Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates

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Abstract

We develop a discrete-time model in which the stock markets of two countries are linked via and with the foreign exchange market. The foreign exchange market is characterized by nonlinear interactions between technical and fundamental traders. Such interactions may generate complex dynamics and recurrent switching between "bull" and "bear" market phases via a well-known pitchfork and period-doubling bifurcation path, when technical traders become more aggressive. The two stock markets are populated by fundamentalists, and prices tend to evolve towards stable steady states, driven by linear laws of motion. A connection between such markets is established by allowing investors to trade abroad, and the resulting three-dimensional dynamical system is analyzed. One goal of our paper is to explore potential spill-over effects between foreign exchange and stock markets. A second, related goal is to study how the bifurcation sequence which characterizes the market with heterogeneous speculators is modified in the presence of interactions with other markets.

Keywords: financial market interactions, nonlinear dynamics and chaos, bifurcation analysis

JEL classification: C61, D84, F31, G15

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1 Introduction

The literature about the dynamics of prices in speculative markets, based on the interaction of boundedly rational heterogeneous agents, has become well developed in recent decades. A considerable portion of this literature has focused, in particular, on the dynamics of financial asset prices. Excellent recent surveys include Hommes (2006), LeBaron (2006) and Lux (2008).

Most models include nonlinear elements. Typically, nonlinearity arises from agents' trading rules or demand functions (e.g. Day and Huang (1990), Chiarella (1992), Rosser et al. (2003)), from evolutionary switching between available strategies, based on certain fitness measures (e.g. Brock and Hommes (1997, 1998)), and from phenomena of contagion and consequent transition of speculators among "optimistic" and "pessimistic" groups (Kirman (1991), Lux (1995, 1997)). Such nonlinearities are of course the mathematical reason for some typical dynamic outcome of these models, such as long-run price oscillations (often characterized by chaotic behavior) around an unstable "fundamental" steady state, the existence of alternative "non-fundamental" equilibria and the emergence of bubbles and crashes.

Within this literature, a large number of models are based on the so-called chartist-fundamentalist approach. We cite, in particular, Day and Huang (1990), Chiarella (1992), Huang and Day (1993), Brock and Hommes (1998), Lux (1998), Chiarella and He (2001), Chiarella et al. (2002), Hommes et al. (2005), De Grauwe and Grimaldi (2005), He and Li (2008). Such models are able to capture - albeit in a stylized way - an important determinant of price fluctuations. Chartists use technical trading rules to forecast prices, and in particular believe in the persistence (and thus exploitability) of bullish and bearish market episodes, and formulate their demands accordingly. In contrast, fundamentalists place their orders by assuming that prices will return towards their "fundamental value", thus playing a stabilizing role in the market. Prices are set as a function of aggregated investors' proposed transactions by assuming market clearing or, more often, the mediation of market makers. Endogenous price dynamics are thus generated by the interplay between the destabilizing forces of technical trading strategies and the mean reverting forces set in action by fundamental traders.

Generally speaking, one central insight of these models is that price movements are at least partially driven by endogenous laws of motion. A second important result is that some of these models have the potential to replicate a number of important stylized facts of financial markets, such as excess volatility, bubbles and crashes, fat tails for the distribution of the returns and volatility clustering. Finally, these models seem to be useful for testing how certain regulatory measures work. For instance, Wieland and Westerhoff (2005) explore the effectiveness of popular central bank intervention strategies, while He and Westerhoff (2005) discuss how price caps may affect the dynamics of commodity markets.

Such literature generally focuses on the dynamics of a single speculative market, driven by the interplay of boundedly rational heterogeneous investors. In particular, most studies of the behavior of asset prices concentrate on the case of a stylized market with a single risky asset and a riskless asset. Recently, the basic ideas have been extended so as to model the dynamics of a market with multiple risky assets, or more generally to explore the dynamics of interacting speculative markets. For instance, Böhm and Wenzelburger (2005) and Chiarella et al. (2005, 2007) establish dynamic setups where prices and returns of multiple risky assets coevolve over time due to dynamic mean-variance portfolio diversification and updating of heterogeneous beliefs. Westerhoff (2005) and Westerhoff and Dieci (2006) model the interactions between different asset markets with fundamental and technical traders, where the connections arise from traders switching between markets, depending on relative profitability. Another related paper is by Corona et al. (2008), in which a model with interacting stock and foreign exchange markets is studied via numerical simulations and calibrated such that it is able to match some statistical properties of actual financial market dynamics. Overall, such models show how interactions may destabilize otherwise stable markets and become a further source of nonlinearity and complex price dynamics, depending on the parameters which characterize agents' behavior. Apart from such initial contributions, however, the dynamic analysis of models of interacting markets within the "heterogeneous agent" approach still remains a largely unexplored research area.

This paper develops and explores a further stylized model of interacting markets, populated by boundedly rational heterogeneous investors, namely the case of two *stock markets* denominated in different currencies, which are linked via and with the related *foreign exchange market*.

The reason of our choice is twofold.

First, there is a sizeable amount of literature about the nature of the connections between stock and foreign exchange markets, but it is characterized by mixed results. For instance, empirical studies on the relationship between stock market returns and exchange rate movements have revealed a certain disagreement about the sign of the correlations or the direction of the causality (see, e.g. Phylaktis and Ravazzolo (2005), Westermann (2004), Granger et al. (2000), and references therein). Moreover, not much research is available on the transmission of volatility between stock markets and currency markets. From a theoretical point of view, there is no consensus about the relationship between stock and exchange rate returns (e.g. “flow-oriented” and “portfolio balance” models predict relationships with different signs). Finally, most of the existing literature takes a long-term macro perspective. We believe that new insight into this topic could be gained by adopting a short-term perspective and focusing on the role of speculative trading within the boundedly rational heterogeneous agent framework. Our contribution - which represents a first step in this direction - in only aimed at illustrating the potential of such an approach, by focusing on a specific model of interacting stock and foreign exchange markets.

Second, and more important, in our simplified model, which obviously neglects several possible channels of interaction between the stock markets and the foreign exchange market, connections arise simply because the trading decisions of stock market traders who are active abroad are based also on expected exchange rate movements, which are influenced by observed exchange rate behavior; in addition, their orders generate transactions of foreign currency and lead to consequent exchange rate adjustments. A model based on such a plain mechanism of interaction is therefore an ideal setup to address the question of potential spillover effects between different speculative markets. In particular, we investigate whether the existence of such connections may contribute to dampening, or amplifying and spreading the price fluctuations which arise in one of the markets due to the interplay of heterogeneous speculators. To do this we assume that - in the absence of interaction - the dynamics in one of the markets (the foreign exchange market) is governed by a chartist-fundamentalist model similar in structure to that developed by Day and Huang (1990). The latter is characterized by a well-known bifurcation route, which has the potential to generate erratic switching between “bull” and “bear” market situations. A link between the three markets is introduced by allowing investors to trade abroad. It turns out that, even in such a simple setup, the role played by market interactions (i.e. whether they are *stabilizing* or rather *destabilizing*) is strongly dependent on some behavioral parameters which govern the intensity of speculative demand. Our findings also allow us to better understand how the bifurcation route described by Day and Huang (1990) is modified in a higher-dimensional model of interconnected markets.

This paper is structured as follows. In Section 2 we describe the details of the model with regard to traders’ demand and price adjustment mechanisms. In particular, Sections 2.1 and 2.2

contain our assumptions about the two stock markets, respectively, whereas Section 2.3 focuses on the exchange rate market. Section 3 describes the resulting three-dimensional, nonlinear dynamical system in discrete-time, and derives analytical results about the steady states of the model and their stability. This is initially done in the case of independent markets (Section 3.1) and subsequently for the full system of interacting markets (Section 3.2). Section 4 discusses the conditions under which market interactions have a stabilizing or destabilizing impact on the dynamics. Such a discussion is based on both the steady-state analysis carried out in Section 3 and the numerical explorations performed in Section 4. Section 5 concludes the paper and suggests possible routes for future research. Mathematical appendices contain the proofs of the propositions and some related discussions.

2 The model

In this section we develop a simple three-dimensional discrete-time dynamic model in which two stock markets (denominated in different currencies) are linked *via* and *with* the foreign exchange market. Let us denote the two stock markets with the superscripts H(ome) and A(broad). In order to highlight the mechanisms by which endogenous dynamics, generated by the interplay of heterogeneous traders, spreads throughout the system of connected markets, we assume that only (national and foreign) fundamental traders are active in each stock market, with fixed proportions. In contrast, we assume the existence of heterogeneous speculators, fundamental traders (or fundamentalists) and technical traders (or chartists), who explicitly focus on the foreign exchange market. Their proportions are assumed to vary over time, depending on market circumstances: the larger the mispricing in the foreign exchange market, the more agents rely on fundamental analysis. For all types of agents, the "beliefs" about future price movements are updated in each period as a function of observed prices.

We focus on a specific mechanism of interaction between such markets. Connections occur in two directions. On the one hand, stock market traders who trade abroad base their demand on both expected stock price and exchange rate movements. On the other hand, they obviously generate transactions of foreign currencies and consequent exchange rate adjustments. For each market, we model the price adjustment process by a log-linear price impact function. Such a function relates the quantity of assets bought or sold in a given time interval and the price change caused by these orders. A similar view is adopted, for instance, in Beja and Goldman (1980), Chiarella (1992) and Farmer and Joshi (2002).

Let us now describe in detail what occurs in each market.

2.1 The stock market in country H

Let us start with a description of the stock market in country H. According to the assumed price impact function, the log stock price (P_t^H) adjustment from time t to time $t+1$ in country H may be expressed as

$$P_{t+1}^H = P_t^H + a^H(D_{F,t}^{HH} + D_{F,t}^{HA}), \tag{1}$$

where a^H is a positive price adjustment parameter and $D_{F,t}^{HH}$, $D_{F,t}^{HA}$ stand for the orders of fundamental traders from countries H and A investing in country H, respectively. Accordingly, if buying (selling) exceeds selling (buying), prices go up (down). The price setting rule (1) may also be interpreted as the stylized behavior of a risk-neutral market maker, who aggregates agents' proposed transactions, clears the market by taking an offsetting long or short position, and then adjusts the price for the next period as a function of excess demand.

The orders placed by fundamental traders (or fundamentalists) from country H are given as

$$D_{F,t}^{HH} = b^H(F^H - P_t^H), \tag{2}$$

where b^H is a positive reaction parameter and F^H is the log fundamental value of stock H. Fundamentalists seek to profit from mean reversion. Hence, these traders submit buying orders when the market is undervalued (and vice versa).

Fundamental traders from abroad may benefit from a price correction in the stock market as well as in the foreign exchange market. The log fundamental value of the exchange rate is denoted by F^S and the log exchange rate by S. Their orders may thus be written as

$$D_{F,t}^{HA} = c^H(F^H - P_t^H + F^S - S_t), \tag{3}$$

where $c^H \geq 0$. Suppose, for instance, that both the stock market and the foreign exchange market are undervalued. Then the foreign fundamentalists take a larger buying position than the national fundamentalists (assuming equal reaction parameters). Should, however, the foreign exchange market be overvalued, then the foreign fundamentalists become more cautious and may even enter a selling position.

2.2 The stock market in country A

Let us now turn to the stock market in country A. We have a set of equations similar to those for stock market H. The log price adjustment is expressed as

$$P_{t+1}^{A} = P_{t}^{A} + a^{A}(D_{F,t}^{AA} + D_{F,t}^{AH}), \tag{4}$$

where $a^A > 0$. The orders of the fundamentalists from country A investing in stock market A amount to

$$D_{F,t}^{AA} = b^{A}(F^{A} - P_{t}^{A}), \tag{5}$$

where $b^A > 0$ and F^A denotes the log-fundamental price of stock market A. The orders of fundamentalists from country H investing in stock market A result in

$$D_{F,t}^{AH} = c^{A}(F^{A} - P_{t}^{A} + S_t - F^{S}), \tag{6}$$

where $c^A \geq 0$. Apart from the notation, the only obvious difference to the case described in the previous section is that here agents take the inverse exchange rates into account. The quantity $-S_t = \ln(1/\exp(S_t))$ is the log of the reciprocal value of the exchange rate, and similarly $-F^S$ is the logarithm of the inverse fundamental rate.

