A LINEAR TIME ALGORITHM FOR EMBEDDING GRAPHS IN AN ARBITRARY SURFACE*

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Abstract. For an arbitrary fixed surface S, a linear time algorithm is presented that for a given graph G either finds an embedding of G in S or identifies a subgraph of G that is homeomorphic to a minimal forbidden subgraph for embeddability in S. A side result of the proof of the algorithm is that minimal forbidden subgraphs for embeddability in S cannot be arbitrarily large. This yields a constructive proof of the result of Robertson and Seymour that for each closed surface there are only finitely many minimal forbidden subgraphs. The results and methods of this paper can be used to solve more general embedding extension problems.

Key words. surface embedding, obstruction, algorithm, graph embedding, forbidden subgraph, forbidden minor

AMS subject classifications. 05C10, 05C85, 68Q20, 68R10

1. Introduction. The problem of constructing embeddings of graphs in surfaces is of practical and of theoretical interest. The practical issues arise, for example, in problems concerning VLSI, and also in several other applications since graphs embedded in low genus surfaces can be handled more easily. Theoretical interest comes from the importance of the genus parameter of graphs and from the fact that graphs of bounded genus naturally generalize the family of planar graphs and share many important properties with them.

There are linear time algorithms that for a given graph determine whether the graph can be embedded in the 2-sphere (or in the plane). The first such algorithm was obtained by Hopcroft and Tarjan [16] in 1974. There are several other linear time planarity algorithms (Booth and Lueker [6], Fraysseix and Rosenstiehl [11], Williamson [36, 37]). Extensions of these algorithms return an embedding (rotation system) whenever a graph is found to be planar [7], or exhibit a forbidden Kuratowski subgraph homeomorphic to K_5 or $K_{3,3}$ if the graph is non-planar [36, 37] (see also [21]). Recently, linear time algorithms have been devised for embedding graphs in the projective plane (Mohar [22]) and in the torus (Juvan, Marinček, and Mohar [19]).

It is known that the general problem of determining the genus [34], or the non-orientable genus [35] of graphs is NP-hard. However, for every fixed surface there is a polynomial time algorithm which checks if a given graph can be embedded in the surface. Such algorithms were found first by Filotti et al. [10]. For a fixed orientable surface S of genus g they discovered an algorithm with time complexity $O(n^{\alpha g+\beta})$ (α, β are constants) which tests if a given graph of order n can be embedded in S. Unfortunately, their algorithms are practically not useful, even in the simplest case when S is the torus. A theoretical estimate on the running time in case of the torus is only $O(n^{188})$. Recently, Djidjev and Reif [9] anounced improvement of the algorithm of [10] by presenting a polynomial time algorithm, for each fixed orientable surface, where the degree of the polynomial is fixed. The basic technique used in [10] and in [9] of embedding a subgraph, attempting to extend this partial embedding, and

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recursively work with discovered forbidden subgraphs for smaller genus surfaces is also used in our algorithm.

For every fixed surface S, an $O(n^3)$ algorithm for testing embeddability in S can be devised using graph minors [27, 31]. Robertson and Seymour recently improved their $O(n^3)$ algorithms to $O(n^2 \log n)$ [28, 29, 30]. An extension which also constructs an embedding is described by Archdeacon in [2]. The running time is estimated to be $O(n^{10})$ but with a little additional care it could be decreased to $O(n^6)$. A disadvantage of these algorithms is that they use the lists of forbidden minors which are not known for surfaces different from the 2-sphere and the projective plane. Even for the projective plane whose forbidden minors are known [1, 13], the algorithms based on checking for the presence of forbidden minors are rather time consuming since their running time estimates involve enormous constants.

In the present paper we describe a linear time algorithm which finds an embedding of a given graph G into a surface S if such an embedding exists. Here S is an arbitrary fixed surface. In case when G cannot be embedded in S, the algorithm returns a subgraph H of G that cannot be embedded in S but every proper subgraph of H admits an embedding in S. A side result of the algorithm is that the returned "minimal forbidden subgraph" H is homeomorphic to a graph with a bounded number of edges (where the bound depends only on S). This yields a constructive proof of the result of Robertson and Seymour [27] that for each closed surface there are only finitely many minimal forbidden subgraphs. A constructive proof for nonorientable surfaces has been published by Archdeacon and Huneke [3], while orientable surfaces resisted all previous attempts. (Recently also Seymour [32] found a constructive proof of that result.)

The results and methods of this paper can be used towards solving a generalization of problems of embedding graphs in surfaces — the so called *embedding extension problems* where one has a fixed embedding of a subgraph K of G in some surface and asks for embedding extensions to G or (minimal) obstructions for existence of such extensions.

The paper is more or less self contained with the exception of using results from [17, 18, 20, 24].

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for some basic operations. This model of computation was introduced by Cook and Reckhow [8]. It is known as the *unit-cost* RAM where operations on integers, whose value is $O(n)$, need only constant time (n is the order of the given graph). The same model of computation is used in many other instances, for example in well-known linear time planarity testing algorithms [16].

2. Basic definitions. We follow standard graph theory terminology as used, for example, in [5]. Let G and H be graphs. We denote by $G - H$ the graph obtained from G by deleting all vertices of $G \cap H$ and all their incident edges. If $F \subseteq E(G)$, then $G - F$ denotes the graph obtained from G by deleting all edges in F.

We will consider 2-cell embeddings of graphs in closed surfaces. They can be described in a purely combinatorial way by specifying:

(1) A *rotation system* $\pi = (\pi_v \, ; \, v \in V(G))$; for each vertex v of the given graph G we have a cyclic permutation π_v of edges incident with v, representing their circular order around v on the surface.
(2) A *signature* $\lambda : E(G) \to \{-1, 1\}$. Suppose that $e = uv$. Following the edge e on the surface, we see if the local rotations π_v and π_u are chosen consistently

or not. If yes, then we have $\lambda(e) = 1$, otherwise we have $\lambda(e) = -1$.

The reader is referred to [14] or [25] for more details. We will use this description as a definition: An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where π is a rotation system and λ is a signature. Having an embedding Π of G, we say that G is Π-*embedded*. If H is a subgraph of G, then the *induced embedding* of H (or the *restriction* of Π to H) is obtained from that of G by ignoring all edges in $E(G)\backslash E(H)$ and by restricting the signature to $E(H)$.

Each embedding Π of G determines a set of closed walks in G, called Π-*facial walks* or simply Π-*faces*, that correspond to traversals of face boundaries of the corresponding topological embedding. Each edge e of G is either contained in exactly two Π-facial walks, or it appears twice in the same Π-facial walk W. In the latter case, e and W are said to be *singular*. Edges e and f incident with the same vertex v of G are Π-*consecutive* if $e = \pi_v(f)$ or $f = \pi_v(e)$. In that case, there is a Π-face F containing e and f as consecutive edges, and we say that the pair $\{e, f\}$ is an *angle* of F.

Suppose that a subgraph K of G is Π-embedded. An embedding $\tilde{\Pi}$ of G is an *extension* of Π if it is an embedding in the same surface as Π and the induced embedding of K is equal to Π. Given a graph G and a Π-embedded subgraph K, we may ask if there is an embedding extension to G. This problem will be referred to as an *embedding extension problem*. An *obstruction* for extensions for such a problem is a subgraph Ω of $G - E(K)$ such that no embedding extension of K to $K \cup \Omega$ exists.

3. Bridges. Let K be a subgraph of G. A K*–bridge* in G (or a *bridge* of K in G) is a subgraph of G which is either an edge $e \in E(G)\backslash E(K)$ with both endpoints in K, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) between this component and K. Each edge of a K–bridge B having an endpoint in K is a *foot* of B. The vertices of $B \cap K$ are the *vertices of attachment* of B, and B is *attached to* each of these vertices. A vertex of K of degree different from 2 is a *main vertex* (or a *branch vertex*) of K. For convenience, if a connected component C of K is a cycle, then we choose an arbitrary vertex of C and declare it to be a main vertex of K as well. A *branch* of K is any path in K (possibly closed) whose endpoints are main vertices but no internal vertex on this path is a main vertex. Every subpath of a branch e is a *segment* of e. If a K-bridge is attached to a single branch e of K, it is said to be *local* (on e). The number of branches of K, denoted by $\mathrm{bsize}(K)$, is the *branch size* of K. If B is a K-bridge in G, then the *size* $\mathrm{bsize}_K(B)$ of B is defined as the number of branches of $K \cup B$ that are contained in B. Note that $\mathrm{bsize}(K \cup B) \le \mathrm{bsize}(K) + 2\,\mathrm{bsize}_K(B)$. A *basic piece* of K is either a main vertex or an *open branch* of K (i.e., a branch with its endpoints removed). If a K-bridge B in G is attached to at least three basic pieces of K, then B is *strongly attached*. Otherwise, it is *weakly attached*.

Suppose that K is Π-embedded. Let B be a K-bridge in G and $\tilde{\Pi}$ an extension of Π to $K \cup B$. Then there is a unique Π-face F that is not a $\tilde{\Pi}$-face, and we say that B is *embedded in* F or that F *contains* B. Clearly, if B is embedded in F, then all basic pieces that B is attached to appear on F. Each basic piece on F has one or more *appearances* (or *occurrences*) on F. The total number of appearances of main vertices on F is the *branch size* of F. We say that the K-bridge B embedded in F is *attached to* an appearance of the basic piece x on F if x contains a vertex x_0 such that the angle in F at this appearance of x_0 on x is not an angle within a $\tilde{\Pi}$-face.

