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A Note on Quasi-*p*-convex Function

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Abstract In this paper, we further study the quasi-*p*-convex function. The concepts of strictly quasi-*p*-convex function and quasi-*p*-convex cone are given and some new fundamental characterizations and operational properties of quasi-*p*-convex function are obtained.

Keywords: *p-convex set, quasi-p-convex function, strictly quasi-p-convex function, quasi-p-convex cone*

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1. Introduction

Convex function is an important kind of function in mathematics, which is widely used in mathematical programming, approximation theory, control theory and other fields. But it is found that the mathematical models of many optimization problems in practical application are non-convex, which prompts us to consider the generalized convex function with weak convexity. Generalized convex function can not only retain some good characteristics of convex function but also relax the requirement for convexity appropriately. So its application scope is wider than convex function. In 1949, De Finetti [1] proposed the first generalized convex function, and in 1953, Fenchel [2] named it quasi-convex function. Yang [3,4,5] further studied the properties of quasi-convex functions. In 2003, Wang et al.[6] proposed the concept of E-quasi-convex function and studied its related properties. In 2011, Lin et al. [7] established several new Hadamard type inequalities for quasi-convex function. In 2013, Zhang [8] defined harmonic quasi-convex function. In 2021, Bai [9] established Simpson type fractional integral inequality for quasi-convex function. In the same year, Sevda et al. [10] defined p-convex functions by using the concept of epigraph on the basis of p-convex sets. In 2022, Gültekin et al. [11] defined quasi-p-convex functions.

In this paper, we further study the quasi-p-convex function. The concepts of strictly quasi-p-convex function and quasi-p-convex cone are given and some new fundamental characterizations and operational properties of quasi-p-convex function are obtained.

2. Preliminaries

Definition 2.1 [13] Let $U \subseteq R^n$ and $0 < p \le 1$. If for each $x, y \in U$, $\lambda x + \mu y \in U$, where $\lambda, \mu \ge 0$, $\lambda^p + \mu^p = 1$, then U is called a p-convex set in R^n.

The definition of p-convexity of U can also be given as

$$\lambda x + \left(1 - \lambda^p\right)^{\frac{1}{p}} y \in U$$

for all $x, y \in U$ and $\lambda \in [0,1]$.

Definition 2.2 [14] Given $f : U \subseteq R^n \to [-\infty, \infty]$, the set

$$\left\{(x,\mu) \middle| x \in U, \mu \in R, \mu \ge f(x)\right\}$$

is called epigraph of f, it is denoted by $epif$.

Definition 2.3 [10] Let $U \subseteq R^n$ and $f : U \to R$ be a function. If the set

$$epif = \left\{(x,\mu) \in R^{n+1} : x \in U, \mu \in R, f(x) \le \mu\right\}$$

is p-convex set, then f is called a p-convex function.

Theorem 2.1 [10] Let $U \subseteq R^n$, $f : U \to R$ be a function and $0 < p \le 1$, then f is a p-convex function if and only if U is a p-convex set and for any $x, y \in U$, $f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ holds, where $\lambda, \mu \ge 0$, $\lambda^p + \mu^p = 1$.

Definition 2.4 [11] Let $p \in (0,1]$.

(i) A function $f : U \to R$ is called quasi-p-convex function if

$$f(\lambda x + \mu y) \le \max\left\{f(x), f(y)\right\}$$

for each $x, y \in U$; $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$ or, equivalently

$$f(y) \le f(x) \Rightarrow f\left(\lambda x + (1-\lambda^p)^{\frac{1}{p}} y\right) \le f(x)$$

for every $x, y \in U$ and for every $\lambda \in [0,1]$.

