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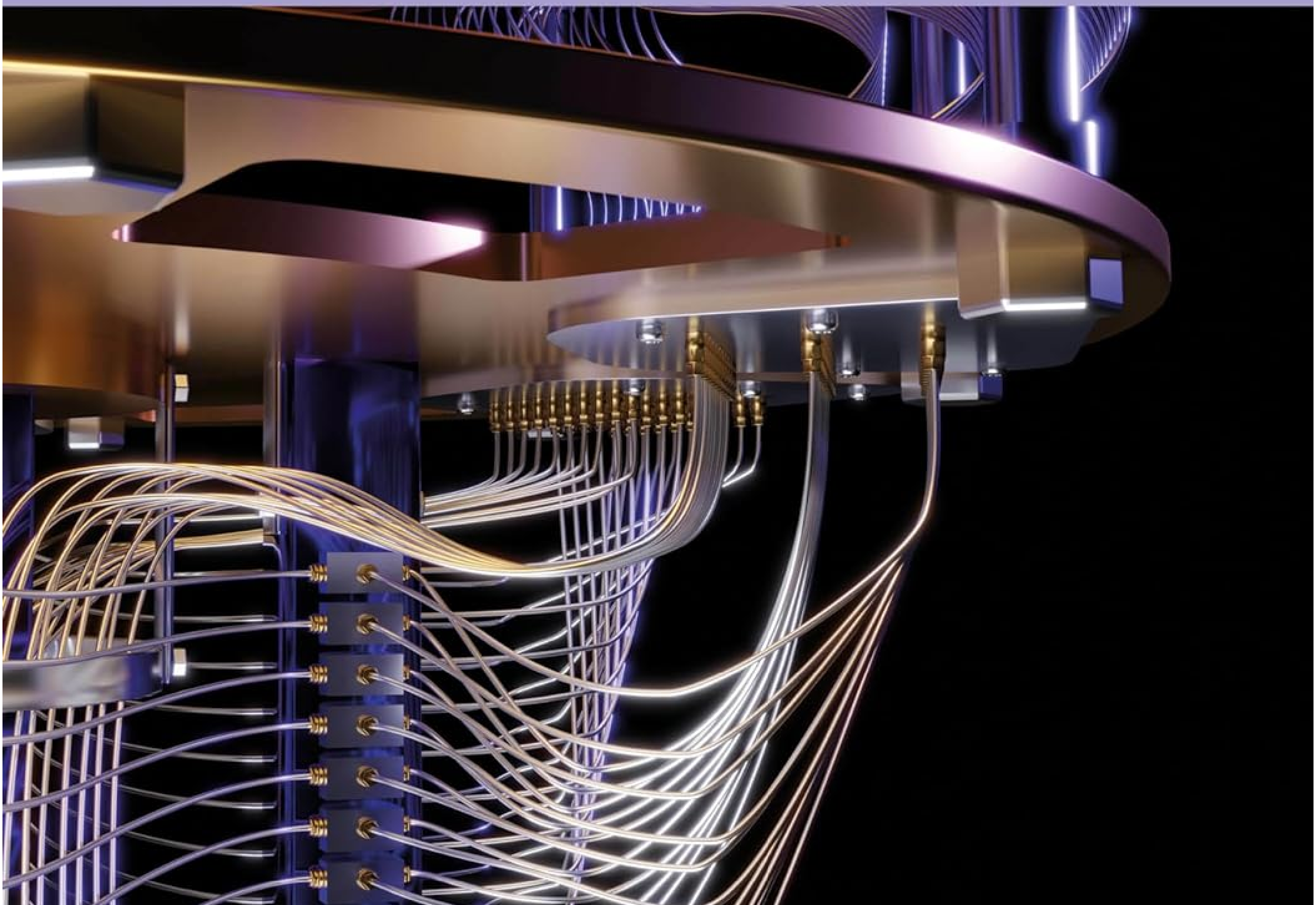
Innovative Quantum Computing

Steven Duplij
Raimund Vogl



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Steven Duplij
Center for Information Processing WWU IT, University of Münster,
48149 Münster, Germany

Raimund Vogl
Center for Information Processing WWU IT, University of Münster,
48149 Münster, Germany

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Preface

This book is devoted to the study of exotic and non-standard mathematical methods in quantum computing. The principal ingredients of quantum computation are qubits and their transformations, which can be provided in different ways: first mathematically, and they can then be further realized in hardware.

In this book we consider various extensions of the qubit concept per se, starting from the obscure qubits introduced by the authors, and other fundamental generalizations. We then introduce a new kind of gate, higher braiding gates, which are implemented for topological quantum computations, as well as unconventional computing, when computational complexity is affected by its environment, which needs an additional stage of computation. Other generalizations are also considered and explained in a widely accessible and easy to understand style.

This book will be useful for graduate students and last year students for additional advanced chapters of lecture courses in quantum computer science and information theory.

Steven Duplij and Raimund Vogl
Münster, Germany
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Author biographies

Steven Duplij

Steven Duplij (Stepan Douplii) is a theoretical and mathematical physicist from the University of Münster, Germany. He was born in Chernyshevsk-Zabaikalsky, Russia, and studied at Kharkov University, Ukraine where he gained his PhD in 1983. While working at Kharkov, he received the title Doctor of Physical and Mathematical Sciences by Habilitation in 1999. Dr Duplij is the editor-compiler of the 'Concise Encyclopedia of Supersymmetry' (2005, Springer), and is the author of more than a hundred scientific publications and several books. He is listed in the World Directory of Mathematicians, Marques Who Is Who In America, the Encyclopedia of Modern Ukraine, the Academic Genealogy of Theoretical Physicists, and the Mathematics Genealogy Project. His scientific interests include supersymmetry and quantum groups, advanced algebraic structures, gravity and nonlinear electrodynamics, constrained systems, and quantum computing.

Raimund Vogl

Raimund Vogl is the CIO of the University of Münster (Germany) and has been the director of the University's IT center since 2007. He holds a PhD in elementary particle physics from the University of Innsbruck (Austria). After completing his PhD studies in 1995, he joined Innsbruck University Hospital as an IT manager for medical image data solutions and moved on to become the deputy head of IT. He is board member and president of EUNIS (European University Information Systems Organisation), and a member of Deutsche Gesellschaft für Medizinische Informatik, Biometrie und Epidemiologie (GMDS) and the Association for Information Systems (AIS). His current research interest in the field of information systems and information management focuses on the management of complex information infrastructures.

Chapter 1

Obscure qubits and membership amplitudes

Nowadays, the development of quantum computing technique is governed by theoretical extensions of its ground concepts (Nielsen and Chuang 2000, Kaye *et al* 2007, Williams and Clearwater 1998). One of these extensions is to allow two kinds of uncertainty, sometimes called randomness and vagueness/fuzziness (for a review, see, Goodman and Nguyen 2002), which leads to the formulation of combined probability and possibility theories (Dubois *et al* 2000) (see, also, Bělohlávek 2002, Dubois and Prade 2000, Smith 2008, Zimmermann 2011). Various interconnections between vagueness and quantum probability calculus were considered in Pykacz (2015), Dvurečenskij and Chovanec (1988), Bartková *et al* (2017), and Granik (1994), including the treatment of inaccuracy in measurements (Gudder 1988, 2005), non-sharp amplitude densities (Gudder 1989), and the related concept of partial Hilbert spaces (Gudder 1986).

