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ENTANGLEMENT MEASURES

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Theory of quantifying entanglement is reviewed in a unified framework. The subject is divided with respect to finite and asymptotic regime, as well as abstract and operational approach. Important measures are presented within four classes according to different methods of construction. The relations between postulates for asymptotic and finite regime are clarified. Many results are formulated based on general properties of the involved functions and classes of states and operations, without referring to entanglement anymore. It is argued that while in finite regime only monotonicity is relevant postulate, for asymptotic regime, one needs, in general, monotonicity, a kind of extensivity and continuity.

Keywords: Entanglement

1. Introduction

Entanglement [1,2] is the corner-stone of the quantum information theory (QIT). Consequently, quantification of entanglement is necessary to understand and develop the theory. The main problem is that we do not understand fully what entanglement is. Rather, we know its mathematical definition as well as its manifestations like violation of Bell's inequalities [3], teleportation [4] or quantum computation [5].

There are, in principle, two approaches to quantifying entanglement: "operational" and "abstract" one, both present in the pioneering paper on entanglement measures [6]. In the first one, entanglement is related to the operational tasks: the system is more entangled if it allows for better performance of some task (impossible without entanglement). One such task is teleportation. By use of a single pair of two qubits in state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and classical communication, a qubit can be transmitted. This is impossible by use of classical communication itself. A mixed state cannot faithfully teleport. However, if Alice and Bob share many pairs of particles each, in state ϱ, then by use of local operations and classical communication, they can obtain a smaller number of pairs in state ψ_+, and perform teleportation. The above procedure is called *distillation* [7]. The number of obtained output pairs per input one is entanglement of distillation. It quantifies entanglement with respect to rate of teleportation. More generally, operational measures are optimal rates of conversion of one form of entanglement into other one.

In abstract approach one says that a state function can be used to quantify entanglement, if it satisfies some natural properties. The fundamental property is [6] that entanglement of two systems, whatever it actually is, cannot be increased without quantum interaction (direct or indirect) between them. If the systems are spatially separated, then entanglement between the quantum systems cannot increase, if only classical communication between Alice and Bob sharing the systems is allowed. The latter sentence expresses the fundamental postulate of entanglement theory - *monotonicity* under local operations and classical communication (LOCC). Development of abstract entanglement theory was

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initiated in Refs. [8,9,10]. In particular, the idea of entanglement measures based on distance from the set of disentangled states of Refs. [9,10] had great impact on entanglement theory. The minimal axioms for entanglement were finally formulated in [11].

Another division of entanglement theory is into two regimes: *finite* and *asymptotic* one. According to the first one (see e.g. [12] and references therein), one would like to quantify entanglement of *single* system (of course a *compound* one). In the second one [13,14], we are interested in entanglement of a *sequence* of systems, or, more generally, *quantum source*†. In the latter regime one works with "asymptotic accuracy" paradigm (as in classical communication theory). Namely, if, for two sequences of systems, their total states converge to each other, one identifies the sequences. This implies that the very axiom of monotonicity will not be sufficient, and some *continuity* will enter the scene [11]. Asymptotic regime is called *thermodynamics of entanglement* [8,10,15]. Its axiomatic treatment was developed in [16].

So far I have referred only to the literature concerning general approach to entanglement theory. However, the latter would be impossible without many concrete results, I will refer to further in the paper.

The purpose of this paper is to gather "phenomenology" of quantifying entanglement around the two divisions: finite-asymptotic, operational-abstract. Subsequently, a connection between operational and abstract measures, both in finite and asymptotic regime will be presented. It allows to evaluate entanglement conversion rates in different processes, and, subsequently, to investigate *irreversibility* in entanglement processing. I will argue that the connection and other results can be obtained without referring to entanglement at all: one simply deals with some functions monotonic under some classes of operations and with sets closed under those classes. The classes and the sets are usually convex, and closed under tensor product. From this point of view, I will show that much can be done, by using monotonicity in finite regime (as pointed out in [11]) and monotonicity, extensivity and continuity in asymptotic regime [16].

I will be mainly concerned with bipartite systems. However I will often point out that some result is valid for multipartite domain, too. Moreover, only finite-dimensional systems will be considered. For recent attempts to quantify continuous variable entanglement see e.g. [19,20].

The paper is organized as follows. First I present some basic classes of states, and operations (Sect. 2). In Sect. 3 I discuss the possible mathematical formulations of monotonicity postulate. Subsequently (Sect. 4) basic entanglement measures are presented, divided into four classes according to four methods of constructing the measures. We present briefly some results on evaluating measures for concrete states. The main results on pure state entanglement are discussed in Sect. 5. Next, I present asymptotic regime (Sect. 6) and discuss appropriate postulates (Sect. 7). Basic theorems of asymptotic regime are presented in Sect. 8. Also some connections between entanglement measures and information-like quantities are briefly discussed.

2. Basic sets of states and classes of operations

2.1. *States*

From the point of view of entanglement theory one is interested in states of *compound* system. We will be concerned with *bipartite* systems, described by Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. It is natural to divide the set of all such states (density matrices) into the set of entangled states $\mathcal{E}$ and its complement, the set of separable states $\mathcal{S}$. A state is entangled if it is not a mixture of product states, otherwise it is separable [21]. However, the concept of distillation suggests other division: Into distillable and non-distillable states. It turns out [18] that there exist entangled states that are non-distillable (one calls them *bound*

† Only the sources of identically, independently prepared systems have been treated so far.

entangled ones). This gives three classes of states: separable, bound entangled ($\mathcal{BE}$) and distillable. The separable states and bound entangled ones are non-distillable, denote them by $\mathcal{ND} = \mathcal{S} \cup \mathcal{BE}$. Actually, the set $\mathcal{ND}$ has not been yet operationally characterized [18,22,23]. Therefore another set is useful: The set of states with positive partial transposition (call it $\mathcal{PPT}$). To define partial transposition T_B of a given state ϱ acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we will use matrix elements of the state in some product basis: $\varrho_{m\mu,n\nu} = \langle m| \otimes \langle\mu|\, \varrho\, |n\rangle \otimes |\nu\rangle$. Here the kets with Latin (Greek) letters form orthonormal basis in Hilbert space describing first (second) system. Now, the partial transposition of ϱ with respect to the system B is defined as

$$\varrho^{T_B}_{m\mu,n\nu} \equiv \varrho_{m\nu,n\mu} \,. \tag{1}$$

The form of the operator ϱ^{T_B} depends on the choice of basis, but its eigenvalues do not. We will say that a state is PPT if $\varrho^{T_B} \geq 0$ i.e. if all the eigenvalues of ϱ^{T_B} are nonnegative (otherwise we will say that a state is NPT)[‡] As shown by Peres [25], separable states are PPT, but for entangled states some eigenvalues can be negative, thus partial transposition can be used to detect entanglement of mixed states. PPT states cannot be distilled [18], so that $\mathcal{PPT} \subset \mathcal{ND}$. Moreover $\mathcal{PPT}$ contains $\mathcal{S}$ as *proper* subset [26]. Most likely $\mathcal{PPT}$ is not equal to $\mathcal{ND}$: there is strong evidence that there are non-distillable states which are not PPT. [§]

2.2. *Important one-parameter families of states*

Werner states - $U \otimes U$ invariant ones The Werner states [21] for $d \otimes d$ systems (i.e. for systems described by the Hilbert space $C^d \otimes C^d$) are given by

$$\varrho_W(p,d) = p\frac{P_-}{N_-} + (1-p)\frac{P_+}{N_+}, \quad 0 \leq p \leq 1 \tag{2}$$

where $P_{-(+)}$ are projectors onto antisymmetric (symmetric) subspace of the total space; $N_\pm = \frac{d^2 \pm d}{2}$ are dimensions of these subspaces. One finds that the state is PPT if and only if $p \leq 1/2$. Under the same condition it is separable. For $p > \frac{3(d-1)}{2(2d-1)}$ it is distillable, while for $\frac{1}{2} \leq p \leq \frac{3(d-1)}{2(2d-1)}$ it is, most likely, bound entangled [22,23]. The Werner states are the only ones that are invariant under operations of the form $U \otimes U$, i.e. they do not change if Alice and Bob apply locally the same unitary transformation.

Isotropic states - $U \otimes U^*$ invariant ones Define first maximally entangled state of a system $\mathcal{H} \otimes \mathcal{H}$ with $\dim \mathcal{H} = d$ (call it $d \otimes d$ system) by

$$P_+(\mathcal{H}) = |\psi\rangle\langle\psi|, \quad \text{with} \quad \psi = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle. \tag{3}$$

(we will call it shortly singlet, even though it is actually not singlet state). The isotropic state [27] (cf. [24]) acting on $d \otimes d$ system is given by

$$\varrho_{iso}(F,d) = \frac{1-F}{d^2-1}I + \frac{Fd^2-1}{d^2-1}P_+(d), \tag{4}$$

‡ Equally well one could take transposition with respect to the system A.

§ For more details concerning qualitative description of entanglement see the article by P. Horodecki & R. Horodecki, in this issue.

where $0 \leq F \leq 1$. Note that $F = \mathrm{Tr}\big[\varrho(F,d)P_+(d)\big]$. As shown in [27] for $F \leq \frac{1}{d}$ the state is separable, PPT, non-distillable; for $F > \frac{1}{d}$ it is nonseparable, distillable, NPT. The isotropic states are the only ones that are invariant under any transformation $U \otimes U^*$ (star denotes complex conjugation).

