Parabola Volume 54, Issue 3 (2018)

Prime Numbers Generated From Highly Composite Numbers

Benny Lim[1]

1 Introduction

It should first be advised that the following piece of research is very much of a recreational and casual nature, conducted in fragments of the author's free time. As such, some (or all) convention may have been forgone.

This paper considers natural numbers of the form $H_n - 1$ and $H_n + 1$ where H_n are *highly composite numbers* (see Section 2 below). We investigate how many of those numbers are prime, and we then draw parallels between this set of prime numbers and others such as the Mersenne primes and the primorial primes. We then discuss prime factorisation of highly composite numbers, before presenting a number of guesses and conjectures, as well as some possible significances of this area of mathematics.

2 Highly Composite Numbers

A highly composite number (HCN) is a positive integer with more factors than any smaller positive integer. Table 2 shows the first five HCNs and how to determine them.

The sequence of HCNs, marked by asterisks (*), is thus $1, 2, 4, 6, 12, \ldots$. From this point, the nth number in this sequence will be denoted by H_n. For instance, $H_5 = 12$. A longer list is given in the Online Encyclopedia of Integer Sequences (A002182) [9].

HCNs were originally studied by the famous mathematician Srinivasa Ramanujan in a 1915 article [7]; see also the later full version [8]. Ramanjun proved, among other results, a lower bound for the number of HCNs at most equal to x, and another famous mathematician, Paul Erdős, later improved these results, in 1944 [2].

3 Prime Neighbours of Highly Composite Numbers

3.1 Primality Testing of Neighbours

With "highly composite primes" sounding ridiculously counter-intuitive, primes of the form $H_n - 1$ and $H_n + 1$ will instead be called *H-primes* for all purposes in this paper.

[1]Benny Lim studied Physics at the National University of Singapore and now works in the banking and finance industry. His passion for math has not waned through the years and he finds prime numbers most interesting. This is the first recreational math paper Benny has written, and he hopes to write more. He may be contacted at bennyljq@gmail.com .

Integer	Number of factors
1*	**1**
2*	**2**
3	2
4*	**3**
5	2
6*	**4**
7	2
8	4
9	3
10	4
11	2
12*	**6**

Table 1: Natural numbers, their number of factors, and the highly composite numbers

We tested the primality of all neighbours $H_n \pm 1$ up to H_{1000} by means of large-prime-number checkers available online [1, 5]. The results for $H_1, \ldots, H_{20}$ are shown in Table 2, where an asterisk (*) next to a number indicates that it is prime. More condensed trends up to H_{1000} will be shown in Graphs 3–5 later in this paper.

3.2 Squared Primes

In addition to the large proportion of neighbours being H-primes from the first 20 HCNs, many of the numbers of the form $H_n + 1$ were squared primes. The largest squared prime found in this investigation (and also the only one not in Table 2) was

$$H_{50} + 1 = 17297281 = 4159^2\,.$$

3.3 Relation to Twin Primes

When both neighbours of a HCN are primes, they form a pair of *twin primes*, namely a pair of primes whose distance is just 2. These sort of pairs are famous, due to the famously unresolved *Twin Prime Conjecture*, posed in 1849 by a short-lived Alphonse de Polignac, stating that there are infinitely many such twin prime pairs. Actually, de Polignac posed the more general conjecture that there are infinitely pairs of primes p and $p + 2k$ for each integer k. Some of the best mathematicians in the world are presently making headlines by their progress on this and related conjectures, including Yitang Zhang who in 2014 provided the first major breakthrough in this area [17], by proving that the conjecture is true for at least some integer $k \leq 70000000$. Terence Tao and others have since improved this result considerably but the Twin Prime Conjecture ($k = 1$) still seems tantalisingly far away from any resolution.

n	H_n	$H_n - 1$	$H_n + 1$
1	1	0	2*
2	2	1	3*
3	4	3*	5*
4	6	5*	7*
5	12	11*	13*
6	24	23*	25 = 5^2
7	36	35	37*
8	48	47*	49 = 7^2
9	60	59*	61*
10	120	119	121 = 11^2
11	180	179*	181*
12	240	239*	241*
13	360	359*	361 = 19^2
14	720	719*	721
15	840	839*	841 = 29^2
16	1260	1259*	1261
17	1680	1679	1681 = 41^2
18	2520	2419	2421*
19	5040	5039*	5041 = 71^2
20	7560	7559*	7561*