Relations between truth values and probabilities were also given in Bolotin (2018). The hardware realization of computations with vagueness was considered in Hirota and Ozawa (1989), and Virant (2000). On the fundamental physics side, it was shown that the discretization of space-time at small distances can lead to a discrete (or fuzzy) character for the quantum states themselves.

With a view to applications of these ideas in quantum computing, we introduce a definition of quantum state that is described by both a quantum probability and a membership function (Duplij and Vogl 2021), and thereby incorporate vagueness/fuzziness directly into the formalism. In addition to the probability amplitude, we will define a membership amplitude, and such a state will be called an obscure/fuzzy qubit (or qudit) (Duplij and Vogl 2021).

In general, the Born rule will apply to the quantum probability alone, while the membership function can be taken to be an arbitrary function of all of the amplitudes fixed by the chosen model of vagueness. Two different models of obscure-quantum computations with truth are proposed below: (1) a Product obscure qubit, in which the resulting amplitude is the product (in $\mathbb{C}$) of the quantum

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amplitude and the membership amplitude; and (2) a Kronecker obscure qubit, for which computations are performed in parallel, so that quantum amplitudes and the membership amplitudes form vectors, which we will call obscure-quantum amplitudes. In the latter case, which we call a double obscure-quantum computation, the protocol of measurement depends on both the quantum and obscure amplitudes. In this case, the density matrix need not be idempotent. We define a new kind of gate, namely, obscure-quantum gates, which are linear transformations in the direct product (not in the tensor product) of spaces: a quantum Hilbert space and a so-called membership space having special fuzzy properties (Duplij and Vogl 2021). We then introduce a new concept of double (obscure-quantum) entanglement, in which vector and scalar concurrences are defined and computed for concrete examples.

1.1 Preliminaries

To establish a notation standard in the literature (see, e.g. Nielsen and Chuang 2000, Kaye *et al* 2007), we present the following definitions. In an underlying d-dimensional Hilbert space, the standard qudit (using the computational basis and Dirac notation) $\mathcal{H}_q^{(d)}$ is given by

$$|\psi^{(d)}\rangle = \sum_{i=0}^{d-1} a_i \,|\, i\rangle, \quad a_i \in \mathbb{C}, \,|\, i\rangle \in \mathcal{H}_q^{(d)}, \tag{1.1}$$

where a_i is a probability amplitude of the state $|\, i\rangle$. (For a review, see, e.g. Genovese and Traina 2008, Wang *et al* 2020.) The probability $_{pi}$ to measure the ith state is $p_i = F_{p_i}(a_1, \ldots, a_n)$, $0 \leqslant p_i \leqslant 1$, $0 \leqslant i \leqslant d-1$. The shape of the functions F_{p_i} is governed by the Born rule $F_{p_i}(a_1, \ldots, a_d) = |\, a_i\,|^2$, and $\sum_{i=0}^{d} p_i = 1$. A one-qudit ($L = 1$) quantum gate is a unitary transformation $U^{(d)}$: $\mathcal{H}_q^{(d)} \to \mathcal{H}_q^{(d)}$ described by unitary $d \times d$ complex matrices acting on the vector (1.1), and for a register containing L qudits quantum gates are unitary $d^L \times d^L$ matrices. The quantum circuit model (Deutsch 1985, Barenco *et al* 1995) forms the basis for the standard concept of quantum computing. Here the quantum algorithms are compiled as a sequence of elementary gates acting on a register containing L qubits (or qudits), followed by a measurement to yield the result (Lloyd 1995, Brylinski and Brylinski 1994).

For further details on qudits and their transformations, see for example the reviews by Genovese and Traina (2008) and Wang *et al* (2020) and the references therein.

1.2 Membership amplitudes

Innovation 1.1. *We define an obscure qudit with d states* via *the following superposition (in place of that given in (1.1))*

$$\left|\psi_{ob}^{(d)}\right\rangle = \sum_{i=0}^{d-1} \alpha_i a_i \,|\, i\rangle, \tag{1.2}$$

where a_i is a (complex) probability amplitude $a_i \in \mathbb{C}$, and we have introduced a (real) membership amplitude α_i, with $\alpha_i \in [0, 1]$, $0 \leqslant i \leqslant d - 1$.

The probability $_{pi}$ to find the ith state upon measurement and the membership function μ_i (of truth) for the ith state are both functions of the corresponding amplitudes, as follows

$$p_i = F_{p_i}(a_0, \dots, a_{d-1}), \quad 0 \leqslant p_i \leqslant 1, \tag{1.3}$$

$$\mu_i = F_{\mu_i}(\alpha_0, \dots, \alpha_{d-1}), \quad 0 \leqslant \mu_i \leqslant 1. \tag{1.4}$$

The dependence of the probabilities of the ith states upon the amplitudes, i.e., the form of the function F_{p_i} is fixed by the Born rule

$$F_{p_i}(a_1, \dots, a_n) = |\, a_i \,|^2, \tag{1.5}$$

while the form of F_{μ_i} will vary according to different obscurity assumptions. In this paper we consider only real membership amplitudes and membership functions—complex obscure sets and numbers were considered in Buckley (1989), Ramot *et al* (2002), and Garrido (2012). In this context, the real functions F_{p_i} and F_{μ_i}, $0 \leqslant i \leqslant d - 1$ will contain complete information about the obscure qudit (1.2).

We impose the normalization conditions

$$\sum_{i=0}^{d-1} p_i = 1, \tag{1.6}$$

$$\sum_{i=0}^{d-1} \mu_i = 1, \tag{1.7}$$

where the first condition is standard in quantum mechanics, while the second condition is taken to hold by analogy. Although (1.7) may not be satisfied, we will not consider that case.

For $d = 2$, we obtain for the obscure qubit the general form, instead of that in (1.2),

$$\left|\psi_{\text{ob}}^{(2)}\right\rangle = \alpha_0 a_0 \,|\, 0\rangle + \alpha_1 a_1 \,|\, 1\rangle, \tag{1.8}$$

$$F_{p_0}(a_0, a_1) + F_{p_1}(a_0, a_1) = 1, \tag{1.9}$$

$$F_{\mu_0}(\alpha_0, \alpha_1) + F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{1.10}$$

The Born probabilities to observe the states $|\, 0\rangle$ and $|\, 1\rangle$ are

$$p_0 = F_{p_0}^{\text{Born}}(a_0, a_1) = |\, a_0 \,|^2, \quad p_1 = F_{p_1}^{\text{Born}}(a_0, a_1) = |\, a_1 \,|^2. \tag{1.11}$$

Innovation 1.2. *The membership functions are*

$$\mu_0 = F_{\mu_0}(\alpha_0, \alpha_1), \quad \mu_1 = F_{\mu_1}(\alpha_0, \alpha_1). \tag{1.12}$$