(ii) A function $f: U \to R$ is called quasi-*p*-concave function if $-f$ is quasi-*p*-convex, i.e., for each $x, y \in U$; $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, $\min\{f(x), f(y)\} \le f(\lambda x + \mu y)$ or, equivalently

$$f(y) \le f(x) \Rightarrow f(y) \le f\left(\lambda x + (1-\lambda^p)^{\frac{1}{p}} y\right)$$

for every $x, y \in U$ and for every $\lambda \in [0,1]$.

Example 2.1 Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$. Define the function $f: U \to R_+ = [0, +\infty)$,

$$f(x_1, x_2, \cdots, x_n) = k(x_1 + x_2 + \cdots + x_n),$$

where $x = (x_1, x_2, \cdots, x_n) \in U$, $k \in R$, then f is a quasi-*p*-convex function.

Proof. Suppose that $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, $0 < p \le 1$, then for each $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots y_n) \in U$,

$$\begin{aligned} f(\lambda x + \mu y) &= f(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \cdots \lambda x_n + \mu y_n) \\ &= k(\lambda x_1 + \mu y_1 + \lambda x_2 + \mu y_2 + \cdots + \lambda x_n + \mu y_n) \\ &= k(\lambda(x_1 + x_2 + \cdots + x_n) + \mu(y_1 + y_2 + \cdots + y_n)) \\ &= \lambda f(x) + \mu f(y) \\ &\le \lambda \max\{f(x), f(y)\} + \mu \max\{f(x), f(y)\} \\ &\le \lambda^p \max\{f(x), f(y)\} + \mu^p \max\{f(x), f(y)\} \\ &= \max\{f(x), f(y)\} \end{aligned}$$

that is, f is a quasi-*p*-convex function.

Theorem 2.2 **[11]** Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$. If $f: U \to R_+$ is a *p*-convex function, then f is a quasi-*p*-convex function.

Definition 2.5 [14] Suppose that K is a subset of R^n, if for each $x \in K$, $\lambda > 0$ such that $\lambda x \in K$, then K is called a cone.

Definition 2.6 [14] A cone is said to be convex cone if it is also a convex set.

Definition 2.7 [12] Let $K \subset R^n$ be a convex cone and $f: K \to R$ be a function. If for each $x \in K$, $t > 0$, $\alpha \in R$ such that $f(tx) = t^\alpha f(x)$, then the function f is called a positive homogeneous function with respect to degree α.

Remark 2.1 [12] (1) Let $\alpha \ne 0$, f be a positive homogeneous function with respect to degree α. If $f(0)$ exists, then $f(0) = 0$.

If $\alpha = 1$, f is called a linear positive homogeneous function.

Theorem 2.3 [14] The function $f: U \subseteq R^n \to R$ is a linear positive homogeneous function if and only if its epigraph $epi\, f$ is a cone in R^{n+1}.

Definition 2.8 [12] Let $f: U \subseteq R^n \to R$, then $x^* \in U$ is called a strict local minimum (maximum) of f, if it exists $\delta > 0$ such that

$$\forall x \in U \cap B_\delta(x^*): x \ne x^*, f(x^*) < f(x) \left(f(x^*) > f(x)\right).$$

Theorem 2.4 **[11]** Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$, then $f: U \to R$ be a quasi-*p*-convex function if and only if for any $\alpha \in R$, the lower level set $L_{\le\alpha} = \{x \in U : f(x) \le \alpha\}$ is a *p*-convex set.

3. Main Results

Definition 3.1 Let $0 < p \le 1$, $U \subseteq R^n$ be a *p*-convex set and $f: U \to R$ be a function. For each $x_1, x_2 \in U$, $x_1 \ne x_2$, if the inequality

$$f(\lambda x_1 + \mu x_2) < \max\{f(x_1), f(x_2)\} \tag{3.1}$$

holds for all $\lambda, \mu > 0$ such that $\lambda^p + \mu^p = 1$, then f is said to be a strictly quasi-*p*-convex function.