2.3. *Classes of operations*

The general problem of quantum operations was considered in [28,29,30]. Essentially new aspects were added quite recently in the context of quantum information processing (cf. [6,11,32,33]). A very elegant, closed description was provided in [34].

All possible physical operations on quantum systems can be divided into

(i) state-to-ensemble operations

(ii) ensemble-to-state operations (mixing)

Here, ensemble is a set of states $\{\varrho_i\}$ with ascribed probabilities $\{p_i\}$ (more generally, ensemble is a probability measure on the set of states). The class (ii) is described by taking convex combination

$$\{p_i, \varrho_i\} \to \varrho_{out} = \sum_i p_i \varrho_i \tag{5}$$

The action of mixing corresponds to erasure of information concerning identity of a member of ensemble. The operations of type (i) we will call simply operations. A special case of operation is *proper operation*. It is state-to-state operation, and is described by trace preserving completely positive (CP) map¶ ‖. In general, the class (i) is described by a family of *suboperations* i.e. trace-decreasing CP maps Λ_i with Kraus operators $V_j^{(i)}$ satisfying

$$\sum_{ij} V_j^{(i)\dagger} V_j^{(i)} = I. \tag{6}$$

The operations thus consist of state transformation plus some classical record. The maps Λ_i describe transformations the state was subjected provided that i was obtained. Given operation, the transformation is of the form

$$\varrho_{in} \to \{p_i, \varrho^i_{out}\} \tag{7}$$

where $p_i = \mathrm{Tr}[\Lambda_i(\varrho_{in})]$ is the probability that the outcome i will occur, while ϱ^i_{out} are outputs of Λ_i, after normalization, i.e. $\varrho^i_{out} = \frac{1}{p_i}\Lambda_i(\varrho_{in})$. The operations can be composed, and tensored with each other [35]. Examples of operations are measurement, or random application of unitary transformations. One can consider *pure* operation, for which suboperations are *extreme* maps, i.e., they are of the form

$$\Lambda_i(\varrho) = V_i \varrho V_i^\dagger. \tag{8}$$

Proper operations are usually strongly impure: no output is read out. Indeed, a proper map has only one suboperation - itself. Hence to be pure it must be of the form (8).

¶A map is CP if it is of the form $\Lambda(\varrho) = \sum_i V_i \varrho V_i^\dagger$. V_i are called Kraus operators of Λ. The condition of preserving trace is equivalent to $\sum_i V_i^\dagger V_i = I$. A CP map is called *trace decreasing* if $\sum_i V_i^\dagger V_i \leq I$.The input and output states can act on different Hilbert space.

‖Sometimes in literature on quantum information proper operations are called deterministic operations, while improper operations – stochastic or probabilistic ones. However in older literature concerning open systems trace preserving CP maps are called stochastic maps.

The only pure proper operations are thus unitary one and isometry (unitary embedding). Example of the latter is adding an ancillary system in some pure state.

It should be emphasized, that only proper operations can be a result of *single run* of experiment. The improper operation and mixing have only statistical meaning, they need many runs. A suboperation cannot be performed itself - in experiment it is always a part of some operation. Thus trying to perform suboperation, one, in general, can succeed only with some probability while the proper operations are performed with certainty. Finally, note that the proper operation can be treated as composition of pure operation with mixing of the resulting ensemble. More generally, any operation can be treated as pure operation followed by partial mixings (the output ensemble is divided into subensembles, and each of them is mixed).

Let us now present basic subclasses of operations (i) in the context of entanglement. Apart from the basic one, i.e. LOCC operations, there are other ones, which sometimes are easier to deal with. The common feature of the classes is that some set that is far from maximally entangled state (it can be $\mathcal{S}$, $\mathcal{PPT}$ or $\mathcal{ND}$) is closed under them. In this way, the possibilities of manipulations of entanglement are restricted, which makes considerations nontrivial.

LOCC operations. This basic class of operations was introduced in [6,7] (cf. [36]). Consider input state of bipartite system, shared by Alice and Bob. Suppose that Alice performs operation on her side of two component system. Such operations, performed exclusively within Alice (or Bob) laboratory we will call *local operations.* Let then Alice communicate the outcome to Bob. Conditional on the outcome, Bob performs his operation, obtains outcome, communicates it to Alice, and so forth. At some stage they stop, left with some ensemble. This complicated action is LOCC operation - the one composed of *local operations* and *classical communication.* To have *proper* LOCC operation (state-to-state transformation) Alice and Bob should forget the outcomes (perform mixing).

It is interesting, that nontrivial proper LOCC operation *cannot* be done without composing state-to-ensemble and ensemble-to-state operations. This is because the operation needs communication between Alice and Bob. Thus they must read outcomes to be able to communicate them and then act conditionally on them, even though they will forget them afterwards. In contrast, any proper operation in *single* laboratory can be done without producing intermediate ensemble: it can be performed coherently, by adding ancilla, applying unitary transformation and removing ancilla (or its part, or part of the initial system). The sets $\mathcal{S}, \mathcal{PPT}, \mathcal{ND}$ are closed under LOCC operations. That is, if the input state belongs to the set, say, $\mathcal{S}$, then so does any of the member of output ensemble.

Separable operations. The class was introduced in [10,35]. An operation is separable, if all its suboperations are of the separable form

$$\Lambda(\varrho) = \sum_k A_k \otimes B_k \varrho A_k^\dagger \otimes B_k^\dagger. \tag{9}$$

Again, all the three sets $\mathcal{S}, \mathcal{PPT}, \mathcal{ND}$ are closed under separable operations.

PPT operations. The LOCC operations have physical rather than mathematical origin. They describe possibilities of Alice and Bob that are in distant laboratories in the situation where sending classical information is cheap while sending quantum one - expensive (quantum states are fragile, while classical ones - robust against disturbance). As mentioned, one can extract the relevant mathematical property of LOCC. Namely, the set of separable states is closed under LOCC class. Consequently, they cannot create singlet from separable states. However, we know that there are larger sets than $\mathcal{S}$ (such as $\mathcal{PPT}, \mathcal{ND}$), that are still closed under LOCC operations. This suggests to consider larger classes of operations. They are allowed to move a state outside $\mathcal{S}$, but one of the considered larger sets should be closed under them.

In this context consider the class of operations (called PPT ones) [35], such that all suboperations Λ_i have the property that the map $T_B \Lambda T_B$ is completely positive. The set $\mathcal{PPT}$ is closed under such operations. Example of such operation is adding a pair in PPT state

$$\varrho_{in} \to \varrho_{out} = \varrho_{in} \otimes \varrho_{PPT} \tag{10}$$

The set is closed under composition and tensor multiplication.

Other possible classes One could consider the set of *all* operations that leave the set $\mathcal{S}$, $\mathcal{PPT}$ or $\mathcal{ND}$ invariant. For example, we have the set of PPT preserving operations that any PPT state convert into PPT one. Such classes are not closed under tensor product. For example, the map interchanging the Alice and Bob systems with each other (i.e. $\Lambda(\varrho) = V\varrho V$ with V being swap operator, $V\psi \otimes \phi = \phi \otimes \psi$) belongs to such class. However, it cannot be tensored with identity. Indeed, such tensoring would correspond to situation, where Alice and Bob have two pairs, and swap is applied to one of them, while nothing is done with the other one. Now, if the initial state was product of two singlets, one singlet at Alice side, the second at Bob's, then after the operation, they will *share* two singlets. Thus entanglement between Alice and Bob was produced out of a product state. Hence swap operation is not a PPT operation. One can show that [37,38] the PPT operations constitute the largest subset of PPT preserving ones, that is closed under tensor multiplication (hence the relation between the sets is similar to the one between positive and completely positive maps).

The considered classes can be still useful, but one must be careful while working with them.

2.4. *Examples of operations: twirlings*

Let Alice apply to the shared bipartite state ϱ a random unitary transformation and send to Bob the information, which transformation was applied. Then let Bob apply the same transformation. Finally let them erase the information, what transformation was applied. The resulting operation

$$\varrho \to \int U \otimes U \, \varrho U^\dagger \otimes U^\dagger \mathrm{d}U \tag{11}$$

is called twirling [6,21]. The resulting state is Werner state, independently of the initial state. There is also $U \otimes U^*$ twirling [27]

$$\varrho \to \int U \otimes U^* \, \varrho U^\dagger \otimes U^{*\dagger} dU \tag{12}$$

The resulting state is always isotropic.

3. Minimal postulates for entanglement measures

Entanglement measures are real positive functions defined usually on the union of all states acting on finite dimensional systems. Below we will list the postulates that certainly should be satisfied by entanglement measures.

3.1. *Monotonicity*

If a state ϱ_{out} was created from ϱ_{in} by use of a proper LOCC operation, then, irrespective of used measure, entanglement of ϱ_{out} cannot exceed that of ϱ_{in}. This basic postulate was announced in Ref. [6]. It says that entanglement is constituted by genuinely quantum correlations, that cannot be increased by use of classical communication.