Table 2: The first 20 HCNs and the primality of their neighbours

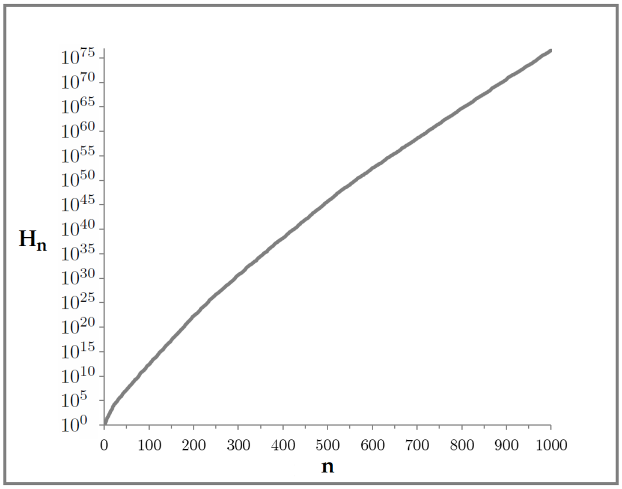
From this context, our investigated HCNs look very interesting: from the first 20 HCNs alone, 25 individual primes and 7 pairs of twin primes can be identified; and from the first 1000 HCNs, there are 210 individual primes and 17 pairs of twin primes.

3.4 Magnitude and Growth of Highly Composite Numbers

To aid in better comprehension of the magnitude of the sequence leading up to H_{1000}, some larger HCNs are shown below in their raw decimal form along with their order of magnitude.

$$
\begin{aligned}
H_{100} &= 2248776129600 \ (\approx 10^{12}) \\
H_{500} &= 5583207648395375171335867419539809268828544000 \ (\approx 10^{45}) \\
H_{1000} &= 338263968832747001135370199977534869763 \ldots \\
&\qquad \text{(continued)} \ldots 7534991987560647925264186336455072000 \ (\approx 10^{76})
\end{aligned}
$$

The HCNs seem to grow exponentially, with an average of 13 HCNs within one order of magnitude (power of 10) for the first 1000 HCNs. Graph 3 shows a plot of H_n against n with H_n on a logarithmic scale.



Graph 3: Growth of HCNs up to H_{1000}

The best fit of this graph gave the equation

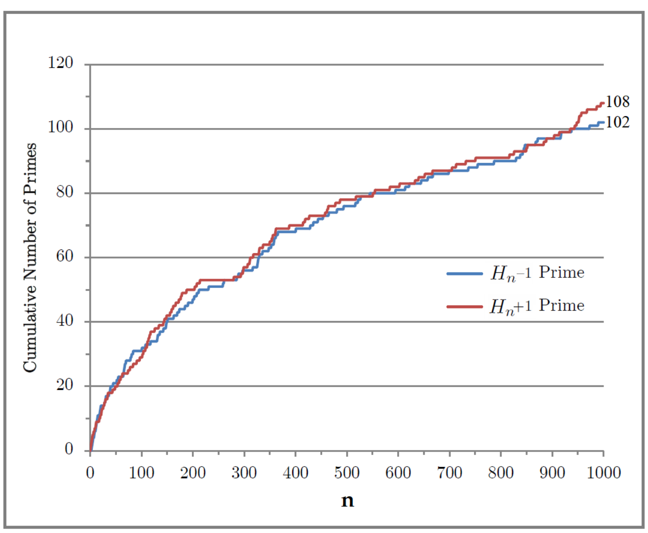
$$H_n = 2 \times 107e^{0.167n} \,,$$

with a least squares coefficient of determination equal to $R^2 = 0.9908$.