Lemma 3.1. *Suppose that there are no local K-bridges in G. Let $\tilde{\Pi}$ be an embedding of G that is an extension of an embedding Π of K. If B is a K-bridge embedded* *in a* Π*-face* F*, we denote by* $q(B)$ *the number of appearances of basic pieces on* F *that* B *is attached to. If* F *is a* Π*-face of branch size* s*, and* $B_1, \ldots, B_k$ *are* K*-bridges embedded in* F*, then*

$$\sum_{i=1}^{k}(q(B_i) - 2) \le 2s - 2\,. \tag{1}$$

Consequently, if $\mathcal{B}$ *is the set of all* K*-bridges in* G*, then*

$$\sum_{B\in\mathcal{B}}(q(B) - 2) \le 4\,\mathrm{bsize}(K). \tag{2}$$

Proof. The proof of (1) is by induction on the number $p \le 2s$ of those occurrences of basic pieces on F that some bridge is attached to. We can assume that $q(B_i) \ge 3$ for $1 \le i \le k$ and that $p \ge 2$. The case $p = 2$ is trivial. If $p > 2$, let B be a strongly attached bridge in F. Let $f_1, \ldots, f_q$ be feet of B attached to distinct basic pieces of K. They divide F into q segments, containing $p_1, \ldots, p_q$ appearances of basic pieces of K (or their parts), respectively. Clearly, $p_1 + \cdots + p_q = p + q$ and $p_i < p$, $i = 1, \ldots, q$. By the induction hypothesis

$$\sum_{i=1}^{k}(q(B_i) - 2) \le (p_1 - 2) + \cdots + (p_q - 2) + (q - 2) = p - 2.$$

This proves (1). The sum of the branch sizes of Π-faces equals $2\,\mathrm{bsize}(K)$. Hence, (2) follows from (1). □

Lemma 3.1 shows, in particular, that too many strongly attached bridges obstruct embedding extensions. Similarly, every weakly attached bridge that is embedded such that it is attached to two or more occurrences of the same basic piece contributes to the left side of (2). Thus, under an embedding extension all except a bounded number of bridges are attached to at most one appearance of the same basic piece. Such embeddings of bridges are *simple*. More generally, and embedding extension is *simple* if all bridges have simple embeddings. In case of simple embeddings, we will use some special subgraphs of K-bridges in G. If B is a K-bridge in G, an E-*graph* in B is a minimal subgraph H of B such that:

- (E1) Any two vertices of $H - K$ are connected by a path in $H - K$.
- (E2) For each branch vertex ζ that B is attached to, H contains a foot incident with ζ. If ζ is an open branch with ends x_1 and x_2 and B is attached to ζ, let ζ_i be the vertex of attachment of B on ζ which is closest to x_i $(i = 1, 2)$. Then H contains feet attached to ζ_1 and ζ_2, respectively (possibly just one if $\zeta_1 = \zeta_2$).
- (E3) Every simple extension of any embedding of K to $K \cup H$ determines a simple extension to $K \cup B$.

The difficult part of this paper is to discover obstructions for simple embedding extensions. The next result somehow simplifies this problem by showing that one can work only with E-graphs of K-bridges in G and that E-graphs are not too large.

Theorem 3.2 (Mohar [24]). *Let* $\mathcal{B}$ *be the set of* K*-bridges in* G*. There is a number* c *depending only on* $\mathrm{bsize}(K)$ *such that each* $B \in \mathcal{B}$ *contains an E-graph* $\tilde{B}$ *with* $\mathrm{bsize}_K(\tilde{B}) \le \mathrm{c}$*. If* $\{B_1, \ldots, B_k\} \subseteq \mathcal{B}$ $(k \ge 1)$ *are arbitrary nonlocal* K*-bridges,* $\tilde{B}_1, \ldots, \tilde{B}_k$ *their corresponding E-graphs, and if* Π *is an embedding of* K*, then any simple extension of* Π *to* $K \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_k$ *can be further extended to a simple extension* *of* Π *to* $K \cup B_1 \cup \cdots \cup B_k$. *Moreover, there is a linear time algorithm that replaces all* K*-bridges* B *in* G *with their E-graphs* $\tilde{B}$.

In [24] it is further proved that the size of E-graphs of weakly attached bridges is at most 12. Moreover, if a weakly attached bridge B has some simple embedding extension, then $\mathrm{bsize}_K(\tilde{B}) \leq 5$.

Theorem 3.2 shows that we can replace every K-bridge B in G by its small E-graph $\tilde{B}$, and simple embedding extension problems do not change. This enables us to consider only obstructions that can be expressed as the union of E-graphs.

4. Restricted embedding extensions. Let K be a subgraph of G and let $\mathcal{P}$ be the set of all basic pieces of K. If B is a K-bridge, let $T \subseteq \mathcal{P}$ be the set of basic pieces of K that B is attached to. We say that B is of *type* T. Suppose that K is Π-embedded in some surface. In general, a bridge of type T can be embedded in two or more faces of K, and in some faces in several different ways. To formalize the essentially different ways of embedding bridges in particular faces, we introduce the notion of embedding schemes. Let F be a Π-face. For $T \subseteq \mathcal{P}$, let $\pi_1, \ldots, \pi_k$ be the appearances of basic pieces from T on F. An *embedding scheme* for the type T in the face F is a subset of $\pi_1, \ldots, \pi_k$ in which at least one appearance of every basic piece from T occurs. An embedding scheme δ is *simple* if each basic piece from T has exactly one appearance in δ. There is a natural partial ordering among the embedding schemes for the type $T \subseteq \mathcal{P}$ in F, induced by the set inclusion: If δ and δ' are embedding schemes for T in the same face F, then $\delta \preceq \delta'$ if every appearance of a basic piece in δ also participates in δ'.

Let B be a K-bridge of type T and δ an embedding scheme for T in a face F. An embedding of B in F is δ-*compatible* (shortly a δ-*embedding*) if B is attached only to appearances of basic pieces from δ. If $\delta \preceq \delta'$, then every δ-embedding is also a δ'-embedding.

An *embedding distribution* $\Delta(T)$ *for a type* $T \subseteq \mathcal{P}$ is a selection of embedding schemes for the type T, possibly in different faces. Suppose that $T_1, T_2, \ldots, T_s$ are all types of K-bridges in G. An *embedding distribution* is a family $\Delta = \{\Delta(T_1), \ldots, \Delta(T_s)\}$ where $\Delta(T_i)$ is an embedding distribution for the type T_i, $i = 1, \ldots, s$. Δ is *simple* if all embedding schemes in $\Delta(T_1), \ldots, \Delta(T_s)$ are simple. Let $\mathcal{B}$ be a set of K-bridges with an embedding extending the given embedding of K. We say that the embedding of $\mathcal{B}$ is Δ-*compatible* (or a Δ-*embedding*) if the embedding of each bridge $B \in \mathcal{B}$ is δ-compatible for some $\delta \in \Delta(T)$, where T is the type of B. The relation $\preceq$ naturally extends from embedding schemes to embedding distributions. The *order* $\mathrm{ord}(\Delta)$ of Δ is equal to the total number of embedding schemes in the embedding distributions $\Delta(T_i)$, $i = 1, \ldots, s$. If Δ is an embedding distribution for the set $\mathcal{B}$ of K-bridges in G and if $\mathcal{B}' \subseteq \mathcal{B}$, then the *restriction* of Δ to $\mathcal{B}'$ is the embedding distribution obtained from Δ by removing the embedding distributions $\Delta(T)$ for those types T that are not present among the bridges in $\mathcal{B}'$. If there is no confusion, the restriction of Δ to $\mathcal{B}'$ is also denoted by Δ.

Now we introduce a formal definition of an *embedding extension problem*, abbreviated EEP. This is a quadruple $\Xi = (G, K, \Pi, \Delta)$ where G is a graph, K is a subgraph of G, Π is an embedding of K, and Δ is an embedding distribution for the K-bridges in G. The EEP is *simple* if Δ is simple. An *embedding extension* (abbreviated EE) for Ξ is an embedding extension of Π to G such that every K-bridge is Δ-embedded. An *obstruction* for Ξ is a set $\mathcal{B}$ of K-bridges or their subgraphs such that $(K \cup \mathcal{B}, K, \Pi, \Delta)$

admits no EE. The *size* $\mathrm{bsize}_K(\mathcal{B})$ of an obstruction $\mathcal{B}$ is

$$\mathrm{bsize}_K(\mathcal{B}) = \sum_{B\in\mathcal{B}} \mathrm{bsize}_K(B).$$

Embedding distributions will be used in the sequel in the following way. For every possible embedding distribution Δ we will try to extend the given embedding of K to a Δ-embedding of G. Embedding distributions will be selected one after another respecting the order $\preceq$. We start with the embedding distribution of order 0, and any bridge is an obstruction for this subproblem. In a general step, we already have obstructions for all embedding distributions $\Delta' \prec \Delta$. Let $\mathcal{B}$ denote their union. Then we try to extend each Δ-embedding of $\mathcal{B}$ to a Δ-embedding of G. Obtaining an embedding, we stop and return the embedding (and our task is complete). Otherwise, an obstruction is obtained. Finally, the obstructions for different embeddings of $\mathcal{B}$ are combined together with $\mathcal{B}$ into a single obstruction for Δ-compatible embedding extensions. We will refer to this process as the *procedure of embedding distribution of types*.

Suppose that we fix an embedding distribution Δ_0. Using the procedure of embedding distribution of types we determine all (minimal) embedding distributions $\Delta \preceq \Delta_0$ for which a Δ-compatible EE exists, and at the same time construct obstructions for all other Δ-embeddings ($\Delta \preceq \Delta_0$). Algorithmically, a problem in the procedure of embedding distribution of types is in bounding the number of Δ-compatible embeddings of the union $\mathcal{B}$ of obstructions for all simpler embedding distributions. By using an operation called *compression* (cf. Section 5), we will be able to achieve that all obstructions have bounded size and hence also bounded number of embeddings. We shall use this approach in the proof of Corollary 5.5.

The procedure of embedding distribution of types can be generalized by introducing the union of EEPs. Suppose that we want to consider embedding extensions where we fix embeddings of some of the bridges. To formalize, we call an EEP $\Xi' = (G, K', \Pi', \Delta')$ a *subproblem* of $\Xi = (G, K, \Pi, \Delta)$ if

(i) K' is the union of K and a set $\mathcal{B}$ of K-bridges in G.
(ii) Π' is an EE of Π.
(iii) The Π'-embedding of every $B \in \mathcal{B}$, viewed as an extension of Π, is Δ-compatible.
(iv) Every Δ'-compatible embedding of a K'-bridge in G, viewed as an EE of the embedding Π, is Δ-compatible.

For $i = 1, \ldots, N$, let $\Xi_i = (G, K_i, \Pi_i, \Delta_i)$ be subproblems of $\Xi = (G, K, \Pi, \Delta)$. Denote by $\mathcal{B}_i$ the set of K-bridges in K_i. We say that Ξ is the *union* of subproblems Ξ_i ($1 \le i \le N$) if for every set $\mathcal{B} \supseteq \cup_{i=1}^N \mathcal{B}_i$ of K-bridges in G, the restriction of Ξ to $K \cup \mathcal{B}$ admits an EE exactly when the restriction to $K \cup \mathcal{B}$ of at least one of Ξ_i does. In this case, an EE for some Ξ_i is also an EE for Ξ, while having obstructions Ω_i for Ξ_i ($1 \le i \le N$), their combination

$$\Omega = \cup_{i=1}^N (\Omega_i \cup \mathcal{B}_i) \tag{3}$$

is an obstruction for Ξ.