Consequently, if we confine to proper operations, the postulate for potential entanglement measure E is

(M1) For any proper LOCC operation Λ and any state ϱ

$$E(\Lambda(\varrho)) \leq E(\varrho) \tag{13}$$

One could impose stronger condition, requiring that *average* entanglement should not increase under any LOCC operations [6,13] **.

(M2) Let an LOCC operation produce ensemble $\{p_i, \varrho_i^{out}\}$ out of initial state ϱ_{in}. Then we require

$$\sum_i p_i E(\varrho_i^{out}) \leq E(\varrho_{in}) \tag{14}$$

The strongest condition arises, if, in addition to (M2), we require also non-increasing under mixing, which is natural condition from physical point of view. One finds [11] that the condition can be written in a much simpler form from mathematical point of view.

(M3.a) Let a *local*, *pure* operation produces ensemble $\{p_i, \varrho_i^{out}\}$ out of ϱ_{in}. Then we require

$$\sum_i p_i E(\varrho_i^{out}) \leq E(\varrho_{in}) \tag{15}$$

(M3.b) E is convex

$$E\left(\sum_i p_i \varrho_i\right) \leq \sum_i p_i E(\varrho_i) \tag{16}$$

This latter postulate will be treated as the fundamental one for any entanglement measure in finite regime (we will argue, that in asymptotic regime it is, in general, too strong).

One can strengthen the conditions (M1) and (M2) by considering larger classes of operations, presented in preceding section. Sometimes it is mathematically more feasible to show that these stronger conditions hold. However, if we add non-increasing under mixing, we will not obtain such simple form as (M3). For separable operations, the condition (M2) plus convexity can be rephrased as monotonicity under *pure* operations plus convexity. For PPT operations even this is impossible. This is because PPT operations that are not separable, cannot be pure. It is easy to see that a pure PPT operation must have suboperations of the form $A \otimes B(\cdot)A^\dagger \otimes B^\dagger$, so that it must be a separable operation (the best way to check it is to act by a suboperation on halves of singlets (cf. [31]) as in example of the last paragraph of sect. 2.3).

Remark. Any of the above conditions implies that E is invariant under product unitary transformations $U_A \otimes U_B$. More generally: *Monotonicity implies invariance under reversible operations.* In finite regime product unitary transformations are the only reversible ones. As we will see, in asymptotic domain there are much more reversible operations.

3.2. *Vanishing on separable states*

Of course, for separable states any entanglement measure must vanish:

(S1) For any separable state ϱ we require $E(\varrho) = 0$.

Why didn't we *started* with this obvious condition? Following [11] we note that monotonicity in any form of previous subsection, ensures $E = const$ for all separable states.

** Of course, entanglement of some members of the output can exceed the initial one. E.g. one can produce with some probability two-qubit singlet state out of $\psi = a|00\rangle + b|11\rangle$ with $a, b > 0$.

This is because separable states can be reversibly converted into each other by LOCC operations. Anticipating a bit, let as require that for some separable state $E(\varrho\otimes\varrho)=2E(\varrho)$. Then we obtain $E=0$ for separable state automatically! In finite regime we will not impose additivity, so that we must set the constant equal to zero ourselves.

A natural question is: shouldn't we require $E=0$ if and *only* if ϱ is separable? If there were only one type of entanglement, then any entangled state should contain at least a small amount of it. However there exists many types of entanglement. Thus, if a measure vanishes for some entangled state, it is not a contradiction, but it implies that the state does not contain the type of entanglement quantified by this particular measure. For example, bound entangled states have zero distillable entanglement: all entanglement they contain is the bound one.

4. Important measures of entanglement

In this section we will present important classes of measures. We will discuss mainly their monotonicity. Other properties will be presented in final subsection and in further sections.

4.1. *Operational measures*

Here we will describe two entanglement measures, *entanglement of distillation* E_D and *entanglement cost* E_C [6] (cf. [35,39]). They have direct quantum communication sense. $E_D(\varrho)$ denotes the maximal number of qubits per pair that can be teleported via pairs in state ϱ. That is, E_D is quantum communication *capacity* of the given source of pairs. E_C is the minimal number of qubits per pair that must be sent to create pairs in the state ϱ. The measures are definitely "asymptotic" objects.

Distillable entanglement. To define distillable entanglement E_D [6,35] of the state ϱ acting on $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$ we consider *protocol* $\mathcal{P}$, i.e., a sequence of proper LOCC operations Λ_n, that map the state $\varrho^{\otimes n}$ of n input pairs into a state σ_n acting on the Hilbert space $\mathcal{H}_n^{out}=(C^2)^{\otimes m_n}\otimes(C^2)^{\otimes m_n}$. Now $\mathcal{P}$ is *distillation protocol* if for high n the final state σ_n approaches the state of m_n two qubit singlets (we will drop the index n at m_n)

$$F\equiv\langle\psi_+(C^2)^{\otimes m}|\sigma_n|\psi_+(C^2)^{\otimes m}\rangle\to 1 \tag{17}$$

(i.e., the fidelity F tends to 1). The asymptotic ratio $D_{\mathcal{P}}$ of distillation via protocol $\mathcal{P}$ is given by

$$D_{\mathcal{P}}(\varrho)\equiv\lim_{n\to\infty}\frac{m}{n} \tag{18}$$

The distillable entanglement is defined by maximum of $D_{\mathcal{P}}$ over all distillation protocols

$$E_D(\varrho)=\sup_{\mathcal{P}} D_{\mathcal{P}}. \tag{19}$$

Remarks. (1) Instead of fidelity condition, one could consider some metric $\mathcal{D}$, and impose $\mathcal{D}(\sigma_n,\psi_+(C^2)^{\otimes m})\to 0$. For Bures metric and the one induced by trace norm, one gets equivalent conditions as the fidelity one. (2) One could consider class of PPT operations. The obtained rate is called PPT-distillable entanglement [35].

Entanglement cost. It is defined in a dual way to E_D. Let a state ϱ acts on $\mathcal{H}\otimes\mathcal{H}$. We start from m pairs of two-qubit singlets and a protocol, i.e. a family of proper LOCC operations Λ_m transforming the singlets into some state σ_n acting on $\mathcal{H}_A^{\otimes n}\otimes\mathcal{H}_B^{\otimes n}$. The state σ_n is a state of n pairs, and is to be closer and closer to $\varrho^{\otimes n}$ which we want to create from input pairs. Thus for the family to be protocol we require

$$\mathcal{D}(\sigma_n, \varrho^{\otimes n}) \to 0 \tag{20}$$

where $\mathcal{D}$ is some metric - for definiteness, let it be the one induced by trace norm $\|A\|_1 = \mathrm{Tr}|A|$.

Again, the asymptotic cost $C_{\mathcal{P}}$ of the production of ϱ via protocol $\mathcal{P}$ is given by

$$C_{\mathcal{P}}(\varrho) \equiv \lim_{n\to\infty} \frac{m}{n} \tag{21}$$

The entanglement cost is defined by minimum of $C_{\mathcal{P}}$ over all protocols

$$E_C(\varrho) = \sup_{\mathcal{P}} C_{\mathcal{P}}. \tag{22}$$

Intuitively, E_D (as well as E_C) should not increase under proper LOCC operations Λ since it is defined as some optimum over them. Since the set of LOCC operations is closed under tensor product, one easily finds that the measures satisfy (M1). Werner showed that it also satisfies (M2) [40]. Most likely E_D is not convex [41] (we will discuss it later).

Asymptotic and finite optimal conversion rates. One can generalize E_D and E_C as follows [6,40]. Suppose that Alice and Bob start with n pairs in state ϱ and (for large n) are able to convert them by LOCC into m pairs in state ϱ'. The optimal asymptotic ratio $\frac{m}{n}$ is called optimal $\varrho \to \varrho'$ conversion rate and is denoted by $R(\varrho \to \varrho')$. The rigorous definition is similar to the definitions of E_D and E_C. In particular, $E_C(\varrho) = 1/R(\psi_+(C^2) \to \varrho)$, $E_D(\varrho) = R(\varrho \to \psi_+(C^2))$.

In finite regime, one can consider optimal probability of conversion [11] $p(\varrho \to \varrho')$. Thus Alice and Bob perform operation on ϱ whose one of outcomes correspond to final state equal to ϱ' (if possible). Then $p(\varrho \to \varrho')$ is maximal probability of obtaining this outcome over all such operations. If such operation does not exist, the optimal conversion probability vanishes. The quantity is especially useful in finite regime for pure states; we will discuss it in more detail in Sect. 5. For fixed ϱ' it satisfies (M3) as a function of ϱ itself. For ϱ mixed and ϱ' pure, it is usually zero. Then we must pass to asymptotic conversion rates.

4.2. *Entanglement measures based on distance*

The distance entanglement measures [10] are based on the natural intuition, that the closer the state is to the set $\mathcal{S}$, the less entangled it is. The measure is minimum distance D between the given state and the states in $\mathcal{S}$:

$$E_{D,\mathcal{S}}(\varrho) = \inf_{\sigma\in\mathcal{S}} D(\varrho, \sigma). \tag{23}$$

Roughly speaking, it turns out that such function is monotonic, if distance measure is monotonic under *all* operations. It is then possible to use known, but so far unrelated, results from literature on monotonicity under completely positive maps. Moreover, it proves that it is not only a technical assumption to generate entanglement measures: monotonicity is a condition of a distance to be a measure of *distinguishability* of quantum states [42,43].