Definition 3.1 can also be expressed as follows:

Let $0 < p \le 1$, $U \subseteq R^n$ be a *p*-convex set and $f: U \to R$ be a function. If for any $x_1, x_2 \in U$, $x_1 \ne x_2$, $f(x_1) \ge f(x_2)$,

$$f\left(\lambda x_1 + (1-\lambda^p)^{\frac{1}{p}} x_2\right) < f(x_1) \tag{3.2}$$

where $\lambda \in (0,1)$, then f is said to be a strictly quasi-*p*-convex function.

Definition 3.2 Let $0 < p \le 1$, $U \subseteq R^n$ be a *p*-concave set and $f: U \to R$ be a function. For each $x_1, x_2 \in U$, $x_1 \ne x_2$, if the inequality

$$\min\{f(x_1), f(x_2)\} < f(\lambda x_1 + \mu x_2) \tag{3.3}$$

holds for all $\lambda, \mu > 0$ such that $\lambda^p + \mu^p = 1$, then f is said to be a strictly quasi-*p*-concave function.

Definition 3.2 can also be expressed as follows:

Let $0 < p \le 1$, $U \subseteq R^n$ be a *p*-convex set and $f: U \to R$ be a function. If for any $x_1, x_2 \in U, x_1 \ne x_2$, $f(x_1) \ge f(x_2)$,

$$f(x_2) < f\left(\lambda x_1 + (1-\lambda^p)^{\frac{1}{p}} x_2\right) \tag{3.4}$$

where $\lambda \in (0,1)$, then f is said to be a strictly quasi-*p*-concave function.

Theorem 3.1 Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$.

(1) If $f: U \to R_+$ is a strictly *p*-convex function, the f is a strictly quasi-*p*-convex function;

(2) If $f: U \to R$ is a strictly quasi-*p*-convex function, then f is a quasi-*p*-convex function.

Proof. (1) From the strictly *p*-convexity of f, for each $x_1, x_2 \in U$, $\lambda, \mu \ge 0$ such that such that $\lambda^p + \mu^p = 1$,

$$f(\lambda x_1 + \mu x_2) < \lambda f(x_1) + \mu f(x_2)$$

$$\le \lambda \max\{f(x_1), f(x_2)\} + \mu \max\{f(x_1), f(x_2)\}$$

$$\le \lambda^p \max\{f(x_1), f(x_2)\} + \mu^p \max\{f(x_1), f(x_2)\}$$

$$= \max\{f(x_1), f(x_2)\},$$

that is, f is a strictly quasi-*p*-convex function.

(2) It comes straight from the definition.

Definition 3.3 A cone is said to be *p*-convex cone if it is also a *p*-convex set.

Remark 3.1 If $p = 1$, *p*-convex cone is a convex cone.

Lemma 3.1 The set $U \subseteq R^n$ is a *p*-convex cone if and only if U is closed for operations of addition and positive multiplication.

Proof. Suppose that $U \subseteq R^n$ is a *p*-convex cone, $0 < p \le 1$, then for each $x, y \in U$, $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$,

$$\lambda x + \mu y \in U.$$

Take $\lambda = \mu = \dfrac{1}{2^{1/p}}$, then

$$\frac{1}{2^{1/p}} x + \frac{1}{2^{1/p}} y \in U.$$

Also U is a cone, so U is obviously closed for operation of positive multiplication. Therefor

$$x + y = 2^{1/p}\left(\frac{1}{2^{1/p}} x + \frac{1}{2^{1/p}} y\right) \in U.$$

That is, U is closed for operation of addition.

Conversely, if U is closed for operations of positive multiplication, then U is obviously a cone. For each $x, y \in U$, $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, then $\lambda x, \mu y \in U$. Also U is closed for operations of addition, so

$$\lambda x + \mu y \in U,$$

that is, U is a *p*-convex set. Thus U is a *p*-convex cone.

