(1 + \sqrt[12]{1.08} - 1)^2 + \ldots\ldots$

So you can see that $a = 1000$, $n = 240$ $r = \sqrt[12]{1.08}$

The sum of 240 of these payments is

$$\frac{1000[(\sqrt[12]{1.08})^{240} - 1]}{\sqrt[12]{1.08} - 1}$$

Susan will only agree to the amount she gets if these two future values are equal. So then, all we do is make them equal and solve the equation for x

$$\frac{1000[(\sqrt[12]{1.08})^{240} - 1]}{\sqrt[12]{1.08} - 1} = x(\sqrt[12]{1.08})^{240}$$

x = €122,077.73

See how much this amount would be worth after 20 years at the given rate.

Applications of sequences and series tasks

Task 15 LCHL

Jack won a lottery; he has been given a choice of two options:

> **Option A:** Receive an annuity of €1500, each month for 25 years. An annuity is a sum of money to be paid in regular intervals.
>
> **Option B:** Take the present value of the annuity (based on an annual growth rate of 10%)

Jack decides to take Option **B** and invests it himself in an account that pays 9% compounded monthly for 20 years.

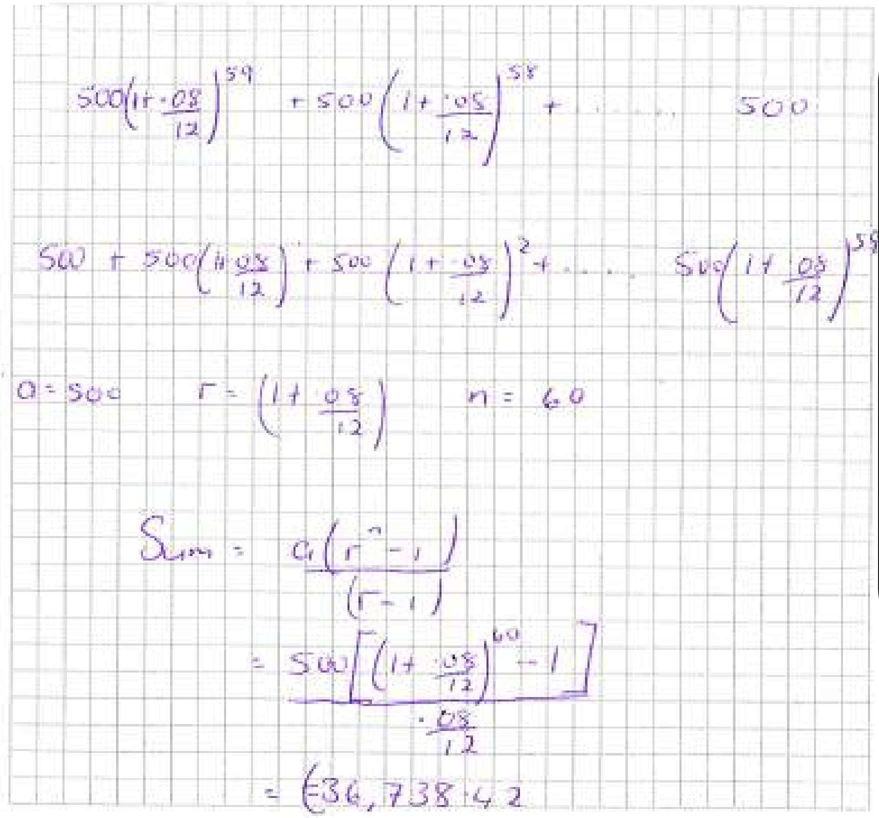
What is the present value of the annuity?

How much will Jack's investment have amounted to after the 20 years?

Task 16 LCHL

At the end of each month a deposit of €500 is made in an account that pays 8% per annum compounded monthly. What will the final amount be after 5 years?

A student calculated this as follows:



$500\left(1+\frac{.08}{12}\right)^{59} + 500\left(1+\frac{.08}{12}\right)^{58} + \ldots\ldots 500$

$500 + 500\left(1+\frac{.08}{12}\right) + 500\left(1+\frac{.08}{12}\right)^{2} + \ldots\ldots 500\left(1+\frac{.08}{12}\right)^{59}$

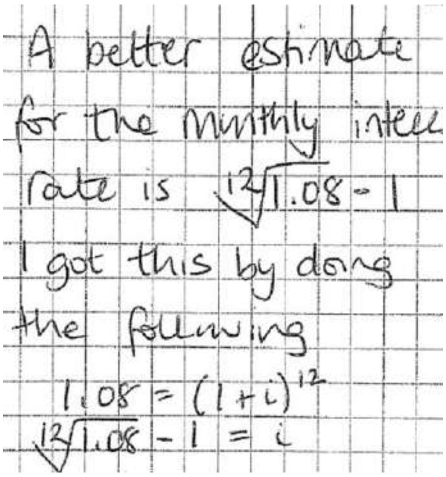
$a = 500 \quad r = \left(1+\frac{.08}{12}\right) \quad n = 60$

$Sum = \frac{a(r^n - 1)}{(r-1)}$

$= \frac{500\left[\left(1+\frac{.08}{12}\right)^{60} - 1\right]}{\frac{.08}{12}}$

$= €36,738.42$

Another student argues:



A better estimate for the monthly interest rate is $\sqrt[12]{1.08} - 1$

I got this by doing the following

$1.08 = (1+i)^{12}$

$\sqrt[12]{1.08} - 1 = i$

What do you think?

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how he solved the problem: Investment 3 (Task 16)

Well, you see***....***There are 60 deposits made in this account…because 5 times 12 is 60. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.

I thought OK… how much will the first €500 accumulate to at the end of the 59 months…… I worked this out by saying it will be €500 + the interest but this will be 8% x 500 / 12 because the interest is compounded monthly. The first payment of €500 will accumulate to an amount of €500$(1 + .08/12)^{59}$. But then another student said that because the interest is compounded monthly you can't just divide by twelve; they calculated the interest rate to be $\sqrt[12]{1.08}-1$. So then I used this rate.

The second payment of €500 will accumulate to an amount of €500$(1 + (\sqrt[12]{1.08}-1))^{58}$.

The third payment will accumulate to €500$(1 + (\sqrt[12]{1.08}-1))^{57}$.

And so on. It's just a geometric sequence.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

So in other words, I need to find the sum of the following series.

€500$(1 +(\sqrt[12]{1.08}-1))^{59}$ + €500$(1 +(\sqrt[12]{1.08}-1))^{58}$ + €500$(1 +(\sqrt[12]{1.08}-1))^{57}$ + . . . + €500

I write that backwards because it's easier to see

€500 + €500$(1 +(\sqrt[12]{1.08}-1))$+ €500$(1 +(\sqrt[12]{1.08}-1))^2$ + . . . + €500$(1 +(\sqrt[12]{1.08}-1))^{59}$

This is a geometric series with ***a*** = €500, ***r*** = $(1 +(\sqrt[12]{1.08}-1))$, and ***n*** = 60 Therefore the sum is

$$\frac{a(r^n-1)}{r-1}$$

$$\frac{500((1+\sqrt[12]{1.08}-1)^{60}-1)}{\sqrt[12]{1.08}-1}$$

Answer = €36, 471.70

Task 17 LCHL

Sonya deposits €300 at the end of each quarter in her savings account. If the account earns 5.75% per annum compound interest how much will she have in 4 years?

Compare, Examine, Discuss and Evaluate

4 years ⇒ 16 deposits

$(1.0575) = (1+i)^4$

$\sqrt[4]{1.0575} = 1 + i$

$i = \sqrt[4]{1.0575} - 1$

$300 + 300\left(1+\sqrt[4]{1.0575}-1\right) + 300\left(1+\sqrt[4]{1.0575}-1\right)^2 + \dots 300\left(1+\sqrt[4]{1.0575}-1\right)^{15}$

$a = 300 \quad r = \sqrt[4]{1.0575} \quad n = 16$

$S = \dfrac{300\left[\left(\sqrt[4]{1.0575}\right)^{16} - 1\right]}{\sqrt[4]{1.0575} - 1} = €5370.19$

Task 18 LCHL

Robert needs €5000 in three years. How much should he deposit at the end of each month in an account that pays 8% per annum in order to achieve his goal?

LCHL

Q. The New Horizons computer company needs to raise money to expand. It issues a 10-year €1,000 bond that pays €30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how he solved the problem.

I wasn't sure what a bond certificate was so I Googled it and basically a bond certificate promises two things – an amount of €1,000 to be paid in 10 years, and a semi-annual payment of €30 for ten years. So, to find the fair market value of the bond, I need to find the present value of the lump sum of €1,000 that you'd receive in 10 years, as well as the present value of the €30 semi-annual payments for the 10 years.

I calculated the present value of the lump sum €1,000 as follows:

$$x\ (1 + (\sqrt{1.07} - 1))^{20} = €1{,}000$$

$$x(1.96715) = €1{,}000$$

$$x = €508.35$$

The present value of the €30 semi-annual payments is $30\left[\dfrac{(1 + (\sqrt{1.07} - 1))^{20} - 1}{\sqrt{1.07} - 1}\right]$

Hence, $$x\ (1 + (\sqrt{1.07} - 1))^{20} = 30\left[\dfrac{(1 + (\sqrt{1.07} - 1))^{20} - 1}{\sqrt{1.07} - 1}\right]$$

$$x = €428.67$$

So

The present value of the lump sum €1,000 = €508.35

The present value of the €30 semi-annual payments = €428.67

Therefore, the fair market value of the bond is €508.35+ €428.67= €937.02

Task 19 LCHL

Mr. Mooney bought his house in 1975, and financed the loan for 30 years at an annual interest rate of 9.8% compounded monthly. His monthly payment was €1260. In 1995, Mr. Mooney decided to pay off the loan. Find the balance of the loan he still owed at that time.

Compare, Examine, Discuss and Evaluate

This is a transcript of a student reflecting on how she solved the investment problem: Investment 4

When I looked at this problem first I thought…I can't answer this because I don't know how much he borrowed to start with…so I even asked my teacher was this information missing from the question…she said no you don't need to know this to work it out so I thought… over the course of the loan Mr Mooney committed to make (30 x12 = 360) 360 payments of €1260. By deciding to pay off the loan in 1995 he still has 120 more payments to make so I thought the present value of these instalments is what the bank should charge him.

So I set about finding the present value of these payments, by putting the two amounts equal.
I used this formula

$$X\left(1+\left(\sqrt[12]{1.098}-1\right)\right)^{120} = \frac{1260\left[\left(1+\sqrt[12]{1.098}-1\right)^{120}-1\right]}{\sqrt[12]{1.098}-1}$$

The left hand side shows the future amount he would owe if he paid the lump sum X, which is its present value. The right hand side shows how much the bank will get if he continues to make the monthly payments.

They must be equal. Thus X = €97,863.68

Making Sense of Financial Mathematics

There are a number of references to financial mathematics in the Leaving Certificate mathematics syllabus

LCFL Strand 3: *solve problems involving*
· finding compound interest
· finding depreciation (reducing balance method)
LCOL Strand 3: *perform calculations involving formulae for compound interest and depreciation (reducing balance method)*
LCHL Strand 3: *use present value when solving problems involving loan repayments and investments*

Perhaps the most insightful learning outcome of all is the following which appears at LCHL Strand 3:

solve problems involving finite and infinite geometric series including applications such as recurring decimals and financial applications, e.g. deriving the formula for a mortgage repayment

Yes, you've got it, financial mathematics of this type is simply an application of geometric patterns.

Consider the concept of ***Compound Interest*** by imagining that you have just found out the following startling news: on the day you were born your godfather, Uncle Nicholas, deposited €5000 in your name in a **trust fund** that pays 6% **APR.**
One of the provisions of the trust fund was that you couldn't touch the money until you turned 18. You are now 18 years 10 months old and you are wondering

- How much money is in the trust fund now?
- How much money will be in the trust fund if you wait until your next birthday to cash it in?
- How much money would be in the trust fund if you left the money there until you retire at age 60?

Represent the money that is in your account in a table starting with the day you were born

Time		Amount of money
Day you were born	0	
1^{st} Birthday	1	
2^{nd} Birthday	2	
3^{rd} Birthday	3	
4^{th} Birthday	4	
5^{th} Birthday	5	
6^{th} Birthday	6	

Can you see a pattern in your completed table?

Does this pattern represent an arithmetic sequence or a geometric one? How do you know?

Can you gerneralise the pattern so that you can find out how much will be in the trust fund on your 60^{th} birthday without extending the table?

Look at the tables A and B below

A

Time		Amount of Mon
Day you were born	0	5000
1st Bday	1	5300
2nd Bday	2	5618
3rd Bday	3	5955.08
4th Bday	4	6312.38

B

Time		Amount of Money
Day you were born	0	5000
1st Bday	1	1.06(5000)
2nd Bday	2	1.06(1.06)(5000)
3rd Bday	3	1.06(1.06)(1.06)(5000)
4th Bday	4	1.06(1.06)(1.06)(1.06)(5000)

What are the differences between the two tables? What are the similarities between them?

Which table makes it easier to generalise the pattern? Why do you think this is the case?

Generalise the pattern. Is it the same for both tables? How do you know?

In the mathematical tables on page 30 there is a formula for calculating compound Interest.

It reads

$$F = P(1+i)^{t}$$

Where F = the final value

P = the Principal

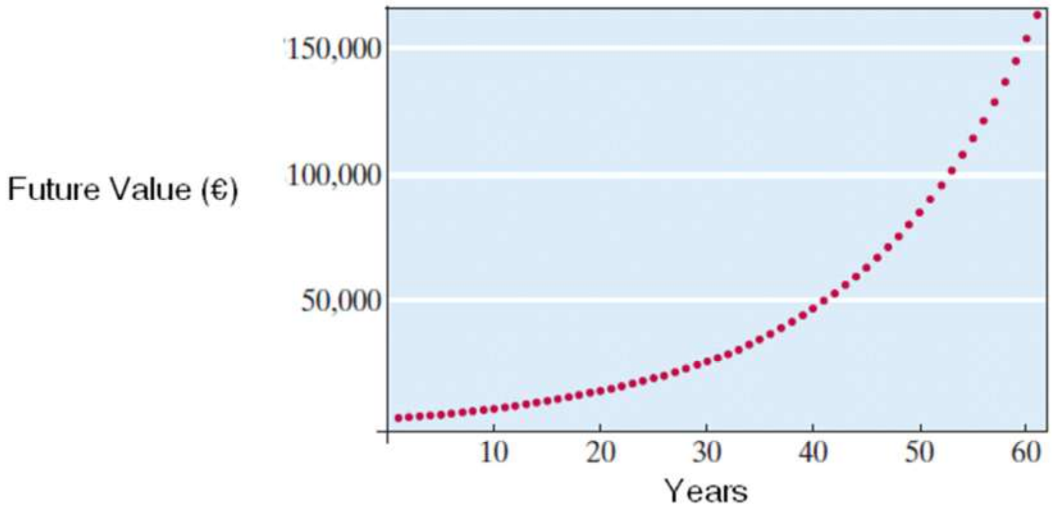
i = the interest rate (annual, or for the relevant period)

t = the time (in years, or other equal intervals of time)

Compare this formula with the generalised expression for the pattern of money in the trust fund.

Are they similar? Can you see the 'F' in your table, ?in your generalised formula? What about the 'P', the 'i', and the 't'?

Below is a graph showing the future value of the trust fund. Are you surprised at the shape of the graph? Why, or why not? Can you find each variable on the graph?



Note that the process of calculating the value of a depreciating asset (reducing balance method) is similar to the process of calculating the value of an interest gaining fixed deposit account, except that the formula has a changed sign:

$$F = P(1 - i)^{t}$$

Can you make out two tables, similar to tables A and B above, showing the value of an asset which cost €5000 if it depreciates each year at a rate of 10%? What is the pattern in this case? Which table makes it easier to generalise the pattern?

Comment on the statement

The future value of an investment at any time t is the nth term of a geometric sequence where P is the initial value of the investment, (1 + i /100) is the common ratio and n is the number of years.

Think …what if Uncle Nicholas had put money into a bank account regularly for you. How could you work out how much it would be worth after a period of time?

Suppose at the start of each year he deposited €500 in an account that pays 3% interest. What will the final amount be after 18 years?

Think about the first €500 he puts in; how many years will it be there for?

The second €500? The third €500 ? for how many years will they be in the account?

How much interest will the first €500 earn? The second €500? The last €500 ?

Ist €500 is in the account for 18 years so will be worth **F = 500(1+0.03)^{18} €851.22**

2nd €500 is in the account for 17 years so will be worth **F = 500(1+0.03)^{17} €826.42**

3rd €500 is in the account for 16 years so will be worth **F = 500(1+0.03)^{16} €802.35**

Can you see the pattern? Is it increasing or decreasing? Why? What will the smallest number be? Why?

So, to calculate how much is in there after 18 years add them all up…or, yes, get the sum of the sequence.

851.22+826.42+803.35+……………………………………………………..+515 =

The tables give you the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$

What is ‘n’ in this situation? What is ‘a’? what is ‘r’?

Now can you answer the question: How much would the money Uncle Nicholas put in the account each year amount to after 18 years?

You might be wondering whether you would have more money if the bank added the interest every month? Try it out and see.

It's really the same situation, except now you have to think about what the interest rate would be if for example the interest was compounded every month instead of every year.

Again it's the same idea: $(1+ i)^{12}$ is now going to be the same as 1.03…provided the 'i' is the monthly rate Why is this? What will be the value of 'n' if a monthly rate is used?

Investigate how much money would be in the account if Uncle Nicholas's bank had compounded the interest monthly.

Now, how would this change if they compounded it daily? hourly? How would the relationship $(1+i)^{12} =1.03$ change? How would the number of terms in the sequence change?

How would things change if the interest was added at the beginning of the month as opposed to the end of the month?

These are all things that you must consider when answering questions about investments and loans. Sometimes, if the question is not very specific, you may have to make assumptions; in other cases, the question may be very specific and you won't need to make any assumptions. Where you do make assumptions, it is advisable to make these clear.

Now, work through the financial maths questions provided. Check the solutions to see if you are working correctly, or to get an insight into the thinking involved (by clicking on the thought bubble).

LCOL

A sum of €3000 is invested in a five-year government bond with an annual equivalent rate (AER) of 3%. Find the value of the investment when it matures in five years' time.

A different investment bond gives 15% interest after 6 years. Calculate the AER for this bond.

LCOL

A machine costing €25,650 depreciates to a scrap value of €500 in 10 years. Calculate:

(i) the annual rate of depreciation.

(ii) The value of the machine at the end of the sixth year.

LCOL

A firm estimates that office equipment depreciates in value by 40% in its first year of use. During the second year it depreciates by 25% of its value at the beginning of that year. Thereafter, for each year, it depreciates by 10% of its value at the beginning of the year

Calculate:

(i) the value after eight years of equipment costing €550 new.

(ii) the value when new of equipment valued at €100 after five years of use.

LCHL Q. Eamon and Sile have just had their first child, Donal. They are planning for his education in eighteen years' time. First, they calculate how much they would like to have in the education fund when Donal is eighteen. Then, they calculate how much they need to invest in order to achieve this. They assume that, in the long run, money can be invested at an inflation-adjusted annual rate of 2%. Your answers throughout this question should therefore be based on a 2% annual growth rate.

(a) Write down the present value of a future payment of €5,000 in one years' time.

(b) Write down, in terms of t, the present value of a future payment of €5,000 in t years' time.

(c) Eamon and Sile want to have a fund that could, from the date of his eighteenth birthday, give Donal a payment of €5,000 at the start of each year for 5 years. Show how to use the sum of a geometric series to calculate the value on the date of his eighteenth birthday of the fund required.

(d) Eamon and Sile plan to invest a fixed amount of money every month in order to generate the fund calculated in part (c). Donal's eighteenth birthday is 18 x 12 = 216 months away.

(i) Find, correct to four significant figures, the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate of 2%.

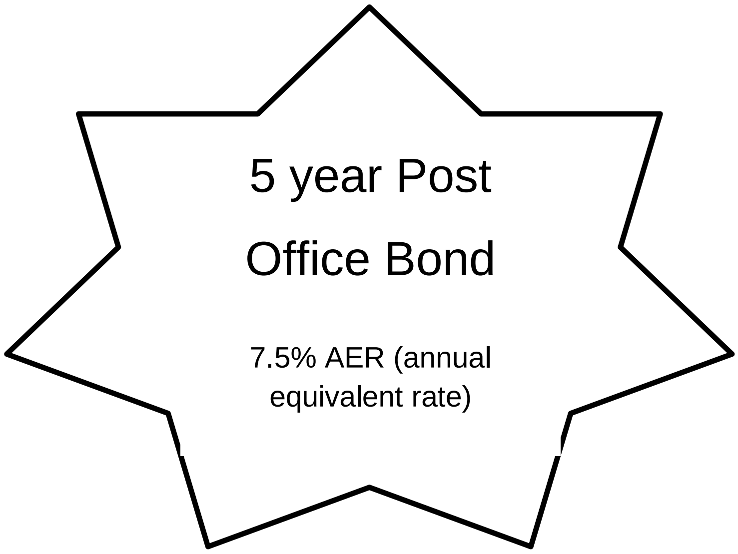
(ii) Write down, in terms of n and P, the value on the maturity date of an education plan of €P made n months before that date.

(iii) If Eamon and Sile make 216 equal monthly payments of €P from now until Donal's eighteenth birthday, what value of P will give the fund he requires?

(e) If Eamon and Sile wait for ten years before starting Donal’s education fund, how much will they then have to pay each month in order to generate the same education fund?

LCFL

Alison has €6,000 to invest; she sees the following advert for a post office bond.



(i) If she invests her €6,000, what will the value of her investment be when it matures in five years' time.

(ii) How much interest will she have made?

(iii) Express her interest as a percentage of her original investment.

LCHL

Carl sees a car on a car dealer's website. He clicks on the section saying "Finance this car" and finds out how much it would cost him to borrow €5,000.

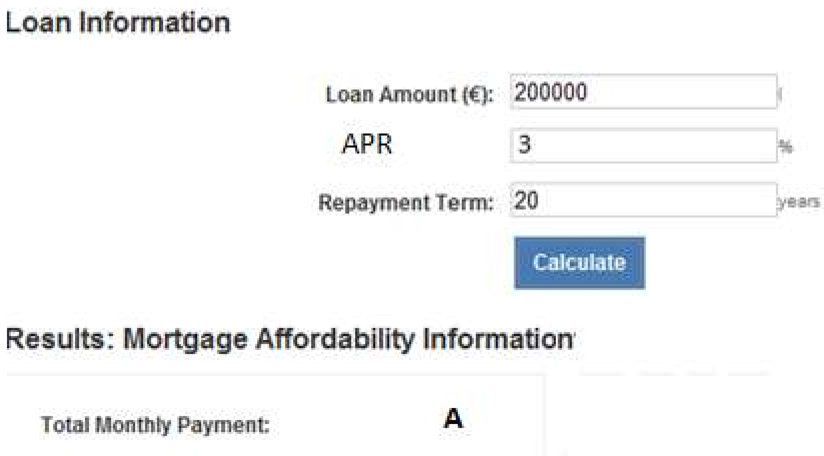
The details are shown in the table:

Fixed Rates				
Over 5 years	APR	Monthly Repayment	Total Repayable	Total Cost of Credit
€5000	14.9%	A	B	C

(i) Calculate the values **A** and **B** and **C.**

LCHL

John purchases a house, which is financed with a 20 year loan of €200,000 at a rate of 3% APR. On the property website where he saw the ad for the house, the mortgage calculator showed the following repayments:



(a) Find the value of the monthly repayment **A.**

LCHL

Tom wants to buy a car in three years' time. He estimates the car will cost €10,000 and so he decides to put a certain amount of money into a special savings deposit account that pays 4% AER compounded monthly. If this is to give him €10,000 in three years' time how much would he need to save each month?

LCHL

Mary deposits €500 each month in her savings account. If the account earns 2.5% AER compounded monthly, how much will she have in 5 years?

LCHL

Sarah has €30,000 in a deposit account that pays 6% AER. Sarah believes that after 15 years this investment will be worth treble its present value. Is this true? If not, find the amount of years correct to the nearest year for this investment to treble.

LCFL

€1500 is invested for 5 years in a savings account which pays compound interest at a rate of 5% per annum provided the €1500 is left invested over the five-year period.

(i) How much money, to the nearest euro, will be in the savings account at the end of the five years if no money is withdrawn from the account?

(ii) If the interest is withdrawn at the end of each year, but the €1500 is left invested, what will be the difference in the total interest earned on the account over the five years?

LCFL

Mary takes out a loan of €7500 from her credit union and agrees to pay back €2000 per year until the loan and interest is paid off. At the end of each year, before the repayments are made, the credit union charges interest at the rate of 7% per annum on the outstanding amount at the start of that year.

(i) Complete the table below showing the balance owed at the start of each year and the amount of interest (to the nearest euro) charged at the end of that year. Assume that the €2000 is paid each year as planned.

Start of Year	**Balance owed**	**Interest due**	**Total Due**
1	€7500	€525	€8025
2	€6025	€422	
3	€4447		
4			
5			

LCOL

The National Treasury Management Agency (NTMA) offers a National Solidarity Bond which earns cumulative interest of 1% for each year the bond is kept. This interest is taxable at 27%. If the bond is kept for at least 5 years, a tax-free lump sum is also paid, as shown in the table

10 Year National Solidarity Bond

At the end of Year	Cumulative 1% Annual Interest[1]		Bonus Tax Free Lump Sum
1	1%	+	0%
2	2%	+	0%
3	3%	+	0%
4	4%	+	0%
5	5%	+	10%
6	6%	+	10%
7	7%	+	22%
8	8%	+	22%
9	9%	+	22%
10	10%	+	40%

(i) What will be the value of a €1000 bond if it is cashed in after 4 years, and the cumulative interest is taxed at 27%?

(ii) €1000 is invested in this bond and the bond is cashed in at the end of the 10 years. If the cumulative interest is taxed at the rate of 27% and the tax-free bonus sum is paid as shown, what is the value of the bond after the 10 years

LCOL

A sum of €3000 is invested in a five-year government bond with an annual equivalent rate (AER) of 3%. Find the value of the investment when it matures in five years' time.

$$F = P(1+i)^t$$

$$F = 3000(1+\cdot 03)^5$$

$$F = €3477\cdot 82$$

A different investment bond gives 15% interest after 6 years. Calculate the AER for this bond.

$$1\cdot 15 = (1+x)^6$$

$$\sqrt[6]{1\cdot 15} = 1+x$$

$$1\cdot 0236 = 1+x$$

$$x = \cdot 0236$$

$$AER = 2\cdot 4\ \%$$

LCOL

A machine cost €25,650 depreciates to a scrap value of €500 in 10 years. Calculate:

(i) the annual rate of depreciation.

$F = P(1-i)^t$

$500 = 25650(1-x)^{10}$

$\frac{500}{25650} = (1-x)^{10}$

$0{\cdot}6745 = 1 - x$

$x = 0{\cdot}32549$

Rate $= 32{\cdot}5\%$

(ii) The value of the machine at the end of the sixth year.

$F = P(1-i)^t$

$x = 25650(1-{\cdot}32549)^6$

$x = €2415{\cdot}56$

$25650(1-{\cdot}325)^6$

$2426{\cdot}10731$

$x = €2426.11$

NB accept either calculation for full credit

LCOL

A firm estimates that office equipment depreciates in value by 40% in its first year of use. During the second year it depreciates by 25% of its value at the beginning of that year. Thereafter, for each year, it depreciates by 10% of its value at the beginning of the year

Calculate:

(i) the value after eight years of equipment costing €550 new.

$F_A = P(1-i)^t$

$F_A = 550(1-0.4)^1$

$= €330$

$\Rightarrow$

$F_B = 330(1-0.25)^1$

$= €247.50$

$F_C = 247.50(1-0.1)^6$

$= 131.53$

Ans €131.53

(ii) the value when new of equipment valued at €100 after five years of use.

$F_A = P(1-0.4)^1$

$= P(0.6)$

$F_B = [P(0.6)](1-0.25)$

$= P(0.6)(0.75)$

$F_C = [P(0.6)(0.75)](1-0.1)^3$

$100 = P(0.6)(0.75)(0.729)$

$100 = P(0.32805)$

$\frac{100}{0.32805} = P \rightarrow$ Ans €304.83

LCHL Q. Eamon and Sile have just had their first child, Donal. They are planning for his education in eighteen years' time. First, they calculate how much they would like to have in the education fund when Donal is eighteen. Then, they calculate how much they need to invest in order to achieve this. They assume that, in the long run, money can be invested at an inflation-adjusted annual rate of 2%. Your answers throughout this question should therefore be based on a 2% annual growth rate.

(a) Write down the present value of a future payment of €5,000 in one years' time.

$$P = \frac{5000}{1 \cdot 02}$$

$$= €4,901 \cdot 96$$

(b) Write down, in terms of *t*, the present value of a future payment of €5,000 in *t* years' time.

$$P = \frac{5000}{(1 \cdot 02)^t}$$

(c) Eamon and Sile want to have a fund that could, from the date of his eighteenth birthday, give Donal a payment of €5,000 at the start of each year for 5 years. Show how to use the sum of a geometric series to calculate the value on the date of his eighteenth birthday of the fund required.

Present value of each €5000

$$\frac{5000}{(1.02)^0} + \frac{5000}{(1.02)^1} + \frac{5000}{(1.02)^2} + \frac{5000}{(1.02)^3} + \frac{5000}{(1.02)^4}$$

5000   4901.96   4805.84   4711.61   4619.23

total = €24038.64

(d) Eamon and Sile plan to invest a fixed amount of money every month in order to generate the fund calculated in part (c). Donal's eighteenth birthday is 18 x 12 = 216 months away.

(i) Find, correct to four significant figures, the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate of 2%.

$$1.02 = (1+i)^{12} \qquad (1.02)^{\frac{1}{12}} = 1+i$$

$$(1.02)^{\frac{1}{12}} - 1 = i \rightarrow 0.0016515 \qquad \text{Ans } 0.1652\%$$

(ii) Write down, in terms of *n* and *P*, the value on the maturity date of an education plan of €*P* made *n* months before that date.

$$F = P(1+i)^n = P(1+0.001652)^n$$

(iii) If Eamon and Sile make 216 equal monthly payments of €*P* from now until Donal's eighteenth birthday, what value of *P* will give the fund he requires?

$$€24038.64 = P(1+i)^1 + P(1+i)^2 + \dots + P(1+i)^{216}$$

$$= P(1.001652)^1 + P(1.001652)^2 + \dots + P(1.001652)^{216}$$

$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{with } a = P(1.001652)$$

$$r = 1.001652$$

$$n = 216$$

$$S_{216} = \frac{P(1.001652)(1.001652^{216}-1)}{(1.001652-1)} = 24038.64$$

$$P[.42908] = 39.71183$$

$$P = €92.55$$

(e) If Eamon and Sile wait for ten years before starting Donal's education fund, how much will they then have to pay each month in order to generate the same education fund?

$$24038.64 = P(1+i)^1 + P(1+i)^2 + \dots + P(1+i)^{96}$$

$$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{with} \quad a = P(1 + 0.001652)$$

$$r = 1.001652$$

$$n = 96$$

$$S_{96} = \frac{P(1.001652)(1.001652^{96} - 1)}{(1.001652 - 1)} = 24038.64$$

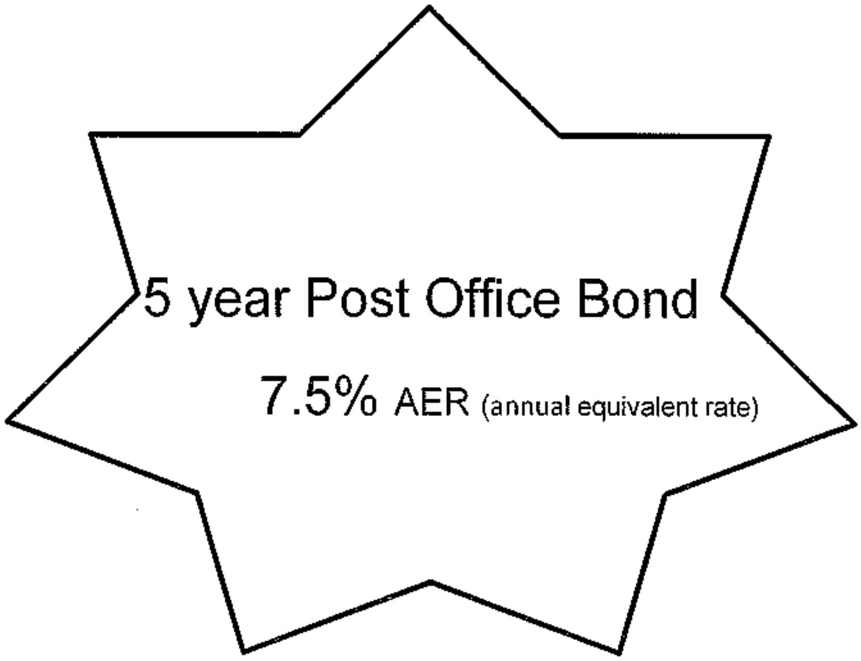
$$P[0.17199] = 39.71183$$

$$P = 230.8961$$

$$P = €230.90$$

LCFL

Alison has €6,000 to invest; she sees the following advert for a post office bond.



(i) If she invests her €6,000, what will the value of her investment be when it matures in five years' time.

Year 1 : 6,000 x 1.075 = 6450
Year 2 : 6450 x 1.075 = 6933.75
Year 3 : 6933.75 x 1.075 = ~~7~~7453.78
Year 4 : 7453.78 x 1.075 = 8012.81
Year 5 : 8012.81 x 1.075 = €8613.78

(ii) How much interest will she have made?

8613.78 − 6,000 = €2613.78

(iii) Express her interest as a percentage of her original investment.

$\frac{2613.78}{6000} = 43.6\%$

LCHL

Carl sees a car on a car dealer's website. He clicks on the section saying "Finance this car" and finds out how much it would cost him to borrow €5,000.

The details are shown in the table:

Fixed Rates

Over 5 years	APR	Monthly Repayment	Total Repayable	Total Cost of Credit
€5000	14.9%	A	B	C

(i) Calculate the values **A** and **B** and **C.**

Monthly Rate → $\sqrt[12]{1.149} = 1.011641575$

Present value of all Repayments = €5000

$$5000 = \frac{A}{\sqrt[12]{1.149}} + \frac{A}{\left(\sqrt[12]{1.149}\right)^2} + \dots + \frac{A}{\left(\sqrt[12]{1.149}\right)^{60}}$$

$$= S_n \text{ of G.P. with } a = \frac{A}{\sqrt[12]{1.149}}; \ r = \frac{1}{\sqrt[12]{1.149}}; \ n = 60$$

$$\therefore 5000 = \frac{\frac{A}{\sqrt[12]{1.149}}\left(1 - \left[\frac{1}{\sqrt[12]{1.149}}\right]^{60}\right)}{1 - \frac{1}{\sqrt[12]{1.149}}}$$

OR use the formula on p. 31 for amortization with $i = \sqrt[12]{1.149} - 1$, $P = 5000$, $t = 60$

€116.26 = A

B = 116.26 × 60 = €6975.60

C = €6976.50 − €5000 = €1975.60

LCHL

John purchases a house, which is financed with a 20 year loan of €200,000 at a rate of 3% APR. On the property website where he saw the ad for the house, the mortgage calculator showed the following repayments:

Loan Information

Loan Amount (€): 200000

APR 3

Repayment Term: 20

Calculate

Results: Mortgage Affordability Information

Total Monthly Payment: **A**

(a) Find the value of the monthly repayment **A.**

Monthly interest rate $= \sqrt[12]{1.03} = 1.00246627$

$i = 0.00246627 \quad P = 200{,}000 \quad t = 240$

$$A = 200{,}000 \left[\frac{0.00246627(1.00246627)^{240}}{(1.00246627)^{240} - 1} \right]$$

$$A = \frac{0.004454358}{0.806111333}(200{,}000)$$

$A = €1105.15$

LCHL

Tom wants to buy a car in three years' time. He estimates the car will cost €10,000 and so he decides to put a certain amount of money into a special savings deposit account that pays 4% AER compounded monthly. If this is to give him €10,000 in three years' time how much would he need to save each month?

Present value of €10,000

$$P = \frac{F}{(1+i)^t} \qquad P = \frac{10{,}000}{(1\cdot04)^3}$$

$$P = €8889\cdot93$$

Monthly rate $= \sqrt[12]{1.04}$

$$A = \frac{8889.93\,(1\cdot0032737 4)^{36}(0\cdot00327374)}{(1\cdot00327374)^{36} - 1}$$

A = €262·18 every month.

LCHL

Mary deposits €500 each month in her savings account. If the account earns 2.5% AER compounded monthly, how much will she have in 5 years?

monthly interest rate : $(1.025)^{\frac{1}{12}} = 1.002059836$

5 yrs ⇒ 60 months $\qquad S_n = \frac{a(r^n - 1)}{r-1}$

$$S_{60} = 500(1.002059836)^1 + 500(1.002059836)^2 + \dots + \dots 500(1.002059836)^{60}$$

$$= \frac{500(1.002059836)\left[1.002059836^{60} - 1\right]}{0.002059836}$$

$= 31963.44$

Ans €31963.44

LCHL

Sarah has €30,000 in a deposit account that pays 6% AER. Sarah believes that after 15 years this investment will be worth treble its present value. Is this true? If not, find the amount of years correct to the nearest year for this investment to treble.

$$30,000(1.06)^{15} = €71,896.75$$

No this is not true.

$$30,000 = \frac{90,000}{(1.06)^t}$$

$$30000(1.06)^t = 90,000$$

$$(1.06)^t = 3$$

$$t = \frac{\log 3}{\log 1.06}$$

$$t = 18.5$$

$t = 19$ years.

LCFL

€1500 is invested for 5 years in a savings account which pays compound interest at a rate of 5% per annum provided the €1500 is left invested over the five-year period.

(i) How much money, to the nearest euro, will be in the savings account at the end of the five years if no money is withdrawn from the account?

(ii) If the interest is withdrawn at the end of each year, but the €1500 is left invested, what will be the difference in the total interest earned on the account over the five years?

(i) $A = 1500 \times (1.05)^5 = 1914.42$ Ans €1914

(ii) Interest $= 1500 \times 0.05 = 75$

$75 \times 5 = 375$

$1914 - 1500 = 414$

$414 - 375 = 39$ Ans €39

LCFL

Mary takes out a loan of €7500 from her credit union and agrees to pay back €2000 per year until the loan and interest is paid off. At the end of each year, before the repayments are made, the credit union charges interest at the rate of 7% per annum on the outstanding amount at the start of that year.

(i) Complete the table below showing the balance owed at the start of each year and the amount of interest (to the nearest euro) charged at the end of that year. Assume that the €2000 is paid each year as planned.

Start of Year	Balance owed	Interest due	Total Due
1	€7500	€525	€8025
2	€6025	€422	€6447
3	€4447	€311	€4758
4	€2758	€193	€2951
5	€951	€67	€1018

LCOL

The National Treasury Management Agency (NTMA) offers a National Solidarity Bond which earns cumulative interest of 1% for each year the bond is kept. This interest is taxable at 27%. If the bond is kept for at least 5 years, a tax-free lump sum is also paid, as shown in the table

10 Year National Solidarity Bond

At the end of Year	Cumulative 1% Annual Interest[1]		Bonus Tax Free Lump Sum
1	1%	+	0%
2	2%	+	0%
3	3%	+	0%
4	4%	+	0%
5	5%	+	10%
6	6%	+	10%
7	7%	+	22%
8	8%	+	22%
9	9%	+	22%
10	10%	+	40%

(i) What will be the value of a €1000 bond if it is cashed in after 4 years, and the cumulative interest is taxed at 27%?

4% = €40
27% = €10.80 } €29.20 net

Ans €1029.20

(ii) €1000 is invested in this bond and the bond is cashed in at the end of the 10 years. If the cumulative interest is taxed at the rate of 27% and the tax-free bonus sum is paid as shown, what is the value of the bond after the 10 years

10% = €100
27% = €27 } €73 net interest

lump sum { 40% = €400 ⇒ 1000 + 400 + 73

Ans €1,473 after 10 yrs

The Derivative

Making sense of Differentiation

Investigating the problem of *speed* can lead to interesting understandings in mathematics.

How can you measure the speed of a moving object at a given instant in time? Or indeed

What is meant by the term *speed?*

Defining speed has wide-ranging implications – not just for solving speed problems but for measuring the rate of change of any quantity.

Your investigative journey will lead you to the key concept of *derivative*, which forms the basis of your study of calculus

How do you measure speed?

The speed of an object at an instant in time is surprisingly difficult to define precisely. Consider the statement. "*At the instant he crossed the finish line in 2009 Usain Bolt was travelling at 28 mph*". How can such a claim be substantiated? A photograph taken at that instant would be no help at all as it would show Usain Bolt motionless. There is some paradox in trying to study Usain's motion at a particular instant in time since, to focus on a single instant, you effectively stop the motion!

Problems of motion were of central concern to Zeno and other philosophers as early as the 5^{th} century B.C. The modern approach, made famous by Newton's calculus, is to stop looking for a simple notion of speed at an instant, and instead to look at speed over small time intervals containing the instant. This method sidesteps the philosophical problems mentioned earlier but introduces new ones of its own.

Gedankenexperiment

The ideas mentioned above can be illustrated in an idealised thought experiment [Gedankenexperiment] that assumes we can measure distance and time as accurately as we want.

Think about the speed of a piece of plasticine that is thrown straight upward into the air at

t = 0 seconds. The plasticine leaves your hand at high speed, slows down until it reaches its maximum height, and then speeds up in the downward direction and finally hits the ground.

Suppose that you want to determine the speed, say, at t = 1 second. The table shows the height, s, of the plasticine above the ground as a function of time.

T (sec)	0	1	2	3	4	5	6
s (m)	2	27	43	49	45	32	9

During the 1st second the plasticine travels 27 – 2 = 25m, and during the second second it travels only 43 – 27 = 16m. Hence, the plasticine travelled faster over the 1st interval, $0 \le t \le 1$, than over the second interval, $1 \le t \le 2$.

Speed v Velocity

Physicists distinguish between velocity and speed. Suppose an object moves along a line. They pick one direction to be positive and say that the *velocity* is positive if it is in this direction, and negative if it is in the opposite direction. For the plasticine, upward is positive and downward is negative. Because *speed* is the magnitude or size of the velocity, it is always positive or zero.

Now … a definition

If *s(t)* is the position of an object at time *t*, then the **average velocity** over the interval $a \le t \le b$ is

$$\textbf{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b)-s(a)}{b-a}$$

In words, the **average velocity** of an object over an interval of time is the net change in position of the object during this interval divided by the change in time (i.e. the time interval).

Test yourself:

Use the table above to calculate the average velocity of the plasticine over the interval $4 \le t \le 5$. What is the significance of the sign of your answer?

Calculate the average velocity of the plasticine over the interval $1 \le t \le 3$.

Why is the average velocity a useful concept? Well, it gives a rough idea of the behaviour of the plasticine. If two pieces of plasticine are thrown into the air, and one has an average velocity of 10 m/s over the interval $0 \le t \le 1$ while the second has an average velocity of 20 m/s over the same interval, the second one is moving faster.

However, average velocity over an interval does not solve the problem of measuring the velocity of the plasticine at *exactly* $t = 1$ second. To get closer to an answer to that question, you have to look at what happens near $t = 1$ in more detail. You must look at the average velocity over smaller intervals on either side of $t = 1$.

The value of the average velocity before t = 1 is slightly more than the average velocity after t = 1 but, as the size of the interval shrinks, the values of the average velocity before I = 1 and the average velocity after t = 1 get closer together. Eventually, in the smallest interval, the two average velocities are the same; this is what we define the ***instantaneous velocity*** at t = 1 to be.

More generally, you can use the same method as for t = *1* to find the ***instantaneous velocity*** at any point t =a:

on small intervals of size h around t = a

$$\text{Average velocity} = \frac{s(a+h)-s(a)}{h}$$

Try to make sense of this definition.

The ***instantaneous velocity*** is the number that the average velocities approach as the intervals decrease in size, that is, as h becomes smaller.

So you can formally define ***instantaneous velocity*** at t = a to be

$$\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}$$

In words, the **instantaneous velocity** of an object at time t = a is given by the limit of the average velocity over an interval of time, as the interval shrinks around a.

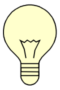
In a time of t seconds, a particle moves a distance of s metres from its starting point, where

$\boldsymbol{s} = 3\boldsymbol{t}^2$.

(a) Find the average velocity between $\boldsymbol{t}$ = 1 and $\boldsymbol{t}$ =1 + $\boldsymbol{h}$ if:

(i) $\boldsymbol{h}$ = 0:1, (ii) $\boldsymbol{h}$ = 0:01, (iii) $\boldsymbol{h}$ = 0:001.

(b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.



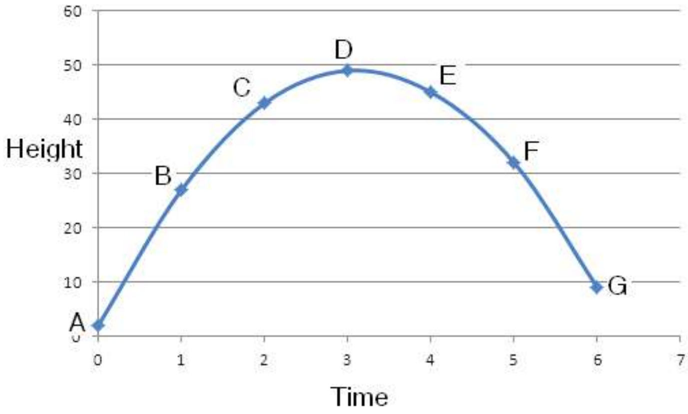
Now think …..How would this question change if ***s*** were a more complex function?

Say, for example, $s = 3t^2 + 4$ or $s = \text{Sin}\,(2t)$

Try it out for yourself. 

Now, back to the plasticine…

The graph shows the height of the plasticine plotted against time.



Try to visualise the average velocity on this graph.

Hint: Suppose ***y*** = **s***(t)* and consider the interval 1 ≤ ***t*** ≤ 2

Look back at the definition of average velocity…

$$\textbf{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b)-s(a)}{b-a}$$

In this situation *b* = 2 and *a* = 1

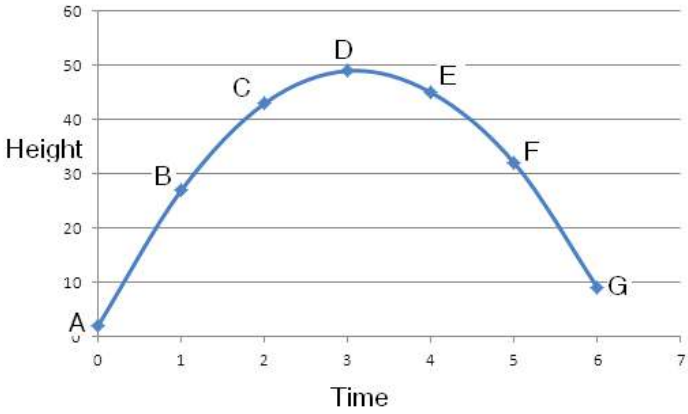
Now look at the *change in position*, *s*(2)-*s*(1) on the graph .Mark it with a line.

Now look at the change in time, 2-1 on the graph .Mark it with a line.

Now can you make sense of this textbook definition?

The **average velocity** over any time interval a ≤ t ≤ b is the slope of the line joining the points on the graph of s(t) which correspond to t = a and t =b

Use this definition to describe the velocity of the plasticine throughout its flight [i.e. is it increasing? Decreasing? Staying the same?] You may like to draw lines on the diagram.



60
50
40
30
20
10
0
Height
A
B
C
D
E
F
G
0
1
2
3
4
5
6
7
Time

Generalising your observations

When thinking about the plasticine you derived an expression for the **average velocity** or the "change in height divided by the change in time". Look back at your diagram and see how these two things are the same.

So now you have derived an expression for the average rate of change of height with respect to time

Average rate of change of height $= \frac{s(a+h)-s(a)}{h}$

with respect to time

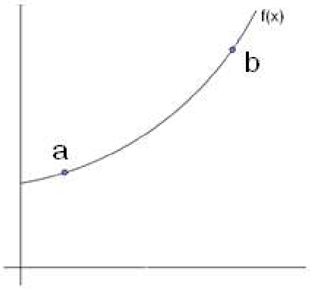
Spend some time thinking about this. Look back at your graphs; where is ***s(a+h) –s(a)*** on your graph? Where is ***h*** on your graph?

This ratio is called the ***difference quotient***. Now, apply the same analysis to any function ***f***, not necessarily a function of time:

Average rate of change of ***f*** over the interval from ***a*** to ***a+h***	$= \frac{f(a+h)-f(a)}{h}$

Time

- Make sense of this definition
- Visualise the definition on the graph below by considering how the function changes from point a to point b, a horizontal distance ***h*** away from a.



Draw a line to represent the numerator, i.e. the distance ***f*** (***a***+***h***) –***f*** (***a***)

Draw a line to represent denominator, i.e. the distance ***h***

Draw the line whose slope is $\frac{numerator}{denominator}$, i.e. ($\frac{\boldsymbol{f}\ (\mathrm{a+h})\ -\boldsymbol{f}\ (\mathrm{a})}{\boldsymbol{h}}$)

Now think about what happens as **h** becomes increasingly smaller.

What happens to this line with slope $\frac{\boldsymbol{f}\ (\mathrm{a+h})\ -\boldsymbol{f}\ (\mathrm{a})}{\boldsymbol{h}}$?

What happens to the slope of this line as **h** becomes increasingly smaller?

In other words, what is $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$?

Make sense of the following definition

> The instantaneous rate of change of a function at a point *a* is defined as the derivative of the function and is written ***f'(a)***
>
> $$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
>
> If the limit exists, then the function ***f*** is said to be ***differentiable*** at *a*.

Rate of change in context

Have you ever noticed that, when you are blowing up a balloon, it seems to blow up faster at the start and then slow down as you blow more air into it?

You can explain this mathematically by examining the rate of change of the radius with respect to the volume.