2.3 The foreign exchange market

In the foreign exchange market, the excess demand consists of orders placed by stock traders who are active abroad and by foreign exchange speculators. The latter group of agents switch between technical and fundamental trading strategies, depending on market conditions. Note first that here we focus on the demand of currency H and we define the exchange rate as the price of one unit of currency H in terms of currency A.[1] An increase in the exchange rate thus means an appreciation

[1] This convention is referred to as "inverse quotation" in the literature.

of currency H. The log exchange rate at time step $t+1$ is determined as

$$S_{t+1} = S_t + d\left[\exp(P_t^H)D_{F,t}^{HA} - \frac{\exp(P_t^A)}{\exp(S_t)}D_{F,t}^{AH} + W_{C,t}D_{C,t}^S + (1-W_{C,t})D_{F,t}^S\right], \tag{7}$$

where d is a positive price adjustment parameter. According to equation (7), the log exchange rate adjustment is proportional to excess demand of currency H, determined by both stock orders of agents who trade abroad and speculative demand in the foreign exchange market. The first two terms in brackets on the right-hand side of equation (7) express the demand generated by stock traders. It is important to note that the stock orders are given in real units. The demand for currency (H or A) of these traders is the product of stock orders times stock prices; in particular, the demand for currency A from traders H investing in stock A, $\exp(P_t^A)D_{F,t}^{AH}$, generates a demand for currency H of the opposite sign, the amount of which is obtained by multiplying the above quantity by the inverse exchange rate.[2] The quantities $D_{C,t}^S$ and $D_{F,t}^S$ denote the orders generated by technical and fundamental foreign exchange speculators, while $W_{C,t}$ and $(1-W_{C,t})$ denote their market shares, respectively.

As in Day and Huang (1990), we assume that orders placed by chartists may be formalized as

$$D_{C,t}^S = e(S_t - F^S), \tag{8}$$

where $e > 0$. According to equation (8), chartists believe in the persistence of a "bull" ("bear") market, and they therefore optimistically buy (pessimistically sell) as long as this is observed. Note that parameter e governs chartists' confidence in the persistence of deviations from fundamentals, and consequently the "intensity" of their speculative demand. This behavioral parameter will be proven to play a crucial role in the following dynamic analysis.

By contrast, fundamentalists seek to exploit misalignments and formulate their demand according to

$$D_{F,t}^S = f(F^S - S_t), \tag{9}$$

where $f > 0$.[3] Following He and Westerhoff (2005), we assume that speculators switch between

[2] Note that this introduces an additional nonlinearity in our model. Dieci and Westerhoff (2008) find that if there are also trend extrapolating chartists in the stock markets (and no speculators in the foreign exchange market) then this "price-quantity" nonlinearity may even be sufficient to create endogenous motion.

[3] Parameters e, f, together with all other parameters governing agents' demand in the stock markets (b^H, c^H, b^A, c^A) may depend, in general, on traders' beliefs about the speed of price correction, the number of traders of each type, their risk aversion, etc.

these two trading rules with respect to market circumstances. The proportion of technical traders is defined as

$$W_{C,t} = \frac{1}{1 + g(F^S - S_t)^2}, \tag{10}$$

which implies that it decreases as the mispricing in the foreign exchange market increases. The rationale for equation (10) is as follows. The more the exchange rate deviates from its fundamental value, the greater the speculators perceive the risk that the bull or bear market might collapse. Hence, fundamental analysis gains in popularity at the expense of technical analysis. Parameter $g > 0$ is a sensitivity parameter. The higher g is, the more sensitive the mass of speculators becomes with regard to a given misalignment. Note that a weighting mechanism similar to (10) has also been assumed in De Grauwe et al. (1993), although based on different arguments and within a quite different model setup.

3 Dynamical system

Equations (1), (4), and (7), which model the price adjustments, combined with equations (2), (3), (5), (6), (8), (9), and (10), which fix the excess demand of traders in the three markets, result in a three-dimensional discrete-time dynamical system with the following structure

$$P^H_{t+1} = G^H(P^H_t, S_t), \quad P^A_{t+1} = G^A(P^A_t, S_t), \quad S_{t+1} = G^S(P^H_t, P^A_t, S_t). \tag{11}$$

Components G^H, G^A and G^S of the map $\mathcal{G} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, which determines the iteration of the system, are expressed respectively as (we omit the time index)

$$G^H(P^H, S) = P^H + a^H(D^{HH}_F + D^{HA}_F), \tag{12}$$

$$G^A(P^A, S) = P^A + a^A(D^{AA}_F + D^{AH}_F), \tag{13}$$

$$G^S(P^H, P^A, S) = S + d\left[\exp(P^H)D^{HA}_F - \frac{\exp(P^A)}{\exp(S)}D^{AH}_F + W_C D^S_C + (1 - W_C)D^S_F\right], \tag{14}$$

where

$$D_F^{HH} = b^H(F^H - P^H), \quad D_F^{HA} = c^H(F^H - P^H + F^S - S),$$
$$D_F^{AA} = b^A(F^A - P^A), \quad D_F^{AH} = c^A(F^A - P^A + S - F^S),$$
$$D_C^S = e(S - F^S), \quad D_F^S = f(F^S - S), \quad W_C = \frac{1}{1 + g(F^S - S)^2}.$$

Note first that equations (12) and (13), which govern stock price adjustments, are linear in the state variables P^H, P^A, and S, while the exchange rate equation (14) is nonlinear due to both the state-dependent weight W_C and the structure of the demand for currency H from stock market traders. A second important remark concerns the role played by parameters c^H and c^A, which determine the strength of interactions between the markets. In the particular case where $c^H = c^A = 0$, the three equations of the system are decoupled, and each market evolves as an independent one-dimensional system. Note that from the point of view of dynamic analysis, two "intermediate" cases exist, where either $c^H = 0$, or $c^A = 0$. In such cases, two of the three equations evolve as an independent two-dimensional system, which makes the model much more tractable analytically than in the full $3D$ case. Analysis of these cases is left to future research. In this paper we take the one-dimensional case $c^H = c^A = 0$ as a starting "reference" case. This will be developed in detail in the next section, before considering the dynamic behavior of the full three-dimensional system in Section 3.2.

3.1 The case of no interactions

In this section we set $c^H = c^A = 0$, which represents the case where no agents trade abroad. The dynamical system takes the following simplified form

$$P_{t+1}^H = P_t^H + a^H b^H (F^H - P_t^H), \tag{15}$$

$$P_{t+1}^A = P_t^A + a^A b^A (F^A - P_t^A), \tag{16}$$

$$S_{t+1} = S_t + d\frac{(S_t - F^S)\left[e - fg(S_t - F^S)^2\right]}{1 + g(S_t - F^S)^2}, \tag{17}$$

i.e. it is described by three independent first-order difference equations. Each stock market is represented by a one-dimensional linear dynamical system. It is easy to check that the fundamental price represents the unique steady state in each stock market, i.e.[4] $\overline{P}^H = F^H$, $\overline{P}^A = F^A$, and that

[4] An overbar denotes steady-state levels for the dynamic variables.

such steady states are globally asymptotically stable, provided that traders or prices do not react too strong, namely $a^H b^H < 2$, $a^A b^A < 2$, respectively. This parameter restriction - which ensures "stable" stock markets in the absence of connections with the foreign exchange market - will be assumed in the rest of the present paper. In contrast, the foreign exchange market evolves according to a one-dimensional nonlinear dynamical system, governed by an equation of "cubic" type. This gives rise to multiple steady states, whose properties are stated in the following

Proposition 1 ***(a)*** *The one-dimensional dynamical system (17) always admits three steady states, the fundamental steady state,* $\overline{S} = F^S$*, and two nonfundamental steady states*

$$\overline{S}_l = F^S - \sqrt{\frac{e}{fg}}, \quad \overline{S}_u = F^S + \sqrt{\frac{e}{fg}} \ ,$$

located in symmetric positions below and above F^S*, respectively.*

(b) *The fundamental steady state is always unstable. If* $df \leq 1$*, the nonfundamental steady states are locally asymptotically stable (LAS), whereas if* $df > 1$ *they are LAS only for* $0 < e < e_{Flip} := f/(df - 1)$*, at which parameter value a period doubling bifurcation occurs.*

Proof. See Appendix *A*.

According to Proposition 1, two non-fundamental steady states exist on either side of the unstable fundamental equilibrium. Such steady states are LAS at least for small values of the exchange rate adjustment parameter, d. If exchange rates adjust strongly to excess demand, the nonfundamental steady states can undergo a Flip bifurcation if parameter e, which captures the intensity of the chartists' speculative demand, is large enough. *Fig. 1* reports bifurcation diagrams associated with the qualitative cases $df \leq 1$ (panels a, b) and $df > 1$ (panels c, d). The chartist parameter e is the bifurcation parameter. For each parameter configuration, the diagrams on the same line report the attractors corresponding to two different initial conditions, one above and one below the fundamental. In the first case (a, b), the attractors are locally stable steady states which do not change qualitatively as e becomes larger, but only increasingly deviate from the fundamental. In the second case (c, d), both steady states lose stability for $e = e_{Flip}$ and are replaced by stable orbits of period 2, which is then followed by a sequence of period-doubling bifurcations and transition to chaos. This sequence is very similar to that illustrated by Day and Huang (1990) in their well-known one-dimensional stylized model of an asset market with heterogeneous investors and a market maker. The periodic orbits, or chaotic intervals, resulting from this sequence of Flip bifurcations are located either above or below the fundamental steady state, depending on the initial condition. At first, chaotic dynamics take place either in the "bull" or the "bear" market region, which are therefore disjoint *trapping*[5] regions, but at some point, log-exchange rates start to wander across both regions: the bifurcation diagrams show a drastic enlargement of the chaotic interval, and the dynamics is characterized by intricate price fluctuations and erratic switching between bull and bear market episodes (panel e). The latter phenomenon is due to a *homoclinic bifurcation* of the repelling fundamental steady state. Without going into details about such bifurcations, here we simply provide a graphical visualization in panel *f*, which represents the map (17) for a particular parameter setting for which homoclinic bifurcation occurs. This kind of bifurcation is strictly related to the noninvertibility of the map, which is characterized by two "critical points", one local maximum and one local minimum. This fact enables a repelling steady state to have further preimages, apart from itself. In the one-dimensional case, such a homoclinic bifurcation occurs precisely at the parameter value for which one of these preimages is a critical point of the map. This is indicated by the arrows in panel *f*. This bifurcation, together with the symmetry[6] of the $1D$ map (17) with respect to the fundamental steady state, determines the "merging" of two disjoint trapping intervals into a unique interval (see Dieci, Bischi and Gardini (2001) and He and Westerhoff (2005) for the analysis of this type of bifurcation arising from one-dimensional economic examples).

***** Fig. 1 approximately here *****

3.2 The case of interacting markets

We now analyze the full system, which is characterized by the existence of stock market traders who trade abroad. This means that at least one of the parameters c^H, c^A is strictly positive. The present section explores in depth the effect of interaction, compared with the results illustrated in the previous section in the benchmark case of independent markets. In particular the following questions are addressed, which arise quite naturally within this model.

The first question concerns the "destabilizing" or "stabilizing" role played by market interac-

[5] I.e. each region is mapped into itself under iteration of (17).

[6] Note that symmetric points (with respect to F^S) are mapped onto symmetric points under iteration of (17). As a consequence, either an attractor is symmetric with respect to F^S or it admits a symmetric attractor with the same stability properties. Note that such a symmetry property is generally lost when switching to the $3D$ model of interacting markets, expressed by (12)-(14).

tions. In particular, we try *(i)* to understand the effect of such interactions on the fundamental equilibrium and its stability in all three markets, *(ii)* to explore the conditions under which further stable equilibria exist, *(iii)* to investigate how far these additional equilibria may deviate from the fundamental value, and *(iv)* to study whether market interactions contribute to an amplification or dampening of price fluctuations around the unstable fundamental steady state, compared with the case of independent markets.