If we assume the Born rule (1.11) for the membership functions as well

$$F_{\mu_0}(\alpha_0, \alpha_1) = \alpha_0^2, \quad F_{\mu_1}(\alpha_0, \alpha_1) = \alpha_1^2, \tag{1.13}$$

which is one of various possibilities depending on the chosen model, then

$$|a_0|^2 + |a_1|^2 = 1, \tag{1.14}$$

$$\alpha_0^2 + \alpha_1^2 = 1. \tag{1.15}$$

Using (1.14)–(1.15) we can parameterize (1.8) as

$$\left|\psi_{\text{ob}}^{(2)}\right\rangle = \cos\frac{\theta}{2}\cos\frac{\theta_\mu}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sin\frac{\theta_\mu}{2}\,|1\rangle, \tag{1.16}$$

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \theta_\mu \leqslant \pi. \tag{1.17}$$

Therefore, obscure qubits (with Born-like rule for the membership functions) are geometrically described by a pair of vectors, each inside a Bloch ball (and not as vectors on the boundary spheres, because '$|\sin|, |\cos| \leqslant 1$'), where one is for the probability amplitude (an ellipsoid inside the Bloch ball with $\theta_\mu = \text{const}_1$) and the other is for the membership amplitude (which is reduced to an ellipse, being a slice inside the Bloch ball with $\theta = \text{const}_2$, $\varphi = \text{const}_3$). However, the norm of the obscure qubits is not constant because

$$\left\langle\psi_{\text{ob}}^{(2)}\middle|\psi_{\text{ob}}^{(2)}\right\rangle = \frac{1}{2} + \frac{1}{4}\cos\left(\theta + \theta_\mu\right) + \frac{1}{4}\cos\left(\theta - \theta_\mu\right). \tag{1.18}$$

In the case where $\theta = \theta_\mu$, the norm (1.18) becomes $1 - \frac{1}{2}\sin^2\theta$, reaching its minimum $\frac{1}{2}$ when $\theta = \theta_\mu = \frac{\pi}{2}$.

Note that for complicated functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$, the condition (1.15) may be not satisfied but the condition (1.7) should nevertheless always be valid. The concrete form of the functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ depends upon the chosen model. In the simplest case, we can identify two arcs on the Bloch ellipse for α_0, α_1 with the membership functions and obtain

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_1}{\alpha_0}, \tag{1.19}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_0}{\alpha_1}, \tag{1.20}$$

such that $\mu_0 + \mu_1 = 1$, as in (1.7).

In Mannucci (2006) and Maron *et al* (2013) a two stage special construction of quantum obscure/fuzzy sets was considered. The so-called classical-quantum obscure/fuzzy registers were introduced in the first step (for $n = 2$, the minimal case) as

$$| s\rangle_f = \sqrt{1-f}\,| 0\rangle + \sqrt{f}\,| 1\rangle, \tag{1.21}$$

$$| s\rangle_g = \sqrt{1-g}\,| 0\rangle + \sqrt{g}\,| 1\rangle, \tag{1.22}$$

where $f, g \in [0, 1]$ are the relevant classical-quantum membership functions. In the second step their quantum superposition is defined by

$$| s\rangle = c_f\,| s\rangle_f + c_g\,| s\rangle_g, \tag{1.23}$$

where c_f and c_g are the probability amplitudes of the fuzzy states $| s\rangle_f$ and $| s\rangle_g$, respectively. It can be seen that the state (1.23) is a particular case of (1.8) with

$$\alpha_0 a_0 = c_f\sqrt{1-f} + c_g\sqrt{1-g}, \tag{1.24}$$

$$\alpha_1 a_1 = c_f\sqrt{f} + c_g\sqrt{g}. \tag{1.25}$$

This gives explicit connection of our double amplitude description of obscure qubits with the approach (Mannucci 2006, Maron *et al* 2013) which uses probability amplitudes and the membership functions. It is important to note that the use of the membership amplitudes introduced here α_i and (1.2) allows us to exploit the standard quantum gates but not to define new special ones, as in Mannucci (2006) and Maron *et al* (2013).

Another possible form of $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ (1.12), with the corresponding membership functions satisfying the standard fuzziness rules, can be found using a standard homeomorphism between the circle and the square. In Hannachi *et al* (2007b) and Rybalov *et al* (2014), this transformation was applied to the probability amplitudes $a_{0,1}$.

Innovation 1.3. *Here we exploit it for the membership amplitudes* $\alpha_{0,1}$

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2\text{sign}\,\alpha_0 - \alpha_1^2\text{sign}\,\alpha_1 + 1}{2}}, \tag{1.26}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2\text{sign}\,\alpha_0 + \alpha_1^2\text{sign}\,\alpha_1 + 1}{2}}. \tag{1.27}$$

So for positive $\alpha_{0,1}$, we obtain (cf Hannachi *et al* 2007b)

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2 - \alpha_1^2 + 1}{2}}, \tag{1.28}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{1.29}$$

The equivalent membership functions for the outcome are

$$\max\left(\min\left(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\right), \min\left(1 - F_{\mu_0}(\alpha_0, \alpha_1)\right), F_{\mu_1}(\alpha_0, \alpha_1)\right), \tag{1.30}$$

$$\min\left(\max\left(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\right), \max\left(1 - F_{\mu_0}(\alpha_0, \alpha_1)\right), F_{\mu_1}(\alpha_0, \alpha_1)\right). \tag{1.31}$$

There are many different models for $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ which can be introduced in such a way that they satisfy the obscure set axioms (Dubois and Prade 2000, Zimmermann 2011).

1.3 Transformations of obscure qubits

Let us consider the obscure qubits in the vector representation, such that

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.32}$$

are basis vectors of the two-dimensional Hilbert space $\mathcal{H}_q^{(2)}$. A standard quantum computational process in the quantum register with L obscure qubits (qudits (1.1)) is performed by sequences of unitary matrices U of size $2^L \times 2^L$ ($n^L \times n^L$), $\mathsf{U}^\dagger \mathsf{U} = \mathsf{I}$, which are called quantum gates (I is the unit matrix). Thus, for one obscure qubit, the quantum gates are 2×2 unitary complex matrices.

Innovation 1.4. *In the vector representation, an obscure qubit differs from the standard qubit (1.8) by a 2×2 invertible diagonal (not necessarily unitary) matrix*

$$\left|\psi_{ob}^{(2)}\right\rangle = \mathsf{M}(\alpha_0, \alpha_1)\,|\psi^{(2)}\rangle, \tag{1.33}$$

$$\mathsf{M}(\alpha_0, \alpha_1) = \begin{pmatrix}\alpha_0 & 0\\ 0 & \alpha_1\end{pmatrix}. \tag{1.34}$$

We call $\mathsf{M}(\alpha_0, \alpha_1)$ a membership matrix which can optionally have the property

$$\mathrm{tr}\mathsf{M}^2 = 1, \tag{1.35}$$

if (1.15) holds.