3.5 Frequency of Prime Neighbours

H-primes are consistently generated even at large magnitudes (up to 10^{75}), as evident from Graph 3 above and Graph 4 below. The largest prime identified in this study is

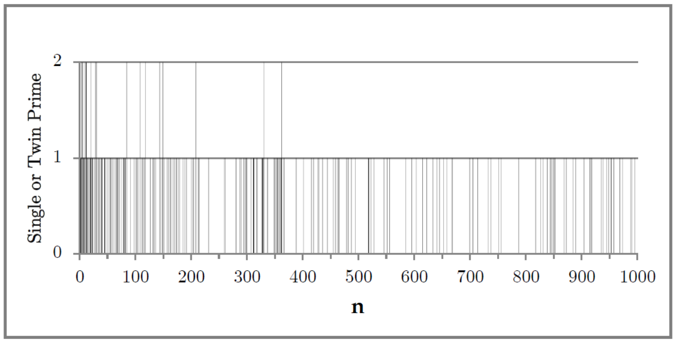
$$\begin{aligned} H_{995} + 1 \quad = \quad & 169131984416373500567685099988767434881 8 \ldots \\ & \text{(continued)}767495993780323962632093168227536000 1 \; (\approx 10^{78}) \,. \end{aligned}$$



Graph 4: Cumulative Frequencies of H-primes up to H_{1000}

3.6 Density of Prime Neighbours

An alternative way to visualise this frequency is through the density of H-primes generated, as shown in Graph 5.



Graph 5: Prime Density of Neighbours up to H_{1000}

The vertical axis indicates if a singular prime is yielded from a particular HCN, or a pair of twin primes. A gradual decline in the density of primes generated is clearly visible. The largest pair of twin primes within the first 1000 HCNs can be seen between H_{350} and H_{400}, specifically

$$H_{362} - 1 = 241271469053348685089061371928479999 \qquad \text{and}$$
$$H_{362} + 1 = 241271469053348685089061371928480001\,.$$

3.7 High Probability of Prime Neighbours

There seems to be a high probability that neighbours of HCNs are primes. This probability will now be compared to that of randomly selected numbers. Specifically, this will be investigated for the range between $H_{801} \approx 10^{64}$ and $H_{1000} \approx 10^{76}$.

According to the Prime Number Theorem[2], the number of primes less than or equal to any real number x is increasingly close to $\frac{x}{\ln x}$ as x grows large. Using this theorem, we can calculate that the approximate percentage of numbers that are primes in the interval between H_{801} and H_{1000} is

$$\frac{1}{\ln(10^{76})} = 0.57144\%\,.$$

From the 200 HCNs in this interval and their 400 neighbours, a total of 28 H-primes were found. If 400 numbers in this interval were picked at random, then there is a lower than 1% chance that 7 or more of these numbers were prime. In other words, the number of primes identified from the 400 HCN neighbours is statistically significant. Indeed, the probability of yielding 28 or more prime numbers from a random selection of 400 numbers in the interval of 10^{64} and 10^{76} is

$$0.00000000000000000018\%\,.$$

[2]For an excellent introduction to this and related theorems, see the article "From binomial coefficients to primes – Chebyshev revisited" by Liangyi Zhao, which also appears in this issue of *Parabola*.

4 Other Sets of Prime Numbers

In this section, comparisons will be made between primes generated from HCNs and primes of other forms, namely Mersenne primes and primorial primes.

4.1 Mersenne Primes

A *Mersenne prime* is a prime number that is one less than a power of two; equivalently, a Mersenne Prime is of the form $M_n = 2_n - 1$; see OEIS A000668 [10]. It is proven that for M_n to be a prime, n must also be a prime.

Mersenne primes are significant in mathematics due to their predictable structure and hence ease of developing efficient algorithms for finding large primes. At the time of this publication, the eight largest known prime numbers are Mersenne primes.

However, even though Mersenne primes could be the most well-known and most sought-after primes, it is evident that their occurrences are few and far between. A simple look at the ten smallest Mersenne primes in Table 6 will indicate this.

Serial Number	n	$M_n = 2_n - 1$
1	2	3
2	3	7
3	5	31
4	7	127
5	13	8191
6	17	131071
7	19	524287
8	31	2147483647
9	61	2305843009213693951
10	89	618970019642690137449562111

Table 6: List of the First 10 Mersenne Primes

As the gaps between successive values of n grow, M_n grows exponentially by virtue of its definition. There are at this time 51 known Mersenne primes, the largest being $2^{82,589,933} - 1$, which is also the largest prime ever known, and was found just over a week ago [4]. It is an unimaginably large number with more than 23 million digits.

This demonstrates the rarity of Mersenne primes, as out of the very many primes known that could be substituted as n in the expression $M_n = 2_n - 1$, only a staggeringly few primes have been proven to produce Mersenne primes.