The second issue concerns the persistence of the bifurcation structure described and discussed in the (one-dimensional) case of no interactions. Namely, we try to understand whether such a structure also "survives" in the (three-dimensional) case of interacting markets. Leaving a rigorous analysis to future research, here we simply aim at providing numerical evidence of the existence of a bifurcation sequence similar in quality to that described in the previous section, and in particular of the homoclinic bifurcation that marks the transition to a regime of erratic switching betweeen bull and bear market phases.

The following Proposition concerns the steady states of the full model. In order to simplify the notation, we introduce the deviations from fundamentals, $x^H := P^H - F^H$, $x^A := P^A - F^A$, $x := S - F^S$, and express the steady states accordingly. We also define $\Phi^H := \exp(F^H)$, $\Phi^A := \exp(F^A)$, $\Phi^S := \exp(F^S)$.

Proposition 2 ***(a)*** *The steady states of the three-dimensional system of interacting markets (11) are given by points $(\overline{x}^H, \overline{x}^A, \overline{x})$ satisfying*

$$\overline{x}^H = -\frac{c^H}{b^H + c^H}\overline{x}, \quad \overline{x}^A = \frac{c^A}{b^A + c^A}\overline{x}, \tag{18}$$

and such that $\overline{x}$ solves

$$x\ \alpha(x) = x\ \beta(x), \tag{19}$$

where

$$\alpha(x) = \frac{e - fgx^2}{1 + gx^2}, \quad \beta(x) = \frac{\Phi^H b^H c^H}{b^H + c^H}\exp\left(-\frac{c^H}{b^H + c^H}x\right) + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A}\exp\left(-\frac{b^A}{b^A + c^A}x\right). \tag{20}$$

(b) *A unique (fundamental) steady state exists ($\overline{x}^H = \overline{x}^A = \overline{x} = 0$) if the chartist parameter e is sufficiently small, or if interactions are sufficiently strong (i.e. for large c^H, c^A), whereas if e is large enough or interactions are sufficiently weak (small c^H, c^A) two further (non-fundamental)* *steady states exist.*

Proof. See Appendix B.

As discussed in Appendix B, *Fig. 2* represents the possible solutions to equation $\alpha(x) = \beta(x)$. The previously discussed situation of no interactions, $c^H = c^A = 0$, corresponds to the case in which $\beta(x) \equiv 0$, where the two curves intersect at the symmetric points $\mp\sqrt{e/(fg)}$. In the case where either c^H or c^A is strictly positive, elementary geometrical considerations suggest that the two curves $\alpha(x)$ and $\beta(x)$ will still intersect each other provided that $\alpha(0) = e$ is sufficiently large with respect to $\beta(0)$, where the latter quantity depends negatively on parameters c^H and c^A. In this case, further steady states exist and this obviously implies the existence of non-fundamental equilibrium prices in all markets, via equations (18). Note, however, that when further steady states exist, their deviation from the fundamental (in the foreign exchange market) is less pronounced than in the case of the absence of interactions.

***** Fig. 2 approximately here *****

To summarize, by establishing a connection between the markets, we obtain a steady-state structure, which can be regarded as "intermediate" between those of the original, independent systems. Which of the original situations prevails in the case of interacting markets depends on the parameters of the model. In particular, we focus on the role played by the chartist parameter e. Under a sufficiently strong chartist extrapolation in the foreign exchange market, the system (and therefore also the two stock markets) will display a structure with multiple equilibrium prices, which is "inherited" from the original structure of the foreign exchange market. In contrast, if chartists extrapolate weakly, the opposite effect takes place, and the structure with a unique stationary state prevails (also in the foreign exchange market). Moreover, the full $3D$ model has such a mixed behavior, also with regard to the stability of the steady states. Roughly speaking, for sufficiently low chartist parameter e, the unique steady state of the full system will also be LAS, as occurs in the (independent) stock markets. This will be specified more precisely by the next Proposition. Moreover, for sufficiently large values of e, two stable non-fundamental steady states will themselves become unstable, and will be replaced by more complex attractors, as in the (independent) foreign exchange market. This will be shown by numerical examples in Section 4. The following Proposition (which is proven in Appendix C) concerns the local stability of the fundamental steady state.

Proposition 3 *Assume that the adjustment and reaction parameters in the stock markets satisfy*

$$a^H(b^H + c^H) < 2, \quad a^A(b^A + c^A) < 2. \tag{21}$$

Then if parameters d, e, are sufficiently small, the unique (fundamental) steady state of the dynamical system (11) is LAS.

Proof. see appendix C.

Proposition 3 basically proves that when the price adjustment parameters to excess buying/selling in the three markets (a^H, a^A, d) are sufficiently small, and when chartist speculation (e) is not too strong, the exchange rate market is stabilized by the connections with stable stock markets, such that an unstable fundamental steady state (among two further non-fundamental equilibia) becomes locally (or even globally) asymptotically stable.

In contrast, for large values of e, two locally stable non-fundamental steady states undergo sequences of bifurcations similar to that illustrated for the one-dimensional independent foreign exchange market in *Figs. 1c,d.* This will be shown in the next section, which contains a broader discussion of the stabilizing/destabilizing role of market interactions.

4 Stabilizing and destabilizing effects of interactions: numerical examples

This section contains numerical examples which illustrate the dynamic behavior of the model and discusses, in particular, the global bifurcations occurring for increasing values of parameter e, which governs the strength of chartists' speculation.

First of all it will be shown that in the case of interacting markets large values of e result in a "stronger instability" than in the case of independent markets, and produces an enlargement of the range of fluctuations. This effect is therefore totally different from the stabilizing impact proven analytically (and observed numerically) for small values of e.

Second, it will be shown that, by taking e as a bifurcation parameter, the full three-dimensional model undergoes a sequence of bifurcations which is similar to that observed for the exchange rate market in the one-dimensional case. In particular, as shown by our analysis, one can easily also detect in the $3D$ case the effects of a homoclinic bifurcation of the fundamental steady state. Such a bifurcation determines the merging of two disjoint "trapping" regions of the phase space, thus giving rise to the typical erratic switching between bull and bear market phases (first described by Day and Huang (1990)), similarly to the one-dimensional case discussed in Section 3.1.

Throughout the examples of this section we use the following common parameter setting: $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $d = 1$, $f = 0.8$, $g = 10000$, $\Phi^H = \Phi^A = \Phi^S = 1$ (so that the log fundamental prices F^H, F^A, F^S are all equal to zero); the remaining parameters c^H, c^A, and e may vary across different examples. Note that parameters d, f, g, and F^S are precisely those used in *Figs. 1a,b*: this means that under the assumed parameter setting, in the absence of interactions, the exchange rate market would be characterized by two coexisting and locally stable non-fundamental steady states for any $e > 0$.

Fig. 3 represents bifurcation diagrams for each of the three dynamic variables P^H, P^A, and S, versus the chartist parameter e. In each panel the asymptotic behaviour in a case with market interactions (with $c^H = c^A = 0.4$) is compared with the corresponding situation with no interactions ($c^H = c^A = 0$, gray dashed line). Thus the figure reports the effect of introducing connections of a certain intensity for different values of e. The three panels located on the right are obtained with a different initial condition from those on the left. Note also that parameters a^H, a^A, b^H, b^A, c^H, c^A, d, Φ^H, Φ^A, and Φ^S (i.e. those that play a role for the linearized system around the fundamental steady state), satisfy all of the restrictions we imposed in Appendix C to derive analytical results about local stability.[7] For a range of low values of parameter e, the stabilizing effect of interactions is clear from the bifurcation diagrams, which confirms our local stability results. As a matter of fact, while the situation remains unchanged in the two stock markets, in the foreign exchange market it changes from the coexistence of two LAS nonfundamental steady states (which surround an unstable fundamental equilibrium) to a unique stable fundamental equilibrium. If we now increase parameter e (in the case $c^H = c^A = 0.4$), the latter loses stability for $e = \beta(0) \simeq 0.6015$. The effect of such a bifurcation is a "pitchfork" scenario[8]: for a certain range of e the phase space is characterized by the coexistence of two stable equilibria that surround the unstable fundamental steady state. For this reason, for such a parameter range the two stock markets are destabilized with respect to the situation of no interactions, and steady state prices deviate from fundamentals. Note, however, that in the foreign exchange market the steady state deviation from the fundamental exchange

[7] One can easily check that conditions (42), (43), (44) are satisfied for any $e > 0$, while condition (41) holds only for $e < e^* \simeq 0.6015$.

[8] Although this is not revealed by the plots in *Fig. 3*, such a bifurcation occurs via a slightly more complicated mechanism than a pitchfork bifurcation, as further discussed in Appendix B.

rate is less pronounced than in the case of independent markets, i.e. a kind of "stabilizing effect" is still at work here. Larger values of e bring about the sequence of period-doubling bifurcations already reported in the one-dimensional case. Within this range of e we can say that all three markets are destabilized with respect to the situation of decoupled dynamics. In particular, for $e \simeq 4.856$ the diagram reports a sudden, drastic enlargement of the chaotic region where asymptotic fluctuations are confined. Such a region "merges" with a coexisting trapping region of the phase space, associated with the second non-fundamental steady state involved in a similar bifurcation sequence. This phenomenon will be further discussed below.

***** Fig. 3 approximately here *****

In *Fig. 4*, the effect of interactions is analyzed from a slighty different perspective. In the upper group of panels we choose a large value of e ($e = 6$). We also set $c^A = 0.2$ and increase c^H. Since this represents the parameter that governs the demand for stock in country H by fundamentalists from country A, we are thus increasing the strength of interactions from A to H. By doing this, we notice a transition to increasingly complex dynamics, that is a "destabilizing effect" similar to that already observed in *Fig. 3*. The results are reported in panels a, b and c, d for log stock price H and the log exchange rate, respectively (the left and right panels are characterized by different initial conditions). By contrast, in the lower panels we select a small value of e ($e = 0.5$). We also fix $c^A = 0.4$ and increase c^H again. In this case we report the opposite effect, i.e. increasing the strength of interactions stabilizes the system (see panels e, f and g, h for P^H and S, respectively). To summarize, the nature of the impact of market interactions (stabilizing or destabilizing) is determined by the level of a crucial behavioral parameter that governs the speculative behavior of the chartists and is strictly related to their confidence in the persistence of deviations from fundamentals.

***** Fig. 4 approximately here *****

Put differently, the fully integrated $3D$ model is able to display the characteristic behavior of each of the starting (independent) markets for different ranges of parameter e. Quite interestingly, as already anticipated by the bifurcation diagrams in *Fig. 3*, even the homoclinic bifurcation of the steady state reported in the one-dimensional foreign exchange market (*Fig. 1c,d*) survives almost identical in the $3D$ model. Here, two attractors lying in two disjoint regions of the three-dimensional phase-space merge into a unique attractor, thus determining a major qualitative change of the dynamics. The situations before and after the bifurcation are represented in *Fig. 5*. Panels a, c and e report the projections of the attractors in the plane of the state variables P^H and S. Before the bifurcation, there are two coexisting attractors (panels a, c) and the asymptotic dynamics of the system depends on the initial state. After the bifurcation the two attractors merge into a unique attractor (panel e). Obviously, this situation is the higher dimensional equivalent of the merging of two coexisting disjoint intervals in *Fig. 1c,d*. Panels b, d and f represent the dynamics of the exchange rate in the time domain before and after bifurcation. While initially the dynamics take place in a specified (bull or bear) market region, depending on the initial condition (panels b, d), after bifurcation the dynamics covers both regions, but still switches between the two pre-existing regions at seemingly unpredictable points in time, thus evolving through a series of bubbles and crashes (panel f).

***** Fig. 5 approximately here *****

Under the same parameter configuration as *Figs. 5e,f*, *Fig 6* (panels a, b, c) represent the time series of all state variables, and shows that the stock prices also jump back and forth between “bull” and “bear” market episodes, triggered by exchange rate fluctuations. Finally, panel d plots in the plane (S_t, S_{t+1}) a trajectory obtained using the same parameters (and with a large number of iterations). If there are no interactions, such a plot would exactly reproduce the $1D$ map underlying the dynamical system (17) (as in *Fig. 1f*). Since markets interact and there is hence feedback from the stock markets to the foreign exchange market, the plot in panel d does not reduce exactly to such a cubic curve. Put differently, the stock markets create some kind of “deterministic noise” for the exchange rate process. Note, however, that the two pictures in panel d and *Fig. 1f* are very similar to each other. Far from being rigorous, we may argue that this fact has something to do with the persistence in the $3D$ model of the original bifurcation structure of the $1D$ model, and of the characteristic bull and bear price dynamics.

