ON THE IRRATIONALITY OF CERTAIN AHMES SERIES

By

P. ERDÖS AND E. G. STRAUS*

[*Received January 22, 1964*]

By an *Ahmes series* we mean a series of reciprocals of positive integers $\Sigma 1/n_k$. In this note we show that the famous series

(1) $$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\ldots+\frac{1}{n_k}+\ldots\ldots$$

with $n_{k+1}=N_k+1=n_k^2-n_k+1$, where N_k denotes the least common multiple (in this case the product) of $n_1, \ldots, n_k$; is typical for Ahmes series with rapidly increasing denominators which represent rational numbers.

Theorem 1. Let $\{n_k\}$ be an increasing sequence of positive integers so that

(*i*) $\limsup n_k^2/n_{k+1}\leqq 1$,

(*ii*) $\{N_k/n_{k+1}\}$ is bounded ;

then the series $\Sigma 1/n_k$ is rational if and only if $n_{k+1}=n_k^2-n_k+1$ for all $k\geqq k_0$ in which case

(2) $$\sum\frac{1}{n_k}=\frac{1}{n_1}+\ldots+\frac{1}{n_{k_0-1}}+\frac{1}{n_{k_0}-1}.$$

Proof. Assume $\Sigma 1/n_k=a/b$, where a and b are integers. Write $bN_k=c_k n_{k+1}-d_k$ with c_k, d_k integers and $0\leqq d_k<n_{k+1}$. Then c_k is positive and bounded by condition (*ii*). Thus, modulo 1,

(3) $$\begin{aligned}0&\equiv(c_k n_{k+1}-d_k)\left(\frac{1}{n_{k+1}}+\frac{1}{n_{k+2}}+\ldots\right)\\&\equiv-\frac{d_k}{n_{k+1}}+\frac{c_k n_{k+1}-d_k}{n_{k+2}}+O\left(\frac{n_{k+1}}{n_{k+3}}\right).\end{aligned}$$

* The second author was supported in part by a grant from the National Science Foundation. This work was done while the authors were participants at a Number Theory Conference at the University of Colorado. The authors are grateful for the opportunity for collaboration given them by this conference.

Hence

(4) $$d_k \equiv c_k \frac{n_{k+1}^2}{n_{k+2}} - d_k \frac{n_{k+1}}{n_{k+2}} + o(1) \pmod{n_{k+1}}$$

But

$$O \leqq c_k \frac{n_{k+1}^2}{n_{k+2}} - \frac{n_{k+1}^2}{n_{k+2}} \leqslant c_k \frac{n_{k+1}^2}{n_{k+2}} - d_k \frac{n_{k+1}}{n_{k+2}} + o(1) \leqq c_k + o(1),$$

so that (4) yields

(5) $$d_k \leqq c_k \text{ for all sufficiently large } k.$$

Now

$$c_{k+1} n_{k+2} - d_{k+1} = bN_{k+1} \leqq n_{k+1} bN_k = c_k n_{k+1}^2 - d_k n_{k+1}$$

and therefore

(6) $$c_{k+1} \leqq c_k \frac{n_{k+1}^2}{n_{k+2}} + o(1) \leqq c_k + o(1)$$

so that $c_{k+1} \leqq c_k$ for all sufficiently large k, which means $c_k = c$=constant for all sufficiently large k. According to (6) this is possible only if

(7) $$\lim n_k^2 / n_{k+1} = 1.$$

Then (4) yields $d_k = c$ for all $k \geqslant k_1$ and (3) becomes

(8) $$\frac{1}{n_{k+1}} = \frac{n_{k+1} - 1}{n_{k+2}} + (n_{k+1} - 1)\left(\frac{1}{n_{k+3}} + \ldots\right),\ k \geqslant k_1;$$

or

$$n_{k+2} = n_{k+1}^2 - n_{k+1} + \frac{n_{k+1}^2}{n_{k+2}} \cdot \frac{n_{k+2}^2}{n_{k+3}} + o(1)$$

$$= n_{k+1}^2 - n_{k+1} + 1 + o(1)$$

so that

(9) $$n_{k+2} = n_{k+1}^2 - n_{k+1} + 1$$

for all sufficiently large k.