A subproblem $\Xi' = (G, K, \Pi, \Delta')$ of $\Xi = (G, K, \Pi, \Delta)$ is *equivalent* to Ξ if for every set $\mathcal{B}$ of K-bridges in G and every Δ-compatible EE of K to $K \cup \mathcal{B}$, there is also a Δ'-compatible EE of K to $K \cup \mathcal{B}$. In such a case, an EE for Ξ' is also an EE for Ξ, and every obstruction for Ξ' is an obstruction for Ξ. Therefore, a solution for Ξ' provides also a solution for Ξ.

We shall use the introduced notions mainly in the following particular case.

LEMMA 4.1. *Let* $\Xi = (G, K, \Pi, \Delta)$ *be an EEP. Let* $\mathcal{B}$ *be a set of* K*-bridges in* G*, and let* $\Pi_1, \ldots, \Pi_N$ *be all* Δ*-embeddings of* $\mathcal{B}$ *extending* Π*. For* $i = 1, \ldots, N$*, let* Δ_i *be the largest embedding distribution for* $(K \cup \mathcal{B})$*-bridges in* G *such that every* Δ_i*-embedding of a* $(K \cup \mathcal{B})$*-bridge is also a* Δ*-embedding, and let* $\Delta'_i \preceq \Delta_i$ *be such an embedding distribution that the EEP* $\Xi_i = (G, K \cup \mathcal{B}, \Pi_i, \Delta'_i)$ *is equivalent to* $(G, K \cup \mathcal{B}, \Pi_i, \Delta_i)$*. Then* Ξ *is the union of subproblems* $\Xi_1, \ldots, \Xi_N$*. In particular, by solving EEPs* $\Xi_1, \ldots, \Xi_N$ *we either get an EE for* Ξ*, or (3) gives an obstruction.*

In our algorithms we shall use Lemma 4.1 only in cases when the number of bridges in $\mathcal{B}$ (and hence also the number N of their Δ-embeddings) is bounded by some constant.

We shall also need the following strengthening of a particular case of Lemma 4.1. Let $\Xi = (G, K, \Pi, \Delta)$ be an EEP and x, y be basic pieces (or segments of basic pieces) of K. Denote by $\mathcal{B}_{x,y}$ the set of K-bridges in G of type $T = \{x, y\}$, and suppose that $\mathcal{B}_{x,y} \neq \emptyset$. If x is a main vertex, put $x_1 = x_2 = x$. If x is an open branch, let x_1 and x_2 be vertices of attachment of bridges in $\mathcal{B}_{x,y}$ that are as close as possible to one and the other end of x, respectively. Define similarly y_1 and y_2. For $i, j \in \{1, 2\}$, we select a bridge $B^{i,j}_{x,y} \in \mathcal{B}_{x,y}$ with the following properties:

(a) $B^{i,j}_{x,y}$ is attached to x_i.

(b) Among all bridges from $\mathcal{B}_{x,y}$ attached to x_i, $B^{i,j}_{x,y}$ has an attachment on y as close to y_j as possible.

(c) Subject to (a) and (b), we select $B^{i,j}_{x,y}$ to be an edge if possible.

Let $\mathcal{B}^{\circ}_{x,y}$ be the set of bridges that contains all bridges $B^{i,j}_{x,y}$ ($i, j \in \{1, 2\}$) and for each $\delta \in \Delta(T)$ such that $\mathcal{B}_{x,y}$ has no δ-embedding, $\mathcal{B}^{\circ}_{x,y}$ contains a pair of bridges from $\mathcal{B}_{x,y}$ whose δ-embeddings overlap. If $\Delta(T)$ is simple, then one can construct $\mathcal{B}^{\circ}_{x,y}$ in linear time by using [23].

LEMMA 4.2. *Assuming the above notation, suppose that* $\Delta(T) = \{\delta_1, \delta_2\}$*. Then* Ξ *is equivalent to the union of subproblems* $\Xi' = (G, K \cup \mathcal{B}^{\circ}_{x,y}, \Pi', \Delta')$*, taken over all* Δ*-compatible EEs* Π' *of* Π *to* $K \cup \mathcal{B}^{\circ}_{x,y}$*, where* Δ' *is the restriction of* Δ *to the remaining bridges with the only exception that* $\Delta'(T)$ *contains only those embedding scheme(s)* δ_i *(*$i \in \{1, 2\}$*) which are used by the bridges from* $\mathcal{B}^{\circ}_{x,y}$ *under the EE* Π'*.*

Proof. It is only to be observed that whenever the embedding of $\mathcal{B}^{\circ}_{x,y}$ uses just one embedding scheme, say δ_1, then all bridges from $\mathcal{B}_{x,y}$ may be assumed to be δ_1-embedded since their embedding obstructs possible embeddings of other bridges no more than the embedding of $\mathcal{B}^{\circ}_{x,y}$. □

Let $\Xi = (G, K, \Pi, \Delta)$ be an EEP. Let $\mathcal{B}$ be the set of all K-bridges in G. Suppose that $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_N$. Denote by Δ_i the restriction of Δ to $\mathcal{B}_i$, $i = 1, \ldots, N$. The EEP $\Xi_i = (K \cup \mathcal{B}_i, K, \Pi, \Delta_i)$ is a *partial problem* of Ξ. We say that Ξ is the *intersection* of partial problems Ξ_i, $i = 1, \ldots, N$, if arbitrary EEs for $\Xi_1, \ldots, \Xi_N$ determine an EE Π_0 for Ξ. More precisely, if there are EEs Π_i for Ξ_i ($i = 1, \ldots, N$), then there is an EE Π_0 for Ξ such that its restriction to $K \cup \mathcal{B}_i$ coincides with Π_i, $i = 1, \ldots, N$.

Having $\Pi_1, \ldots, \Pi_N$, one can determine Π_0 in linear time as described below. We shall assume that $\operatorname{bsize}(K)$ and N are bounded by a constant since this will hold in our applications (although this assumption is not essential). The number of Π-faces is bounded by $2 \operatorname{bsize}(K)$. Therefore, it suffices to describe the algorithm for an arbitrary Π-face F of K. Let $\mathcal{B}'_i \subseteq \mathcal{B}_i$ ($1 \leq i \leq N$) be the bridges that are Π_i-embedded in F. Select an orientation of F. For $B \in \mathcal{B}'_i$, let $v_0, \ldots, v_{q-1}$ be its consecutive attachments on F. If e is a foot of B attached to v_j, then we put

$\mathrm{next}(e) = v_{j+1}$ where the index is taken modulo q. The function next can easily be computed in linear time (for all bridges at the same time). Now, consider an appearance of a vertex v on F, and let $\{e_1, e_2\}$ be the angle on F at this appearance of v. We may also assume that F is oriented so that e_1 precedes e_2. The local rotation Π_0 at v between e_1 and e_2 is now easily determined by a merging: we proceed through the lists $L_i = (\Pi_i(e_1), \Pi_i^2(e_1), \Pi_i^3(e_1), \ldots, e_2)$, $i = 1, \ldots, N$, and insert in the rotation of Π_0 at v the initial edge e from that list L_i which has the largest $\mathrm{next}(e)$, i.e., the distance along F from v to $\mathrm{next}(e)$ (in the given direction) is maximal. If there is more than one candidate for e, there are exactly two of them, and one of them belongs to a K-bridge with more than two attachments, the other to a bridge with two attachments. In such a case we select the former one. It can be shown that this procedure gives the desired embedding Π_0. The details are left to the reader.

5. Simple embedding extensions. In this section we will consider only simple embeddings of bridges and simple EEPs. We may assume the following:

(a) Each bridge has been replaced by its small E-graph (cf. Theorem 3.2).
(b) Every K-bridge in G has at least one simple embedding extending some embedding of K. (Otherwise, its E-graph is a small obstruction and we may stop.) In particular, if some bridge is attached only to two vertices of K, its E-graph is just a branch.
(c) There are no local bridges.
(d) Multiple branches between the same pair of vertices of K have been replaced by a single one.
(e) There are at most $4\,\mathrm{bsize}(K)$ strongly attached bridges. (Otherwise we get an obstruction of bounded size by Lemma 3.1.)

We shall refer to above assumptions (a)–(e) as *Property (E)* of K.

Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP where K has Property (E). We shall now consider some special subproblems of Ξ. Suppose that $\mathcal{B}$ is a set of K-bridges and $\Xi' = (G, K \cup \mathcal{B}, \Pi', \Delta')$ is a subproblem of Ξ. Then Ξ' is *2-restricted* if every K-bridge B in G, $B \notin \mathcal{B}$, has at most two Δ'-compatible embeddings extending the embedding Π'.

Suppose that we have a set of vertices $W_0 \subseteq V(K)$. Let W_1 be the union of W_0 and all main vertices of K. Denote by $\mathcal{S}$ the set of connected components of $K - W_1$. Suppose that we replace the paths in $\mathcal{S}$ by new pairwise disjoint paths in $G - W_1$ joining the same ends as the original paths. Then the new subgraph K' of G is homeomorphic to K and the homeomorphism $K \to K'$ is the identity on the stars of vertices in W_1. The types of bridges with respect to K and K' are in the obvious correspondence and so are the embeddings of K and K' and the embedding schemes for their bridges. Suppose that G contains exactly the same types of K-bridges and K'-bridges. Then the replacement of K by K' is called a *compression* with respect to W_0.

THEOREM 5.1 (JUVAN AND MOHAR [20]). *There is a function* $\mathrm{c}_1 : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ *such that the following holds. Let* $\Xi = (G, K, \Pi, \Delta)$ *be a 2-restricted subproblem of an EEP, and let* W_0 *be a set of vertices of* K*. If there is no* Δ*-compatible EE, then there is a compression* $K \mapsto K'$ *with respect to* W_0 *such that the modified EEP* $\Xi' = (G, K', \Pi, \Delta)$ *admits an obstruction* $\mathcal{B}$ *such that* $\mathrm{bsize}_{K'}(\mathcal{B}) \le \mathrm{c}_1(|W_0|, \mathrm{bsize}(K))$*. Moreover, there is an algorithm with time complexity* $O(\mathrm{c}_1(|W_0|, \mathrm{bsize}(K))\,|V(G)|)$ *that either finds an EE for* Ξ*, or performs the compression* $K \mapsto K'$ *and returns an obstruction* $\mathcal{B}$ *for* Ξ' *as described above.*

The compression combined with the procedure of embedding distribution of types will be our main tool that will be used in order to guarantee that the obstructions constructed by our algorithms are not too large.