Lemma 3.2 For $0 < p \le 1$, let $U \subseteq R^n$ be a *p*-convex cone, $f: U \to R$ be a positive homogeneous function of degree *p*. Then f is a *p*-convex function if and only if for any $x, y \in U$,

$$f(x+y) \le f(x) + f(y).$$

Proof. Suppose f is a *p*-convex function, then $epif$ is a *p*-convex set. Also f is a positive homogeneous function of degree *p*, for each $(x, \kappa) \in epif$, $\lambda > 0$, then

$$f(\lambda x) = \lambda^p f(x) \le \lambda^p \kappa.$$

Specially, take $p = 1$, then $f(\lambda x) \le \lambda \kappa$, namely $\lambda(x, \kappa) \in epif$, so the $epif$ is a cone.

From lemma 3.1, for each $(x, f(x)), (y, f(y)) \in epif$,

$$(x + y, f(x) + f(y)) \in epif,$$

namely

$$f(x+y) \le f(x) + f(y).$$

Conversely, for each $(x, \kappa), (y, \nu) \in epif$, then

$$f(x+y) \le f(x) + f(y) \le \kappa + \nu,$$

namely

$$(x + y, \kappa + \nu) \in epif,$$

so $epif$ is closed for operation of addition. Also because f is a positive homogeneous function of degree *p*, taking $p = 1$, by Theorem 2.3, we known that $epif$ is closed for operation of positive multiplication. From Lemma 3.1, $epif$ is a *p*-convex set, so f is a *p*-convex function.

Theorem 3.2 For $0 < p \le 1$, let $U \subseteq R^n$ be a *p*-convex cone and $f : U \to R$ be a positive homogeneous function of degree *p*. If for all $0 \ne x \in U$, $f(x) > 0$, then f is a quasi-*p*-convex function if and only if f is a *p*-convex function.

Proof. For all $0 \ne x \in U$, by Theorem 2.2, a *p*-convex function is obviously is a quasi-*p*-convex function.

Conversely, if f is a quasi-*p*-convex function, by the positive homogeneity of degree *p* of f and Lemma 2, it just need to prove that f satisfies the inequality in Lemma 2.

Let any $x_1, x_2 \in U - \{0\}$, $y_1 = f(x_1) > 0,\ y_2 = f(x_2) > 0$. Since f is a positive homogeneous function of degree *p*, namely

$$f(tx) = t^p f(x), \forall t > 0,$$

it follows that

$$f\left(\frac{x_1}{y_1^{1/p}}\right) = f\left(\frac{x_1}{(f(x_1))^{1/p}}\right) = \frac{f(x_1)}{f(x_1)} = 1,\ f\left(\frac{x_2}{y_2^{1/p}}\right) = \frac{f(x_2)}{f(x_2)} = 1.$$

Given the quasi-*p*-convexity of f, we obtain

$$f\left(\lambda \frac{x_1}{y_1^{1/p}} + \mu \frac{x_2}{y_2^{1/p}}\right) \le \max\left\{ f\left(\frac{x_1}{y_1^{1/p}}\right), f\left(\frac{x_2}{y_2^{1/p}}\right)\right\},\ \lambda^p + \mu^p = 1$$

$$f\left(\lambda \frac{x_1}{y_1^{1/p}} + \mu \frac{x_2}{y_2^{1/p}}\right) \le 1,\ \lambda^p + \mu^p = 1.$$

Let $\lambda = \left(\frac{y_1}{y_1 + y_2}\right)^{1/p},\ \mu = \left(\frac{y_2}{y_1 + y_2}\right)^{1/p}$, then

$$f\left(\left(\frac{y_1}{y_1+y_2}\right)^{\frac{1}{p}} \frac{x_1}{y_1^{1/p}} + \left(\frac{y_2}{y_1+y_2}\right)^{\frac{1}{p}} \frac{x_2}{y_2^{1/p}}\right) \le 1$$

$$\frac{f(x_1 + x_2)}{y_1 + y_2} \le 1$$

$$\frac{1}{f(x_1) + f(x_2)} f(x_1 + x_2) \le 1$$

$$f(x_1 + x_2) \le f(x_1) + f(x_2).$$

If either x_1 or x_2 is zero, for example $x_1 = 0$, remark 1 show that $f(x_1) = 0$, then

$$f(x_1 + x_2) = f(x_2) = f(x_1) + f(x_2).$$

This completes the proof.