The last statement of the theorem is now obvious since

$$\frac{1}{n_{k_0} - 1} = \frac{1}{n_{k_0}} + \frac{1}{n_{k_0+1}} + \cdots + \frac{1}{n_{k-1}}$$

for all $k > k_0$.

We now wish to examine to what extent the conditions (*i*) and (*ii*) of the theorem are necessary. It is clear that the mere finiteness of $\limsup n_k^2/n_{k+1}$ does not suffice. As a trivial example consider the series $\Sigma 1/(an_k)$ where a is a positive integer and $\Sigma 1/n_k$ is the series in (1). Here obviously $\lim n_k^2/n_{k+1}=a$ while condition (*ii*) remains valid. A somewhat less trivial example with $\limsup n_k^2/n_{k+1}=a$, an integer, and $\liminf n_k^2/n_{k+1}=1$ is given by the series $1/a=\Sigma 1/n_k$ where $n_1=a+1$ and n_{2k+1} is the least integer so that $1/n_1+\ldots+1/n_{2k+1}<1/a$ while n_{2k} is the least integer so that $1/n_1+\ldots+1/n_{2k-1}+1/(n_{2k}-a+1)<1/a$. Then $n_{2k+1}\equiv 1 \pmod a$ and $n_{2k}\equiv 0 \pmod a$ with $n_{2k}=n_{2k-1}^2-n_{2k-1}+a$ and $n_{2k+1}=(n_{2k}^2/a)-n_{2k}+1$ so that $\lim n_{2k-1}^2/n_{2k}=1$ while $\lim n_{2k}^2/n_{2k+1}=a$ and $N_k/n_{k+1}\leqq a^{-[k/2]+1}n_1\ldots n_k/n_{k+1}$ is bounded. It would be easy to modify the rule of construction so that $\{n_k\}$ satisfies no algebraic recursion relation. However, if we strengthen condition (*ii*) somewhat we can obtain information about the behaviour of n_k^2/n_{k+1}.

Theorem 2. Let $\{n_k\}$ satisfy

(*i*) $\{n_k^2/n_{k+1}\}$ is bounded ;

(*ii*) $\{N_k^*/n_{k+1}\}$ is bounded, $N_k^*=n_1n_2\ldots n_k$.

If $\Sigma 1/n_k$ is rational then $\{n_k^2/n_{k+1}\}$ has only a finite number of limiting values all of which are rational and $\liminf n_k^2/n_{k+1}\leqq 1$.

Proof. We proceed as in the proof of Theorem 1 replacing N_k by N_k^*. The proof of the boundedness of d_k from (4) remains valid, while (6) becomes

$$c_{k+1}=c_k\frac{n_{k+1}^2}{n_{k+2}}+o(1) \tag{6'}$$

so that all limiting values of $\{n_k^2/n_{k+1}\}$ are rational numbers whose numerators and denominators do not exceed the bound of $\{bN_k^*/n_{k+1}\}$. If $\liminf n_k^2/n_{k+1}=1+\delta>1$ then

$$\frac{N_k^*}{n_{k+1}} = \frac{1}{n_1} \cdot \frac{n_1^2}{n_2} \cdot \frac{n_2^2}{n_3} \cdots \frac{n_k^2}{n_{k+1}} > C(1+\delta)^k$$

where C is a positive constant, contrary to condition (ii').

As to condition (ii), it may well be that Theorem 1 remains valid without it. Its main use in the proof lies in the derivation that c_k is constant for large k from the inequality (6). This derivation can be made under weaker hypotheses.

Theorem 3. Let $\{n_k\}$ satisfy (i) and

$$(ii'') \qquad \limsup \frac{N_k}{n_{k+1}} \left(\frac{n_{k+1}^2}{n_{k+2}} - 1 \right) \leqq 0,$$

then $\Sigma 1/n_k$ is rational if and only if $n_{k+1} = n_k^2 - n_k + 1$ for all $k \geqq k_0$.