From the first 1000 prime numbers, only 20 are Mersenne primes, with the largest being $2^{4423} - 1$. This "success rate" of finding Mersenne primes by exhaustion pales in comparison to the 210 primes found by testing the neighbours of the first 1000 highly composite numbers.

Even from a small study, the extremes in the comparative densities of H-primes and Mersenne primes could prove useful in furthering our understanding and/or achievements in Number Theory, especially in the search for large primes, finding a non-Mersenne record-breaking prime, and so on.

4.2 Primorial Primes

Primorial primes are a relatively unknown set of primes. They are of the form $p_n\# - 1$ (OEIS A057704) [11] and $p_n\# + 1$ (OEIS A014545) [12] where p_n is the n^{th} prime, and $p_n\#$ is the product of the first n primes.

To demonstrate, consider $n = 3$: here, $p_3 = 5$ and $p_3\# = 2 \times 3 \times 5 = 30$, and $p_3\# - 1 = 29$ and $p_3\# + 1 = 31$ happen to both be (primorial) primes.

This set of primes bears a similarity to primes yielded from highly composite numbers, as they both rely on the fact (or suspicion) that neighbours of numbers comprising singular or multiple primorials are far more likely to be prime than other numbers. This similarity in the prime factorisation of primorials and HCNs will be further discussed in the next section.

As with Mersenne primes, the frequency and growth of primorial primes are unimpressive compared to those generated from highly composite numbers. The first few primorial primes in sequence are

$$\begin{gathered}2, 3, 5, 7, 29, 31, 211, 2309, 2311, 30029, 200560490131,\\ 304250263527209,\ 23768741896345550770650537601358309, \dots .\end{gathered}$$

Note the extremely fast growth rate and resulting low density.

5 Prime Factorisation of Highly Composite Numbers

In general, the prime factorisation of a highly composite number is a product of primorials. For example, the 30th HCN is factorised as

$$\begin{aligned}110880 &= 2^5 \times 3^2 \times 5 \times 7 \times 11\\ &= (2)^3 \times (2 \times 3) \times (2 \times 3 \times 5 \times 7 \times 11)\\ &= p_1\#^3 \times p_2\# \times p_5\# .\end{aligned}$$

The factorisations of larger HCNs often involve one large primorial and several significantly smaller ones. The primorial factorisation of the 15000th HCN is

$$H_{15000} = p_1\#^5 \times p_2\#^3 \times p_3\#^2 \times p_5\# \times p_8\# \times p_{19}\# \times p_{230}\#.$$

Understanding how primorials are involved in the prime factorisation of highly composite numbers exposes some predictability in their structure. This predictability and (relative) simplicity is reminiscent to that of the Mersenne primes, which could prove helpful in furthering research into these primes.

6 Guesses and Conjectures

It should be obvious by this point that this paper lacks the mathematical rigour in its approach that comes with properly written and peer reviewed research papers. So, as per the theme of everything preceding this section, several hand-waving guesses (conjectures if this were a professional research paper) regarding the properties of highly composite numbers and their neighbourly primes are made.

Guess 0. *There are infinitely many highly composite numbers.*

This Guess is labelled the 0^{th} because it is probably true and easily proven; just multiply more prime factors to the largest existing HCN, and a number with more factors will be generated.

Guess 1. *There are infinitely many prime neighbours $H_n \pm 1$.*

Guess 2. *There are infinitely many twin primes of the form $H_n - 1$ and $H_n + 1$ for the same n.*

This sharpens the Twin Prime Conjecture.

Guess 3. *The number $\pi_H(n)$ of H-primes $H_k \pm 1$ for $k \leq n$ is of the form $\pi_H(n) = \alpha n^{\beta}$, where α and β are constants.*

This guess was inspired by the best fit for Graph 4 which gave the power function

$$\pi_H(n) = 2.3149n^{0.5595}\,.$$

Guess 4. *Neither of the forms $H_n - 1$ or $H_n + 1$ is more prevalently prime.*

This Guess is inspired by Graph 4. It appears that the number of primes among $H_n - 1$ and among $H_n + 1$ seem to converge for large n, probably in the form stated in Guess 3. A total of 102 and 108 primes were generated from $H_n - 1$ and $H_n + 1$, respectively, for $1 \leq n \leq 1000$; these are fairly similar numbers.