More precisely, consider a given state ϱ and a state $\sigma \in \mathcal{S}$. Suppose that an operation applied to those states produces ensembles $\{p_i, \varrho_i\}$ and $\{q_i, \varrho_i\}$ respectively. Then the distance is required to satisfy

$$\sum_i p_i D(\varrho_i, \sigma_i) \le \mathcal{D}(\varrho, \sigma). \tag{24}$$

Note, that by this very assumption we obtain that $D(\varrho,\varrho)$ is a constant independent of ϱ (this is in strict analogy to the similar case in sect. 3.2). This is, because for arbitrary states ϱ and σ, there is a proper operation transforming ϱ into σ. The constant will be the value of the entanglement measure on separable states. To set the value zero, is to require $D(\varrho,\varrho)=0$ which is obvious condition for distances. Then monotonicity (24) will ensure non-negativity of the distance. Let us stress, however, that we do not need constitutive properties of distances like symmetry or triangle inequality.

It is immediate to see that monotonicity of a distance implies (M2) for associated measure. The clue is that $\mathcal{S}$ is closed under LOCC operations. Take σ_0 saturating the infimum (23), so that $E_{D,\mathcal{S}}(\varrho)=D(\varrho,\sigma_0)$. For simplicity, consider proper LOCC operation Λ applied to ϱ. The distance will satisfy $D(\Lambda(\varrho),\Lambda(\sigma_0))\le D(\varrho,\sigma_0)$. Since $\Lambda(\sigma_0)$ is still in $\mathcal{S}$, the new distance is no smaller than $E_{D,\mathcal{S}}(\Lambda(\varrho))$ which is *minimum* distance. If the distance is in addition doubly convex then the arising measure is convex, satisfying then (M3).

Once a distance, good in the above sense, was chosen, one can consider different measures by changing the sets closed under LOCC operations. In this way we obtain $E_{D,\mathcal{PPT}}$ [35] or $E_{D,\mathcal{ND}}$. The measure involving set $\mathcal{PPT}$ is much easier to evaluate. The greater the set, the smaller the measure is, so that we have

$$E_{D,\mathcal{ND}} \le E_{D,\mathcal{PPT}} \le E_{D,\mathcal{S}}. \tag{25}$$

In Ref. [10] two distances were shown to satisfy (24) and double convexity: Bures one $D_B(\varrho,\sigma)=2-2\sqrt{F}(\varrho,\sigma)$ where $F(\varrho,\sigma)=[\mathrm{Tr}(\sqrt{\varrho}\sigma\sqrt{\varrho})^{1/2}]^2$ is fidelity [44,45] and relative entropy $S(\varrho|\sigma)=\mathrm{Tr}\varrho(\log\varrho-\log\sigma)$. The measure based on the latter, called *relative entropy of entanglement* turned out to be one of the fundamental measures, as the relative entropy is one of the most important functions in quantum information theory (see [46,47]). Depending on choice of set we will denote them by $E_{R,\mathcal{X}}$ with $\mathcal{X}=\mathcal{S},\mathcal{PPT}$ or $\mathcal{ND}$. Relative entropy of entanglement turned out to be powerful upper bound for entanglement of distillation [35,10].

4.3. *Convex roof measures*

Here we consider the following method: one starts by imposing a measure E_p on pure states, an then extends it to mixed ones by so-called *convex roof*: [49,48]

$$E(\varrho)=\inf\sum_i p_i E(\psi_i),\quad \sum_i p_i=1,\quad p_i\ge 0. \tag{26}$$

where the infimum is taken over all ensembles $\{p_i,\psi_i\}$ for which $\varrho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$. If E_p is continuous then infimum is reached on a particular ensemble [48]. The ensemble we call *optimal*. Thus E is equal to average over optimal ensemble.

One easily checks that E must be convex. Actually, convex roof measures are the largest functions that are (i) convex (ii) compatible with given values for pure states [48]. Thus they are as flat as only possible without violation of convexity. The prototype for the method was *entanglement of formation* E_F introduced in [6], where $E(\psi)$ is von Neumann entropy of the reduced density matrix of ψ. It constituted first upper bound for distillable entanglement, and was intended to be entanglement cost. It is indeed the case for pure states, but we still do not know if $E_F=E_C$ in general. In Ref. [6] monotonicity of E_F was shown. In Ref. [11] general proof for monotonicity of all possible convex-roof measures was established. We follow the latter approach.

For the convex roof measure E, again, the very definition significantly simplifies the condition of monotonicity. As said, convexity is satisfied almost automatically. The condition (M3a) is reduced to monotonicity for pure states. To see it, consider ϱ with optimal ensemble $\{p_i, \psi_i\}$. Take any local pure operation. It transforms initial state ϱ as follows

$$\varrho \to \{q_k, \sigma_k\}, \quad q_k = \mathrm{Tr} V_k \varrho V_k^\dagger, \quad \sigma_k = \frac{1}{q_k} V_k \varrho V_k^\dagger \tag{27}$$

The members of the ensemble $\{p_i, \psi_i\}$ transform into ensembles of pure states (because operation is pure)

$$\psi_i \to \{q_k^i, \psi_k^i\}, \quad q_k^i = \mathrm{Tr}(V_k |\psi_i\rangle\langle\psi_i| V_k^\dagger), \quad \psi_k^i = \frac{1}{\sqrt{q_k^i}} V_k \psi_i. \tag{28}$$

One finds that $\sigma_k = \frac{1}{q_k} \sum_i p_i q_k^i |\psi_k^i\rangle\langle\psi_k^i|$.

Now we want to show that the initial entanglement $E(\varrho)$ is no less than final average entanglement $\overline{E} = \sum_k q_k E(\sigma_k)$, assuming, that for pure states E is monotonic under the operation. Since σ_k is a mixture of ψ_k^i's, then due to convexity of E we have

$$E(\sigma_k) \leq \frac{1}{q_k} \sum_i p_i q_k^i E(\psi_k^i) \tag{29}$$

Thus $\overline{E} \leq \sum_i p_i \sum_k q_k^i E(\psi_i^k)$. Due to monotonicity on pure states we have $\sum_k q_k^i E(\psi_k^i) \leq E(\psi_i)$ hence we obtain $\overline{E} \leq \sum_i p_i E(\psi_i)$. However, the ensemble $\{p_i, \psi_i\}$ was optimal, so that the latter term is equal simply to $E(\varrho)$. This ends the proof. Note that the presented reasoning is valid for multipartite setting.

Thus, for any function monotonic for pure states, its convex roof is monotonic for *all* states. How general is this implication? Of course it applies to multipartite setting. As far as different classes of operations are concerned, note that we have used the fact that monotonicity is equivalent to convexity plus *pure* operations (consequently the pure state is converted into pure one). Then the result applies to separable operations, but not to PPT ones (cf. discussion after (M3) postulate). From the above result one can easily get monotonicity of E_F under separable operations [38]. Indeed, we already know, that relative entropy of entanglement is monotonic under separable operations. In particular, it is monotonic for pure states. However for the latter, it is equal to E_F, so that we have monotonicity of E_F for pure states, which then extends to mixed states. Of course, E_F can increase under PPT operations. For example, changing separable state into entangled PPT one is legitimate PPT operation. The output state has $E_F > 0$ as it is entangled. This is compatible with the fact, that we couldn't apply the above reasoning to PPT operations.

In Sect. 5 we will discuss the question of monotonicity for pure states. We will see that any concave, expansible function of reduced density matrix of ψ satisfies (M3a). For example [11] one can take $E_\alpha(\psi)$ given by Renyi entropy $\frac{1}{1-\alpha} \log_2 \mathrm{Tr}(\varrho^\alpha)$ of the reduction. For $\alpha = 1$ it gives E_F, while for $\alpha = 0$, the so-called Schmidt rank [50]. Finally, for two qubits the measure called *concurrence* was introduced for pure states [51]. In [52] closed expression for its convex roof extension was found, and formula for E_F was derived in two-qubit case.

4.4. *Negativities*

A couple of measures, so far unrelated to each other, were put into elegant unified scheme in Ref. [53]. The idea comes from a measure introduced in [54] called robustness of entanglement (see also [55]). For given ϱ consider decomposition into two separable state $\varrho = a_+ \varrho_+ - a_- \varrho_-$ with $a_\pm \geq 0$. Robustness of entanglement is minimal possible a_- that can be achieved in this way. To obtain other measure one can take $\varrho_\pm$ to be

PPT ones. Interestingly, one can consider decomposition of ϱ into operators from *any* compact set $\mathcal{B}$ of Hermitian operators of trace one, with the property that its real linear span gives all Hermitian operators. One then defines $\mathcal{B}$-norm as $\|\varrho\|_{\mathcal{B}} = \min\{a_+ + a_-\}$ and corresponding *negativity* $\mathcal{N}(\varrho) = \min\{a_-\}$. Since the involved operators have unit trace we have $||\varrho||_{\mathcal{B}} = 1+2\mathcal{N}(\varrho)$. It is not hard to see that $\|\cdot\|_{\mathcal{B}}$ is indeed a norm, hence it is convex. Now, if the set $\mathcal{B}$ is closed under class of operations, then the negativities satisfy (M3a) under this class, as well. Indeed, let the operation produce ensembles $\{p_i, \varrho_i\}$, $\{p_i^{\pm}, \varrho_i^{\pm}\}$ out of of ϱ and $\varrho_\pm$ respectively. Applying the operation to optimal decomposition of ϱ we obtain decompositions $\varrho_i = a_+ p_i^+/p_i \varrho_i^+ + a_- p_i^-/p_i \varrho_i^-$. Due to required property of the set, $\varrho_i^{\pm}$ belong to the set, so that $\mathcal{N}(\varrho_i) \leq a_- p_i^-/p_i$, hence the final average negativity satisfies $\overline{\mathcal{N}} \leq \sum_i p_i a_- p_i^-/p_i = a_- \leq \mathcal{N}(\varrho)$.