There is another important special instance of EEPs. Suppose that K has Property (E) and that there is a Π-face F that contains two singular branches e and f. Suppose that $F = AeBfCe^{-}Df^{-}$ where e^{-} and f^{-} denote the traversal of e and f, respectively, in the opposite direction and where A, B, C, D are open segments of F between the appearances of e and f. Let $\mathcal{B}$ be a set of K-bridges in G, each of which has an attachment in the interior of e or f. Suppose also that Δ is a simple embedding distribution for bridges in $\mathcal{B}$ such that for each of the types, the embedding schemes allow all together at most one appearance of each basic piece distinct from e, f. Then the EEP $\Xi = (K \cup \mathcal{B}, K, \Pi, \Delta)$ and every EE subproblem of Ξ, $\Xi_0 = (K \cup \mathcal{B}, K \cup \mathcal{B}_0, \Pi_0, \Delta_0)$ $(\mathcal{B}_0 \subseteq \mathcal{B})$, is a *corner EEP*. Every bridge in $\mathcal{B}$ is attached to the interior of e or f. Therefore, under any EE for Ξ (or Ξ_0), all bridges from $\mathcal{B}$ are embedded in the face F. The following nontrivial result has been proved by Marinček, Juvan, and Mohar [18].

THEOREM 5.2 (JUVAN, MARINČEK, AND MOHAR [18]). *There is a constant* c_0 *such that every corner EEP is the union of at most* c_0 *2-restricted EE subproblems.*

The difficult part of the proof of Theorem 5.2 consists of showing that $\mathcal{B}$ contains a subset $\mathcal{B}_0$ of at most 30 bridges such that for every Δ-embedding of $\mathcal{B}_0$ in F that gives rise to a subface F' of F, which contains a singular segment of e and a singular segment of f, the following holds. For one of the singular branches of F', say ε, the bridges $\mathcal{B}_\varepsilon \subseteq \mathcal{B} \backslash \mathcal{B}_0$ that are attached to ε admit a Δ-embedding (extending the embedding of $\mathcal{B}_0$) such that no Δ-embedding of any of the remaining bridges from $\mathcal{B}$ is obstructed by this embedding. Consequently, the subproblem with such an embedding of $\mathcal{B}_0$ is equivalent to a subproblem where the bridges from $\mathcal{B}_\varepsilon$ have the corresponding fixed embedding. Under this subproblem, F' can be considered as not having two singular branches. Therefore we say that $\mathcal{B}_0$ *removes the double* $\{e, f\}$*-singularity*. Having $\mathcal{B}_0$ with the above property, one can see that each subproblem with a fixed Δ-embedding of $\mathcal{B}_0$ is the union of 2-restricted subproblems. It is shown in [18] that $\mathcal{B}_0$ and additional representatives for further reductions to 2-restricted EEPs can be obtained in linear time. By applying the generalized procedure of embedding distribution of types with compression, this yields:

THEOREM 5.3. *There is a function* $\mathrm{c}_2 : \boldsymbol{N} \to \boldsymbol{N}$ *such that the following holds. Let* $\Xi = (G, K, \Pi, \Delta)$ *be a corner EEP with corresponding singular branches* e *and* f*, and let* W_0 *be a set of vertices of* K*. If there is no* Δ*-compatible EE, then there is a compression* $K \mapsto K'$ *with respect to* W_0 *such that the modified corner EEP* $\Xi' = (G, K', \Pi, \Delta)$ *admits an obstruction* $\mathcal{B}$ *of bounded size,* $\mathrm{bsize}_{K'}(\mathcal{B}) \le \mathrm{c}_2(|W_0|)$*. Moreover, there is an algorithm with time complexity* $O(\mathrm{c}_2(|W_0|)\,|V(G)|)$*, that either finds an EE for* Ξ*, or performs a compression* $K \mapsto K'$ *(by changing only segments of* e *and* f*) and returns an obstruction* $\mathcal{B}$ *for* Ξ' *as described above.*

Proof. By Theorem 5.2, Ξ is the union of a bounded number of 2-restricted subproblems $\Xi_i = (G, K \cup \mathcal{B}_0, \Pi_i, \Delta_i)$, $1 \le i \le s$. Moreover, as shown in [18], $\mathcal{B}_0$ and the corresponding subproblems Ξ_i can be generated in linear time, and by using compression with respect to W_0, also the size of $\mathcal{B}_0$ is bounded by certain constant. Let W_1 be the union of W_0 and the set of vertices of attachment of all bridges in $\mathcal{B}_0$. For $i = 1, \ldots, s$, we solve the 2-restricted subproblem Ξ_i by using Theorem 5.1 and perform compression with respect to W_i. Obtaining an EE we stop. Otherwise, let $\mathcal{B}_i$ be the resulting obstruction (of bounded size). It may happen that after the compression $K \mapsto K'$, some K'-bridges in G become large. Therefore we apply the procedure from [24] in order that K' and its bridges satisfy Property (E). We then define W_{i+1} as the union of W_i and vertices of attachment of all bridges in $\mathcal{B}_i$. This choice guarantees that the compression at the ith step does not change any of the previous obstructions $\mathcal{B}_j$ $(j < i)$ and that $\mathcal{B}_j$ remains an obstruction for Ξ_j although the subgraph K has been changed. One can think of a corner EEP as being an embedding into the torus of a graph homeomorphic to K_4. Since $\mathrm{bsize}(K_4) = 6$, Theorem 5.1 implies that the size of $\mathcal{B}_i$ is bounded by $\mathrm{c}_1(|W_i|, 6)$. Since s is bounded by the constant c_0 from Theorem 5.2, it follows that $|W_i|$ and $\mathrm{bsize}_{K'}(\mathcal{B}_i)$ are bounded for each i.

After s steps we either find an EE or we stop with a compressed graph K' and the corresponding obstruction $\mathcal{B}' = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_s$ for Ξ' composed of E-graphs of K'-bridges in G. □

Suppose that we have an EEP $\Xi = (G, K, \Pi, \Delta)$, and that $\mathcal{B}$ is an obstruction for all EEPs $\Xi' = (G, K, \Pi, \Delta')$ for which $\Delta' \prec \Delta$. Consider all possible Δ-compatible embedding extensions of Π to $K \cup \mathcal{B}$. Then Ξ is the union of subproblems, in each of which $\mathcal{B}$ has a fixed embedding. In each of these subproblems, for every type T of K-bridges and each embedding scheme $\delta \in \Delta(T)$, there is a bridge of type T in $\mathcal{B}$ that is δ-embedded since otherwise, the embedding of $\mathcal{B}$ would be Δ'-compatible for some $\Delta' \prec \Delta$. Such a bridge is called a *representative* for δ (with respect to the chosen subproblem), and we say that $\mathcal{B}$ is a *complete set of representatives* for Ξ.

The next result will enable us to apply Theorems 5.1 and 5.3 in solving general simple EEPs.

THEOREM 5.4. *Let K be a subgraph of G with Property (E). Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP and suppose that no edge of K appears on a Π-facial walk twice in the same direction. Suppose that $\mathcal{B}_0$ is a complete set of representatives for Ξ and that $K \cup \mathcal{B}_0$ also has Property (E). Then there is a number c_3 depending only on* $\mathrm{bsize}(K \cup \mathcal{B}_0)$ *such that each subproblem $\Xi_0 = (G, K \cup \mathcal{B}_0, \Pi_0, \Delta_0)$ of Ξ is equivalent to the union of at most c_3 EE subproblems, each of which is the intersection of a 2-restricted EEP and at most* $\mathrm{bsize}(K)/2$ *corner EEPs. The decompositions of Ξ_0 to subproblems and of these to corresponding partial problems can be performed in $O(\mathrm{c}_3|V(G)|)$ time.*

Proof. Let $\mathcal{B}_0'$ be the set of K-bridges consisting of $\mathcal{B}_0$, all strongly attached $(K \cup \mathcal{B}_0)$-bridges, and all bridges $\mathcal{B}^\circ_{x,y}$, where x, y are arbitrary basic pieces of $K \cup \mathcal{B}_0$, and bridges $\mathcal{B}^\circ_{x,y}$ are defined before Lemma 4.2. Since $K \cup \mathcal{B}_0$ has Property (E), the size of $\mathcal{B}_0'$ is bounded by a function of $\mathrm{bsize}(K \cup \mathcal{B}_0)$. Lemma 4.2 implies that Ξ_0 is the union of subproblems $\Xi' = (G, K \cup \mathcal{B}_0', \Pi', \Delta')$ taken over all Δ_0-embeddings of $\mathcal{B}_0' \backslash \mathcal{B}_0$ extending the embedding Π_0 where every 2-restricted type of $(K \cup \mathcal{B}_0)$-bridges in Ξ_0 has its representatives for embedding schemes in Δ'. It suffices to see that every such subproblem Ξ' is the union of a bounded number of subproblems, each of which is equivalent to the intersection of a 2-restricted EEP and at most $\mathrm{bsize}(K)/2$ corner problems.

First, we shall prove that Ξ' is equivalent to the union of a bounded number of subproblems of the form $\Xi'' = (G, K \cup \mathcal{B}_0'', \Pi'', \Delta'')$ where $\mathcal{B}_0''$ consists of $\mathcal{B}_0'$ and some additional bridges. The number of these additional bridges is bounded (depending on $\mathrm{bsize}(K)$).