The volume V of a sphere is given by $V = \frac{4}{3}\pi r^3$

Rearranging gives $r = \sqrt[3]{\frac{3V}{4\pi}}$

Examine the average rate of change of the radius with respect to V over the intervals $0.5 \le V \le 1$ and $1 \le V \le 1.5$ to see what happens to the rate of change of the radius as the volume increases.

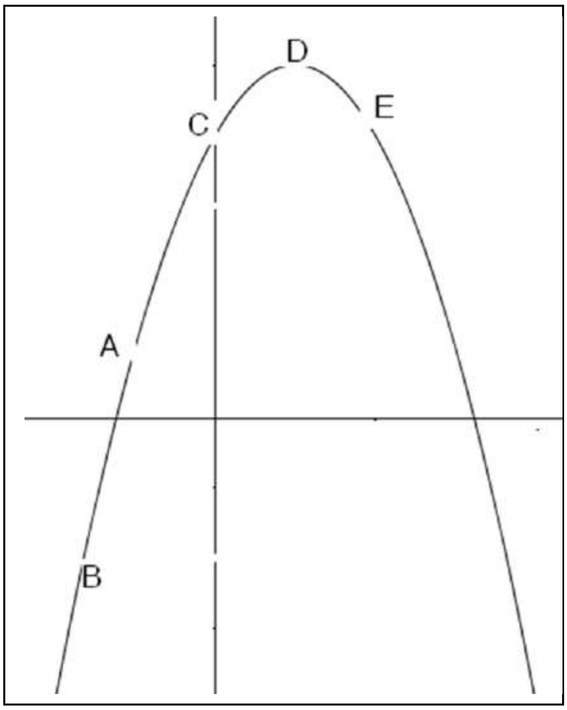
It should start to decrease, thus explaining the observation that when you blow up a balloon it seems to blow up faster at the start and then appears to slow down as you blow more air into it

Try it out for yourself

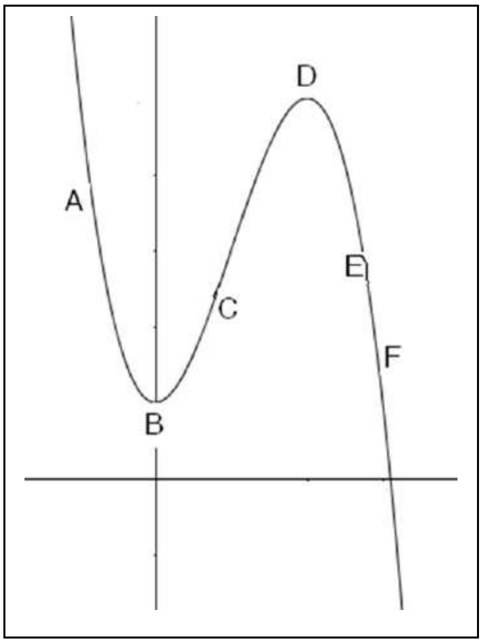
Tasks

Match the points labelled on the given curves with the slopes in the table.

Slope	Point
–4	
4	
0	
10	
8	



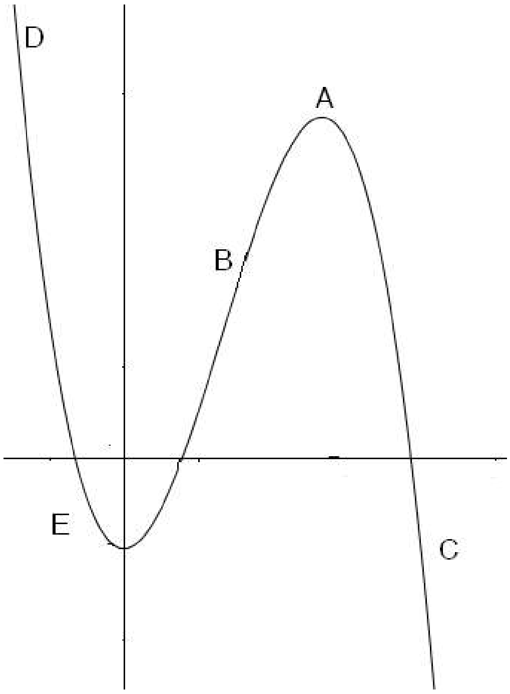
Slope	Point
–9	
0	
–10	
3	
–3.75	



For the function shown,

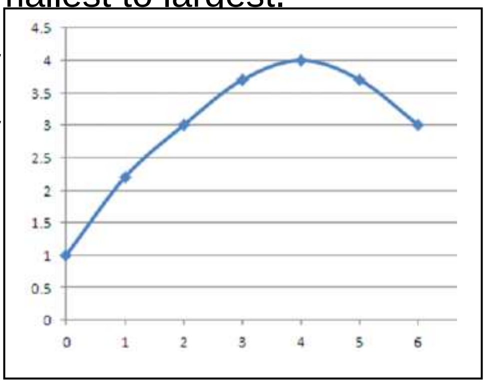
(a) At what labelled points is the slope of the graph (i) positive (ii) negative?

(b) At which labelled point does the graph have (i) the greatest slope (ii) the least slope?



The graph of ***f***(t) in the diagram below gives the position of a particle at time t. List the following quantities in order, smallest to largest.

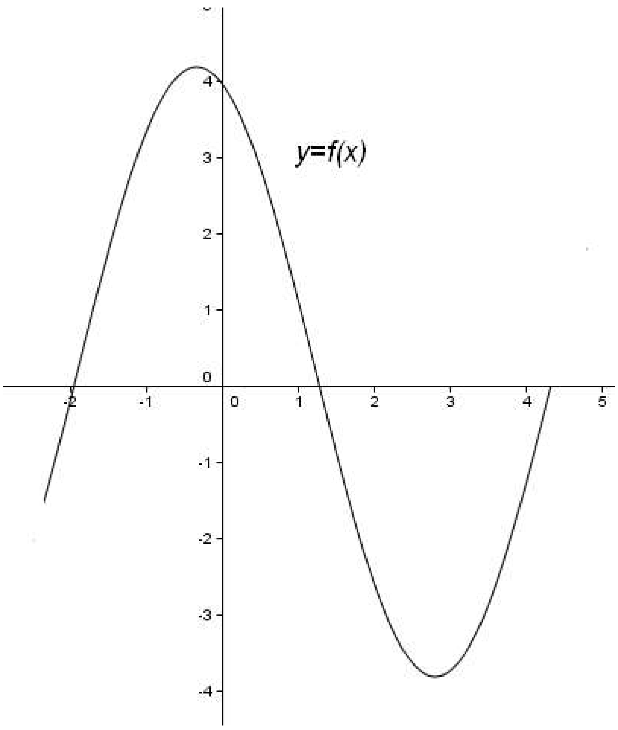
- A, average velocity between t = 1 and t
- B, average velocity between t = 5 and t
- C, instantaneous velocity at t = 1,
- D, instantaneous velocity at t = 3,
- E, instantaneous velocity at t = 5,
- F, instantaneous velocity at t = 6.

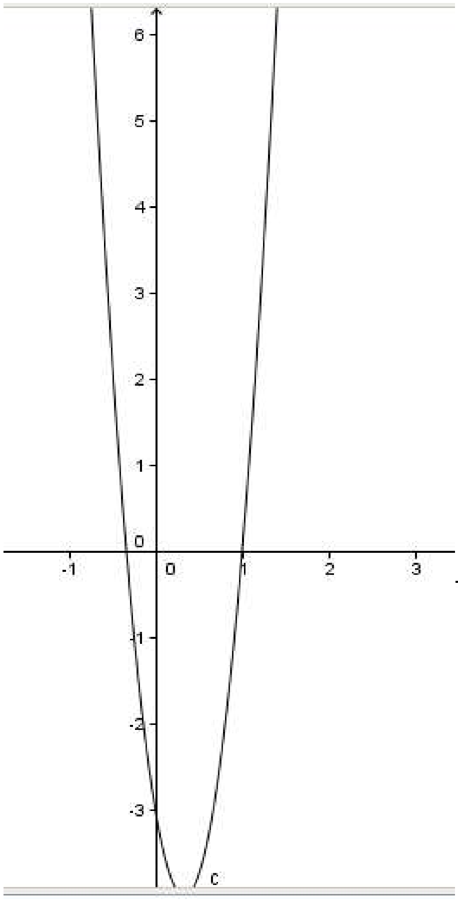


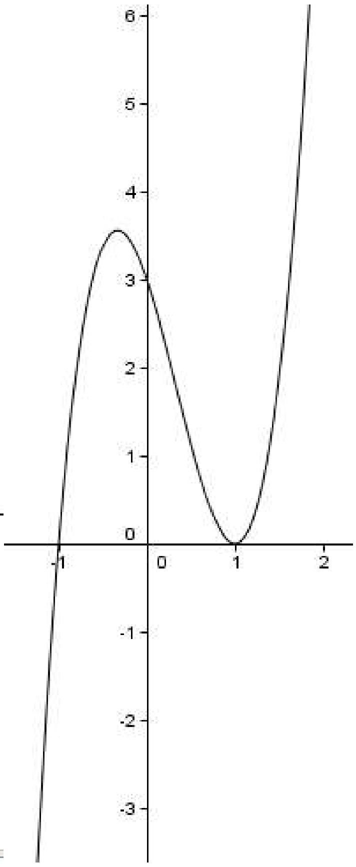
An object moves at varying velocity along a line and ***s*** = ***f***(*t*) represents the particle's distance from a point as a function of time, *t*. Sketch a possible graph for ***f*** if the average velocity of the particle between ***t*** = 2 and ***t*** = 6 is the same as the instantaneous velocity at

t = 5.

Estimate the derivative of the function ***f***(x) shown in each graph below at *x* = –2, –1, 0, 1, 2, 3, 4







The table gives values of ***c***(***t***), the concentration (µg/cm^3) of a drug in the bloodstream, at time ***t*** (min). Construct a table of estimated values for ***c'(t***), the rate of change of ***c(t)*** with respect to time

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
c(***t***)(µg/cm^3)	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

Compare, Examine, Discuss and Evaluate

$$C'(t) = \frac{C(t+h) - C(t)}{h}$$

$$C'(0) = \frac{C(0{\cdot}1) - C(0)}{\cdot 1} = \frac{\cdot 89 - \cdot 84}{\cdot 1} = 0{\cdot}5 \text{ mg/cm}^3$$

$$C'(\cdot 1) = \frac{C(0{\cdot}2) - C(0{\cdot}1)}{\cdot 1} = \frac{0{\cdot}94 - 0{\cdot}89}{\cdot 1} = 0{\cdot}5 \text{ mg/cm}^3$$

$$C'(2) = \frac{C(0{\cdot}3) - C(0{\cdot}2)}{\cdot 1} = \frac{0{\cdot}98 - 0{\cdot}94}{\cdot 1} = 0{\cdot}4 \text{ mg/cm}^3$$

$$C'(3) = \frac{C(0{\cdot}4) - C(0{\cdot}3)}{\cdot 1} = \frac{1{\cdot}00 - 0{\cdot}98}{\cdot 1} = 0{\cdot}2 \text{ mg/cm}^3$$

$$C'(\cdot 4) = \frac{C(0{\cdot}5) - C(0{\cdot}4)}{\cdot 1} = \frac{1{\cdot}00 - 1{\cdot}00}{\cdot 1} = 0 \text{ mg/cm}^3$$

$$C'(5) = \frac{C(0{\cdot}6) - C(0{\cdot}5)}{\cdot 1} = \frac{\cdot 97 - 1{\cdot}00}{\cdot 1} = -0{\cdot}3 \text{ mg/cm}^3$$

$$C'(6) = \frac{C(0{\cdot}7) - C(0{\cdot}6)}{\cdot 1} = \frac{\cdot 9 - \cdot 97}{\cdot 1} = -0{\cdot}7 \text{ mg/cm}^3$$

$$C'(7) = \frac{C(0{\cdot}8) - C(0{\cdot}7)}{\cdot 1} = \frac{\cdot 79 - \cdot 9}{\cdot 1} = -1{\cdot}1 \text{ mg/cm}^3$$

$$C'(8) = \frac{C(0{\cdot}9) - C(0{\cdot}8)}{\cdot 1} = \frac{\cdot 63 - \cdot 79}{\cdot 1} = -1{\cdot}6 \text{ mg/cm}^3$$

$$C'(\cdot 9) = \frac{C(1{\cdot}0) - C(0{\cdot}9)}{\cdot 1} = \frac{\cdot 41 - \cdot 63}{\cdot 1} = -2{\cdot}2 \text{ mg/cm}^3$$

Task:

In your groups discuss the definitions of the following

- Domain of a function
- Range of a function

Write a note in your learning journal that will help you to remember these definitions.

(i) Find the domain and range of the following functions. Justify your answer algebraically and graphically

- $\boldsymbol{f}(x) = x^2+2$
- $\boldsymbol{f}(t) = \frac{1}{t+2}$
- $\boldsymbol{g}(s) = \sqrt{3-x}$

(ii) *You are told that the height* $\boldsymbol{h}$ *of a certain projectile as a function of time in seconds is given by* $\boldsymbol{h} = 20\boldsymbol{t} - 4.9\boldsymbol{t}^2$. Find the domain and range of this function.

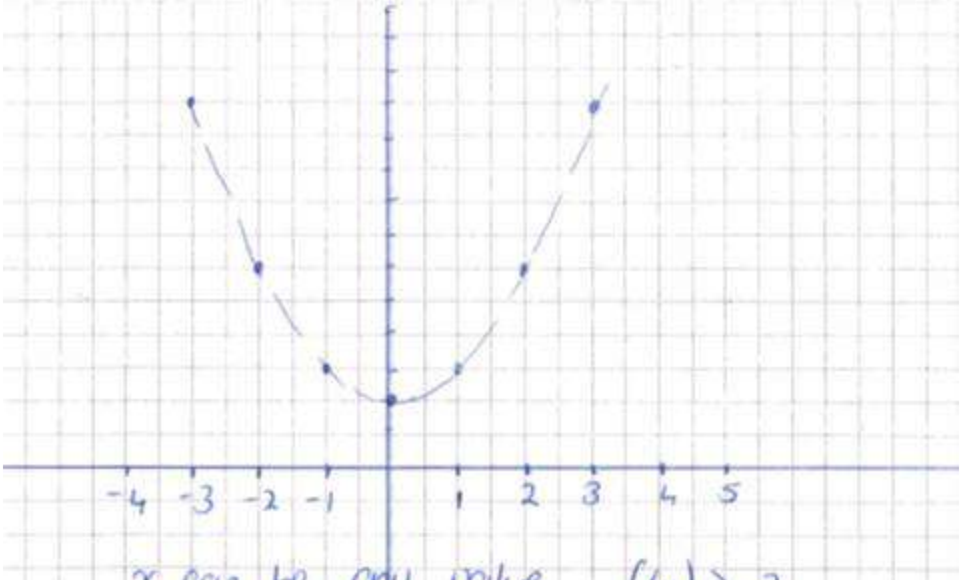
We decided that the domain is a set and it is made up of only numbers that will make the function work. The function is working when the output is Real. The range is a set too it is all the numbers that come out when all the numbers in the domain go into the function.

(i) $f(x) = x^2 + 2$

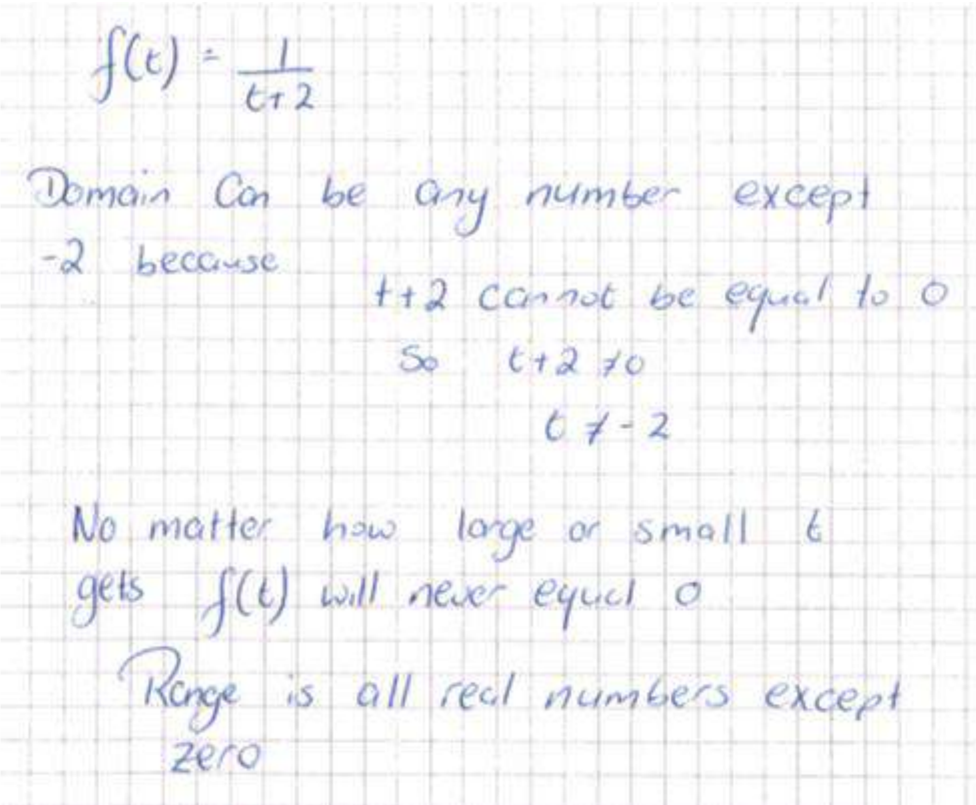
The domain is any Real number because there are no restrictions on x

x^2 is never negative so $x^2 + 2$ is always greater than 2

So the range of $f(x)$ is all real numbers $f(x) \geq 2$



x can be any value $f(x) \geq 2$



$f(t) = \frac{1}{t+2}$

Domain can be any number except -2 because

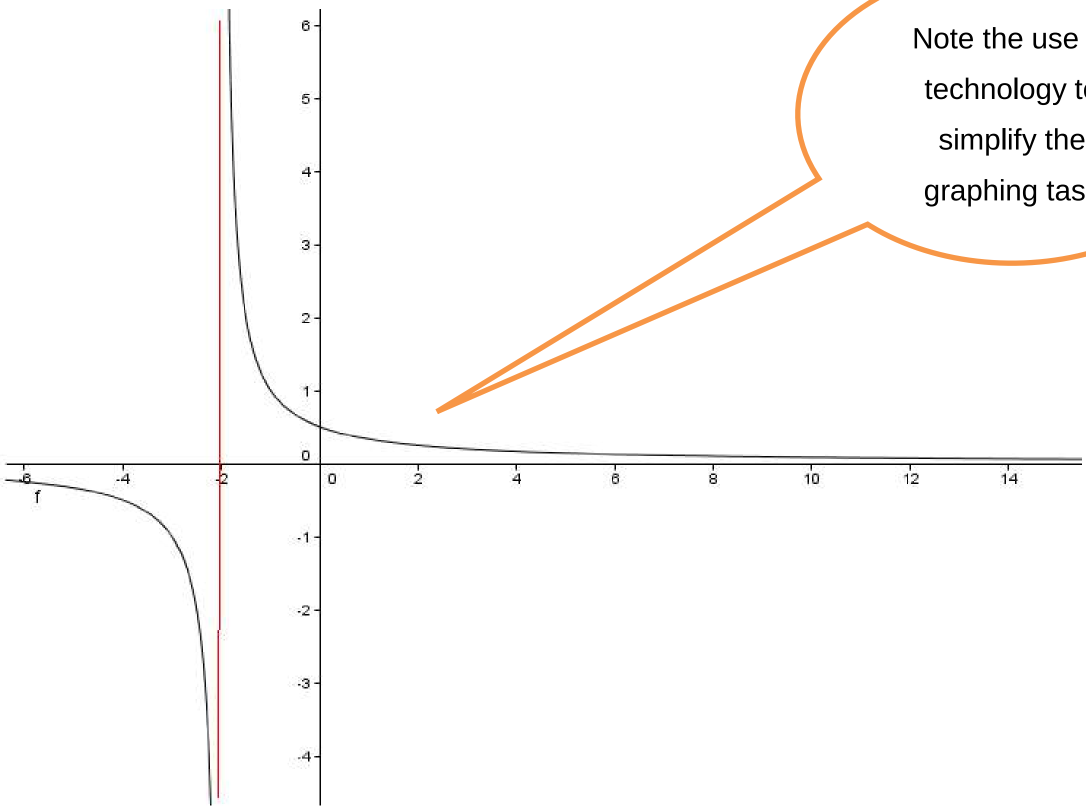
t+2 cannot be equal to 0

So $t+2 \neq 0$

$t \neq -2$

No matter how large or small t gets f(t) will never equal 0

Range is all real numbers except zero



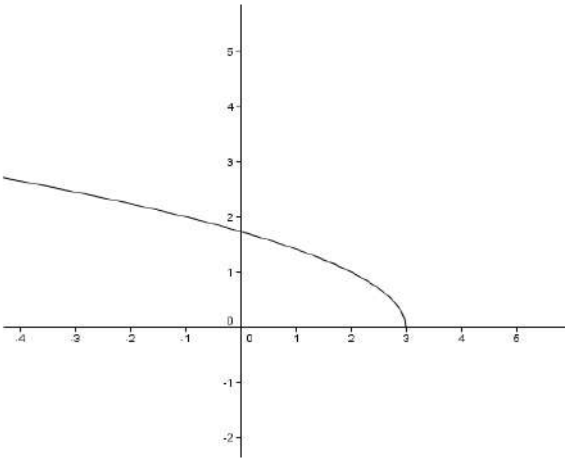
Note the use of technology to simplify the graphing task

$g(s) = \sqrt{3 - x}$

$3 - x \geq 0$
$-x \geq -3$
$x \leq 3$

Domain is all Real numbers $x \leq 3$

$g(s) \geq 0$ is the range



time values can't be negative and the projectile hits the ground and stops it doesn't go underground. It hits the ground when

$$20t - 4.9t^2 = 0$$

$$(20 - 4.9t)t = 0$$

$$t = 0 \quad \text{or} \quad 20 - 4.9t = 0$$

$$t = 4.08 \text{secs}$$

So Domain is

all real values of t Where $0 \leq t \leq 4.08$

The Max height it goes to is

When $20 - 9.8t = 0$

$$9.8t = 20$$

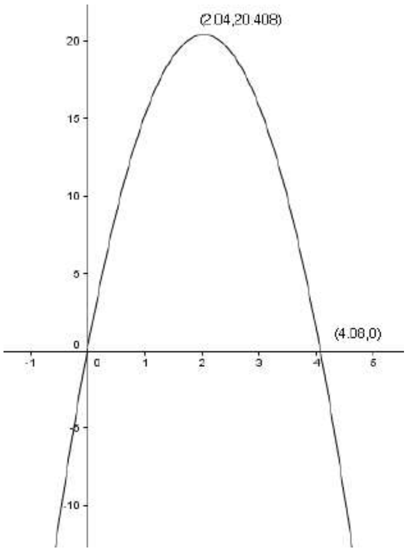
at $t = 2.04$ sec

So

$$h = 20(2.04) - 4.9(2.04)^2$$

$$= 20.408$$

Range is all real values $0 \leq h \leq 20.408$



Note to students: The students whose work is displayed above used Geogebra as a tool to help their understanding of the concepts under investigation. Can you use Geogebra? Have you downloaded this onto your PC at home?

Look at the strategies the students used to decide on the domains and ranges of the functions; can you generalise the strategies used? Write a note in your journal outlining how you might find the domain and range of a function.

Extension: Now that you have a means of finding the domain and range of given functions, consider the reverse process. If you were given the range and domain of a function would you be able to sketch the graph of the function?

Strand 5 Resources – Leaving Certificate

In Strand 5, you extend your knowledge of patterns and relationships from Strands 3 and 4 and build on your experience of these strands in the junior cycle. You should now be able to make connections between coordinate geometry, algebra, functions and calculus.

The following pages contain activities related to functions and calculus. Try the concept of slope presentation first and as you work through the activities think about how these connect with other areas of the mathematics course.

Concepts in calculus

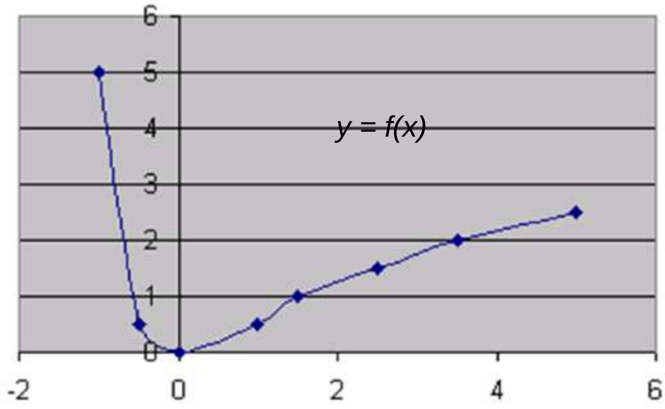
One of the aims of Project Maths is to help students develop conceptual understanding of mathematics. If you have conceptual understanding you will be able to

- generalise from particular examples
- apply and adapt ideas to new situations
- approach problems visually, numerically or algebraically and convert easily from one representation to another
- associate meaning with results
- connect old ideas with new ideas
- understand the limitations of an idea.

One can assess conceptual understanding by using problems/tasks that require you to do all of these things. Some problems/tasks related to functions are shown below. You may find these problems/tasks difficult and you may not be able to quickly see a solution strategy. Do not worry; these concept problems, by very definition, are not routine but designed to place you in new situations where you will have to apply and adapt what you have learned.
You may find it helpful to work through these questions with a study partner, The emphasis here is not on obtaining the correct answer but rather on making you think and discuss and on the reasoning and sense-making opportunities the problems afford you.

Examples of student work are included for a selection of the tasks, Try the tasks yourself before you look at other students' work. We invite you to **Compare, Examine, Discuss and Evaluate** the solution strategies provided.

Q The diagram shows the graph of the function $y = f(x)$ for $-1 \leq x \leq 5$. Approximate the x-value(s) for which



(a) $f'(x) = 0$

(b) $f'(x) < 0$

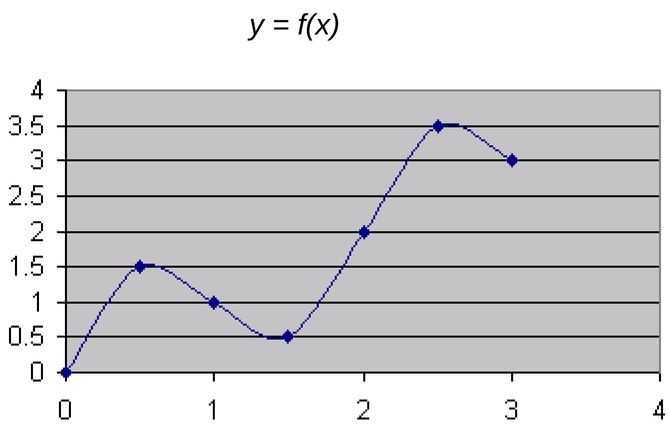
(c) $f'(x) > 0$

(d) $f''(x) < 0$

(e) $f''(x) > 0$

Q The diagram shows the graph of the function $y = f(x)$ for $0 \leq x \leq 4$.

Approximate the x-value(s) for which



(a) $f'(x) = 0$

(b) $f'(x) < 0$

(c) $f'(x) > 0$

(d) $f''(x) < 0$

(e) $f''(x) > 0$

What is unusual about these two questions? You are asked to examine $f'(x)$ and $f''(x)$ of given functions. What does *$f'(x)$* mean*?* What does *$f''(x)$* mean*?* How can you find *$f'(x)$* or *$f''(x)$* of a function when you are only given the graph of the function?

This is the challenge in these task; they are *assessing* how well you understand the concept of the derivative. You will only be able to answer them if you fully understand this concept.

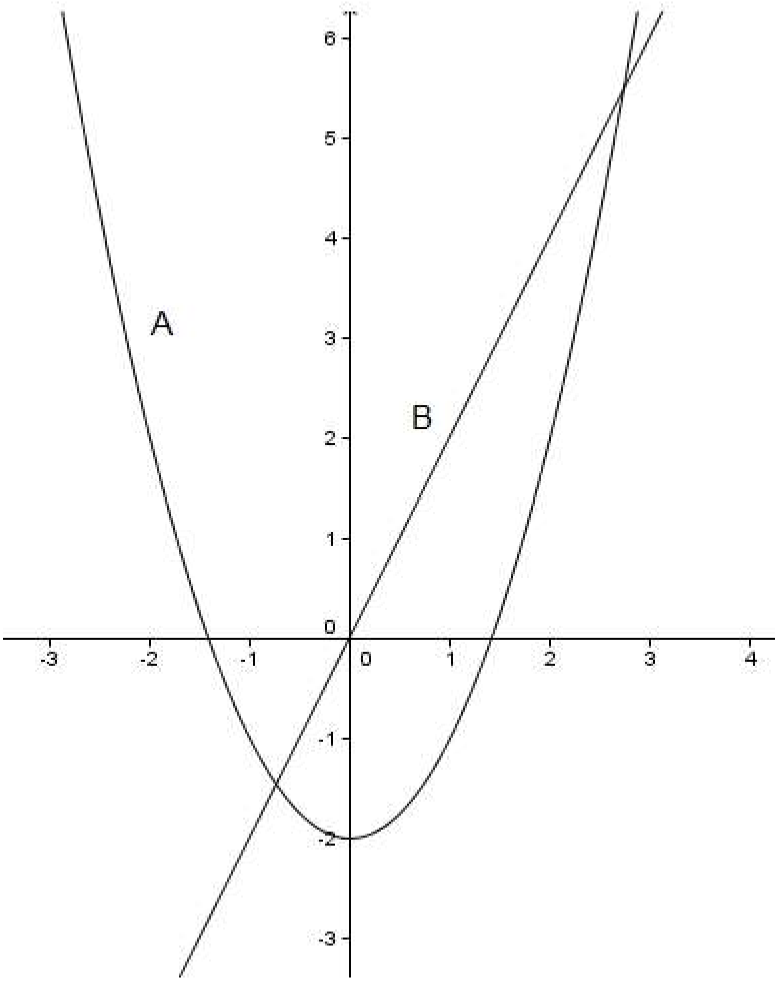
So; what if you don't fully understand the concept, how can you help yourself develop an understanding of the concept of the derivative?

- Work your way through *The concept of slope presentation*
- Read the document entitled *The Derivative: making sense of differentiation*
- Investigate with GeoGebra. You can download GeoGebra free at www.geogebra.org

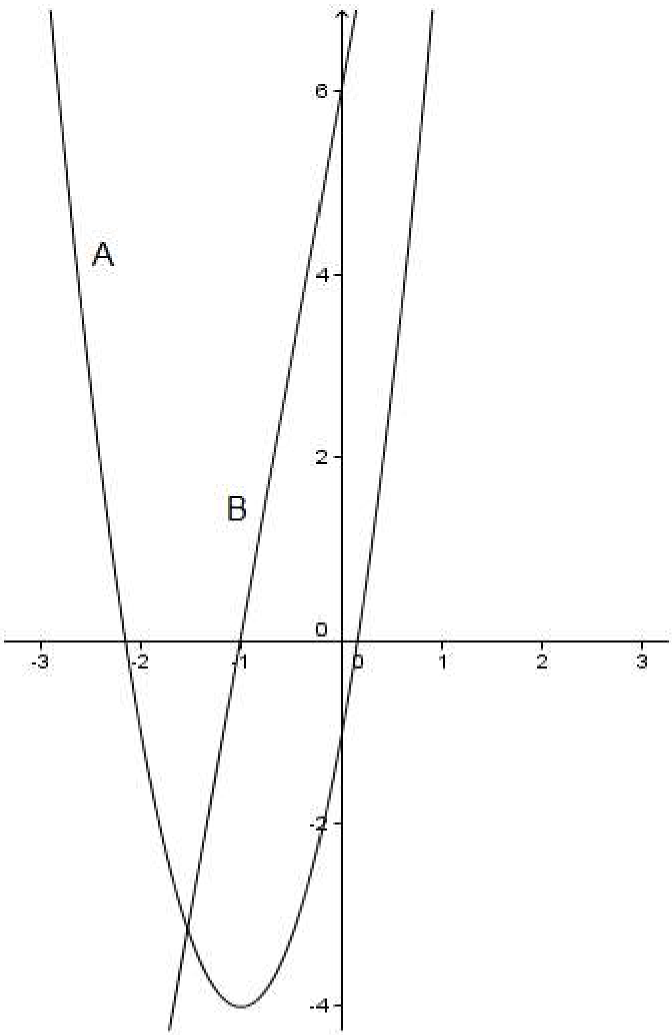
As you can see in order to answer these questions you need to have a lot of understanding. It is not a matter of simply getting the right answer; the question really does require you to show deep conceptual understanding. Now that you have a better understanding of the derivative can you extend this understanding to the second derivative?

Other things you need to consider about the questions are what does it mean

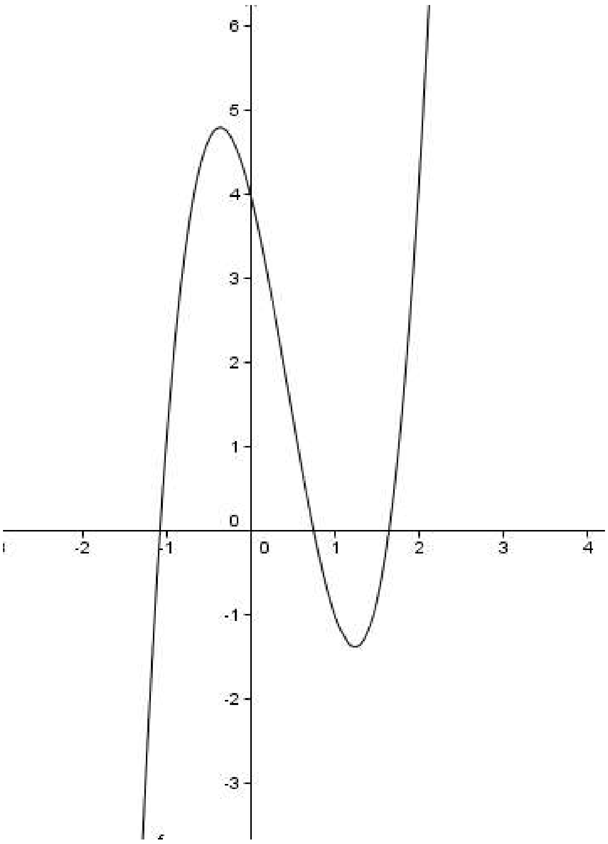
Q. The diagram shows the graph of a function and its derivative. Which is which? Give at least three reasons to support your choice.



Q. The diagram shows the graph of a function and its derivative. Which is which? Give at least two reasons to support your choice.

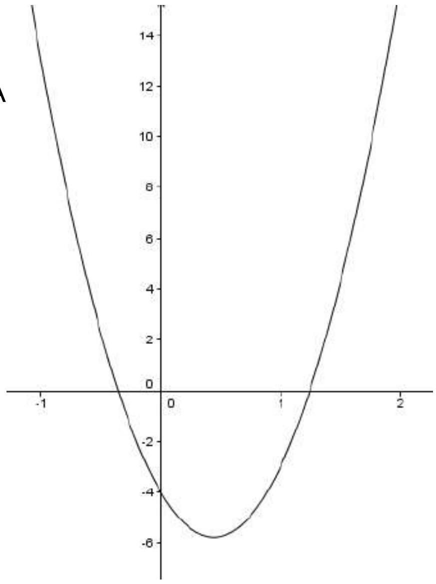


Q. The diagram shows the function f(x)

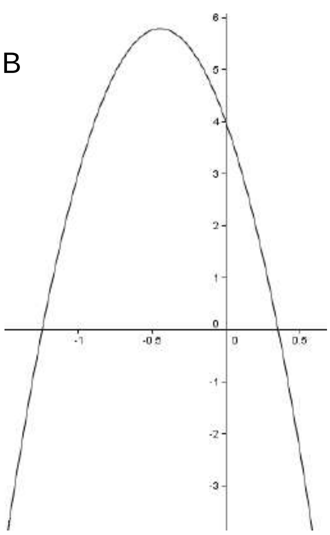


Which of the graphs below: A, B, C, D shows the derivative of *f(x)?* Give 3 reasons for your answer.

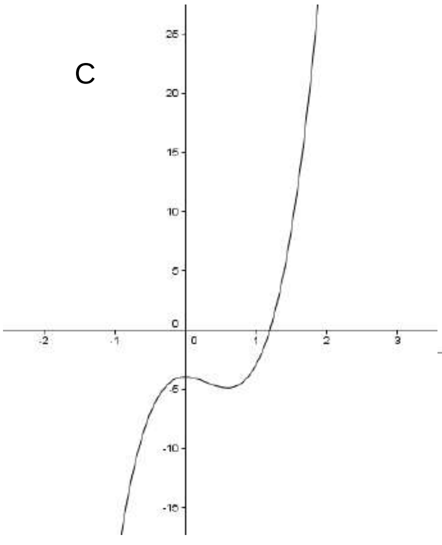
A



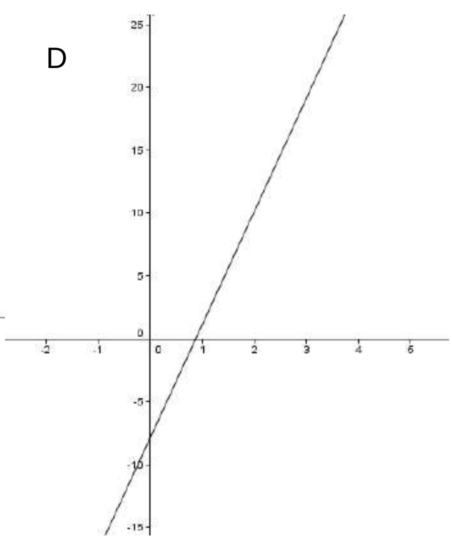
B



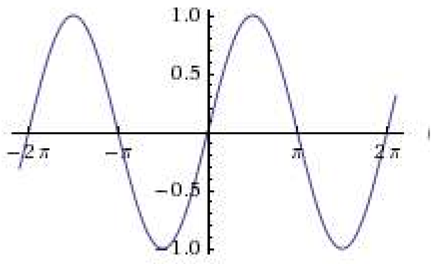
C



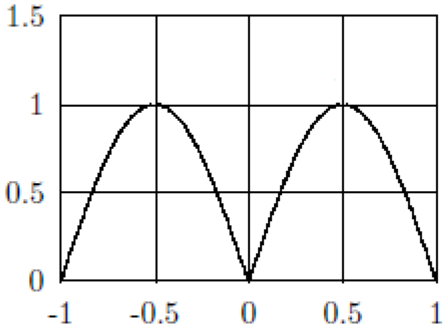
D



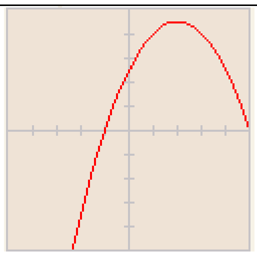
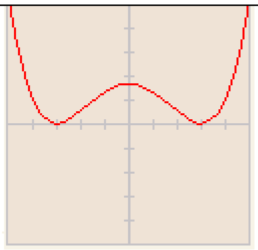
Q Sketch the derivative of the function shown

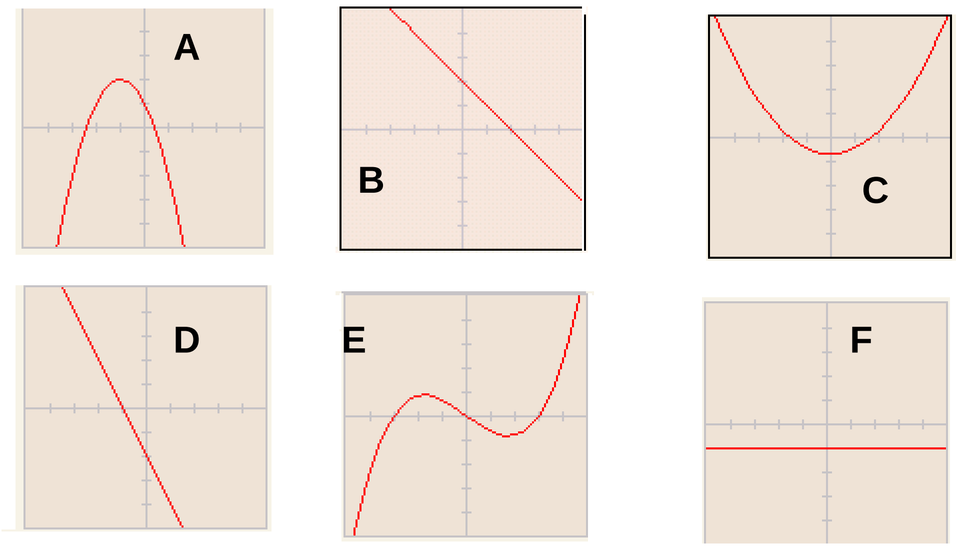


Q Consider the function y = h(x) whose graph is shown below. Find the critical points of h and determine intervals on which h′(x) is (i) positive and (ii) negative. Give reasons for your answer in each case.



Q The graphs of three functions are shown in the table. Six other graphs, labelled A, B, C, D, E and F, are shown below the table. Complete the table by inserting the appropriate letter in each of the empty cells.

Function	First Derivative	Second derivative
		
		
		



Q The table below gives the values of a function f and its first and second derivatives at selected values of x. Determine which row gives the data for f, which row gives the data for f′, and which row gives the data for f′′. Explain your reasoning.

x	0.00	0.33	0.66	1.00	1.33	1.66	2.00	2.33	2.66	3.00
A(x)	0.00	0.64	1.14	1.38	1.28	0.84	0.08	- 0.89	- 1.91	- 2.83
B(x)	0.00	0.11	0.41	0.84	1.30	1.66	1.82	1.69	1.22	0.42
C(x)	2.00	1.78	1.16	0.24	- 0.83	- 1.85	- 2.65	- 3.07	- 3.00	- 2.40

Q Suppose a car is driving on a straight road and that its velocity is positive for the first hour and then negative for the next 20 minutes. What can you conclude?

Q Suppose that the function m(t) gives the area of the lighted portion of the moon as seen from the front of your school. How can you express the times at which the moon is waxing (getting larger) and at which the moon is waning (getting smaller)?

Q. The table below gives the values of the function f at selected points. Find a reasonable approximation for f′ (1).

x	0.8	0.9	1.0	1.1	1.2
f (x)	1.67	1.85	2.03	2.21	2.38

Q For each of the following sentences identify

a function whose second derivative is being discussed

what is being said about the concavity of that function

The FTSE reacted to the latest report that the rate of inflation was slowing down.

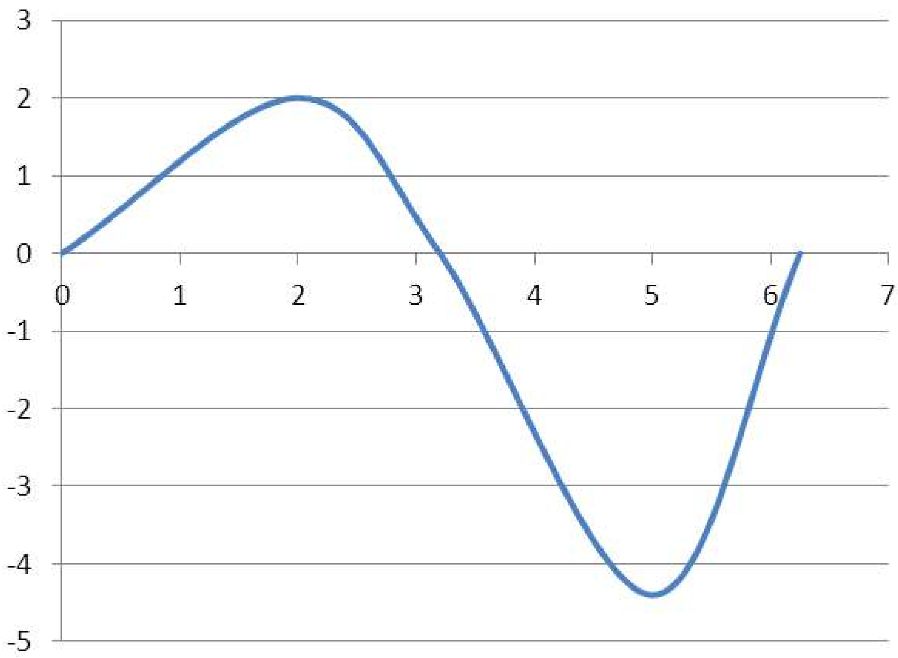
When he saw the light turn amber he hit the accelerator.

As the swine flu vaccination programme rolled out, the rate of new infections decreased dramatically.

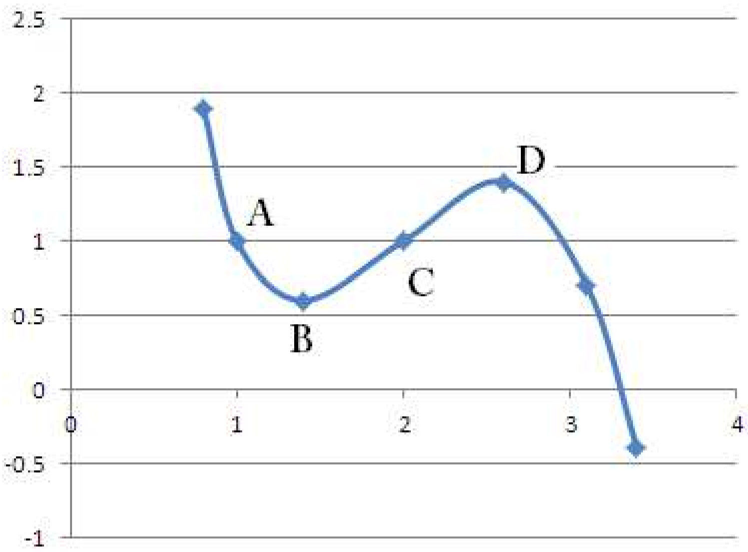
Q The graph of the function f is shown below. Referring to this graph, arrange the

following quantities in ascending order.

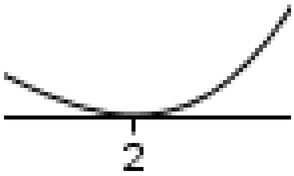
f′ (1), f′ (5), f (5), f (7), f′ (3)



Q. The graph shows a function y = f(x). At which labelled point(s) might it be possible that d2y/dx2 = dy/dx? Explain your reasoning.



Q A polynomial function p has degree 3. A part of its graph near the point (2, 0) is shown below.



Which one of the following could be the rule for the polynomial p? Give reasons for your answer.

p(x) = x(x+2)2

p(x) = (x–3)3

p(x) = x2(x–2)

p(x) = (x–1)(x–2)2

p(x) = –x(x–2)2

Q. The table below gives the values of the function f at selected points. Find a reasonable approximation for f′(1).

x	0.8	0.9	1.0	1.1	1.2
f(x)	1.67	1.85	2.03	2.21	2.38

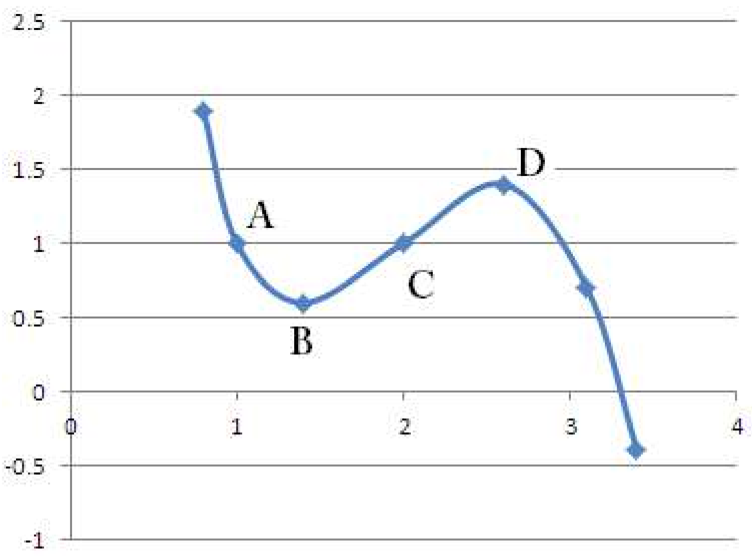
Compare, Examine, Discuss and Evaluate

When I was first confronted with this question I thought: "I just can't do it. I have never seen anything like this before." Then my teacher told me to think about what I know about functions and their derivatives.

I remembered that a function can be represented in a table and a graph, and in a story or pattern. I can see the table and it looks like it is going up. I checked and I saw that it was going up by the same amount each time, 0.18. Now I know that this is a linear equation.

I felt more confident then and I went to Excel and put in the points and yes it is a straight line and I could see the slope was 0.18. Since the differentiation of the function is the slope I can say that a reasonable approximation for f′ (1) is 0.18 and it won't change regardless of x because it is constant.