***** Fig. 6 approximately here *****

Although we have discussed such phenomena using a particular parameter setting, they can easily be detected for a wide region of the parameter space. We do not push ahead with the analysis of such bifurcation mechanisms here, but leave further exploration to future research. As already discussed in Section 3.1, in the one-dimensional system (17) such phenomena are due to a homoclinic bifurcation of the repelling fundamental steady state, strictly related to the fact that the map is noninvertible, with two "critical points" (one local minimum and one local maximum). In the $1D$ case much can be said about this bifurcation, on analytical grounds. Of course, a similar analysis in the $3D$ case seems to be impossible: apart from the higher dimension of the dynamical system, the $3D$ map $\mathcal{G}$ defined by (12)-(14) is no longer symmetric with respect to the fundamental steady state. However, as already noted, when one of the two parameters c^H or c^A is equal to zero, and the other is strictly positive, then two of the three state variables evolve as an independent two-dimensional system, and in this case it is possible to understand several dynamic phenomena of the full $3D$ system by in fact studying a two-dimensional model. In this case computer-assisted proofs of this homoclinic bifurcation can be provided, based on the analytical properties of the so-called "critical curves" of non-invertible maps of the plane (see Mira et al. (1996)), which represents the two-dimensional analogue of the critical points for one-dimensional maps. An initial study in this direction is provided by Dieci, Gardini, Tramontana and Westerhoff (2008).

5 Conclusions

Financial markets are characterized by highly volatile prices and repeatedly display severe bubbles and crashes. The chartist-fundamentalist approach offers a number of endogenous explanations for these challenging phenomena, by stressing the interplay between the destabilizing impact of technical trading strategies and the mean reverting price behavior set in action by fundamentalists. The goal of our paper is to analyse the effect of such a basic determinant of price fluctuations - and in particular to explore the emergence of bull and bear market dynamics - within a system of internationally connected financial markets. This is achieved by studying a stylized deterministic dynamic model of the interactions between stock and foreign exchange markets. In our model the two stock markets are nonlinearly interwoven - by construction - *via* and *with* the foreign exchange market. Such connections are only due to the existence of stock market traders who trade abroad: their orders are based on both price and exchange rate misalignments, and their demand for foreign assets also triggers exchange rate adjustments. While the stock markets are modelled as simply as possible by means of linear equations, the foreign exchange market "works" in a nonlinear way. The reason for this nonlinearity is that foreign exchange speculators switch between competing linear trading rules. The model results in a three-dimensional discrete-time dynamical system. The focus of our analysis is on how exchange rate movements generated by the interplay of heterogeneous speculators in the foreign exchange market spill over into the stock markets, and feed back again into the foreign exchange market.

Beyond the specific example we have chosen to analyze the effect of market interactions, this paper addresses the question of how the behavior of "stable" markets, in which prices are close to their fundamental values, may be affected by connections to an "unstable" market, characterized by systematic deviations from the fundamental, the interplay between heterogeneous speculators and, in particular, the destabilizing action of chartists who bet on the persistence of bull or bear market dynamics. One may wonder which of the original situations prevails once the connections have been introduced, or whether the behavior of the resulting integrated system stays at some intermediate level. We have reported "mixed" results, despite the simplicity of the model. We have shown - by means of both analytical study and numerical experiments - that market interactions may destabilize stock markets, but may also play a stabilizing effect on the foreign exchange market and on the whole system of interacting markets. The nature of the effect is strictly related to the parameters of the model, in particular to parameter e, which governs the speculative demand of chartists. Our findings on the impact of market interactions may be summarized as follows.

- The analytical results about steady states and their stability (Section 3.2) reveal a possible stabilizing effect of market interactions for low values of the price adjustment parameters and of the chartist parameter e. Although in the model with independent markets the fundamental steady state is always unstable and coexists with further non-fundamental steady states (which are LAS for sufficiently small values of e), the model with interacting markets displays a unique stable fundamental steady state, at least when e is sufficiently small. Put differently, if the strength of chartist speculation in the foreign exchange market is not too strong, establishing a connection with stable markets results in the stabilization of the whole system of interacting markets, with prices settling down on fundamentals in the two stock markets and the foreign exchange market. Apart from our analytical findings, this stabilizing effect is also confirmed by the numerical experiments performed in Section 4 (in particular the leftmost part of the bifurcation diagrams versus e, in *Fig. 3*, and the diagrams in *Figs. 4e-h*, where e is fixed at a small value and the strength of interaction in increased).
- For larger values of e, two further non-fundamental steady states exist. This represents a case

where market interactions have a mixed effect on the steady state structure. One the one hand, interactions destabilize the stock markets, where a unique globally stable fundamental steady state is replaced by two locally stable non-fundamental steady states. In other words, if speculation in the foreign exchange market is sufficiently strong, connections with stable markets are no longer able to bring the whole system back to the fundamentals. On the contrary, they destabilize the stock markets, too, meaning that prices tend to deviate from the fundamentals in the long run. On the other hand, in the foreign exchange market a kind of stabilizing effect is still in action, at least as long as e is not too high, in the sense that steady-state misalignments from the fundamental exchange rate are less pronounced than in the case of decoupled markets (see the middle part of the bifurcation diagrams in *Figs. 3e,f*).

- For large values of e, numerical simulation reveals a destabilizing effect of interactions. In both stock markets and the foreign exchange market, previously stable (fundamental or non-fundamental) steady states are replaced by periodic motion or complex endogenous dynamics. The possible deviations from the fundamentals are wider than in the case of no interaction. The amplitude of fluctuations increases with e (as shown in the rightmost part of the bifurcation diagrams in *Fig. 3*) and with the "strength" of interactions (one example is the bifurcation diagrams in *Figs. 4a-d*). As long as e remains within a given range, fluctuations occur either in the bull or in the bear market regions (depending on initial conditions). Afterwards, the interval in which asymptotic fluctuations of prices takes place is drastically enlarged, as an effect of a global bifurcation occurring for large e. This brings about a typical alternance of "bull and bear" market dynamics, with repeating bubbles and crashes around fundamentals in both the foreign exchange and the stock markets. Markets that contain neither technical traders nor behavioral nonlinearities may then switch between bull and bear episodes, due to quite natural interaction with more speculative markets. Our numerical analysis suggests that the emergence of such a price behavior may be due to bifurcation mechanisms strictly related to those governing similar phenomena for one-dimensional systems.

The model explored here is not only an exercise to address general questions or an occasion to study particular global bifurcations in a higher dimensional context. It may also be interesting *per se* in that it captures the way in which a characteristic mechanism of interaction between stock markets and the foreign exchange market may contribute to how price fluctuations are spread or absorbed. Of course our choice to make endogenous motion start from the foreign exchange market is purely conventional, and the same results could be obtained as well by assuming that heterogeneous speculators operate in one of the two stock markets. Our model thus indicates that stock market volatility (or exchange rate volatility) may be caused to some extent by exchange rate changes (or stock price movements), and suggests how such volatility may be related to the strength of interactions and the intensity of speculative demand. An immediate generalization of our setup would be that of introducing heterogeneous investors in all markets, together with exogenous noise on agents' demand and fundamental prices, in order to conduct a more thorough analysis on "how much" of the price volatility in each market can be ascribed to the link with other speculative markets. A further and related interesting extension is to explore the impact of regulatory measures, such as financial market liberalizations and central bank interventions, under different assumptions about agents' behavioral parameters. This can be done, for instance, along the lines of Wieland and Westerhoff (2005).

References

Beja, A. and Goldman, M.B. (1980): On the dynamic behavior of prices in disequilibrium. Journal of Finance, 35, 235-248.

Böhm, V. and Wenzelburger, J. (2005): On the performance of efficient portfolios. Journal of Economic Dynamics and Control, 29, 721-740.

Brock, W. and Hommes, C.H. (1997): A rational route to randomness. Econometrica, 65, 1059-1095.

Brock, W. and Hommes, C.H. (1998): Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics Control, 22, 1235-1274.

Chiarella, C. (1992): The dynamics of speculative behaviour. Annals of Operations Research, 37, 101-123.

Chiarella, C., Dieci, R. and Gardini, L. (2002): Speculative behaviour and complex asset price dynamics: A global analysis. Journal of Economic Behavior and Organization, 49, 173-197.

Chiarella, C., Dieci, R. and Gardini, L. (2005): The dynamic interaction of speculation and diversification. Applied Mathematical Finance, 12, 17-52.

Chiarella, C., Dieci, R. and He, X.-Z. (2007): Heterogeneous expectations and speculative behaviour in a dynamic multi-asset framework. Journal of Economic Behavior and Organization, 62, 408–427.

Chiarella, C. and He, X.-Z. (2001): Asset price and wealth dynamics under heterogeneous expectations. Quantitative Finance, 1, 509-526.

Corona, E., Ecca, S., Marchesi, M. and Setzu, A. (2008): The interplay between two stock markets and a related foreign exchange market: a simulation approach. Computational Economics, in press.

Day, R. and Huang, W. (1990): Bulls, bears and market sheep. Journal of Economic Behavior and Organization, 14, 299-329.

De Grauwe, P., Dewachter, H. and Embrechts, M. (1993): Exchange rate theories. Chaotic models of the foreign exchange market. Blackwell, Oxford.

De Grauwe, P. and Grimaldi, M. (2005): Heterogeneity of agents, transactions costs and the exchange rate. Journal of Economic Dynamics and Control, 29, 691-719.

Dieci, R., Bischi, G.I. and Gardini, L. (2001): From bi-stability to chaotic oscillations in a macroeconomic model. Chaos, Solitons and Fractals, 12, 805-822.

Dieci, R., Gardini, L., Tramontana, F. and Westerhoff, F. (2008): The emergence of "bull and bear" dynamics in a nonlinear $3D$ model of interacting markets. Working paper, University of Urbino.

Dieci, R. and Westerhoff, F. (2008): Nonlinear interaction between stock prices and exchange rates. Working Paper, University of Bologna.

Farebrother, R.W. (1973): Simplified Samuelson conditions for cubic and quartic equations. Manchester School of Economics and Social Studies, 41, 396-400.

Farmer, J. D. and Joshi, S. (2002): The price dynamics of common trading strategies. Journal of Economic Behavior and Organization, 49, 149-171.

Granger, C.W.J., Huang, B.-N. and Yang, C.-W. (2000): A bivariate causality between stock prices and exchange rates: evidence from recent Asianflu. Quarterly Review of Economics and Finance, 40, 337-354.

He, X.-Z. and Li, Y. (2008): Heterogeneity, convergence and autocorrelations. Quantitative Finance, 8, 59 - 79.

He, X.-Z. and Westerhoff, F. (2005): Commodity markets, price limiters and speculative price dynamics. Journal of Economic Dynamics and Control, 29, 1577-1596.

Hommes, C.H. (2006): Heterogeneous agent models in economics and finance. In: Tesfatsion, L. and Judd, K. (eds.): Handbook of computational economics Vol. 2: Agent-based computational economics. North-Holland, Amsterdam.

Hommes, C.H., Huang, H. and Wang, D. (2005): A robust rational route to randomness in a simple asset pricing model. Journal of Economic Dynamics and Control, 29, 1043-1072.

Huang, W. and Day, R. (1993): Chaotically switching bear and bull markets: The derivation of stock price distributions from behavioral rules. In: Day, R. and Chen, P. (eds.): Nonlinear dynamics and evolutionary economics. Oxford University Press, Oxford, 169-182.

Kirman, A. (1991): Epidemics of opinion and speculative bubbles in financial markets. In: Taylor, M. (ed.): Money and financial markets. Blackwell, Oxford, 354-368.