Recall that $\mathcal{B}_0'$ contains all strongly attached $(K \cup \mathcal{B}_0)$-bridges in G. Because of Property (E), $\mathcal{B}_0'$ contains all $(K \cup \mathcal{B}_0)$-bridges that are attached to two main vertices of K. Let $B \notin \mathcal{B}_0'$ be a K-bridge of type $T = \{e, v\}$ where e is an open branch and

v is a main vertex of K. Let e_B be the smallest closed segment of e containing all vertices of attachment of B to e. Suppose that F is a Π'-face in which B can be Δ'-embedded. Since B is not a strongly attached $(K \cup \mathcal{B}_0)$-bridge, e_B is contained in an open branch $e' \subseteq e$ of $K \cup \mathcal{B}_0$. Denote by ε an appearance of e' in F. Let $v_1, \ldots, v_l$ be the appearances of v on F. Since $\mathcal{B}_0$ is a complete set of representatives, $\Delta'(T)$ contains at most two embedding schemes using ε and one of $v_1, \ldots, v_l$. Moreover, any embedding extension of Π' to a subset of K-bridges in G can be changed so that all bridges of type T in F that are attached to ε are attached just to one appearance of v in F. This implies that Ξ' is equivalent to a subproblem $\Xi'' = (G, K \cup \mathcal{B}_0', \Pi', \Delta'')$ where K-bridges that are not attached to two open branches of K have at most two admissible embeddings.

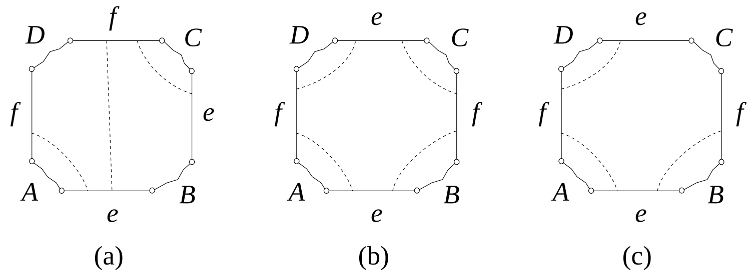

FIG. 1. *The possibilities for more than two embedding schemes*

It remains to see how we control embeddings of K-bridges that are attached to two open branches of K. For most pairs e, f of open branches, K-bridges of type $\{e, f\}$ will have at most two Δ'-embeddings. This may not be the case only when segments of both e and f appear twice on the same Π'-facial walk. Possible cases are shown in Figure 1 with dotted curves indicating the embedding schemes in Δ' that contain appearances of e or f. By assumption, each of the branches e and f appears on the facial walk once in each direction. Therefore we can speak about the left and the right side of e and the top or bottom of f (with respect to the presentation in Figure 1). We shall assume that the face F shown in Figure 1 is a Π-face, and we shall have in mind that there is a collection of K-bridges from $\mathcal{B}_0'$ that are Π'-embedded in F but not explicitly shown.

Let us first consider pairs $T = \{e, f\}$ which correspond to case (a) of Figure 1. In each of such cases we shall either conclude that bridges of type T admit at most two Δ''-embeddings (possibly after restricting to an equivalent subproblem), or we will find a bridge B whose presence in $\mathcal{B}_0''$ would guarantee the same as in the former possibility. Since there are only a bounded number of pairs T, we can afterwards add all such bridges B to $\mathcal{B}_0''$ and then start again from the beginning. The presence of the added bridges in $\mathcal{B}_0''$ will now guarantee that the former possibility always occurs.

Let $B_1 = B_{e,f}^{i,j}$ $(i, j \in \{1, 2\})$ be the K-bridge corresponding to the rightmost attachment e_i on e and the topmost attachment f_j on f. Note that $B_1 \in \mathcal{B}_0 \cup \mathcal{B}_{x,y}^{\circ} \subseteq \mathcal{B}_0'$ for some $x \subseteq e$, $y \subseteq f$. Assume first that B_1 is Π'-embedded in F so that it is attached to the right occurrence of e. Then B_1 is attached to f at its upper occurrence since the other possibility is not Δ''-compatible. Let y be an attachment of B_1 to f. By the choice of B_1, if y is not the only attachment of B_1 on f, then each of the bridges of type T admits at most two Δ''-embeddings extending Π', and we are done. If y is the only attachment, then we can have bridges of type T with three distinct Δ''-embeddings. However, the set $\mathcal{B}'$ of such bridges has only one attachment on f;

it is equal to y. (Another possibility for bridges with three embeddings in F includes bridges of type T whose only attachment on e is e_i. Though, this case is excluded since the left-right embeddings in F are not Δ''-compatible.) The two occurrences of y on F separate F into two segments. If no bridge from $\mathcal{B}'_0$ is embedded in F such that it is attached to the interior of each of these segments, then every EE of Π' to a subset of K-bridges can be changed so that no bridge from $\mathcal{B}'$ is attached to the left occurrence of y (say). In other words, Ξ'' is equivalent to a subproblem where each bridge of type T has only two allowed embeddings (and we shall assume that this subproblem is already Ξ''). On the other hand, if there is a Π'-embedded bridge $B_2 \in \mathcal{B}'_0$ in F that separates the two occurrences of y, there is only one possibility for a bridge of type T to have three possible Δ''-embeddings. Such a bridge B must be attached only to two vertices, and so it is just a branch by Property (E). In this case we shall add B in $\mathcal{B}''_0$. Then we will be able to forget about B having three distinct embeddings on the expense of a few additional subproblems to be solved.

The second possibility is when B_1 is attached to the lower occurrence of e and the left occurrence of f. Now, the only bridges of type T with more than two possible Δ''-embeddings have their only vertex on e equal to e_i. We conclude in the same way as we did in the first case, using e_i instead of y.

The third possibility is when B_1 is embedded so that it is attached to the lower and the upper occurrence of e and f, respectively. In this case, there are two ways that bridges could have more than two Δ''-embeddings extending Π'. If $B^{i,3-j}_{e,f} \neq B_1$, then one of the two possibilities is excluded. The remaining one is essentially the same as the second possibility treated above. On the other hand, if $B^{i,3-j}_{e,f} = B_1$, then we have a situation that is essentially the same as the first case above. In each case we know how to act.

Let us now consider cases (b) and (c) of Figure 1. In case (c), it may happen that there is an embedding scheme in Δ'' containing an appearance of a basic piece in the segment C and the left occurrence of f (or the bottom occurrence of e). In such a case, bridges of type $\{e, f\}$ may be assumed to have only two possible embeddings. This is established in the same way as above (by possibly adding a new bridge to $\mathcal{B}''_0$ or restricting to an equivalent subproblem). We assume from now on that this is not the case.

We say that $\mathcal{B}''_0$ *removes the double* $\{e, f\}$*-singularity* if no subface F' of F contains singular branches $e' \subseteq e$ and $f' \subseteq f$ such that there exist $(K \cup \mathcal{B}''_0)$-bridges attached to each of e' and f'. If the Π''-embedded bridges $\mathcal{B}''_0$ do not remove the double $\{e, f\}$-singularity, then $\{e, f\}$ is a *corner pair* for Ξ''. Since $\mathcal{B}_0$ is a complete set of representatives, distinct corner pairs are disjoint. Therefore there are at most $\mathrm{bsize}(K)/2$ corner pairs. If case (b) or (c) applies for $T = \{e, f\}$ and T is not a corner pair, then reductions from [18] (by possibly extending $\mathcal{B}''_0$ or restricting Δ'' to an equivalent subproblem) can be used to get subproblems where all K-bridges of type T have at most two Δ''-embeddings. This will be assumed in the sequel as already done.

If $\{e, f\}$ is a corner pair, let $\mathcal{B}^{e,f}_1$ be the set of K-bridges in G of type $\{e, f\}$ that are not in $\mathcal{B}''_0$. Let $\mathcal{B}_2$ be the set of K-bridges that are not in $\mathcal{B}''_0$ and that are not in $\mathcal{B}^{e,f}_1$ for any corner pair $\{e, f\}$. Furthermore, let $\mathcal{B}^{e,f}_2$ contain all K-bridges from $\mathcal{B}_2$ that have an attachment on e or f and have at most one Δ''-embedding extending the embedding Π'' of $K \cup \mathcal{B}''_0$. Similarly, let $\mathcal{B}_1$ contain those bridges from $\mathcal{B}^{e,f}_1$, taken over all corner pairs $\{e, f\}$, which have at most one Δ''-embedding extending Π''.

Consider the EEPs

$$\Xi_1^{e,f} = (K \cup \mathcal{B}_0'' \cup \mathcal{B}_1^{e,f} \cup \mathcal{B}_2^{e,f}, K \cup \mathcal{B}_0'', \Pi'', \Delta_1^{e,f}) \tag{4}$$

where $\{e, f\}$ is a corner pair and $\Delta_1^{e,f}$ is the restriction of Δ'' to $\mathcal{B}_1^{e,f} \cup \mathcal{B}_2^{e,f}$. Let

$$\Xi_2 = (K \cup \mathcal{B}_0'' \cup \mathcal{B}_1 \cup \mathcal{B}_2, K \cup \mathcal{B}_0'', \Pi'', \Delta_2) \tag{5}$$

be the partial problem of Ξ'' restricted to $\mathcal{B}_1 \cup \mathcal{B}_2$. We claim that Ξ'' is the intersection of partial problems $\Xi_1^{e,f}$ (taken over all corner pairs $\{e, f\}$) and Ξ_2. Suppose not. Since different corner pairs do not obstruct each other, there is an EE Π_1 for some $\Xi_1^{e,f}$ and an EE Π_2 for Ξ_2 that cannot be combined into an EE for Ξ''. This means that a Π_1-embedded bridge $B_1 \in \mathcal{B}_1^{e,f} \backslash \mathcal{B}_1$ and a Π_2-embedded bridge $B_2 \in \mathcal{B}_2 \backslash \mathcal{B}_2^{e,f}$ overlap. Since $\mathcal{B}_0$ is a complete set of representatives, B_1 overlaps only with bridges that are attached to e or to f. Since $B_2 \notin \mathcal{B}_0''$, we may assume that B_2 is of type $\{f, x\}$ where $x \subseteq A$. See Figure 2 where the cases (a) and (b) from below are distinguished.

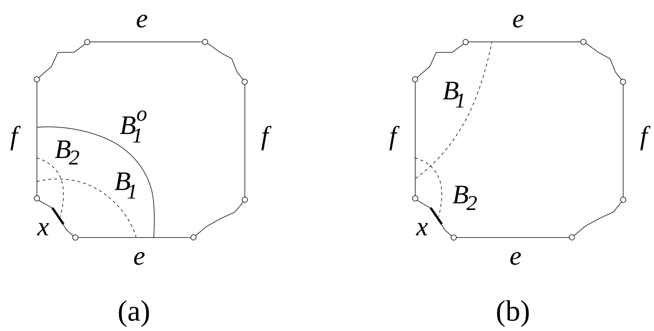

FIG. 2. *B_1 and B_2 overlap*

An embedding of $B \in \mathcal{B}_1^{e,f}$ is an *embedding in the corner* α if B is attached to the lower occurrence of e and the left occurrence of f. Similarly we define embeddings in corners β, γ, δ as those that are using the lower/right, upper/right, or upper/left occurrences of e/f, respectively. In the obvious way we also classify embeddings of bridges of type $\{f, x\}$ to be in corners α, β, γ, or δ. We may assume that B_2 is Π_2-embedded in the corner α.