Let us introduce the orthogonal commuting projection operators

$$\mathsf{P}_0 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \mathsf{P}_1 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{1.36}$$

$$\mathsf{P}_0^2 = \mathsf{P}_0, \ \ \mathsf{P}_1^2 = \mathsf{P}_1, \ \ \mathsf{P}_0\mathsf{P}_1 = \mathsf{P}_1\mathsf{P}_0 = \mathsf{0}, \tag{1.37}$$

where $\mathsf{0}$ is the 2×2 zero matrix. Well-known properties of the projections are that

$$\mathsf{P}_0 \mid \psi^{(2)}\rangle = a_0 \mid 0\rangle, \qquad \mathsf{P}_1 \mid \psi^{(2)}\rangle = a_1 \mid 0\rangle, \tag{1.38}$$

$$\langle\psi^{(2)} \mid \mathsf{P}_0 \mid \psi^{(2)}\rangle = \mid a_0 \mid^2, \qquad \langle\psi^{(2)} \mid \mathsf{P}_1 \mid \psi^{(2)}\rangle = \mid a_1 \mid^2. \tag{1.39}$$

Innovation 1.5. *The membership matrix (1.34) can be defined as a linear combination of the projection operators with the membership amplitudes as coefficients*

$$\mathsf{M}(\alpha_0, \alpha_1) = \alpha_0\mathsf{P}_0 + \alpha_1\mathsf{P}_1. \tag{1.40}$$

We compute

$$\mathsf{M}(\alpha_0, \alpha_1) \left| \psi_{\text{ob}}^{(2)} \right\rangle = \alpha_0^2 a_0 \mid 0\rangle + \alpha_1^2 a_1 \mid 1\rangle. \tag{1.41}$$

We can therefore treat the application of the membership matrix (1.33) as providing the origin of a reversible but non-unitary obscure measurement on the standard qubit to obtain an obscure qubit—cf the mirror measurement (Battilotti and Zizzi 2004, Zizzi 2005) and also the origin of ordinary qubit states on the fuzzy sphere (Zizzi and Pessa 2014).

An obscure analog of the density operator (for a pure state) is the following form for the density matrix in the vector representation

$$\rho_{\text{ob}}^{(2)} = \left| \psi_{\text{ob}}^{(2)} \right\rangle \left\langle \psi_{\text{ob}}^{(2)} \right| = \begin{pmatrix} \alpha_0^2 \mid a_0 \mid^2 & \alpha_0 a_0^* \alpha_1 a_1 \\ \alpha_0 a_0 \alpha_1 a_1^* & \alpha_1^2 \mid a_1 \mid^2 \end{pmatrix} \tag{1.42}$$

with the obvious standard singularity property $\det \rho_{\text{ob}}^{(2)} = 0$. But $\text{tr}\rho_{\text{ob}}^{(2)} = \alpha_0^2 \mid a_0 \mid^2 + \alpha_1^2 \mid a_1 \mid^2 \neq 1$, and here there is no idempotence $\left(\rho_{\text{ob}}^{(2)}\right)^2 \neq \rho_{\text{ob}}^{(2)}$, which can distinct $\rho_{\text{ob}}^{(2)}$ from the standard density operator.

1.4 Kronecker obscure qubits

We next introduce an analog of quantum superposition for membership amplitudes, called 'obscure superposition' (cf Cunha *et al* 2019, and also Toffano and Dubois 2017).

Innovation 1.6. *Quantum amplitudes and membership amplitudes will here be considered separately in order to define an obscure qubit taking the form of a double superposition (cf (1.8), and a generalized analog for qudits (1.1) is straightforward)*

$$\mid \boldsymbol{\Psi}_{\text{ob}}\rangle = \frac{\mathsf{A}_0 \mid 0\rangle + \mathsf{A}_1 \mid 1\rangle}{\sqrt{2}}, \tag{1.43}$$

where the two-dimensional vectors

$$\mathsf{A}_{0,1} = \begin{bmatrix} a_{0,1} \\ \alpha_{0,1} \end{bmatrix} \tag{1.44}$$

are the (double) obscure-quantum amplitudes of the generalized states $|0\rangle, |1\rangle$.

For the conjugate of an obscure qubit we put (informally)

$$\langle \mathbf{\Psi}_{ob}| = \frac{\mathsf{A}_0^\star \langle 0| + \mathsf{A}_1^\star \langle 1|}{\sqrt{2}}, \tag{1.45}$$

where we denote $\mathsf{A}_{0,1}^\star = \begin{bmatrix} a_{0,1}^* & \alpha_{0,1} \end{bmatrix}$, such that $\mathsf{A}_{0,1}^\star \mathsf{A}_{0,1} = |a_{0,1}|^2 + \alpha_{0,1}^2$. The (double) obscure qubit is normalized in such a way that, if the conditions (1.14)–(1.15) hold, then

$$\langle \mathbf{\Psi}_{ob}|\mathbf{\Psi}_{ob}\rangle = \frac{|a_0|^2 + |a_1|^2}{2} + \frac{\alpha_0^2 + \alpha_1^2}{2} = 1. \tag{1.46}$$

Innovation 1.7. *A measurement should be made separately and independently in the probability space and the membership space, which can be represented using an analog of the Kronecker product.*

Indeed, in the vector representation (1.32) for the quantum states and for the direct product amplitudes (1.44) we should have

$$|\mathbf{\Psi}_{ob}\rangle_{(0)} = \frac{1}{\sqrt{2}}\mathsf{A}_0 \otimes_K \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathsf{A}_1 \otimes_K \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.47}$$

where the (left) Kronecker product is defined by (see (1.32))

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_K \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \alpha\begin{pmatrix} c \\ d \end{pmatrix} \end{bmatrix} = \begin{bmatrix} a(c\mathsf{e}_0 + d\mathsf{e}_1) \\ \alpha(c\mathsf{e}_0 + d\mathsf{e}_1) \end{bmatrix},$$
$$\mathsf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathsf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathsf{e}_{0,1} \in \mathcal{H}_q^{(2)}. \tag{1.48}$$

Informally, the wave function of the obscure qubit, in the vector representation, now lives in the four-dimensional space of (1.48), which has two two-dimensional spaces as blocks. The upper block, the quantum subspace, is the ordinary Hilbert space $\mathcal{H}_q^{(2)}$, but the lower block should have special (fuzzy) properties, if it is treated as an obscure (membership) subspace $\mathcal{V}_{memb}^{(2)}$. Thus, the four-dimensional space, where lives $\left|\Psi_{ob}^{(2)}\right\rangle$, is not an ordinary tensor product of vector spaces because of (1.48) and the vector A on the lhs has entries of different natures, i.e., the quantum amplitudes $a_{0,1}$ and the membership amplitudes $\alpha_{0,1}$. Despite the unit vectors in $\mathcal{H}_q^{(2)}$ and $\mathcal{V}_{memb}^{(2)}$ having the same form (1.32), they belong to different spaces (because they are vector spaces over different fields). Therefore, instead of (1.48), we introduce a Kronecker-like product $\tilde{\otimes}_K$ by