Usually it is hard to find minimal a_-. Fortunately, if we choose the set $\mathcal{B}$ of all unit trace operators that are positive under partial transpose, we obtain a measure introduced in [56] - sum of negative eigenvalues of partially transposed ϱ. In terms of norms we have that $\|\varrho\|_{\mathcal{B}} = \|\varrho^{PT}\|_1$ in this case. The above result of [53] shows that it is indeed a good measure. In this way we obtain the first calculable measure of entanglement. One can consider logarithmic negativities $\log(2\mathcal{N}+1)$, which will usually not be convex. The logarithmic negativity $E_{\mathcal{N}} = \log\|\varrho^{PT}\|$ is additive, it is not convex anymore, but still satisfies (M1). Moreover, it is additive.

In Ref. [57] a measure based on the so called *cross-norm* was proposed. Within the present framework, the cross norm is $\mathcal{S}$-norm, hence the cross-norm measure is closely related to robustness of entanglement.

Finally in Ref. [58] two different concepts were combined, to give the following measure

$$E_{R+\mathcal{N}} = \inf_\sigma \left(S(\varrho|\sigma) + \log\|\sigma^{PT}\|_1\right) \tag{30}$$

where the infimum is taken under the set of *all* states. One easily finds that the measure satisfies (M1).

4.5. *Summary*

Thus, we have, in principle, four kinds of measures. All of them involve some optimization. Let us roughly summarize the concepts of obtaining monotonicity. For operational measures, monotonicity is immediate consequence of definition. The price is that they are extremely hard to calculate. Monotonicity of other measures is implied by their properties. Distance measures and negativities base on invariance of some sets of states under LOCC operations. Convex-roof measures being extensions of measures defined for pure states, inherit monotonicity from the latter measures.

4.6. *Evaluating measures*

Here I will review a couple of results of calculating or estimating entanglement measures. As said, $E_{\mathcal{N}}$ is easily calculable for any state (it suffice to find eigenvalues of partial transpose of the latter). Entanglement of formation is efficiently calculable for two-qubits [52]. In higher dimensions it can be evaluated for states with high symmetries [60,61]. In particular for Werner states one has [6,61]

$$E_F(\varrho_W(p,d) = H(1/2 - \sqrt{p(1-p)}) \tag{31}$$

where $H(x) = -x\log x - (1-x)\log(1-x)$ (we give the values of measures for $p \geq 1/2$ for Werner states, and $F \geq 1/d$ or isotropic ones, as otherwise the states are separable). Distance measures are very easy to calculate for Werner and isotropic states by exploiting their symmetries [35]. It turns out that

$$E_{R,\mathcal{S}}(\varrho_W(p,d)) = S(\varrho_W(p,d)|\varrho_W(1/2,d))$$
$$E_{R,\mathcal{S}}(\varrho_{iso}(F,d)) = S(\varrho_{iso}(p,d)|\varrho_{iso}(1/d,d)). \tag{32}$$

Consequently, we have

$$E_{R,\mathcal{S}(\mathcal{PPT})}(\varrho_W(p,d)) = 1 - H(p), \tag{33}$$

$$E_{R,\mathcal{S}(\mathcal{PPT},\mathcal{ND})}(\varrho_{iso}(F,d)) = \log d - (1-F)\log(d-1) - H(F) \tag{34}$$

In [61] a surprising result was obtained, concerning possible additivity of E_R. It turned out that E_R of two copies of Werner states with $p = 1$, is, for large d almost the same as for one copy. Thus, the relative entropy of entanglement can be strongly non-additive.

It is worth to recall also the values of $E_{\mathcal{N}}$ for the considered states.

$$E_{\mathcal{N}}(\varrho(F,d)) = \log dF \tag{35}$$

$$E_{\mathcal{N}}(\varrho_W(p,d)) = \log\left[\frac{2}{d}(2p-1)+1\right] \tag{36}$$

Concerning the operational measures, we know that $E_C = E_F^\infty \equiv \lim_{n\to\infty}\frac{1}{n}E_F(\varrho^{\otimes n})$ [39]. If E_F were additive (which is a long-standing open problem) then it would be equal to E_C. E_D is bounded from above by E_F [6]. For pure states $E_D = E_F = E_C = E_{R,\mathcal{X}} = S(\varrho_A)$ where ϱ_A is reduced density matrix of the given pure state [13,10]; $\mathcal{X} = \mathcal{S},\mathcal{ND},\mathcal{PPT}$. We will return to this coincidence while discussing *uniqueness theorem* in sect. 8.3. In [62] it was found that for some some bound entangled state (i.e. with $E_D = 0$) $E_C > 0$. One expects that the only mixed states with $E_D = E_C$ are the mixtures of locally orthogonal states [15]. Apart from such mixtures[††], the value measure E_D is known only for one family of mixed states - mixtures of two maximally entangled two qubit states [35]

$$E_D(\varrho) = 1 - S(\varrho) \tag{37}$$

In general, for mixtures of two-qubit maximally entangled states we have $E_D \geq 1 - S$ by the so-called *hashing protocol* [6]. For higher dimension powerful tools for evaluating E_D were provided in [58]. One knows several upper bounds for E_D [6,10,35,64,53,58]. The best known bound is $E_{R+\mathcal{N}}$ provided in [58]. For Werner states it is equal to regularization of $E_{R,\mathcal{S}}$ given by $E_{R,\mathcal{S}}^\infty = \lim_n \frac{1}{n}E_{R,\mathcal{S}}(\varrho^{\otimes n})$ calculated in [59]:

$$E_{R,\mathcal{S}}^\infty = E_{R+\mathcal{N}} = \begin{cases} 1 - H(p) & \frac{1}{2} \leq p \leq \frac{1}{2}+\frac{1}{d} \\ \log\left(\frac{d-2}{d}\right) + p\log\left(\frac{d+2}{d-2}\right) & \frac{1}{2}+\frac{1}{d} \leq p \leq 1 \end{cases} \tag{38}$$

In Ref. [16] a general method of obtaining bounds was provided: roughly speaking, any entanglement measure, satisfying some continuity and additivity assumptions is upper bound for E_D. We will discuss it in more detail later.

5. Finite regime, pure states

5.1. *All measures for pure states*

In Ref. [11] it was shown that measures for pure states are in one-to-one correspondence to functions f of density matrix satisfying

[††] Sometimes it is not immediate that a given state is a mixture of locally orthogonal pure states. In Ref. [63] it was shown if Alice and Bob share n singlet pairs, and will forget which particle comes from which pair, the resulting state is a such mixture.

(i) f is symmetric, expansible function [‡‡] of eigenvalues of ϱ

(ii) f is concave function of ϱ

(by expansibility we mean $f(x_1,\ldots,x_k,0,\ldots,0)=f(x_1,\ldots,x_k)$). In this way all possible entanglement measures for pure states were found. More precisely we have the following theorem

Theorem 1. (Vidal [11])

- (direct) Let E_p, defined for pure states, satisfy $E_p(\psi)=f(\varrho_A)$, where ϱ_A is reduction of ψ, and f satisfies (i) and (ii). Then there exists an entanglement measure E (satisfying M3) coinciding with E_p on pure states (E is convex-roof extension of E_p).

- (converse) Let E satisfy (M3). Then $E(\psi)=f(\varrho_A)$ for some f satisfying (i) and (ii).

We will recall here the proof of the "direct" part.

Proof. We are to show that convex-roof extension E of E_p satisfies (M3). E is convex by definition, hence it remains to show (M3a). As discussed in previous section, it suffices to show it for pure states. Consider then any pure operation on, say, Alice side (for Bob's one, the proof is the same) which produces ensemble $\{p_i,\psi_i\}$ out of state ψ. We want to show that the final average entanglement $\overline{E}_p=\sum_i p_i E_p(\psi_i)$ does not exceed the initial entanglement $E_p(\psi)$. In other words, we need to show $\sum_i p_i f(\varrho_A^{(i)})\leq f(\varrho_A)$, where $\varrho_A^{(i)}$ are reductions of ψ_i on Alice's side. We note that due to Schmidt decomposition of ψ, reductions ϱ_A and ϱ_B have the same non-zero eigenvalues. Thus, $f(\varrho_A)=f(\varrho_B)$, due to (i). Similarly $f(\varrho_A^{(i)})=f(\varrho_B^{(i)})$. Thus it remains to show that $\sum_i p_i f(\varrho_B^{(i)})\leq f(\varrho_B)$. How ϱ_B is related to $\varrho_B^{(i)}$? Well, due to no-superluminal-signalling, the mixture produced by Alice's action must have the same Bob's reduction as the initial state. The former reduction is $\sum_i p_i\varrho_B$, the latter one is ϱ_B. We conclude that $\varrho_B=\sum_i p_i\varrho_B^{(i)}$ (which is usually not true for Alice part!). Thus our question reduces to the inequality $\sum_i p_i f(\varrho_B^{(i)})\leq \sum_i p_i f(\varrho_B^{(i)})$. This is however true, due to concavity of f.∎

Let us consider this latter result from a different point of view. The result could be stated as follows: any function satisfying (i) and (ii) does not increase on average under pure measurement. If we define information about the state as $I=I_0-f$, where I_0 is some constant, then the physical contents of the statement is: *The average information gain after pure measurement is always nonnegative.* Moreover, if monotonicity of information we take as necessary property of information, we obtain that any candidate for information must be function satisfying (i) and convexity. In this context, a version of the result was proven by Lindblad, where f was von Neumann entropy, while the class of measurements was confined to orthogonal ones. Here, we have a back-on-the-envelope proof for fully general setting! Moreover, the proof is no longer technical. This illustrates the power of "entanglement approach" to quantum mechanics.