LeBaron, B. (2006): Agent-based computational finance. In: Tesfatsion, L. and Judd, K. (eds.): Handbook of computational economics Vol. 2: Agent-based computational economics. North-Holland, Amsterdam.

Lux, T. (1995): Herd behavior, bubbles and crashes. Economic Journal, 105, 881-896.

Lux, T. (1997): Time variation of second moments from a noise trader/infection model. Journal of Economic Dynamics and Control, 22, 1-38.

Lux, T. (1998): The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions. Journal of Economic Behavior and Organization, 33, 143-165.

Lux, T. (2008): Financial power laws: Empirical evidence, models and mechanisms. In: Cioffi-Revilla, C. (ed.): Power laws in the social sciences: Discovering complexity and non-equilibrium dynamics in the social universe. Cambridge University Press, Cambridge, in press.

Mira, C., Gardini, L., Barugola, A. and Cathala, J.C. (1996): Chaotic Dynamics in Two-Dimensional Noninvertible Maps. World Scientific, Singapore.

Phylaktis, K. and Ravazzolo, F. (2005): Stock prices and exchange rate dynamics. Journal of International Money and Finance, 24, 1031-1053.

Rosser, B., Ahmed, E. and Hartmann, G. (2003): Volatility via social flaring. Journal of Economic Behavior and Organization, 50, 77-87.

Westerhoff, F. (2004): Multiasset market dynamics. Macroeconomic Dynamics, 8, 596-616.

Westerhoff, F. and Dieci, R. (2006): The effectiveness of Keynes-Tobin transaction taxes when heterogeneous agents can trade in different markets: A behavioral finance approach. Journal of Economic Dynamics and Control, 30, 293-322.

Westermann, F. (2004): Does the Euro affect the dynamic interactions of stock markets in

Europe? Evidence from France, Germany and Italy. European Journal of Finance, 10, 139-148.

Wieland, C. and Westerhoff, F. (2005): Exchange rate dynamics, central bank interventions and chaos control methods. Journal of Economic Behavior and Organization, 58, 117-132.

Appendix A: Proof of Proposition 1

(a) The one-dimensional nonlinear map (17), which governs the exchange rate dynamics in the case of $c^H = c^A = 0$, can be rewritten as

$$S_{t+1} = S_t + d\left[f(F^S - S_t) + \frac{(e+f)(S_t - F^S)}{1 + g(S_t - F^S)^2}\right]. \tag{22}$$

By introducing deviations from the fundamental value, $x := S - F^S$, equation (22) takes the form $x_{t+1} = h(x_t)$, where

$$h(x) = x + d\left[-fx + \frac{(e+f)x}{1+gx^2}\right] = (1-df)x + \frac{d(e+f)x}{1+gx^2}. \tag{23}$$

For strictly positive e, f, g, map $h(x)$ admits three fixed points, solutions of $h(x) = x$, given by $\overline{x} = 0$, $\overline{x}_l = -\sqrt{e/(fg)}$, $\overline{x}_u = \sqrt{e/(fg)}$. It follows that the dynamical system (22) has three steady states, namely $\overline{S} = F^S$, $\overline{S}_l = F^S - \sqrt{e/(fg)}$, $\overline{S}_u = F^S + \sqrt{e/(fg)}$.

(b) In order to study the stability properties of the steady states, we consider the derivative of map (23)

$$\frac{dh(x)}{dx} = 1 - df + d(e+f)\frac{1-gx^2}{(1+gx^2)^2}. \tag{24}$$

Evaluation of (24) at the fundamental steady state yields

$$\left.\frac{dh}{dx}\right|_{x=0} = 1 + de > 1,$$

which reveals that the fundamental steady state is always unstable. With regard to the non-fundamental steady states $\overline{x}_l = -\sqrt{\frac{e}{fg}}$, $\overline{x}_u = \sqrt{\frac{e}{fg}}$, note first that $g\overline{x}_l^2 = g\overline{x}_u^2 = e/f$. Therefore we obtain

$$\left.\frac{dh}{dx}\right|_{x=\overline{x}_l} = \left.\frac{dh}{dx}\right|_{x=\overline{x}_u} = 1 - \frac{2def}{e+f} < 1.$$

The non-fundamental steady states are thus LAS if and only if $1 - \frac{2def}{e+f} > -1$, i.e.

$$e(df - 1) < f. \tag{25}$$

By taking the chartist parameter e as a bifurcation parameter, the stability condition (25) holds for any $e > 0$ if $df \leq 1$. In the opposite case, $df > 1$, the stability condition is satisfied only for $e < e_{Flip} := f/(df - 1)$, at which value a period doubling bifurcation occurs.■

Appendix B: Proof of Proposition 2

(a) The steady states $(\overline{P}^H, \overline{P}^A, \overline{S})$ of the dynamical system (11) are the solutions of the following system of equations

$$\overline{P}^H = G^H(\overline{P}^H, \overline{S}), \quad \overline{P}^A = G^A(\overline{P}^A, \overline{S}), \quad \overline{S} = G^S(\overline{P}^H, \overline{P}^A, \overline{S}), \tag{26}$$

which can be rewritten as follows

$$(b^H + c^H)(F^H - \overline{P}^H) + c^H(F^S - \overline{S}) = 0, \tag{27}$$

$$(b^A + c^A)(F^A - \overline{P}^A) + c^A(\overline{S} - F^S) = 0, \tag{28}$$

$$\begin{aligned} c^H(F^H - \overline{P}^H + F^S - \overline{S})\exp(\overline{P}^H) - c^A(F^A - \overline{P}^A + \overline{S} - F^S)\exp(\overline{P}^A - \overline{S}) + \\ + \frac{e(\overline{S} - F^S)}{1 + g(F^S - \overline{S})^2} + \frac{fg(F^S - \overline{S})^3}{1 + g(F^S - \overline{S})^2} = 0. \end{aligned} \tag{29}$$

Note that from (27) and (28) it follows, respectively, that

$$\overline{P}^H - F^H = \frac{c^H}{b^H + c^H}(F^S - \overline{S}), \quad \overline{P}^A - F^A = \frac{c^A}{b^A + c^A}(\overline{S} - F^S). \tag{30}$$

The quantities $(F^H - \overline{P}^H + F^S - \overline{S})$ and $(F^A - \overline{P}^A + \overline{S} - F^S)$ in (29) can therefore be rewritten, respectively, as

$$(F^H - \overline{P}^H + F^S - \overline{S}) = -\frac{b^H}{b^H + c^H}(\overline{S} - F^S),$$

$$(F^A - \overline{P}^A + \overline{S} - F^S) = \frac{b^A}{b^A + c^A}(\overline{S} - F^S),$$

while the quantities $\exp(\overline{P}^H)$ and $\exp(\overline{P}^A - \overline{S})$ in the same equation become

$$\exp(\overline{P}^H) = \Phi^H \exp\left[-\frac{c^H}{b^H + c^H}(\overline{S} - F^S)\right],$$

$$\exp(\overline{P}^A - \overline{S}) = \frac{\Phi^A}{\Phi^S} \exp\left[-\frac{b^A}{b^A + c^A}(\overline{S} - F^S)\right],$$

where $\Phi^H := \exp(F^H)$, $\Phi^A := \exp(F^A)$, $\Phi^S := \exp(F^S)$. By introducing deviations from fundamentals, $x^H := P^H - F^H$, $x^A := P^A - F^A$, $x := S - F^S$, equation (30) simplifies into (18), while (29) can be rewritten as

$$\frac{e\overline{x} - fg\overline{x}^3}{1 + g\overline{x}^2} - \frac{c^H b^H}{b^H + c^H}\overline{x}\ \Phi^H \exp\left(-\frac{c^H}{b^H + c^H}\overline{x}\right) - \frac{c^A b^A}{b^A + c^A}\overline{x}\frac{\Phi^A}{\Phi^S}\exp\left(-\frac{b^A}{b^A + c^A}\overline{x}\right) = 0,$$

or $x\left[\alpha(\overline{x}) - \beta(\overline{x})\right] = 0$, that is equation (19), where $\alpha(x)$ and $\beta(x)$ are defined by (20).

(b) Equation (18) determines equilibrium stock prices as functions of equilibrium exchange rates, where the latter are the solutions of equation (19). Equation (19) always admits the solution $x = 0$, which corresponds to the fundamental steady state $x^H = x^A = x = 0$. Further non-fundamental steady states are related to the possible solutions of $\alpha(x) = \beta(x)$. We discuss the conditions for their existence by means of *Fig. 2*. Note first that in the case $c^H = c^A = 0$, quantity $\beta(x)$ is identically equal to zero, so that we are back to the case of no interactions where two symmetric exchange rate equilibria exist, $x = \mp\sqrt{e/(fg)}$ (panel *a*). If either c^H or c^A is strictly positive, then $\beta(x)$ is represented by a (strictly decreasing) negative exponential function[9], and the graphical visualization of equation $\alpha(x) = \beta(x)$ is as in the panels from *b* to *d*. The bell-shaped function $\alpha(x)$ intersects the vertical axis at level $\alpha(0) = e$, while $\beta(x)$ intersects at the level

$$\beta(0) = \frac{\Phi^H b^H c^H}{b^H + c^H} + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A}\ . \tag{31}$$

Note that the smaller c^H, c^A are, the lower the ordinate $\beta(0)$ of the intersection. Therefore, as long as e is sufficiently large (panel *c*) or market interactions are sufficiently weak, i.e. c^H or c^A are small enough (panel *d*), equation $\alpha(x) = \beta(x)$ will still admit two solutions (and the system will then have three steady states) similarly to the case of no interaction. In particular, $e \geq \beta(0)$ represents a sufficient condition for the existence of multiple solutions. In the opposite case (e sufficiently small, or c^H, c^A large enough), there will be no solutions to $\alpha(x) = \beta(x)$ (panel *b*) and the system will admit a unique steady state.■

We add a few comments on the proof of this result. Consider, for instance, the transition from panel *b* to panel *c*, which is obtained by increasing parameter e for fixed values of c^H, c^A. The exact

[9]The graph of $\beta(x)$, however, looks approximately flat at the scale of *Fig. 2*.

bifurcation value of e at which two new steady states appear can only be computed numerically. Since $\alpha(x)$ is represented by a bell-shaped curve, symmetric with respect to the vertical axis, and $\beta(x)$ is negatively sloped, the tangency between $\alpha(x)$ and $\beta(x)$, and the subsequent crossing, will take place for strictly positive x, and the two new solutions will both be positive, initially. For $e = \beta(0)$, one of the two solutions coincides with $x = 0$, whereas for higher e, the two solutions will have opposite signs. The bifurcation mechanism is in fact that of a *saddle-node* bifurcation, followed by a *transcritical* bifurcation of the fundamental steady state. If $\beta(x)$ is sufficiently flat (as is the case in *Fig. 2*), the two bifurcations may occur very close to each other. In any case, the "final" result is similar to that of a *pitchfork* bifurcation.