Q. At which labelled point(s) might it be possible that d2y/dx2 = dy/dx
Explain your reasoning.



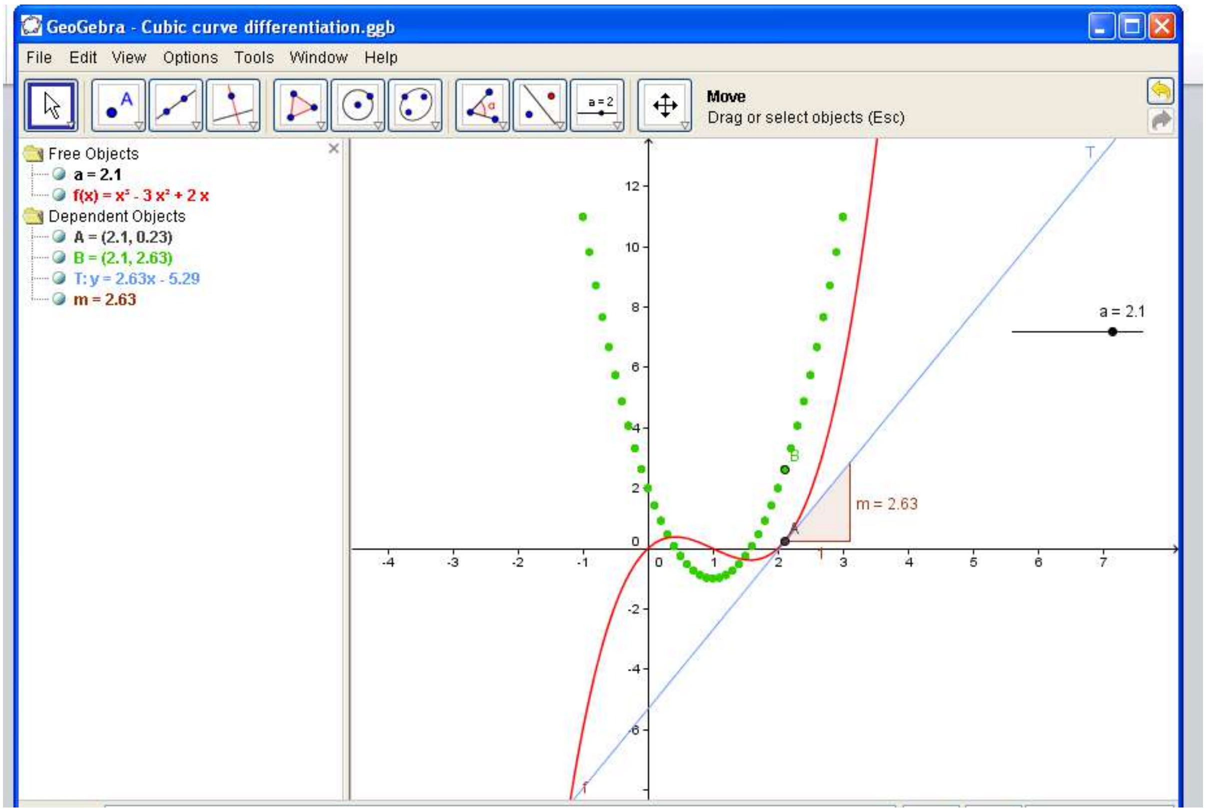
Compare, Examine, Discuss and Evaluate

When my group first looked at this question we thought: right,…where do we start? I remembered what our teacher said ...make a list of things you know about functions – so we got to work.

This is a cubic function because it has a max and a min so that means the equation is in the form y = ax3+ bx2 + cx + d
When you differentiate this once you get a quadratic and when you differentiate it twice you get a linear one or a line

Josh remembered that when the graph is concave up the 2nd derivative is positive and when it's concave down the 2nd derivative is negative

But then I thought: oh no, maybe it's the other way round. So we thought: how can we decide? I said let's go to geogebra and find out. We were able to investigate lots with Geogebra . We discovered that Josh was right when he said when the graph is concave up the 2nd derivative is positive and when it's concave down it's negative. We could also see that when the function is increasing the first derivative is positive and when the function is decreasing the first derivative is negative



It took us a long time to come to this conclusion but Geogebra helped loads. The teacher told us to write about what we had learned in our learning journal. I thought that was a good idea 'cos we really had to think about it .We also learned that C was an interesting point as we couldn't decide if it was concave up or down. We argued both. The teacher told us that this is an interesting point because it is where the concavity changes; when the 2nd derivative changes from positive (concave up) to negative (concave down) it is called a point of inflection and the 2nd derivative is 0 there.

So we thought all the way to C the 2nd derivative is positive and all the way from C the 2nd derivative is negative. From A to B the function is decreasing and the 1st derivative is negative and from B to D the function is increasing so the 1st derivative is positive. In our graph C is a point of inflection ‘cos the 2nd derivative changes sign from positive to negative.

In the end we decided there were no labelled points where it was possible that the 1st and 2nd derivative were the same. At A the function is decreasing which means the 1st derivative is negative and concave up which means the 2nd derivative is positive. At B the function is a minimum which means the 1st derivative is 0. Also, it is concave up which means the 2nd derivative is positive.
C is the point of inflection which means the 2nd derivative = 0. It is not a max or min so the 1st derivative is not zero; it is a positive number ‘cos the function is increasing. D is a max and like B the 1st derivative is 0. But this time the 2nd derivative is negative ’cos it is concave down.

It took us ages to come to these conclusions but I really think I have it now.

We think the 1st and 2nd derivatives might be the same at points between B and C since the 1st and 2nd derivatives are both positive.

Having worked through this material you should have noticed that the questions required you to do a lot of thinking and it probably took you a long time to complete each question. You should have been busy thinking back to things you have done before and thinking how you could adapt those ideas to the new situations presented by the questions. You should have found that more than ever you were being required to attach meaning to your results. This is because they were concept questions which have been carefully designed to not only assess how well you have understood the concepts but also to give you an opportunity to reason and make sense of the new material in light of what you already know.

Think about how you approach the task; were you like the students whose work was **featured**? Were you lost in the beginning without a clue? The students whose work was featured mentioned a lot of strategies that helped them make sense of the tasks.

*"**My teacher said make a list of things you know about functions"***

*"**Geogebra helped loads**"*

What do you know about functions? Could you spot a linear function in a context? In a table? In a graph? In a generalised equation? Could you spot a quadratic function in a context? In a table? In a graph? In a generalised equation? Can you tell when a function is increasing? Decreasing?

Could you spot an exponential function in a context? In a table? In a graph? In a generalised equation? What would the first derivative of each of these functions look like on a graph? In an equation? What would the second derivative look like?

Have a look at the concept of slope presentation.

Set A: Review Materials – Junior Certificate Strand 1 and Strand 2

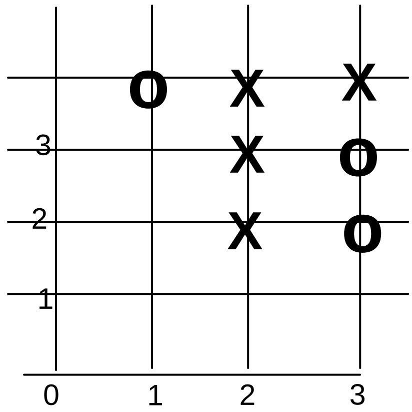
Before you attempt these questions have a look at the "understanding statistics" document.

This set of questions, compiled in two documents, is intended to help you review your work as you prepare for Paper 2 in the Junior Certificate examination. The questions are not intended to be exact matches of what will come up in the exam but they should give you a flavour of how the concepts can be examined in context. Other questions and activities can be found in the Mathematics Resources for Students on the student zone at www.ncca.ie/projectmaths

JCFL

Melissa and Sean are playing a game

Melissa has to make a line of 4 **X** to win.



Put an **X** on the grid to make a winning line for Melissa

Write the co-ordinates of each **X** in this winning line.

(….. , …..) (….. , …..) (….. , …..) (….. , …..)

Look at the numbers in the co-ordinates of these points.

What do you notice?

Is the point (1, 6) on Melissa's winning line?

How do you know?

Where can you put the X so that there are 4 in a row?

Try out different places ….you may extend the grid if you like. Now decide where you would put the X so that there are 4 in a row.

Now try to remember how to label points on a co-ordinate grid. How far did you go out along the x axis? This is the x-coordinate.

How far did you go up or down along the y axis? This is the y-coordinate.

Can you see a pattern between the x and y coordinates?

It might help if you were to put them in a table

x-coordinate	y-coordinate

Now think about the point (1, 6) is this on the winning line? How would you know?

One way to find out is to put the point (1, 6) in your table and see does it fit with the pattern you saw before.

If it doesn’t fit with the pattern you saw why do you think this is? Try to explain.

Can you think of another way to make a decision about whether or

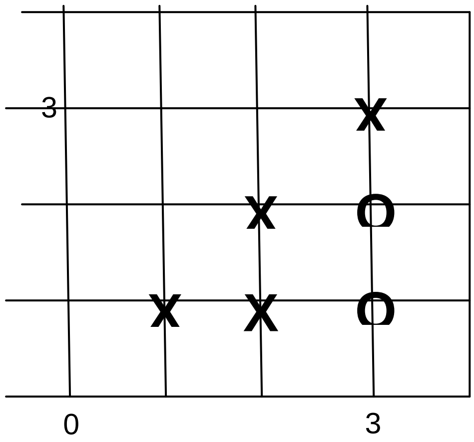
JCOL

Melissa and Sean are playing a game

Melissa has to make a line of 4 **X** to win

Put an **X** on the grid to make a winning line for Melissa.

Write the co-ordinates of the four **X** in this winning line.



(….. , …..) (….. , …..) (….. , …..) (….. , …..)

Look at the numbers in the co-ordinates of these points. What do you notice?

Is the point (6, 7) on Melissa's winning line?

How do you know?

What is the relationship between the x and y coordinates of all points on Melissa's winning line?

Where can you put the X so that there are 4 in a row? What is different about this question and the FL question above?

Try out different places ….you may extend the grid if you like. Now, where would you put the X so that there are 4 in a row?

Now try to remember how to label points on a co-ordinate grid. How far did you go out along the x axis? This is the x-coordinate.

How far did you go up or down along the y axis? This is the y-coordinate.

Can you see a pattern between the x and y coordinates?

It might help if you were to put them in a table

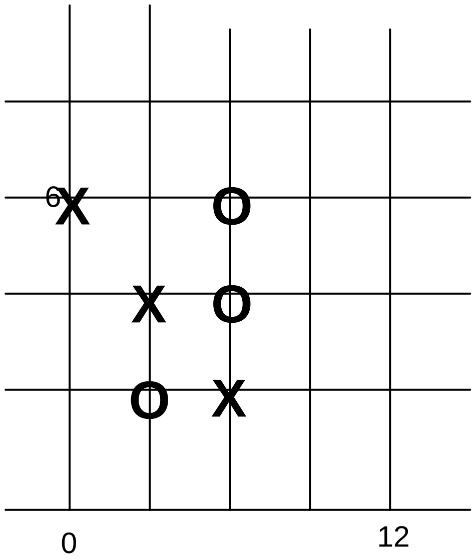
x-coordinate	y-coordinate

Now think about the point (6,7); is this on the winning line? How would you know?

One way to find out is to put the point (6,7) in your table and see does it fit with the pattern you saw before.

If it doesn’t fit with the pattern you saw why do you think this is? Try to explain

JCHL



Melissa and Sean are playing a game

Melissa has to make a line of 4 **X** to win.

Put an **X** on the grid to make a winning line for Melissa

Write the co-ordinates of the four **X** in this winning line.

(….. , …..) (….. , …..) (….. , …..) (….. , …..)

What is the relationship between the x and y coordinates of all points on Melissa's winning line?

If the x coordinate of a point on this line is 25 what should the Y coordinate be?

Explain how you arrived at your answer.

Scaling the axes is a challenge in this question, look at the axes and see why this is the case. What is different about this question and the FL and OL questions above?

Can you see a pattern between the x and y coordinates?

It might help if you were to put them in a table

x-coordinate	y-coordinate

Now think about other points on this winning line; they should fit with this pattern. Try to generalise the pattern you see; this will give you the equation of the line. Can you find the equation of the line in any other way?

Compare the two methods.

Remember the equation of a line is just the generalisation of the pattern that exists between the x and y coordinates of the points on a line. Once you know this generalised pattern you can find any points on the line and make predictions about the line.

Q. 2 JCHL

Joe and Sophie were investigating the relationship between the current flowing through a wire and the voltage across the wire. They performed an experiment and recorded their results in the table.

Voltage (Volts)	Current (Amps)
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6

Plot their results on a coordinate grid.

What is the relationship between the x and y coordinates? Generalise this relationship and write it in the form of an algebraic formula.

If the voltage across the wire was 10 volts, what do you think the current flowing through the wire would be? Explain your thinking.

When you plot your points on the grid decide what type of a relationship exists between the current flowing through the wire and the voltage across it. Is it a linear relationship? How would you know? Is it a quadratic relationship? How would you know? Is it an exponential relationship? How would you know?

Click on the *concept of slope* presentation for help with this question.

When you have decided on the type of relationship that exists between the current flowing through a wire and the voltage across it you can generalise this relationship; again the *concept of slope* presentation should help you with this.

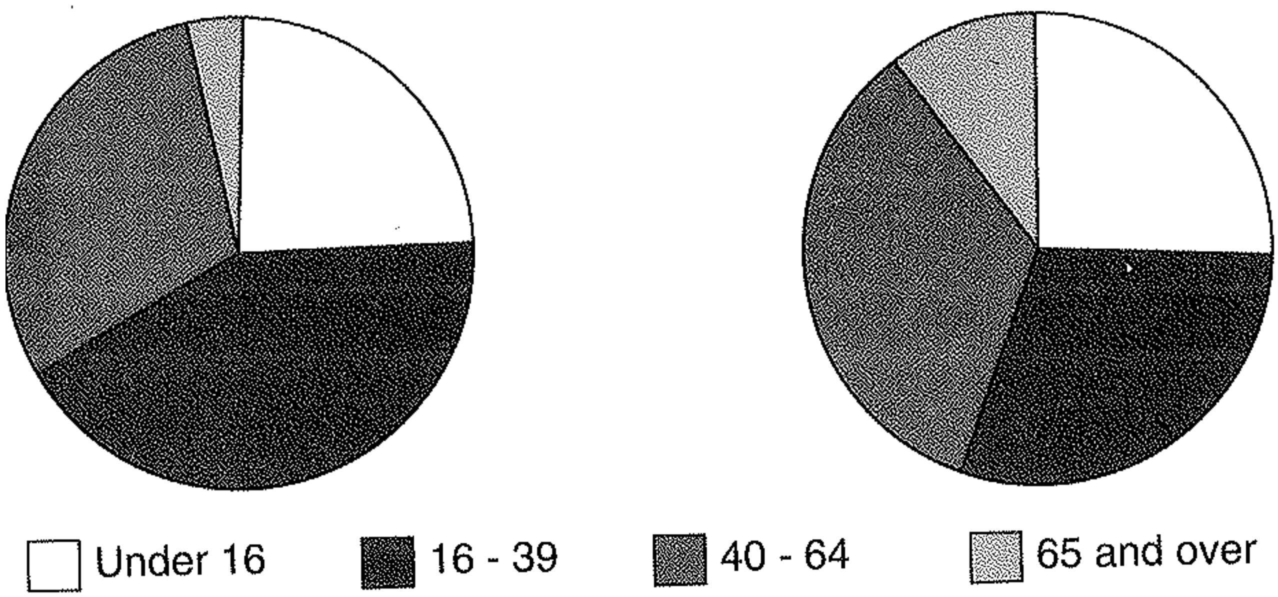
Once you have generalised the relationship or know the equation you can answer lots of questions about the relationship between other points that lie on the line.

Q. 3   JCHL

These pie charts show information from a survey of people’s ages.

400 people in Dublin and 800 people in Cork were surveyed.

Dublin   Cork



John looked at the charts and said

“There is roughly the same number of people under 16 in Cork as there is in Dublin”

Looking at the charts, why do you think John said this?

Do you agree with John? Give a reason why you do or do not agree with him.

Display the data in a way that will make it easier for John to compare the two surveys.

Explain why you made this choice.

This question was designed to promote discussion about pie charts and the information that they can give you. When you discuss things with your friends it gives you an opportunity to get a good idea about what they are thinking in their heads. Sometimes you are all thinking the same thing; sometimes when you hear what others think it makes you think again about your own ideas .You might say “Gosh I never thought about it like that” or “I never really knew that”; when this happens you are able to **refine** your ideas to take into consideration those of your friends. At other times you might disagree and think “No that is not what this is about” and you will **defend** your ideas to your friends. Both of these types of reactions, **reflection/refinement** and **defending,** are a very important part of the learning experience. When your teacher engages in discussion with you he/she gets an idea of what is in your head and he/ she will be able to help you change/refine or extend your thinking. That is why you will find you are doing a lot more discussing these days in Maths class.

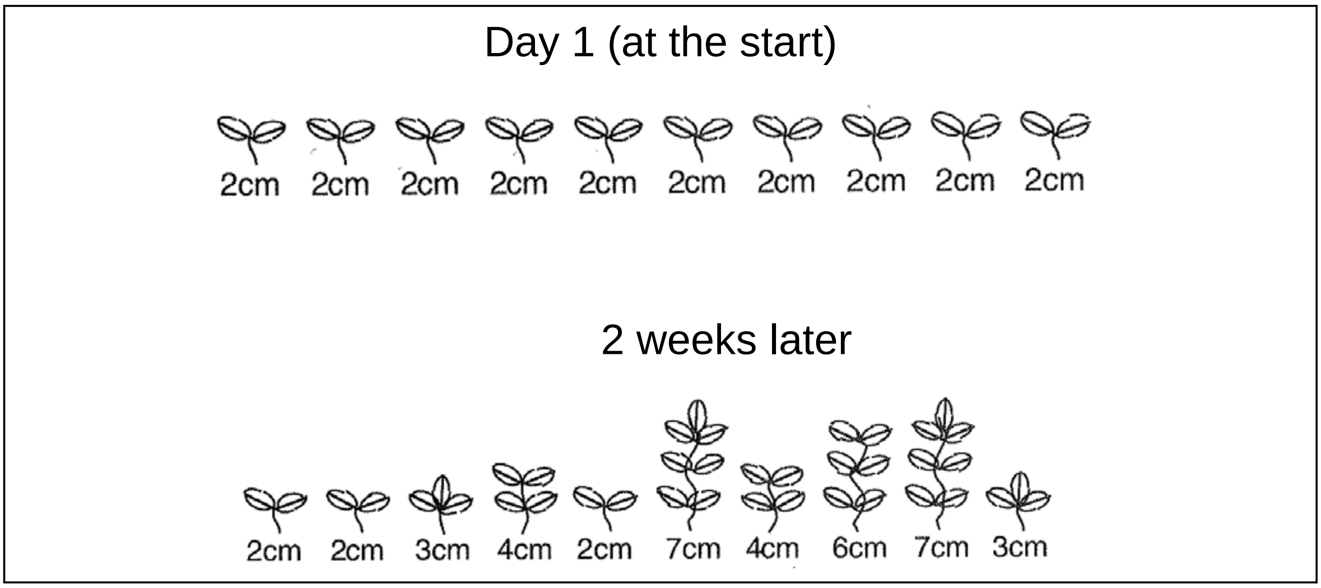
Now back to this question. Do you agree with John? Exactly what information is contained in the sections of a pie chart? Does it contain exact amounts? or proportions? If it contains exact amounts, then is John right? If it contains proportions then is John right? Can you see why John may or may not be right? Is the fact that 400 people were surveyed in Dublin and 800 surveyed in Cork significant? If so, how?

Q. 4 JCHL

An advert says

Wondergrow doubles the height of your plants in 2 weeks

Susie uses Wondergrow on her plants.



a) Does Wondergrow really double the height of the plants?

Use the **mean** and **range** to explain your answer.

b) If Susie chose one of her plants at random what is the probability that it would have doubled in height after 2 weeks?

c) Circle the statement which you think is most accurate. Use Susie's data to explain your choice.

i. It is **impossible** that your plants will double in height after 2 weeks if you use Wondergrow.
ii. It is **unlikely** that your plants will double in height after 2 weeks if you use Wondergrow.
iii. It is **likely** that your plants will double in height after 2 weeks if you use Wondergrow.
iv. It is **certain** your plants will double in height after 2 weeks if you use Wondergrow.

This question encourages you to think about statistical claims and to use evidence from data to agree with or disagree with a claim.

Take a first look at the data; what are you first instincts? Does wondergrow double the height of any of the plants? All of the plants? Some of the plants?

What does the **mean** height tell you? Calculate the **mean** height before and after the treatment with wondergrow. What has wondergrow done to the **mean** height of the plants?

What about the **range** of heights? What was the **range** of heights before the treatment with wondergrow? and after?

What does the **range** tell you about the heights of the plants?

Looking at the data; how likely is it that if you use wondergrow it will double the height of your plants after 2 weeks?

Certain? Why? Why not?

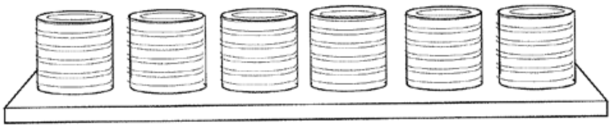
Impossible? Why? Why not?

Likely? Why? Why not?

Unlikely? Why? Why not?

Q. 5 JCFL

Kai has 6 tins of paint but the labels have come off.



He knows that he had

White, Magnolia, Yellow, Rose, Midnight Sun, and **Cream**

Kai picks a tin.

He thinks that the probability that he will pick **magnolia** is 1/6.

Kai is **right.**

Explain why.

Yetunde knows that students in his school liked to watch these sports:

Soccer Gaelic Football Rugby

He says:

"*The probability that the next person I meet likes to watch Rugby is 1/3, because there are three sports*"

Do you agree with Yetunde? Explain why.

Think! How many cans of Magnolia paint are there?

How many cans of paint are there altogether?

Can you see now why Kai is right when he says the probability of choosing a can of magnolia paint is 1/6?

Think about your school; if you wanted to know the probability of a student liking soccer, rugby or Gaelic football how would you go about finding out?

Would you have to survey the students? Or would you agree with Yetunde there is no need to survey the students because there are three sports so the probability of someone liking soccer must be 1/3?

Q. 6 JCHL

During May 2010, 110 cars were taken to a car testing station.

The results showed that 36 had defective brakes and lights, 42 had defective brakes, and 47 had defective lights. A car will not pass the test if it has one or more of these defects.

Display the information in a Venn diagram.

What is the probability that a car chosen at random

a) Failed the test

b) Passed the test

c) Had exactly one defect.

Q. 7 JCOL

Sarah, Jo, Alan and Amy want to find out what people think and do about child labour.

They are preparing a questionnaire.

Here are some questions they suggest:

Sarah: Are you a member of a human rights organisation? Yes/No **Jo:** Are children important? Yes/No **Alan:** Don't you agree that making young people work is very, very cruel? Yes/No **Amy:** Do you buy products from shops that sell goods manufactured by children? Yes/No

Choose two of these questions that **you** think ***should not be used***.

Whose questions are they?

Explain why you think these two questions should not be used

Write an extra question that **you** would use. People should be able to answer the question with ‘Yes’ or ‘No’.

Q. 8 JCOL

A librarian asks you to do a survey of the people using the library. She does not want you to talk to any of the members.

She chooses these five headings:

Time of day, Sex, Age, What was borrowed, Reason for borrowing the item

She gives you this record sheet:

Time of day	Sex	Age		What was borrowed				Reason for borrowing the item
	M/F	Under 40	40 or over	CD	DVD	Novel	Ref Book	

It is easy to use the heading “**Sex**” because it is usually easy to see if a person is male or female.

Pick one heading which is harder to use for collecting information.

Explain why it is harder to use

For the rest of this question you will need to think about surveys which **you** have done.

Think back to a sheet which **you** have used to collect information.

What were you collecting information about?

Write down one heading you used. It can not be the same as any of the librarian's headings.

Was the heading easy to use for collecting information?

Explain why.

Think about designing questionnaires. It is likely that you have done a statistical investigation in class and may have had to ask people questions in order to get information or data. **Bias** is something you should always consider when you are asking people questions. The way you ask the question can influence the answers that people give, this is known as bias. If you ask a biased question your data is **unreliable** and you can’t really be sure that is what the person who answered really thinks.

Q. 9 JCFL

The youth club is planning a trip

This is what each person chooses

Cinema	Sarah, Amy, Mags, John, Eamonn, Sean, Padraig, Mary, Steven, Anne, Erica, Paul
Bowling	Ross, Charlie, Roy Bernie, Amanda, Adrian, Hannah, Erin
Quasar	Brendan, Pete, Lauren, Gavin, Paul, Ciaran

Display this data in a way that will allow you to answer the questions below.

Where do most people want to go?

The Youth leader decides to ask everyone to write their choice on a piece of card and places these in a hat.

The Youth leader pulls 1 piece of card from the hat. This is where they will all go. What is the probability that Adrian will get his choice?

Q. 10 JCHL

Rosin and Peter wanted to see which of the two restaurants in town gives the best value for money.

They decided to visit each restaurant over a two week period, order a meal and record the number of chips on their plates. The results are recorded below

Lucy's Lunches	Number of chips on the plate													
	33	34	34	35	34	32	34	33	36	30	32	33	34	35
Dave's Diner	39	26	25	42	35	47	42	39	24	30	37	42	26	25

Display the results in a way that will allow you to compare the two sets of data.

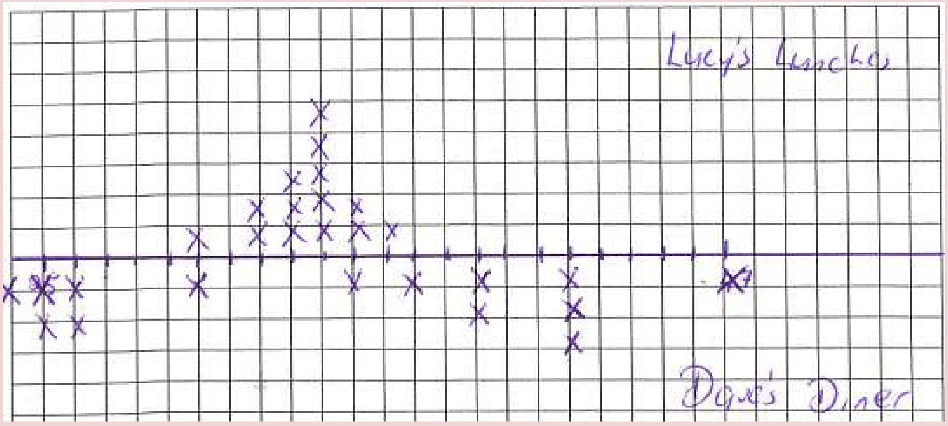
Help Rosin and Peter use their data to decide which restaurant gives the better value for money.

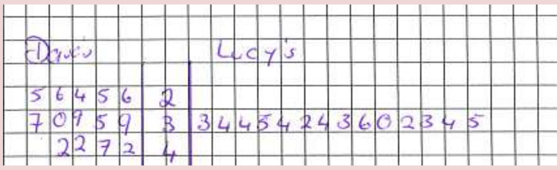
How would **you** investigate which restaurant gives the best value for money?

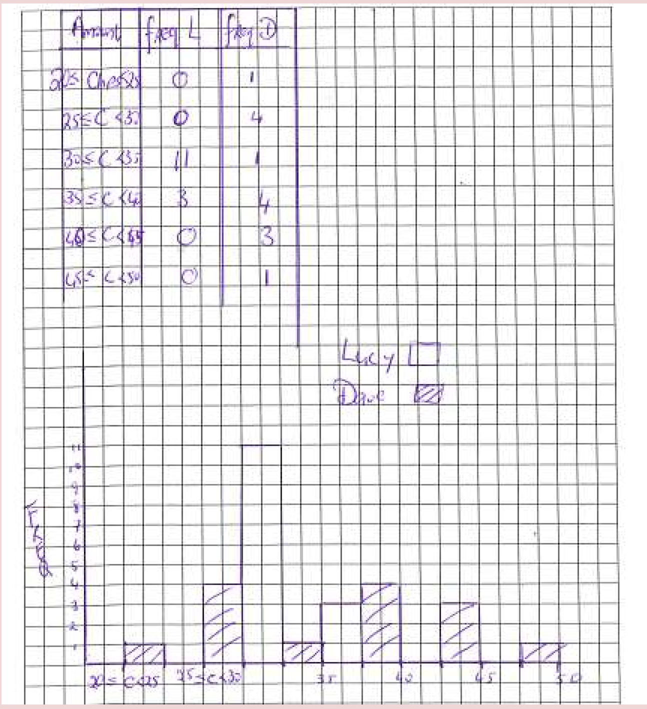
Once you have gathered the data remember you need to display it in a way that allows you to see patterns in the variation.

Think about the different displays you have used throughout the JC course. Think about what makes each of these displays useful.

Look at the displays below that other students made of the data. Which do you think is most useful and why? How would you display this data?







Q. 11 JCHL

A group of students was investigating the weight of coins.

They weighed a **sample** of 48 *two-cent coins* and recorded the weights to the nearest .01g in the table.

Weight of a two-cent coin (g)							
3.08	2.95	3.10	2.97	3.02	3.10	2.84	3.00
3.10	3.12	3.03	2.85	3.09	3.05	3.15	3.09
3.06	3.05	3.11	3.07	3.02	3.05	3.06	3.18
3.05	3.14	4.52	3.43	3.00	3.09	3.07	2.94
3.05	3.15	3.15	3.00	3.04	3.07	3.06	3.17
3.13	3.05	3.11	3.12	3.03	3.09	3.00	3.01

a) None of the 48 coins weighs the same. What do you think may be a cause of this variation?

b) Display the data in a way that will allow you to describe it. What do you think is a typical weight for a two-cent coin? Explain your reasoning.

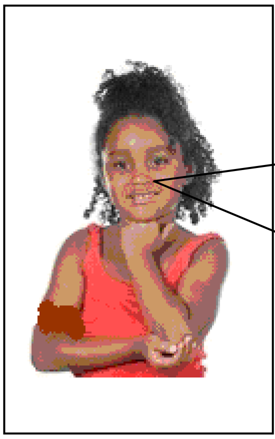
c) Based on the data in the table what do you think the weight of a 49th two-cent coin will be? Are you more confident to give an actual value or a range of values? Explain your thinking.

d) Are there any unexpected values in this data set? How do you know?

Q. 12 JCOL

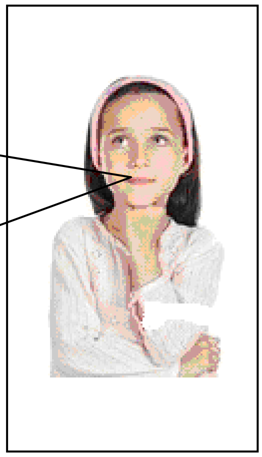
Sarah, Ellie and Samir were measuring the length of the science lab. Sarah used a **metre stick**. Ellie and Samir used a **measuring tape**.

Each group of students measured the length of the lab 6 times and recorded the measurements to the nearest cm in a table



Well each time I worked out how many paces it took for me to walk down the lab. Then I measured the length of a pace with the metre stick and multiplied that by the number of paces and wrote it in the table.

Samir and I worked together. He held the tape against the wall and I walked to the opposite wall and read the measurements. Then we changed, I stayed at the wall and Samir walked down and took the reading; we measured it 6 times.



Measuring Instrument	Length of lab (cm)					
Metre stick	850	870	910	880	915	885
Measuring tape	889	888	889	889	888	888

a) Why do you think there are differences in the measurements in the table?

b) Which method gave a more accurate measurement of the length of the science lab? Explain why you think this is the case.

c) If you were asked to state the length of that science lab, what answer would you give? Explain why.

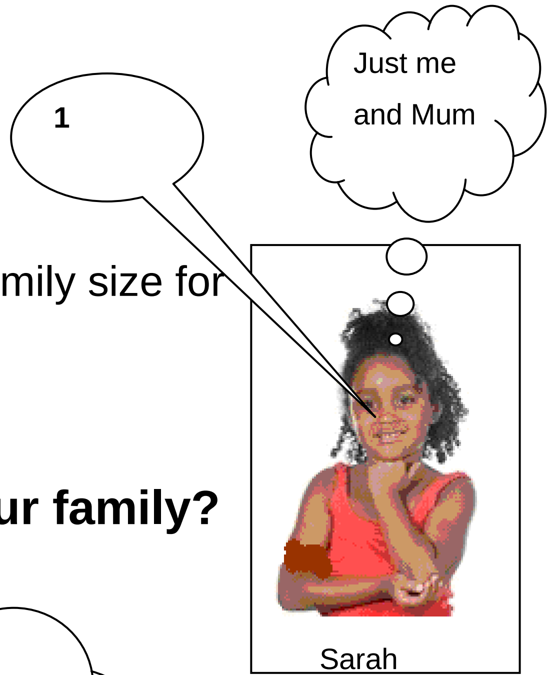
Q. 13 JCOL

Esperanza was investigating family sizes.

She wanted to find out what was a typical family size for people in her class

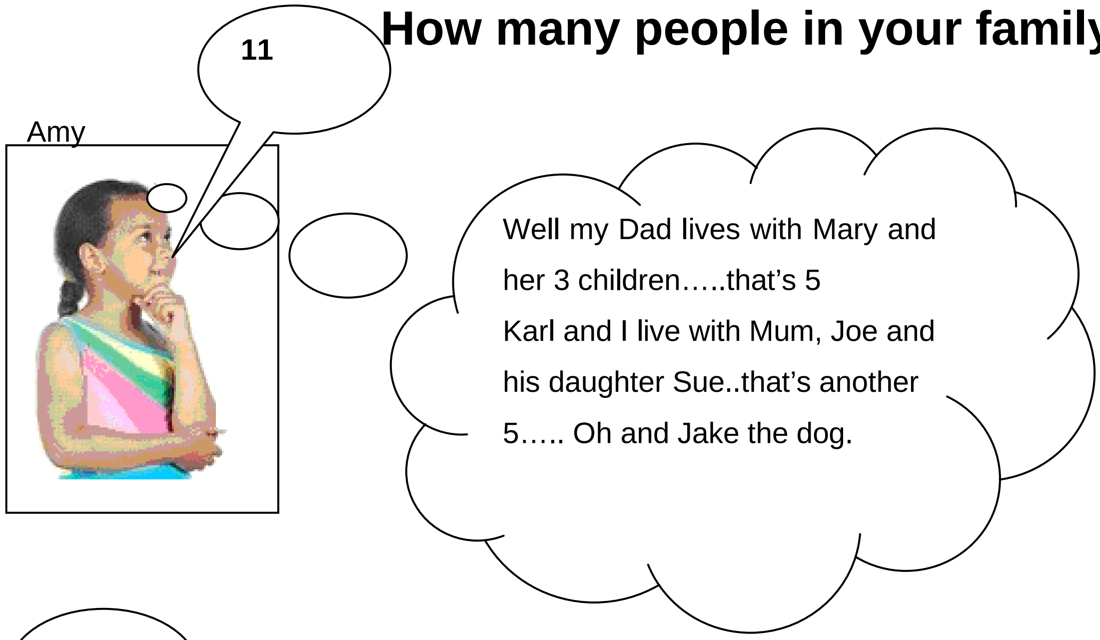
She asked four classmates:

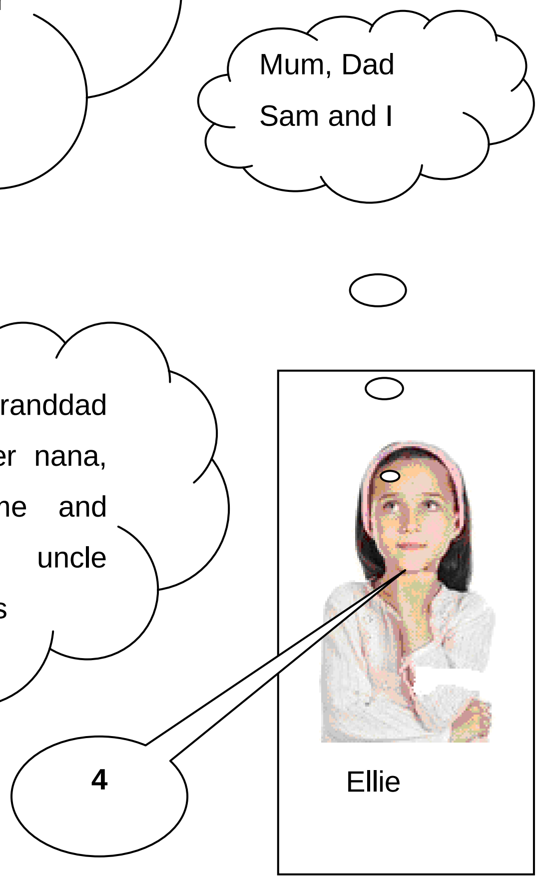
How many people in your family?



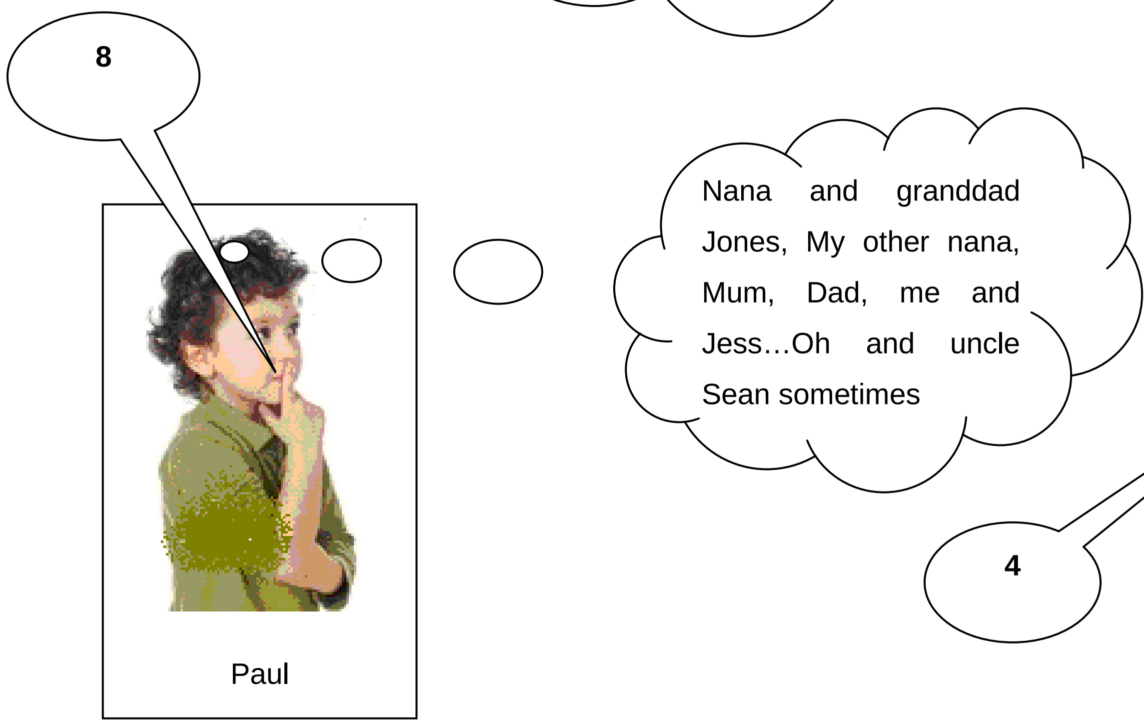
Sarah

Amy





Ellie



Paul

Esperanza's data is not **reliable**? Explain why

If you wanted to find out the typical family size of people in **your** class what would **you** do to make sure the data you gathered was **reliable**?

Below is such data gathered by a group of 3rd Year students.

3,3,3,4,4,4,4,4,5,5,5,6,7,10,4,2,2,5,5,8,4,3,3,3,4,4,4,3,3,3

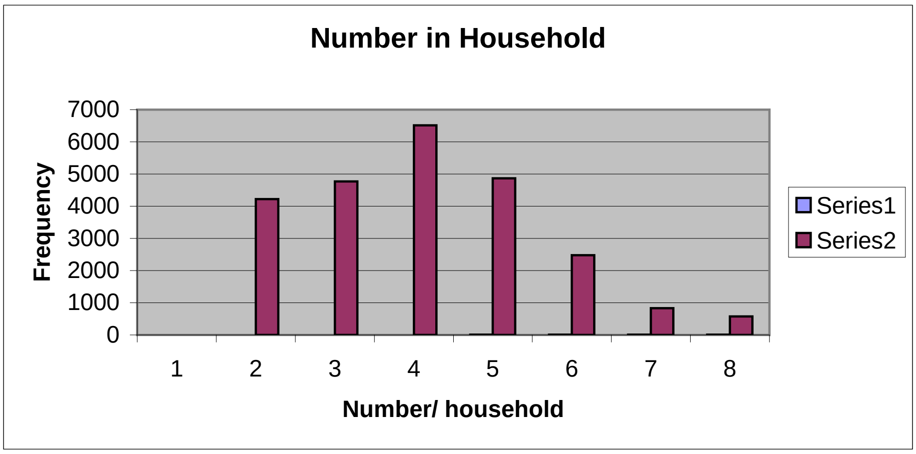
Display this data in a way that allows you to see a pattern in the data.

What would you say is the typical family size for students from this 3rd year group?

Justify your answer with evidence from the data.

The diagram shows the distribution of household sizes from households in the Carlow area.

Household size	Frequency
2	4218
3	4773
4	6512
5	4870
6	2478
7	833
8	572



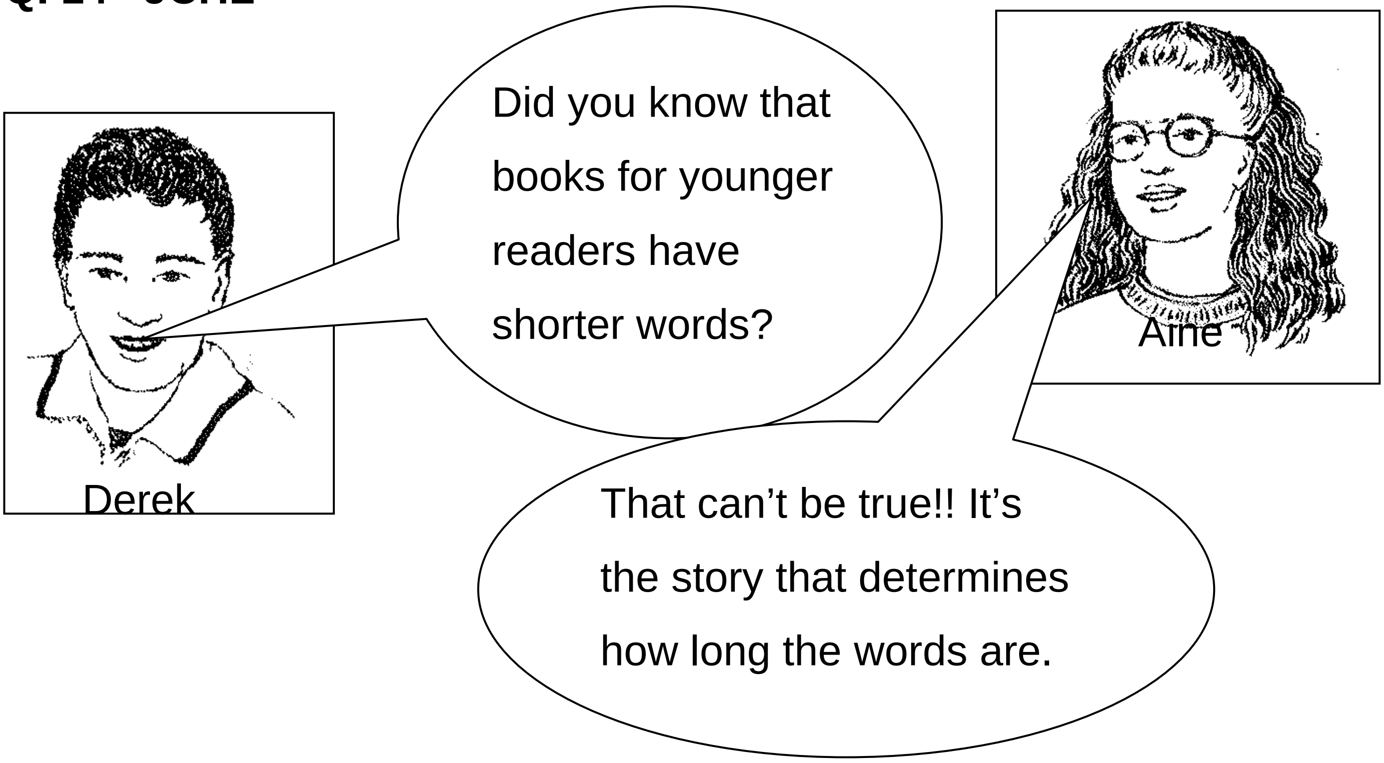
Compare the set of data from Carlow with the 3rd year set.

How likely is it that the 3rd year group surveyed all live in Carlow?

Justify your answer with evidence from the data.

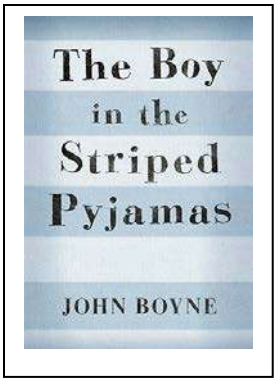
I think it is (impossible, unlikely, likely, certain) that the 3rd year group lived in Carlow because

Q. 14 JCHL



Use the following extracts from ***Charlie and the chocolate factory*** and ***The boy in the striped pyjamas*** to see if there is any evidence that the words in ***The boy in the striped pyjamas*** are longer than the words in the ***Charlie and the chocolate factory.***

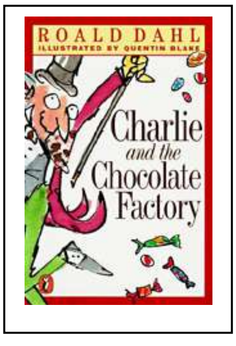
Extract 1



The boy in the striped pyjamas

…One afternoon, when Bruno came home from school, he was surprised to find Maria, the family's maid — who always kept her head bowed and never looked up from the carpet — standing in his bedroom, pulling all his belongings out of the wardrobe and packing them in four large wooden crates, even the things he'd hidden at the back that belonged to him and were nobody else's business. ….

Extract 2



Charlie and the chocolate factory

….. he did. He told all the workers that he was sorry, but they would have to go home. Then, he shut the main gates and fastened them with a chain. And suddenly, Wonka's giant chocolate factory became silent and deserted. The chimneys stopped smoking, the machines stopped whirring and from then on, not a single chocolate or sweet was made. Not a soul went in or out…..

What would **you** do differently if you were going to look for evidence to support Derek’s theory?

Think about

- how you would select your sample of words from both books
- the size of your sample.

How does the display help you decide on the *typical* value?

Do the different contexts make it easier or more difficult to state the *typical* value?

How do the *mean*, *mode*, *median* and *range* relate to the *typical* value?

Q. 15   JCFL

Samil drops a tray with these objects on it.

They fall on a wooden floor

How likely are they to break?

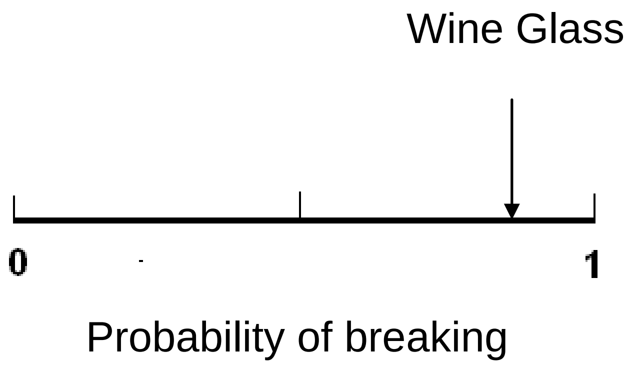
Put them all in order

Most Likely ……………………………

……………………………

……………………………

Least likely ……………………………



The arrow on the number line shows the probability of the wine glass breaking

Explain why this is a sensible place to put the arrow.

Put more arrows on the line to show the probability of the other objects breaking.

Think about the probability of the calculator breaking.

Roughly how far along the line did you put its arrow?

Write this as a **decimal,** a **percentage** or a **fraction**.

Q. 16 LCOL

8 out of every 10 men in Ireland are colour blind.

What is the probability that a man in Ireland picked at random is not colour blind?

Two men in Ireland are picked at random. What is the probability that

a) they are both colour blind?

b) at least one is colour blind?

Q. 17 JCHL

Devise a game of chance that can be played in school to raise money for charity.

Your game must involve **two independent events,** for example, ‘tossing two coins’ or ‘rolling a die and tossing a coin’.

- Invent a clear set of rules for your game. You should clearly state the conditions for **winning**, **losing** and getting your **money back**.
- Give an example of how you might **“win”** the game, how you might **“lose”** the game, and how you might just get your “**money back”.**
- Decide on how much you will charge to play the game and how much a player will get if they win the game.
- Create a sample space showing **all** possible outcomes.
- Calculate the probability of winning the game.
- Assuming that 250 students play the game, calculate the profit you are likely to make.
- Will you definitely make this profit? Explain why, or why not.

Examine this piece of student work.

Roll a dice and Pick a card

- Get 6 and Ace — Win €10
- Get Odd and Ace — get money back
- Anything Else — Lose

	1	2	3	4	5	6
A	1A	2A	3A	4A	5A	6A
NA	1NA	2NA	3NA	4NA	5NA	6NA

P(Win) = $\frac{1}{12}$

P(Money back) = $\frac{3}{12}$

P(Lose) = 8/12

240 play at €1 each €240

P(Win) = $\frac{1}{12}$ $\frac{1}{12} \times 240 = 20$ 20 win €10 = €200

P(Money back) = $\frac{3}{12}$ $\frac{3}{12} \times 240 = 60$ €60

€260

It is likely that this game will cost us €20

I think I'll change the rules that you only win €1

So P(Win) = $\frac{1}{12} \times 240 = 20$ €20

and you get money back if you get 1 and an Ace

P(Money back) = $\frac{1}{12}$ $\frac{1}{12} \times 240 = 20$ €20

It is likely this time the game will make €200. We won't definitely win this because this is only the theoretical probability. This matches the experimental one over loads of trials 240 is a lot but 1000 might be more likely to definitely get the €200 But it will be close

What do you think of this piece of work? What would you do differently?

Q. 18 JCOL

Sarah and Caoimhe were raising money to help buy a scanner for the local hospital. They created a game of chance called ***Score 10 to win 10*.** They charged **€1** to play and the prize for winning was **€10.**

Rules: Spin a spinner numbered 1-4 and throw a die.

- If the total is odd get your money back.
- **Score 10** and Win 10.

Create a sample space showing all the **possible outcomes** Identify those outcomes that are a "win" and those that will get the money back.

Calculate the probability of a player winning the game.

Calculate the probability of a player getting their money back.

If 240 students play the game, how much will Sarah and Caoimhe raise for the hospital scanner?

Change the rules so as to increase the profits, assuming the same number of students play.