5.2. *Entanglement measures and conversion probability*

Another important step in finite regime in Ref. [11] for development of entanglement measures was realizing the connection between axiomatically defined measures, and operational ones, i.e., conversion probabilities. For asymptotic regime this was anticipated in [67] and obtained in [16] (see sect. 8.1)We have the following theorem

‡‡ As mentioned earlier, entanglement measures are not functions of states acting only on some definite Hilbert space, but on the union of all finite-dimensional states. Similarly, our function acts on union of n dimensional simplexes for all n, rather than on some k dimensional simplex.

Theorem 2. [11] For any measure E satisfying M3, and S1, the following inequality holds

$$p(\varrho \to \varrho') \leq \frac{E(\varrho)}{E(\varrho')} \tag{39}$$

Corrolary. Since $p(\varrho \to \varrho')$ as a function of ϱ is a monotone itself, we have $p(\varrho \to \varrho') = \min_E \frac{E(\varrho)}{E(\varrho')}$, where the minimum is taken over all monotones E.

This theorem gives a general hint how to employ the axiomatic theory of entanglement measures to evaluate interesting operational quantities.

Proof. Consider optimal operation converting ϱ into ϱ'. The final average E is certainly no smaller than $pE(\varrho')$ were p is probability of obtaining ϱ'. However, the process is assumed to be optimal, hence $p = p(\varrho \to \varrho')$. Thus the final entanglement is no smaller than $p(\varrho \to \varrho')E(\varrho')$. Due to monotonicity, this cannot exceed the initial entanglement $E(\varrho)$. ∎

The result is simple, but very important. Basing on it and on Nielsen result [68] Vidal provided formula for conversion probability for any two pure states (see the article by Vidal & Nielsen, this issue). The last result allowed to discover the surprising effect of catalysis in entanglement processing [70].

On the other hand, the above theorem, though very powerful for pure states, is not very useful in mixed state domain, as it is usually impossible to convert some mixed state (or pure one) into another mixed state with perfect accuracy, as required in the definition of $p(\varrho \to \varrho')$ (see [71,72]). Then conversion probability is zero, and the inequality becomes trivial. This motivates to use "asymptotic accuracy" and "collective operations" paradigm introduced in [6]. One considers many copies in the same state, apply collective operations and require that for high number of copies, the resulting state will approach the desired one (or some number of the copies of the latter one). We will see that it is still possible to derive the connection between abstract and operational measures within this asymptotic regime [16].

6. Asymptotic regime

As we have seen, even only for pure states there is a plenty of valid entanglement measures. All of them are important ones. Then, what about mixed states? The situation is much worse - the finite regime approach does not allow to see any ordered structure: it constitutes a cloud hiding some general features of the structure of entanglement. Such a general feature is, e.g., distillability. As discussed in sect. 2.1 the set of entangled states is then divided into two ones - bound entangled and free entangled. The borderline between those sets will certainly say us much about entanglement. However, it would not emerge in a natural way in finite regime considerations.

The asymptotic regime, that is believed to lead to better understanding of entanglement, is called *thermodynamics of entanglement*. To build the engine we do not need to know microscopic dynamics of all molecules, only several parameters are relevant, like temperature or pressure. Similarly, in entanglement domain, the plenty parameters of finite regime are killed in the asymptotic limit [8]. For example, as shown in [13] if we consider a large number n of pairs in the same state ψ, then out of all entanglement measures, the limit $n \to \infty$ only *one* measure survives! Then it is very tempting to develop this approach.

How the reduction of parameters is achieved in asymptotic regime? Mainly due to the fact that we can operate collectively on many identically prepared copies. Besides, since we allow imperfect conversions much more operations are reversible (for perfect conversions, the only reversible LOCC operations are product unitary ones).

Interestingly, unlike in statistical physics, where we go, in a sense, from reversible dynamics to irreversibility, for pure bipartite states we will go from very often and natural irreversibility in finite regime [68], to full reversibility in asymptotic domain. However, "thermodynamic limit" will not kill irreversibility for *mixed* states. The survived irreversibility is connected with bound entanglement - and is a result of mixing of quantum and classical information levels (cf. [74]). In multipartite case, even for pure states there is a basic irreversibility, [14,76] and it is, in turn, the reflection of the multipartite entanglement, fundamentally different from bipartite one (in general, for each n we have genuine n partite entanglement that cannot be reversibly transformed into $n-1$-partite entanglement).

6.1. *States and sources*

Let us now state more precisely what objects are of interest in asymptotic regime. In finite one those are states. Here, instead, we will consider *quantum sources.* A quantum source is a *compatible family* of states ϱ_n acting on Hilbert space $\mathcal{H}^{(1)}\otimes\ldots\otimes\mathcal{H}^{(n)}$. Compatibility means $\mathrm{Tr}_{\mathcal{H}^{(n)}}\varrho_n=\varrho_{n-1}$. We can imagine a source emitting systems, such that the first n emitted systems is in joint state ϱ_n. In our case the systems will be pairs so that $\mathcal{H}^{(i)}=\mathcal{H}_A\otimes\mathcal{H}_B$. The simplest example of source is stationary memoryless one, for which $\varrho_n=\varrho^{\otimes n}$. Then the subsequently emitted systems are completely uncorrelated, and the state of each system is the same.

We will be interested in entanglement of *source*, instead of a *pair*, as in finite case. Of course that entanglement will be infinite, so we will pass to intensive parameter, dividing entanglement by "volume" - number of pairs. Thus, given entanglement measure E from finite regime, we can calculate its *density* (or *mean*) E^∞ for a source $Q=\{\varrho_n\}$ as follows

$$E^\infty(Q)=\lim_{n\to\infty}\frac{E(\varrho_n)}{n} \tag{40}$$

For memoryless source we can write $E^\infty(\varrho)$. One finds that $E^\infty(\varrho)$ is what was called in literature *regularization* of E. Note that ϱ does not stand for a state of a pair, any longer, but it stands for a symbol of *infinite* number of pairs. Thus, E^∞ is not entanglement *of* a pair, but *per* pair.

It seems that some entanglement parameters of source do not need to come from some finite regime measure. For example distillable entanglement is defined directly for source. It says with what rate can we convert source ϱ into source $\psi_+(C^2)$. In general, we can consider asymptotic conversion rate $R(Q\to Q')$.

Having distinguished between source entanglement and state entanglement, we should first note that in the case of memoryless source, the former one, treated *as a function of* ϱ should not be required to be a good measure in finite regime. Let us exhibit an example. It is very natural to consider convexity of entanglement as a fundamental postulate [11]. Well, erasure of some information cannot increase quantum correlations. However, quite recently it was argued that E_D as a function of state need not be convex. Namely, in Ref. [41] it was shown that if for some states $E_D(\varrho_1)+E_D(\varrho_2)=0$ but $E_D(\varrho_1\otimes\varrho_2)>0$, then E_D is non-convex (moreover, a strong evidence was provided that such states exist). Following the reasoning of [41] take $\varrho_i'=\varrho_i\otimes|i\rangle\langle i|$ where $|i\rangle$'s are orthogonal states on Alice side, and ϱ_i satisfy the conditions above. The process of mixing of primed states is then locally reversible: Alice can measure the "flags" $|i\rangle$ and get to know which state is actually shared. Then $E_D(\varrho_{mix})$ with $\varrho_{mix}=p\varrho_1'+(1-p)\varrho_2'$ is nonzero. Indeed, having n pairs in the state ϱ_{mix}, Alice measures the flags, and communicates to Bob the results. They are left on average with np pairs in state ϱ_1 and $n(1-p)$ pairs ϱ_2. If $p\le 1-p$ they have np "double" pairs in the total state $\varrho_1\otimes\varrho_2$, and van distill them, according to our assumption. Thus we have $E_D(\varrho_{mix})>pE_D(\varrho_1)+(1-p)E_D(\varrho_2)=0$. We could conclude, that E_D is not a good measure. However, E_D is one of the central quantities showing capability of entanglement, hence it must be a good measure. The other way around is to give up a condition of convexity for entanglement measures. This seems to be even more unreasonable.