Appendix C: Proof of Proposition 3

The Jacobian matrix of the system (11), evaluated at the fundamental steady state $\mathbf{F} := (F^H, F^A, F^S)$, is the following

$$D\mathcal{G}(\mathbf{F}) = \begin{bmatrix} 1 - a^H(b^H + c^H) & 0 & -a^H c^H \\ 0 & 1 - a^A(b^A + c^A) & a^A c^A \\ -d\Phi^H c^H & d\dfrac{\Phi^A}{\Phi^S} c^A & 1 + d\left[e - \Phi^H c^H - \dfrac{\Phi^A}{\Phi^S} c^A\right] \end{bmatrix}. \tag{32}$$

For notational purposes, it is convenient to define the following aggregate parameters

$$q^H := 1 - a^H(b^H + c^H), \quad q^A := 1 - a^A(b^A + c^A), \tag{33}$$

$$u^H := a^H(c^H)^2 d\Phi^H, \quad u^A := a^A(c^A)^2 d\frac{\Phi^A}{\Phi^S}, \tag{34}$$

$$k = 1 - d\left[\Phi^H c^H + \frac{\Phi^A}{\Phi^S} c^A\right]. \tag{35}$$

With some algebra, the characteristic polynomial of $D\mathcal{G}(\mathbf{F})$ can be rewritten as

$$\mathcal{P}(\lambda) = \lambda^3 + m_1\lambda^2 + m_2\lambda + m_3, \tag{36}$$

where

$$
\begin{aligned}
m_1 &= -(q^H + q^A + k + de), \\
m_2 &= (q^H + q^A)(k + de) + q^H q^A - (u^H + u^A), \\
m_3 &= q^H u^A + q^A u^H - q^H q^A (k + de).
\end{aligned}
$$

Note that from our assumption (21) and from the fact that a^H, b^H, a^A, b^A, are strictly positive and c^H, c^A are non-negative, it follows that $\left|q^H\right| < 1$, $\left|q^A\right| < 1$, which implies $\left|q^H q^A\right| < 1$, $\left|q^H + q^A\right| < 2$. The following set of inequalities imposed on the coefficients of the characteristic polynomial (36) provide a necessary and sufficient condition for all the eigenvalues of (32) to be of modulus smaller than unity (Farebrother (1973)), which implies a locally asymptotically stable steady state:

$$1 + m_1 + m_2 + m_3 > 0, \tag{37}$$

$$1 - m_1 + m_2 - m_3 > 0, \tag{38}$$

$$1 - m_2 + m_3(m_1 - m_3) > 0, \tag{39}$$

$$m_2 < 3. \tag{40}$$

Conditions (37)-(40) can be rewritten in terms of the parameters of the model, respectively

$$(1 - q^H)(1 - q^A) - u^H(1 - q^A) - u^A(1 - q^H) - (1 - q^H)(1 - q^A)(k + de) > 0, \tag{41}$$

$$(1 + q^H)(1 + q^A) - u^H(1 + q^A) - u^A(1 + q^H) + (1 + q^H)(1 + q^A)(k + de) > 0, \tag{42}$$

$$
\begin{aligned}
&1 - q^H q^A + u^H + u^A - (q^H u^A + q^A u^H)^2 - (q^H u^A + q^A u^H)(q^H + q^A) - \\
&- \left[(q^H u^A + q^A u^H)(1 - 2q^H q^A) + (q^H + q^A)(1 - q^H q^A)\right](k + de) + \\
&+ q^H q^A (1 - q^H q^A)(k + de)^2 > 0,
\end{aligned} \tag{43}
$$

$$q^H q^A - (u^H + u^A) + (q^H + q^A)(k + de) < 3. \tag{44}$$

We now discuss conditions (41)-(44) and show, in particular, that they are simultaneously satisfied under assumption (21), provided that parameters d, e are not too large. Note that parameters u^H,

u^A, and k depend on d, according to (34) and (35), respectively. In the following we assume that d is as small as necessary, so that in particular quantity k, defined in (35), is strictly positive. In general we will regard the left-hand side of each of conditions (41)-(44) as a function of parameter $e > 0$ for sufficiently small values of d and fixed values of the other parameters (satisfying all of the assumed restrictions).

Condition (41) can be rewritten as

$$(1-q^H)(1-q^A)de < (1-q^H)(1-q^A)(1-k) - u^H(1-q^A) - u^A(1-q^H).$$

In the above equation, the term on the right-hand side is strictly positive for any $d > 0$. To prove this, simply note that

$$0 \leq \frac{a^H c^H}{(1-q^H)} = \frac{a^H c^H}{a^H(b^H+c^H)} < 1, \quad 0 \leq \frac{a^A c^A}{(1-q^A)} = \frac{a^A c^A}{a^A(b^A+c^A)} < 1$$

and therefore

$$(1-k) = d\left[\Phi^H c^H + \frac{\Phi^A}{\Phi^S} c^A\right] > d\left[\Phi^H c^H \frac{a^H c^H}{(1-q^H)} + \frac{\Phi^A}{\Phi^S} c^A \frac{a^A c^A}{(1-q^A)}\right] = \frac{u^H}{(1-q^H)} + \frac{u^A}{(1-q^A)}.$$

Moreover, since the quantity $(1-q^H)(1-q^A)$ is strictly positive, too, it follows that condition (41) is satisfied for sufficiently small e, namely

$$e < \frac{1}{d}\left[1 - k - \frac{u^H}{(1-q^H)} - \frac{u^A}{(1-q^A)}\right] = \Phi^H \frac{b^H c^H}{b^H + c^H} + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A} := e^*. \tag{45}$$

Condition (42) can be rewritten as

$$(1+q^H)(1+q^A)(1+k) - u^H(1+q^A) - u^A(1+q^H) + (1+q^H)(1+q^A)de > 0.$$

Since quantities $1 + q^H = 2 - a^H(b^H + c^H)$ and $1 + q^A = 2 - a^A(b^A + c^A)$ are strictly positive under our assumptions, the coefficient of the term containing parameter e is strictly positive, too. The sum of the remaining terms on the left-hand side is strictly positive if

$$1 + k - \left(\frac{u^H}{(1+q^H)} + \frac{u^A}{(1+q^A)}\right) > 0,$$

which results in

$$d < 2\left[\Phi^H c^H \frac{2 - a^H b^H}{2 - a^H(b^H + c^H)} + \frac{\Phi^A}{\Phi^S} c^A \frac{2 - a^A b^A}{2 - a^A(b^A + c^A)}\right]^{-1} := d^*.$$

Under such a restriction on d, condition (42) is satisfied for any $e > 0$.

Condition (43) is much more complicated to deal with, but it can be noted that for $d, e \to 0$ (in which case $k \to 1$, $u^H, u^A \to 0$), the left-hand side becomes

$$(1 - q^H q^A)(1 - q^H)(1 - q^A),$$

which is strictly positive under our assumptions (21). By continuity arguments, (43) will then be satisfied for sufficiently small values of d and e.

Condition (44) is obviously satisfied when $q^H + q^A \leq 0$, i.e. $a^H(b^H + c^H) + a^A(b^A + c^A) \geq 2$. Assume now $q^H + q^A > 0$, in which case the left-hand side of (44) strictly increases with e. Since our assumptions imply $\left|q^H q^A\right| < 1$, $0 < q^H + q^A < 2$, $0 < k < 1$, we obtain

$$q^H q^A - (u^H + u^A) + (q^H + q^A)k < 3.$$

It follows that when $q^H + q^A > 0$, equation (44) is satisfied for sufficiently small e, namely for

$$e < \frac{1}{d}\left[\frac{3 + (u^H + u^A) - q^H q^A}{q^H + q^A} - k\right] := e^{**}. \tag{46}$$

Our assumptions about aggregate parameters q^H, q^A, also imply that threshold e^{**} defined in equation (46) is larger than quantity e^* in (45). To see this, note that (since $\left|q^H q^A\right| < 1$, $0 < q^H + q^A < 2$) the following inequalities hold

$$de^{**} + k = \frac{3 + (u^H + u^A) - q^H q^A}{q^H + q^A} > 1 > 1 - \frac{u^H}{(1 - q^H)} - \frac{u^A}{(1 - q^A)} = de^* + k.$$

This implies that condition (46) is fulfilled whenever (45) holds, and therefore condition (44) is redundant.■

Our analysis proves that the fundamental steady state is LAS for sufficiently small d and e. It also provides a rough picture of how local bifurcations may occur. Assume that d is as small as necessary (in particular $d < d^*$) and assume also that by increasing parameter e condition (43) is satisfied at least for $0 < e \leq e^*$ (this is actually the case of the parameter setting we chose for the numerical examples in *Fig. 3*). In this case, the steady state loses stability for $e = e^*$. Note that bifurcation value e^* is equal to quantity $\beta(0)$ (see equation (31) in Appendix B). This means that in this case the loss of stability occurs precisely when one of the two newborn nonfundamental steady states collides with the fundamental steady state. Numerical and graphical analysis suggests that this contact corresponds to a transcritical bifurcation of the fundamental steady state (see also Appendix *B*).

Figure captions

Figure 1: The case of no market interactions. Panels a and b present bifurcation diagrams for the log exchange rate, for $d = 1$, $0 < e < 6$, $f = 0.8$ and two different initial conditions. Panels c and d show the same but now for $d = 1.5$ and $f = 1$. Panel e depicts the evolution of the log exchange rate in the time domain for $d = 1.5$, $e = 5.3$, and $f = 1$. Panel f shows the $1D$ map (17) for $d = 1.5$, $e \simeq 5.257$ and $f = 1$. The gray arrows in this panel indicate that the maximum is in fact a pre-image of the repelling fundamental steady state (and is therefore the minimum). The remaining parameters are $g = 10000$ and $F^S = 0$.

Figure 2: A characterization of the non-fundamental steady states. In the four panels we plot functions $\alpha(x)$ (black solid line) and $\beta(x)$ (gray dashed line) for different parameter combinations. Panel a: $c^H = 0$, $c^A = 0$, and $e = 0.5$. Panel b: $c^H = 0.4$, $c^A = 0.4$, and $e = 0.5$. Panel c: $c^H = 0.4$, $c^A = 0.4$, and $e = 0.7$. Panel d: $c^H = 0.1$, $c^A = 0.4$, and $e = 0.5$. The remaining parameters are $b^H = 1$, $b^A = 1.5$, $d = 1$, $f = 0.8$, $g = 10000$ and $F^H = F^A = F^S = 0$.

Figure 3: No market interactions versus market interactions. The six panels present bifurcation diagrams for the log stock price in country H (panels a and b), the log stock price in country A (panels c and d) and the log exchange rate (panels e and f) for two different sets of initial conditions, respectively. The parameters are: $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $c^H = 0.4$, $c^A = 0.4$, $d = 1$, $0 < e < 6$, $f = 0.8$, $g = 10000$ and $F^H = F^A = F^S = 0$. The superimposed gray dashed lines illustrate the reference case of "no market interactions", i.e. $c^H = 0$ and $c^A = 0$, for the same values of the other parameters.

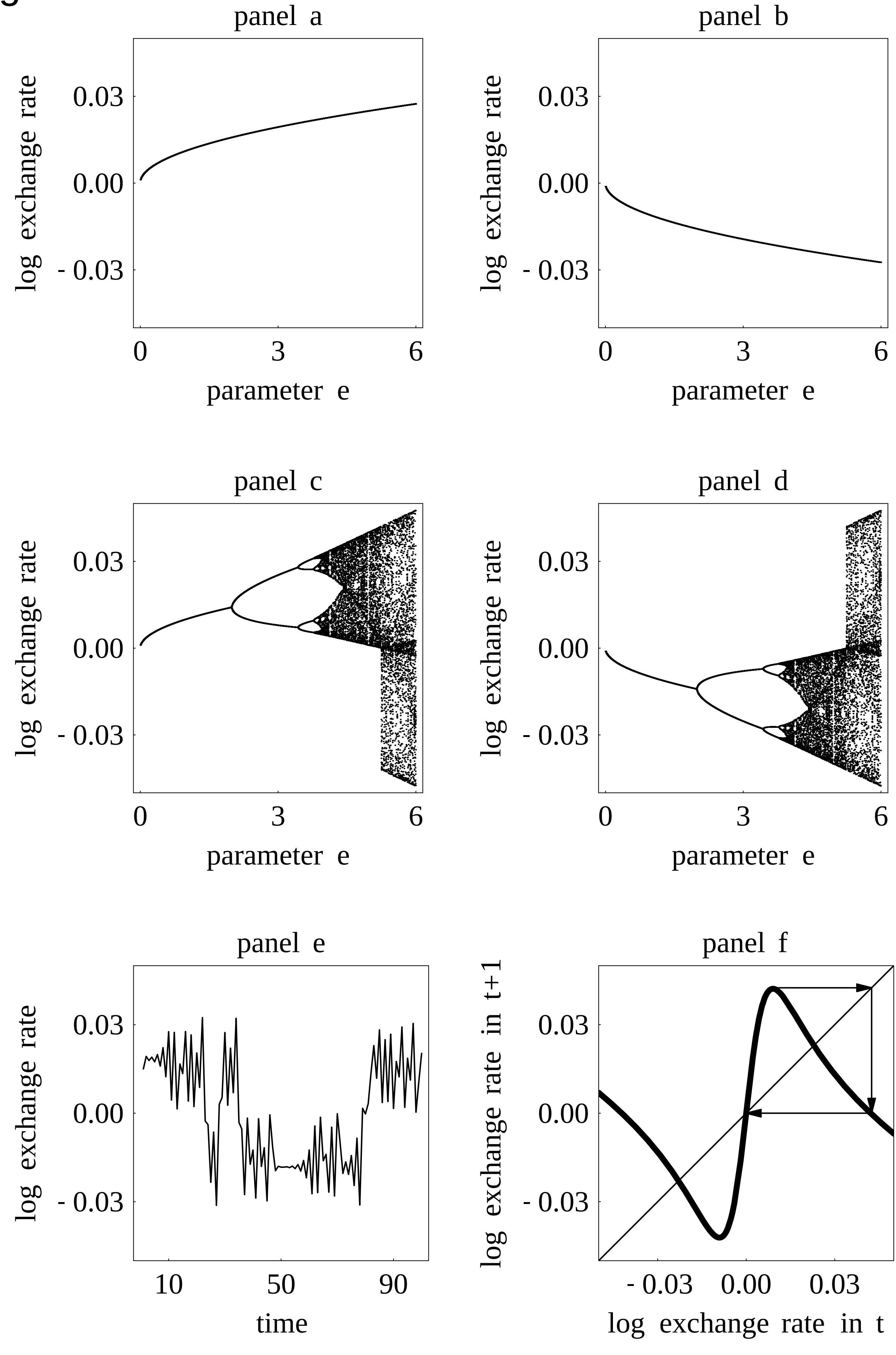
Figure 4: The destabilizing/stabilizing effect of market interactions. Panels a and b (c and d) show bifurcation diagrams for the log stock price in country H (the log exchange rate) for two