Q. 19 JCFL

The table below shows the main causes of death in Ireland in the years **2000**, **2001** and **2002.**

						Numbers
	2000		2001		2002	
Principal cause	**Males**	**Females**	**Males**	**Females**	**Males**	**Females**
			Republic of Ireland			
Circulatory diseases	6,449	6,217	6,109	5,777	5,886	5,709
Respiratory diseases	2,326	2,537	2,156	2,316	2,118	2,212
Cancer	4,079	3,587	4,038	3,594	4,066	3,433
Road traffic accidents	326	109	316	95	269	85
Suicides	395	91	429	90	371	80
All other	2,617	2,658	2,643	2,649	2,507	2,612
Total deaths	**16,192**	**15,199**	**15,691**	**14,521**	**15,217**	**14,131**

What was the main cause of death of **males** in Ireland in 2002?

Newspaper reports in 2002 stated that the number of **male suicides** was on the increase in Ireland. Is there evidence from the table to support this claim?

What has happened to the total number of deaths in Ireland in the period from 2000 to 2002?

Newspaper reports claim that more **young Irish males** commit suicide than **young Irish females.** Is there evidence in the table to support this claim?

Q. 20 JCOL

A lottery takes place twice a week in Ireland. **Seven** balls are randomly chosen from **45.** If you match the first six balls drawn then you win or share the jackpot.

What is the probability of drawing ball number **5**?

What is the probability of drawing an **odd numbered** ball?

What is the probability of drawing an **even numbered** ball?

Below is a table showing the frequency in brackets that the balls have appeared to date.

1 (58)	**2**(69)	**3**(70)	**4**(68)	**5**(68)	**6**(53)	**7**(59)	**8**(53)	**9**(65)
10(63)	**11**(55)	**12**(58)	**13**(74)	**14**(51)	**15**(66)	**16**(48)	**17**(62)	**18**(58)
19(58)	**20**(54)	**21**(69)	**22**(61)	**23**(61)	**24**(64)	**25**(40)	**26**(58)	**27**(67)
28(66)	**29**(59)	**30**(55)	**31**(58)	**32**(68)	**33**(58)	**34**(51)	**35**(60)	**36**(70)
37(55)	**38**(65)	**39**(72)	**40**(57)	**41**(54)	**42**(55)	**43**(39)	**44**(70)	**45**(53)

Write a list of the 5 most frequent numbers and the 5 least frequent numbers.

People often think that ***13*** is an unlucky number and will not choose it. Is there any evidence in the table to suggest that ***13*** is an unlucky number? Explain your reasoning.

Q. 21 JCFL

The table shows the total rainfall that fell in Ireland in the **month of July** over a 51 year period from 1958 to 2008.

Year	Total Rainfall (mm)	Year	Total Rainfall (mm)	Year	Total Rainfall (mm)
1958	110	1975	28	1992	69
1959	45	1976	83	1993	60
1960	140	1977	26	1994	65
1961	52	1978	51	1995	70
1962	68	1979	47	1996	37
1963	24	1980	39	1997	54
1964	47	1981	36	1998	54
1965	79	1982	9	1999	35
1966	37	1983	18	2000	44
1967	84	1984	31	2001	30
1968	16	1985	107	2002	68
1969	44	1986	58	2003	46
1970	68	1987	33	2004	38
1971	63	1988	80	2005	84
1972	41	1989	10	2006	18
1973	79	1990	48	2007	119
1974	100	1991	26	2008	112

If 130mm of rain fell in Ireland in July 2009, complete the table below showing the total rainfall for each of the decades listed.

Years	Total Rainfall (mm)
1960-1969	
1970-1979	
1980-1989	
1990-1999	
2000-2009	

Display your data in a way that allows you to see a pattern in the variation.

Is there any evidence to support the claim; *Julys in Ireland are getting wetter?*

Q. 22 JCHL

The table shows the number of hours per day spent by 3rd year and TY students playing on a games console.

Number of hours spent playing on a games console	Number of TY Students	Number of 3rd Year Students
1		
2	1	1
3	2	3
4	1	1
5	1	2
6	5	2
7		3
8		
9	1	3
10		1
11		3
12		2
13	3	3
14	1	1
15	4	
16	4	3
17	2	1
18	4	2
19	4	4
20	3	2
21	2	
22	3	
23	1	
24		
25	1	4

Display the data in a way that allows you to compare the two groups.

Which group of students spends more time playing on a games console? Give evidence from the data to support your answer.

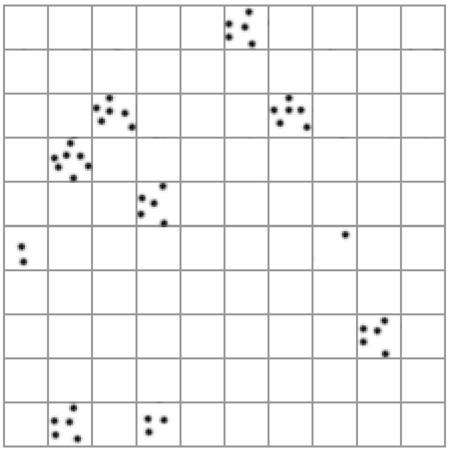
Q. 23 JCHL

Photographs taken from satellites help officials keep track of the number of different objects on the earth below.

They could be keeping track of the number of animals in remote areas, or counting the number of sheep on a farmer's land to ensure that they are claiming the correct subsidy.

Below is a photograph of a sub-Saharan region in Africa, populated by gazelle. The area is divided into 100 sub-regions. Some of the sub-regions are obscured by cloud.

A gazelle



Based on the number of gazelle in this sample, make an estimate of the number of gazelle in the entire region.

Is it possible that there are more gazelle than your estimate?

Is it possible that there are less gazelle than your estimate?

How might a smaller sample size have affected your estimate? Consider some smaller samples from the photograph to help in explaining your thinking.

Q. 24 JCFL

The table below reports the approximate lowest frequency and approximate highest frequency of the hearing ranges for humans and five other animals. Frequency is measured in Hertz (Hz), which is 1 vibration per second.

Animal	Lowest frequency	Highest frequency
Human	64	23,000
Cat	45	64,000
Dog	67	45,000
Gerbil	100	60,000
Goldfish	20	3,000
Parakeet	200	8,500

Which animal has the smallest hearing range?

What animals can hear lower frequencies than a parakeet can?

Which animal can hear the highest frequencies?

What animal has a hearing range more than twice that of a human?

Q. 25   JCFL

The table below shows some cloud formations and their recorded distances above the Earth.

Cloud Type	**Distance above the Earth (miles)**
Altocumulus	4
Altostratus	5
Cirrostratus	6
Cirrus	7
Cumulonimbus	2
Cumulus	3
Stratus	1

What is the **median** distance above the Earth of the cloud formations listed above?

Q. 26 JCOL

Sam asked the 29 students in 3rd year how many times they were absent from school last term. The results are shown in the table below.

Unfortunately a blot covers part of the table.

Number of days absent	Frequency
0	3
1	10
2	9
3	
4	
More than 4	1

a) (i) What might the table look like if the blot was not there?

Give two possible answers.

Number of days absent	Possible Frequency 1	Possible Frequency 2
0	3	3
1	10	10
2	9	9
3		
4		
More than 4	1	1

(ii) How many possibilities are there, other than the two you have shown?

b) (i) Working from Sam's original table, calculate (if possible) the mode, median, mean and range of the data.

Q. 27 JCOL

The plot shows the heart rates of a group of 3rd years half way through their PE class.

Heart Rate (Beats/min)

8	7 7 9
9	6 7 7 7
10	4 5 6 6 8
11	1 3 4 9 9
12	2 5 5 8
13	0 0 3 4 7
14	3 5
15	7
16	9

11 | 3 = 113

Optimum heart rate is between 110 and 140 beats per minute.

How many heart rates shown on the plot are between 110 and 140 beats per minute?

Guidelines state that, if the heart rate exceeds 165 beats per minute, exercise should be stopped immediately.

Should any of these students stop exercising immediately? Explain your answer

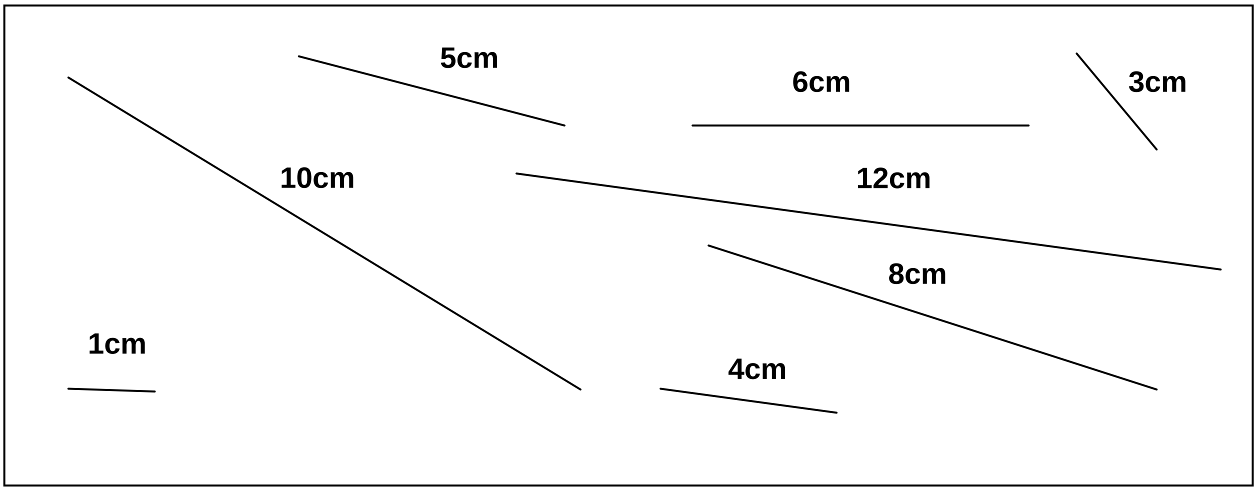
Set B: Review Materials – Junior Certificate
Strand 1 and Strand 2

Before you attempt these questions have a look at the "Geometry and Trigonometry Tutorial"
https://emea67395290.adobeconnect.com/_a858841383/maths/
the Using Geometry and Trigonometry to solve problems presentation
https://emea67395290.adobeconnect.com/_a858841383/p15113229/presentation
and the *concept of slope* presentation.

This set of questions; compiled in two documents are intended to help you as you review your work in preparation for Paper 2 in the Junior Certificate examination. They are not intended to be exact matches of what will come up in the exam but they should give you a flavour of how the concepts can be examined in context. Other questions and activities can be found in the Mathematics Resources for Students on the student zone at www.ncca.ie/projectmaths

Task 1: JCOL

Jason had these sticks.



He wanted to make two right-angled triangles.

He picked up three sticks and found he could **not** make a right-angled triangle.

(a) Which three sticks might Jason have picked up?

(b) Why did these three sticks not make a right-angled triangle? Use a theorem from your geometry course to help you explain.

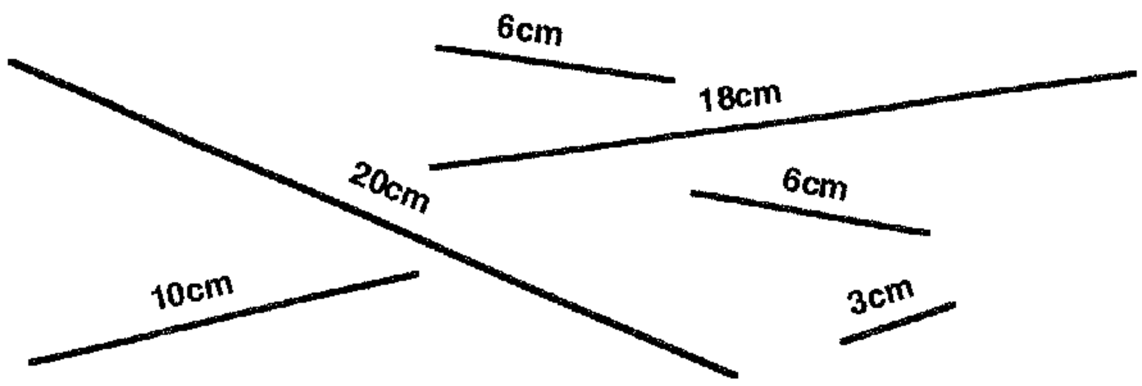
(c) Choose three sticks that will make a right-angled triangle.

(d) Choose three other sticks which will also make a right-angled triangle.

(e) Show how you know that, in each of these cases, the sticks will make a right-angled triangle.

Extension to LCOL

Jason had these sticks.



He wanted to make two triangles.

He picked up three sticks, but he could **not** make a triangle with them.

(a) Which three sticks might Jason have picked up?

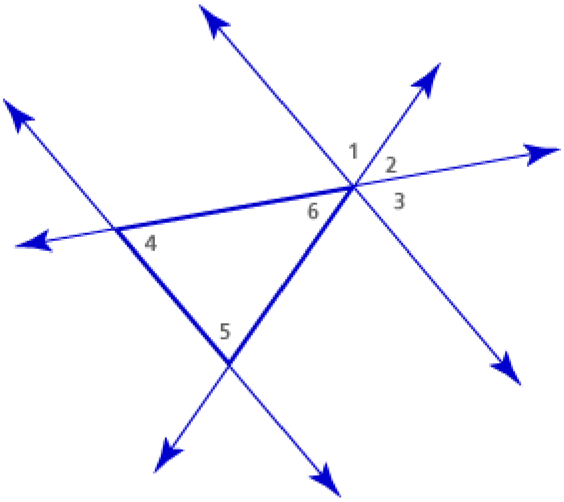
(b) Why did these three sticks not make a triangle? Use a theorem from your geometry course to help you explain.

(c) Choose three sticks that will make a triangle?

(d) Choose three other sticks that will also make a triangle?

Accurately construct one of the triangles using the measurements that you have chosen. Show all your construction marks.

Task 2: JCHL



(a) In the picture above, what is the sum of angles numbered 1, 2 and 3? Explain the reasoning that led to your answer.

(b) In the picture above, the angle numbered 1 is equal in measure to one of the angles in the triangle. Which one?

(c) In the picture above, angle 2 is equal in measure to one of the angles in the triangle. Which one?

(d) In the picture above, the angle numbered 3 is equal in measure to one of the angles of the triangle. Which one?

(e) Use your answers to questions (a)-(d) to explain why the sum of the angles numbered 4, 5 and 6 is 180°.

Explain why this would be true for any triangle, and not just the one pictured. Use the following guide.

Given:

To prove:

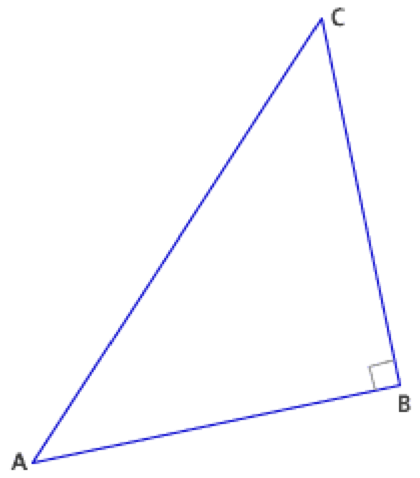
Proof:

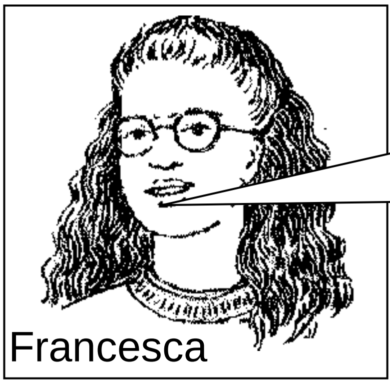
Note: The proof of this theorem is not examinable. However, you should be able to set out your explanation using the sequence of thinking that was involved in the task above.

Task 3: JCOL

Francesca and Leo were dissecting shapes and rearranging them to form new shapes. One of their tasks is shown below

Transform this right-angled triangle into a rectangle by dissecting it and rearranging the parts.

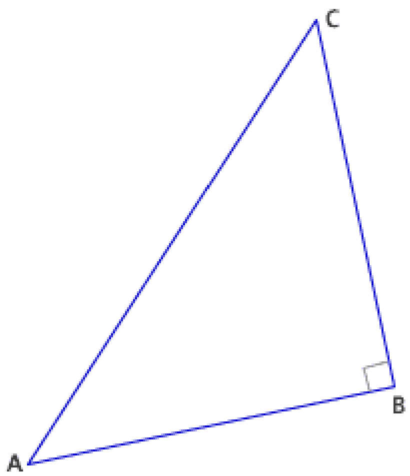




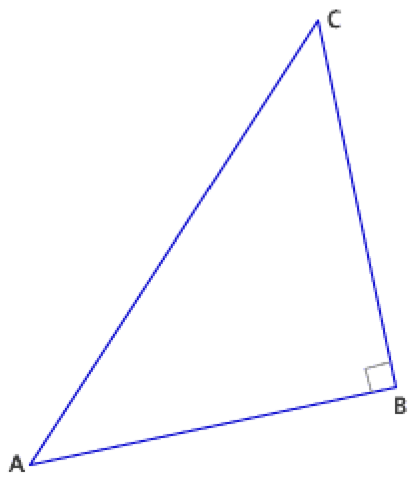
Construct the perpendicular bisector of CB. Label the midpoint of CB as the point D and the point of intersection of the bisector and AC as the point E.

Yes I see…If we cut off the little triangle CED and sort of turn it so that CE lines up with AE. They are equal ‘cos of that **theorem.** Then we will have a rectangle.

On the diagram below accurately follow Francesca's instructions. Show all construction marks clearly.



(a) Accurately complete Leo's instructions in the box below



(b) What theorem was Leo referring to when he said " *..CE will line up with AE. They are equal 'cos of that theorem…"*

Why are CE and AE equal?

(c) Leo says that the re-arranged shapes will make a rectangle.

Do you agree with Leo?

Explain your thinking. You will need to write down some **properties** of a rectangle and show how the figure Leo ends up with has these **properties**.

Task 4: JCHL

Calculate the height of

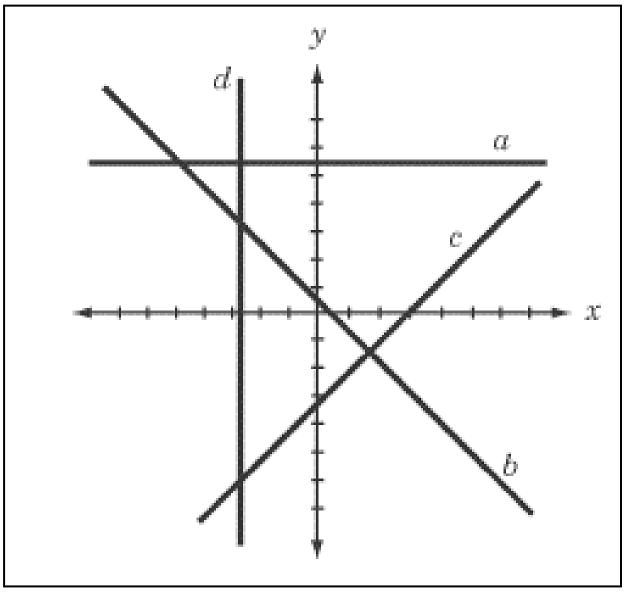
(a) an equilateral triangle of side length x

(b) an isosceles triangle of side lengths x and y

What are the properties of an isosceles triangle? Sketch the triangle and mark the length of each of the sides.

Task 4: JCHL

Of the four lines pictured below, one has a slope of 0, one has a slope of 1, another has a slope of –1, and another has an undefined slope. Complete the table to show which is which.

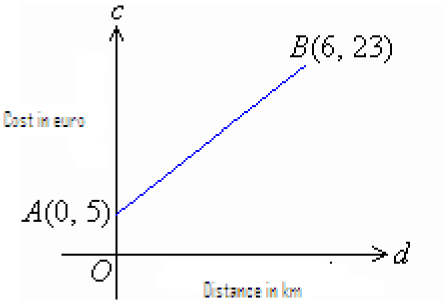


Line *a* has slope:	
Line *b* has slope:	
Line *c* has slope:	
Line *d* has slope:	

Give reasons for your choices.

Task 5: JCHL

The cost of transporting documents by courier can be represented by the following straight line graph.



Use the graph to help you work out how the courier charges customers.

How much would the courier charge to transport documents a distance of 30km?

Eoin’s company is working to a tight budget. They need to transport documents 15km across the city.

Eoin gets a quote from another company who claim to be cheaper. Their advertisement reads

Cheapest courier in town
We charge €3.50/ km and no standing charge.

Which courier should Eoin ask to transport his documents across the city? Justify your decision by comparing the prices charged for this job by both companies.

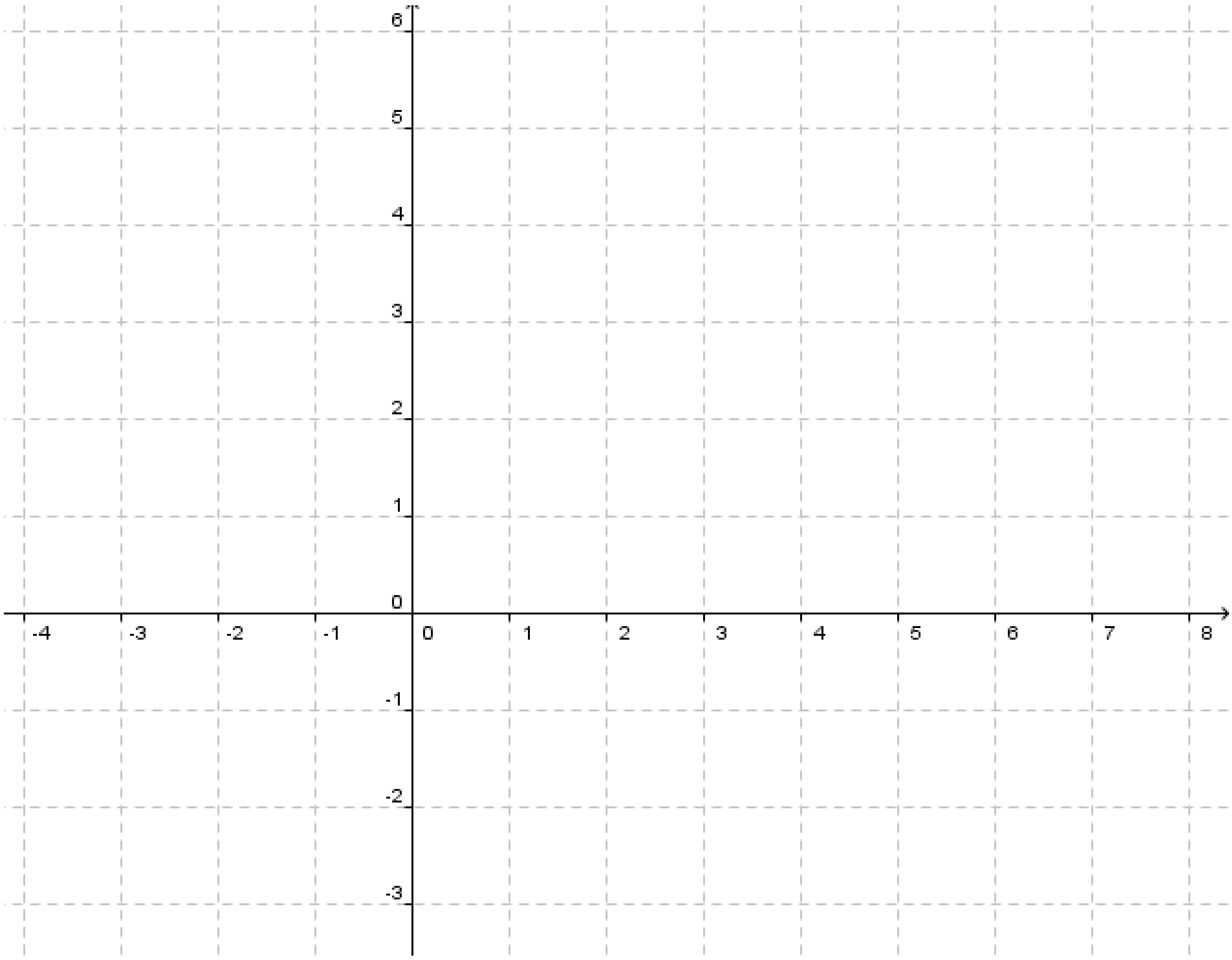
Can the second company stand by their claim of being the cheapest courier in town?

Justify your answer by referring to a graphical representation of each company's charges.

Task 6: JCFL

Draw the following shapes on the coordinate axes.

- a square
- a right angled triangle
- an isosceles triangle
- a parallelogram



Write down the co-ordinates of the **vertices** of each shape

Square (….., ….)   (….., ….)   (….., ….)   (….., ….)

Right- angled triangle (….., ….)   (….., ….)   (….., ….)

Isosceles triangle (….., ….)   (….., ….)   (….., ….)

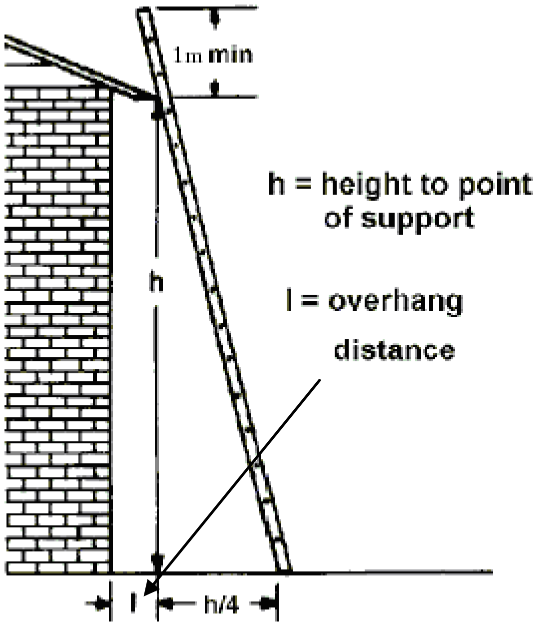
Parallelogram (….., ….)   (….., ….)   (….., ….)   (….., ….)

Task 7: JCOL

You're locked out of your house and the only open window is on the second floor, **7m** above the ground. You need to borrow a ladder from one of your neighbours. There's a bush along the edge of the house, so you'll have to place the bottom of the ladder **3m** from the house. What length of ladder do you need to reach the window?

Sketch a mathematical diagram. Use straight lines to represent the **wall of the house**, the **ladder** and the **ground**. Mark each line with the correct measurement. If you do not know the measurement mark it x.

Use your geometry to calculate the length of the ladder needed.



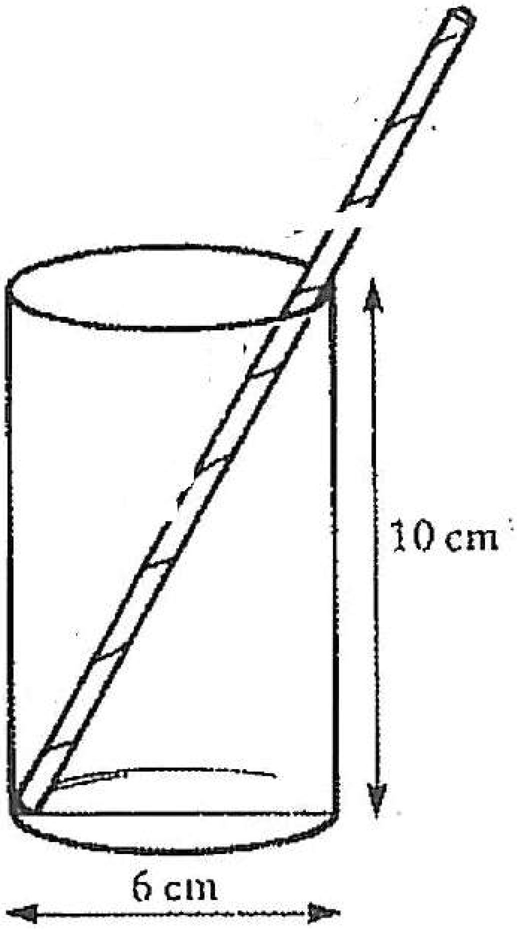
The diagram is taken from a safety manual that accompanies a particular brand of ladder.

You can see from the diagram that the length of a ladder that you would choose for a particular job depends on the height of the object it will be leaning up against.

Generalise this relationship. Use surds in your answer.

Does the angle the ladder makes with the horizontal depend on the height of the object it is leaning up against? Explain your answer.

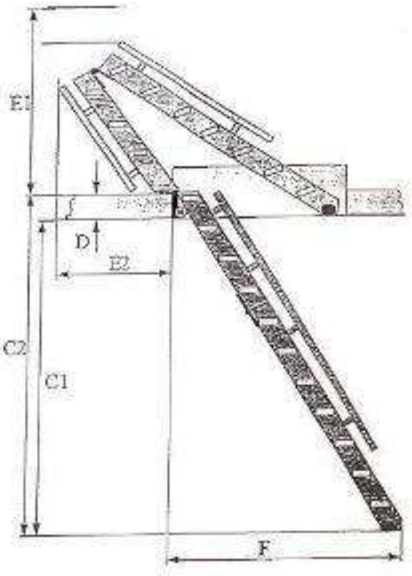
Task 8: JCHL



The straw is 20cm long. Calculate the length of the straw sticking out from the top of the glass.

Task 9: JCHL

An installation guide for the *Sandringham Electric Attic ladder* is shown below

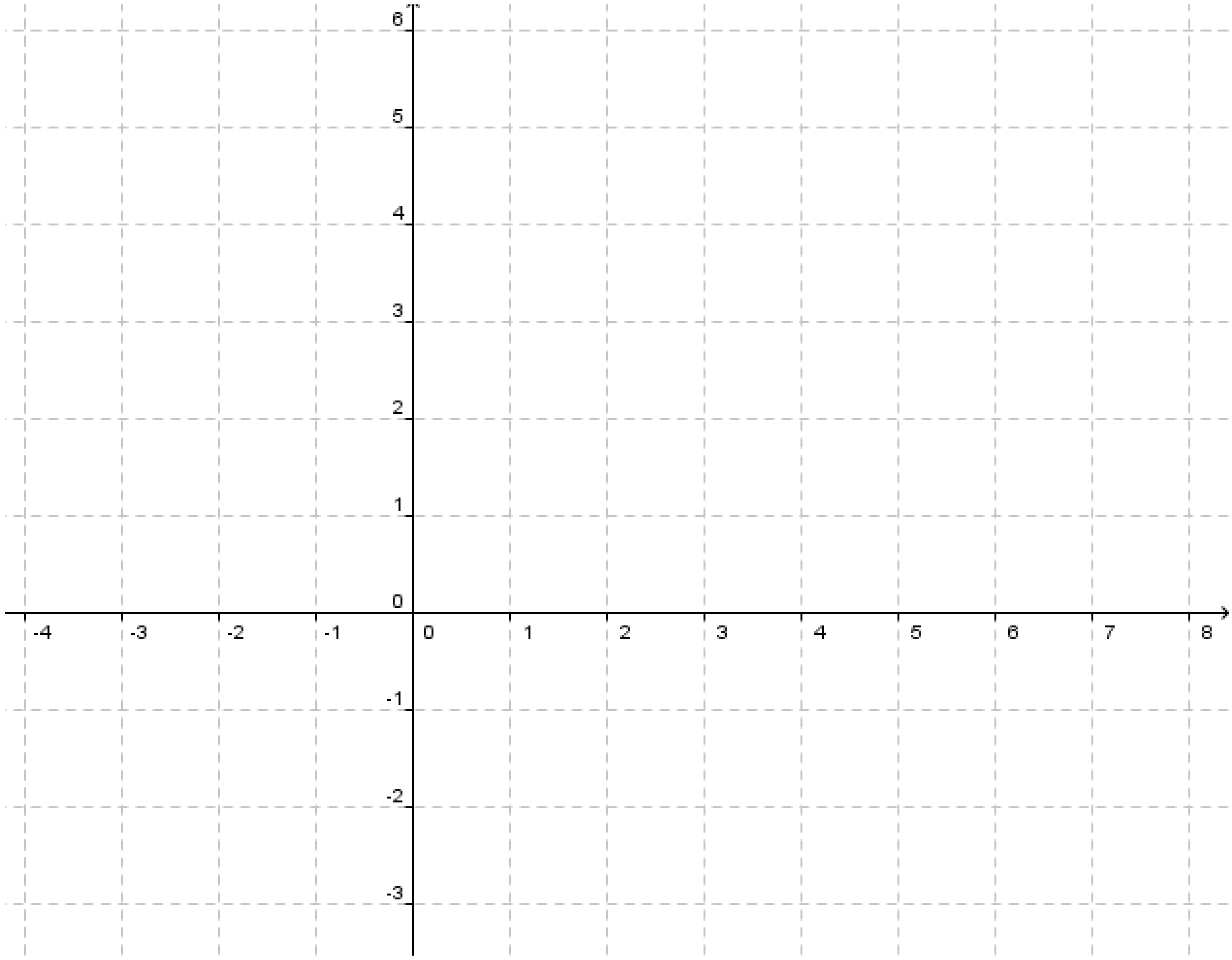


Ladder Size	Floor to Floor Height C_2	Storage swing and Height	Horizontal Distance F
	250cm	145cm	159cm
Length [illegible] cm	260cm	155cm	166cm
[illegible]	270cm	[illegible] cm	173cm
	280cm	175cm	180cm
Length 150cm	[illegible] cm	146cm	166cm
[illegible]	280cm	166cm	[illegible] cm

Some ink has spilled on the table. Use your mathematics to find the lengths covered by the ink blots. If you are unable to calculate a particular missing length, explain why you are unable to do so.

Task 10: JCFL

Plot the points (-1, 2), (5,2), (-1,-1) (5,-1) on the grid.



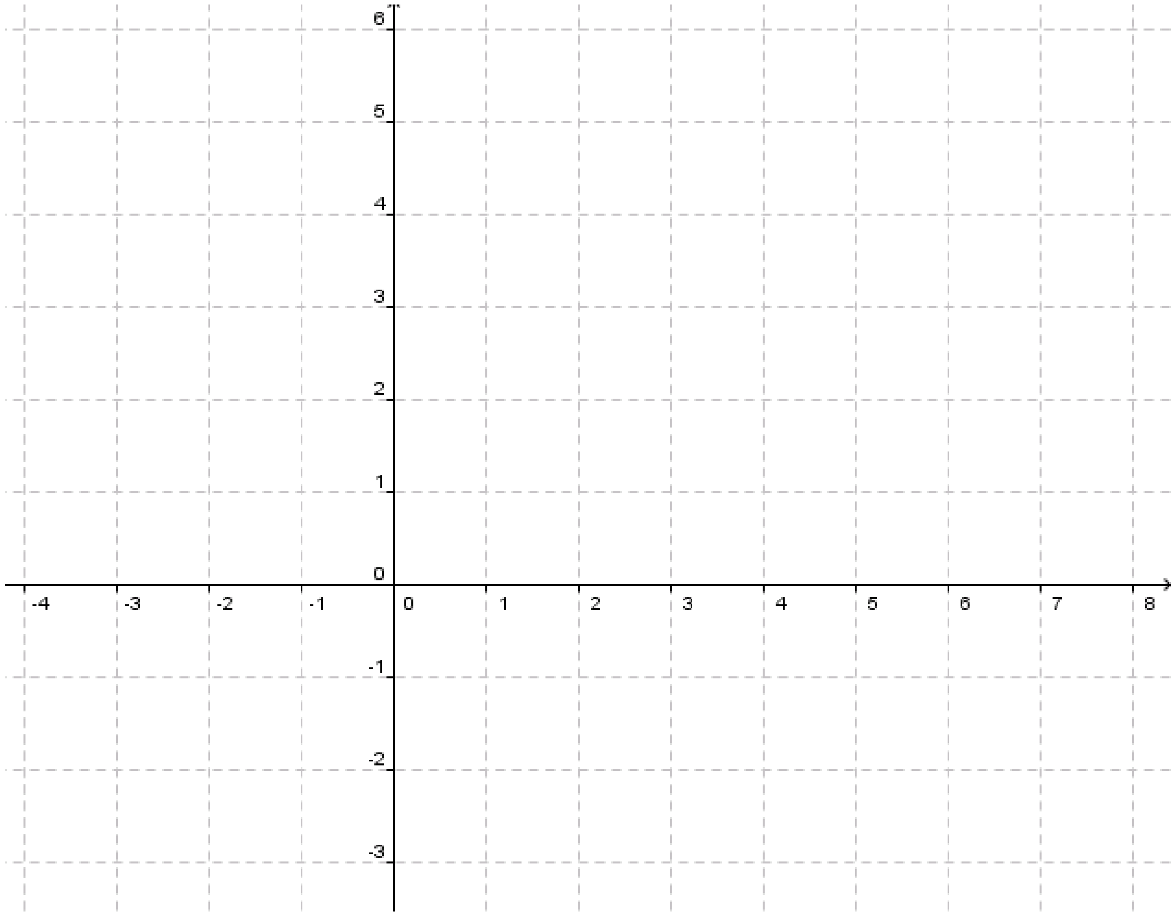
Join them to form a shape.

What is the name given to this shape?

Write down two properties of that shape.

Task 11: JCOL

Plot the points (-1, 2), (5, 2), (-1, -1) (5, -1) on the grid.



Join them to form a shape.

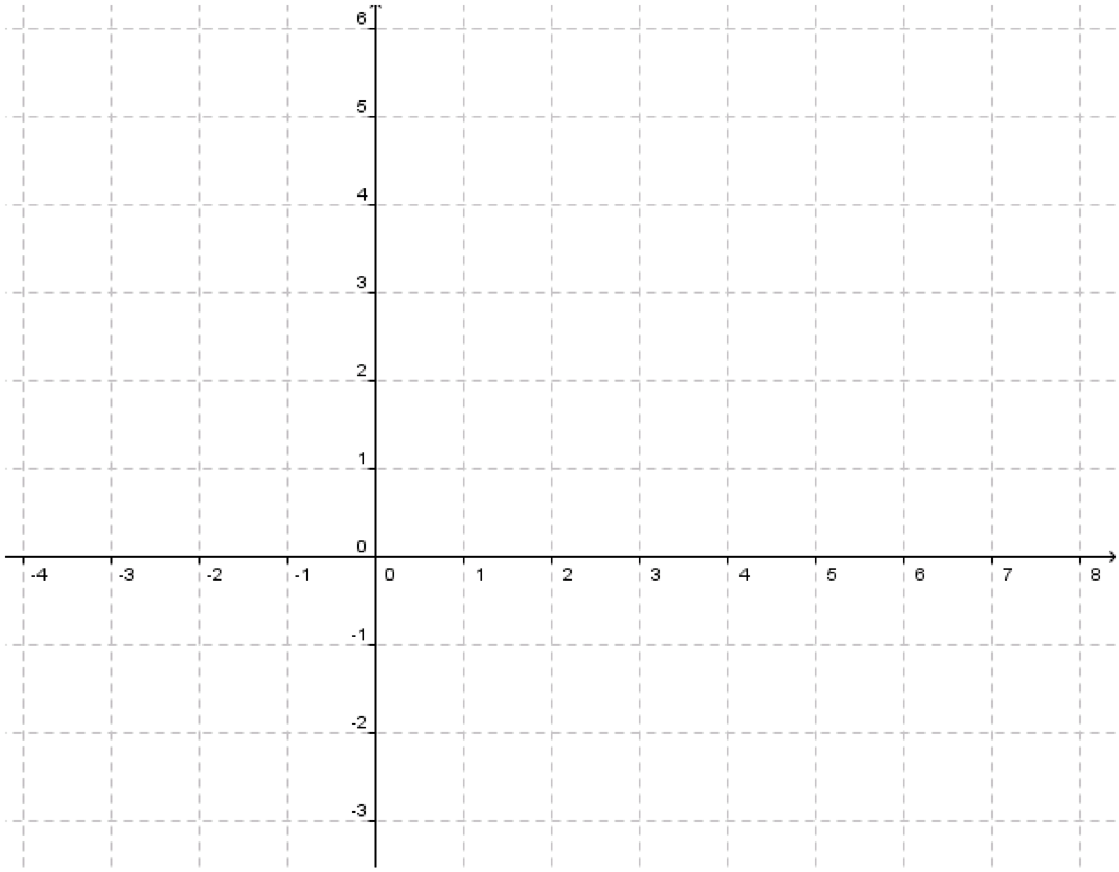
What is the name given to this shape?

Write down two properties of that shape.

Prove that the shape you have made on the grid has those properties.

Task 12: JCHL

Plot the points (8, 2), (8,-4), (2,-1) on the grid.



Join them to form a shape.

What is the name given to this shape?

Write down two properties of that shape.

Prove that the shape you have made on the grid has those properties

Task 11: JCFL

Say which of the following is true by ticking the correct box

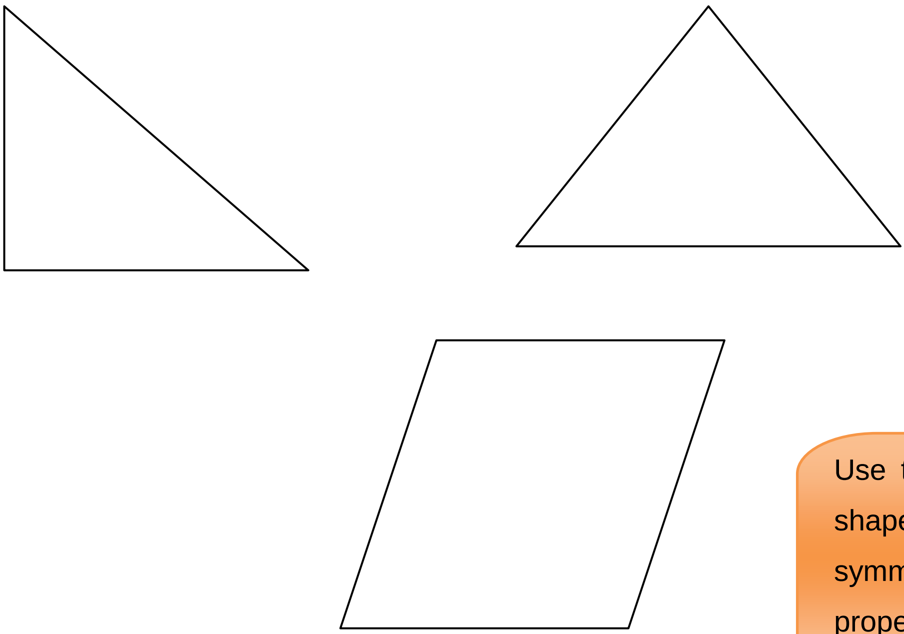
In the diagram below:

One **F** is the image of the other after an axial symmetry ☐

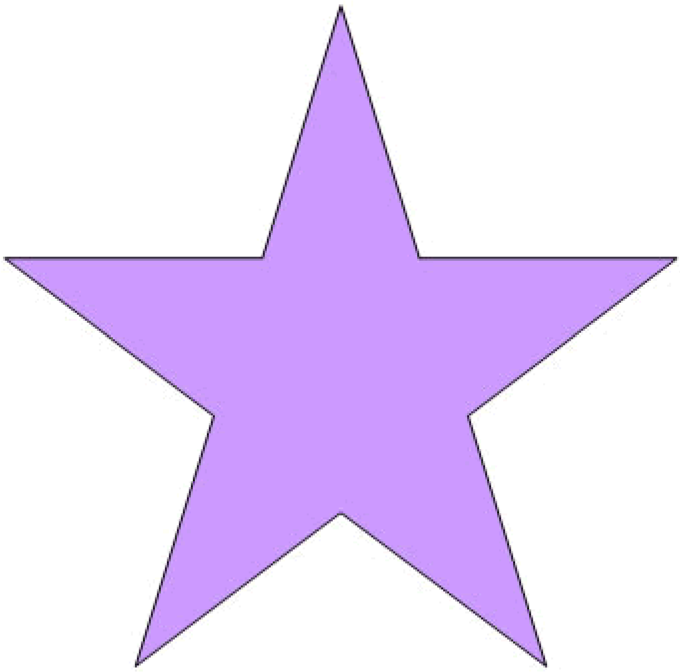
One **F** is the image of the other after a central symmetry ☐

One **F** is the image of the other after a translation ☐

Draw as many lines as symmetry as possible for each figure below.



Use tracing paper or fold the shape to help find the lines of symmetry. Look for patterns or properties of these lines.

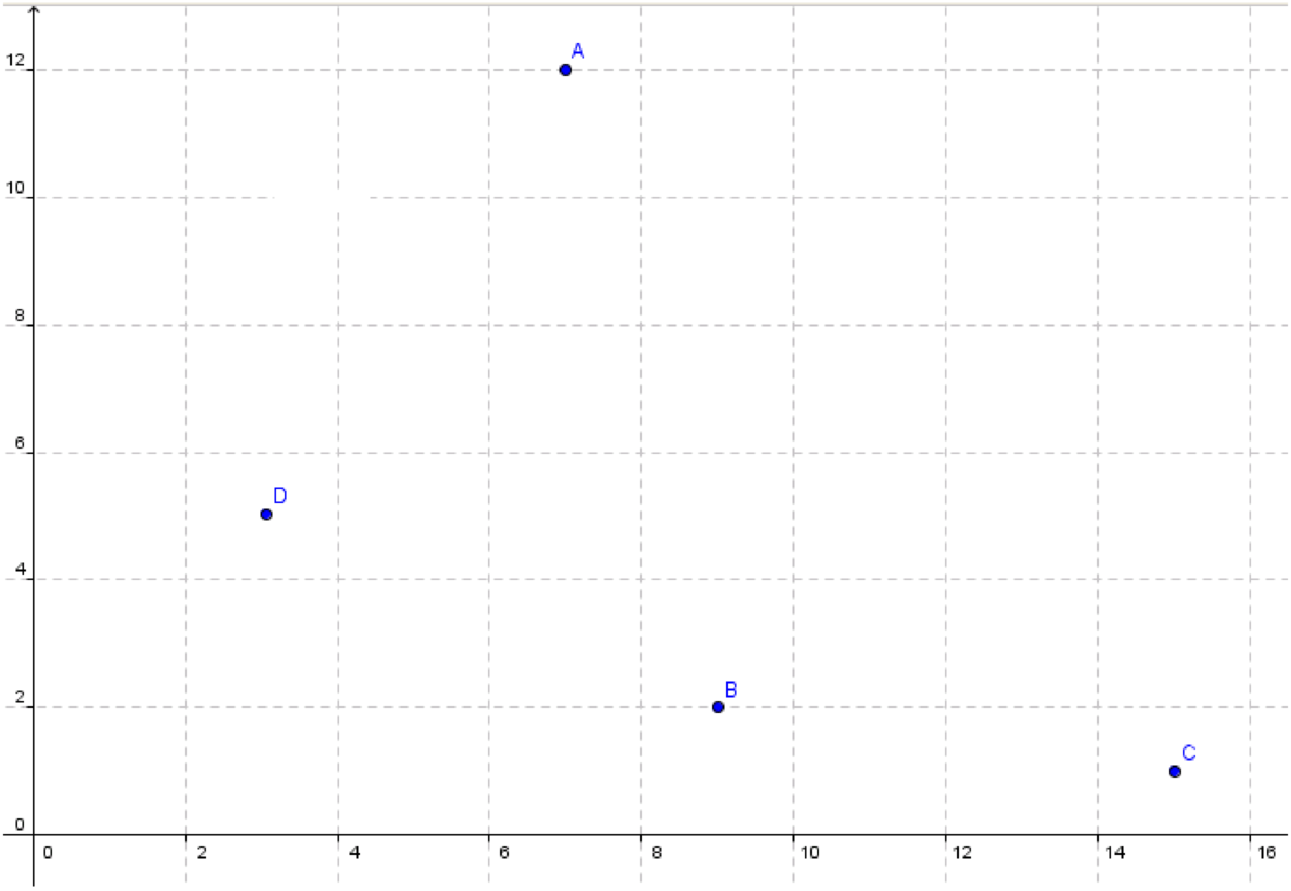


Task 12: JCHL

A monorail similar to the one shown was planned for an amusement park.

The original plans had the supports located as shown on the grid below.

A (7, 12) B (9, 2) C (15, 1) D (3, 5)



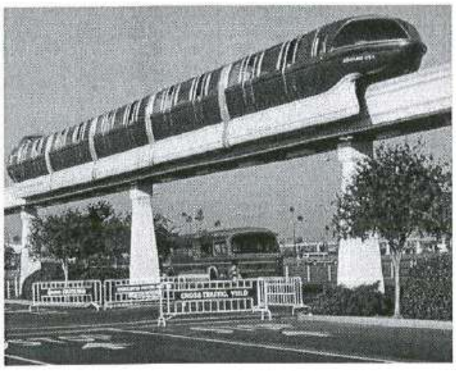
In order to make room for a car park, engineers have decided to demolish the supporting pillar C and relocate it.

They have also decided that, on the plans, the new support pillars should be able to form a parallelogram

Plot the new location of the supporting pillar and write its coordinates. Label it $\mathbf{C_1}$.

Use the definition or properties of a parallelogram to verify that the new layout is a parallelogram. You must use the slopes of the sides, the lengths of the lines or both to verify your answer.