The answer is, that it is misunderstanding to treat E_D as a good measure of entanglement of *individual pair.* E_D is entanglement of a source. This implies that the requirement of convexity of E_D treated as a function of ϱ has nothing to do with convexity in finite regime. To see it, let us apply convexity to states $\varrho_1 = \varrho^{\otimes n}$ and $\varrho_2 = \sigma^{\otimes n}$ for some finite regime measure E. We obtain

$$E(p\varrho^{\otimes n} + (1-p)\sigma^{\otimes n}) \leq pE(\varrho^{\otimes n}) + (1-p)E(\sigma^{\otimes n}). \tag{41}$$

We see that it is much different than

$$E((p\varrho + (1-p)\sigma)^{\otimes n}) \leq pE(\varrho^{\otimes n}) + (1-p)E(\sigma^{\otimes n}). \tag{42}$$

Now, requirement of convexity of E_D treated as a function of ϱ is similar to the latter condition which is artificial and completely unjustified. Thus, how convexity translates from finite regime to asymptotic one? Note that we can think of E_D as if it were entanglement of $\varrho^{\otimes n}$ with large n. Thus the inequality (41) applies. Consequently, the proper asymptotic convexity condition is that the rate of distillation of the source $pQ + (1-p)Q'$ with $Q = \{\varrho^{\otimes n}\}, Q' = \{\varrho^{\otimes n}\}$ does not exceed $pE_D(\varrho) + (1-p)E_D(\sigma)$. Clearly, mixture of sources is not a source of mixtures. Mixture of sources means that there is only on kind of states, Alice and Bob not knowing which one, while the source of mixtures emits different types of states.

7. Postulates for asymptotic regime

Henceforth we will be interested only in *memoryless* sources, as only this case has been treated in the literature so far. Accordingly, we will deal with two categories of entanglement measures: (i) entanglement of source (which can, but do not have to come from some finite regime measure) *treated as a function of state* (ii) entanglement of state, which gives raise to entanglement of source via regularization. Both categories deserve separate treatment. As far as (ii) is concerned, *stronger* postulates than in finite regime should be required. In the case (i) the postulates should be instead *weaker*. For example, E_F is convex and belong to (ii). E_C is its regularization, belongs to (i), and would need not be convex, however it is convex [17]. E_D belongs to (i), we do not know if it is regularization of some measure. As mentioned, it is expected not to be convex. We will see that E_F is asymptotically continuous, but E_C and E_D most likely are not. In general, there can be measures of type (i), still satisfying strong properties. Then they will constitute a strong tool.

Let us emphasise that in the following we will treat all measures (either of type (i) or (ii)) as function of state.

7.1. *Monotonicity*

As argued in sect. 6.1 for entanglement of source we should give up convexity. It is yet not clear if we should impose (M1) or (M2). It seems that (M1) should be enough. Entanglement of state should, in principle, satisfy (M3). However, it is reasonable to consider measures satisfying only (M2) or even (M1), as useful tools.

7.2. *Continuity*

The paradigm of "asymptotic accuracy" implies continuity constraints on finite regime measures that can be useful for asymptotic regime. This was first observed in [11], and developed in ([16] and [73,75,17]). The transformation of ϱ into σ means that $\varrho^{\otimes n}$ was transformed into σ'_m approaches $\sigma^{\otimes m}$ for large m. Thus we, in a sense, identify the states that asymptotically converge to one another. Consequently, we require corresponding entanglement densities to converge, as well.

(C1) For any sequences ϱ_n, σ_n acting on $\mathcal{H}_n = \mathcal{H}_n^A \otimes \mathcal{H}_n^B$ we have

$$\|\varrho_n - \sigma_n\| \to 0 \Rightarrow \frac{E(\varrho_n) - E(\sigma_n)}{\log \dim \mathcal{H}_n}. \tag{43}$$

(C2) One can weaken the condition (C1) by requiring $\mathcal{H}_n = \mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$. Then the measure need not be continuous as a function on fixed Hilbert space.

One can also consider the following conditions:

(CP1) Same as (C1) with ϱ_n being pure states.

(CP2) Same as (C1) with both ϱ_n and σ_n being pure states.

A useful condition is continuity on isotropic state combined with some *normalization* [16]:

(C3) For family of isotropic states $\varrho(F_n, d_n)$ with $F_n \to 1$, $d_n \to \infty$ (i.e. approaching maximally entangled state) we require

$$\frac{E(\varrho(F_n, d_n))}{\log d_n} \to 1 \tag{44}$$

The latter condition is easy to check, by direct calculating the given measure for isotropic state and performing the limit.

It is reasonable to impose postulates of type (C) for state entanglement measures (i.e. type (ii)). For source entanglement (type (i)), in principle none of them should be imposed, but we will see that usually properties of type (CP) are satisfied.

It is known that $E_{R,\mathcal{X}}$ satisfies (C1) for any compact convex set $\mathcal{X}$ including maximally mixed state [75] (the structure of tensor product is irrelevant here). E_F satisfies (C1), too [73].

Convex roof measures based on Renyi entropies and negativities fail to satisfy (C1) [11]. As a matter of fact, any measure that is not equal to E_F for pure states must not satisfy (C1) (cf. sect. 8.3). $E_{\mathcal{N}}$ satisfies (C3).

7.3. *Extensivity*

A measure useful for asymptotic regime, must exhibit a kind of extensivity [8]. As important postulate we consider

(A1) For any state ϱ the following limit exists

$$E^\infty(\varrho) = \lim_{n\to\infty} E(\varrho^{\otimes n}) \tag{45}$$

In practice, one could weaken it by requiring $\limsup_n \frac{E(\varrho^{\otimes n})}{n} < \infty$. (Of course, we are interested only in measures for which the limit gives nonzero value at least for some states). Evaluating regularizations E^∞ of known measures is now a challenge for entanglement theory. This extremely difficult task is necessary to obtain insight into asymptotic behaviour of entanglement. First example of performing regularization was done in Ref. [59] for E_R. For some states it turned out that E_R was additive, so that $E_R^\infty = E_R$ [35]. At present, even a single nontrivial example of state for which we would know whether E_F needs regularization or not is not known!

Stronger conditions than (A1) are the following:

(A2) (additivity) $E(\varrho \otimes \sigma) = E(\varrho) + E(\sigma)$ (then $E^\infty = E$)

(A3) (partial additivity) $E(\varrho^{\otimes n}) = nE(\varrho)$.

Entanglement of *source* satisfies (A1) and (A3) automatically: ϱ and $\varrho \otimes \varrho$ represent the same source. The full additivity (A2) is too strong: most likely distillable entanglement is not additive [41]. Measure basing on Bures distance satisfies (A1), however the limit vanishes for all states, as the distance is bounded by a constant. Also negativities vanish after regularization. One can obtain extensivity for those measures by playing with logarithm (e.g. $E_{\mathcal{N}}$ is additive) at a price of losing convexity and (M2).

A plausible postulate for source entanglement would be the following one

(A4) $E(\varrho) \geq E(\varrho_1) + E(\varrho_2)$ where ϱ is a state of two pairs shared by Alice and Bob and ϱ_i are the states of each pair (reductions of ϱ).

It is not clear to what extent it holds. Certainly it is satisfied by entanglement of distillation.

8. Basic theorems in asymptotic domain

8.1. *Entanglement measures and conversion rates*

We will first exhibit a theorem on extreme measures which was proven originally in [16] and improved in [17]. It was inspired by previous results [48,67]. In [17] the assumptions were weakened and some variations of the theorem were proven.

Theorem 3. Any real-valued function E of state, satisfying M1, CP1, A3, and being in addition convex on pure decompositions, satisfies

$$E_D(\varrho) \leq E(\varrho) \leq E_C(\varrho) \tag{46}$$

Thus E_D and E_C are, in a sense, extreme measures. The measures known to satisfy all the assumptions are regularized relative entropy of entanglement, $E^\infty_{R,\mathcal{S}}$ or $E^\infty_{R,\mathcal{PPT}}$, and entanglement cost E_C itself. Concerning E_D the only trouble is with $CP1$ - we do not know if it holds. Most likely it would be satisfied if the hashing distillation method [6] worked for higher dimension. Note that if $E_D = E_C$ then under assumptions of the theorem, all the measures coincide (cf. sect. 8.3).

Let us now formulate the theorem binding axiomatic measures and asymptotic conversion rates i.e. the asymptotic regime analogue of Theorem 2. It was proven (in a bit less general form) in Ref. [17].

Theorem 4. Let E satisfy (M1), (C1) and (A0), and $R(\varrho \to \sigma) < \infty$ Then the following inequality holds

$$E^\infty(\sigma) R(\varrho \to \sigma) \leq E^\infty(\varrho). \tag{47}$$

Sketch of the proof. Take large n and consider optimal conversion map (proper operation) $\Lambda_n(\varrho^{\otimes n}) = \sigma'_m$ with σ'_m approaching $\sigma^{\otimes m}$. We will estimate the initial entanglement. Due to monotonicity (M1) of E we have

$$\frac{E(\varrho^{\otimes n})}{n} \geq \frac{E(\sigma'_m)}{n} = \frac{E(\sigma'_m)}{m}\frac{m}{n} \tag{48}$$

The fraction $\frac{m}{n}$ tends to $R(\varrho \to \sigma)$ as $n \to \infty$. Since E is asymptotically continuous (C1) $\frac{E(\sigma'_m)}{m}$ approaches $\frac{E(\sigma^{\otimes m})}{m}$ which tends to $E^\infty(\sigma)$.∎

The contents of the theorem is intuitively clear: The left hand side of the inequality is the final entanglement while the right-hand-side is the initial one, per input pair. The theorem is very general, in particular, it works for multipartite case. The proof is also extremely transparent. In fact, we do not make use of structure of tensor product. The key ingredient is that E is *monotonic under some class of operations*, and that R is defined as some *optimum over the same class*. Continuity is needed only because we deal with imperfect conversions. If the optimum is infinity, i.e., if some states can be created for free the E^∞ must automatically vanish on such states.