Guess 5. *There are infinitely squared primes among the neighbours $H_n + 1$.*

7 Significance

The following "significances" of this research arise exclusively from my optimism and speculation, so here is to hoping that most of them are justified and warranted. Generating discussion on the validity of these significances (or lack thereof) is also one of the purposes of this paper.

7.1 Density

The significantly higher frequency of primes from HCNs than that of primes from Mersenne primes, as seen from this relatively small study, is exciting, since this could pave the way for more comprehensive studies which could produce primes larger than the largest Mersenne primes with higher consistency. There is hopeful potential for this new set of primes for practical applications such as digital cryptography and for advancing Number Theory.

7.2 Twin Primes

This small section could appear overambitious and even devalue the countless man-hours of research into the lower bound of prime gaps. However, there is a very real possibility of an avenue into this well-documented niche (see Guess 2). If enough interest could be generated for serious research into Guess 2, then we might advance towards cracking the long-standing Twin Prime Conjecture.

7.3 Largest Prime

Over the period 1992 – 2018, there have been 18 record-holding primes in terms of magnitude, and all of them were Mersenne primes. Thirteen of the latest were discovered by the Great Internet Mersenne Prime Search (GIMPS) [4]. The computational ease of prime-checking for Mersenne numbers was mentioned briefly above. There are still many challenges to overcome if we wish to investigate large primes generated from highly composite numbers with similar computational simplicity; therefore, as with the previous point, we may not be far from a record-breaking non-Mersenne prime, should enough interest be generated for research into this subject.

7.4 Introduction of a New Set of Prime Numbers

For documentation purposes, this new set (or sets: $H_n - 1$ and $H_n + 1$) of prime numbers resulting directly from highly composite numbers should be recognised, named, and categorised. This would perhaps grant legitimacy for those interested in delving further into the properties of these primes.

8 Further Challenges

There remain several obvious avenues that may be taken to further this research, They remain unexplored within the context of this paper since I have concluded that it will be too arduous at this point to go it alone. My intention is instead to lay out each of these avenues and their complementary challenges, such that they may be undertaken by the occasional enthusiastic (and more technically capable) reader of this article.

8.1 Generating Large Highly Composite Numbers

There is currently no easy way of generating the HCNs in sequence, though a list of the first 779674 HCNs was computed by the German mathematician Achim Flammenkamp [3]. As described above, we do however know some of the basic structure of HCNs in terms of prime factorisation. Given this fact, I propose that generating large numbers which are probable-HCNs is not difficult. There are just some parameters to work out: the ratios between large and small primorials perhaps.

8.2 Experimenting with Superior Highly Composite Numbers

There is however an easy way of generating *superior highly composite numbers* [15] in sequence. In particular, let

$$e_p(x) = \left\lfloor \frac{1}{\sqrt[x]{p} - 1} \right\rfloor$$

for any prime number p and positive real x. Then

$$s(x) = \prod_{p \text{ prime}} p^{e_p(x)}$$

is a superior highly composite number.

These numbers form a subset of HCNs and grow much more quickly in size than HCNs. It is unclear as of now if the neighbours of superior highly composite numbers possess the same probable-prime property as that of the HCNs. Recalling 8.1, perhaps a similar but more lenient floor function could generate a bigger set of probable-HCNs (containing all the HCNs), as compared to this stricter floor function which only generates a subset of HCNs.

8.3 Improving Primality Testing of Neighbours

The Mersenne primes are verified using the Lucas-Lehmer primality test [14] which is the fastest and most computationally efficient known test for Mersenne primes (though not for other primes in general). A big challenge is to find a test that could come close to the efficiency of Lucas-Lehmer for Mersenne primes. Thus far, candidates that could be used to tackle primes of the forms of HCN neighbours are the Pocklington-Lehmer primality test [16] for H_n+1, and a suitable unnamed $n+1$ test from The Prime Pages [13] for $H_n - 1$.

9 Conclusion and Acknowledgements

This paper was written in the hope of highlighting to the mathematical community something that has not yet received much attention but which has the potential to be vastly interesting and possibly important. Ditching convention, I would like to acknowledge the following sources, without which I could not have written the article:

- Brady Haran's videos on the YouTube channel *Numberphile* [6] which informed me about highly composite numbers in the first place;
- free-to-use online large-prime-number checkers, namely the Alpertron [1] and Number World [5]; and most importantly,
- the immeasurable editorial support rendered by Dr Thomas Britz, without whom this paper would never have been worthy of being published.

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