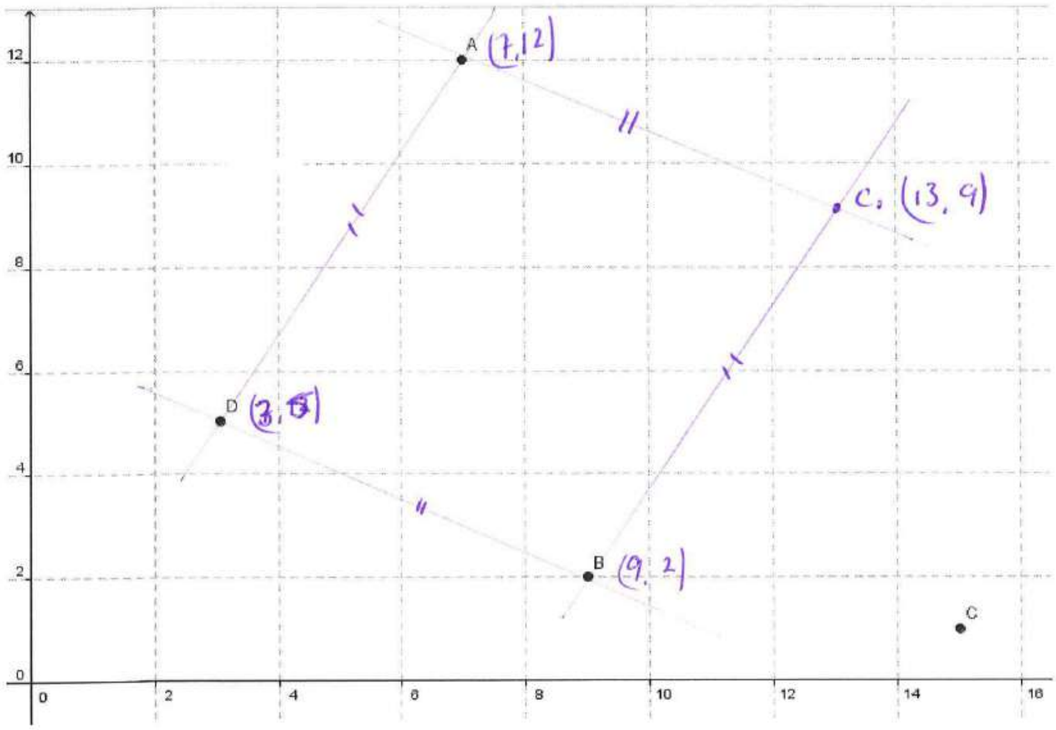
JCHL



A monorail similar to the one shown was planned for an amusement arcade.

The original plans had the supports located as shown on the grid below.

A (7,12) B(9,2) C (15,1) D (3,5)



2 pairs of // sides
Opposite sides equal in length

Use the definition or properties of a parallelogram to verify that the new layout is a parallelogram. You must use the slopes of the sides, the lengths of the lines or both to verify your answer.

AD || BC, if slopes are equal

Slope AD $= \frac{12-5}{7-3} = \frac{7}{4}$ Slope BC, $= \frac{9-2}{13-9} = \frac{7}{4}$

DB || AC, if slopes are equal

Slope DB $= \frac{2-5}{9-3} = \frac{-3}{6} = \frac{-1}{2}$ Slope AC, $= \frac{9-12}{13-7} = \frac{-3}{6} = \frac{-1}{2}$

$|AD| = \sqrt{(12-5)^2 + (7-3)^2}$
$= \sqrt{49 + 16}$
$= \sqrt{65}$

$|BC,| = \sqrt{(13-9)^2 + (9-2)^2}$
$= \sqrt{16 + 49}$
$= \sqrt{65}$

$|AC,| = \sqrt{(13-7)^2 + (9-12)^2}$
$= \sqrt{36 + 9}$
$= \sqrt{45}$

$|DB| = \sqrt{(9-3)^2 + (2-5)^2}$
$\sqrt{36 + 9}$
$= \sqrt{45}$

Task 13: JCHL

Linda Armstrong is a professional forester.

In order to calculate how much wood is in a forest she must measure the height of the trees.

We don't count and measure every tree in the forest. That would simply take too long. Foresters measure a few bits of the forest and, on the basis of those bits, estimate what the whole forest contains

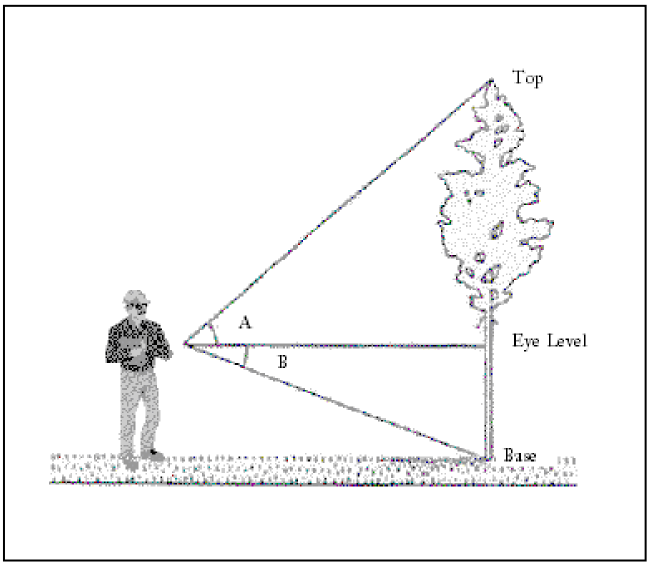
Suggest a way that Linda could choose which "bits" to **sample.**

Foresters need to monitor the growth of trees. They measure their heights each year and can determine the **yearly tree growth.**

You can determine the tree's height by using trigonometry. If you measure the horizontal distance between yourself and the tree, and measure the angles leading to the tree's top and its base, using a simple instrument called a clinometer, you have enough information to calculate the tree's height.

Linda used this technique and obtained the measurements in the table

Angle A	55°
Angle B	25°
Distance from Linda to Tree	2.5m



Use trigonometry to calculate the height of the tree.

Linda wanted to compare the growth of trees on a tree farm with the growth of trees in a forest. The stem and leaf plot shows the yearly growth, in cm, of a selection of trees in both the tree farm and the forest.

Tree Farm		Forest
1	1	0 1 3
3 3	2	1 5 7
7 2 1	3	0 1 3 8 9 9
9 8 0	4	2 3 4 4 8
1 0	5	0 1 3 7

| 2 | 5 = 25cm

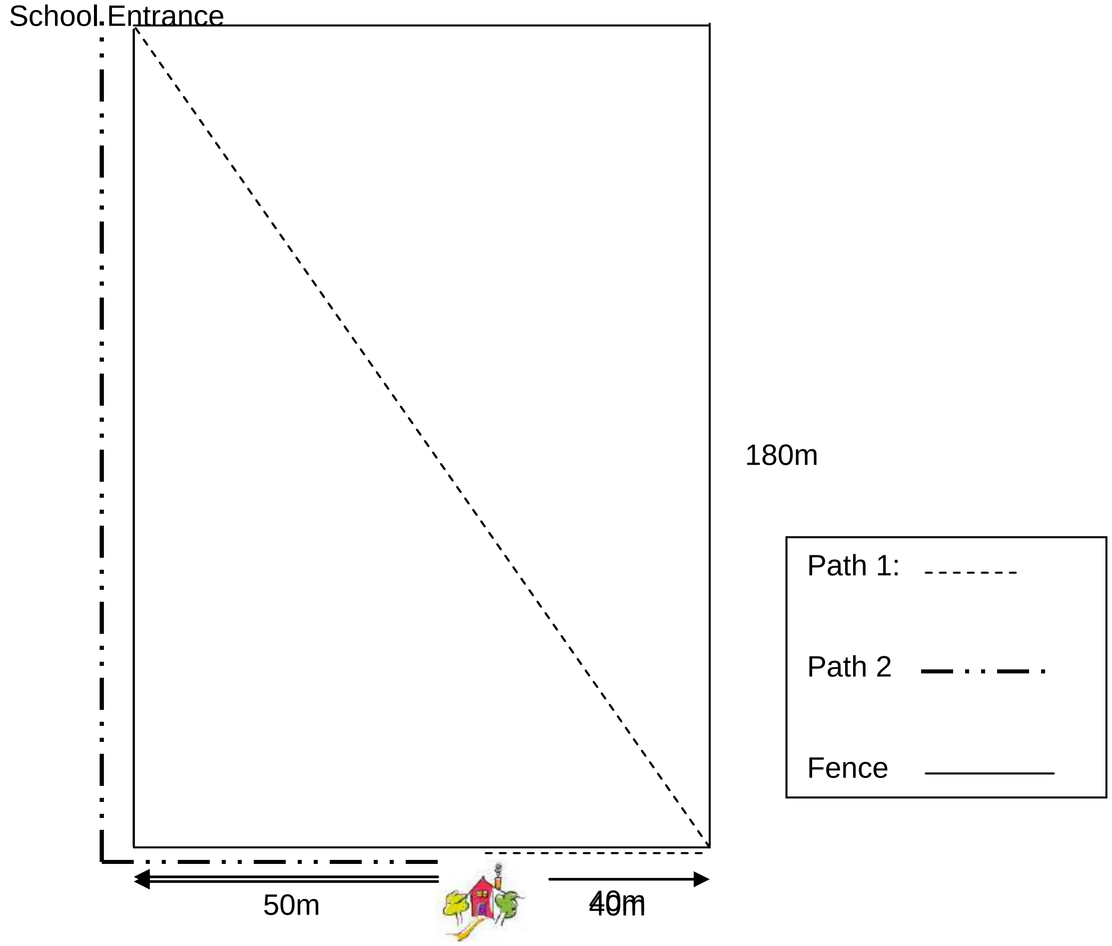
1 | 5 | = 51cm

What is the difference between the **median** yearly growth in cm of the selection of trees from the forest and those from the tree farm?

Is there any evidence to suggest that the trees on the farm grow quicker than the trees in the forest?

Task 14: JCFL

Mark's house is located near the **perimeter fence** of his school playing field.



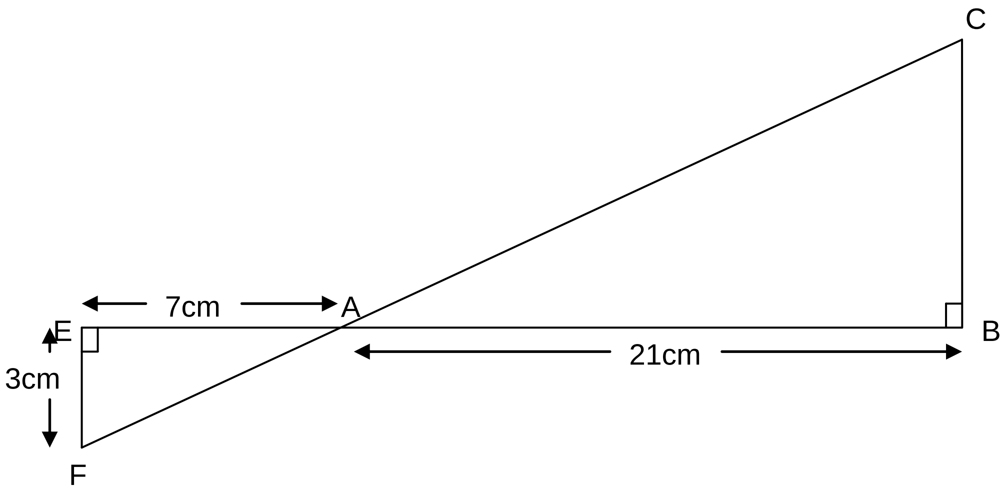
There are two paths Mark can take to school. He can walk along the fence, go through the gate to the playing field and walk across the field (Path 1), or walk around the perimeter fence (Path 2).

What is the difference in distance between the two paths?

Task 15: JCOL

In the diagram below, line segments CF and BE intersect at A.

Is ΔAEF similar to ΔACB?

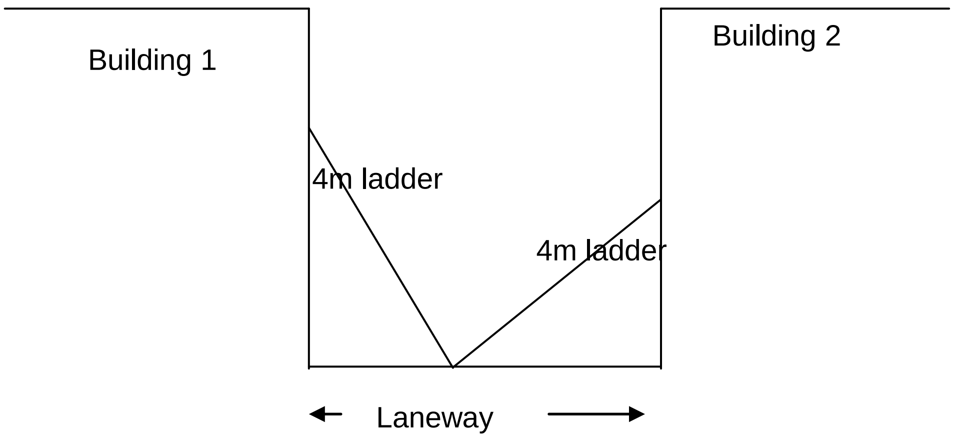


Give a reason for your answer.

Calculate the lengths AF, AC and CB.

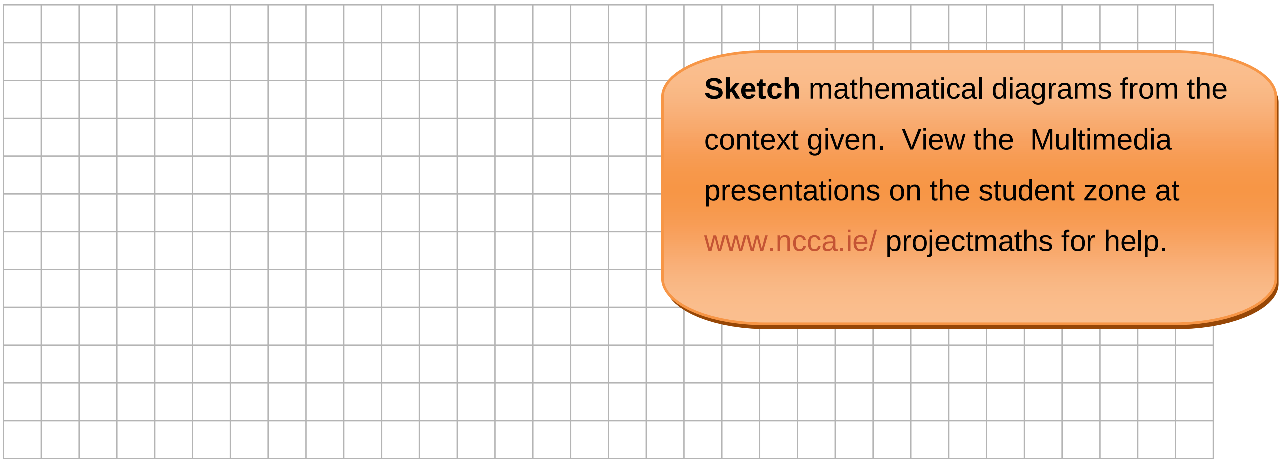
Task 16: JCOL

Jack placed a 4m ladder in a laneway between two buildings



When he tilted the ladder one way it reached 2.5m up the wall of building 1 and when tilted the other way it reached 1.5m up the wall of building 2.

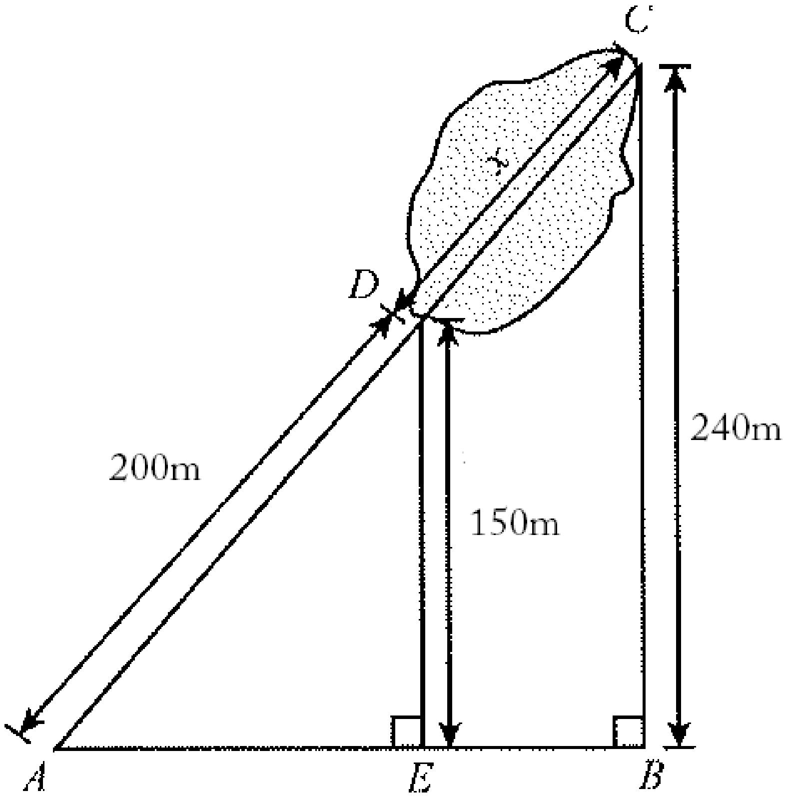
What is the width of the laneway?



Task 17: JCOL

A surveyor wants to determine the distance across a lake.

She is unable to make the measurements directly.



She will use triangles ADE and ACB.

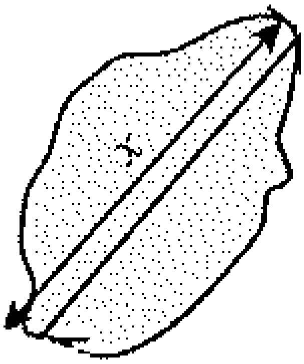
Explain why, in geometric terms, triangles ADE and ACB are similar.

Create a ratio that can be used to find the distance *x* across the lake. Use this ratio and the measurements given in the diagram to calculate x, the distance across the lake.

Task 18: JCHL

A surveyor wants to determine the distance across a lake.

She is unable to make the measurements directly.



Suggest some measurements she could make and how she could use these to determine the distance across the lake.

Task 19: JCHL

The JCDecaux advertising agency sought a building that was tall enough to accommodate an **18m** high rectangular billboard.

An employee of the company thought he had found a building that would work. He is 2m tall and, on the morning he examined the building, he cast a shadow 0.5m long. The building cast a shadow 4m long.

Determine whether or not the building will accommodate the billboard.

There are a number of ways to answer this. Visit the student tzone at www.ncca.ie/projectmaths and look at the multimedia presentations of how learners approach such problems.

Recommendations to Teachers and Students

- Use the syllabus as the main reference document in preparing for the examination. (Find your syllabus at www.ncca.ie/projectmaths). The examinations will reflect the aim, objectives, and learning outcomes of the syllabus, and will support the development of the key skills of the senior cycle curriculum.

- Remember that the learning outcomes at Ordinary Level are additional to those at Foundation Level, and that those at Higher Level are additional to those at Ordinary and Foundation levels. Accordingly, give due regard to the outcomes listed for the level(s) below the one you are dealing with. Similarly, as the Leaving Certificate syllabus builds on the knowledge and skills developed at Junior Certificate, ensure that you can recall and apply those skills too.

- Try to develop understanding of all mathematical methods employed. Skills will transfer much more readily to unfamiliar scenarios when they are based on understanding. Furthermore, you may be explicitly asked to explain or justify the methods you employ.

- Use the resources provided by the Project Maths Development Team and the NCCA. The examinations are designed on the assumption that candidates have engaged with these activities or ones of a similar type. These materials also help in interpreting the syllabus.

- Engage in activities that draw together skills and understanding from more than one area of the course.

- Be prepared for the unfamiliar. A high level of achievement in mathematics is characterised by the ability to bring insightful knowledge and well-developed skills to bear on new problems. It is not helpful to try to second-guess every conceivable type of problem that might be encountered, in order to learn off the correct method for doing each. It is more productive – both for the achievement of the objectives of the syllabus and for success in the examinations – to develop generic problem-solving skills and to have had plenty of experience in engaging with tasks that vary considerably in their level of familiarity. Teachers should make a concerted effort to expose students to problems that are not like ones they have encountered before, in order to develop their problem-solving skills.

- Ensure that basic skills are not neglected. These too are specified as syllabus outcomes and will be tested directly. Furthermore, problem solving is only possible when the basic tools needed to address the problems are readily available. Fundamental skills in arithmetic, algebra, and geometry need to be continually attended to.

- Ensure you understand the concept of a mathematical proof, that you can use valid reasoning to justify conclusions, and that you can identify and rectify deficiencies in arguments presented by others. Ensure also that you are able to reproduce whatever formal proofs are specified in the syllabus as being directly examinable.

- Be familiar with the terminology and language of the subject. When engaged in discussion and exploration, use the correct terms and seek clarification of any words that are unfamiliar.

- Read questions carefully. Information on examination papers is concise, careful, and deliberate, and it is easy to miss or misread a critical piece of information. Give careful consideration to the question before you begin answering it.

- Use common sense when thinking about questions, and reflect on your answers. If an answer seems unreasonable, this may assist in locating a mistake. Knowledge and skills that have been acquired outside the mathematics classroom are valid and useful.

- Do not be put off or upset if a problem is not working out. Some problems are intended to be challenging. When an examination task is non-routine, then you will be well rewarded for exploring the problem in a reasoned way and applying plausible lines of attack, even if you do not ultimately fully solve the problem.

- Show all your work. Partial credit will be awarded for any substantive work of merit.

- Communicate your thinking as clearly as possible, whether you are solving a mathematical problem or offering a text-based answer.

- Even if you are not asked to draw a diagram, it can often be a very helpful first step. You may gain some credit for the diagram. More importantly, the way forward with the problem very often becomes much clearer when the given information is presented on a diagram.

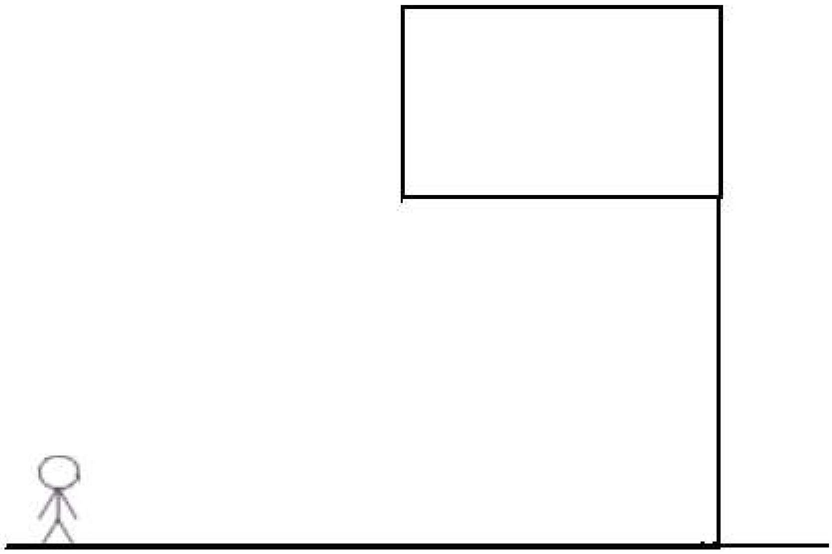
- Attempt all parts of the questions you are doing. The examiner will always search for merit in what you write. But if you write nothing, you cannot get any marks.

- If you make more than one attempt at a question, make it clear which attempt is your final version. However, you should also ensure that your other attempts remain legible. In most circumstances, you will get credit for your best attempt, even if it has been cancelled in favour of another.

- Ensure that you are thoroughly familiar with your own calculator and capable of using it efficiently and intelligently. Make sure that your calculator conforms to the rules governing the use of calculators in the State examinations, and that it has a sufficient range of features to meet your needs during the examination.

Question **(Suggested maximum time: 8 minutes)**

(a) Some students were measuring the height of a flagpole near the school. They had a measuring tape and a **clinometer.**



The following measurements were taken

Height of student	1.5 m
Distance from Student to Flagpole	2 m
Angle of elevation of top of flagpole (θ)	

(a) Add these measurements to the diagram and show how the students could use them to calculate the height of the flagpole.

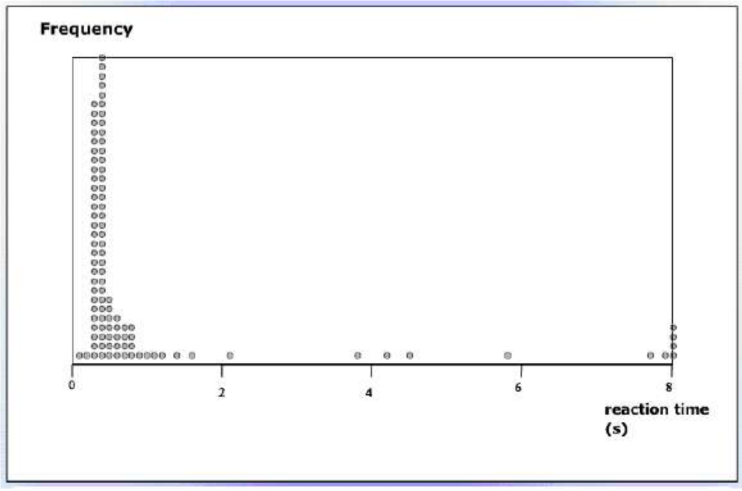
(b) The students calculated the height of the flag pole and found it to be 9.9 m. Unfortunately, before they could hand in their work, an ink blot spilled on it and covered the angle value. They did not want to go out and measure it again. Sophie suggested they work backwards to find the missing angle.

Find the missing angle by working backwards. You will need to use the table below.

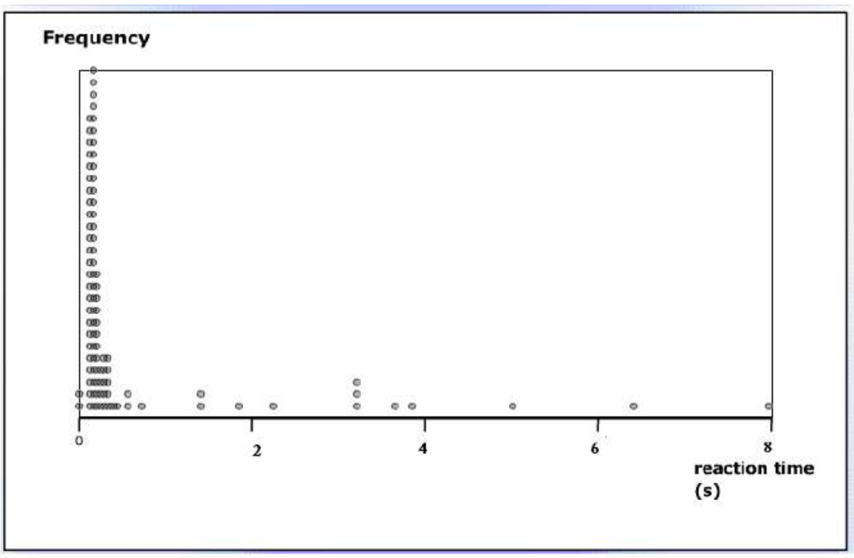
Angle θ	**Tan θ**
38	.7813
37	.7536
36	.7265
35	.7002
34	.6745
33	.6494

Question **(Suggested maximum time: 10 minutes)**

(a) 100 students taking part in a *Censusatschool* survey had their reaction times for a task checked online. The results are shown below.



The students were allowed to practise the task and their reaction times were tested again; the results are shown below.

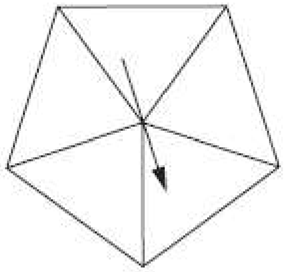


Do you think that this evidence suggets that reaction times can improve with practice?
Explain your thinking.

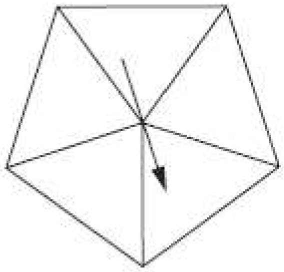
Question **(suggested maximum time: 8 minutes)**

On each spinner write five numbers to make the statements correct.

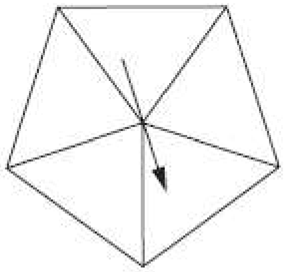
(i) It is *certain* that you will get a number less than 6.



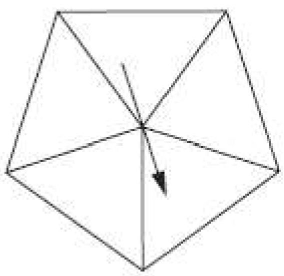
(ii) It is *more likely* that you will get an even number than an odd number.



(iii) It is *impossible* that you will get a multiple of 2.

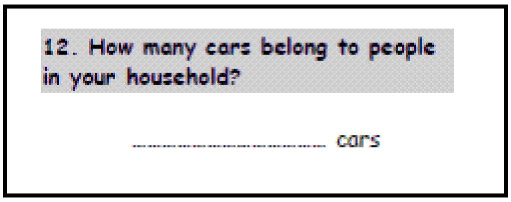


(iv) It is *likely* you will get a prime number.



Question **(Suggested maximum time: 12 minutes)**

The following question was asked on the phase 10 *Censusatschool* questionnaire.


12. How many cars belong to people in your household?

.... cars

The data below are from groups of students chosen at random from Ireland and South Africa.

No of Cars per Household	
Ireland	South Africa
1	1
1	2
2	0
1	0
1	2
2	0
2	0
2	1
3	1
1	1
1	1
3	1
2	3
5	2
1	2
3	2
6	1
5	1
2	1
3	1
2	1
1	3
2	3
1	2
1	1
1	0
2	1
2	1
1	1
2	1

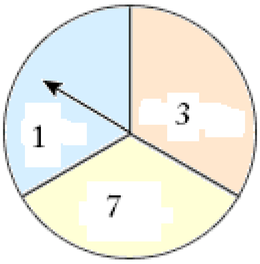
(a) Display the data in a way that allows you to compare the two groups.

(b) What do you notice about these two groups of students? Is there any evidence that households in one country have more cars than the other?.Explain your answer.

Question **(Suggested maximum time 25 mins)**

Sophie and Amy were designing a game of chance to raise money for charity in their school. They agreed on the following:

- they would call the game ***Spin and Win***
- they would charge 50 cent to play
- the rule would be: roll a die and spin the spinner shown and add the totals**.**



(a) Create a *sample space* showing all the possible outcomes.

However, Sophie and Amy had different ideas about which outcomes would result in a win, a loss or the player getting their money back.

Sophie's Idea

Money back: Get total of 13
Win €1: Even number total
Lose: Anything else

Amy's Idea

Money back: Even number total
Win €1: Odd total, but not Prime
Lose: Anything else

(b) What is the probability of winning if you play using Sophie's idea?

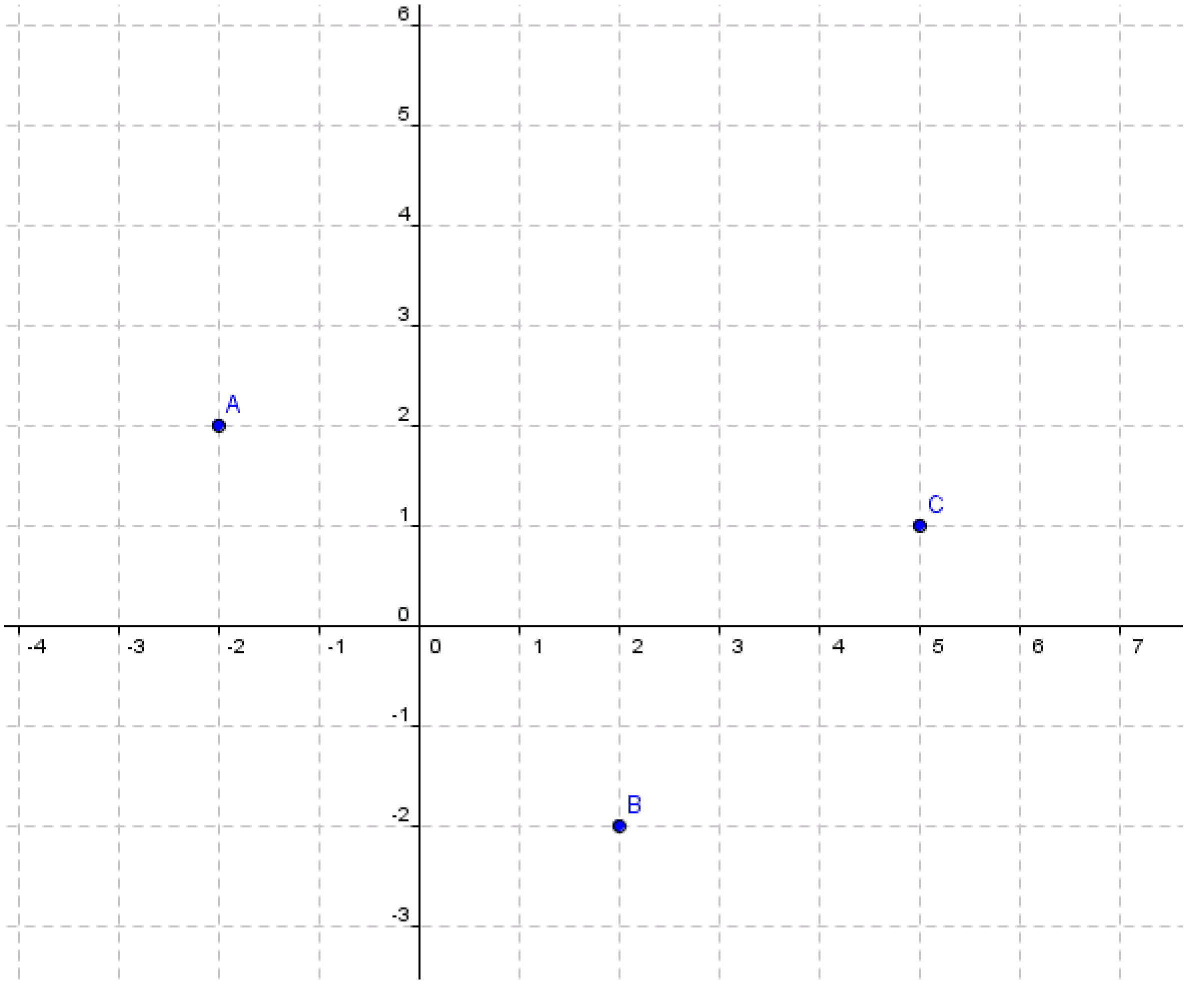
(c) What is the probability of getting your money back if you play using Amy's idea?

(d) If you play using Amy's idea, is the probability of winning **greater than** or **less than** the probability of winning if you play using Sophie's idea? Explain your reasoning.

(e) If 180 people play the game using Sophie's idea, how much are they likely to raise for charity? Show how you worked out your answer.

Question **(Suggested maximum time: 8 minutes)**

Points *A*, *B* and *C* are marked on the coordinate grid below.



(a) Write down the coordinates of *A*, *B* and *C*

(b) Point *D* is such that *ABCD* is a rectangle. Mark *D* accurately on the diagram.

(c) Write down a property of a rectangle.

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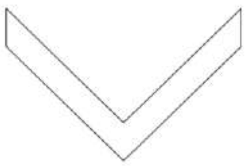
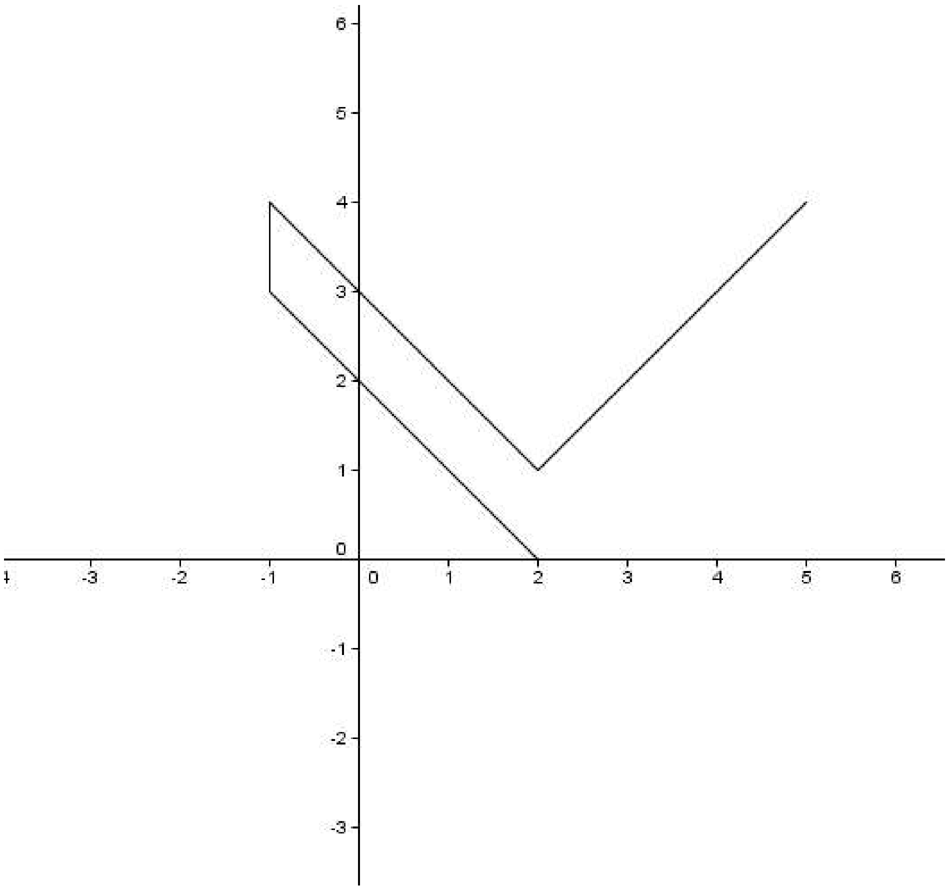
Question **(Suggested maximum time: 7 minutes)**

John is drawing plans for a logo. The logo is in the shape of the letter V as shown.



(a) On the diagram plot three points that John will need to join in order to complete the Logo.

(b) Draw the axis of symmetry on the logo.



Question **(Suggested maximum time 12 minutes)**

(a) Construct a triangle ABC, where $|AB| = 6$ cm $|AC| = 8$ cm and $|BC| = 10$ cm.

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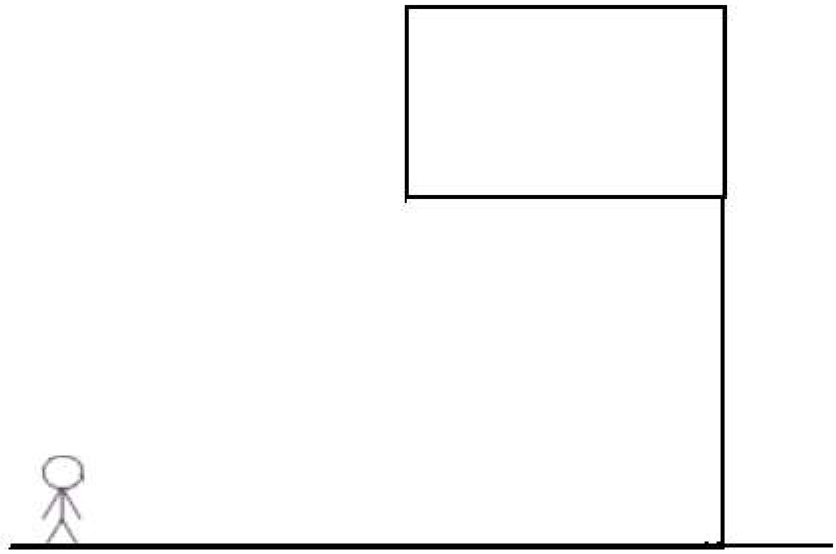

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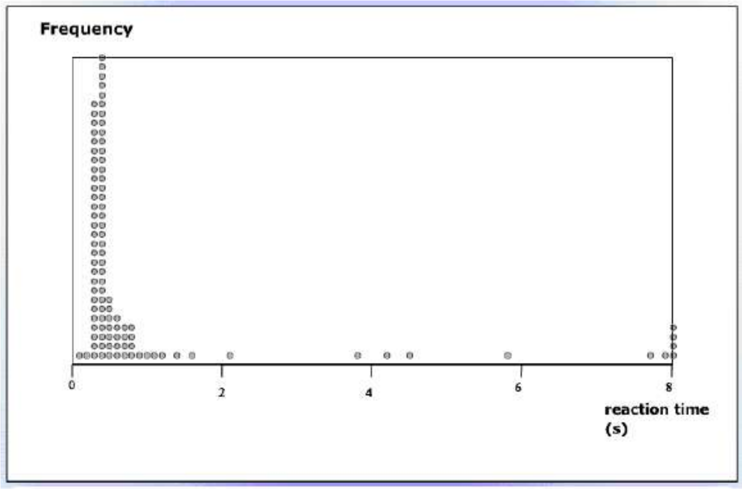

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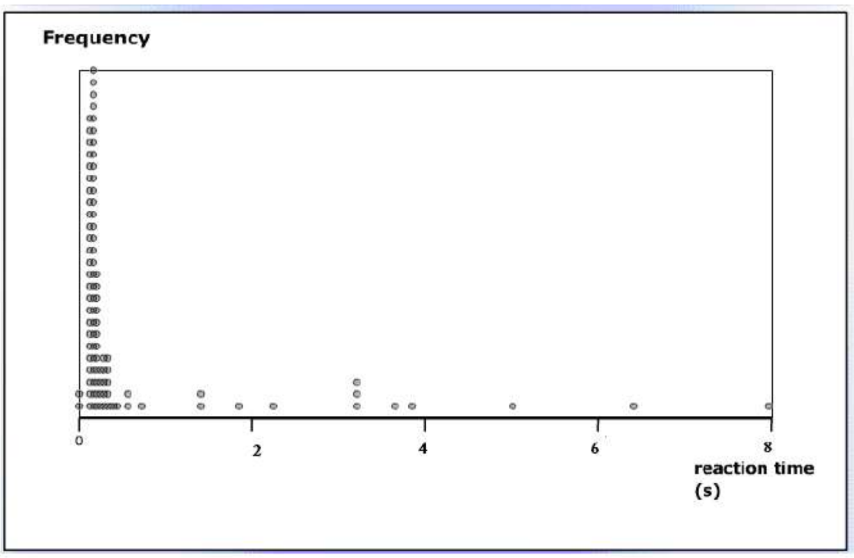

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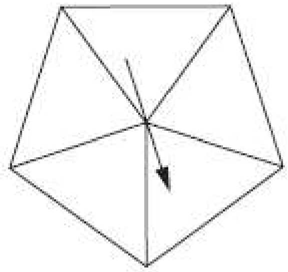

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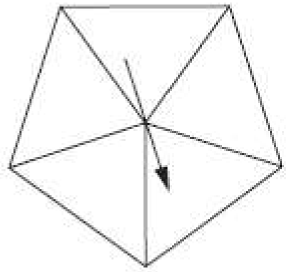

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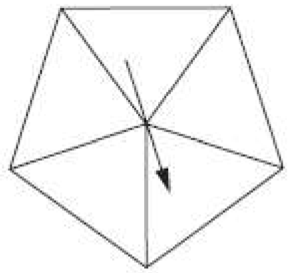

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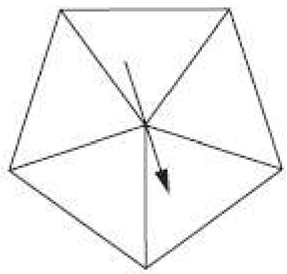

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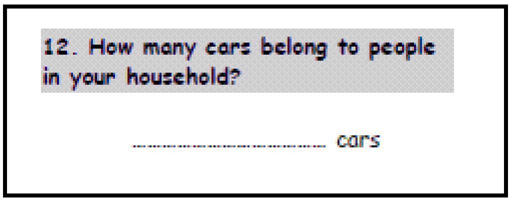
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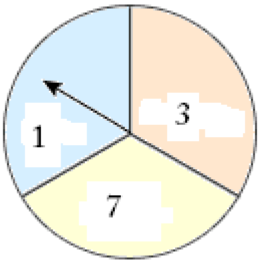

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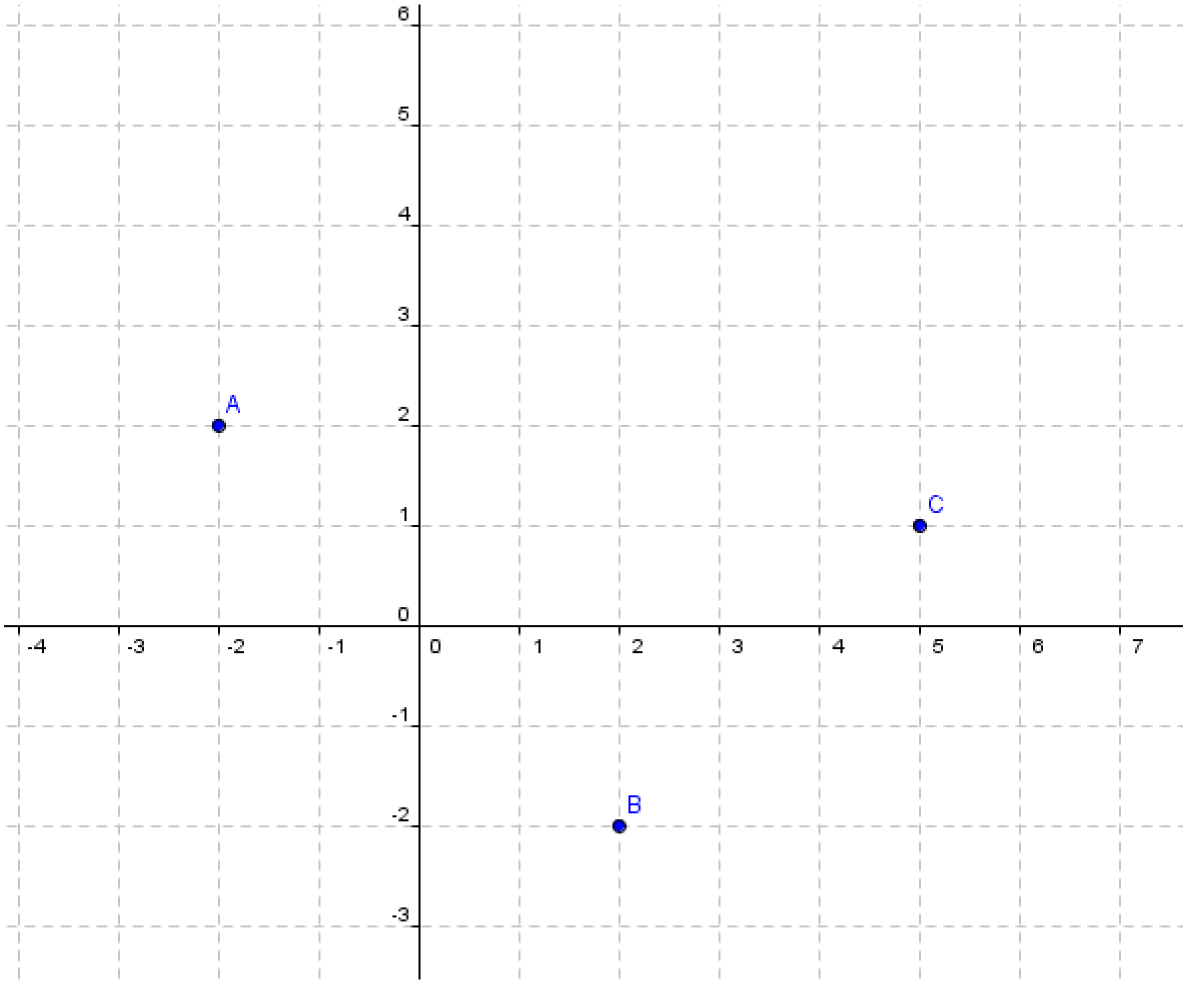

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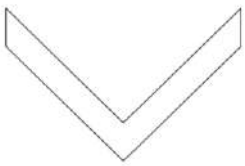

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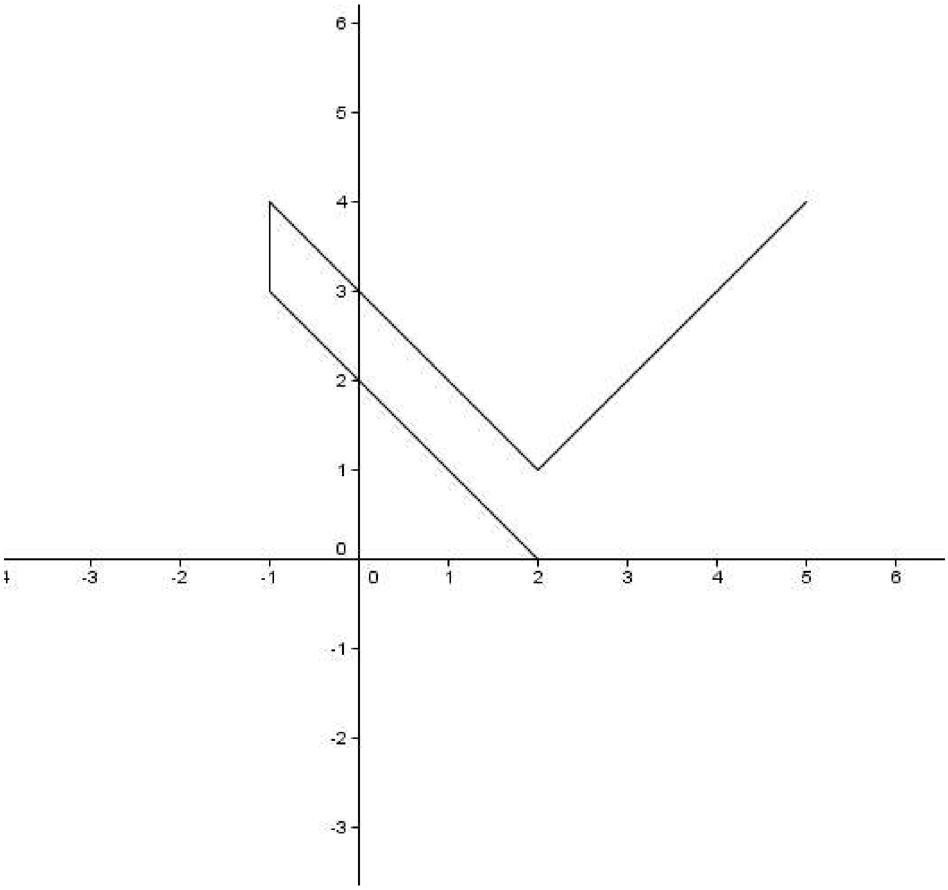

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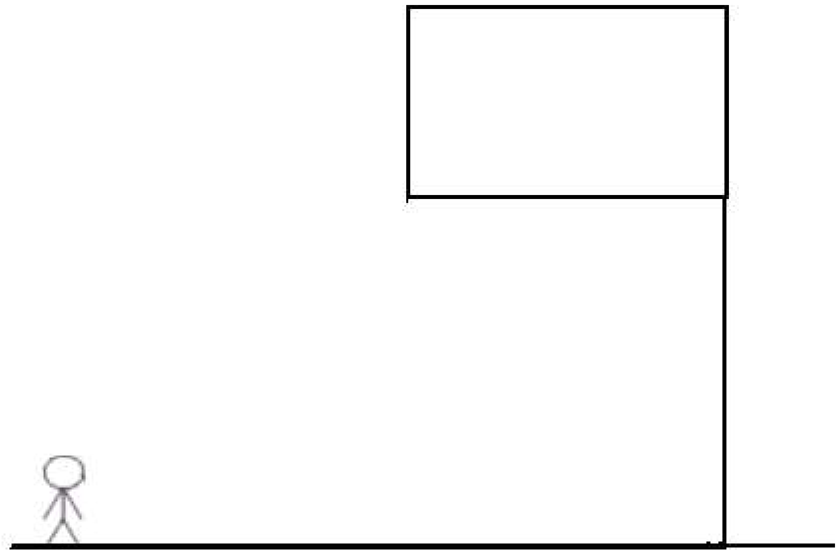

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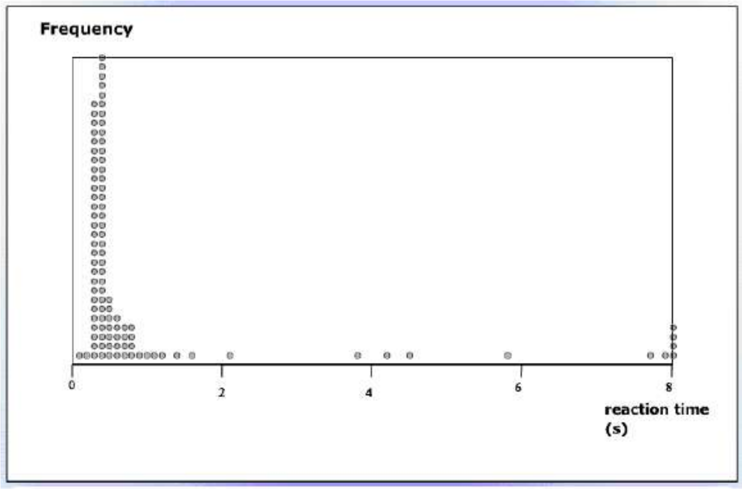

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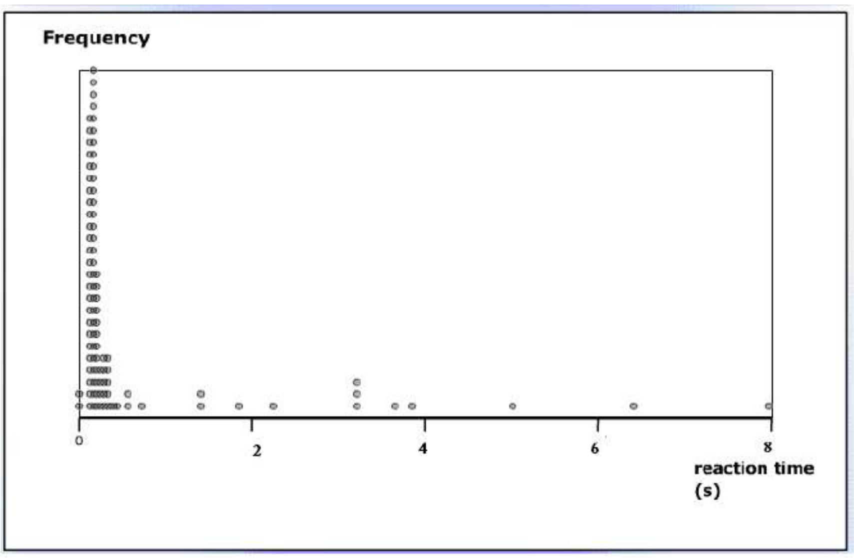

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Explain your thinking.