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \tilde{\otimes}_K \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a(c\mathsf{e}_0 + d\mathsf{e}_1) \\ \alpha(c\varepsilon_0 + d\varepsilon_1) \end{bmatrix}, \tag{1.49}$$

$$\mathsf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathsf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathsf{e}_{0,1} \in \mathcal{H}_q^{(2)}, \tag{1.50}$$

$$\varepsilon_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)}, \quad \varepsilon_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)}, \quad \varepsilon_{0,1} \in \mathcal{V}_{memb}^{(2)}. \tag{1.51}$$

In this way, the obscure qubit (1.43) can be presented in the from

$$\begin{aligned} |\boldsymbol{\Psi}_{ob}\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \mathsf{e}_0 \\ \alpha_0 \varepsilon_0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \mathsf{e}_1 \\ \alpha_1 \varepsilon_1 \end{bmatrix}. \end{aligned} \tag{1.52}$$

Therefore, we call the double obscure qubit (1.52) a Kronecker obscure qubit to distinguish it from the obscure qubit (1.8). It can be also presented using the Hadamard product (the element-wise or Schur product)

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_H \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} ac \\ \alpha d \end{bmatrix} \tag{1.53}$$

in the following form

$$|\boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \mathsf{A}_0 \otimes_H \mathsf{E}_0 + \frac{1}{\sqrt{2}} \mathsf{A}_1 \otimes_H \mathsf{E}_1, \tag{1.54}$$

where the unit vectors of the total four-dimensional space are

$$\mathsf{E}_{0,1} = \begin{bmatrix} \mathsf{e}_{0,1} \\ \varepsilon_{0,1} \end{bmatrix} \in \mathcal{H}_q^{(2)} \times \mathcal{V}_{memb}^{(2)}. \tag{1.55}$$

The probabilities $p_{0,1}$ and membership functions $\mu_{0,1}$ of the states $|\,0\rangle$ and $|\,1\rangle$ are computed through the corresponding amplitudes by (1.11) and (1.12)

$$p_i = |\,a_i\,|^2, \quad \mu_i = F_{\mu_i}(\alpha_0, \alpha_1), \quad i = 0, 1, \tag{1.56}$$

and in the particular case by (1.13) satisfying (1.15).

By way of example, consider a Kronecker obscure qubit (with a real quantum part) with probability p and membership function μ (measure of trust) of the state $|\,0\rangle$, and of the state $|\,1\rangle$ given by $1-p$ and $1-\mu$, respectively. In the model (1.19)–(1.20) for μ_i (which is not Born-like) we obtain

$$|\,\Psi_{\text{ob}}\rangle = \frac{1}{\sqrt{2}}\left[\begin{matrix}\begin{pmatrix}\sqrt{p}\\0\end{pmatrix}\\ \begin{pmatrix}\cos\frac{\pi}{2}\mu\\0\end{pmatrix}^{(\mu)}\end{matrix}\right] + \frac{1}{\sqrt{2}}\left[\begin{matrix}\begin{pmatrix}0\\\sqrt{1-p}\end{pmatrix}\\ \begin{pmatrix}0\\\sin\frac{\pi}{2}\mu\end{pmatrix}^{(\mu)}\end{matrix}\right]$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}\mathsf{e}_0\sqrt{p}\\ \varepsilon_0\cos\frac{\pi}{2}\mu\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\mathsf{e}_1\sqrt{1-p}\\ \varepsilon_1\sin\frac{\pi}{2}\mu\end{bmatrix}, \tag{1.57}$$

where e_i and ε_i are unit vectors defined in (1.50) and (1.51).

This can be compared, e.g., with the classical-quantum approach (1.23), and Mannucci (2006) and Maron *et al* (2013), in which the elements of the columns are multiplied, while we consider them independently and separately.

1.5 Obscure-quantum measurement

Let us consider the case of one Kronecker obscure qubit register $L = 1$ (see (1.47)), or using (1.48) in the vector representation (1.52). The standard (double) orthogonal commuting projection operators, Kronecker projections, are (cf (1.36))

$$\mathbf{P}_0 = \begin{bmatrix}\mathsf{P}_0 & 0\\ 0 & \mathsf{P}_0^{(\mu)}\end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix}\mathsf{P}_1 & 0\\ 0 & \mathsf{P}_1^{(\mu)}\end{bmatrix}, \tag{1.58}$$

where 0 is the 2×2 zero matrix, and $\mathsf{P}_{0,1}^{(\mu)}$ are the projections in the membership subspace $\mathcal{V}_{\text{memb}}^{(2)}$ (of the same form as the ordinary quantum projections $\mathsf{P}_{0,1}$ (1.36))

$$\mathsf{P}_0^{(\mu)} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}^{(\mu)}, \quad \mathsf{P}_1^{(\mu)} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^{(\mu)}, \quad \mathsf{P}_0^{(\mu)}, \mathsf{P}_1^{(\mu)} \in \operatorname{End}\mathcal{V}_{\text{memb}}^{(2)}, \tag{1.59}$$

$$\mathsf{P}_0^{(\mu)2} = \mathsf{P}_0^{(\mu)}, \;\; \mathsf{P}_1^{(\mu)2} = \mathsf{P}_1^{(\mu)}, \;\; \mathsf{P}_0^{(\mu)}\mathsf{P}_1^{(\mu)} = \mathsf{P}_1^{(\mu)}\mathsf{P}_0^{(\mu)} = 0. \tag{1.60}$$

For the double projections we have (cf (1.37))

$$\mathbf{P}_0^2 = \mathbf{P}_0, \;\; \mathbf{P}_1^2 = \mathbf{P}_1, \;\; \mathbf{P}_0\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_0 = \mathbf{0}, \tag{1.61}$$

where $\mathbf{0}$ is the 4×4 zero matrix, and $\mathbf{P}_{0,1}$ act on the Kronecker qubit (1.58) in the standard way (cf (1.38))

$$\mathbf{P}_0 \mid \boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \mathsf{e}_0 \\ \alpha_0 \varepsilon_0 \end{bmatrix} = \frac{1}{\sqrt{2}} \mathsf{A}_0 \otimes_H \mathsf{E}_0, \tag{1.62}$$

$$\mathbf{P}_1 \mid \boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \mathsf{e}_1 \\ \alpha_1 \varepsilon_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \mathsf{A}_1 \otimes_H \mathsf{E}_1. \tag{1.63}$$

Observe that for Kronecker qubits there exist in addition to (1.58) the following orthogonal commuting projection operators

$$\mathbf{P}_{01} = \begin{bmatrix} \mathsf{P}_0 & 0 \\ 0 & \mathsf{P}_1^{(\mu)} \end{bmatrix}, \quad \mathbf{P}_{10} = \begin{bmatrix} \mathsf{P}_1 & 0 \\ 0 & \mathsf{P}_0^{(\mu)} \end{bmatrix}, \tag{1.64}$$

and we call these the crossed double projections. They satisfy the same relations as (1.61)

$$\mathbf{P}_{01}^2 = \mathbf{P}_{01}, \ \mathbf{P}_{10}^2 = \mathbf{P}_{10}, \ \mathbf{P}_{01}\mathbf{P}_{10} = \mathbf{P}_{10}\mathbf{P}_{01} = \mathbf{0}, \tag{1.65}$$

but act on the obscure qubit in a different (mixing) way than (1.62), i.e.,

$$\mathbf{P}_{01} \mid \boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \mathsf{e}_0 \\ \alpha_1 \varepsilon_1 \end{bmatrix}, \tag{1.66}$$

$$\mathbf{P}_{10} \mid \boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \mathsf{e}_1 \\ \alpha_0 \varepsilon_0 \end{bmatrix}. \tag{1.67}$$