Since $B_1 \notin \mathcal{B}_0''$, there is a $(K \cup \mathcal{B}_0)$-bridge $\tilde{B}_1 \in \mathcal{B}_0''$ that is of the same type $\{e_1, f_1\}$, $e_1 \subseteq e$, $f_1 \subseteq f$, as B_1 and of the form $B_{f_1,e_1}^{i,j}$ where $(f_1)_i$ refers to the lowest attachment on f_1. Similarly, there is $\tilde{B}_2 \in \mathcal{B}_0''$ of the same type $\{e_2, f_2\}$ as B_2 and of the form $B_{f_2,e_2}^{k,l}$ where $(f_2)_k$ refers to the topmost attachment on f_2. Now we distinguish two cases.

Case (a). *Embeddings of bridges of type $\{e, f\}$ in corner α are $\Delta_1^{e,f}$-compatible:* There is a representative $B_1^\circ \in \mathcal{B}_0$ that is Π''-embedded (and hence also Π_1-embedded and Π_2-embedded) in the corner α. Therefore B_1 is Π_1-embedded in α as well. Since $\tilde{B}_1 \in \mathcal{B}_0''$, it does not overlap with B_2 under the embedding Π_2. Since $\{e, f\}$ is a corner pair, $\tilde{B}_1$ and B_1° do not remove the double $\{e, f\}$-singularity. Hence $\tilde{B}_1$ is embedded in the corner β. Consequently, $\tilde{B}_2$ is not Π''-embedded in β. If $\tilde{B}_2$ is embedded in the corner γ or δ, B_1° and $\tilde{B}_2$ remove the double $\{e, f\}$-singularity. Hence, $\tilde{B}_2$ is Π''-embedded in α. This implies that B_1 cannot be embedded in the corner α, a contradiction.

Case (b). *Embeddings in corner α are not $\Delta_1^{e,f}$-compatible:* Let $B_\beta, B_\gamma, B_\delta$ be representatives from $\mathcal{B}_0$ that are Π''-embedded in corners β, γ, δ, respectively. We shall distinguish four cases according to where $\tilde{B}_2$ is Π''-embedded.

Subcase α. $\tilde{B}_2$ is in the corner α. This contradicts the fact that B_1 is Π_1-embedded in α.

Subcase β. $\tilde{B}_2$ being in β, $\tilde{B}_1$ is in α or in δ. Since embeddings in α are not $\Delta_1^{e,f}$-compatible, $\tilde{B}_1$ is in δ. This eliminates the possibility for B_2 being Π_2-embedded in the corner α, and we are done.

Subcase γ. $\tilde{B}_2$ is in γ. Denote by y the lowest attachment of B_γ to f. When adding the sets $\mathcal{B}^\circ_{x',y'}$ $(x' \subseteq x, y' \subseteq f)$ into $\mathcal{B}'_0 \subseteq \mathcal{B}''_0$ and restricting Δ'' according to Lemma 4.2 we have assured that there are representatives for all embedding schemes of $(K \cup \mathcal{B}_0)$-bridges of such types $\{x', y'\}$. Since $\{e, f\}$ is a corner pair for Ξ'', such bridges with embeddings in corner α have all their attachments to f strictly below y. Therefore, they all belong to $\mathcal{B}_2^{e,f}$. In particular, this holds for the bridge B_2, and we have a contradiction.

Subcase δ. $\tilde{B}_2$ is in δ. Let y be the lowest attachment of B_δ to f. We conclude as above.

This proves that Ξ'' is the intersection of corner problems $\Xi_1^{e,f}$ and Ξ_2. Let us observe that the ≤ 2 embeddings of bridges from $\mathcal{B}_2$ are determined by their types as $(K \cup \mathcal{B}''_0)$-bridges. Therefore, Ξ_2 can be formulated as a 2-restricted EEP, and the proof is complete. □

The assumption in Theorem 5.4 that no edge appears on a Π-facial walk twice in the same direction is not essential. We have decided to use it since it eliminates a few cases in the proof and since this condition will be automatically satisfied at the time when applying the theorem. Let us also mention that with a slightly modified proof of Theorem 5.4, one can achieve c_3 being bounded only by a function of $\mathrm{bsize}(K)$.

COROLLARY 5.5. *Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP and let W_0 be a subset of vertices of K. Suppose that K has Property (E) and that no edge of K appears on a Π-facial walk twice in the same direction. There is a function $\mathrm{c} : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ and an algorithm with time complexity $O(\mathrm{c}(|W_0|, \mathrm{ord}(\Delta))|V(G)|)$ that either finds a Δ-compatible EE or returns a subgraph K' of G obtained by a compression with respect to W_0 and a set of at most $\mathrm{c}(|W_0|, \mathrm{ord}(\Delta))$ E-graphs of K'-bridges in G that form an obstruction for the corresponding EEP $\Xi' = (G, K', \Pi, \Delta)$.*

Proof. The proof is by induction on $\mathrm{ord}(\Delta)$. If $\mathrm{ord}(\Delta) = 0$, then any K-bridge in G is an obstruction for Ξ. Hence, a Δ-embedding exists if and only if $K = G$. Suppose now that $\mathrm{ord}(\Delta) > 0$. There are $\mathrm{ord}(\Delta)$ embedding distributions $\Delta_1, \Delta_2, \ldots$ that are strictly simpler than Δ and are maximal with this property. Inductively, we first solve the subproblem $\Xi_1 = (G, K, \Pi, \Delta_1)$ taking care of the set W_0. An EE makes us happy and we stop. Otherwise, we compress K with respect to W_0. Let K_1 be the new subgraph of G and $\mathcal{B}_1$ an obstruction of bounded size as guaranteed by the induction hypothesis. Let W_1 be the union of W_0 and the set of vertices of attachment of bridges from $\mathcal{B}_1$. Now we replace W_0 by W_1 and solve the subproblem $\Xi_2 = (G, K_1, \Pi, \Delta_2)$, taking care of the set W_1. We either stop, or we get a new graph K_2 (after a compression with respect to W_1) and an obstruction $\mathcal{B}_2$ of bounded size. In the latter case we extend W_1 into W_2 by adding all attachments of bridges from $\mathcal{B}_2$. Continuing, we either find an EE, which is a Δ-embedding as well, or we stop after $\mathrm{ord}(\Delta)$ steps with a subgraph K' of K that is a compression of K with respect to W_0. At the same time we get an obstruction $\mathcal{B}_0 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$. Now, since $\mathcal{B}_0$ is an obstruction for all simpler EEPs, it is a complete set of representatives for Ξ.

Since Ξ is the union of subproblems, taken over all Δ-embeddings of $\mathcal{B}_0$, and since $\mathcal{B}_0$ has bounded size, we can consecutively apply Theorem 5.4 combined with Theorems 5.1 and 5.3, and for each of these subproblems perform a compression with respect to attachments of E-graphs in all previously obtained obstructions. An upper bound on $\mathrm{c}(|W_0|, \mathrm{ord}(\Delta))$ is easy to obtain by our inductive approach, and we leave the details to the reader. □

6. Embedding graphs in an arbitrary surface. In this section we prove the final result of this paper that embeddability in any fixed surface S can be decided in linear time. Our algorithm not only verifies if such an embedding exists. If it does, such an embedding is constructed. If not, the algorithm identifies a subgraph of G that cannot be embedded in S but every proper subgraph can. Such a subgraph is called a *minimal forbidden subgraph* for embeddability in S. We define the *Euler genus* of S as $2 - \chi(S)$ where $\chi(S)$ is the Euler characteristic of S.

THEOREM 6.1. *Let S be a fixed closed surface. There is a constant c and a linear time algorithm that for an arbitrary given graph G either:*

(a) finds an embedding of G in S, or

(b) identifies a minimal forbidden subgraph $K \subseteq G$ for embeddability in S. The branch size of K is bounded by c.

Remark. In case (a), our algorithm constructs an embedding in the surface of the smallest Euler genus (and the same orientability characteristic as S). Such an embedding determines a (possibly not 2-cell) embedding in S. If one insists on 2-cell embeddings in S, there is a polynomial time solution using an algorithm for the maximum genus [12] (which turns out to be trivial for nonorientable surfaces, cf., e.g., [26]).

A corollary of Theorem 6.1 is the result of Robertson and Seymour [27] that the set of minimal forbidden minors (or subgraphs) is finite for each surface. It is worth mentioning that our proof is constructive while the proof in [27] is only existential.

COROLLARY 6.2 (ROBERTSON AND SEYMOUR [27]). *For every surface S there is a finite list of graphs such that an arbitrary graph G can be embedded in S if and only if G does not contain a subgraph homeomorphic to one of the graphs in the list.*

The rest of the paper is devoted to the proof of Theorem 6.1. Let us just point out that in case (b) it suffices to find a subgraph K of bounded branch size (in terms of the Euler genus of S) since such a subgraph is easily changed to a minimal one in constant time (for example, by considering all subgraphs of K, up to homeomorphism, and all their embeddings).

Denote by g the Euler genus of S. If S is orientable, our algorithm determines the smallest $h \le g$ such that G can be embedded in the orientable surface of Euler genus h (or proves that such an h does not exist). If S is nonorientable, then we will determine the surface (or two surfaces) with the smallest Euler genus $h \le g$ in which G can be embedded (or show that G cannot be embedded in S). If such minimal Euler genus h is even, there is a nonorientable surface $\tilde{S}_h$ as well as an orientable surface S'_h with Euler genus h. If G can be embedded in $\tilde{S}_h$ and $h \le g$, then it can also be embedded in S. If G has an embedding in S'_h, then changing the sign of an arbitrary edge which is not a cutedge of G gives an embedding in $\tilde{S}_{h+1}$. Hence, any outcome determines the nonorientable genus of G.

The orientable genus of G is equal to the sum of the genera of its blocks [4] and a minimum genus embedding is a simple combination of minimal embeddings of the blocks. A similar reduction works in the nonorientable case [33]. Since the blocks can be determined in linear time, we may assume from now on that the graph G is 2-connected.