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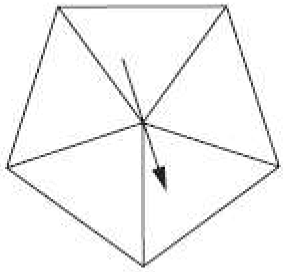

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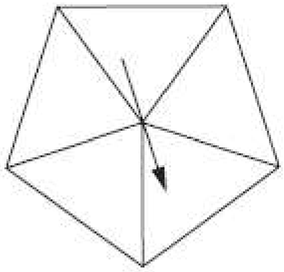

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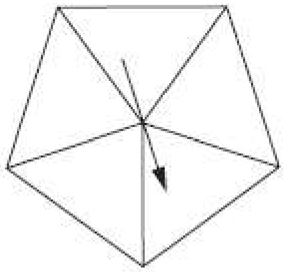

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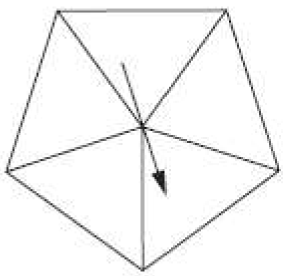


Question **(Suggested maximum time: 12 minutes)**

The following question was asked on the phase 10 *Censusatschool* questionnaire.

> 12. How many cars belong to people in your household?
>
> ……………………… cars

The data below are from groups of students chosen at random from Ireland and South Africa.

No of Cars per Household	
Ireland	South Africa
1	1
1	2
2	0
1	0
1	2
2	0
2	0
2	1
3	1
1	1
1	1
3	1
2	3
5	2
1	2
3	2
6	1
5	1
2	1
3	1
2	1
1	3
2	3
1	2
1	1
1	0
2	1
2	1
1	1
2	1

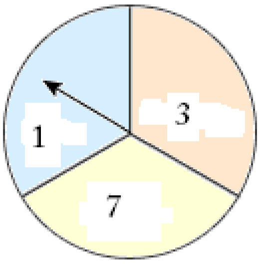
(a) Display the data in a way that allows you to compare the two groups.

(b) What do you notice about these two groups of students? Is there any evidence that households in one country have more cars than the other?.Explain your answer.

Question **(Suggested maximum time 25 mins)**

Sophie and Amy were designing a game of chance to raise money for charity in their school. They agreed on the following:

- they would call the game ***Spin and Win***
- they would charge 50 cent to play
- the rule would be: roll a die and spin the spinner shown and add the totals**.**



(a) Create a *sample space* showing all the possible outcomes.

However, Sophie and Amy had different ideas about which outcomes would result in a win, a loss or the player getting their money back.

Sophie's Idea

Money back: Get total of 13
Win €1: Even number total
Lose: Anything else

Amy's Idea

Money back: Even number total
Win €1: Odd total, but not Prime
Lose: Anything else

(b) What is the probability of winning if you play using Sophie's idea?

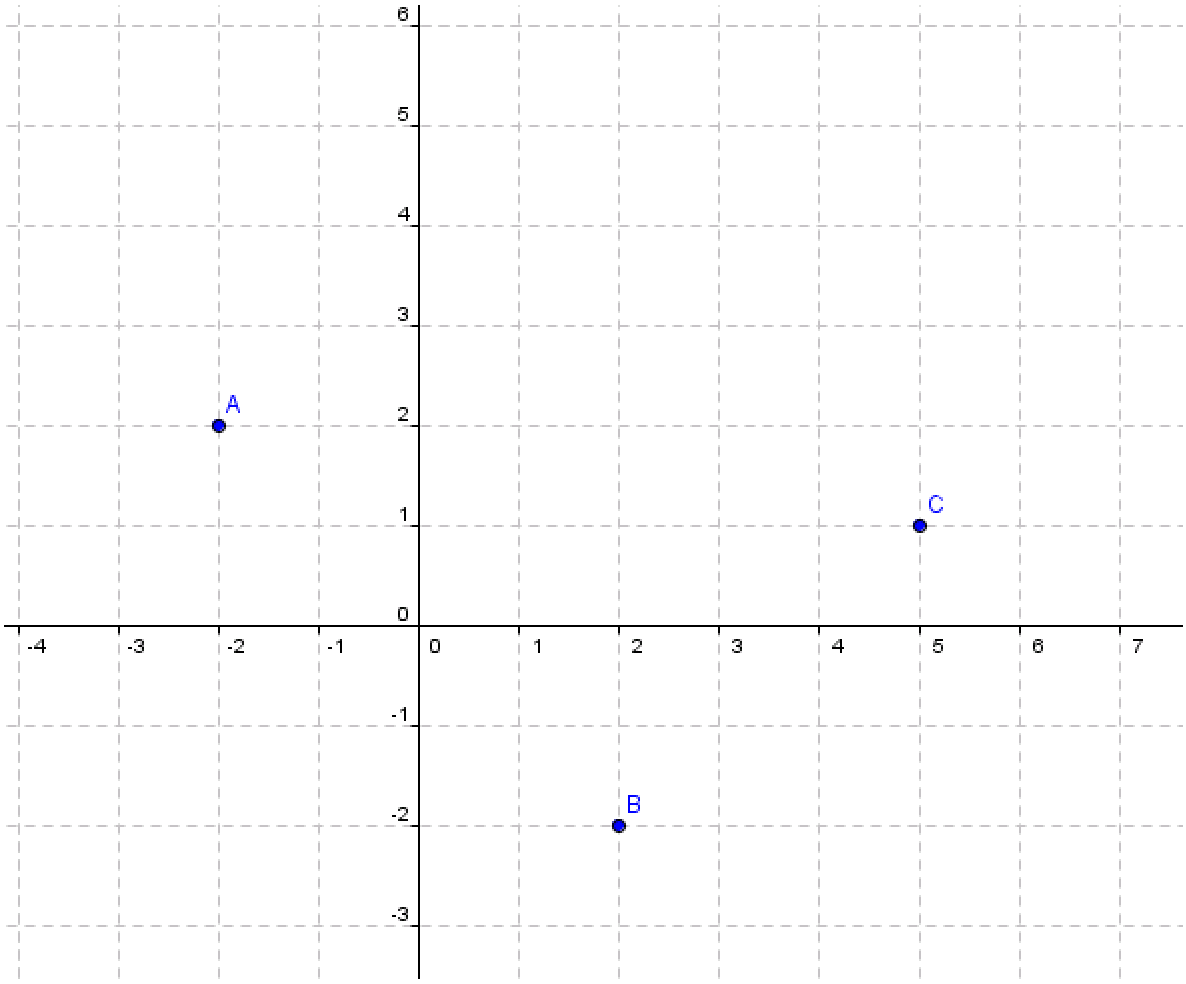
(c) What is the probability of getting your money back if you play using Amy's idea?

(d) If you play using Amy's idea, is the probability of winning **greater than** or **less than** the probability of winning if you play using Sophie's idea? Explain your reasoning.

(e) If 180 people play the game using Sophie's idea, how much are they likely to raise for charity? Show how you worked out your answer.

Question **(Suggested maximum time: 8 minutes)**

Points *A*, *B* and *C* are marked on the coordinate grid below.



(a) Write down the coordinates of *A*, *B* and *C*

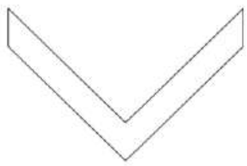
(b) Point *D* is such that *ABCD* is a rectangle. Mark *D* accurately on the diagram.

(c) Write down a property of a rectangle.

(d) Show that the rectangle *ABCD* has the property that you wrote down in part **(c)**.

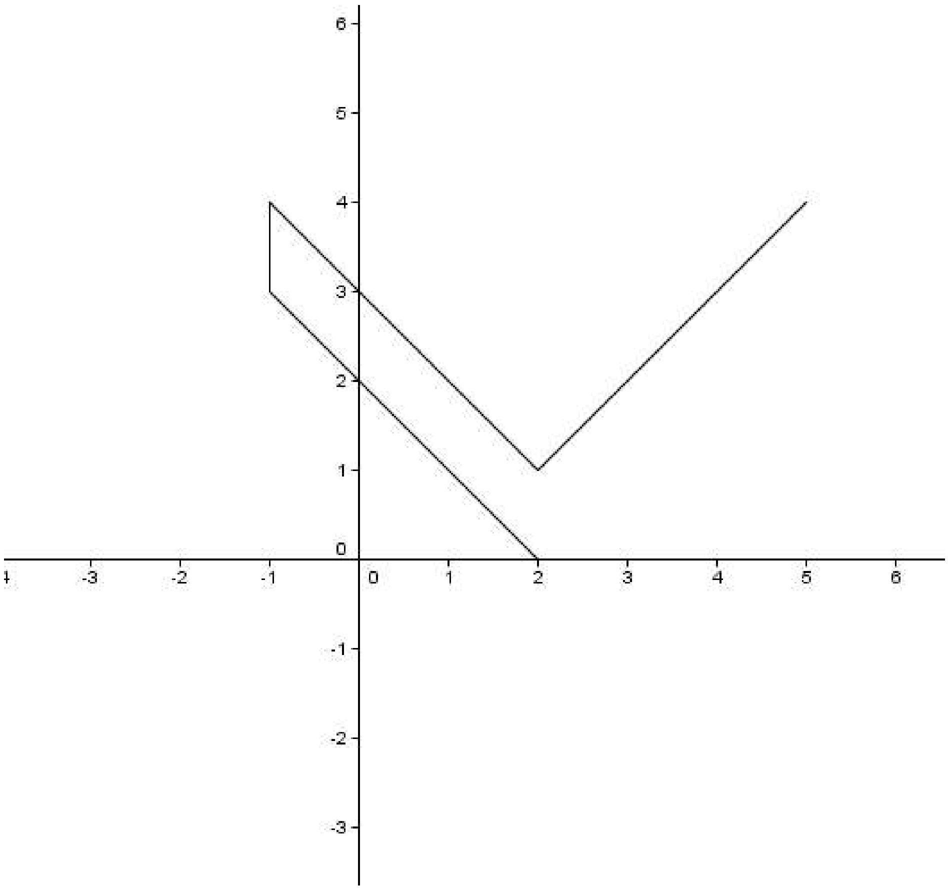
Question **(Suggested maximum time: 7 minutes)**

John is drawing plans for a logo. The logo is in the shape of the letter V as shown.



(a) On the diagram plot three points that John will need to join in order to complete the Logo.

(b) Draw the axis of symmetry on the logo.



Question **(Suggested maximum time 12 minutes)**

(a) Construct a triangle ABC, where $|AB| = 6$ cm $|AC| = 8$ cm and $|BC| = 10$ cm.

(b) What type of a triangle is this? Mathematically prove that this is so.

Section A	Concepts and Skills

Question

(a) Show that the equation

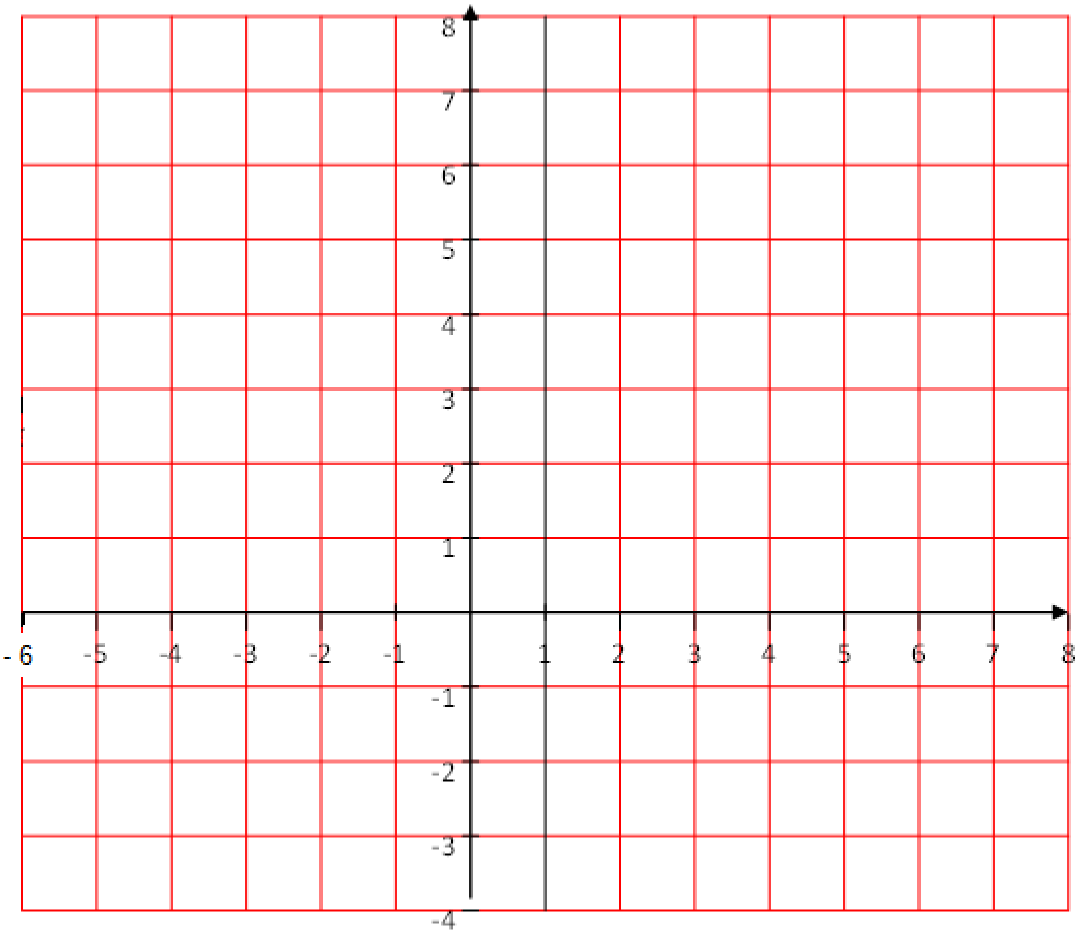
$$15\cos^2 x = 13 + \sin x$$

may be written as a quadratic equation in $\sin x$.

(b) Solve the quadratic equation for $\sin x$, and hence solve for all values of x where $0° \leq x \leq 360°$. Give your answer(s) correct to the nearest degree.

Question

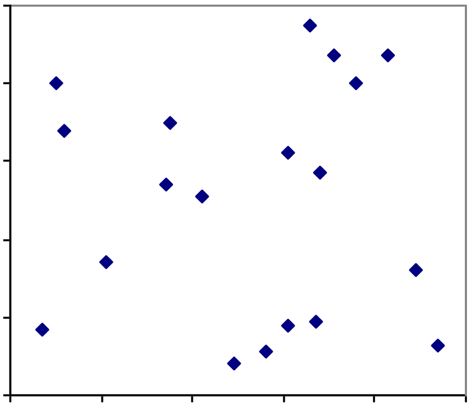
(a) On the grid provided draw circle p whose equation is $x^2 + y^2 - 4x - 2y - 5 = 0$.



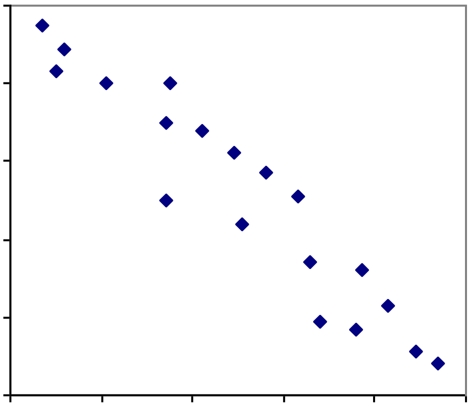
(b) Use **two different** methods to determine whether the line $l: 3x + y + 3 = 0$ is tangent to this circle *p*.

Question

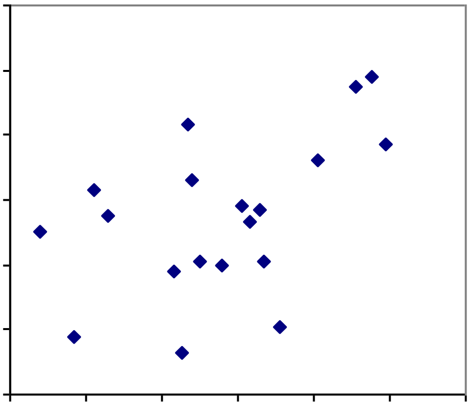
(a) For each of the four scatter plots below, estimate the correlation coefficient.



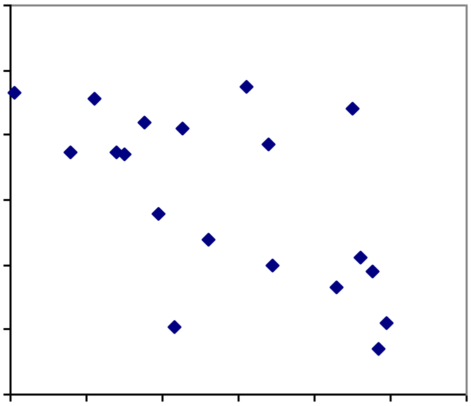
Correlation ≈



Correlation ≈



Correlation ≈



Correlation ≈

(b) Using your calculator, or otherwise, find the correlation coefficient for the data given in the table.

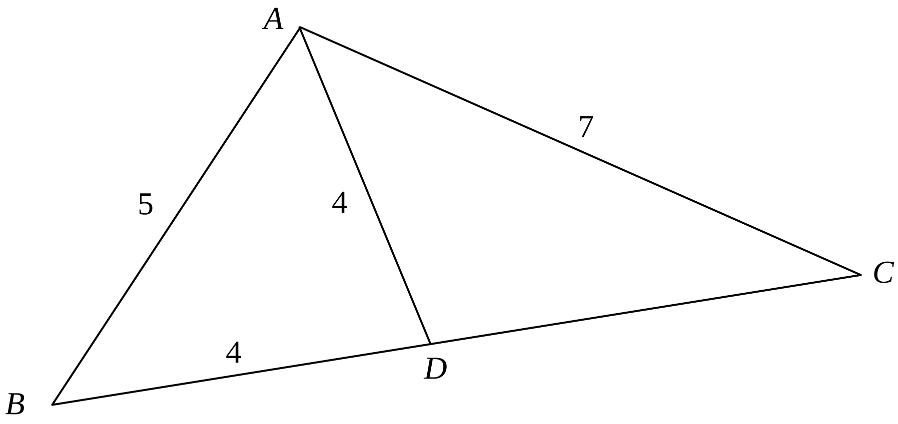
Give your answer correct to two decimal places.

x	y
0.0	0.5
5.0	1.3
5.2	3.3
6.1	6.7
9.3	4.5
9.5	4.6
9.9	6.5

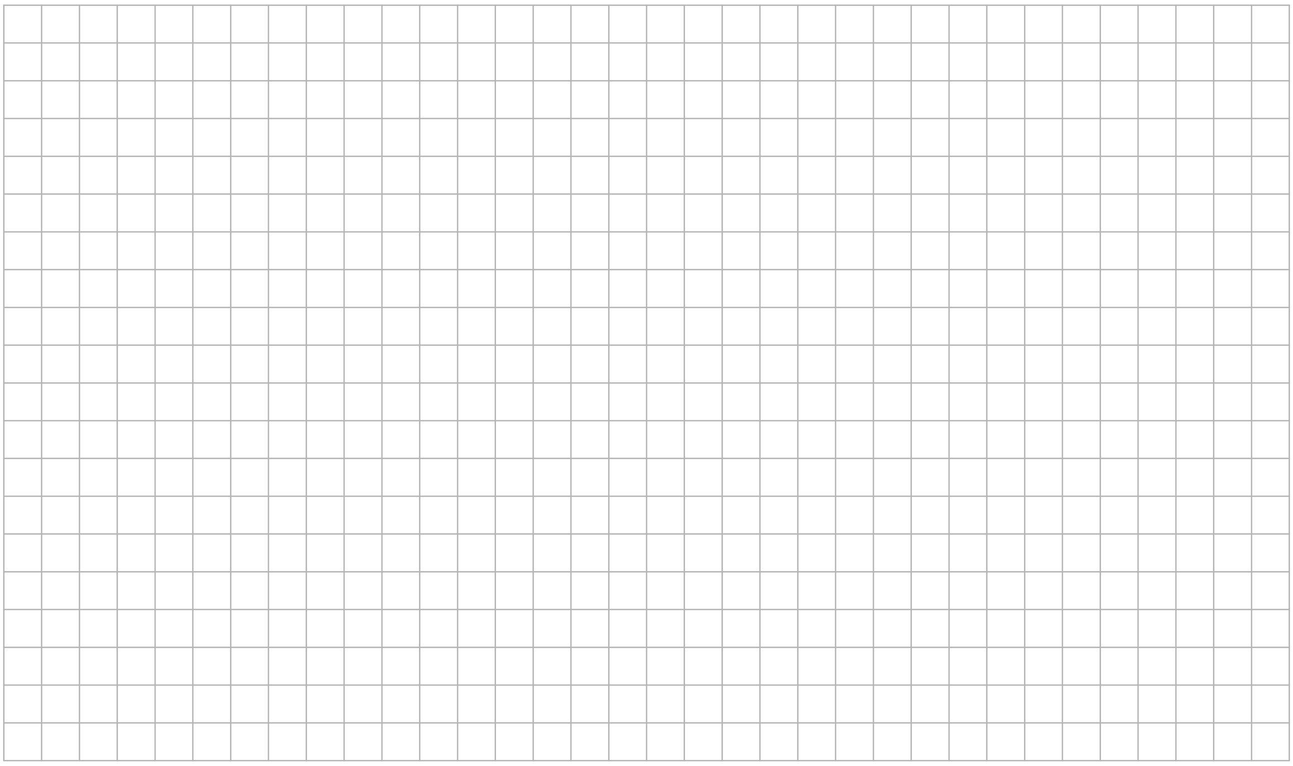
Answer:

Question

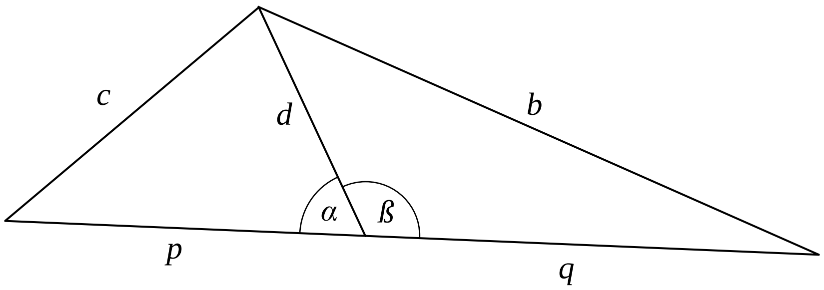
(a) *ABC* is a triangle, and *D* is a point on [*BC*].
The lengths |*AB*|, |*AD*|, |*AC*| and |*BD*| are as shown in the diagram.



Find |*DC*|, correct to one decimal place.



(b) Consider the diagram below.



Express $\cos\alpha$ **and** $\cos\beta$ **in terms of the labelled lengths**

(ii) Show that $pb^2 + qc^2 = \; p+q \quad pq+d^2$

Question

(a) The following formula relates to the *binomial distribution*.

$$P(X = r) = \binom{n}{r} p^r q^{n-r}$$

(i) State what each of the letters *p*, *q*, *n*, and *r* represents in the formula above.

p is ____

q is ____

n is ____

r is ____

(ii) Describe the type of experiment that results in a random variable that has a binomial distribution.

(b) In a certain type of archery competition, Laura hits the target with an average of two out of every three shots. The shots are independent of each other. During one such competition, she has ten shots at the target.

(i) Find the probability that Laura hits the target exactly nine times.

Give your answer correct to three decimal places.

(ii) Find the probability that Laura hits the target fewer than nine times.

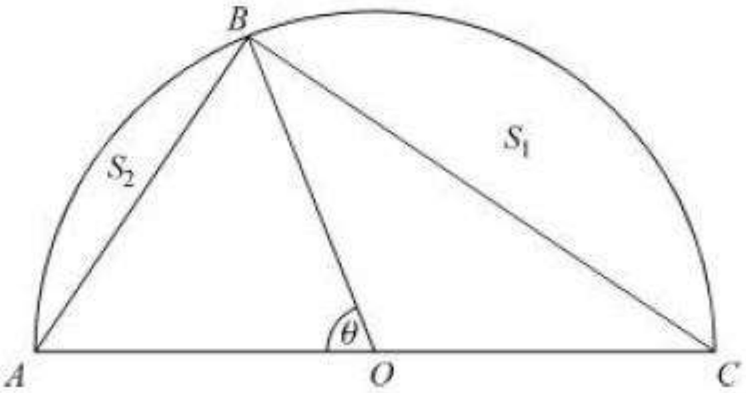
Give your answer correct to three decimal places.

Question

You work for a market research company and you have been asked to find out the percentage of small and medium sized Irish companies that have email and internet access. Describe how you would find out this information.

In your answer, you should mention the **population**, the **sampling frame**, how you would select a **sample**, the **size of the sample** you would select and why you would choose this sample size. You should describe how you will gather the data and the potential **biases** you may encounter.

Question



The diagram shows a semicircle *ABC* on [*AC*] as diameter. The mid-point of [*AC*] is *O*, and angle *AOB* = θ radians, where $0 < \theta < \frac{\pi}{2}$. The area of the segment S_1 cut off by the chord *BC* is twice the area of the segment S_2 bounded by the chord *AB*.

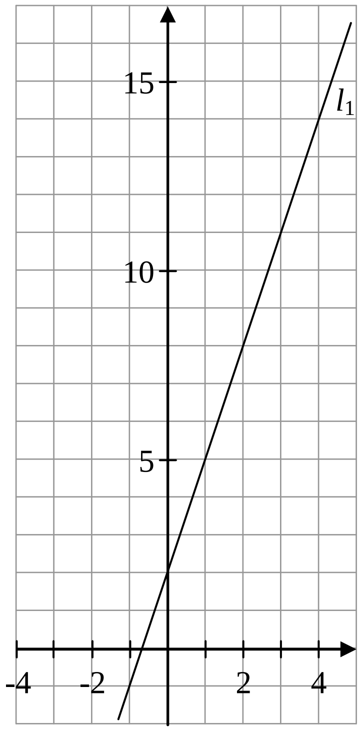
Show that $3\theta = \pi + \sin\theta$

Question

The line l_1 in the diagram has slope 3 and y-intercept 2.

(a) Write down the equation of this line, in the form $y = mx + c$.

(b) On the diagram, draw and label the lines l_2 and l_3, where:
l_2 has slope 3 and y-intercept 7
l_3 has slope 1 and y-intercept 8.



(c) On the diagram, draw and label the line l_4, which is perpendicular to l_1 and passes through the point (0, 4).

(d) Determine whether l_4 passes through the point $(27, -4)$.

Question

P is the point (0, 7) and *Q* is the point (8, 11).

(a) Find the equation of the circle with diameter *PQ*.

(b) Find the equation of the tangent at *Q*.

(c) This tangent crosses the x-axis at the point *R*. Find the co-ordinates of *R*.

Question

20% of the bolts produced by a machine are defective.

(a) Find the probability that, in a group of five bolts randomly selected from a batch produced by the machine, at most two are defective.

(b) A shipment of 250 packets of 5 bolts produced by this machine is inspected. A packet is rejected if it has more than two defective bolts. Show that approximately 14 packets are expected to be rejected.

Question

ABC is an equilateral triangle inscribed in a circle centre O. A radius is drawn from *O* through the midpoint of *AB* to meet the circumference of the circle at *D*.

(a) Construct this diagram accurately, showing all construction marks.

(b) Prove that *ODA* is equilateral.

Question

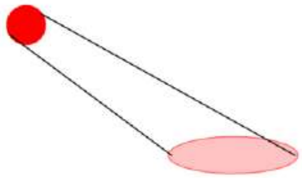
ABC *is* an isosceles triangle such that $|AB| = |AC|$ and D is a point on AB such that $CD \perp AB$. Represent this on a diagram.

Show that $|\angle BCD| = ½ |\angle BAC|$

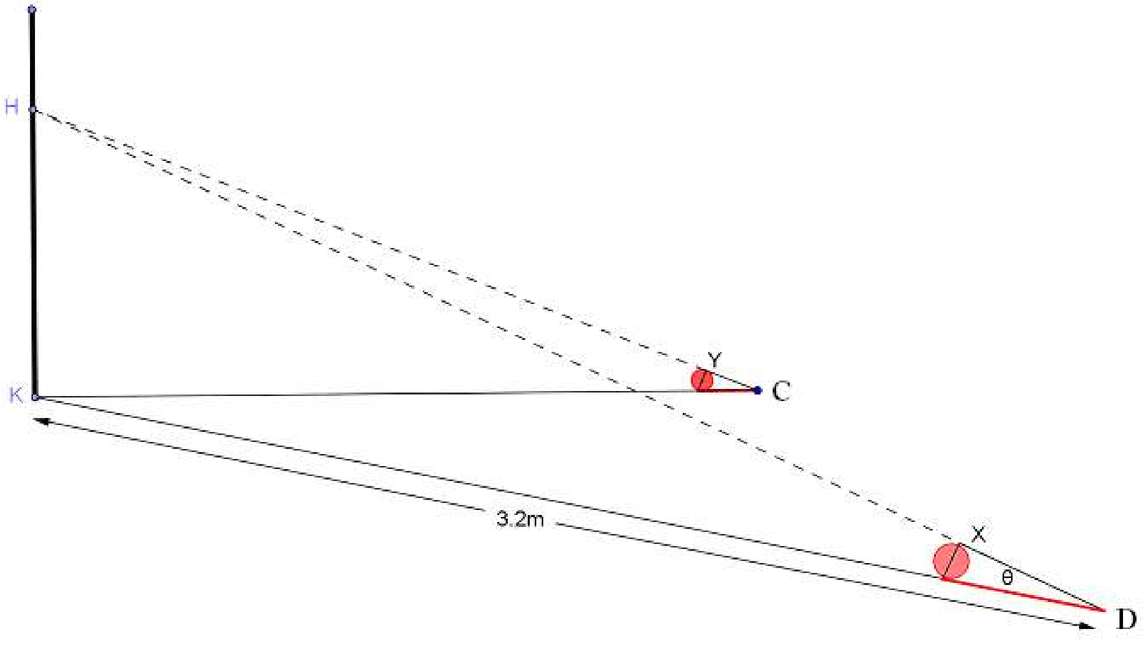
Section B	Contexts and Applications

Question

(a) Forensic investigators encounter crime scenes containing traces of blood. A spherical drop of blood makes an elliptical spatter when it hits the ground at an angle**.**



Investigators have mapped out the crime scene below showing two blood spatters on the floor at C and D. They measure a spatter in order to calculate the angle of impact and trace this back to an approximate starting place (assuming the blood drops travel in a straight line).



Since they know the drop started as a sphere, the width of the spatter drop will be the same as its diameter. They record the measurements of the blood spatter X:

Blood spatter from drop X

Estimate the victim's height given that the blood originated from a chest wound. Show all your working and state any assumptions you make.

Question

(a) In a component factory, machine A produces 30% of the output, machine B 25% and machine C the remainder.

Over a period of time, 1% of the output from machine A is found to be defective, 1.2% from machine B and 2% from machine C.

(i) On a given day, the three machines produce a total of 10,000 components. How many components are likely to be defective?

(ii) A quality controller selects a component at random from that day's output and finds that it is defective. What is the probability that this component was produced by machine B?

(b) Cereal packets filled by a machine have a mean of 200 grams with a standard deviation of 10 grams.

(i) In a sample of 300 packets, how many can be expected to weigh more than 215 grams? Show all your work.

(ii) The machine is adjusted and another sample of 300 is checked. If the expected value of the number of packets that weigh more than 215 grams is now 10, calculate the new mean, assuming that the standard deviation after adjustment is the same as before?

(c) A company states that 20% of the visitors to its website purchase at least one of their products. A sample of 400 site visitors is checked and the number who purchased a product is found to be 64.

(i) Calculate the margin of error in this case.

(ii) Based on this sample, should the company's claim be accepted? Explain your reasoning.

Question

Some research was carried out into the participation of girls and boys in sport. The researchers selected a simple random sample of fifty male and fifty female teenagers enrolled in GAA clubs in the greater Cork area. They asked the teenagers the question: *How many sports do you play?*

The data collected were as follows:

Boys	Girls
0, 4, 5, 1, 4, 1, 3, 3, 3, 1,	3, 3, 3, 1, 1, 3, 3, 1, 3, 3,
1, 2, 2, 2, 5, 3, 3, 4, 1, 2,	2, 2, 4, 4, 4, 5, 5, 2, 2, 3,
2, 2, 2, 3, 3, 3, 4, 5, 1, 1,	3, 3, 4, 1, 6, 2, 3, 3, 3, 4,
1, 1, 1, 2, 2, 2, 2, 2, 3, 3,	4, 5, 3, 4, 3, 3, 3, 4, 4, 3,
3, 3, 3, 3, 3, 3, 3, 3, 3, 3	1, 1, 3, 2, 1, 3, 1, 3, 1, 3

(a) Display the data in a way that gives a picture of each distribution.

(b) State **one difference** and **one similarity** between the distributions of the two samples.

Difference:

Similarity:

(c) Do you think that there is evidence that there are differences between the two populations? Explain your answer.

Note: you are not required to conduct a formal hypothesis test.

Answer: ____

Justification:

(d) The researchers are planning to repeat this research on a larger scale. List **two** improvements they could make to the design of the research in order to reduce the possibility of *bias* in the samples. Explain why each improvement you suggest will reduce the likelihood of bias.

Question

The *Wonder Building* is an arched building that does not need any support inside, due partly to the fact that its shape is an arc of a circle.

The photograph shows a *Wonder Building* being used in Antarctica.

The arc for a *Wonder Building* can be a full semicircle or less than a semicircle. It cannot be more than a semi-circle. The "span" of the building is the total width from one side of the arch to the other.

(a) A particular *Wonder Building* has a span of 30 metres and a height of 10 metres. Find the radius of the arc.

(b) A customer wants a building with a span of 18 metres and a height of 10 metres.

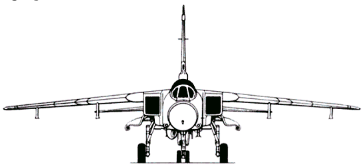
(i) What arc radius would be required to give such a building?

(ii) Explain why the *Wonder Building* that the customer wants is not possible.

(c) An air force needs a *Wonder Building* to house a *Tornado* military jet.

The dimensions of the aircraft are as follows:

- Wingspan: 14 metres
- Height: 6 metres
- Height of wingtips above ground: 2 metres.

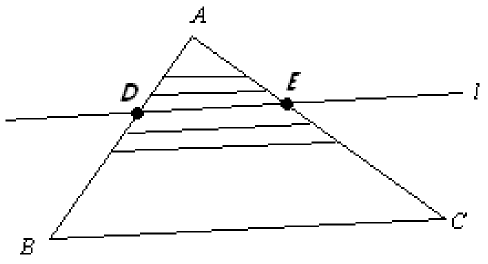


The shelter must be at least 0·5 metres above the top of the tail, and at least 1 metre clear horizontally of the wingtips.

For the shelter to have the exact clearance required, find the radius of the arc.

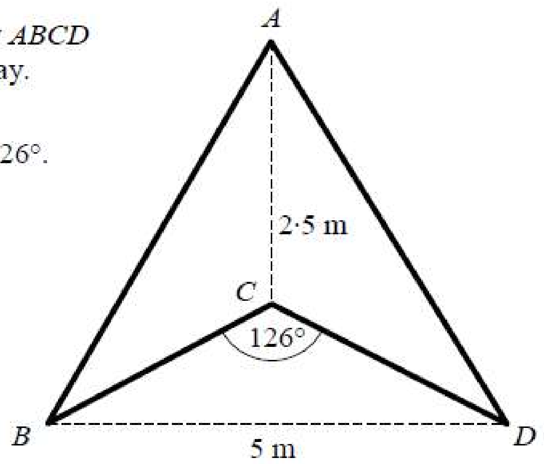
Question

(a) Let ΔABC be a triangle. The line l is parallel to BC and cuts $|AB|$ in the ratio s:t, where s and t are natural numbers. Prove that l also cuts AC in the ratio s:t.



(b) The diagram represents a large symmetrical arrow *ABCD* that is painted on the ground at one end of a runway.

The distance *BD* is 5 metres and distance *CA* is 2·5 metres, as shown. The angle *ADB* measures 126°.



(i) Find the length of the perimeter of the arrow.
Give your answer correct to one decimal place.

(ii) *Cat's eyes* are reflective devices used in road markings. They are being laid along the perimeter of the arrow. One is placed at each vertex, and others are placed at intervals of no more than half a metre along the perimeter. How many are needed?

Section A Concepts and Skills

Question

(a) Show that the equation

$$15\cos^2 x = 13 + \sin x$$

may be written as a quadratic equation in sin *x***.**

$\cos^2 x = 1 - \sin x$

$15(1 - \sin^2 x) = 13 + \sin x$

$15 - 15\sin^2 x = 13 + \sin x$

$15\sin^2 x + \sin x - 2 = 0$

(b) Solve the quadratic equation for sin *x*, and hence solve for all values of *x* where $0° \leq x \leq 360°$.
Give your answer(s) correct to the nearest degree.

Let $x = \sin x$

$15x^2 + x - 2 = 0$

$(5x + 2)(3x - 1) = 0$

$5\sin x = -2$ $3\sin x = 1$

$\sin x = \frac{-2}{5}$ $\sin x = \frac{1}{3}$

Reference angle $23.6°$ 19.47

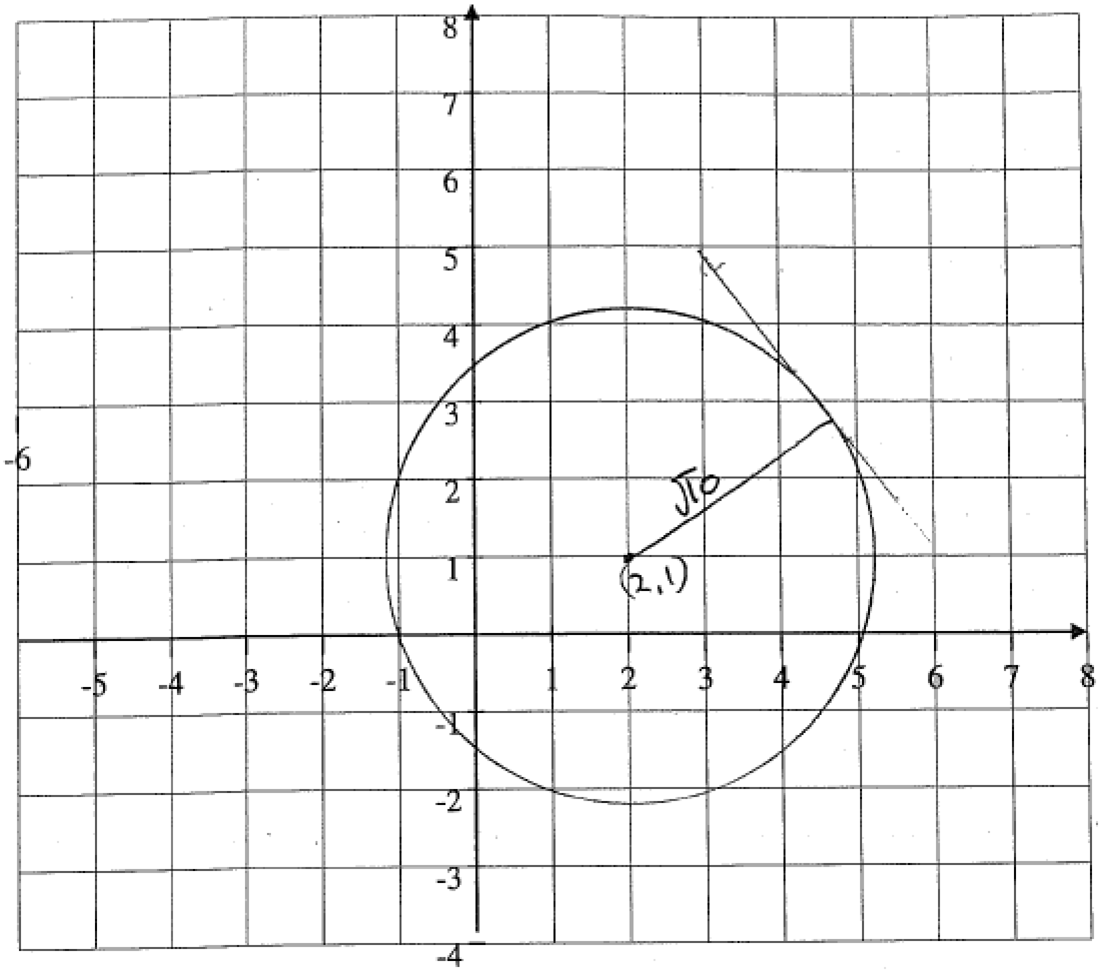
$180° + 23.6° = 203.6°$ $180° - 19.47 =$

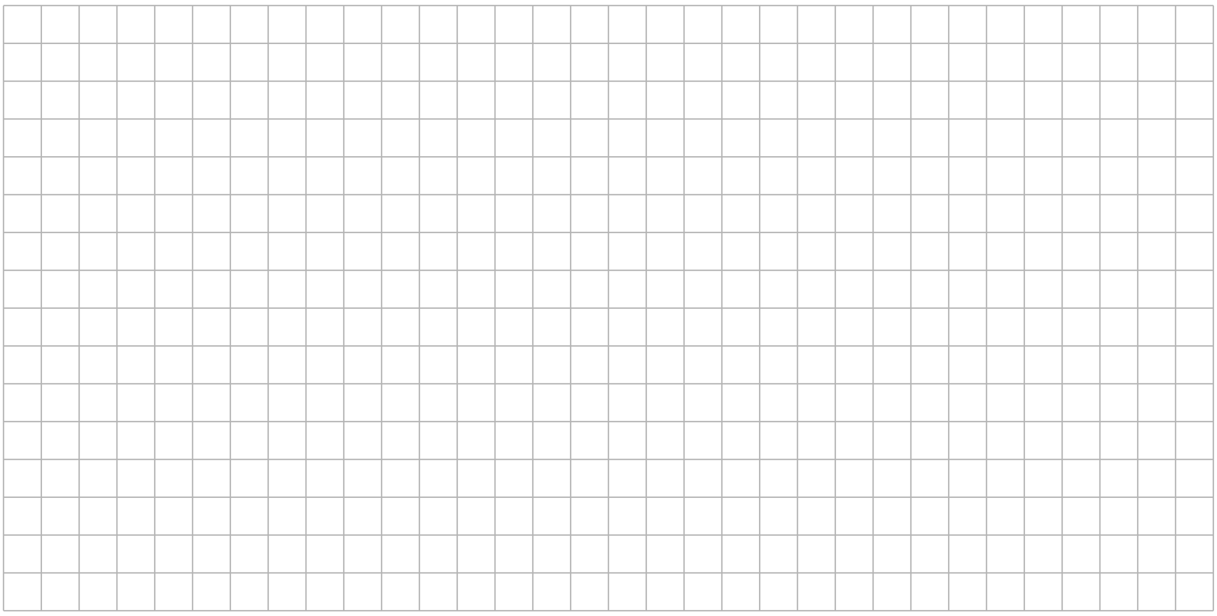
$360 - 23.6° = 336.4°$ $160.53°$

ans $= \{19°, 161°, 204°, 336°\}$

Question

(a) On the grid provided draw circle p whose equation is $x^2 + y^2 - 4x - 2y - 5 = 0$.





(b) Use **two different** methods to determine whether the line $l: 3x + y + 3 = 0$ is tangent to this circle *p*.

(2, 1) $3x + y + 3 = 0$

$\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|3(2) + 1(1) + 3|}{\sqrt{3^2 + 1^2}} = \frac{|6 + 1 + 3|}{\sqrt{10}} = \frac{10}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{10\sqrt{10}}{10} = \sqrt{10}$

∴ The Distance from the centre to line is $\sqrt{10}$ the same as the radius.

(2)

$3x + y + 3 = 0$

$y = -3x - 3$

$x^2 + y^2 - 4x - 2y - 5 = 0$

$x^2 + (-3x - 3)^2 - 4x - 2(-3x - 3) - 5 = 0$

$x^2 + 9x^2 + 9x + 9x + 9 - 4x + 6x + 6 - 5 = 0$

$x^2 + 9x^2 + 20x + 10 = 0$

$10x^2 + 20x + 10 = 0$

$x^2 + 2x + 1 = 0$

$(x + 1)(x + 1)$

$x = -1 \quad x = -1$

$3x + y + 3 = 0$

$3(-1) + y + 3 = 0$

$y = 3 - 3$

$y = 0$

$3x + y + 3 = 0$

$3(-1) + y + 3 = 0$

$y = 0$

(−1, 0)

only on one line (tangent)

(−1, 0)

$3x + y + 3 = 0$

$3(-1) + 0 + 3 = 0$

$0 = 0$

∴ it is on the line

$x^2 + y^2 - 4x - 2y - 5 = 0$

$-1^2 + 0^2 - 4(-1) - 2(0) - 5 = 0$

$1 + 4 - 5 = 0$

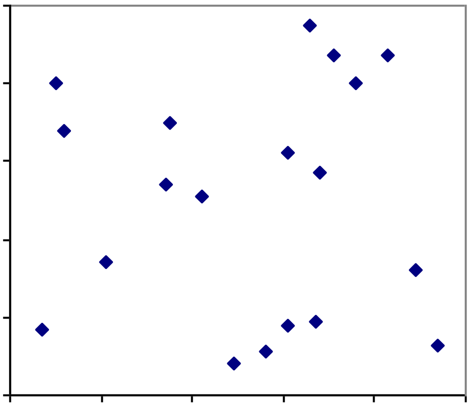
$0 = 0$

∴ it is on the ~~lin~~ circle

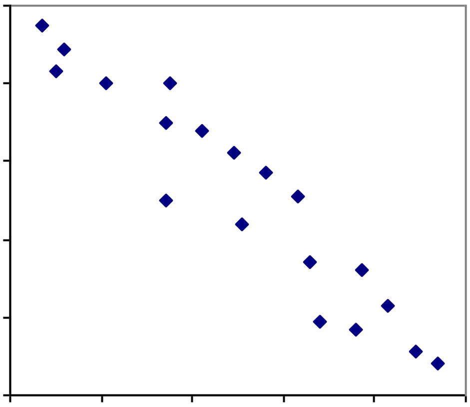
∴ There is only one solution and fits on both line and circle concluding that it is a tangent to the circle.