The multiplication of the crossed double projections (1.64) and the double projections (1.58) is given by

$$\mathbf{P}_{01}\mathbf{P}_0 = \mathbf{P}_0\mathbf{P}_{01} = \begin{bmatrix} \mathsf{P}_0 & 0 \\ 0 & 0 \end{bmatrix} \equiv \mathbf{Q}_0, \quad \mathbf{P}_{01}\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_{01} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{P}_1^{(\mu)} \end{bmatrix} \equiv \mathbf{Q}_1^{(\mu)}, \tag{1.68}$$

$$\mathbf{P}_{10}\mathbf{P}_0 = \mathbf{P}_0\mathbf{P}_{10} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{P}_0^{(\mu)} \end{bmatrix} \equiv \mathbf{Q}_0^{(\mu)}, \quad \mathbf{P}_{10}\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_{10} = \begin{bmatrix} \mathsf{P}_1 & 0 \\ 0 & 0 \end{bmatrix} \equiv \mathbf{Q}_1, \tag{1.69}$$

where the operators $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_0^{(\mu)}$, $\mathbf{Q}_1^{(\mu)}$ satisfy

$$\mathbf{Q}_0^2 = \mathbf{Q}_0, \ \ \mathbf{Q}_1^2 = \mathbf{Q}_1, \ \ \mathbf{Q}_1\mathbf{Q}_0 = \mathbf{Q}_0\mathbf{Q}_1 = \mathbf{0}, \tag{1.70}$$

$$\mathbf{Q}_0^{(\mu)2} = \mathbf{Q}_0^{(\mu)}, \ \ \mathbf{Q}_1^{(\mu)2} = \mathbf{Q}_1^{(\mu)}, \ \ \mathbf{Q}_1^{(\mu)}\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{1.71}$$

$$\mathbf{Q}_1^{(\mu)}\mathbf{Q}_0 = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1 = \mathbf{Q}_1\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{1.72}$$

and we call these 'half Kronecker (double) projections'.

These relations imply that the process of measurement when using Kronecker obscure qubits (i.e. for quantum computation with truth or membership) is more complicated than in the standard case.

To show this, let us calculate the obscure analogs of expected values for the projections above. Using the notation

$$\bar{\mathbf{A}} \equiv \langle \mathbf{\Psi}_{\text{ob}} \mid \mathbf{A} \mid \mathbf{\Psi}_{\text{ob}} \rangle. \tag{1.73}$$

Then, using (1.43)–(1.45) for the projection operators $\mathbf{P}_i$, $\mathbf{P}_{ij}$, $\mathbf{Q}_i$, $\mathbf{Q}_i^{(\mu)}$, $i, j = 0, 1$, $i \neq j$, we obtain (cf (1.39))

$$\bar{\mathbf{P}}_i = \frac{\mid a_i \mid^2 + \alpha_i^2}{2}, \quad \bar{\mathbf{P}}_{ij} = \frac{\mid a_i \mid^2 + \alpha_j^2}{2}, \tag{1.74}$$

$$\bar{\mathbf{Q}}_i = \frac{\mid a_i \mid^2}{2}, \qquad \bar{\mathbf{Q}}_i^{(\mu)} = \frac{\alpha_i^2}{2}. \tag{1.75}$$

So follows the relation between the obscure analogs of expected values of the projections

$$\bar{\mathbf{P}}_i = \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_i^{(\mu)}, \quad \bar{\mathbf{P}}_{ij} = \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_j^{(\mu)}. \tag{1.76}$$

Taking the ket corresponding to the bra Kronecker qubit (1.52) in the form

$$\langle \mathbf{\Psi}_{\text{ob}} \mid = \frac{1}{\sqrt{2}}\left[a_0^*(1 \ \ 0), \ \alpha_0(1 \ \ 0)\right] + \frac{1}{\sqrt{2}}\left[a_1^*(0 \ \ 1), \ \alpha_1(0 \ \ 1)\right], \tag{1.77}$$

a Kronecker (4×4) obscure analog of the density matrix for a pure state is given by (cf (1.42))

$$\rho_{\text{ob}}^{(2)} = \mid \mathbf{\Psi}_{\text{ob}} \rangle \langle \mathbf{\Psi}_{\text{ob}} \mid = \frac{1}{2}\begin{pmatrix} \mid a_0 \mid^2 & a_0 a_1^* & a_0\alpha_0 & a_0\alpha_1 \\ a_1 a_0^* & \mid a_1 \mid^2 & a_1\alpha_0 & a_1\alpha_1 \\ \alpha_0 a_0^* & \alpha_0 a_1^* & \alpha_0^2 & \alpha_0\alpha_1 \\ \alpha_1 a_0^* & \alpha_1 a_1^* & \alpha_0\alpha_1 & \alpha_1^2 \end{pmatrix}. \tag{1.78}$$

If the Born rule for the membership functions (1.13) and the conditions (1.14)–(1.15) are satisfied, then the density matrix (1.78) is non-invertible because $\det \rho_{\text{ob}}^{(2)} = 0$ and has unit trace $\text{tr}\rho_{\text{ob}}^{(2)} = 1$ but is not idempotent $\left(\rho_{\text{ob}}^{(2)}\right)^2 \neq \rho_{\text{ob}}^{(2)}$ because it holds for the ordinary quantum density matrix (Nielsen and Chuang 2000).

1.6 Kronecker obscure-quantum gates

In general, (double) obscure-quantum computation with L Kronecker obscure qubits (or qudits) can be performed by a product of unitary (block) matrices $\mathbf{U}$ of the (double size to the standard one) size $2 \times (2^L \times 2^L)$ (or $2 \times (n^L \times n^L)$), $\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}$ (here $\mathbf{I}$ is the unit matrix of the same size as $\mathbf{U}$). We can also call such computation a quantum computation with truth (or with membership).

Let us consider obscure-quantum computation with one Kronecker obscure qubit. Informally, we can present the Kronecker obscure qubit (1.52) in the form

$$| \mathbf{\Psi}_{\text{ob}} \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{1.79}$$

Innovation 1.8. *The state* $| \mathbf{\Psi}_{\text{ob}} \rangle$ *can be interpreted as a vector in the direct product (not tensor product) space* $\mathcal{H}_{\text{q}}^{(2)} \times \mathcal{V}_{\text{memb}}^{(2)}$*, where* $\mathcal{H}_{\text{q}}^{(2)}$ *is the standard two-dimensional Hilbert space of the qubit, and* $\mathcal{V}_{\text{memb}}^{(2)}$ *can be treated as the membership space, which has a different nature from the qubit space and can have a more complex structure.*

For discussion of similar spaces, see for example Dubois *et al* (2000), Bělohlávek (2002), Smith (2008), and Zimmermann (2011). In general, one can consider obscure-quantum computation as a set of abstract computational rules, independently of the introduction of the corresponding spaces.