different sets of initial conditions. The parameter setting is $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $0 < c^H < 0.5$, $c^A = 0.2$, $d = 1$, $e = 6$, $f = 0.8$, $g = 10000$ and $F^H = F^A = F^S = 0$. In the bottom four panels we repeat these computations but now use $0.15 < c^H < 0.25$, $c^A = 0.4$ and $e = 0.5$.

Figure 5: The emergence of bull and bear market dynamics. In panels a and c the log exchange rate is plotted against the log stock price of country H for two different sets of initial conditions. Panels b and d present the corresponding evolution of the log exchange rate in the time domain. The parameter setting is $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $c^H = 0.4$, $c^A = 0.4$, $d = 1$, $e = 4.6$, $f = 0.8$, $g = 10000$ and $F^H = F^A = F^S = 0$. In panels e and f we do the same but now use one set of initial conditions and assume that $e = 5.3$.

Figure 6: Bull and bear market dynamics in action. Panels a, b and c show the evolution of the log stock price of country H, the log stock price of country A and the log exchange rate in the time domain, respectively. In panel d the log exchange rate at time step $t + 1$ is plotted against the log exchange rate at time step t. The parameter setting is $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $c^H = 0.4$, $c^A = 0.4$, $d = 1$, $e = 5.3$, $f = 0.8$, $g = 10000$ and $F^H = F^A = F^S = 0$.

Figure 1



panel a
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e
panel b
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e
panel c
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e
panel d
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e
panel e
log exchange rate
0.03
0.00
- 0.03
10
50
90
time
panel f
log exchange rate in t+1
0.03
0.00
- 0.03
- 0.03
0.00
0.03
log exchange rate in t