Question

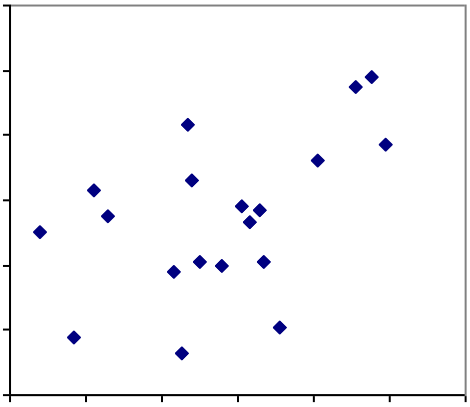
(a) For each of the four scatter plots below, estimate the correlation coefficient.



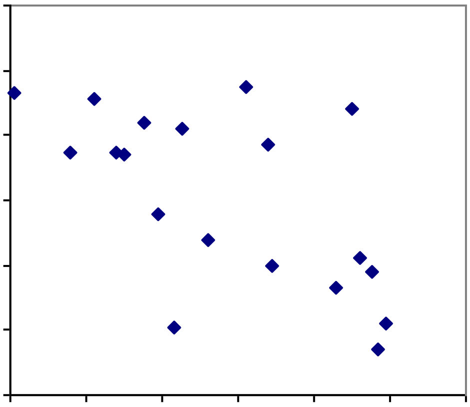
Correlation ≈ 0



Correlation ≈ -0.9



Correlation ≈ 0.5



Correlation ≈ 0.6

(b) Using your calculator, or otherwise, find the correlation coefficient for the data given in the table.

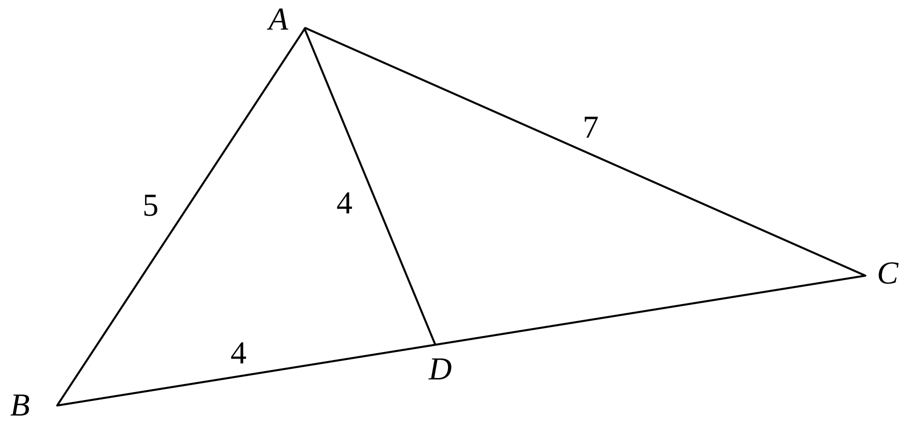
Give your answer correct to two decimal places.

x	y
0.0	0.5
5.0	1.3
5.2	3.3
6.1	6.7
9.3	4.5
9.5	4.6
9.9	6.5

Answer: 0.76

Question

(a) ABC is a triangle, and D is a point on $[BC]$.
The lengths $|AB|$, $|AD|$, $|AC|$ and $|BD|$ are as shown in the diagram.



Find $|DC|$, correct to one decimal place.

$\triangle ABD \quad 5^2 = 16 + 16 - 2(4)(4)\cos D$

$\cos D = 7/32 \quad D = 77^\circ$

$\angle CDA = 103^\circ$

$7^2 = 4^2 + x^2 - 2(4)(x)\cos 103^\circ$

$49 = 16 + x^2 - 8x\cos 103^\circ$

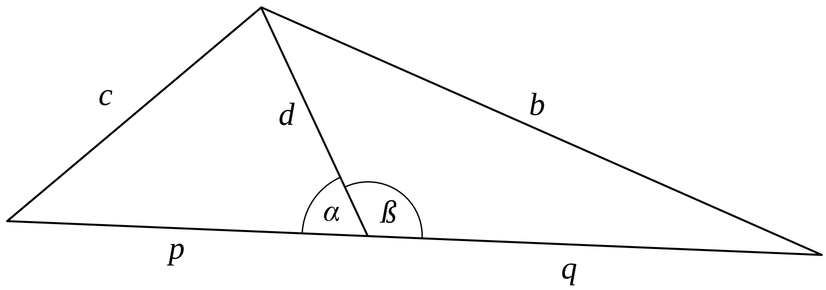
$x^2 + 1.8x - 33 = 0 \quad a = 1 \quad b = 1.8 \quad c = 33$

$\dfrac{-1.8 \pm \sqrt{3.24 + 132}}{2} = \dfrac{-1.8 \pm \sqrt{135.24}}{2}$

$x = 4.9$ or -6.7

$\therefore \; x = 4.9 \quad |DC| = 4.9$

(b) Consider the diagram below.



Express $\cos\alpha$ and $\cos\beta$ in terms of the labelled lengths

$c^2 = d^2 + p^2 - 2(d)(p)\cos\alpha$ $\quad$ $b^2 = q^2 + d^2 - 2(q)(d)\cos\beta$

$c^2 - d^2 - p^2 = -2dp\cos\alpha$ $\quad$ $b^2 - q^2 - d^2 = -2qd\cos\beta$

$\frac{c^2 - d^2 - p^2}{-2dp} = \cos\alpha$ $\quad$ $\frac{b^2 - q^2 - d^2}{-2qd} = \cos\beta$

(ii) Show that $pb^2 + qc^2 = (p+q)(pq+d^2)$

$\cos\alpha = -\cos\beta$

$\frac{c^2 - p^2 - d^2}{-2pd} = \frac{b^2 - q^2 - q^2}{-2dq}$

$-2dqc^2 + 2dqp^2 + 2d^2q = 2pdb^2 - 2pd^2 - 2pdq^2$

$qp^2 + qd^2 + pd^2 + pq^2 = pb^2 + qc^2$

$p(pq + d^2) + q(pq + d^2) = pb^2 + qc^2$

$(p+q)(pq + d^2) = pb^2 + qc^2$

Question

(a) The following formula relates to the *binomial distribution*.

$$P(X = r) = \binom{n}{r} p^r q^{n-r}$$

(i) State what each of the letters *p*, *q*, *n*, and *r* represents in the formula above.

(i) State what each of the letters *p*, *q*, *n*, and *r* represents in the formula above.

p is The probability of success

q is The probability of failure

n is Number of trials

r is Number of successes.

(ii) Describe the type of experiment that results in a random variable that has a binomial distribution.

Repeated experiments that are independent of each other where there are two possible outcomes, success or failure and where the probability of success is the same for each trial.

(b) In a certain type of archery competition, Laura hits the target with an average of two out of every three shots. The shots are independent of each other. During one such competition, she has ten shots at the target.

(i) Find the probability that Laura hits the target exactly nine times.

Give your answer correct to three decimal places.

$p = 2/3 \quad q = 1/3 \quad n = 10 \quad r = 9$

$P(9 \text{ hits}) = \binom{10}{9}\left(\frac{2}{3}\right)^9\left(\frac{1}{3}\right)^1 = \frac{10 \times 2^9}{3^{10}} = 0.0867$

$P(9 \text{ hits}) = 0.087$

(ii) Find the probability that Laura hits the target fewer than nine times.

Give your answer correct to three decimal places.

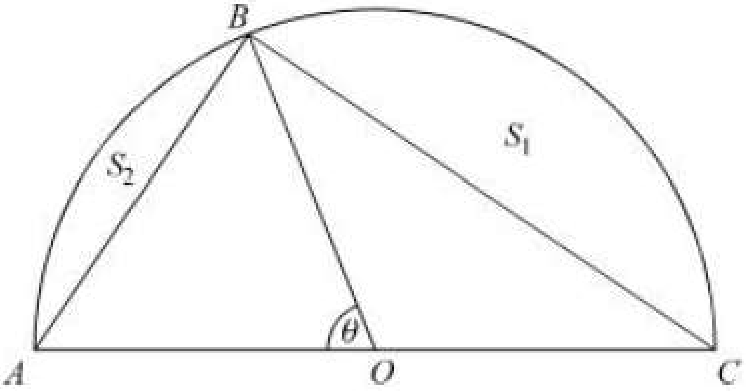
$P(10 \text{ hits}) = \left(\frac{2}{3}\right)^{10} = 0.0173$

$P(9 \text{ hits}) = 0.0867$

$P(<9) = 1 - (0.0867 + 0.0173) = 0.8960$

Thus fewer than 9 times $\Rightarrow$ 0.896

Question



The diagram shows a semicircle *ABC* on [*AC*] as diameter. The mid-point of [*AC*] is *O*, and angle *AOB* = θ radians, where $0 < \theta < \frac{\pi}{2}$. The area of the segment S_1 cut off by the chord *BC* is twice the area of the segment S_2 bounded by the chord *AB*.

Show that $3\theta = \pi + \sin\theta$

Area S_1 = 2 Area S_2

Area S_1 = Area of a sector − area of ΔBOC

$\frac{1}{2}|oc|^2(180-\theta) - \frac{1}{2}|oc||oB|\sin(180-\theta)$

$\frac{1}{2}|oc|^2(180-\theta) - \frac{1}{2}|oc|^2\sin(180-\theta)$

Area of S_2 = Area of sector − area of ΔBoA

$\frac{1}{2}|oa|^2(\theta) - \frac{1}{2}|oa||oB|\sin\theta$

$\frac{1}{2}|oa|^2(\theta) - \frac{1}{2}|oa|^2\sin\theta$

$\frac{1}{2}|oc|^2[(\pi-\theta) - \sin(\pi-\theta)] = 2[\frac{1}{2}|oa|^2(\theta-\sin\theta)]$

$[\pi-\theta - \sin(\pi-\theta)] = 2(\theta-\sin\theta)$

$[\pi-\theta - \sin\pi\cos\theta + \cos\pi\sin\theta] = 2\theta - 2\sin\theta$

$\pi - \theta - 0 - \sin\theta = 2\theta - 2\sin\theta$

$-\theta - 2\theta = -\pi - \sin\theta$

$3\theta = \pi + \sin\theta$

Question

20% of the bolts produced by a machine are defective.

(a) Find the probability that, in a group of five bolts randomly selected from a batch produced by the machine, at most two are defective.

$P(r) = \binom{n}{r} p^r q^{n-r}$ p = success, q = failure

$P(0) + P(1) + P(2)$

$P(0) = \binom{5}{0}(0.2)^0(0.8)^5 = 0.32768$

$P(1) = \binom{5}{1}(0.2)^1(0.8)^4 = 0.4096$

$P(2) = \binom{5}{2}(0.2)^2(0.8)^3 = 0.2048$

Ans = 0.94208

(b) A shipment of 250 packets of 5 bolts produced by this machine is inspected. A packet is rejected if it has more than two defective bolts. Show that approximately 14 packets are expected to be rejected.

1 − P(at most two) = 0.05792

P(rejected) = 0.05792

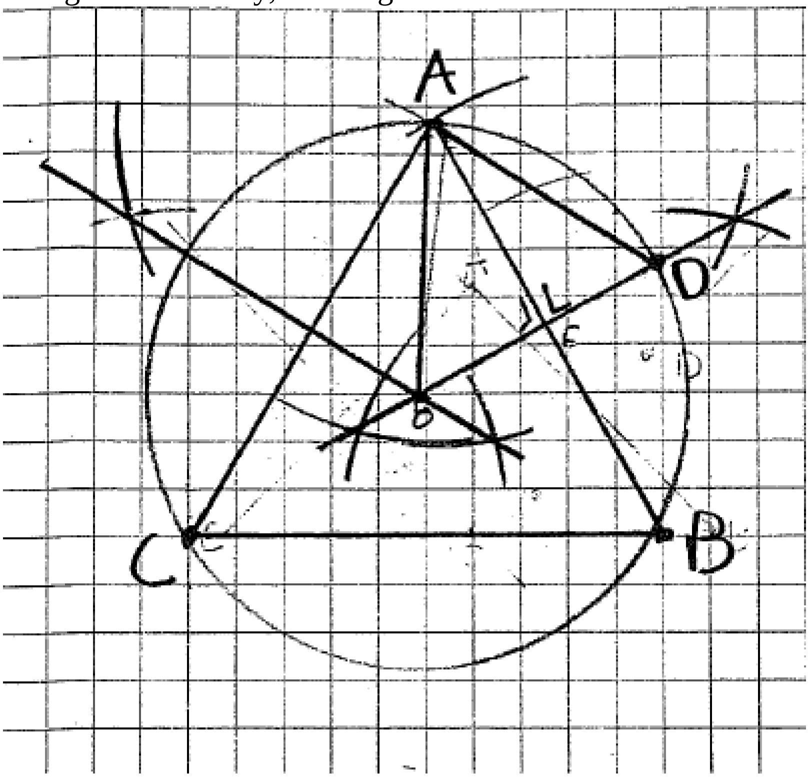
$250 \times 0.05792 = 14.48$

Approx 14 packets

Question

ABC is an equilateral triangle inscribed in a circle centre O. A radius is drawn from *O* through the midpoint of *AB* to meet the circumference of the circle at *D*.

(a) Construct this diagram accurately, showing all construction marks.



(b) Prove that *ODA* is equilateral.

Let E be the midpoint of AB

Δ AOE = Δ BOE |OA| = |OB| radii
|AE| = |EB| midpoint E
|OE| = |OE| common

|∠AOB| = 120° on same arc as |∠ACB| = 60°

But ∠AOE = ∠BOE ⇒ ∠AOE = 60°

|OA| = |OD| both radii   ΔOAD isosceles

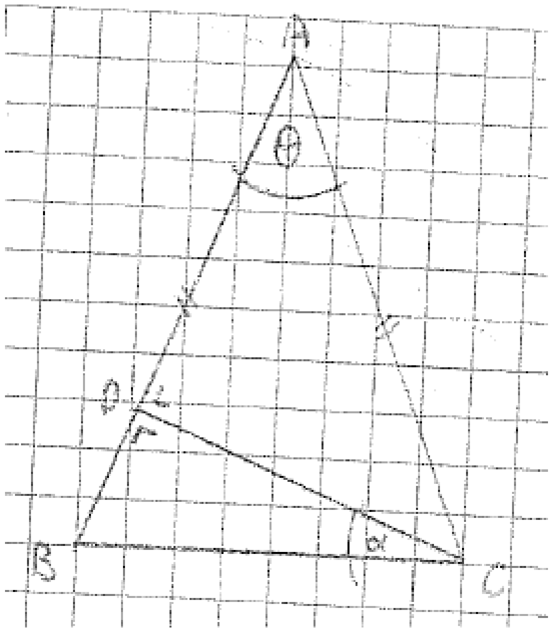
|∠OAD| = |∠ODA| = ½(180° − 60°) = 60°

∴ ODA is equilateral

Question

ABC is an isosceles triangle such that $|AB| = |AC|$ and *D* is a point on *AB* such that $CD \perp AB$. Represent this on a diagram.

Show that $|\angle BCD| = \frac{1}{2}|\angle BAC|$



$\angle B = \angle C$ $\qquad$ $\angle B = \frac{1}{2}(180° - \angle BAC)$

$\angle B = 90° - \frac{1}{2}|\angle BAC|$

$\angle B = 180° - 90° - |\angle BCD|$

$= 90° - |\angle BCD|$

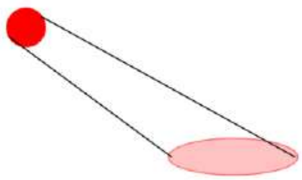
$90° - |\angle BCD| = 90° - \frac{1}{2}|\angle BAC|$

$|\angle BCD| = \frac{1}{2}|\angle BAC|$

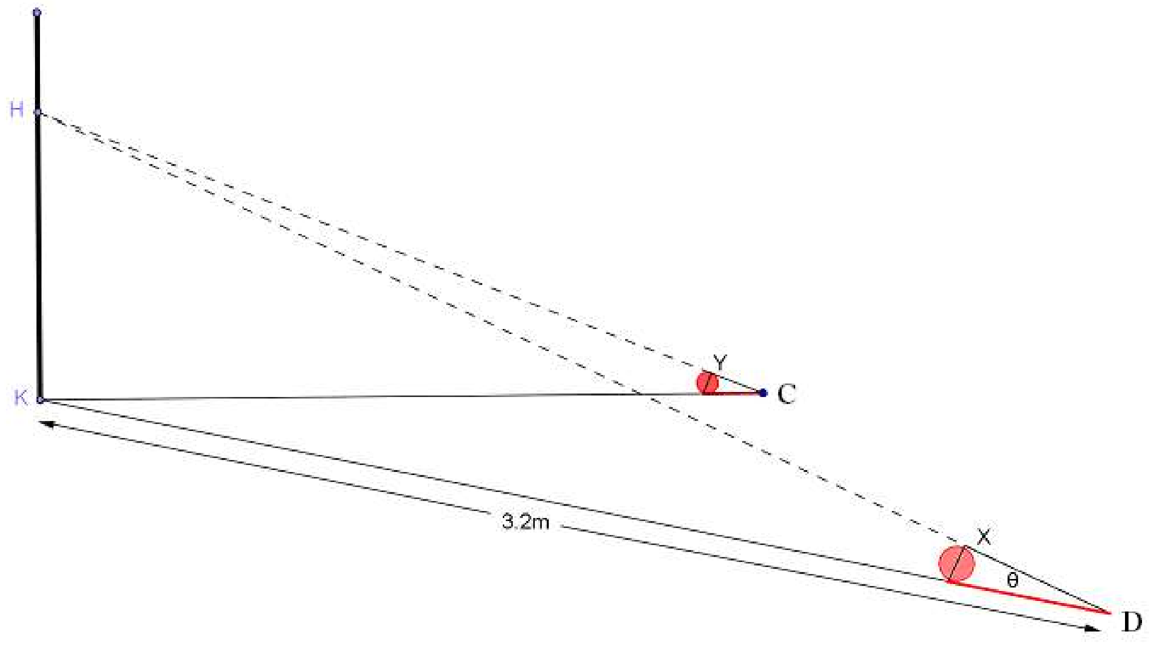
Section B	Contexts and Applications

Question

(a) Forensic investigators encounter crime scenes containing traces of blood. A spherical drop of blood makes an elliptical spatter when it hits the ground at an angle**.**



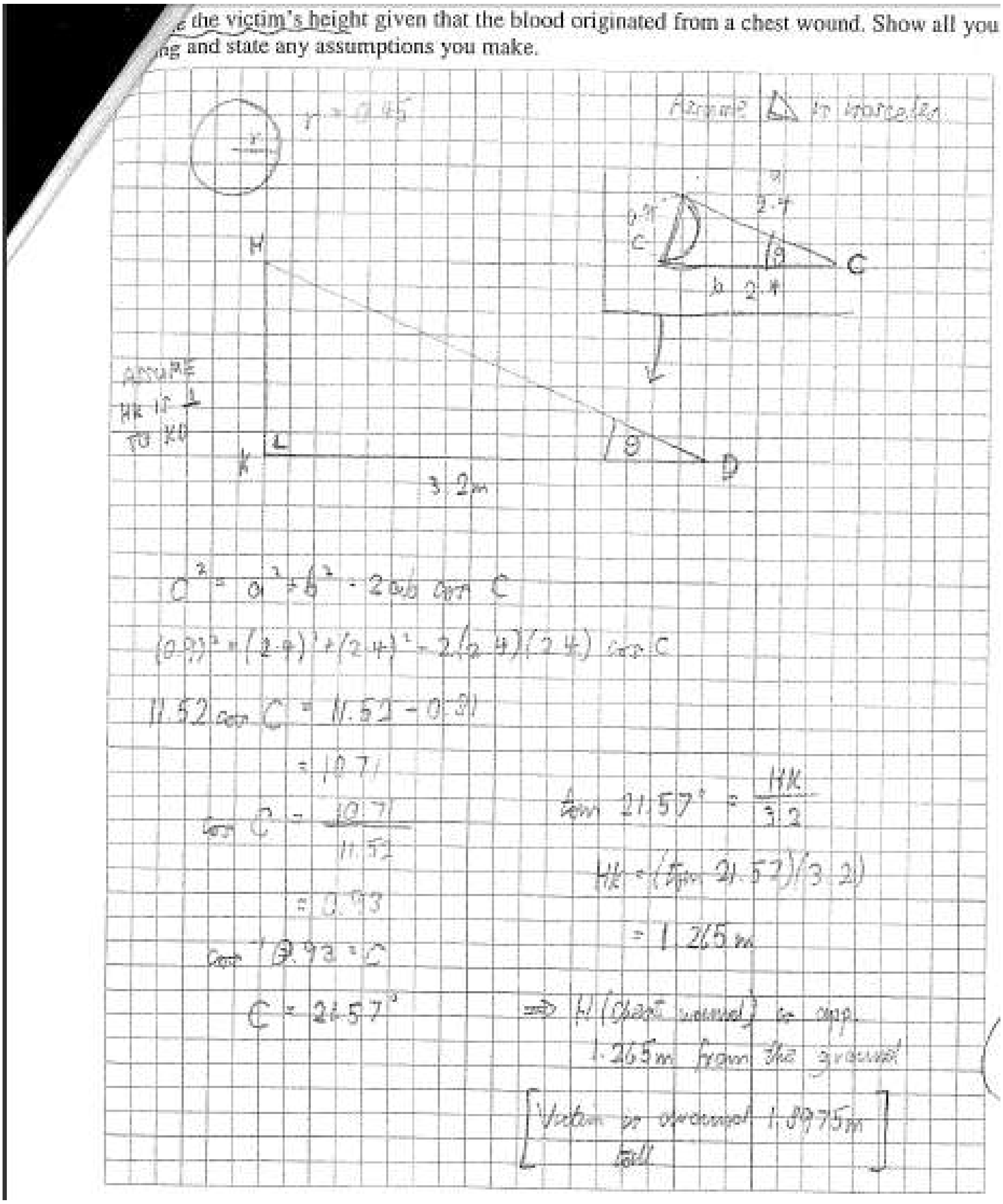
Investigators have mapped out the crime scene below showing two blood spatters on the floor at C and D. They measure a spatter in order to calculate the angle of impact and trace this back to an approximate starting place (assuming the blood drops travel in a straight line).



Since they know the drop started as a sphere, the width of the spatter drop will be the same as its diameter. They record the measurements of the blood spatter X:

Blood spatter from drop X

Estimate the victim's height given that the blood originated from a chest wound. Show all your working and state any assumptions you make.



Question

(a) In a component factory, machine A produces 30% of the output, machine B 25% and machine C the remainder.

Over a period of time, 1% of the output from machine A is found to be defective, 1.2% from machine B and 2% from machine C.

(i) On a given day, the three machines produce a total of 10,000 components. How many components are likely to be defective?

A => 0.3 B => 0.25 C = 0.45

= 3000 units 2500 units 4500 units

3000 × 1% 2500 × 1.2% 4500 × 2%

= 30 = 30 = 90

30 + 30 + 90 = 150 units are likely to be defective.

(ii) A quality controller selects a component at random from that day's output and finds that it is defective. What is the probability that this component was produced by machine B?

Probability its from B given its defective.

Defective overall = 150

Defective from B = 30

P (Produced from B) = $\frac{30}{150}$ = 20%

(b) Cereal packets filled by a machine have a mean of 200 grams with a standard deviation of 10 grams.

(i) In a sample of 300 packets, how many can be expected to weigh more than 215 grams? Show all your work.

$\bar{x}$ = 200grams σ = 10 grams   x = 215

$z = \frac{x - \bar{x}}{\sigma} = \frac{215-200}{10} = 1.5$

z (1.5) = 0.9332

more than 215 = 1 - 0.9332 = 0.0668 = 6.68%

300 × 6.68% = = 20.04 = 20 packets

(ii) The machine is adjusted and another sample of 300 is checked. If the expected value of the number of packets that weigh more than 215 grams is now 10, calculate the new mean, assuming that the standard deviation after adjustment is the same as before?

σ = 10

$\frac{10 \text{ packets}}{300}$ = 0.0333 or 3.33%

1 - 0.0333 = 0.9667

z = 1.835

$z = \frac{x - \bar{x}}{\sigma}$ $-\bar{x}$ = -196.65

$1.835 = \frac{215 - \bar{x}}{10}$ $\bar{x}$ = 196.45

$\bar{x}$ = 18.35 - 215

(c) A company states that 20% of the visitors to its website purchase at least one of their products. A sample of 400 site visitors is checked and the number who purchased a product is found to be 64.

(i) Calculate the margin of error in this case.

$\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{400}} = 0.05$ or 5%

(ii) Based on this sample, should the company's claim be accepted? Explain your reasoning.

$\frac{64}{400} = 16\%$

$16 - 5\% \leq p \leq 16 + 5\%$

$11\% \leq p \leq 21\%$

Yes the companys claim can be supported. They claim 20% of visitors make a purchase and this is between the limits seen when the margin of error due to their sample size is taken into account.

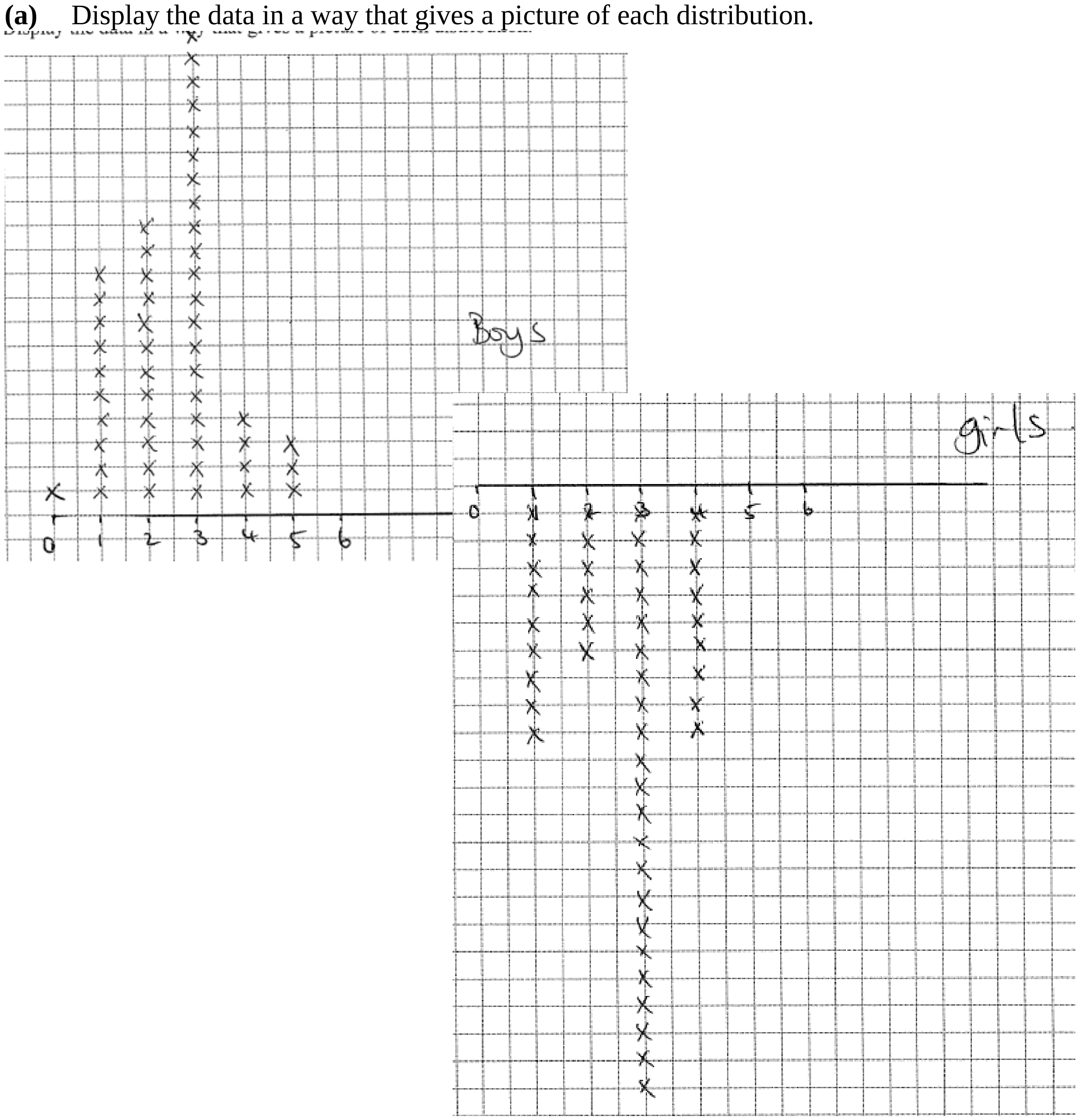
Question

Some research was carried out into the participation of girls and boys in sport. The researchers selected a simple random sample of fifty male and fifty female teenagers enrolled in GAA clubs in the greater Cork area. They asked the teenagers the question: *How many sports do you play?*

The data collected were as follows:

Boys	Girls
0, 4, 5, 1, 4, 1, 3, 3, 3, 1,	3, 3, 3, 1, 1, 3, 3, 1, 3, 3,
1, 2, 2, 2, 5, 3, 3, 4, 1, 2,	2, 2, 4, 4, 4, 5, 5, 2, 2, 3,
2, 2, 2, 3, 3, 3, 4, 5, 1, 1,	3, 3, 4, 1, 6, 2, 3, 3, 3, 4,
1, 1, 1, 2, 2, 2, 2, 2, 3, 3,	4, 5, 3, 4, 3, 3, 3, 4, 4, 3,
3, 3, 3, 3, 3, 3, 3, 3, 3, 3	1, 1, 3, 2, 1, 3, 1, 3, 1, 3

(a) Display the data in a way that gives a picture of each distribution.



(b) State **one difference** and **one similarity** between the distributions of the two samples.

Difference: 12 out of 50 girls play more than 3 sports, whereas only 7 out of 50 boys play more than 3 sports

Similarity: the mode is the same for both.

(c) Do you think that there is evidence that there are differences between the two populations? Explain your answer.

Note: you are not required to conduct a formal hypothesis test.

Answer: No

Justification: I think the samples are so similiar that it is unlikely to be due to chance.

(d) The researchers are planning to repeat this research on a larger scale. List **two** improvements they could make to the design of the research in order to reduce the possibility of *bias* in the samples. Explain why each improvement you suggest will reduce the likelihood of bias.

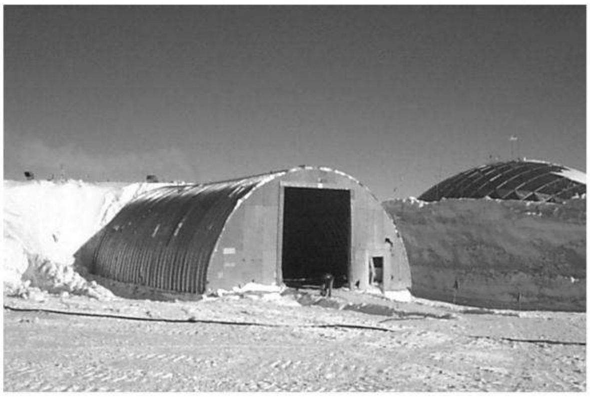
They could sample people other than those in GAA clubs as this could result in potential bias.
They could include urban + rural areas in a wider area than just Cork

Question

The *Wonder Building* is an arched building that does not need any support inside, due partly to the fact that its shape is an arc of a circle.

The photograph shows a *Wonder Building* being used in Antarctica.

The arc for a *Wonder Building* can be a full semicircle or less than a semicircle. It cannot be more than a semi-circle. The "span" of the building is the total width from one side of the arch to the other.



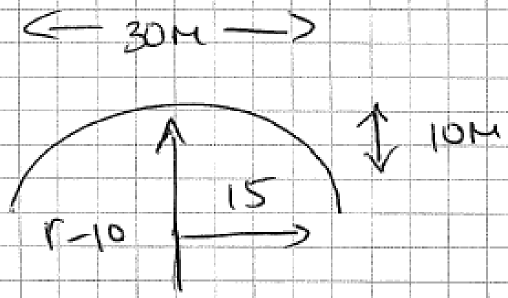
(a) A particular *Wonder Building* has a span of 30 metres and a height of 10 metres. Find the radius of the arc.

$r^2 = 15^2 + (r-10)^2$

$r^2 = 225 + r^2 - 20r + 100$

$20r = 325$

$r = 16.25$ M



(b) A customer wants a building with a span of 18 metres and a height of 10 metres.

(i) What arc radius would be required to give such a building?

$r^2 = 9^2 + (r-10)^2$

$r^2 = 81 + r^2 - 20r + 100$

$20r = 181$

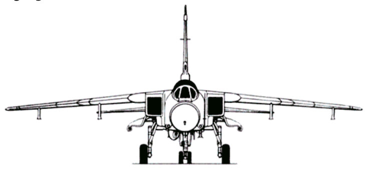
$r = 9.05$M

(ii) Explain why the *Wonder Building* that the customer wants is not possible.

$r-10$ can't be negative

(c) An air force needs a *Wonder Building* to house a *Tornado* military jet.

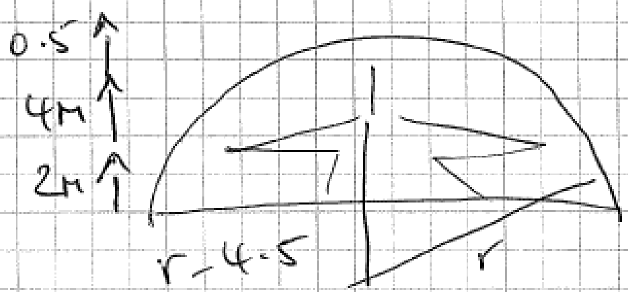
The dimensions of the aircraft are as follows:



- Wingspan: 14 metres
- Height: 6 metres
- Height of wingtips above ground: 2 metres.

The shelter must be at least 0·5 metres above the top of the tail, and at least 1 metre clear horizontally of the wingtips.

For the shelter to have the exact clearance required, find the radius of the arc.



$r^2 = 8^2 + (r-4.5)^2$

$= 64 + r^2 - 9r + 20.25$

$9r = 84.25$

$r = 9.3611$ metres.

Section A	Concepts and Skills

Question

A computer is going to choose a letter at random from the text of an English novel.
The table shows the probabilities of the computer choosing the various vowels.

Vowel	**A**	**E**	**I**	**O**	**U**
Probability	**0.06**	**0.13**	**0.07**	**0.08**	**0.03**

(a) What is the probability it will **not** choose a vowel?

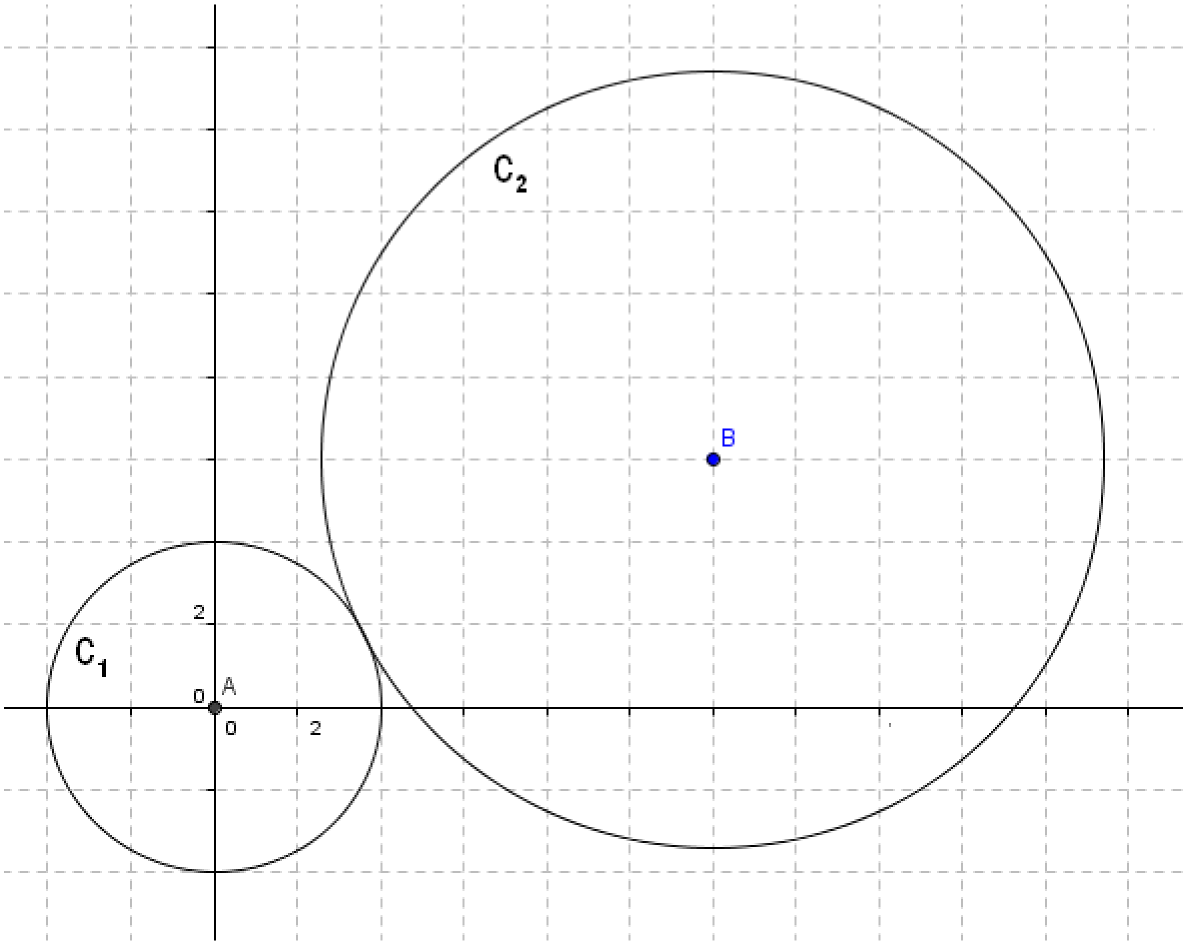
(b) The probability that the computer will choose the letter **T** is **0.09.**

The computer chooses a letter at random, and then a second, and then a third letter.
What is the probability that these letters will be **E, A** and **T** (in that order)?

(c) How many ways can these three letters be arranged? Show each arrangement.

Question

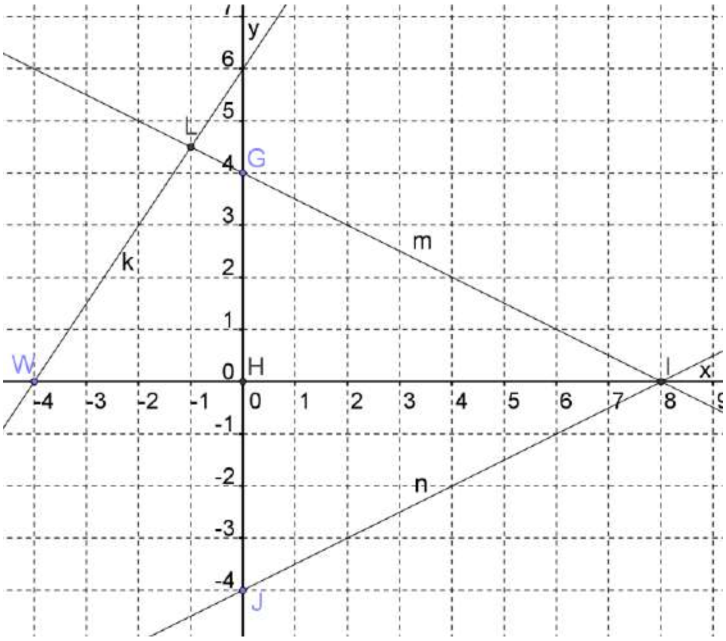
(a) The diagram shows two touching circles; c_1 and c_2. Using the diagram to estimate the centres and radii as accurately as you can, find the equations of the two circles.



(b) It is claimed that the line with equation $x - y + 6 = 0$ is a tangent to both circles.
By performing suitable calculations, decide whether this claim is true or false.
Explain your answer.

Question

In this diagram state whether each of the following statements is true or false (by placing a ✓ in the appropriate box) and in each case give a reason for your answer.



a) $k \perp m$

True	False

b) area ΔGIJ = 32 sq. units

True	False

c) the equation of k is $y = -\frac{2}{3}x + 6$

True	False

d) $m \perp n$

True	False

e) the line $y = -2x + 1$ is perpendicular to n

True	False

f) the line $y = 2x$ is parallel to m

True	False

g) ΔGIJ is an isosceles triangle

True	False

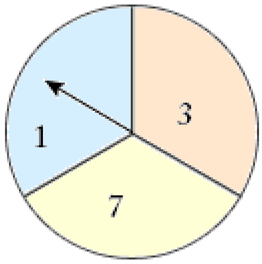
h) the x-axis is the bisector of $\angle GIJ$

True	False

Question

Sophie and Amy were designing a game of chance to raise money for charity in their school. They agreed on the following:

- they would call the game ***Spin and Win***
- they would charge **50 cent** to play
- the rule would be: roll a six-sided die and spin the spinner shown and **add the totals**.



However, they each had different ideas about which outcomes would result in a win, a loss or the player getting their money back.

Sophie's Idea
Money back: Get total of 13

Win €1: Even number total

Lose: Anything else

Amy's Idea
Money back: Even number total

Win €1: Odd total, but not prime

Lose: Anything else

(a) By calculating the expected value of the profit or loss for each idea, write an argument to support **either** Sophie **or** Amy's idea.

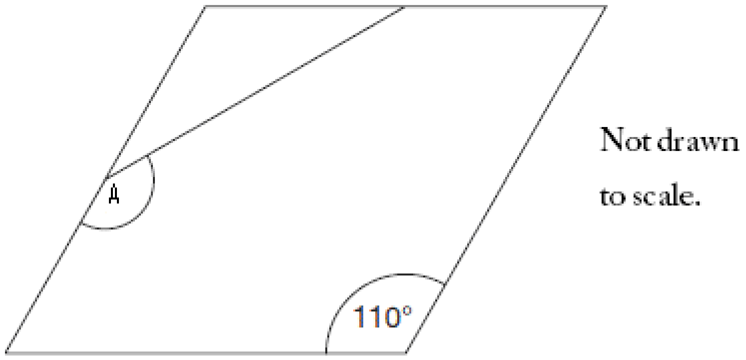
Without changing the rules, give your own idea for ***win***, ***lose*** and ***money back*** that would generate more money for the charity. Justify your idea.

Question

Construct an equilateral triangle. Prove that the inscribed circle and the circumcircle have the same centre.

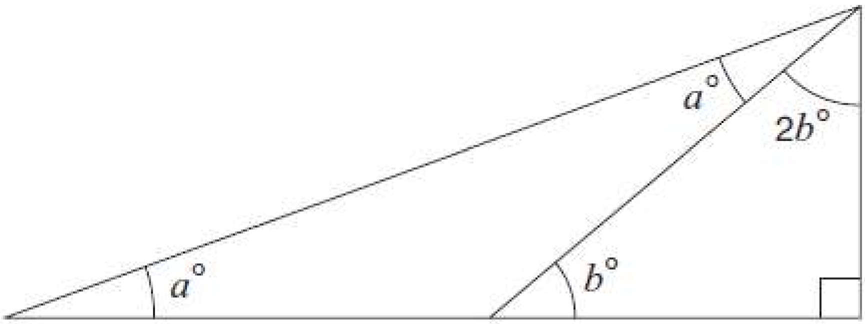
Question

(a) The diagram shows a rhombus (that is, a parallelogram with four sides of equal length). The midpoints of two of its sides are joined with a straight line segment.



Calculate the size of angle *A*. Show how you found your answer.

(b)



Not drawn to scale.

Find the value of ***a*.** Show how you found your answer.

$\angle GIJ$

Question

Sarah is on a TV game show called ***Take the Money and Run***.

She has won €10,000 so far. She now has four options:

Option 1: Leave the show with €10,000 – that is, ***Take the Money and Run***

Option 2: Play on and take a 50% chance of winning €50,000

Option 3: Play on and take a 30% chance of winning €75,000

Option 4: Play on and take a 20% chance of winning €100,000

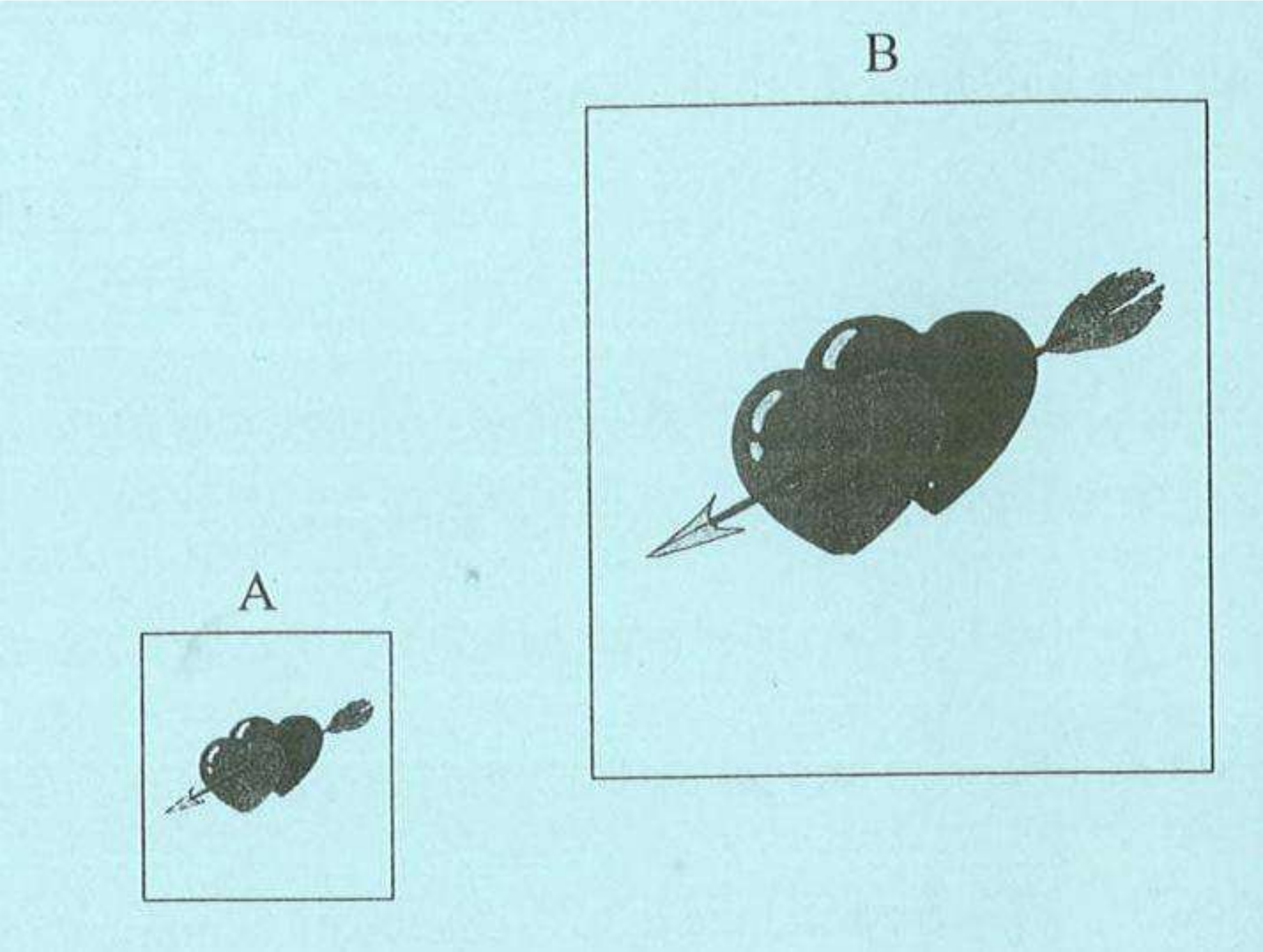
If she plays on, and does not win the higher amount she loses the €10,000.

(a) Calculate the ***Expected Value*** of Sarah's winnings for each of the four options.

(b) What would you advise Sarah to do and why?

Question

The ray method was used to enlarge a design for a Valentine card. The original is labelled A and the image is labelled B.



(a) Find the centre of the enlargement.

(b) Find the scale factor of the enlargement. Show your work.

(c) Calculate the ratio $\frac{Area\ of\ drawing\ B}{Area\ of\ drawing\ A}$. Give your answer correct to one decimal place.

Question

Two spinners, each with four equal segments numbered 1 to 4, are spun.

(a) Using a list, table, tree diagram, or otherwise, show all the **possible outcomes.**

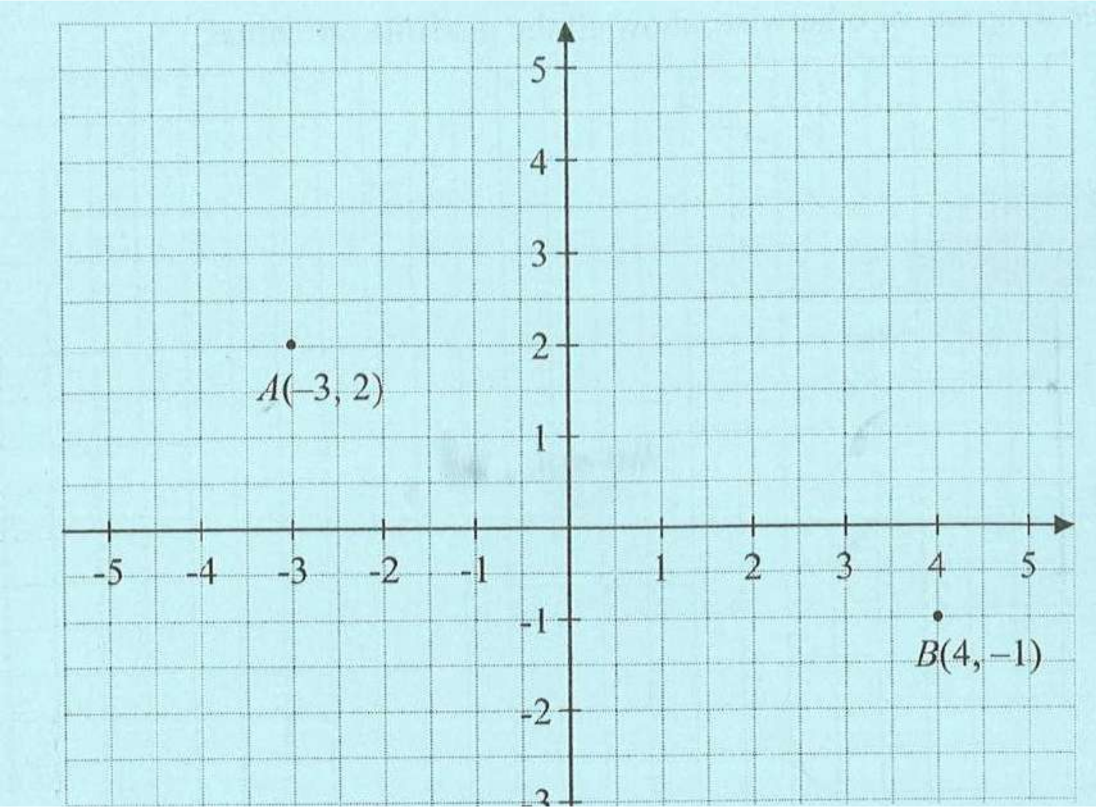
(b) If the spinners are fair, what is the probability of getting two fours?