An obscure-quantum gate will be defined as an elementary transformation on an obscure qubit (1.79) and is performed by unitary (block) matrices of size 4×4 (over $\mathbb{C}$) acting in the total space $\mathcal{H}_{\text{q}}^{(2)} \times \mathcal{V}_{\text{memb}}^{(2)}$

$$\mathbf{U} = \begin{pmatrix} \mathsf{U} & 0 \\ 0 & \mathsf{U}^{(\mu)} \end{pmatrix}, \quad \mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}, \tag{1.80}$$

$$\mathsf{U}\mathsf{U}^\dagger = \mathsf{U}^\dagger \mathsf{U} = \mathsf{I}, \quad \mathsf{U}^{(\mu)}\mathsf{U}^{(\mu)\dagger} = \mathsf{U}^{(\mu)\dagger}\mathsf{U}^{(\mu)} = \mathsf{I}, \quad \mathsf{U} \in \text{End}\mathcal{H}_{\text{q}}^{(2)}, \, \mathsf{U}^{(\mu)} \in \text{End}\,\mathcal{V}_{\text{memb}}^{(2)}, \tag{1.81}$$

where $\mathbf{I}$ is the unit 4×4 matrix, I is the unit 2×2 matrix, and U and $\mathsf{U}^{(\mu)}$ are unitary 2×2 matrices acting on the probability and membership subspaces, respectively. The matrix U (over $\mathbb{C}$) will be called a quantum gate, and we call the matrix $\mathsf{U}^{(\mu)}$ (over $\mathbb{R}$) an obscure gate. We assume that the obscure gates $\mathsf{U}^{(\mu)}$ are of the same shape as the standard quantum gates, but they act in the other (membership) space and have only real elements (see, e.g. Nielsen and Chuang 2000). In this case, an obscure-quantum gate is characterized by the pair $\{\mathsf{U}, \mathsf{U}^{(\mu)}\}$, where the components are known gates (in various combinations), e.g., for one qubit gates: Hadamard, Pauli-X (`NOT`) ,Y,Z (or two qubit gates e.g. `CNOT`, `SWAP`, etc). The transformed qubit then becomes (informally)

$$\mathbf{U}\mid\boldsymbol{\Psi}_{\rm ob}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathsf{U}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\mathsf{U}^{(\mu)}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{1.82}$$

Thus, the quantum and the membership parts are transformed independently for the block diagonal form (1.80). Some examples of this can be found, e.g., in Domenech and Freytes (2006), Mannucci (2006), and Maron *et al* (2013). Differences between the parts were mentioned in Kreinovich *et al* (2011). In this case, an obscure-quantum network is physically realized by a device performing elementary operations in sequence on obscure qubits (by a product of matrices), such that the quantum and membership parts are synchronized in time; for a discussion of the obscure part of such physical devices, see Hirota and Ozawa (1989), Kóczy and Hirota (1990), Virant (2000), and Kosko (1997). Then, the result of the obscure-quantum computation consists of the quantum probabilities of the states together with the calculated level of truth for each of them (see, e.g. Bolotin 2018).

For example, the obscure-quantum gate $\mathbf{U}_{\rm H,NOT} = \{\text{Hadamard}, \texttt{NOT}\}$ acts on the state E_0 (1.55) as follows

$$\mathbf{U}_{\rm H,NOT}\mathsf{E}_0 = \mathbf{U}_{\rm H,NOT}\begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\mathsf{e}_0 + \mathsf{e}_1) \\ \varepsilon_1 \end{bmatrix}. \tag{1.83}$$

It would be interesting to consider the case when $\mathbf{U}$ (1.80) is not block diagonal and try to find possible physical interpretations of the non-diagonal blocks.

1.7 Double entanglement

Let us introduce a register consisting of two obscure qubits ($L = 2$) in the computational basis $\mid \mathrm{ij}'\rangle = \mid \mathrm{i}\rangle \otimes \mid \mathrm{j}'\rangle$, as follows

$$\left|\boldsymbol{\Psi}_{\rm ob}^{(n=2)}(L=2)\right\rangle = \mid\boldsymbol{\Psi}_{\rm ob}(2)\rangle = \frac{\mathsf{B}_{00'}\mid 00'\rangle + \mathsf{B}_{10'}\mid 10'\rangle + \mathsf{B}_{01'}\mid 01'\rangle + \mathsf{B}_{11'}\mid 11'\rangle}{\sqrt{2}}, \tag{1.84}$$

determined by two-dimensional vectors (encoding obscure-quantum amplitudes)

$$\mathsf{B}_{ij'} = \begin{bmatrix} b_{ij'} \\ \beta_{ij'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1', \tag{1.85}$$

where $b_{ij'} \in \mathbb{C}$ are probability amplitudes for a set of pure states and $\beta_{ij'} \in \mathbb{R}$ are the corresponding membership amplitudes. By analogy with (1.43) and (1.46), the normalization factor in (1.84) is chosen so that

$$\langle\boldsymbol{\Psi}_{\rm ob}(2)|\boldsymbol{\Psi}_{\rm ob}(2)\rangle = 1, \tag{1.86}$$

if (cf (1.14)–(1.15))

$$|b_{00'}|^2 + |b_{10'}|^2 + |b_{01'}|^2 + |b_{11'}|^2 = 1, \tag{1.87}$$

$$\beta_{00'}^2 + \beta_{10'}^2 + \beta_{01'}^2 + \beta_{11'}^2 = 1. \tag{1.88}$$

A state of two qubits is entangled if it cannot be decomposed as a product of two one-qubit states, and otherwise it is separable (see, e.g. Nielsen and Chuang 2000).

Innovation 1.9. *We define a product of two obscure qubits (1.43) as*

$$|\Psi_{ob}\rangle \otimes |\Psi'_{ob}\rangle = \frac{A_0 \otimes_H A'_0 |00'\rangle + A_1 \otimes_H A'_0 |10'\rangle + A_0 \otimes_H A'_1 |01'\rangle + A_1 \otimes_H A'_1 |11'\rangle}{2}, \tag{1.89}$$

where $\otimes_H$ *is the Hadamard product (1.53).*

Comparing (1.84) and (1.89), we obtain two sets of relations, for probability amplitudes and for membership amplitudes

$$b_{ij'} = \frac{1}{\sqrt{2}} a_i a_{j'}, \tag{1.90}$$

$$\beta_{ij'} = \frac{1}{\sqrt{2}} \alpha_i \alpha_{j'}, \quad i, j = 0, 1, \quad j' = 0', 1'. \tag{1.91}$$

In this case, the relations (1.14)–(1.15) give (1.87)–(1.88).