Figure 2

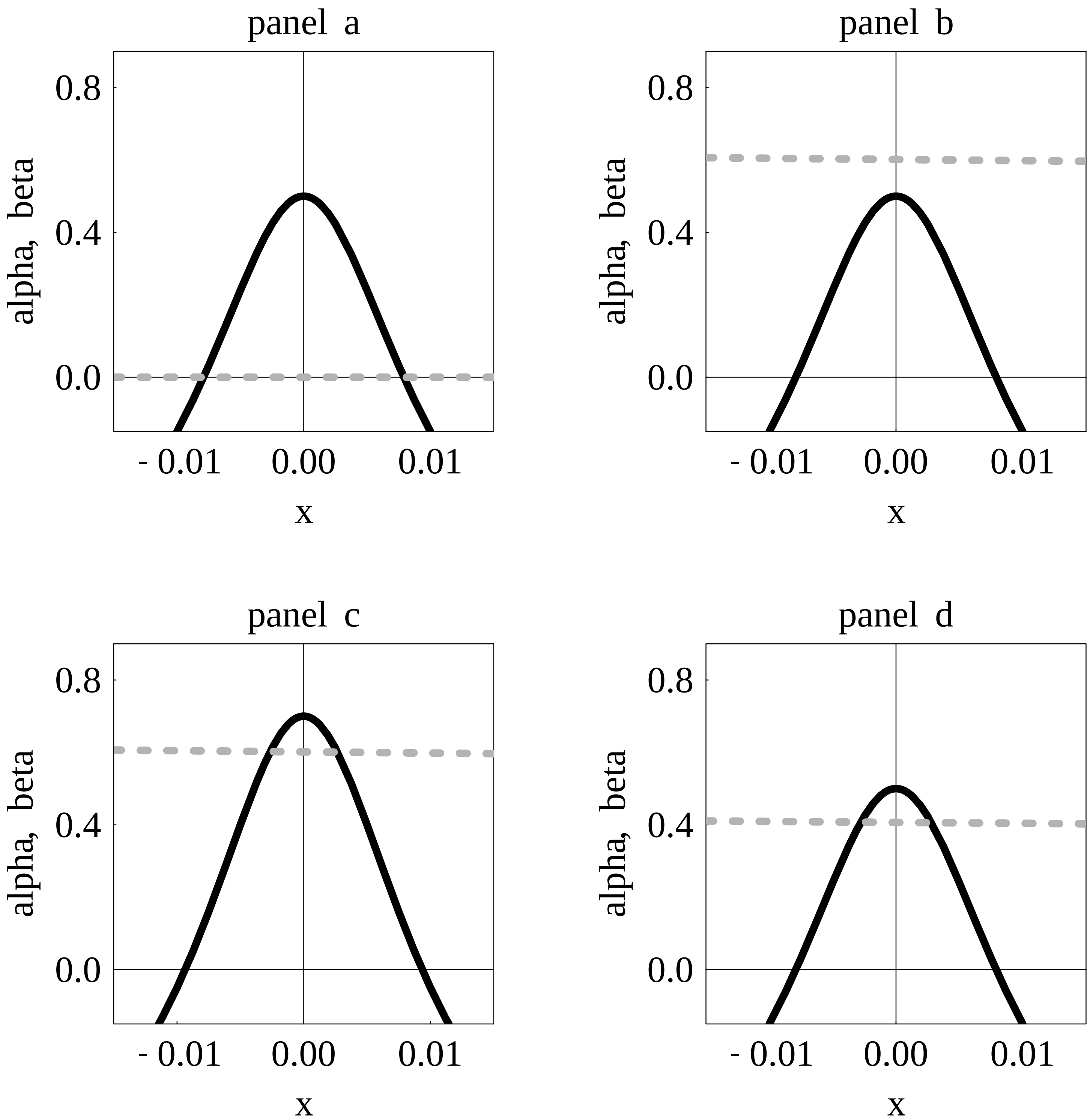


Figure 3

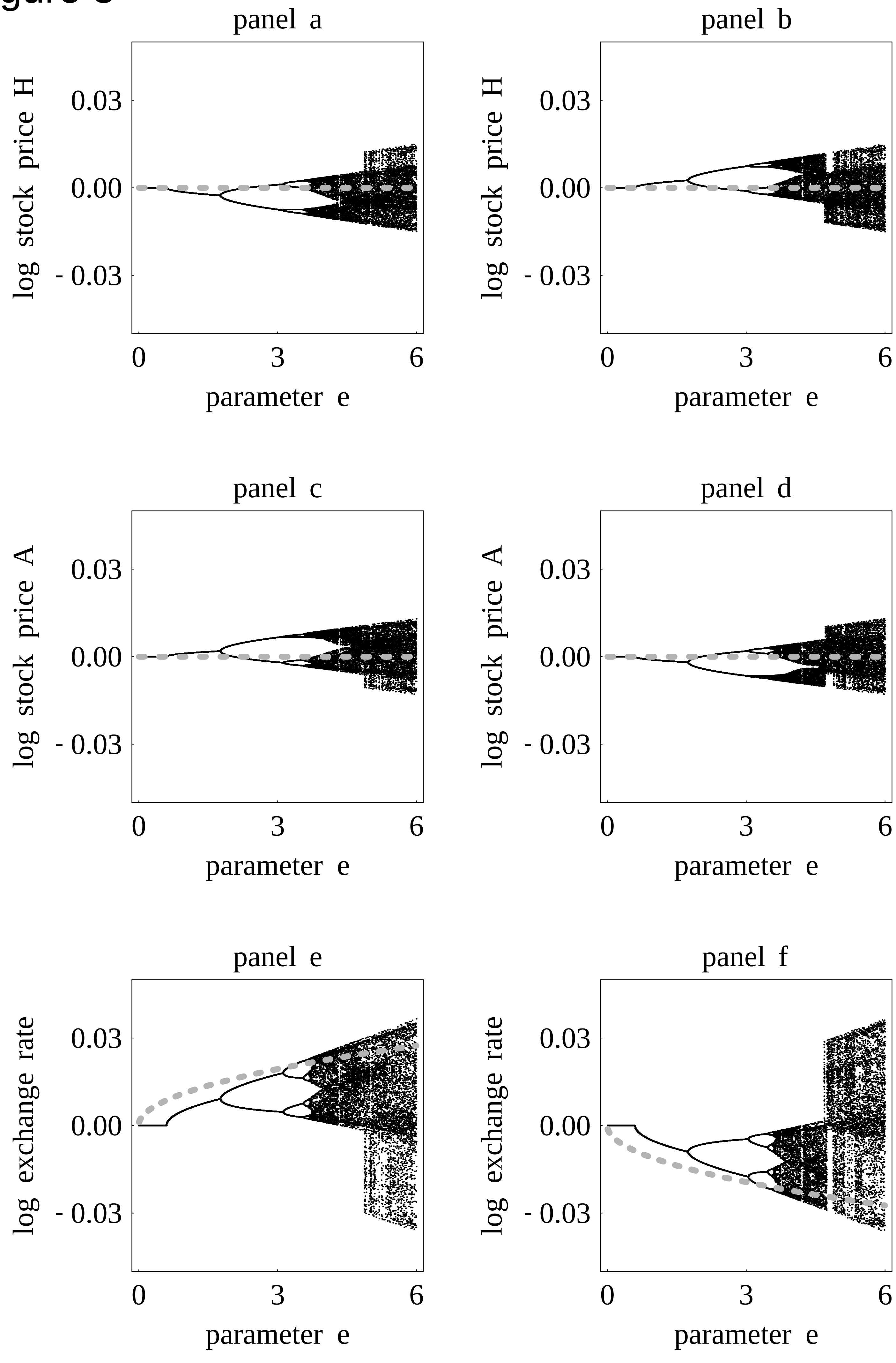
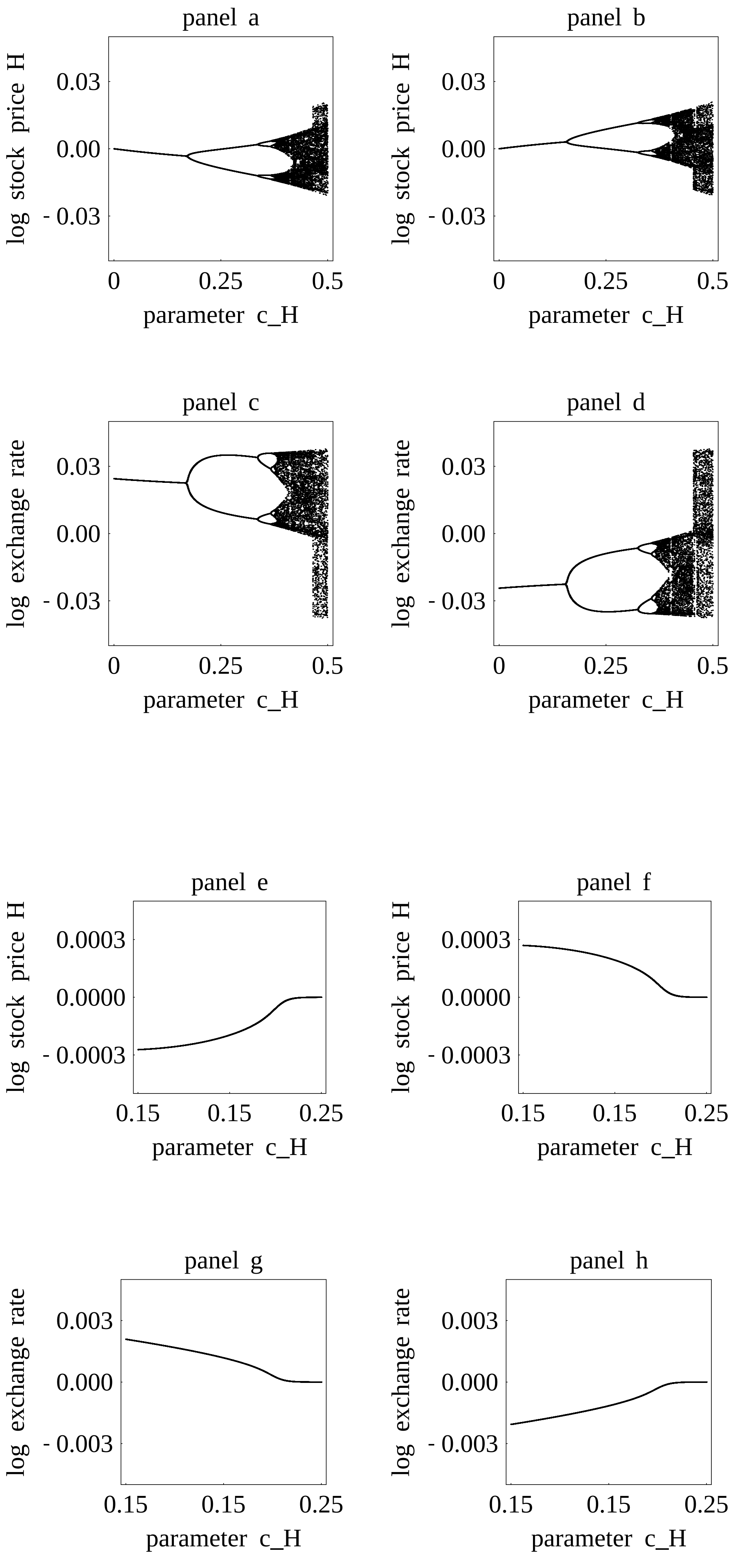
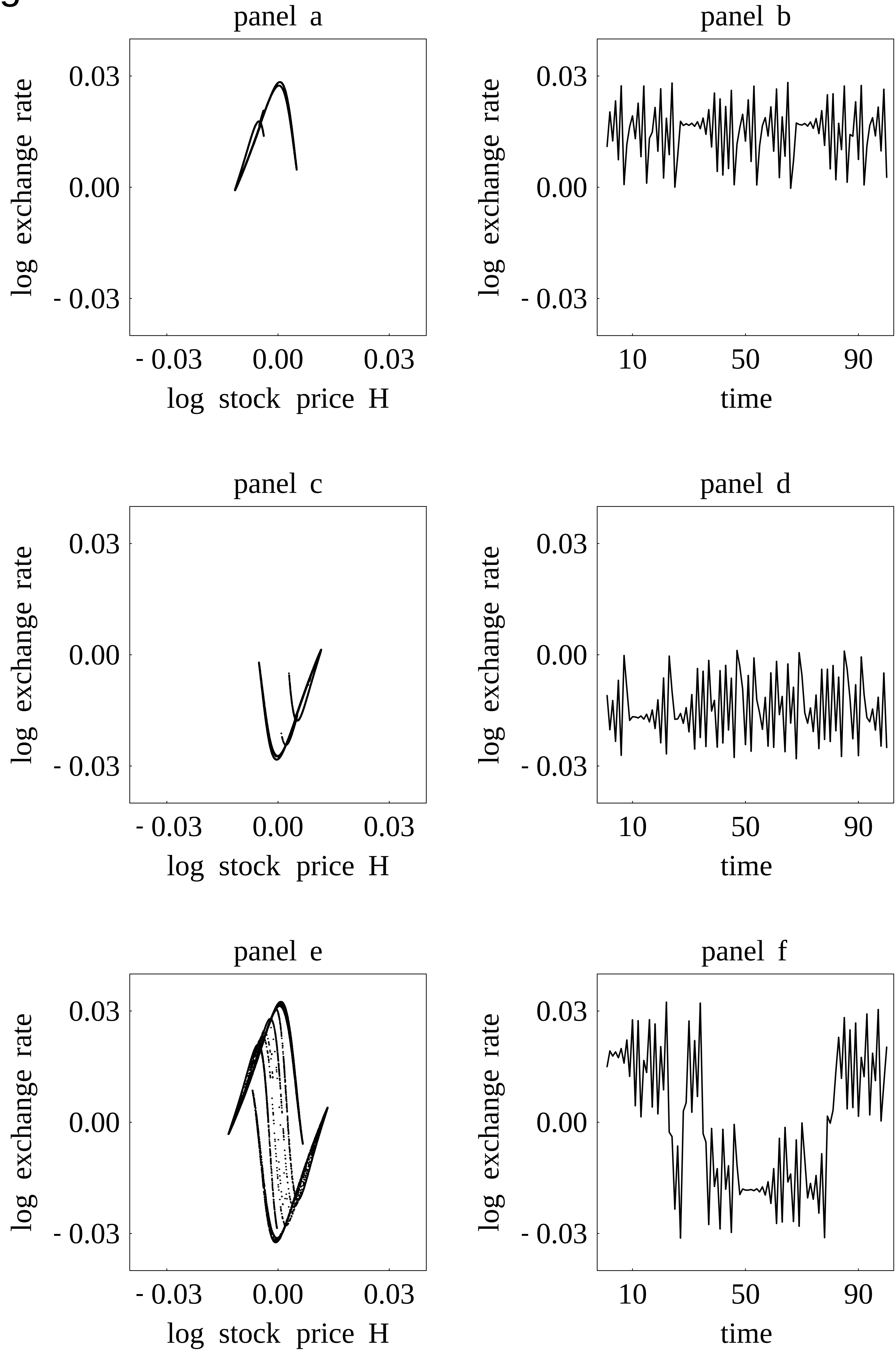

panel a
log stock price H
0.03
0.00
- 0.03
0
3
6
parameter e
panel b
log stock price H
0.03
0.00
- 0.03
0
3
6
parameter e
panel c
log stock price A
0.03
0.00
- 0.03
0
3
6
parameter e
panel d
log stock price A
0.03
0.00
- 0.03
0
3
6
parameter e
panel e
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e
panel f
log exchange rate
0.03
0.00
- 0.03
0
3
6
parameter e

Figure 4



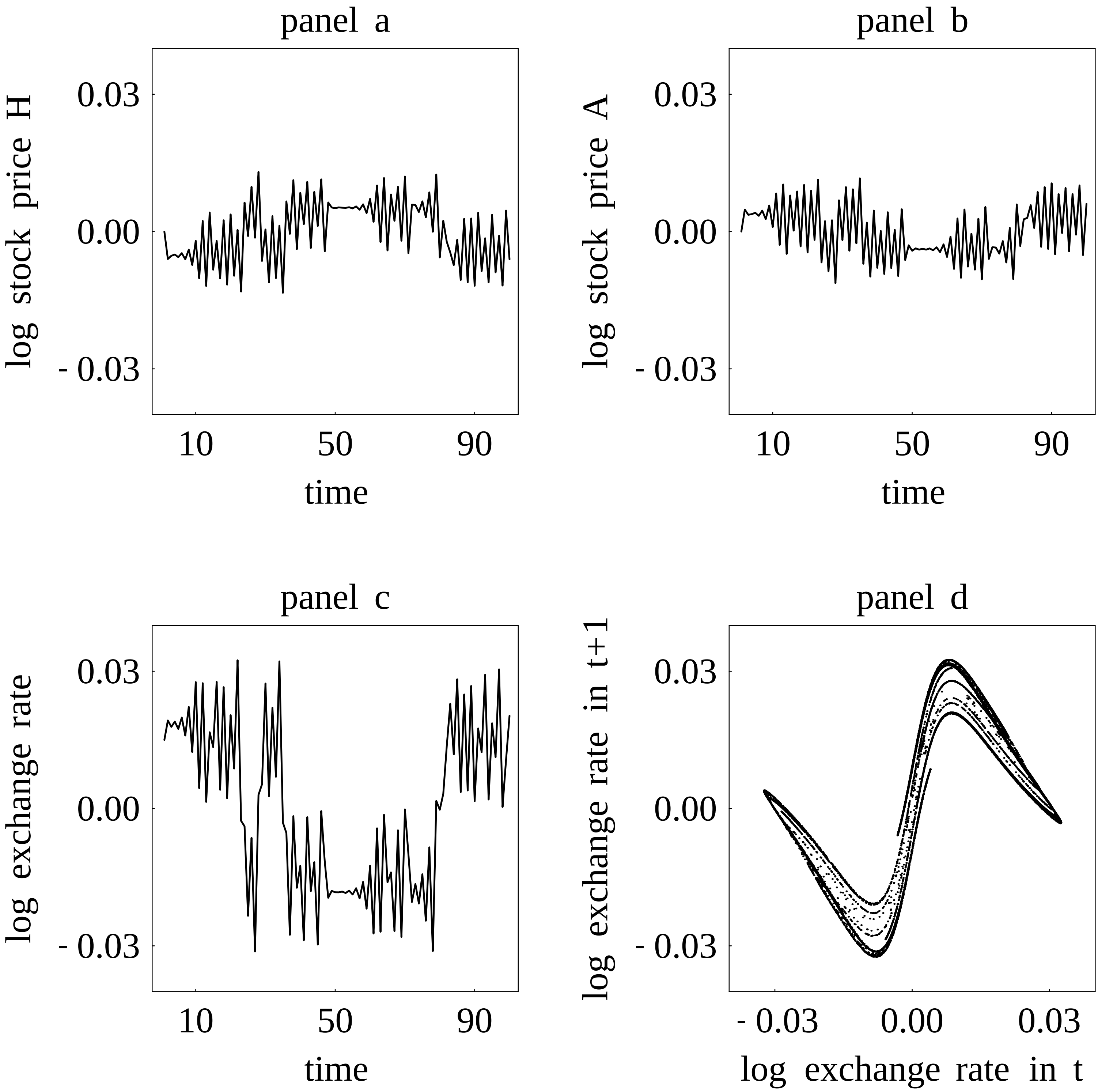
panel a
log stock price H
0.03
0.00
- 0.03
0
0.25
0.5
parameter c_H
panel b
log stock price H
0.03
0.00
- 0.03
0
0.25
0.5
parameter c_H
panel c
log exchange rate
0.03
0.00
- 0.03
0
0.25
0.5
parameter c_H
panel d
log exchange rate
0.03
0.00
- 0.03
0
0.25
0.5
parameter c_H
panel e
log stock price H
0.0003
0.0000
- 0.0003
0.15
0.15
0.25
parameter c_H
panel f
log stock price H
0.0003
0.0000
- 0.0003
0.15
0.15
0.25
parameter c_H
panel g
log exchange rate
0.003
0.000
- 0.003
0.15
0.15
0.25
parameter c_H
panel h
log exchange rate
0.003
0.000
- 0.003
0.15
0.15
0.25
parameter c_H

Figure 5



panel a
panel b
panel c
panel d
panel e
panel f
log exchange rate
0.03
0.00
- 0.03
- 0.03
0.00
0.03
log stock price H
10
50
90
time

Figure 6


panel a
log stock price H
0.03
0.00
- 0.03
10
50
90
time
panel b
log stock price A
0.03
0.00
- 0.03
10
50
90
time
panel c
log exchange rate
0.03
0.00
- 0.03
10
50
90
time
panel d
log exchange rate in t+1
0.03
0.00
- 0.03
- 0.03
0.00
0.03
log exchange rate in t