Theorem 3.3 Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$ and $f : U \to R$ be a quasi-*p*-convex function. If $x^* \in U$ is a strict local minimum of f, it is also a strict global minimum of f, and the set U^* of all minimal points of f is *p*-convex set.

Proof. Let $x^* \in U$ be a strict local minimum of f, if x^* is not a strictly global minimum of f, then it exists $\bar{x} \in U : \bar{x} \ne x^*$ such that $f(\bar{x}) \le f(x^*)$.

By using the quasi-*p*-convexity of f, for $\lambda \in [0,1]$, we have

$$f\left(\lambda \bar{x} + (1 - \lambda^p)^{\frac{1}{p}} x^*\right) \le f(x^*). \tag{3.5}$$

It exists small enough λ such that

$$x = \lambda \bar{x} + (1 - \lambda^p)^{\frac{1}{p}} x^* \in U \cap B_\delta(x^*),$$

namely $f(x) \le f(x^*)$, it contradicts that x^* is a strict local minimum of f. So x^* is the strict global minimum of f.

Now assume that $U^* \ne \varnothing$, let m be the minimum value of f on U, it is noticed that

$$U^* = \{x \in U : f(x) = m\} = \{x \in U : f(x) \le m\} = L_{\le m}$$

By the quasi-*p*-convexity of f and Theorem 2.4, the lower level set $L_{\le m}$ is a *p*-convex set. Thus U^* is also a *p*-convex set.

Theorem 3.4 Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$, and $f : U \to R$ be a quasi-*p*-convex function. Then for each $k > 0$, then kf a quasi-*p*-convex function on U.

Proof. For all $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, it can be given by conditions

$$(kf)(\lambda x + \mu y) = kf(\lambda x + \mu y)$$
$$\le k \max\{f(x), f(y)\} = \max\{kf(x), kf(y)\}.$$

Thus kf is a quasi-*p*-convex function.

Corollary 3.1 If the function $f : U \to R$ is a strictly quasi-*p*-convex function, then kf a strictly quasi-*p*-convex function on U.

Proof. It is clear from Theorem 3.4.

Theorem 3.5 Let $U \subseteq R^n$ be a *p*-convex set, $0 < p \le 1$. If $f_i : U \to R$ is quasi-*p*-convex functions for $i = 1, 2, \cdots, m$, then $f = \sum_{i=1}^m a_i f_i$ is a quasi-*p*-convex function where $a_i \ge 0$.

Proof. Let $x, y \in U$, $f_i(x) \geq f_i(y)$, so $f(x) > f(y)$. For $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, we have

$$f(\lambda x + \mu y) = \sum_{i=1}^{m} a_i f_i(\lambda x + \mu y)$$

$$\leq \sum_{i=1}^{m} a_i \max\{f_i(x), f_i(y)\} = \sum_{i=1}^{m} a_i f_i(x) = f(x).$$

This is, f is a quasi-*p*-convex functions.

Theorem 3.6 If function $f: U \to R$ is a quasi-*p*-concave function. For each $k > 0$, then kf a quasi-*p*-concave function on U.

Proof. For all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, it can be given by conditions

$$(kf)(\lambda x + \mu y) = kf(\lambda x + \mu y)$$

$$\geq k \min\{f(x), f(y)\} = \min\{kf(x), kf(y)\}.$$

Thus kf is a quasi-*p*-concave function.

Corollary 3.2 If the function $f: U \to R$ is a strictly quasi-*p*-concave function, then kf a strictly quasi-*p*-concave function on U.

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