Two obscure-quantum qubits are entangled if their joint state (1.84) cannot be presented as a product of one qubit states (1.89), and in the opposite case the states are called totally separable. It follows from (1.90)–(1.91) that there are two general conditions for obscure qubits to be entangled

$$b_{00'} b_{11'} \neq b_{10'} b_{01'}, \quad \text{or } \det \mathbf{b} \neq 0, \ \mathbf{b} = \begin{pmatrix} b_{00'} & b_{01'} \\ b_{10'} & b_{11'} \end{pmatrix}, \tag{1.92}$$

$$\beta_{00'} \beta_{11'} \neq \beta_{10'} \beta_{01'}, \quad \text{or } \det \boldsymbol{\beta} \neq 0, \ \boldsymbol{\beta} = \begin{pmatrix} \beta_{00'} & \beta_{01'} \\ \beta_{10'} & \beta_{11'} \end{pmatrix}. \tag{1.93}$$

The first equation (1.92) is the entanglement relation for the standard qubit, while the second condition (1.93) is for the membership amplitudes of the two obscure qubit joint state (1.84). The presence of two different conditions (1.92)–(1.93) leads to new additional possibilities (which do not exist for ordinary qubits) for partial entanglement (or partial separability), when only one of them is fulfilled. In this case, the states can be entangled in one subspace (quantum or membership) but not in the other.

The measure of entanglement is numerically characterized by the concurrence. Taking into account the two conditions (1.92)–(1.93), we propose to generalize the notion of concurrence for two obscure qubits in two ways. First, we introduce the vector obscure concurrence

$$\mathsf{C}_{\mathrm{vect}} = \begin{bmatrix} C_{\mathrm{q}} \\ C^{(\mu)} \end{bmatrix} = 2\begin{bmatrix} |\det \mathbf{b}| \\ |\det \beta| \end{bmatrix}, \tag{1.94}$$

where $\mathbf{b}$ and β are defined in (1.92)–(1.93), and $0 \leqslant C_{\mathrm{q}} \leqslant 1$, $0 \leqslant C^{(\mu)} \leqslant 1$.

Innovation 1.10. *The corresponding scalar obscure concurrence can be defined as*

$$C_{\mathrm{scal}} = \sqrt{\frac{|\det \mathbf{b}|^2 + |\det \beta|^2}{2}}, \tag{1.95}$$

such that $0 \leqslant C_{\mathrm{scal}} \leqslant 1$. *Thus, two obscure qubits are totally separable, if* $C_{\mathrm{scal}} = 0$.

For instance, for an obscure analog of the (maximally entangled) Bell state

$$|\boldsymbol{\Psi}_{\mathrm{ob}}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|00'\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|11'\rangle\right) \tag{1.96}$$

we obtain

$$\mathsf{C}_{\mathrm{vect}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_{\mathrm{scal}} = 1. \tag{1.97}$$

A more interesting example is the intermediately entangled two obscure qubit state, e.g.,

$$|\boldsymbol{\Psi}_{\mathrm{ob}}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|00'\rangle + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{5}}{4} \end{bmatrix}|10'\rangle + \begin{bmatrix} \frac{\sqrt{3}}{4} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}|01'\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{4} \end{bmatrix}|11'\rangle\right), \tag{1.98}$$

where the amplitudes satisfy (1.87)–(1.88). If the Born-like rule (as in (1.13)) holds for the membership amplitudes, then the probabilities and membership functions of the states in (1.98) are

$$p_{00'} = \frac{1}{4}, \quad p_{10'} = \frac{1}{16}, \quad p_{01'} = \frac{3}{16}, \quad p_{11'} = \frac{1}{2}, \tag{1.99}$$

$$\mu_{00'} = \frac{1}{2}, \quad \mu_{10'} = \frac{5}{16}, \quad \mu_{01'} = \frac{1}{8}, \quad \mu_{11'} = \frac{1}{16}. \tag{1.100}$$

This means that, e.g., that the state $|10'\rangle$ will be measured with the quantum probability 1/16 and the membership function (truth value) 5/16. For the entangled obscure qubit (1.98) we obtain the concurrences

$$\mathsf{C}_{\text{vect}} = \begin{bmatrix} \frac{1}{2}\sqrt{2} - \frac{1}{8}\sqrt{3} \\ \frac{1}{8}\sqrt{2}\sqrt{5} - \frac{1}{4}\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.491 \\ 0.042 \end{bmatrix}, \tag{1.101}$$
$$C_{\text{scal}} = \sqrt{\frac{53}{128} - \frac{1}{16}\sqrt{5} - \frac{1}{16}\sqrt{2}\sqrt{3}} = 0.348.$$

In the vector representation (1.49)–(1.52), we have

$$|\, \mathsf{ij}' \rangle = |\, \mathsf{i} \rangle \otimes |\, \mathsf{j}' \rangle = \begin{bmatrix} \mathsf{e}_i \otimes_{\mathrm{K}} \mathsf{e}_{j'} \\ \varepsilon_i \otimes_{\mathrm{K}} \varepsilon_{j'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1', \tag{1.102}$$

where $\otimes_{\mathrm{K}}$ is the Kronecker product (1.48), and e_i, ε_i are defined in (1.50)–(1.51). Using (1.85) and the Kronecker-like product (1.49), we put (informally, with no summation)

$$\mathsf{B}_{ij'} \,|\, \mathsf{ij}' \rangle = \begin{bmatrix} b_{ij'}\mathsf{e}_i \otimes_{\mathrm{K}} \mathsf{e}_{j'} \\ \beta_{ij'}\varepsilon_i \otimes_{\mathrm{K}} \varepsilon_{j'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1'. \tag{1.103}$$

To clarify our model, we show here a manifest form of the two obscure qubit state (1.98) in the vector representation

$$|\, \boldsymbol{\Psi}_{\text{ob}}(2) \rangle = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \\ \frac{\sqrt{5}}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{4}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \\ \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \\ \frac{1}{4}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}^{(\mu)} \end{bmatrix} \right). \tag{1.104}$$

Innovation 1.11. *The states above may be called 'symmetric two obscure qubit states'. However, there are more general possibilities, as may be seen from the rhs of (1.103) and (1.104), when the indices of the first and second rows do not coincide. This would allow more possible states, which we call 'non-symmetric two obscure qubit states'. It would be worthwhile to establish their possible physical interpretation.*

These constructions show that quantum computing using Kronecker obscure qubits can involve a rich structure of states, giving a more detailed description with additional variables reflecting vagueness.

1.8 Conclusions

We have proposed a new scheme for describing quantum computation bringing vagueness into consideration, in which each state is characterized by a measure of truth. A membership amplitude is introduced in addition to the probability amplitude in order to achieve this, and we are led thereby to the concept of an obscure qubit. Two kinds of these are considered: the product obscure qubit, in which the total amplitude is the product of the quantum and membership amplitudes; and the Kronecker obscure qubit, where the amplitudes are manipulated separately. In the latter case, the quantum part of the computation is based, as usual, in Hilbert space, while the truth part requires a vague/fuzzy set formalism, which can be performed in the framework of a corresponding fuzzy space. Obscure-quantum computation may be considered as a set of rules (defining obscure-quantum gates) for managing quantum and membership amplitudes independently in different spaces. In this framework, we obtain not only the probabilities of final states but also their membership functions, i.e., how much trust we should assign to these probabilities. Our approach considerably extends the theory of quantum computing by adding the logic part directly to the computation process. Future challenges could lie in the direction of development of the corresponding logic hardware in parallel with the quantum devices.

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