Note that (i) and (ii) imply (ii'') but (i) and (ii'') do not imply (ii).

Proof. Condition (i) implies that

$$\frac{N_k}{n_{k+1}} \leqq \frac{N_k^*}{n_{k+1}} = \frac{1}{n_1} \cdot \frac{n_1^2}{n_2} \cdot \frac{n_2^2}{n_3} \cdots \frac{n_k^2}{n_{k+1}} < C(1+\delta)^k$$

for any $\delta > 0$. Thus $c_k = o(e^{\delta k}) = o(n_k)$ and (4) remains valid. Now, by (ii''), we have

$$(10) \qquad c_k \frac{n_{k+1}^2}{n_{k+2}} = c_k + c_k \left(\frac{n_{k+1}^2}{n_{k+2}} - 1 \right)$$

$$= c_k + O\left(\frac{N_k}{n_{k+1}} \left(\frac{n_{k+1}^2}{n_{k+2}} - 1 \right) \right) \leqq c_k + o(1)$$

so that (5) remains valid. As before we then get (6) from (10). The rest of the argument is unchanged.

Example 1. The series $\Sigma 1/n_k$, where

$$n_k = a^{2^k} + b_k, \ a, b_k \text{ integers}, \ a > 1$$

so that $\Sigma |b_k| a^{-2^k} < \infty$, is irrational. [1]

Proof. We have

$$\frac{n_k^2}{n_{k+1}} (1 + 2b_k a^{-2^k} + b_k^2 a^{-2^{k+1}}) / (1 + b_{k+1} a^{-2^{k+1}}) \to 1$$

so that condition (*i*) is satisfied. Also

$$N_k^*/n_{k+1}=a^{-1}\prod_{l=1}^{k}(1+b_l a^{-2^l})/(1+b_{k+1}a^{-2^{k+1}})$$

which is bounded so that condition (*ii*) is satisfied. Thus, according to Theorem 1, if $\Sigma 1/n_k$ were rational we would have $n_{k+1}=n_k^2-n_k+1$ for all sufficiently large k, or

$$b_{k+1}=2a^{2^k}b_k+b_k^2-a^{2^k}-b_k+1$$

so that $b_k\neq 0$ implies for sufficiently large k

$$|b_{k+1}|>a^{2^k}. \tag{11}$$

Since $b_k=0$ implies $b_{k+1}=-a^{2^k}+1\neq 0$, we may assume $b_k\neq 0$ and applying (11) repeatedly get

$$|b_{k+l}|>a^{2^{k+l}-2^k}$$

so that

$$|b_{k+l}|a^{-2^{k+l}}>a^{-2^k}$$

does not tend to 0 as $l\to\infty$, contrary to hypothesis.

Example 2. The Ahmes series $\Sigma 1/n_k$ where $n_{k+1}=n_k^2+an_k+b$ is rational if and only if $a=-1$ and $b=1$.

Example 3. If $\{n_k\}$ satisfies (*i*) and there is a prime p so that $n_k n_{k+1}\cdots n_{k+l}\equiv 0 \pmod p$ for a fixed l and all k then $\Sigma 1/n_k$ is irrational.

Proof. We have to verify that N_k/n_{k+1} is bounded. But

$$N_k\leq p^{-[k/l]+1}N_k^*=p^{-[k/l]+1}\cdot\frac{1}{n_1}\cdot\frac{n_1^2}{n_2}\cdots\frac{n_k^2}{n_{k+1}}n_{k+1}$$

$$<Cp^{-[k/l]+1}(1+\epsilon)^k n_{k+1}$$

for any $\epsilon>0$. Choosing $1+\epsilon<p^{1/l}$ we get $N_k/n_{k+1}<p^2C$.

REFERENCE

1. J.W. Golomb, *A Special Case*, On the sum of the reciprocals of the Fermat numbers and related irrationalities, Can. Journal of Math. 15(1963), 475-478.

University of British Columbia and
University of California, Los Angeles