Finally, note that if we are interested in conversion from ϱ to σ the continuity is needed *only for the state* $\sigma^{\otimes n}$. For example, we will see that to obtain bounds for E_D we will need continuity only for isotropic state. Conversely, if we want to prove anything concerning asymptotic rate R, we must deal with E continuous for states of interest.

Finally, let us note that the condition of *reversibility* of the asymptotic conversion of ϱ into σ takes form

$$R(\varrho \to \sigma) R(\sigma \to \varrho) = 1 \tag{49}$$

(here one puts $0 \cdot \infty = 0$). For example if ϱ is two-qubit singlet then we recover the condition $E_D = E_C$. Note that if $R(\varrho \to \sigma) = R(\sigma \to \varrho) = \infty$ then the question of reversibility does not make sense.

8.2. *Example: GHZ-3EPR conversion*

Entanglement measures have been successfully applied to search of irreversibility in multipartite entanglement in [14,76,77]. Here we will consider irreversibility of conversion between tripartite GHZ state and 3EPR one shared by Alice, Bob and Charlie (shown in [76]). The first one is given by $\frac{1}{\sqrt{3}}(|000\rangle + |111\rangle)$. The 3EPR state is tripartite but six-qubit state: each pair of parties share one EPR pair (pair of two qubits in singlet state). In Ref. [76] the following measure on pure states was considered:

$$E_1(\psi_{ABC}) = S(\varrho_{BC}) + E_{R,\mathcal{S}}(\varrho_{BC}), \tag{50}$$

where ϱ_{BC} is partial trace of ψ over Alice part. More precisely it was shown that (M3a) holds for this measure (for tripartite LOCC operations). The entropy $S(\varrho_{BC})$ is monotonic, as well (as it is monotonic as a function of bipartite system A-BC). One can extend both measures to mixed states by convex roof. The second one is thus entanglement of formation with respect to $A-BC$ division of the system, denote it be E_2. As proven in [73] E_2 satisfies (C1). We do not know if E_1 extended to mixed states satisfies (C1). We will assume it is true [†] One finds that for GHZ state we have $E_1^\infty = E_1 = 1$, $E_2^\infty = E_2 = 1$, while for $3EPR$ state $E_1^\infty = E_1 = 3$, $E_2^\infty = E_2 = 2$. Putting these values into formula (47) we obtain

$$R(GHZ \to 3EPR) \leq \frac{E_1(GHZ)}{E_1(3EPR)} = \frac{1}{3} \tag{51}$$

and

$$R(3EPR \to GHZ) \leq \frac{E_2(3EPR)}{E_2(GHZ)} = 2 \tag{52}$$

Now, if we go from GHZ to $3EPR$ and then back to GHZ we obtain the optimal $GHZ \to GHZ$ through $3EPR$ conversion rate bounded by $\frac{2}{3}$. Thus $\frac{1}{3}$ ebits of entanglement is necessarily irreversibly dissipated. There is no reversible process joining $3EPR$ and GHZ. Note that the bound (51) can be achieved: from GHZ state, EPR pair can be created at any two parties. Thus we obtain $R(GHZ \to 3EPR) = 1/3$.

8.3. *Uniqueness theorem*

For pure bipartite states we have full reversibility: there is only one measure good in asymptotic domain: entanglement of formation. It was argued in Ref. [8] and improved in [11] by adding continuity. More self-contained formulation was presented in [73]. Finally, necessary and sufficient conditions for a measure to be unique were found [17].

Theorem 5. Let E be a real-valued function of pure states. Then, the following conditions are equivalent:

1) $E = cE_F$ where c is some constant. 2) E satisfies (i) additivity, (ii) continuity (CP2), (iii) monotonicity: $E(\psi) \geq E(\Lambda(\psi))$ for any LOCC proper operation Λ for which $\Lambda(\psi)$ is pure

Remark. Note that as monotonicity we require analogue of (M1) for pure states.

To prove the "necessary" part, one checks that E_F satisfies the conditions (i)-(iii). The idea of the proof of sufficiency is to show that any function satisfying the conditions must lie between E_C and E_D and then use the fact that the latter ones coincide for pure states,

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† Following Ref. [69] one can show that as far as we consider conversions between pure states, the condition (CP1) is sufficient to have formula (47).

which was proven in [13] (see [73,17] for the proof compatible with rigorous definitions of E_D and E_C).

If we had $E_C = E_D$ for all states, then in asymptotic limit, there would be unique measure, and the conversions would be *reversible*. However as shown in [62] for some bound entangled mixed states we have irreversibility, i.e. $E_D < E_C$.

8.4. *Bounds for entanglement of distillation*

Since E_D is of prime interest in asymptotic theory of bipartite entanglement being capacity of noisy teleportation channel, one would like to evaluate it. Since, however, it is defined by a complicated optimization, what we can do is to provide tighter and tighter upper and lower bound. The lower bounds are obtained by designing distillation protocol that realizes the bound. Upper bounds are provided by use of axiomatic entanglement measures. (First upper bound was provided in [6] - it was entanglement of formation. The second one, which appeared to be tight in some cases was relative entropy of entanglement [35,10].

Let us present a powerful theorem obtained in Ref. [16] which allows for easy production of upper bounds for E_D. The theorem follows, in fact, from the proof of Theorem 4.

Theorem 5. Any function E satisfying (i) monotonicity (M1), (ii) continuity for isotropic state (CI) (iii) $E(\varrho^{\otimes n}) \leq nE(\varrho)$ is upper bound for E_D.

Remark. If E does not satisfy (iii), then under the assumptions of theorem we have $\limsup \frac{E(\varrho^{\otimes n}}{n}) \geq E_D$.

Proof. By (iii) we have

$$E(\varrho) \geq \frac{1}{n} E(\varrho^{\otimes n}). \tag{53}$$

Since the only relevant parameters of the output of the process of distillation are the dimension of the output Hilbert space and fidelity F (see definition of distillable entanglement), we can consider distillation protocol ended by $U \otimes U^*$ twirling [27], that results in isotropic final state. By monotonicity, distillation does not increase E, hence

$$\frac{1}{n} E(\varrho^{\otimes n}) \geq \frac{1}{n} E(\varrho_{iso}(F_{d_n}, d_n)) \tag{54}$$

Now, in the limit of large n, distillation protocol produces $F \to 1$ and $(\log_2 d_n)/n \to E_D(\varrho)$, hence by continuity the right hand side of the inequality tends to $E_D(\varrho)$. Thus we obtain that $E(\varrho) \geq E_D(\varrho)$.∎

Using the above result and applying the formulas for values of the measures on isotropic state one immediately checks that relative entropy of entanglement, logarithmic negativity and the combined measure $E_{\mathcal{N}+R}$ of Ref. [58] are upper bounds for E_D.

8.5. *Entanglement measures and information-like quantities*

As we have seen, measures of source entanglement coincide on pure states. One can view it as a result of two facts: (i) $E_D = E_C = S(\varrho_X)$ for pure states, as shown in [13] (cf. [17]); (ii) the theorem 3, saying that E_D and E_D are extreme measures. This is the first link between measures and von Neumann entropy - which is a measure of quantum information content of the quantum source [78]. For mixed states, the measures good in asymptotic regime are likely to satisfy inequality

$$E \geq I_X, \tag{55}$$

$X = A, B$, where $I_X = \max\{0, S(\varrho_X) - S(\varrho)\}$ is coherent information [79], a quantity closely related to capacity of quantum noisy channel. As a matter of fact coherent information is expected to play the role of quantum counterpart of Shannon mutual information. One knows that separable states have $I_X = 0$ [80]. For E_F the inequality follows from convexity of I_A [15]. For E_R the proof is much more involved [65]. From the inequality $E_R \geq I_X$ together with trivial one $E_R \leq S(\varrho_X)$ one obtains $E_R = S(\varrho_X)$ for pure states. In Ref. [66] the *hashing inequality* was conjectured $E_D^{\rightarrow} \geq I_B$, where $E_D^{\rightarrow}$ is one-way distillable entanglement (classical communication only from Alice to Bob is allowed [6]). Since $E_D^{\rightarrow}$ is no greater than E_D (Alice and Bob have to use only one-way communication to distill the state) and the latter is lower bound for asymptotic entanglement measures, the hashing inequality would imply inequality (55).

The von Neumann entropy can be also used to quantify *classical* information about quantum state. In Ref. [63] it was argued that during mixing average entanglement loss ΔE should not exceed average loss of the information ΔI about the system. This can be presented in the form of the following inequality

$$E(\varrho) - \sum_i p_i E(\varrho_i) \leq S(\varrho) - \sum_i p_i S(\varrho_i). \tag{56}$$

One easily finds [66] that (55) implies the inequality for pure states ϱ_i.

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