(c) Jason thinks that one of the spinners is not fair.

Describe an experiment that he could do to find out whether the spinner is fair.

Question

Two points *A*(- 3, 2) and *B*(4, - 1) are shown on the diagram below. Plot two suitable points *C* and *D* so that *ABCD* is a parallelogram. Label the points and write down their coordinates.

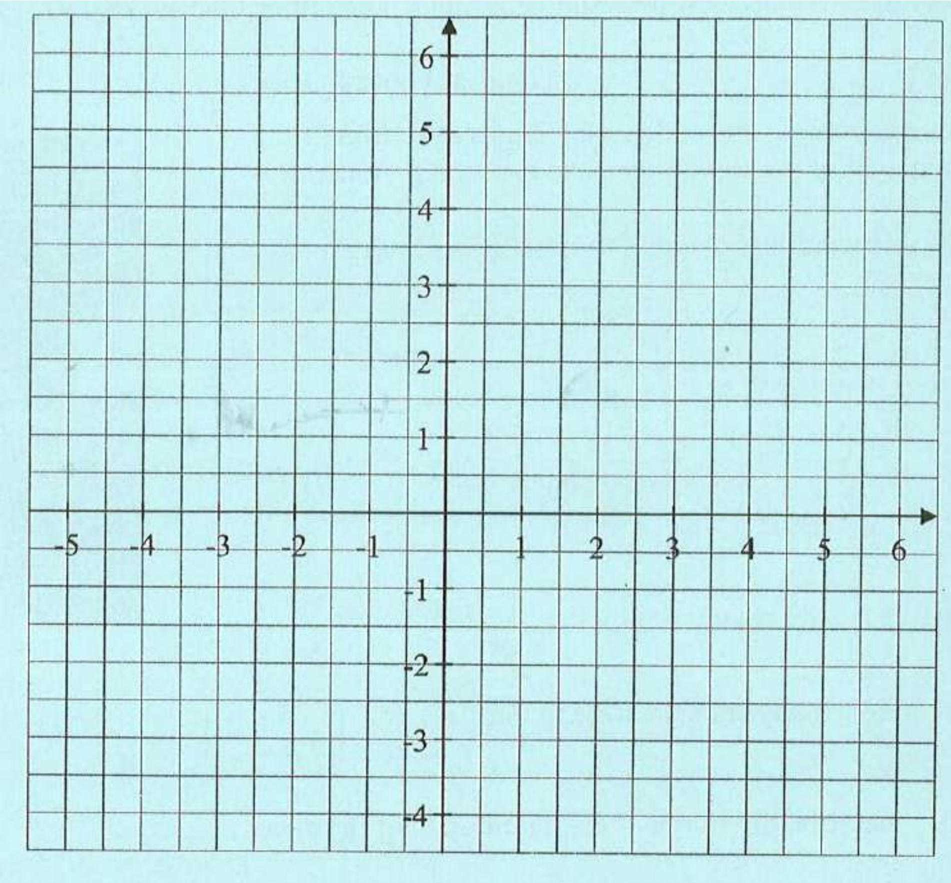


(b) By performing suitable calculations, show that the figure you have drawn is indeed a parallelogram.

(c) Verify that the diagonals of the parallelogram bisect each other.

Question

(a)On the diagram below, show the triangle *ABC*, where *A* is (-4, 1) *B* is (-2, 5) and *C* is (6, 1)



(b) Find D, the midpoint of [AC], and label this point on the diagram.

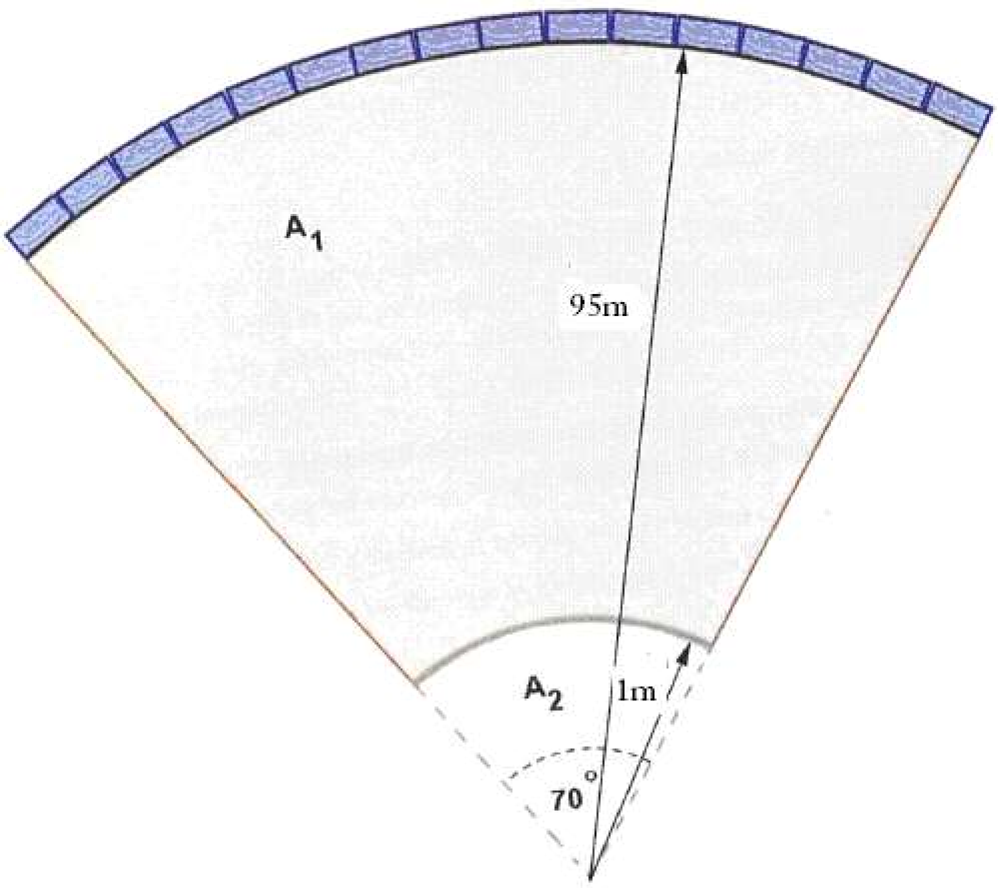
(c)Hence, construct on the diagram the circle with diameter [AC].

(d) Show that the angle $\angle ABC$ is a right angle.

Section B	Contexts and Applications

Question

(a) The modern or Olympic *hammer throw* is an athletic throwing event where the object is to throw a heavy metal ball attached to a wire and handle. In the diagram below A_2 represents a portion of the *throwing circle* and A_1 represents the area in which the hammer should land. The diagram is not drawn to scale.



(i) A net is to be erected at the end of the landing area. The foundation consists of a single row of bricks; each brick is 41cm long. How many bricks will be needed to lay the foundations?

(ii) The area A_1 will be planted with grass. A *10kg* bag of lawn seed covers approximately $220m^2$. How many bags of grass seed must be bought?

Show all your work and state any assumptions you make.

Question

The lengths of the ring fingers of 30 Irish students chosen randomly from amongst those who completed the *censusatschool* phase 9 questionnaire are displayed below. The measurements are in cm.

7.5	8	7	6	7.5
8.3	6.5	8	5	9
7.3	8.5	7	7	9
7.2	6.5	7	10	9
3	4	6.6	6	8
7	8	7	7.5	8.4

(a) Use the data to investigate whether ring finger lengths are normally distributed. Explain your answer.

(b) Sharon measured the length of her ring finger and found it to be 11.3cm. Her boyfriend says her finger length is most unusual; Sharon disagrees. By calculating the mean and standard deviation of the distribution above, present evidence to support either Sharon's argument, or that of her boyfriend.

Question

(a) A teacher asked the students in her class to estimate the height of the church opposite the school in metres.

The stem-and-leaf diagram shows all the results

Key: 3|5 represents 35m

3	5 9
4	0 2 6 8 8
5	3 3 5 7 7 9
6	0 5 5 5
7	4 8
8	2 7

(i) How many students are in the class?____________

(ii) Describe the **shape** of the distribution of the data

(iii) What was the median estimate? _____________

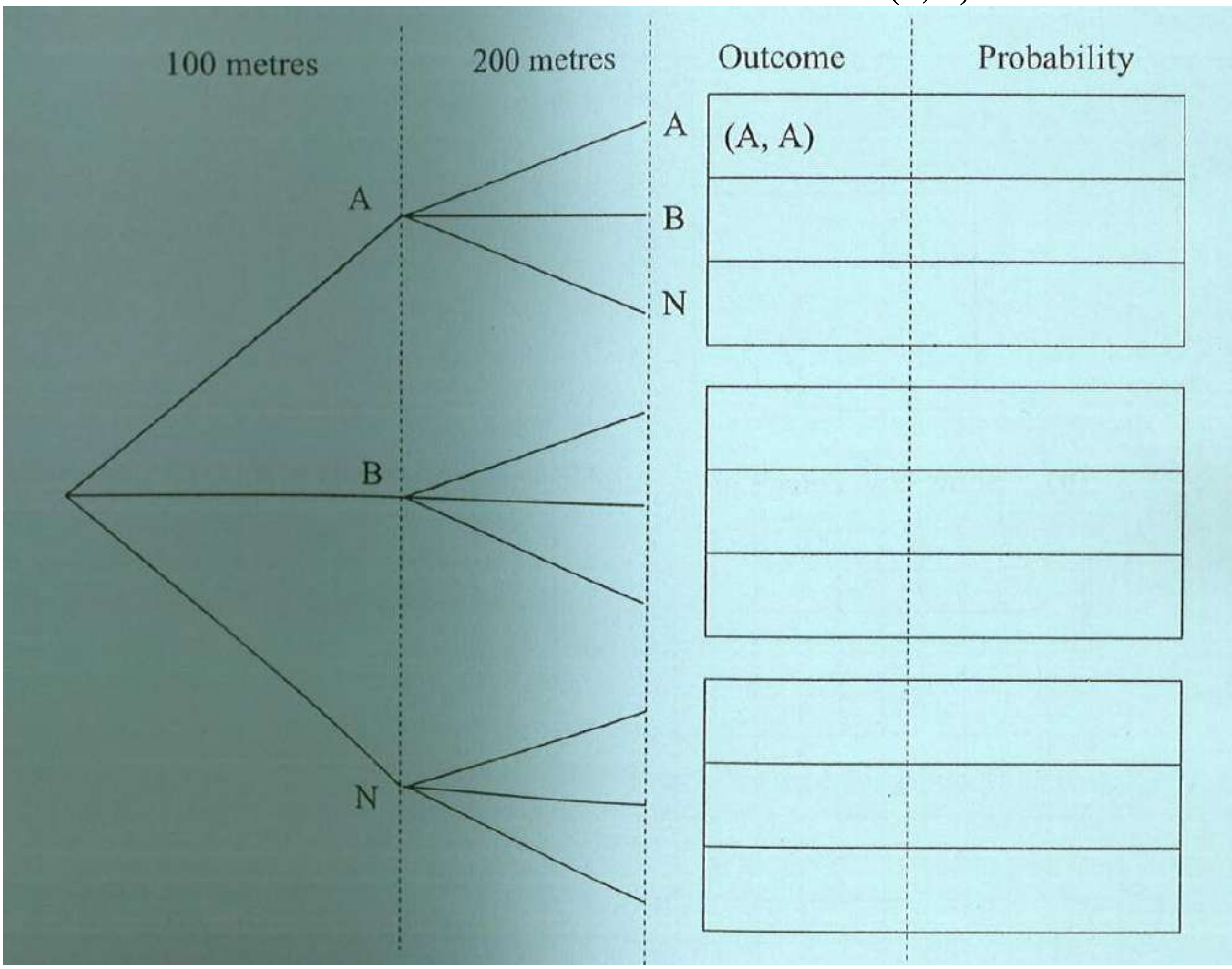
(iv) Explain the answer to part (iii) to someone who does not know what the word "median" means.

(b) Alex and Bobby are running in the final of a 100m race and a 200m race.
The probabilities of each of them winning each race are given in the table below.
The probability that neither of them wins the 100m race is also given.

	Alex	**Bobby**	**Neither**
100 metre race	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{7}{12}$
200 metre race	$\frac{1}{4}$	$\frac{3}{8}$	

(i) Complete the table above, by inserting the probability that someone other than Alex or Bobby wins the 200 metre race.

(ii) Using the tree diagram or otherwise, complete the list of outcomes below. For example, the outcome that Alex wins the first race and the second race is recorded as (A, A) as shown.

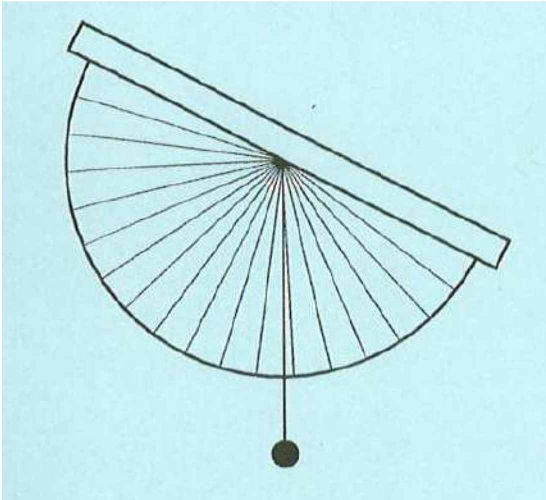


(c) What is the probability that Alex and Bobby win a race each?

Question

The students mentioned in **(a)** above went to measure the height of the church.

(a) Peter explained his group's method:
"We made a clinometer from a protractor, a pen tube, some thread and a weight.
We measured the distance from here to the church and it was 92 metres.
We made sure the ground was flat,then we used the clinometer to look up at the top of the spire of the church. The weight had moved from 90° to 65°, so we knew the angle up was 25°. We worked out the height from that. But we had to remember to add on my height of 1.8 metres at the end."



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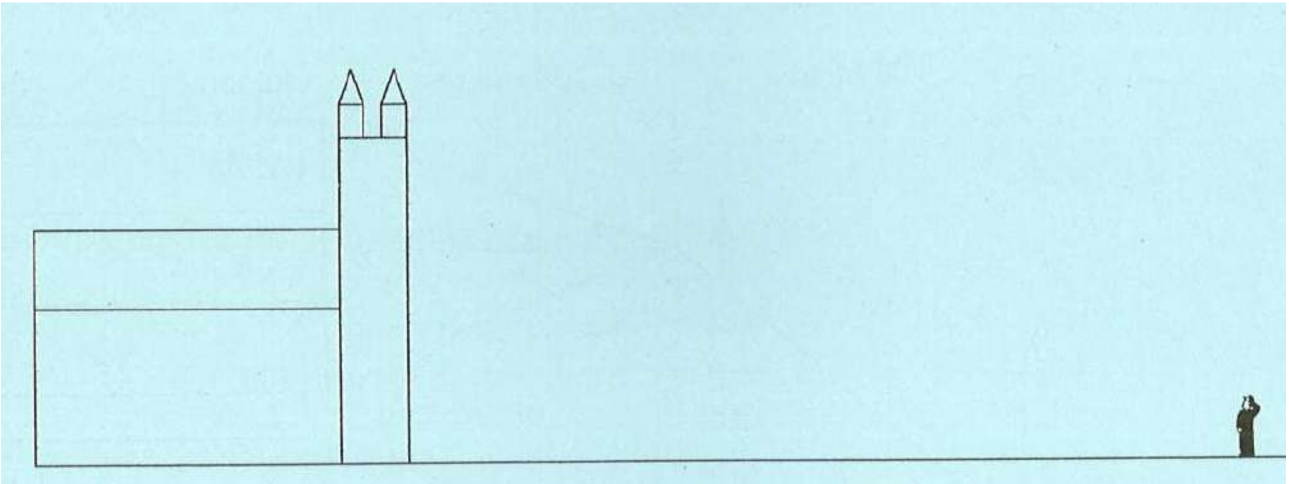
(i) On the diagram below, show the measurements that Peter's group made.



(ii) Show how Peter's group used these measurements to find the height of the church.

b) Hannah was in a different group from Peter. She explained her group's method for finding the height of the church:

"It was really sunny and we used the shadows cast by the sun.
Amy stood with her back to the sun and we used a tape measure to measure Amy's shadow along the ground from the tips of her toes to the top of her shadow's head. We also measured Amy's height and recorded the results in the table.

Then we recorded the length of the shadow cast by the church. We measured along the ground from the base of the church out to the end of its shadow and recorded this measurement."

Amy's Shadow	2 m
Church's Shadow	69.4 m
Amy's Height	1.7 m

Show how Hannah's group used their results to calculate the height of the church.

(c)The church is actually 50 metres high. Calculate the percentage error in each groups result.

Question

Oxygen levels in a polluted river were measured at randomly selected locations before and after a clean-up. These results were given in the table:

Before (mg/l)				After (mg/l)			
20 23 2	25 23 10	20 10 11	9 11 5 11	26 11 3	10 15 8	10 11 11	9 11 4 13

(a) Construct a back-to-back stem-and-leaf plot of the above data.

(b) State **one difference** and **one similarity** between the distributions of the measurements before and after cleanup.

Difference:

Similarity:

Question

Noel and Sarah were taking part in a mathematics competition with other students from the *ProjectMaths* schools. They were finding the area of the face of the triangular sculpture shown below.

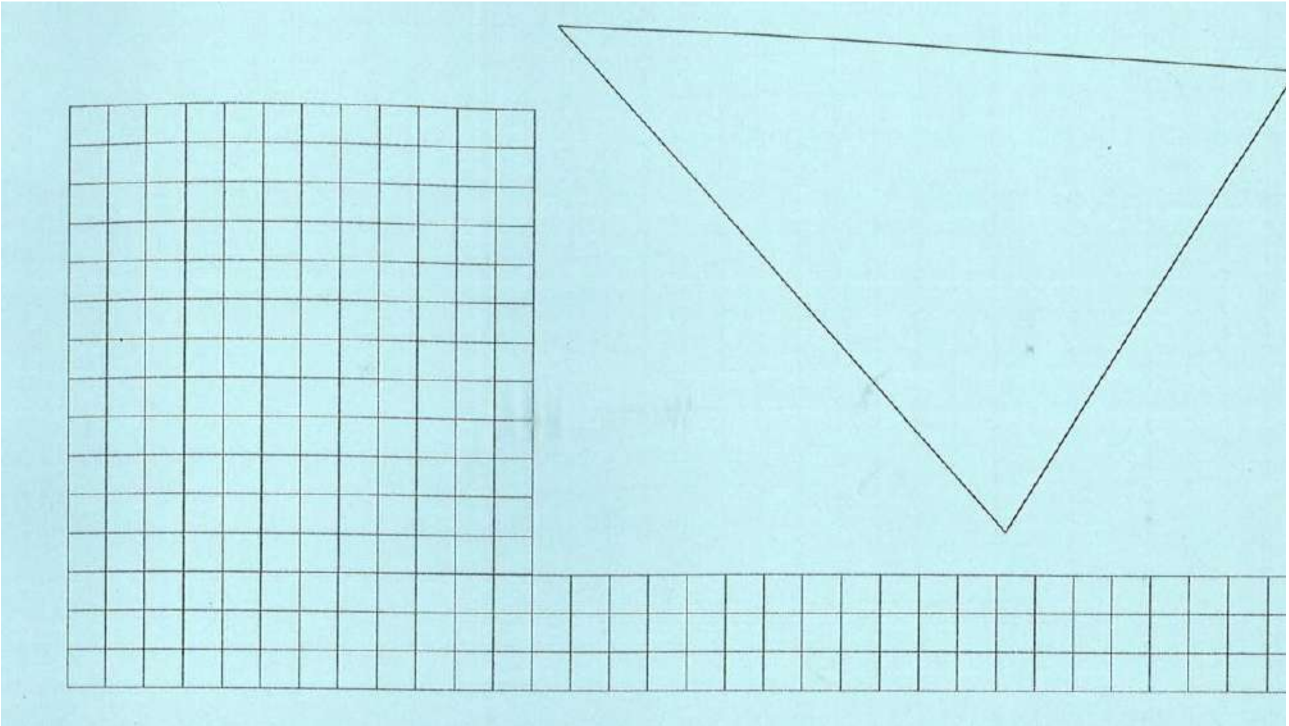
Noel said: "We should measure the height and base of the triangle. Then use the formula that says the area is half the base by the height."

Sarah said: "Okay. How do we know which side is the base?"

Noel said: "it doesn't matter, because of the theorem we did."

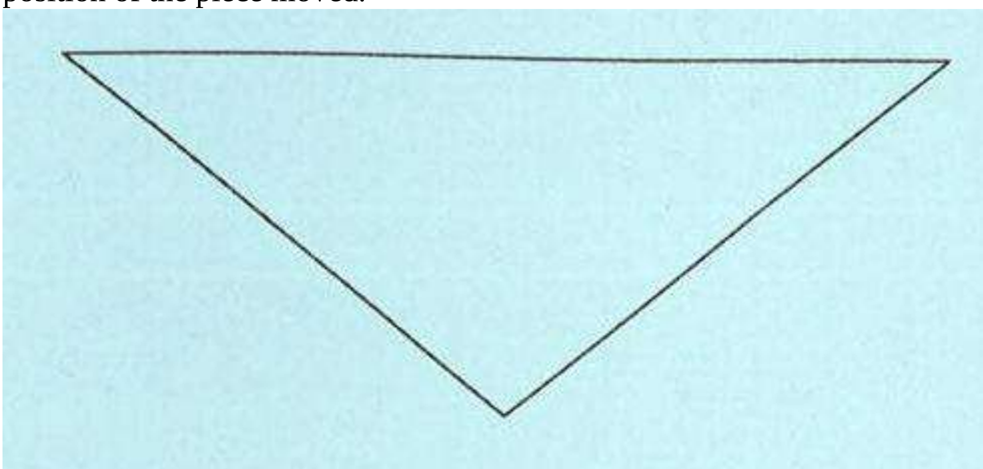
(i) State the theorem that Noel is talking about.

(ii) Noel and Sarah trace the triangle on the photograph onto a page to find it's area. Their drawing is shown here. By making suitable measurements on the drawing, verify the theorem you stated in part (a).



(c) Suppose that the drawing was a true representation of the face of the sculpture. If each centimetre in the drawing represents 70cm in reality find the area of the face of the sculpture.

(d) The true shape of the face of the sculpture is shown below. The people who made it have changed their mind and now want a parallelogram instead!
Show how the triangle could be turned into a parallelogram by making one cut and moving one of the two pieces. You should make it clear exactly where the cut is to be made, and show the new position of the piece moved.



Section A	Concepts and Skills

Question

A computer is going to choose a letter at random from the text of an English novel.
The table shows the probabilities of the computer choosing the various vowels.

Vowel	**A**	**E**	**I**	**O**	**U**
Probability	**0.06**	**0.13**	**0.07**	**0.08**	**0.03**

(a) What is the probability it will **not** choose a vowel?

A = 0·06
E = 0·13
I = 0·07
O = 0·08
U = 0·03
·37

·37 × 100 = 37%

100% − 37 = 63%

Ans = 63%

(b) The probability that the computer will choose the letter **T** is **0.09.**

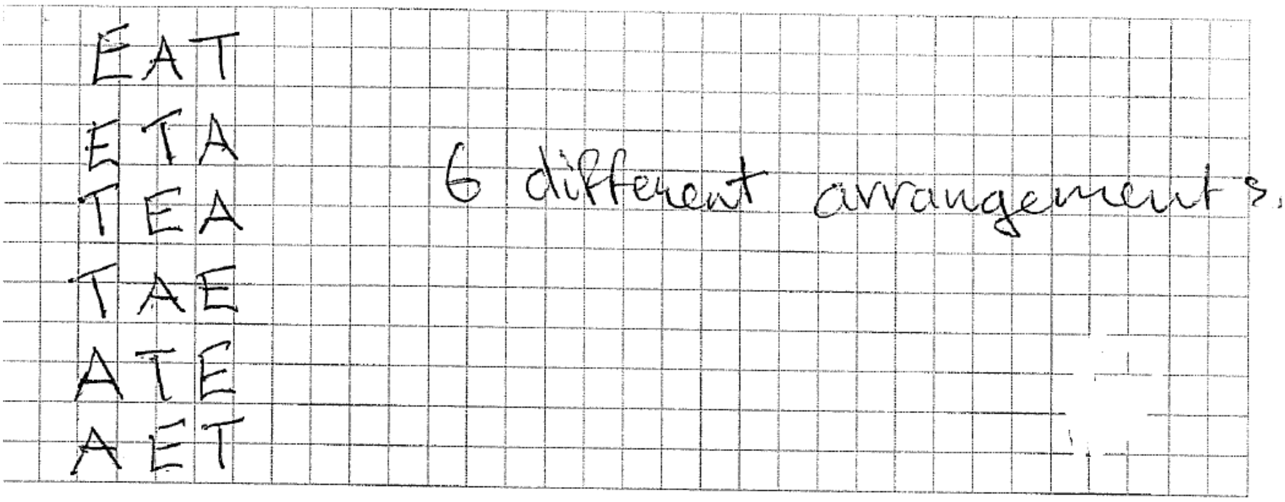
The computer chooses a letter at random, and then a second, and then a third letter.
What is the probability that these letters will be **E, A** and **T** (in that order)?

E × A × T

0·13 × 0·06 × 0·09 = $\frac{351}{500,000}$

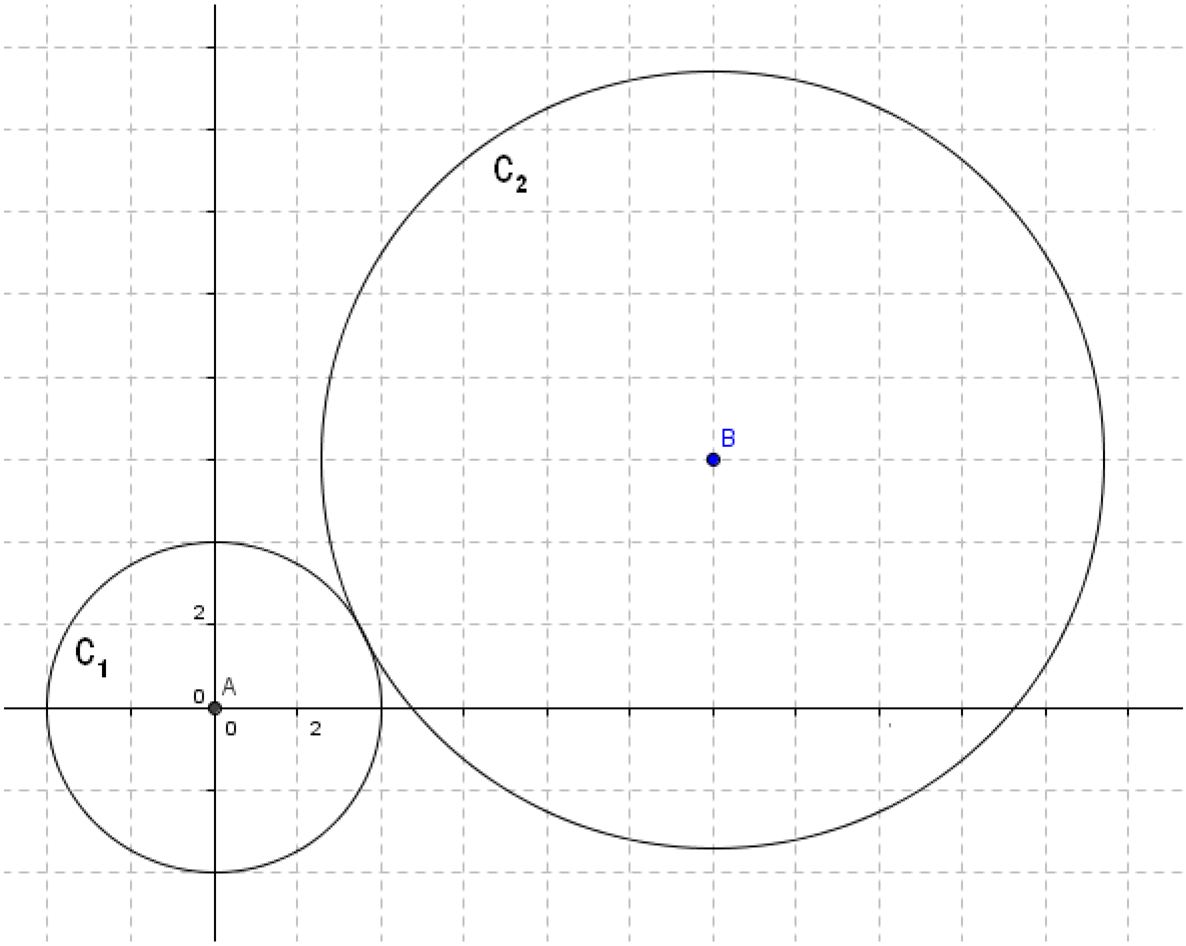
$\frac{351}{500,000}$ × 100 = ·0702 %

(c) How many ways can these three letters be arranged? Show each arrangement.



Question

(a) The diagram shows two touching circles; c_1 and c_2. Using the diagram to estimate the centres and radii as accurately as you can, find the equations of the two circles.



$C_1 - R = 4$, $C = (0,0)$

$C_2 - R = 9.5$, $C = (12, 6)$

C_1 $(x-h)^2 + (y-k)^2 = r^2$

$x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = r^2$

$x^2 - 2x(0) + (0) + y^2 - 2y(0) + (0)^2 = 16$

$x^2 + y^2 = 16$

C_2 $x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = r^2$

$x^2 - 2x(12) + (12)^2 + y^2 - 2y(6) + (6)^2 = (9.5)^2$

$x^2 - 24x + 144 + y^2 - 12y + 36 = 90.25$

$x^2 + y^2 - 24x - 12y - 89.75 = 0$

(b) It is claimed that the line with equation $x - y + 6 = 0$ is a tangent to both circles. By performing suitable calculations, decide whether this claim is true or false. Explain your answer.

$x = y - 6$ $(y-6)^2 + y^2 = 16$

$y^2 - 12y + 36 + y^2 = 16$

$2y^2 - 12y + 52 = 0$

$y^2 - 6y + 26 = 0$

More than one point of contact → not a tangent

$(x-12)^2 + (y-6)^2 = 90.25$

$(y-6-12)^2 + (y-6)^2 = 90.25$

$(y-18)^2 + (y-6)^2 = 90.25$

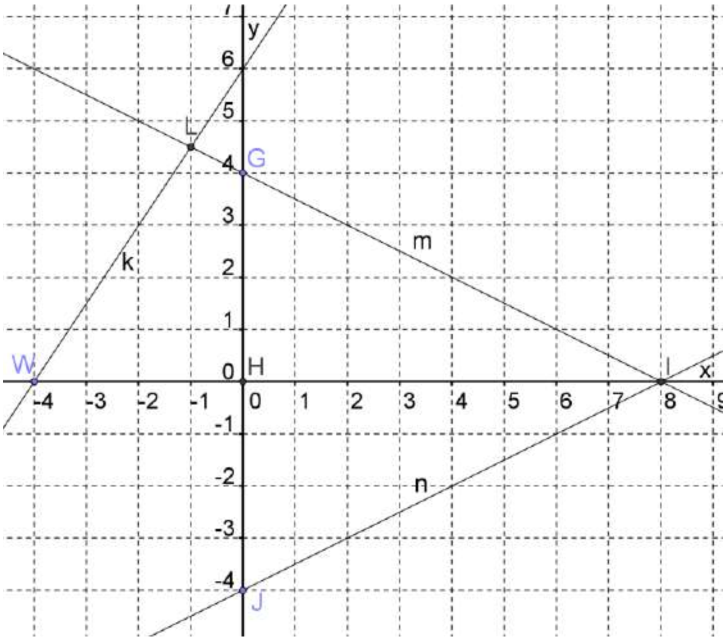
$y^2 - 36y + 324 + y^2 - 12y + 36 = 90.25$

$2y^2 - 48y + 269.75 = 0$ More than one point of contact

$y^2 - 24y + 134.875 = 0$ Not a tangent

Question

In this diagram state whether each of the following statements is true or false (by placing a ✓ in the appropriate box) and in each case give a reason for your answer.



a) $k \perp m$

True	False
	x

Slope $k = \frac{y_2 - y_1}{x_2 - x_1}$

$y_2 = 6$, $y_1 = 0$, $x_1 = -4$, $x_2 = 0$

$= \frac{6-0}{0-4}$

$= -\frac{6}{4}$

Slope $m = \frac{y_2 - y_1}{x_2 - x_1}$

$y_2 = 4$, $y_1 = 0$, $x_2 = 0$, $x_1 = 8$

$\frac{4-0}{0-8}$

$= -\frac{4}{8}$

b) area $\Delta GIJ = 32$ sq. units

True	False
x	

$\frac{1}{2}$ base × perpendicular height

4 × 8 = 32 sq units

c) the equation of k is $y = -\frac{2}{3}x + 6$

True	False
	x

$y - y_1 = m(x - x_1)$

$y - 0 = -\frac{6}{4}(x - (-4))$

$y = -\frac{6}{4}x - 6$

d) $m \perp n$

True	False
	x

Slope of M $= -\frac{1}{2}$

Slope of n $= \frac{1}{2}$

$\therefore$ M $\not\perp$ n

e) the line $y = -2x + 1$ is perpendicular to *n*

True	False
x	

m = -2
Slope n = 1/2
-2 × 1/2
= -1

f) the line $y = 2x$ is parallel to *m*

True	False
	x

m = 2
Slope m = -1/2

g) ΔGIJ is an isosceles triangle

True	False
x	

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
$\sqrt{(0-8)^2+(4-0)^2}$
$\sqrt{64+16} = \sqrt{80}$

$\sqrt{80} = 8.94$
$\sqrt{(0-8)^2+(-4-0)^2}$
$= \sqrt{80}$

|GJ| = 8

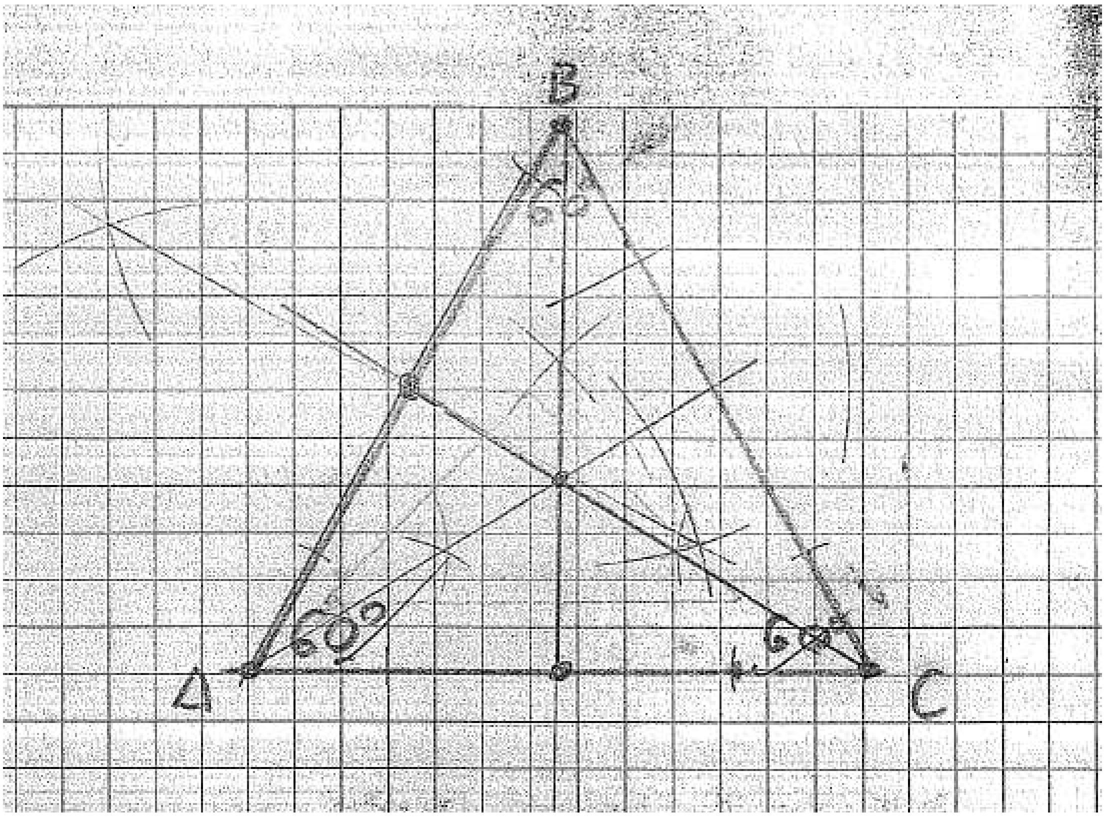
h) the *x*-axis is the bisector of $\angle GIJ$

True	False

tan A = -4/8 tan B = 4/8
A = B
∴ x-axis bisects ∠GIJ

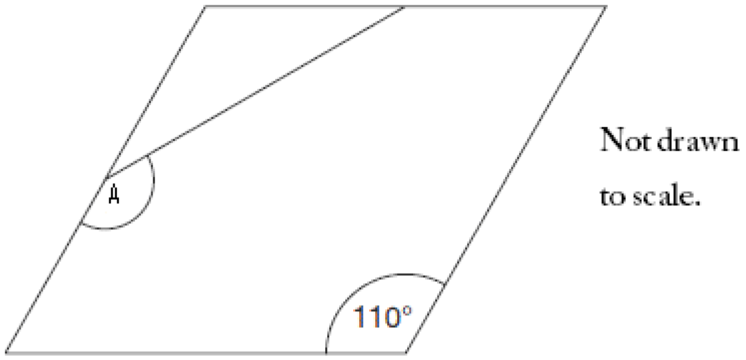
Question

Construct an equilateral triangle. Prove that the inscribed circle and the circumcircle have the same centre.

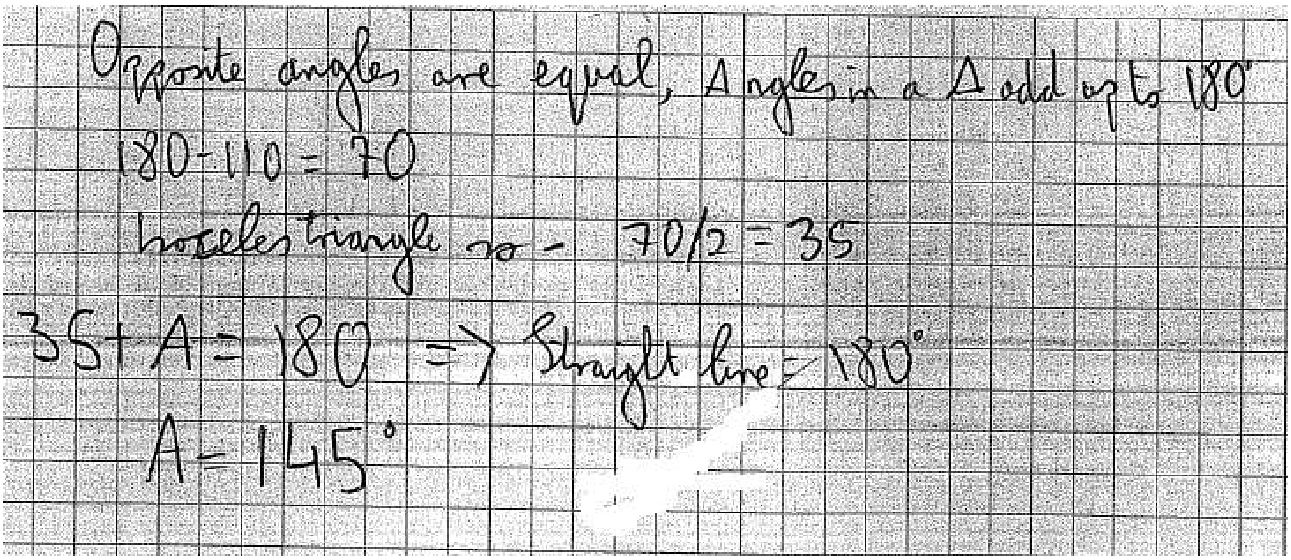


Question

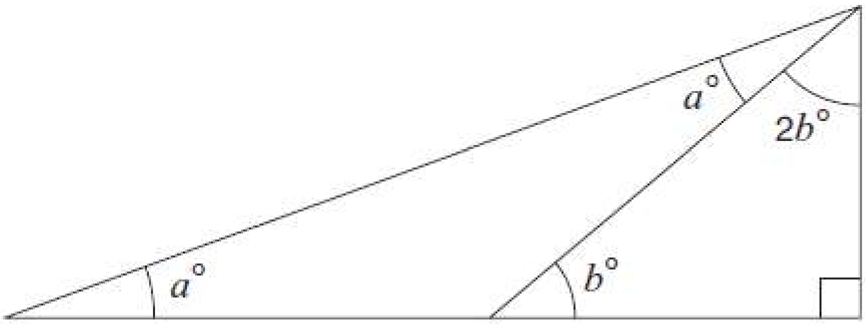
(a) The diagram shows a rhombus (that is, a parallelogram with four sides of equal length). The midpoints of two of its sides are joined with a straight line segment.



Calculate the size of angle *A***.** Show how you found your answer.



(b)



Not drawn
to scale.

Find the value of ***a.*** Show how you found your answer.

∴ = 180° − 90° = 90°

3b = 90°

b = 90°/3

b = 30°

Straight line = 180° − 30°
= 150°

3 angle in triange = 180°

∴ 180° = 150° + 2a

180° − 150° = 2a

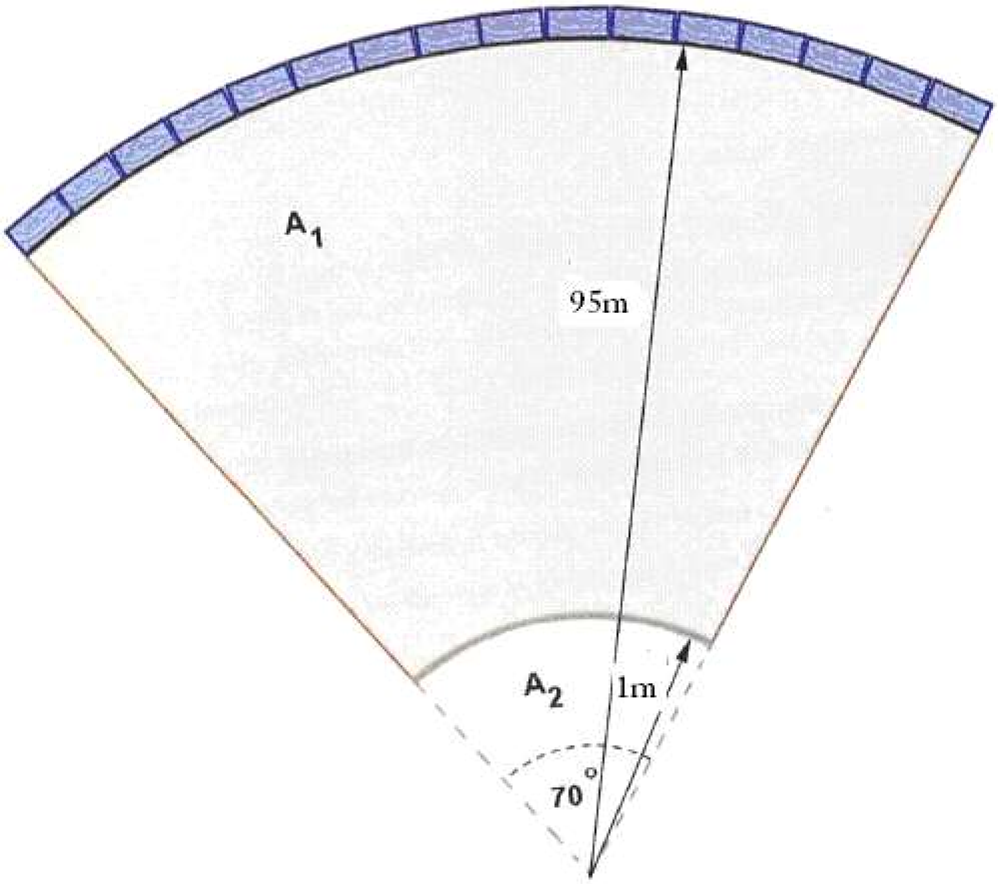
30° = 2a

15° = a

Section B	Contexts and Applications

Question

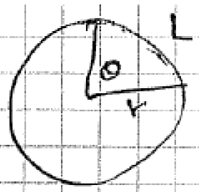
(a) The modern or Olympic *hammer throw* is an athletic throwing event where the object is to throw a heavy metal ball attached to a wire and handle. In the diagram below A_2 represents a portion of the *throwing circle* and A_1 represents the area in which the hammer should land. The diagram is not drawn to scale.



(i) A net is to be erected at the end of the landing area. The foundation consists of a single row of bricks; each brick is 41cm long. How many bricks will be needed to lay the foundations?

(ii) The area A_1 will be planted with grass. A *10kg* bag of lawn seed covers approximately $220m^2$. How many bags of grass seed must be bought?

Show all your work and state any assumptions you make.



$$L = 2\pi r \left[\frac{\theta}{360^\circ}\right]$$

$$L = (2)(\pi)(95)\left[\frac{70^\circ}{360^\circ}\right]$$

cm 11606.4

$L = 116.064 \ m^2$

$L \approx L$

$11606.4 \div 41$

$= 283.08$

284 bags needed

Area of A_1 = $A = \pi r^2 \left[\frac{\theta}{360^\circ}\right]$

$$A = \pi (95)^2 \left[\frac{70^\circ}{360^\circ}\right]$$

$$A = 5513.058$$

Area of A_2 = $A = \pi r^2 \left[\frac{\theta}{360}\right]$

$$A = \pi (1)^2 \left[\frac{70}{360}\right]$$

(10)

$$A = .06108$$

$A_1 - A_2 = 5513.058 - .06108 = 5512.99 \div 220$

$= 25.059$

Ans = 26 bags

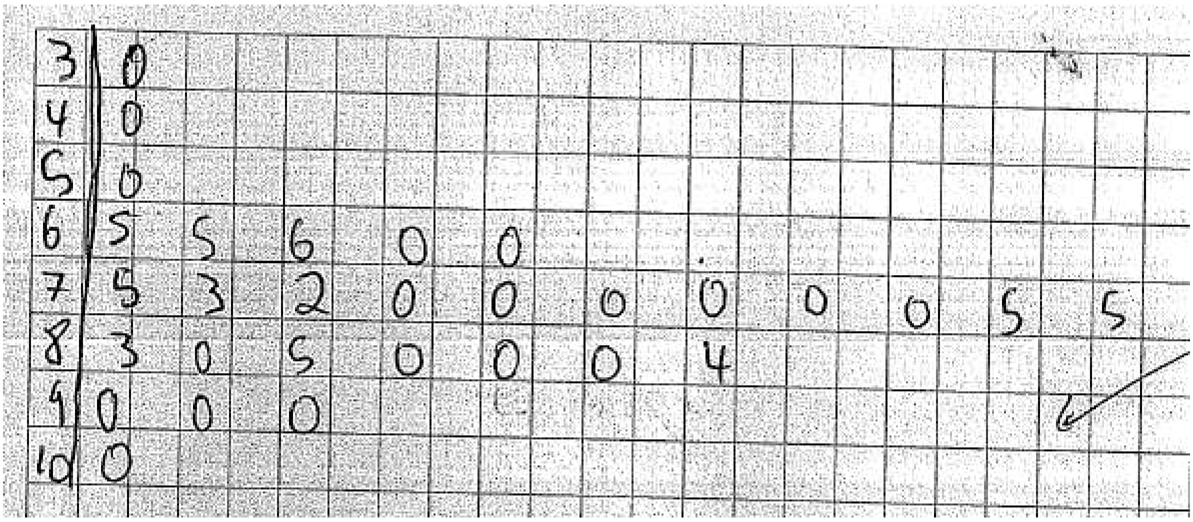
page | running

Question

The lengths of the ring fingers of 30 Irish students chosen randomly from amongst those who completed the *censusatschool* phase 9 questionnaire are displayed below. The measurements are in cm.

7.5	8	7	6	7.5
8.3	6.5	8	5	9
7.3	8.5	7	7	9
7.2	6.5	7	10	9
3	4	6.6	6	8
7	8	7	7.5	8.4

(a) Use the data to investigate whether ring finger lengths are normally distributed. Explain your answer.



(b) Sharon measured the length of her ring finger and found it to be 11.3cm. Her boyfriend says her finger length is most unusual; Sharon disagrees. By calculating the mean and standard deviation of the distribution above, present evidence to support either Sharon's argument, or that of her boyfriend.

Mean = 7.28

Standard deviation = 1.46

Sharon's ring size is Most unusual as it lies between the second and third standard deviation.

Mathematics (Project Maths – Phase 2)

Pre-Leaving Certificate – Ordinary Level Paper 2
February 2011