If G is 2-connected and $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{x, y\} \subseteq V(G)$ and each of G_1 and G_2 contains at least two edges, then we say that $\{x, y\}$ is a *separating pair*. In such a case, let the graph G_i' be obtained from G_i by adding the edge xy if it is not already present in G_i $(i = 1, 2)$. The added edge xy is called the *virtual edge* of G_i'. If G_1' is 3-connected, then G_1' is a *pendant 3-connected block* of G. If G_2' is planar, then every embedding of G_1' can be changed into an embedding of G in the same surface after replacing the edge $xy \in E(G_1')$ by G_2 using a planar embedding of G_2'. In such a case we call the operation of replacing G by G_1' a *2-reduction*. We can consider the graph G_1' as being a subgraph of G by using a path in G_2 from x to y instead of the new edge xy. Therefore, any obstructions in G_1' give rise to obstructions of the same branch size in G. By using linear time algorithms of Hopcroft and Tarjan to determine the 3-connected components of G [15] and for testing planarity [16], we can perform all possible 2-reductions in linear time. At the same time we locate all pendant 3-connected blocks in G, and for each such block B we find a Kuratowski subgraph $H_B \subseteq B$. If possible, we choose H_B so that it does not contain the virtual edge of B.

We shall assume from now on that G is a 2-connected graph in which no 2-reductions are possible. In particular, G is simple and has no vertices of degree 2. The following lemmas will be used to bound the number of pendant 3-connected blocks.

LEMMA 6.3. *Suppose that $K = L \cup H$ where H is a subgraph of K homeomorphic to a Kuratowski graph and that $L \cap H$ is either empty, one or two vertices, a segment of a branch of H, or a segment of a branch of L. If g' is the Euler genus of K and g is the Euler genus of L, then $g' \geq g + 1$.*

Proof. By the additivity of the Euler genus, the result is clear when $L \cap H$ is empty or a single vertex. Otherwise, let x and y be the two vertices of of $L \cap H$ or the ends of the segment of a branch (of L or of H) in $L \cap H$, respectively. Since $L \subseteq K$, we have $g \leq g'$.

If $g' = g$, consider an embedding Π' of K with Euler genus g. It is an extension of an embedding Π of L. Since Π is an embedding of L of minimal Euler genus, no Π-facial walk W contains two vertices that appear on W in the interlaced order. (If not, one could change Π to an embedding with smaller Euler genus.) This immediately excludes the case when $L \cap H = \{x, y\}$. Similarly, if $L \cap H$ is contained in a branch of H: Since K_5 and $K_{3,3}$ are 3-connected, there is a single L-bridge in K. It is attached to x and y only, and it does not have a simple EE. Therefore, x and y appear interchangeably on a Π-facial walk, a contradiction. The remaining case is when $L \cap H$ is a segment σ of a branch e of L. Let $L' = L - \operatorname{int} \sigma$. Since K_5 and $K_{3,3}$ are 3-connected, there are one or two L'-bridges in K. In the latter case, one of the L'-bridges is just a segment of a branch of H, and by replacing σ with that branch we can appeal to the previous case treated above. So, we may assume that there is a single L'-bridge in K; it is equal to H. If the branch e is contained in two Π-facial walks, the embedding extension of $\Pi | L'$ to $L' \cup H = K$ gives a contradiction as above. On the other hand, if e is singular, it appears on the facial walk twice in opposite direction and hence the embedding of K yields an embedding of H in the cylinder, a contradiction. □

LEMMA 6.4. *Let G be a 2-connected graph. Suppose that $G = K \cup B_1^- \cup \cdots \cup B_s^-$ $(s \geq 5)$ where K has a branch e containing vertices $x_1, y_1, x_2, y_2, \ldots, x_s, y_s$ (in that* *order; $x_i \neq y_i$ but possibly $y_i = x_{i+1}$) such that $K \cap B_i^-$ is equal to the segment of e from x_i to y_i ($1 \le i \le s$). and if $B_i^- \cap B_{i+1}^- \neq \emptyset$ for some $1 \le i < s$, then $B_i^- \cap B_{i+1}^- = \{y_i\} = \{x_{i+1}\}$. Suppose, moreover, that each of the graphs B_i^- is planar but $B_i^- + x_iy_i$ is nonplanar ($1 \le i \le s$). Let $G' = K \cup B_1^- \cup B_3^- \cup B_5^-$. If Π' is a minimum genus or a minimum Euler genus embedding of G' in the surface S, then Π' can be changed into an embedding of G in S.*

Proof. Since $B_i^- + x_iy_i$ is not planar and since G (and hence also K) contains a path from x_i to y_i that is edge-disjoint from B_i^-, B_i^- contains a Π'-noncontractible cycle C_i ($i = 1, 3, 5$). The graphs $B_1^-, \ldots, B_s^-$ are planar and distinct B_i^-, B_j^- intersect in at most one vertex which belongs to e. This implies that there is an embedding Π_0 of $e \cup B_1^- \cup \cdots \cup B_s^-$ of genus 0. If C_1 is 2-sided, then Π' restricted to K and Π_0 are easily combined into an embedding of G in S. Similarly, if C_3 or C_5 is 2-sided. On the other hand, if C_1, C_3, C_5 are all 1-sided, then the Euler genus of Π' restricted to $K - e$ is smaller than the Euler genus of Π' by at least three since C_1, C_3, C_5 are disjoint. Now, the same surgery as used in the 2-sided case yields an embedding of G whose Euler genus increases by at most two. This is a contradiction to minimality of Π'. □

Our next goal is to find a 2-connected subgraph K of G such that no K-bridges in G are local. First we construct an intermediate graph K_0. If G is 3-connected, then we let K_0 be a Kuratowski subgraph of G. Otherwise, for each pendant 3-connected block B of G, let K_B be its subgraph obtained by the following construction. Let H_B be a Kuratowski subgraph of B and let $\{x, y\}$ be the separating pair of G corresponding to B. If H_B contains the virtual edge xy, then put $K_B = H_B - xy$. Otherwise, let K_B be obtained from H_B by adding two disjoint paths (possibly of length 0) from $\{x, y\}$ to H_B. The graphs K_B are easily constructed in linear time by standard techniques mentioned earlier in this paper. Now, we start by taking $K_0 = K_{B_0}$ where B_0 is an arbitrary pendant 3-connected block of G. We shall extend K_0 in several steps. Note that K_0 may become 2-connected only after the next step. In each of these steps we first check if there is a pendant 3-connected block B such that either K_B is edge-disjoint from the current graph, or $K_B \supseteq H_B$. If so, we add K_B and two disjoint paths from its separating set to the current graph. If one of such paths passes through a pendant 3-connected block Q, we make sure that inside Q it uses only edges of K_Q. By Lemma 6.3, the new graph K_0 has larger Euler genus than the previous one, so this case occurs at most g times (or else we get a small forbidden subgraph for embeddability in S and stop). After $O(g)$ such steps, each of the remaining pendant 3-connected blocks B has the property that B^- (B without its virtual edge) is planar and that $K_0 \cap K_B$ is a segment of a branch e of K_0. We say that B is pendant on e.

Consider the bridges $B_1, \ldots, B_s$ that are pendant on e, in the order as their segments $K_0 \cap B_i^-$ appear on e. By Lemma 6.4 we may assume that $s \le 4$ (possibly after changing the graph G by replacing B_2, B_4, and $B_6, \ldots, B_s$ by corresponding segments of e). Now we add the graphs K_{B_i}, $i = 1, \ldots, s$, into K_0. Note that in this case, there is no need to add corresponding linking paths. We repeat the same for all branches e of K_0, and then our construction stops. Since we make all together $O(g)$ steps, we can afford to spend $O(n)$ time for each step, hence there is no problem in achieving linear time complexity in the construction of K_0.

The graph K_0 constructed above is 2-connected and $\mathrm{bsize}(K_0)$ is bounded. For each branch e of K_0, let $\mathrm{local}(e, K_0)$ be the union of e and all local K_0-bridges on e. If $\{x, y\}$ is a separating pair of G, then each component of $G - \{x, y\}$ intersects some pendant 3-connected block and hence contains a main vertex of K_0. This property of

K_0 enables us to use a linear time algorithm from [17] to achieve one of the following:

(a) We get a path e' in local(e, K_0) joining the ends of e such that the graph $K_0' = K_0 - e + e'$ has no local bridges on e'. Note that local$(f, K_0') =$ local(f, K_0) for all branches $f \neq e$ of K_0', and that local$(e', K_0') = e'$.

(b) We get a subgraph $K_e \subseteq$ local(e, K_0) that is homeomorphic to a Kuratowski graph. In this case we delete e from K_0, and then add K_e and paths in local(e, K_0) from the ends of e to K_e so that the resulting graph K_0' is 2-connected. Note that this step increases the branch size of the graph at most by 13.

We repeat the procedure with the new graph K_0' and all its branches f for which local$(f, K_0') \neq f$. Lemma 6.3 shows that after a bounded number of steps we either stop with a 2-connected graph $K \subseteq G$ such that there are no local K-bridges in G (which we assume henceforth), or we find a subgraph of G of bounded branch size that cannot be embedded in S.

Having constructed K as explained above, the algorithm continues by induction on the genus g of S (or the Euler genus g of S if S is nonorientable). Recursively, we have either found an embedding in a surface of (Euler) genus smaller than g (in which case we stop), or we got a 2-connected subgraph K of G that cannot be embedded in any surface with (Euler) genus smaller than g. By the induction hypothesis (or by the above construction if $g = 0$), bsize(K) is bounded. Therefore, K has only a bounded number of embeddings in S (and each of them is 2-cell). Existence of an embedding of G in S is thus equivalent to the existence of an EE with respect to a bounded number of EEPs corresponding to particular embeddings of K in S. By solving all these problems (and successively performing compressions, if necessary, and taking care that vertices of attachment of bridges in previously obtained obstructions are not changed during later compressions), we either get an embedding of G in S, or the union of obstructions for the EEPs gives a subgraph $\tilde{K}$ of bounded branch size that cannot be embedded in S. If we will use $\tilde{K}$ in further processing, we just make sure that there are no local $\tilde{K}$-bridges. This can be done in the same way as in the construction of the initial subgraph K.

It remains to see how we solve an EEP $\Xi = (G, K, \Pi, \Delta)$ where Δ contains all embedding schemes that are possible under the given embedding Π of K in the surface S. Let us first verify that no edge of K appears on a Π-facial walk F traversed twice in the same direction. This is clear if S is orientable. If S is nonorientable, changing the signature on such an edge would change Π into an embedding with the same facial walks except that F splits into two facial walks. This contradicts the fact that Π is an embedding of K with minimal Euler genus.

We will construct a sequence of graphs $K_0, K_1, \ldots$ such that $K_0 = K$ and K_{i+1} is obtained (after a compression) from K_i by adding an obstruction for simple embedding extensions. Let us describe the construction of K_{i+1} ($i = 0, 1, 2, \ldots$) in more details. First of all, we replace each K_i-bridge in G by its E-graph. This can be done in linear time by Theorem 3.2. By using Corollary 5.5, we get in linear time the set $\mathcal{B}_i$ of K_i-bridges in a compressed obstruction for simple embedding extensions of K_i to G, taken over all EEs of Π to K_i. Of course, having found an EE, we stop and by Theorem 3.2 we also get an EE of K_0 to G. Assuming that no EE has been found, and assuming inductively that the branch size of K_i is bounded, also bsize$_{K_i}(\mathcal{B}_i)$ is bounded (Corollary 5.5). We now define $K_{i+1} = K_i \cup \mathcal{B}_i$ and observe that there are no K_{i+1}-bridges that are local on a branch of K_{i+1} contained in K_i. On the other hand, bridges that are local on branches from $\mathcal{B}_i$ can be eliminated by the algorithm

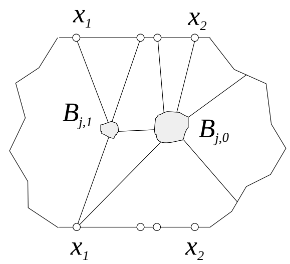

FIG. 3. $B_{j,1}$ *does not increase* q

from [17] similarly as at the very beginning of our algorithm. After doing that, we stop if $K_{i+1} = G$ or if K_{i+1} has no embeddings in S.

Note that for each i, $\mathcal{B}_i \neq \emptyset$ (or we stop with an embedding). Therefore, the above process terminates after a finite number of steps. We claim that the number of steps cannot be too large. Let $B_1, \dots, B_k$ be the K_0-bridges in K_i $(i \geq 1)$. (When constructing K_i, we may have used a compression and thus have changed $K_{i-1}, K_{i-2}, \dots, K_0$. But a compression is a graph homeomorphism which is identity on the neighborhoods of main vertices of K_{i-1}, and hence we can also view $K_0, K_1, \dots$ as being subgraphs of the changed graph K_{i-1}.) Since $\mathcal{B}_0, \dots, \mathcal{B}_{i-1}$ always consist of E-graphs with respect to $K_0, \dots, K_{i-1}$, respectively, each B_j $(1 \leq j \leq k)$ can be written as $B_j = B_{j,0} \cup B_{j,1} \cup \cdots \cup B_{j,i-1}$ where $B_{j,l} = B_j \cap \mathcal{B}_l$, $l = 0, \dots, i-1$. Let us consider an embedding Π_i of K_i in S as an EE of the embedding Π of K_0. Then an E-graph in some $B_{j,0}$ is nonsimply embedded. This implies that B_j is attached to at least three appearances of basic pieces of K_0. Consider the sum

$$\sum_{r=1}^{k} (q(B_r) - 2) \tag{6}$$

where $q(B_r)$ is defined in Lemma 3.1. Now, $q(B_j)$ contributes at least 1 to (6). Let us now consider the induced embedding of Π_i to K_2 as an extension of the embedding of K_1. Since $\mathcal{B}_1 \subseteq K_2$ is an obstruction for simple extensions of K_1, there is an E-graph B in some $B_{j,1}$ that is not simply embedded. We claim that we can choose B such that $q(B_{j,0} \cup B_{j,1}) \geq q(B_{j,0} \cup B) > q(B_{j,0})$ (in all three cases viewed as K_0-bridges). If this is not the case, then B is attached only to $B_{j,0}$ and to the same appearances of basic pieces of K_0 as $B_{j,0}$. No basic piece in $B_{j,0} \backslash K_0$ is singular under the considered embedding of K_1. Hence $B_{j,1}$ is attached to two appearances of a basic piece x' of K_1, and if $x \supseteq x'$ is the basic piece of K_0 containing x', then $B_{j,0}$ is attached to the corresponding appearances of x. Since $B_{j,0}$ is an E-graph of a K_0-bridge, it contains feet at extreme attachments x_1, x_2 of B_j on x. We have shown above that no edge of K_0 appears on a Π-facial walk twice in the same direction. It follows that the embedding of $B \subseteq B_{j,1}$ is as shown in Figure 3 and that x' is an extreme attachment of B_j, say $x' = x_1$. However, this embedding can easily be changed so that B is not attached to the upper occurrence of x_1 (say), without affecting possible embeddings of other bridges from $\mathcal{B}_1$. After doing the same with other candidates for B, we get a contradiction with $\mathcal{B}_1$ being an obstruction for simple embeddings.

The same proof can be carried further, for embeddings of K_3, K_4, etc. We conclude that the sum (6) is at least i. Now, Lemma 3.1 implies that $i \leq 4\,\mathrm{bsize}(K_0)$. The proof is complete.

REFERENCES

[1] D. Archdeacon, *A Kuratowski Theorem for the projective plane*, J. Graph Theory 5 (1981) 243–246.

[2] ———, *The complexity of the graph embedding problem*, in "Topics in Combinatorics and Graph Theory," R. Bodendiek and R. Henn (Eds.), Physica-Verlag, Heidelberg, 1990, pp. 59–64.

[3] ———, P. Huneke, *A Kuratowski theorem for nonorientable surfaces*, J. Combin. Theory Ser. B 46 (1989) 173–231.

[4] J. Battle, F. Harary, Y. Kodama, J. W. T. Youngs, *Additivity of the genus of a graph*, Bull. Amer. Math. Soc. 68 (1962) 565–568.

[5] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1981.

[6] K. S. Booth, G. S. Lueker, *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-trees*, J. Comput. System Sci. 13 (1976) 335–379.

[7] N. Chiba, T. Nishizeki, S. Abe, T. Ozawa, *A linear algorithm for embedding planar graphs using PQ-trees*, J. Comput. System Sci. 30 (1985) 54–76.

[8] S. A. Cook, R. A. Reckhow, *Time bounded random access machines*, J. Comput. Syst. Sci. 7 (1976) 354–375.

[9] H. Djidjev, J. H. Reif, *An efficient algorithm for the genus problem with explicit construction of forbidden subgraphs*, 23rd Annual ACM Symposium on Theory of Computing, New Orleans, LA, May 1991, pp. 337–347.

[10] I. S. Filotti, G. L. Miller, J. Reif, *On determining the genus of a graph in* $O(v^{O(g)})$ *steps*, in "Proc. 11th Ann. ACM STOC," Atlanta, Georgia (1979) pp. 27–37.

[11] H. de Fraysseix, P. Rosenstiehl, *A depth-first search characterization of planarity*, Ann. Discr. Math. 13 (1982) 75–80.

[12] M. L. Furst, J. L. Gross, L. A. McGeoch, *Finding a maximum-genus graph imbedding*, J. Assoc. Comput. Mach. 35 (1988) 523–534.

[13] H. Glover, J. P. Huneke, C. S. Wang, 103 *graphs that are irreducible for the projective plane*, J. Combin. Theory, Ser. B 27 (1979) 332–370.

[14] J. L. Gross, T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

[15] J. E. Hopcroft, R. E. Tarjan, *Dividing a graph into triconnected components*, SIAM J. Comput. 2 (1973) 135–158.

[16] ———, ———, *Efficient planarity testing*, J. ACM 21 (1974) 549–568.

[17] M. Juvan, J. Marinček, B. Mohar, *Elimination of local bridges*, Math. Slovaca 47 (1997) 85–92.

[18] ———, ———, ———, *Obstructions for simple embeddings*, submitted.

[19] ———, ———, ———, *Embedding a graph into the torus in linear time*, submitted.

[20] M. Juvan, B. Mohar, *Extending 2-restricted partial embeddings of graphs*, submitted.

[21] A. Karabeg, *Classification and detection of obstructions to planarity*, Lin. Multilin. Algebra 26 (1990) 15–38.

[22] B. Mohar, *Projective planarity in linear time*, J. Algorithms 15 (1993) 482–502.

[23] ———, *Obstructions for the disk and the cylinder embedding extension problems*, Combin. Probab. Comput. 3 (1994) 375–406.

[24] ———, *Universal obstructions for embedding extension problems*, submitted.

[25] ———, C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Univ. Press, to appear.

[26] G. Ringel, *The combinatorial map color theorem*, J. Graph Theory 1 (1977) 141–155.

[27] N. Robertson, P. D. Seymour, *Graph minors. VIII. A Kuratowski theorem for general surfaces*, J. Combin. Theory, Ser. B 48 (1990) 255–288.

[28] ———, ———, *Graph minors. XIII. The disjoint paths problem*, J. Combin. Theory,

Ser. B 63 (1995) 65–110.

[29] ———, ———, *Graph minors. XXI. Graphs with unique linkages*, preprint, 1992.

[30] ———, ———, *Graph minors. XXII. Irrelevant vertices in linkage problems*, preprint, 1992.

[31] ———, ———, *An outline of a disjoint paths algorithm*, in: "Paths, Flows, and VLSI-Layout" (B. Korte, L. Lovász, H. J. Prömel and A. Schrijver eds.), Springer-Verlag, Berlin, 1990, pp. 267–292.

[32] P. D. Seymour, *A bound on the excluded minors for a surface*, submitted.

[33] S. Stahl, L. W. Beineke, *Blocks and the nonorientable genus of graphs*, J. Graph Theory 1 (1977) 75–78.

[34] C. Thomassen, *The graph genus problem is NP-complete*, J. Algorithms 10 (1989) 568–576.

[35] ———, *Triangulating a surface with a prescribed graph*, J. Combin. Theory Ser. B 57 (1993) 196–206.

[36] S. G. Williamson, *Embedding graphs in the plane — algorithmic aspects*, Ann. Discrete Math. 6 (1980) 349–384.

[37] ———, *Depth-first search and Kuratowski subgraphs*, J. ACM 31 (1984) 681–693.